20/06/24

Lecture-12

12/05/21

Measuring the Power Spectrum

- There is no measure we only have to Estimates
- Estimates require Samples

* Now as a designer when for example a telecom guy will design a Telecom Signal which should stay in a certain BW. so it must not spread out of that BW.

– Even when we are designing a clock Generator, we would like to add filter to the clock. so that we can spread the spectrum

We know periodic signals have Spiky Spectrum we will see it in lecture-13 we are forced to spread it because we have some interference mask that are given to us as specification and we have to stay below certain interference mask. and then we have to compute the spectrum of the signal and it should inside the mask.

* → We would also like to verify the design by measuring the Spectrum. The problem is that Power Spectrum is a statisti-cal Quantity They cannot be measured.

- i.e We can't measure. How many Avg. volts are in a wire.

- But we can measure How many volts are here at the time instant.

* Therefore, only thing we could do is estimating the power spectrum.

ex: We cannot by looking at a coin & tell it is Fair coin for tossing.

What we can do is do repeated trails of tossing the coin and see How many times we have tails (or) Heads and making a statistic of this will give mean estimate of the Average which should be equal for both sides of the Coin. Similarly we can take measurements to estimate the Power Spectrum.

→ (Q) What kind of measurements we have to take? to estimate the Power Spectrum

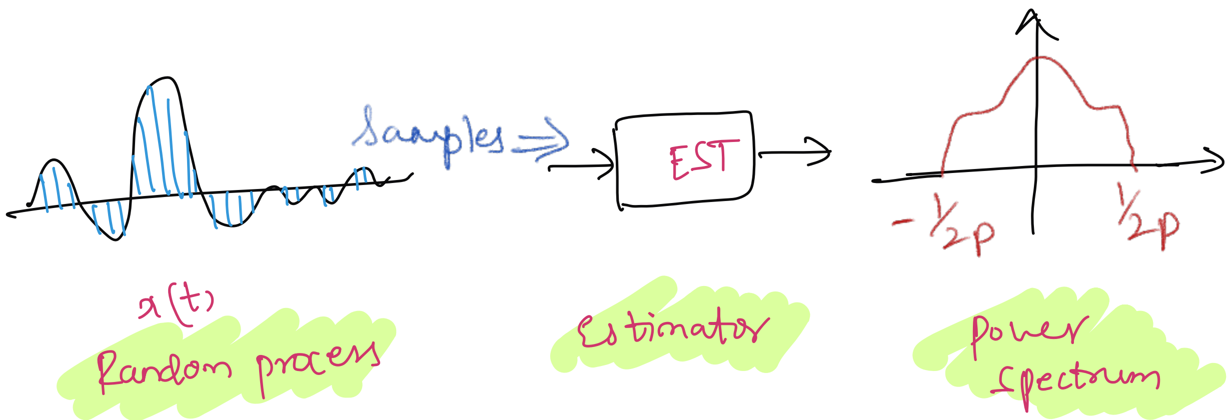
Ans The Good news is we can do it by taking samples at instances of the Random Process.

- From Sampling Theory we already know that If the samples collected follow Shannon-Nyquist criteria we can completely construct the signal.

** Similary There is Analogous Sampling theorem for Power Spectrum

- Assumption that if the Power spectrum is contained inside $\left[-\frac{1}{2p}, \frac{1}{2p}\right]$ i.e it is Band limited.

Then we can Estimate the Power Spectrum starting from samples with sampling period 'p'



→ General Estimation

$$x_0, \dots, x_{n-1} \longrightarrow \boxed{EST} \longrightarrow \hat{A}$$

estimates A

Average Squared Error $= \mathcal{E} = E\left[(\hat{A}-A)^2\right]$

$m_{\hat{A}} = E[\hat{A}]$

$$\Rightarrow E\left[\left(\hat{A} - m_{\hat{A}} + m_{\hat{A}} - A\right)^2\right]$$

$$\Rightarrow \underbrace{E\left[\left(\hat{A}-m_{\hat{A}}\right)^2\right]}_{\text{Variance}} + E\left[\left(m_{\hat{A}}-A\right)^2\right]$$

$$+ 2E\left[\left(\hat{A}-m_{\hat{A}}\right)\left(m_{\hat{A}}-A\right)\right]$$

$$2\left(m_{\hat{A}}-A\right)\underbrace{E\left[\hat{A}-m_{\hat{A}}\right]}_{=0}$$

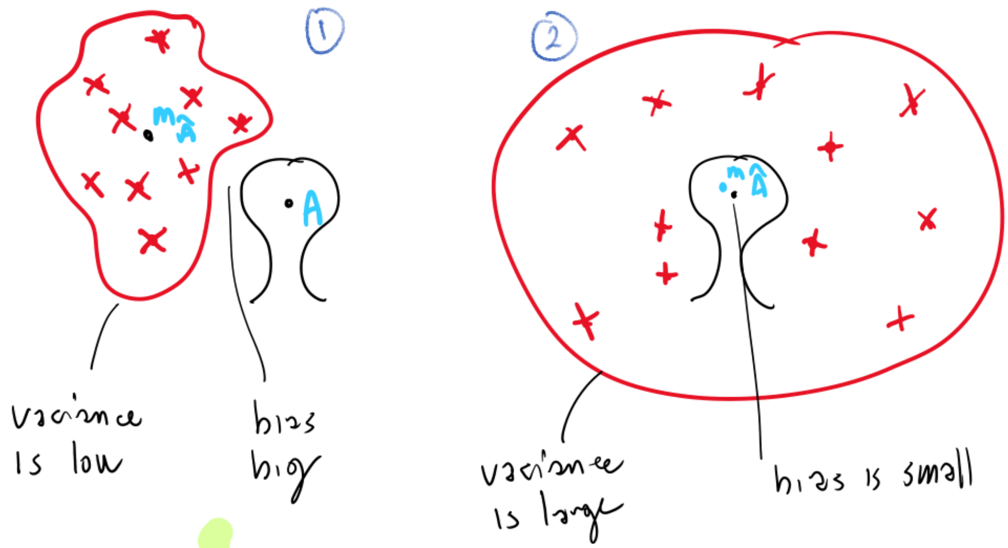
$$\therefore \boxed{\mathcal{E} = \sigma_{\hat{A}}^2 + \left(m_{\hat{A}}-A\right)^2}$$

Variance of the estimator

Bias of the estimator

Since both are squared terms inorder to reduce error we have to reduce both Variance & Bias.

In General there are two cases that are possible



Bias - Variance Dilemma , Trade-off

$$\varepsilon = (m_{\hat{A}} - A)^2 + \sigma_{\hat{A}}^2$$

$$\downarrow\uparrow \qquad \uparrow\downarrow$$

if we try and reduce Bias, the variance will increase and vice versa.

* We are going to study THREE estimators

1. Periodogram
2.

(I) Periodogram

Power spectrum for Discrete time process

$$\boxed{S_x^{w}(f) = \underset{p}{P L_{Y}}(f) \underbrace{\lim_{n \to \infty}} \frac{1}{2N+1} E\left[\left| \mathcal{F}[L_N x](f) \right|^2 \right]}$$

cannot be done here

$n \to \infty$: can't wait for eternity

E : No because we are only observing samples and can't know the probability with which they appear.

$\therefore$ for Periodigram

$$\hat{S}_x^{w}(f) = P L_{Y_p}(f) \frac{1}{2N+1} \left[\left| \mathcal{F}(L_N x(t))(f) \right|^2 \right]$$

This is just an estimation

it is biased

$$\boxed{m_{\hat{A}} \neq A}$$

i.e $E\left[\hat{S}_x^{w}(f) \right] \neq S_x^{w}(f)$

It has large Variance

$$\boxed{\sigma^2_{\hat{S}_x^{w}(f)} \neq 0}$$

→ The only Good news is if we reintroduce the limit i.e $\lim_{N\to\infty} \hat{m}_A - A = 0$

i.e if we take more samples the Bias will go to zero. It's Asymptotically Unbiased estimator

- But the variance will not go to zero even we increase the no. of samples.

$$\lim_{N\to\infty} \sigma^2_{\hat{S}^w_x(f)} \neq 0$$

inconsistent estimator.

∴ Periodigram is Asymptotically Unbiased and Inconsistent.

→ Computation of the bias for stationary process

We are computing for stationary process because by Wiener Kinchiv Theorem for stationary process

Average Correlation = Correlation.

∴ TRUE POWER SPECTRUM for stationary process

$$S^w_x(f) = P\, L_{1/p}(f)\ \mathcal{F}[C_x(\tau)]$$

where $C_x(\tau) = R_x(0,\tau) = E\left[x^*(t)\, x(t+\tau) \right]$

$$\Rightarrow P\, L_{1/p}(f) \sum_{\tau=-\infty}^{+\infty} C_x(\tau) e^{-2\pi i p \tau f}$$

We have to compare the TRUE SPECTRUM with expected value.

Average Estimation:

$$m_{\hat{S}_x^{aw}(f)} = E\left[\hat{S}_x^{aw}(f) \right]$$

$$\Rightarrow P\, L_{1/p}(f)\; E\left[\frac{1}{2N+1} \left| \mathcal{F}(L_N x](f) \right|^2 \right]$$

$$\Rightarrow P L_{1/p}(f)\, E\left[\frac{1}{2N+1} \left| \sum_{t=-N}^{N} x(t)\, e^{-2\pi i p t f} \right|^2 \right]$$

$$\Rightarrow P L_{1/p}(f)\, E\left[\frac{1}{2N+1} \sum_{t=-N}^{N} \sum_{s=-N}^{N} x^*(t)\, x(s)\, e^{-2\pi i p s f} e^{+2\pi i p t f} \right]$$

$$\Rightarrow P\, L_{1/p}(f)\, E\left[\frac{1}{2N+1} \sum_{t=-N}^{N} \sum_{s=-N}^{N} x^*(t)\, x(s)\, e^{-2\pi i p f (s-t)} \right]$$

$$\Rightarrow P L_{\gamma_p}(f) \; \frac{1}{2N+1} \sum_{t=-N}^{N} \sum_{s=-N}^{N} \underbrace{E\left[x^*(t)\,x(s)\right]} e^{-2\pi i p f \overbrace{(s-t)}^{}_{\text{'d'}}}$$

$\because$ The process is stationary The expectation is Invariant any Translation in time

$$C_x(\tau) = R_x(0,\tau) = E\left[x^*(t)\,x(t+\tau)\right]$$

Similary

$$C(s-t) = R_x(0, s-t)$$

$$C(d) = E\left[x^*(0)\,x(s-t)\right]$$

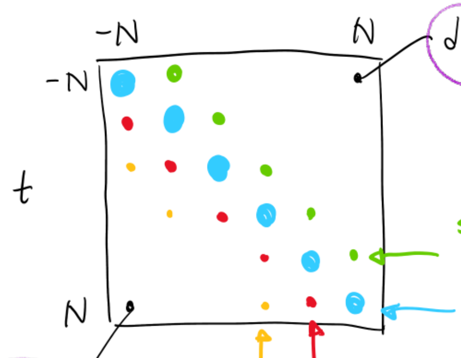
$$= E\left[x^*(t)\,x(s)\right]$$

$$C_x(\underbrace{s-t}_{d}) = R_x(0, s-t)$$

$$= E\left[x^*(0)\,x(s-t)\right]$$

$$= E\left[x^*(t)\,x(s)\right]$$

double sum analysis :



d = 2N maximum value of s-t

d = 1
s-t = 1

s-t = 0
d = 0

minimum value of s-t

d = -2N

s-t = -2
d = -2

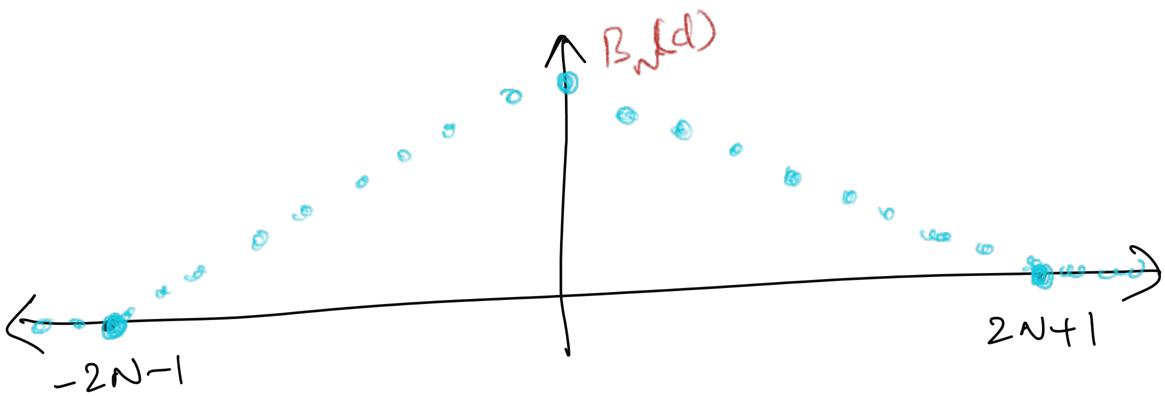
s-t = -1
d = -

From above figure it would be easy to execute the double sum if we sum the elements along the Diagonal.

$$\boxed{\hat{S}_x^{m\omega}(f) \Rightarrow P\, L_{1/p}(f)\, \frac{1}{2N+1} \sum_{d=-2N}^{2N} C_x(d)\, e^{-2\pi i p f d}\, \underbrace{(2N+1-|d|)}_{\text{No. of Boxes along the diagonal}}}$$

$$\boxed{\Rightarrow P\, L_{1/p}(f) \sum_{d=-2N}^{2N} C_x(d) \underbrace{\left(1-\frac{|d|}{2N+1}\right)}_{B_N(d)} e^{-2\pi i p f d}}$$

$B_N(d)$ is the New function we define.



$$\Rightarrow \hat{S}^{\omega}_{x}(f) \; p\, L_{1/p}(f) \sum_{d=-\infty}^{\infty} C_x(d)\, B_N(d)\, e^{-2\pi i p f d}$$

We extended the limits because outside $2N'$ $B_N(d) = 0$ $\therefore$ above expression is valid

to compare with

$$\Rightarrow p\, L_{1/p}(f) \sum_{t=-\infty}^{\infty} C_x(t)\, e^{-2\pi i p f t} \; S^{\omega}_{x}(f)$$

In the function $B_N(d)$ if $N \to \infty$

$$\left(1 - \frac{|d|}{2N+1}\right) \Rightarrow 1$$

(in any finite interval)

$\therefore$ if we increase the no. of samples

Bias will decrease.

$\rightarrow$ "Trick" to reduce the Variance of an Estimation.

$$A \longrightarrow \hat{A}_J \qquad J = 0, \ldots \; n-1$$

i.e Multiple Independent Estimators.

Assumptions $E[\hat{A}_J] = m_{\hat{A}} \quad \forall J$

$$E\left[(\hat{A}_J - m_{\hat{A}})^2\right] = \sigma_{\hat{A}}^2 \quad \forall J$$

all estimators have same Average & variance.

$$\boxed{\hat{A} = \frac{1}{n}\sum_{j=0}^{n-1}\hat{A}_J}$$ a New Estimator

$$E[\hat{A}] = \frac{1}{n}\sum_{j=0}^{n-1}\underbrace{E[\hat{A}_J]}_{m_{\hat{A}}} = m_{\hat{A}}$$

* → But when it comes to variance

$$E\left[(\hat{A} - m_{\hat{A}})^2\right] = E\left[\left(\frac{1}{n}\sum_{j=0}^{n-1}\hat{A}_J - \frac{1}{n}\sum_{j=0}^{n-1}m_{\hat{A}}\right)^2\right]$$

$$\Rightarrow E\left[\left(\frac{1}{n}\sum_{j=0}^{n-1}(\hat{A}_J - m_{\hat{A}})\right)^2\right]$$

$$\Rightarrow \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} E\left[(\hat{A}_j-\mu_{\hat{A}})(\hat{A}_k-\mu_{\hat{A}})\right]$$

$\because$ we assume two estimators are independent $\therefore$ Their Covariance must be zero

i.e

$$\boxed{J \neq K \quad E\left[(\hat{A}_j-\mu_{\hat{A}})(\hat{A}_k-\mu_{\hat{A}})\right] = 0}$$

$$\boxed{J = K \quad \underbrace{E\left[(\hat{A}_j-\mu_{\hat{A}})(\hat{A}_j-\mu_{\hat{A}})\right]}_{= \sigma_{\hat{A}}^2}}$$

$$\boxed{E\left[(\hat{A}-\mu_{\hat{A}})^2\right] \Rightarrow \frac{1}{n^2}\sum_{j=0}^{n-1}\sigma_A^2 \Rightarrow \frac{n\sigma_A^2}{n^2} \Rightarrow \frac{\sigma_A^2}{n}}$$

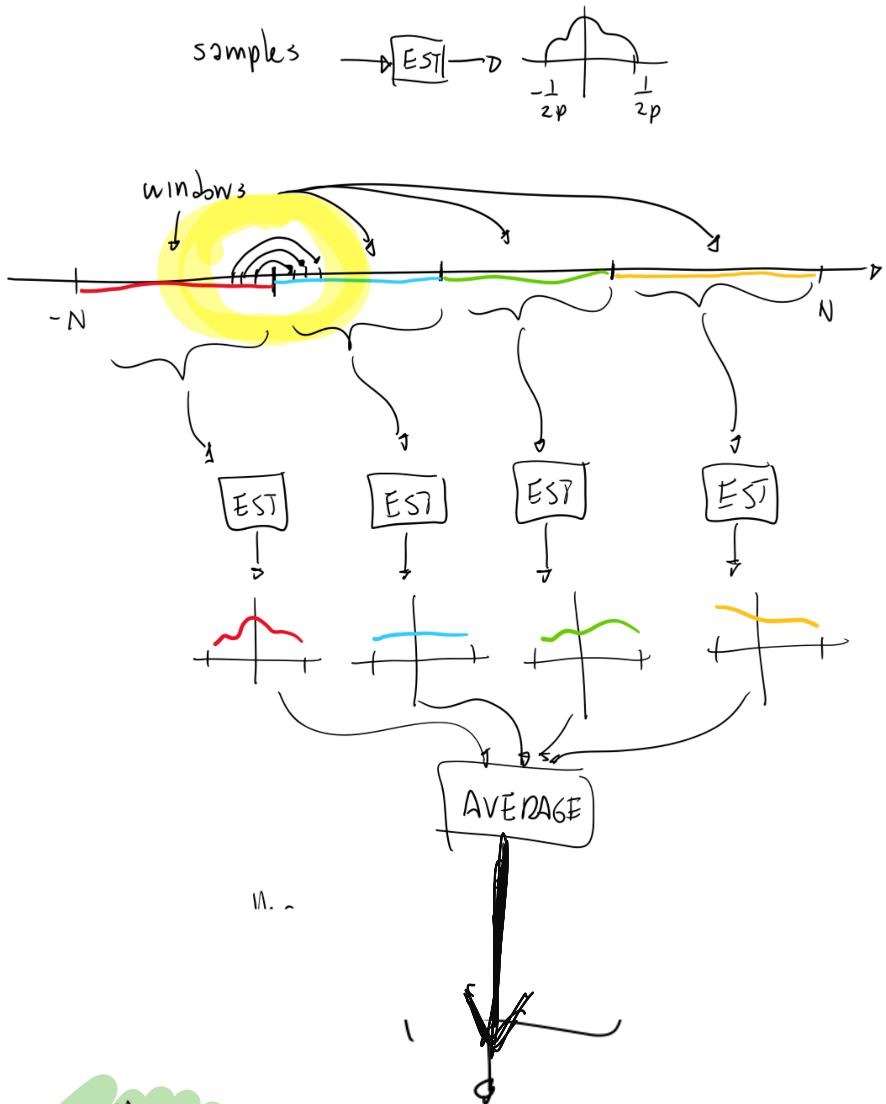
Here variance can be reduced by dividing how many estimations can be made.

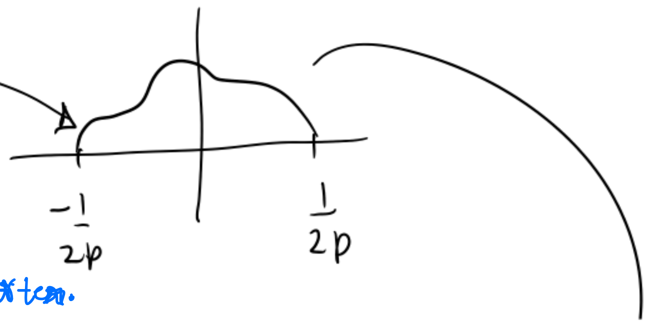
→ We can split the No. of Samples into Windows and do seperate Estimation for each window.

Regardless of no. of samples we will get an Estimation

However no. of samples ↑ estimation ↑

Application to periodogram





smaller variance

∵ we used '4' different Estimators even Though samples per estimator were only Quarter.

** But if the Estimators are Not independent i.e if the samples of 1^{st} window are correlated with 2^{nd} window as shown in the figure.

→ Q) How to avoid this problem?

Ans Mixing

If we describe a RP in a window and another window. and if we make the two windows apart the dependence b/w two windows disappears.

This is the Gist

→ *

* problem #1 : windows are independent

variance of compound estimator

Samples are not fully Independent

$$\frac{\sigma^2_{\hat{A}}}{n}$$

variance of each separate each separate estimator

number of windows

non independent windows $= \dfrac{\sigma^2_{\hat{A}}}{n^{\alpha}}$ $\alpha < 1$

This occurs due to dependency b/w various samples

∴ for NonIndependent window variance $= \dfrac{\sigma_{\hat{A}}^2}{n^\alpha}$ $\quad \alpha < 1$

Here ∵ $\alpha < 1$ we would like to increase the no. of windows so that variance decreases.

window overlap → decrease α
→ increase n

usually overlapping windows are also considered and the balance b/w α, n is achieved

* problem #2

N samples in total

W windows

no overlap (for example)

$\Downarrow$

each estimation uses $\frac{N}{W}$ samples

$W\uparrow \rightarrow \sigma^2 \downarrow$

$W\uparrow \rightarrow \frac{N}{W} \downarrow$ each estimation has less sample

$\rightarrow$ bias $\uparrow$

* Since we are windowing Bias will increase. So is there any way possible to put this Bias error in a feature of the spectrum that we don't care about. That would be clever.

→ * Modified Periodogram

BIAS – VARIANCE DILEMMA

"Escape" from the dilemma

modified periodogram

has windows → bias↑ → put the bias where it does not harm

periodogram

$$\hat{S}_x^w(f) = pL_{\frac{1}{p}}(f)\frac{1}{2N+1}\left|\sum_{t=-N}^{N} x(t)e^{-2\pi i t p f}\right|^2$$

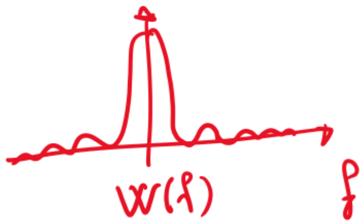
what if we could multiply each sample by w_t be the transformation.

→ * modified periodogram

$$\hat{S}_x^w(f) = p\, L_{\frac{1}{p}}(f) \frac{1}{2N+1} \left| \sum_{t=-N}^{N} w_t\, x(t)\, e^{-2\pi i t p f} \right|^2$$

apodization function

given w_t →



$$\boxed{E[\hat{S}_x^w(f)] = S_x^w(f) * W(f)}$$

no bias $\Longleftrightarrow$ $W(f) = \delta(f)$

Multiplication in time domain = Convolution in Frequency Domain

Fourier Transform