03/05/24

Lecture-10

28/04/21

Gaussian Vectors & Processes

→

Gaussian $\begin{cases} \text{Random vectors} \\ \text{Stochastic Process} \end{cases}$

$x \in \mathbb{R}^n$ 'x' is a Gaussian vector with 'n' components and all Real RVs

$$\boxed{f_x(\xi) = \frac{1}{\sqrt{(2\pi)^n \det K_x}} \, e^{-\frac{1}{2}\underbrace{(\xi - m_x)^t K_x^{-1} (\xi - m_x)}_{\text{Quadratic Form}}}}$$

Formula that defines the PDF of a Gaussian Vector

- The Formula for PDF contains two main ingredients

  (i) a vector m_x → vector of Averages

  (ii) a Covariance Matrix K_x

* $m_x = E[x] = E\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} E[x_0] \\ E[x_1] \\ \vdots \\ E[x_{n-1}] \end{bmatrix} = \begin{bmatrix} m_{x_0} \\ m_{x_1} \\ \vdots \\ m_{x_{n-1}} \end{bmatrix}$

$$K_x = E\left[(x-m_x)(x-m_x)^t\right]$$ Covariance Matrix

$$\Rightarrow E\left[\begin{bmatrix} x_0-m_{x_0}\\ x_1-m_{x_1}\\ \vdots \\ x_{n-1}-m_{x_{n-1}}\end{bmatrix}\begin{bmatrix} x_0-m_{x_0} & \cdots & x_{n-1}-m_{x_{n-1}}\end{bmatrix}\right]$$

$$\Downarrow$$

$$\Rightarrow \begin{bmatrix} \overbrace{E\left[(x_0-m_{x_0})(x_0-m_{x_0})\right]}^{\sigma_{x_0}^2} & & & \\ \underbrace{E\left[(x_1-m_{x_1})(x_0-m_{x_0})\right]}_{\gamma_{x_1 x_0}} & \sigma_{x_1}^2 & & \\ & & \ddots & \sigma_{x_{n-1}}^2 \\ \underbrace{E\left[(x_{n-1}-m_{x_{n-1}})(x_0-m_{x_0})\right]}_{\gamma_{x_{n-1} x_0}} & & & E\left[(x_{n-1}-m_{x_{n-1}})(x_{n-1}-m_{x_{n-1}})\right] \end{bmatrix}$$

- The Diagonal entries have same RVs, so on the diagonal we have set of Variances $\sigma_{x_0}^2, \sigma_{x_1}^2, \ldots, \sigma_{x_{n-1}}^2$

- Out of the Diagnal we get all possible combinations of Expectation to two different RVs

→ Now, that we know about two main ingredients in the PDF of the Gaussian vector $x \in \mathbb{R}^n$

- We want the K_x to be Non Singular because we need it in the Inverse form.

- Importantly, We have a Quadratic form as the power of the exponential 'e'

$$\text{Quadratic Form: } \underbrace{(y-m_x)^t}_{\text{vector}} \underbrace{K_x^{-1}}_{\text{Matrix}} \underbrace{(y-m_x)}_{\text{vector}}$$

$$\boxed{f_x(y) = \frac{1}{\sqrt{(2\pi)^n \det K_x}} e^{-\frac{1}{2}(y-m_x)^t K_x^{-1}(y-m_x)}}$$

it is a constant, it is needed to be sure that the Integral of $f_x(y)$ over all the $\mathbb{R}^n$ is equal to ONE.

Remember the above Formula!

From the Formula we are able derive all the Properties of Gaussian Vectors

→ $z \in \mathbb{C}^n$ $\quad z = x + iy \quad \underbrace{x \in \mathbb{R}^n \quad y \in \mathbb{R}^n}$

complex

f_{xy} → 'x' is Gaussian
→ 'y' is Gaussian

Extension of above definition of PDF of a Gaussian Vector to Complex Domain.

$z = x + iy$ $\quad$ 'x', 'y' } Gaussian Random vector of Real & Imaginary part

The equation we would get would be

$$f_z(z) = \frac{1}{\sqrt{\pi^{2n} \det\begin{pmatrix} K_z & J_z \\ J_z^{*} & K_z^{*} \end{pmatrix}}} \; e^{-\frac{1}{2}\begin{bmatrix} z - m_z \\ z^* - m_z^* \end{bmatrix}^{+} \begin{bmatrix} K_z & J_z \\ J_z^{+} & K_z^{+} \end{bmatrix}^{-1} \begin{bmatrix} z - m_z \\ z^* - m_z^* \end{bmatrix}}$$

Composed matrix

No need to remember, because we can simply it.

* addition to K_z covariance Matrix we have

J_z

$$m_z = E[z]$$

$$K_z = E\left[(z - m_z)(z - m_z)^{+}\right]$$

Hermitian i.e conjugate + Transpose

$$J_z = E\left[(z-m_z)(z-m_z)^t\right]$$ → only Transposition

Psuedo covariance Matrix

→ We want to get rid of J_z ∴ If we limit ourselves to Gaussian complex (Circular) RVs then $\boxed{J_z = 0}$

Lucky point of the above trick is, most of the Gaussian vectors fall into the above case.

∴ for $J_z = 0$ in the above generalized formula

$$\boxed{f_z(z) = \frac{1}{\pi^n \det K_z} e^{-(z-m_z)^\dagger K_z^{-1}(z-m_z)}}$$

This structure is similar to the Real RV Gaussian PDF formula.

→ The meaning of $J_z = 0$

$$J_z = E\left[(z-m_z)(z-m_z)^t\right]$$

$z = x+iy$

⇓

The expectation is linear ⟹ $\boxed{m_z = m_x + i m_y}$

$$0 = J_z = E\left[\left[(x-m_x)+i(y-m_y)\right]\left[(x-m_x)+i(y-m_y)\right]^t\right]$$

$$\Rightarrow E\left[(x-m_x)(x-m_x)^t\right] + iE\left[(y-m_y)(x-m_x)^t\right]$$

$$\underbrace{\quad}_{K_x} \quad + i\, \underbrace{E\left[(x-m_x)(y-m_y)^t\right]}_{K_{xy},\ K_{xy}^t}$$

$$- \underbrace{E\left[(y-m_y)(y-m_y)^t\right]}_{K_y}$$

$$\Rightarrow \underbrace{K_x - K_y}_{\text{Real}} + i\, \underbrace{(K_{xy} + K_{xy}^t)}_{\text{Imaginary}} = 0$$

$$K_x - K_y = 0 \qquad K_{xy} + K_{xy}^t = 0$$

$$\Rightarrow \boxed{K_x = K_y} \qquad \boxed{K_{xy} = -K_{xy}^t}$$

covariances of 'x', 'y' are similar

These two quantities have opposite sign

$$\underbrace{\begin{cases} (K_{xy})_{JK} = E\left[(x_J - m_{x_J})(y_K - m_{y_K})\right] \\ (K_{xy}^t)_{JK} = E\left[(x_K - m_{x_K})(y_J - m_{y_J})\right] \end{cases}}$$

cross covariances of 'x', 'y' are antisymmetric

* Properties of Gaussian Variables

→ For Gaussian circular and Real Gaussian vectors which we encounter 99% of the time we will discuss many properties

ex: $\mathbb{R}^2$

$x \in \mathbb{R}^2 \qquad x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \qquad \gamma_{x_0 x_1} = 0$ Zero Covariance

(I) Zero Covariance:

For Jointly Gaussian RVs which are entries of a Gaussian vector being Independent and Having Zero Covariance is the same

Note: • Covariance is an index of predictability when we restrict ourselves to Linear prediction.

i.e if $\gamma_{x_0 x_1} = 0$ then we can't make predictions using Linear predictors

• When we discussed Independence it was unpredictability by any means. (i.e with Non Linear Filters) also.

$$\therefore \quad f_x(\xi) = \frac{1}{\sqrt{(2\pi)^2 \det K_x}} \, e^{-\frac{1}{2}(\xi - m_x)^T K_x^{-1} (\xi - m_x)}$$

$$K_x = \begin{bmatrix} \sigma_{x_0}^2 & 0 \\ 0 & \sigma_{x_1}^2 \end{bmatrix}$$

We will restrict to Formula of Real RV PDF because both complex & Real formulas are almost the same!

i.e if x_0, x_1 are independent we can seperate the Joint PDF into two seperate Individual PDF

$$m_x = \begin{bmatrix} m_{x_0} \\ m_{x_1} \end{bmatrix} \qquad K_x = \begin{bmatrix} \sigma_{x_0}^2 & 0 \\ 0 & \sigma_{x_1}^2 \end{bmatrix}$$

$$f_{x_0 x_1}(\xi_0, \xi_1) = \frac{1}{\sqrt{(2\pi)^2 \sigma_{x_0}^2 \sigma_{x_1}^2}} e^{-\frac{1}{2} \begin{bmatrix} \xi_0 - m_{x_0} \\ \xi_1 - m_{x_1} \end{bmatrix}^t \begin{bmatrix} \frac{1}{\sigma_{x_0}^2} & 0 \\ 0 & \frac{1}{\sigma_{x_1}^2} \end{bmatrix} \begin{bmatrix} \xi_0 - m_{x_0} \\ \xi_1 - m_{x_1} \end{bmatrix}}$$

$$\Downarrow$$

$$f_{x_0 x_1}(\xi_0, \xi_1) = \frac{1}{\sqrt{(2\pi)^2 \sigma_{x_0}^2 \sigma_{x_1}^2}} e^{-\frac{1}{2} \underbrace{\begin{bmatrix} \xi_0 - m_{x_0} \\ \xi_1 - m_{x_1} \end{bmatrix}^t \begin{bmatrix} \frac{1}{\sigma_{x_0}^2} & 0 \\ 0 & \frac{1}{\sigma_{x_1}^2} \end{bmatrix} \begin{bmatrix} \xi_0 - m_{x_0} \\ \xi_1 - m_{x_1} \end{bmatrix}}}$$

$$(\xi_0 - m_{x_0}) \frac{1}{\sigma_{x_0}^2} (\xi_0 - m_{x_0}) +$$

$$(\xi_1 - m_{x_1}) \frac{1}{\sigma_{x_1}^2} (\xi_1 - m_{x_1})$$

$$= \underbrace{\frac{1}{\sqrt{2\pi \sigma_{x_0}^2}} e^{-\frac{1}{2} \frac{(\xi_0 - m_{x_0})^2}{\sigma_{x_0}^2}}}_{f_{x_0}(\xi_0)} \underbrace{\frac{1}{\sqrt{2\pi \sigma_{x_1}^2}} e^{-\frac{1}{2} \frac{(\xi_1 - m_{x_1})^2}{\sigma_{x_1}^2}}}_{f_{x_1}(\xi_1)} = f_{x_0 x_1}(\xi_0, \xi_1)$$

→ We know

for RV 'α' $\quad \psi_\alpha(\xi) = E\left[e^{2\pi i \alpha \xi}\right]$

$$\Rightarrow \mathcal{F}^{-1}[f_\alpha](\xi)$$

$$= \mathcal{F}[f_\alpha](-\xi)$$

Similarly

R.V 'α' $\in \mathbb{R}^n$
$\in \mathbb{C}^n$

$$\psi_\alpha(\xi) = E\left[e^{2\pi i \xi^t \alpha}\right]$$

This becomes '+' transpose conjugate if $\alpha \in \mathbb{C}^n$

$$= \mathcal{F}^{-1}[f_\alpha](\xi)$$

$$= \mathcal{F}[f_\alpha](-\xi)$$

→ Therefore if we begin from

$$f_x(\xi) = \frac{1}{\sqrt{(2\pi)^n \det K_x}} e^{-\frac{1}{2}(\xi - m_x)^t K_x^{-1} (\xi - m_x)}$$

PDF

$\Updownarrow$

MGF

$$\psi_x(\omega) = e^{-2\pi^2 \omega^t K_x \omega + 2\pi i m_x^t \omega}$$

vector variable of MGF of RV 'x'

Quadratic form

phase factor

* Both PDF & MGF have same structure i.e both are exponential with a Quadratic form

All the properties we derive from now on depend on the above results

Note: The corresponding b/w PDF & MGF is the basis for Pauli's exclusion principle in QM.

→ It is a Great property of Gaussian RV's that they have same structure in the both F.T & non transformed domain.

→ Gaussian & Linearity

$x \in \mathbb{R}^n$ is Gaussian

'A' any Matrix

$w = Ax$ is also Gaussian

Gaussian is preserved by linear Transformation

∴ The Fourier Transform on Gaussian variable preserves Gaussianity

i.e $f(x) \longrightarrow F(\omega)$

Both are Gaussian

To prove that 'W' is Gaussian variable, we will prove that using MGF of 'W' is also Gaussian.

$$\psi_W(\omega) = E\left[e^{2\pi i \omega^t W}\right] = E\left[e^{2\pi i \omega^t A x}\right]$$

$$= E\left[e^{2\pi i (A^t \omega)^t x}\right]$$

This is the definition of MGF of 'x' with T/F variable $A^t\omega$

$$\Rightarrow \psi_x(A^t\omega)$$

$$\Rightarrow e^{-2\pi^2 (A^t\omega)^t K_x (A^t\omega) + 2\pi i\, m_x^t (A^t\omega)}$$

using above formula

$$\Rightarrow e^{-2\pi^2 \omega^t \underbrace{A K_x A^t}_{?} \omega + 2\pi i\, (\underbrace{A m_x}_{?})^t \omega}$$

$$\Rightarrow e^{-2\pi^2 \omega^t K_\omega \omega + 2\pi i m_\omega^t \omega}$$

Covariance Matrix of 'w'

expectation of w

- $m_\omega = E[\omega] = E[Ax] = A E[x] = A m_x$

- $K_\omega = E\left[(\omega - m_\omega)(\omega - m_\omega)^t\right]$

$$\Rightarrow E\left[(Ax - Am_x)(Ax - Am_x)^t\right]$$

$$\Rightarrow A\, E\left[\underbrace{(x - m_x)(x - m_x)^t}_{K_x}\right] A^t$$

$$\boxed{K_\omega \Rightarrow A\, K_x A^t}$$

In this case we can write the PDF of 'w' directly if we know the K_x & m_x of the input

$\because$ $\omega = Ax$

$\rightarrow$ The Reverse is also TRUE! with an assumption

let $A = a^t = [a_0 \;.\,.\,.\,.\,.\; a_{n-1}]$

$\forall a \quad y = a^t x$ is Gaussian

Here the o/p 'y' is Gaussian, using that from that we can infer 'x' is also a Gaussian variable.

Consider

$$E\left[e^{i2\pi \underbrace{a^t x}_{y}}\right] = \varphi_x(a) \quad \text{--- (1)}$$

This can also be written as

$$E\left[e^{i2\pi y 1}\right] \Rightarrow E\left[e^{i2\pi 1 y}\right] = \varphi_y(1)$$

Here we deliberately introduced '1' to get the MGF of 'y' at '1'. We do it because we know 'y' is Gaussian

$$\varphi_y(1) = e^{-2\pi^2 1^t \sigma_y^2 1 + 2\pi i m_y^t 1}$$

$$\Rightarrow e^{-2\pi^2 1 \sigma_y^2 1 + 2\pi i m_y^t 1}$$

$$\varphi_y(1) \Rightarrow e^{-2\pi^2 \sigma_y^2 + 2\pi i m_y^t}$$

Now since we have only scalar 'y' $\therefore$ in the place of covariance matrix we have the variance 'y'

$\because y = a^t x$

$$m_y = E[y] = E[a^t x] = a^t E[x] = a^t m_x$$

$$\sigma_y^2 = E[(y-m_y)^2] = E[(a^t x - a^t m_x)(a^t x - a^t m_x)]$$

collect & Transpose

$$\Rightarrow a^t E[(x-m_x)(x-m_x)^t]\, a$$

$$\Rightarrow a^t K_x a$$

$$\boxed{\begin{aligned} \psi_x(a) &= E\left[e^{i 2\pi a^t x}\right] = e^{-2\pi^2 1 \sigma_y^2 1 + 2\pi i m_y^t} \\ &\text{substituting } \sigma_y^2 \text{ & } m_y \text{ from above} \\ &\Rightarrow e^{-2\pi^2 a^t K_x a + 2\pi i m_x^t a} \end{aligned}}$$

$\Downarrow$

This has the structure of Gaussian MGF ∴ 'x' is Gaussian

& is true for any value of 'a'

∴ Zero covariance b/w Gaussian RV's makes them Independent of each other.

(II) Gaussian & Higher order Correlations

$$E\left[x_0^{r_0} x_1^{r_1} \cdots x_{n-1}^{r_{n-1}}\right]$$

Correlation of Higher order

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \in \mathbb{R}^n$$

is Gaussian

& $r_0, r_1, \ldots, r_{n-1}$ are integers ≥ 0

ex: Higher correlations are used to characterize Interferences etc.

→ We want to calculate the expectation of every component of Gaussian vector raised to a power with non negative integers.

$$\text{Expectation} = \int \cdots \int \boxed{}\; f_{x_0, \ldots, x_{n-1}}(\;\;)\, dx_0 \ldots dx_{n-1}$$

The function that we would get.

PDF of the Gaussian vector

• Since, the PDF of Gaussian vector depends only on m_x & K_x from the expression we have to remember till we die

$\therefore$ The Formula to define Higher order correlation of elements of Gaussian vector depends on m_x & K_x

If $m_x = 0$, then the final expression contains only the elements of "K_x"

Recipe to calculate the formula for Higher order Correlations: with $M_x = 0$

• If $R = \sum_{j=0}^{n-1} r_j$, 'R' is ODD then $E[\ ] = 0$

(The sum of exponents)

• If 'R' is Even $x_0^{r_0} x_1^{r_1} \cdots x_{n-1}^{r_{n-1}}$

define

$$\underbrace{(x_0 x_0 x_0 \cdots x_0)}_{r_0 \text{ times}} \underbrace{(x_1 x_1 \cdots x_1)}_{r_1 \text{ times}} \cdots \cdots \underbrace{(x_{n-1} x_{n-1} \cdots x_{n-1})}_{r_{n-1}}$$

Define: $q : \{0, \dots, R-1\} \longrightarrow \{0, \dots, n-1\}$

$$\text{S.t} \quad \prod_{j=0}^{n-1} x_j^{r_j} = \prod_{j=0}^{R-1} x_{q(j)}$$

define $q : \{0, \dots, R-1\} \to \{0, \dots, n-1\}$

such that $\prod_{j=0}^{n-1} x_j^{r_j} = \prod_{j=0}^{R-1} x_{q(j)}$

$x_0^2 x_1^3 x_2$ $\quad n=3 \quad R=6 \quad q : \{0, \dots, 5\} \to \{0, \dots, 2\}$

j	$q(j)$	
0	0	x_0
1	0	x_0
2	1	x_1
3	1	x_1
4	1	x_1
5	2	x_1

$= x_0^2 x_1^3 x_2$

2	1	x_1
3	1	x_1
4	1	x_1
5	2	x_2

$= x_0\, x_1\, x_2$

Permutations: $p: \{0, \dots R-1\} \to \{0, \dots, R-1\}$ bijection #permutations $= R!$

class of permutation $\mathbb{P}$ contains permutation satisfying

property 1) $p(2k+1) > p(2k) \quad \forall k$

property 2) $p(2k+2) > p(2k) \quad \forall k$

k	$p(k)$
0	1
1	2
2	4
3	5
4	3
5	0

violates property 2

violates property 1

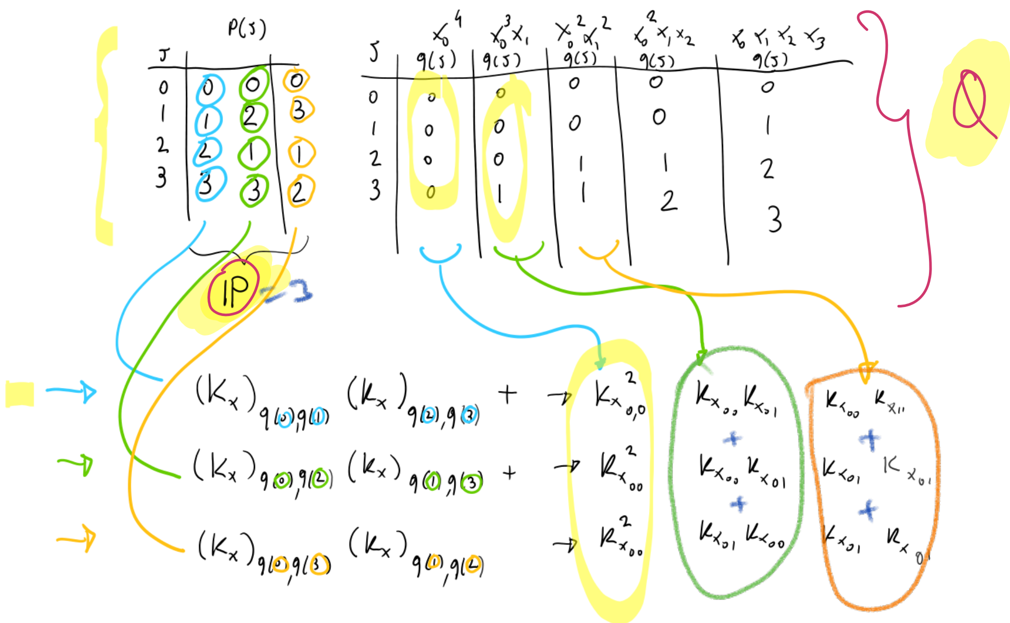
$$E\left[x_0^{r_0} x_1^{r_1} \cdots x_{n-1}^{r_{n-1}}\right] = \sum_{p \in \mathbb{P}}' \prod_{j=0}^{\frac{R}{2}-1} (K_x)_{g(p(2j)),\, g(p(2j+1))}$$

$R = \sum_{j=0}^{n-1} r_j$ is even

Example $R = 4$

- $E[x_0^4] = 3\,(K_x)_{0,0}^2 \leftarrow 3\left(\sigma_{x_0}^2\right)^2$
- $E[x_0^3 x_1^1] = 3\,(K_x)_{0,0}(K_x)_{0,1} \leftarrow 3\,\sigma_{x_0}^2\,\gamma_{x_0 x_1}$
- $E[x_0^2 x_1^2] = (K_x)_{00}(K_x)_{11} + 2(K_x)_{01}^2 \leftarrow \sigma_{x_0}^2 \sigma_{x_1}^2 + 2\gamma_{x_0 x_1}^2$
- $E[x_0^2 x_1 x_2] = (K_x)_{00}(K_x)_{11} + 2\,(K_x)_{01}(K_x)_{02}$
- $E[x_0 x_1 x_2 x_3] = (K_x)_{01}(K_x)_{23} + (K_x)_{02}(K_x)_{13} + (K_x)_{03}(K_x)_{12}$

check on your own



* White Gaussian Process, Power Spectrum *

Definition of Gaussian Stochastic Process:

- We know how to characterize "Stochastic Process". There are Three ways.

  1) Joint probability characterization

  2) Correlation & Covariance

  3) Projections (by far the easiest of all)

- From Projection characterization we can derive PDF's & Correlations

- So, we are considering processes like "collection of waveforms" & "Probability attached to each of the waveforms"

- We know that these waveforms form a Vectorspace and in that vector space we take "projections along special waveforms that we have choosen" and are deterministic.

→ Gaussian Stochastic process $x(t)$

$x(t)$ iff $\forall \phi \in L_2$ i.e

is a Gaussian stochastic process

$$\int |\phi(t)|^2 < \infty$$

Finite Energy

$$\boxed{P = \int \phi(t) x(t) \, dt}$$

projection of $x(t)$ along $\phi(t)$ waveform

This projection is also Gaussian

Here the projections are Scalars.

- A set of waveforms ϕ_J s.t $\int |\phi_J(t)|^2 dt < \infty$

$$\boxed{P_J = \int \phi_J(t) x(t) \, dt}$$

Projection is a Scalar product b/w ϕ_J & $x(t)$

We know that if 'x' is Gaussian

Then $y = Ax$ is also Gaussian

like we have seen in last Class

i.e $y = a^t x$ $A = a^t = [a_0, a_1, \dots a_{n-1}]$

(This is Scalar Product)

$|||^{ly}$ $\boxed{P_J = \int \phi_J(t)\, x(t)\, dt}$ is also a Scalar product

- Here we choose waveforms as vectors & scalar product as defined above.

Note:
We make use of projection & the definition of Scalar product as above because we generally donot have access to PDF's. (There is a Good Reason for not doing so i.e because of White Gaussian Noise)

$\therefore$ The definition of Stochastic Process based on the definition of projection

$$P = \int \phi_J(t)\, x(t)\, dt \text{ is Gaussian}$$

- Generally we have many ϕ_J's

$\therefore$ A set of waveforms ϕ_J s.t $\int |\phi_J(t)|^2 < \infty$

$$P_J = \int \phi_J(t)\, x(t)\, dt$$

$$\begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_{n-1} \end{bmatrix} \rightarrow \text{is a Gaussian vector}$$

$\therefore$ if $x(t)$ is a Gaussian Process then each P_J is Gaussian

* We need to prove that if all P_J are Gaussian the vector $\begin{bmatrix} P_0 \\ \vdots \\ P_{n-1} \end{bmatrix}$ is also Gaussian i.e all the projections put together also behave Gaussian.

- let's build a linear combination of P_J

$$\sum_J \alpha_J P_J = \sum_J \alpha_J \int \phi_J(t)\, x(t)\, dt$$

$$\Rightarrow \int \left[\sum_J \alpha_J \phi_J(t) \right] x(t)\, dt$$

Sum of Finite energy function will result in Finite energy functions.

$\phi_J \in L_2$

$\sum_J \alpha_J \phi_J \in L_2$

$\Downarrow$

$\phi \in L_2$

We can call the Linear combination of all ϕ_J as ϕ

$$\Rightarrow \int \phi(t)\, x(t)\, dt$$ is Gaussian

$\because$ $x(t)$ is Gaussian process

* Since any linear combination of Projection is Gaussian $\therefore$ we can say the vector of Gaussian Projections is also Gaussian.

$$\therefore P = \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_{n-1} \end{bmatrix} \text{ is Gaussian} \Rightarrow m_p, K_p$$

Note: If we have a vector of Gaussian variables we know there is PDF which is completely characterized by a Vector of Averages m_p & Covariance Matrix K_p

→ Average

$$m_p = E[P] = \begin{bmatrix} E[P_0] \\ E[P_1] \\ \vdots \\ E[P_{n-1}] \end{bmatrix}$$

- $$E[P_J] = E\left[\int \phi_J(t)\, x(t)\, dt\right]$$

$\phi_J(t)$ is a constant function

$$\Rightarrow \int \phi_J(t)\, \underbrace{E[x(t)]}_{m_{x(t)}}\, dt$$

$$E[P_J] \Rightarrow \int \phi_J(t)\, m_{x(t)}\, dt$$

This is a scalar

This is one of the component of m_J and each component is a projection over ϕ_J the average of the process $m_{x(t)}$

→ Covariance Matrix K_p

P, m_p are vectors

$$K_p = E\left[(P - m_p)(P - m_p)^t\right]$$

let's compute $(K_p)_{JK}$ element of the Matrix

$$\boxed{(K_p)_{JK} \Rightarrow E\left[(P_J - m_{pJ})(P_K - m_{pK})^t\right]}$$

$$\Rightarrow E\left[\left(\underbrace{\int \phi_J(t)\,x(t)\,dt}_{P_J} - \underbrace{\int \phi_J(t)\,m_{x(t)}\,dt}_{m_{pJ}}\right)\left(\underbrace{\int \phi_K(t)\,x(t)\,dt}_{P_K} - \underbrace{\int \phi_K(t)\,m_{x(t)}\,dt}_{m_{pK}}\right)\right]$$

$$\Rightarrow E\left[\left(\int \phi_J(t)\left(x(t) - m_{x(t)}\right)dt\right)\left(\underbrace{\int \phi_K(s)\left(x(s) - m_{x(s)}\right)ds}_{\text{Here we are just changing Integration variable to avoid confusion}}\right)\right]$$

$$\Rightarrow \iint \phi_J(t)\, \phi_K(s)\, \underbrace{E\left[\left(x(t) - m_{x(t)}\right)\left(x(s) - m_{x(s)}\right)\right]}_{K_x(t,s)}\, dt\, ds$$

Covariance Function of the process 'x'

$$\boxed{\Rightarrow \iint \phi_J(t)\, \phi_K(s)\, K_x(t,s)\, ds\, dt = (K_p)_{JK}}$$

This will tell us what is any of the element of the Covariance Matrix of (K_p) vector of projections (P).

paths & ideas :

- usually when we process (operation) a stochastic $x(t)$ process, we don't make calculation on the stochastic process itself but instead we use projections. We extract the projections apply

Gains corresponding to our filtering requirements and then mix everything together to get the output.

∴ for processing we act on the projection of the stochastic process

∴ The Processing of a Stochastic process $x(t)$ is completely defined by its 2^{nd} order statistical Features

$m_x(t)$ → Average

$K_x(t,s)$ → Covariance function

- With these two we can build

$$m_p \ \& \ K_p \quad \forall \, p = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix}$$

$\boxed{\therefore \text{ 'p' is Gaussian, hence it is completely characterized in statistical terms}}$

Note: Power Spectrum is a 2^{nd} order feature of a Gaussian process. We will see it in the next lecture.