16/05/24 Lecture - 11 12/05/21

Definition & properties of Gaussian Process (Continued)

→ We are studying, How to extend the properties of Gaussian RVs to the Gaussian processes

→ we were able to understand that our definition of Gaussian process goes through the use of projections and if a process obeys The assumption of being a Gaussian process Then it's projections when I align them in a single vector will form a Random Vectors with its elements Being Jointly Gaussian.

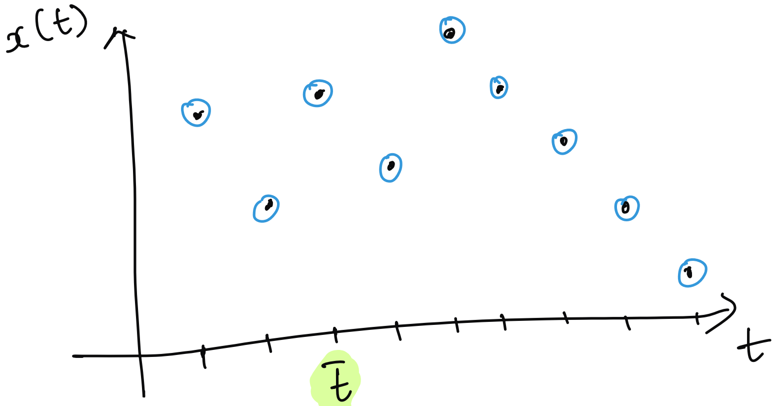
→ Q) What about The samples of a Gaussian process?

Are The samples (scalars) we get are Gaussian?

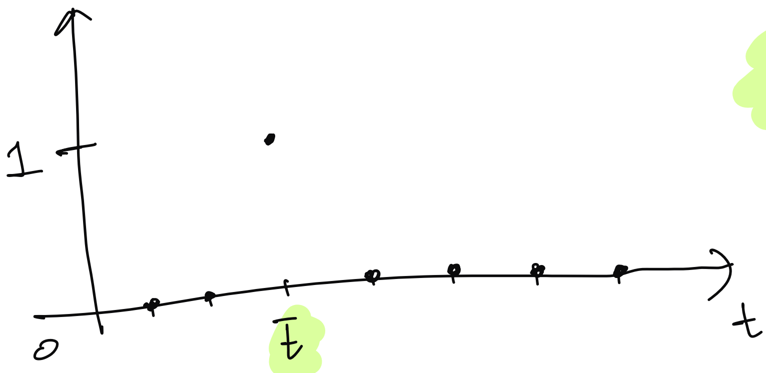
- We would like the samples to have a Gaussian distribution and this requires a little attention, because This is True for

Discrete time Sampling of DT Gaussian Processes

Discrete time case



Process



$\delta_{t,\bar{t}}$ Delta function

$$p = \sum_{t=-\infty}^{+\infty} \delta_{t,\bar{t}}\, x[t] = x[\bar{t}]$$

When we project the instance of the process onto the Delta centered in $\bar{t}$

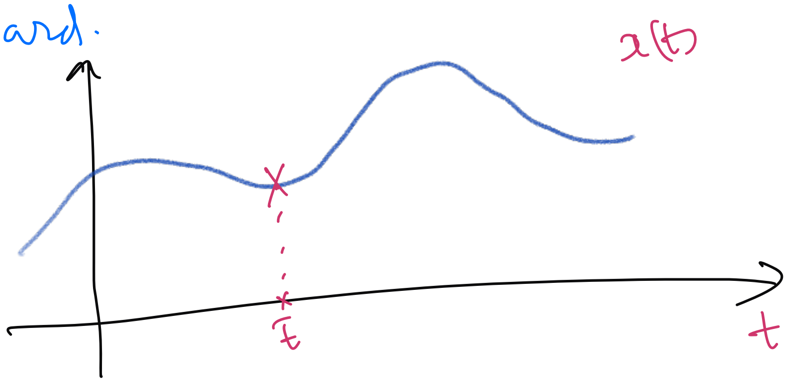
$$P = \underbrace{\sum_{t=-\infty}^{\infty} \delta_{t,\bar{t}}\, x[t]}_{\text{projection}} = \underbrace{x[\bar{t}]}_{\text{sampling}}$$

$$\sum_{t=-\infty}^{+\infty} \delta_{t,\bar{t}}^{2} = 1 < +\infty \quad \text{ok!}$$

This is a Finite Energy function

$\therefore$ There is no problem with Discrete Time

But, for the Continuous time this is not so straight forward.



Now, if we project our CT Gaussian RP onto the Dirac Delta function. Do the samples have Finite Energy?

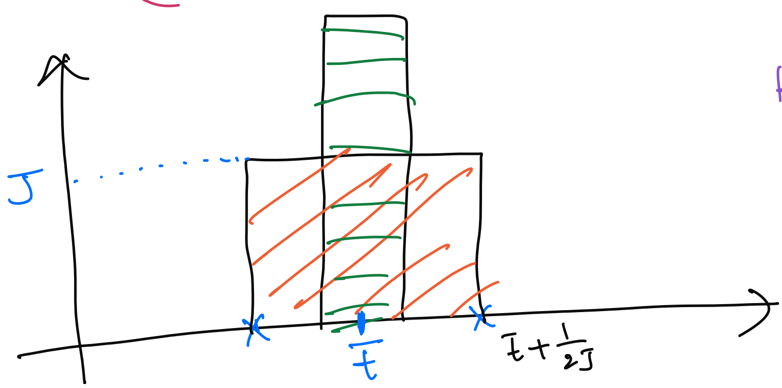
$$p = \underbrace{\int \delta(t-\bar{t})\, x(t)\, dt}_{\text{projection}} = \underbrace{x(\bar{t})}_{\text{sampling}}$$

$$\int \delta^2(t-\bar{t})\, dt = +\infty$$

Not OK!

** So, We will add an additional assumption that $\gamma_x(t,s)$ is continuous Function!

(covariance function)



Height = J

width = 1/J

Approximation of Dirac Delta

$$\phi_{J,\bar{t}}(t) = \begin{cases} 0 & \text{if } |t-\bar{t}| > \frac{1}{2J} \\ J & \text{if } |t-\bar{t}| \leq \frac{1}{2J} \end{cases}$$

$\bar{t}-\frac{1}{2J}$

$\phi_{J,\bar{t}}$
Finite energy function $\forall J$

$$\int \phi^2_{J,\bar{t}}(t)\,dt < +\infty$$

$$P_J = \int \phi_{J,\bar{t}}(t)\, x(t) \Rightarrow \text{is Gaussian}$$

if $x(t)$ is Gaussian

→ Now we also consider the sample of our RP $x(t)$ at $t=\bar{t}$ i.e $x(\bar{t})$ and we wait to evalute

$$E\left[\left(P_J - x(\bar{t})\right)^2\right] \rightarrow \text{Average Squared Error}$$

and if this value goes to zero when $J \rightarrow \infty$ then we know that there are sequence of Gaussian Radom variables that tend to be our samples.

$$E\left[\left(\int \phi_{J,\bar{t}}(t)\,x(t)\,dt - \underbrace{x(\bar{t})\,dt}\right)^2\right]$$

$$\int \phi_{J,\bar{t}}(t)\,x(\bar{t})\,dt$$

$$\Rightarrow x(\bar{t}) \underbrace{\int \phi_{J,\bar{t}}(t)\,dt}$$

This is equal to 1

$$\Rightarrow E\left[\left(\int \phi_{J,\bar{t}}(t)\left(x(t) - x(\bar{t})\right)dt\right)^2\right]$$

The square of an integral is an double Integral and in this case we use two variables t, s

$$\Rightarrow E\left[\iint \phi_{J,\bar{t}}(t)\left[x(t) - x(\bar{t})\right] \phi_{J,\bar{t}}(s)\left[x(s) - x(\bar{t})\right] dt\,ds\right]$$

$$\Rightarrow \iint \phi_{J,\bar{t}}(t)\,\phi_{J,\bar{t}}(s)\,E\left[\left(x(t) - x(\bar{t})\right)\left(x(s) - x(\bar{t})\right)\right] dt\,ds$$

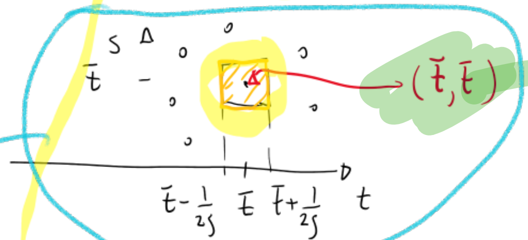
$$\Rightarrow \int_{-\infty}^{+\infty}\!\!\int \phi_{J,\bar{t}}(t)\,\phi_{J,\bar{t}}(s)\Big\{E\left(x(t)\,x(s)\right) - E\left[x(t)\,x(\bar{t})\right] - E\left[x(\bar{t})\,x(s)\right] + E\left[x(\bar{t})\,x(\bar{t})\right]\Big\}\,dt\,ds$$

$\Rightarrow$

Here we are evaluating the correlation function at two different time instances

Here we have correlation function at the same time instant

$$= \iint_{-\infty}^{+\infty} \varphi_{S,\bar t}(t)\,\varphi_{S,\bar t}(s)\,\Big\{E[x(t)x(s)] - E[x(t)x(\bar t)] - E[x(\bar t)x(s)] + E[x(\bar t)x(\bar t)]\Big\}\,dt\,ds$$



The square shrinks as $S \to \infty$ and t^*, s^* are inside the square centered at $\bar t$

$$= \iint_{\bar t - \frac{1}{2S}}^{\bar t + \frac{1}{2S}} S^2 \big[C_x(t,s) - C_x(t,\bar t) - C_x(\bar t, s) + C_x(\bar t,\bar t)\big]\,dt\,ds$$

R_x is continuous

$\Downarrow$

C_x is continuous

$\Downarrow$

the integrand is continuous

intermediate value theorem

$$= \frac{1}{\cancel{S^2}}\,\cancel{S^2}\big[C_x(t^*,s^*) - C_x(t^*,\bar t) - C_x(\bar t, s^*) + C_x(\bar t,\bar t)\big]$$

$$= \frac{1}{\cancel{S^2}} \cancel{S^2} \left[C_x(t^*, s^*) - C_x(t^*, \bar{t}) - C_x(\bar{t}, s^*) + C_x(\bar{t}, \bar{t}) \right]$$

$S \to \infty \qquad \frac{1}{S} \to 0 \qquad t^*, s^* \to \bar{t}$

C_x is continuous $\to$

$$S \to \infty \qquad \begin{aligned} C_x(t^*, s^*) &\to C_x(\bar{t}, \bar{t}) \\ C_x(t^*, \bar{t}) &\to C_x(\bar{t}, \bar{t}) \\ C_x(\bar{t}, s^*) &\to C_x(\bar{t}, \bar{t}) \end{aligned}$$

$$= 0 = E\left[(P_S - x(\bar{t}))^2 \right]$$

is Gaussian

→ Since $x(\bar{t})$ are any way Gaussian, and Now the Error b/w The projection & The $x(\bar{t})$ is zero our projections of CT RP $x(t)$ are also Gaussian.

* This comes in Handy in the analysis of White Gaussian process.

→ White Gaussian Noise

WGN is a Gaussian stochastic process

$\because$ $\boxed{p = \int \phi(t)\, x(t)}$ ⇒ is Gaussian

WGN ⇒ $\boxed{m_p = 0}$ & $\boxed{\sigma_p^2 = \frac{N_0}{2} \|\phi\|^2}$

NORM

Average of the projection is zero

$\Rightarrow \frac{N_0}{2} \int |\phi(t)|^2 dt$

energy of the $\phi(t)$

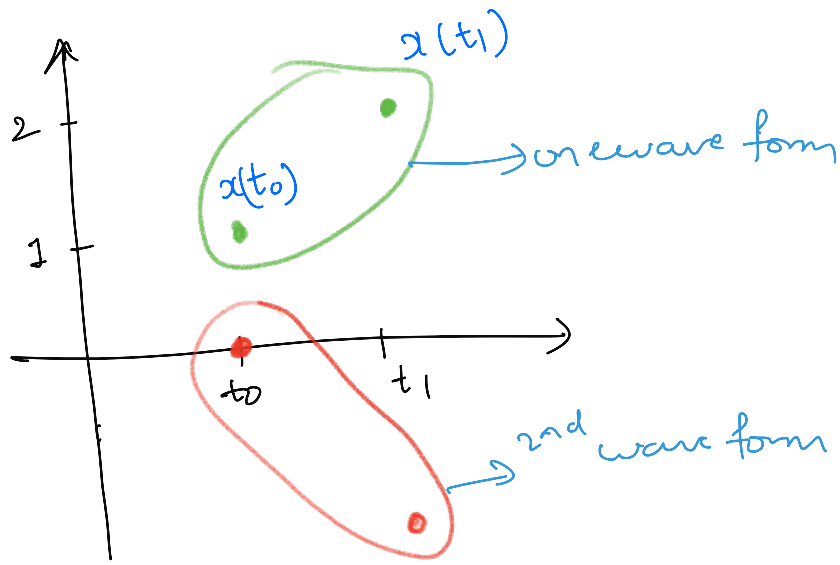
So it only depends on the energy of the $\phi(t)$

$\therefore$ if we take two $\phi(t)$ which are sine & cosine, so the σ_p^2 of the two projections made using sine & cosine will be same because sine & cosine have equal Average energy.

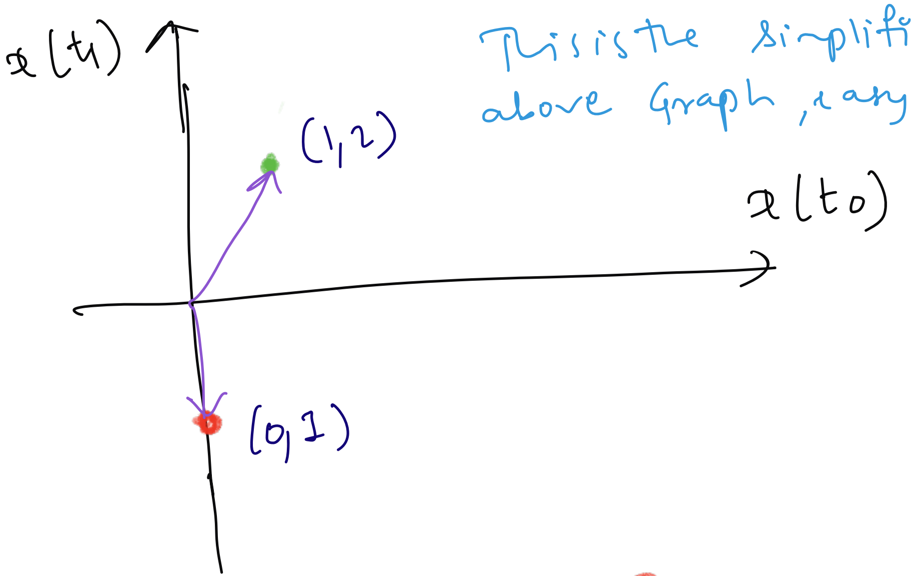
- This dependency of σ_p^2 only on the Energy makes the WGN a special object full of virtues & problems

- From geometrical point of view WGN is seen as a sphere i.e from any direction it looks the same from statistical point of view.

ex: WGN in Discrete Time processes



- They are discrete waveforms because they depend on different instances of time i.e t_0, t_1



This is the simplification of the above Graph, easy to analyze

Q) What are ϕ's ? in our simplified environment

$$\phi_0 = \begin{bmatrix} \phi_{00} \\ \phi_{01} \end{bmatrix} \qquad \phi_1 = \begin{bmatrix} \phi_{10} \\ \phi_{11} \end{bmatrix}$$

$$\|\phi_0\|^2 = 1 \qquad \|\phi_1\|^2 = 1$$

energy = 1

$$\phi_{00}^2 + \phi_{01}^2 = 1$$

i.e ϕ_0 is a unit vector with tip of the arrow is on the unit circle

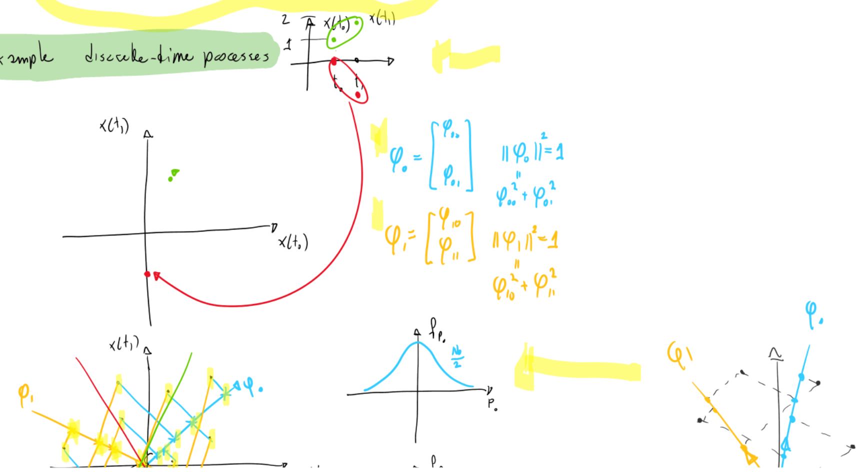
ϕ_1 is also the same

∴ if project on the ϕ_0 (or) ϕ_1 we get the same Gaussian vector because both have the same energy

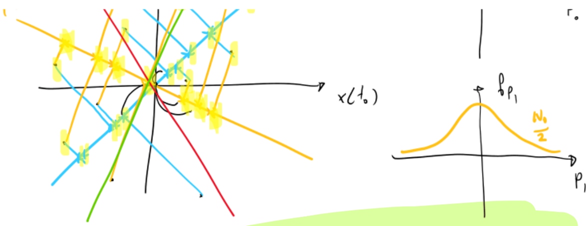
⇓

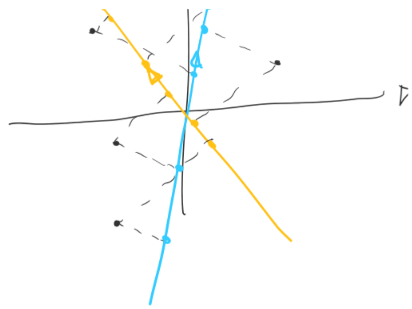
Gaussian White noise is a "sphere"

Example discrete-time processes



White Gaussian processes, power spectrum Pagina 5





$$\int |\varphi_0(t)|^2 dt = \int |\varphi_1(t)|^2 dt = \sigma^2_{P_0} = \sigma^2_{P_1}$$

P_0 and P_1 are the same r.v.

Consequence of "Whiteness"

if the φ' are ⊥ lar

$\varphi_0 \perp \varphi_1 \iff \int \varphi_0(t)\,\varphi_1(t)\,dt = 0$

$p_0 = \int \varphi_0(t)\,x(t)\,dt \qquad p_1 = \int \varphi_1(t)\,x(t)\,dt$

$$p = \int \underbrace{[\varphi_0(t)+\varphi_1(t)]}_{\varphi(t)}\,x(t)\,dt = p_0 + p_1$$

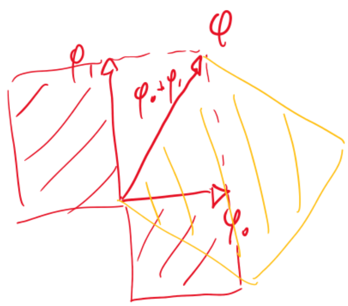
$$\sigma_p^2 = E[p^2] = E[(p_0+p_1)^2] = \overset{\sigma_{p_0}^2}{E[p_0^2]} + \overset{\sigma_{p_1}^2}{E[p_1^2]} + \underbrace{2E[p_0 p_1]}_{=0}$$

$$= \frac{N_0}{2}\|\varphi_0+\varphi_1\|^2 = \frac{N_0}{2}\left(\|\varphi_0\|^2+\|\varphi_1\|^2\right) = \frac{N_0}{2}\|\varphi_0\|^2 + \frac{N_0}{2}\|\varphi_1\|^2$$

Pythagorean Theorem

$$\int |\varphi_0(t)+\varphi_1(t)|^2\,dt =$$

$$\underbrace{\int |\varphi_0(t)|^2\,dt}_{\|\varphi_0\|^2} + \underbrace{\int |\varphi_1(t)|^2\,dt}_{\|\varphi_1\|^2} + 2\underbrace{\int \varphi_0(t)\,\varphi_1(t)\,dt}_{0}$$



$\varphi_0 \perp \varphi_1 \rightarrow$ $\left.\begin{array}{l} E[p_0 p_1] = 0 \\ E[p_0] = E[p_1] = 0 \end{array}\right\} \rightarrow \gamma_{p_0 p_1} = 0$

$p_0\ p_1$ are jointly Gaussian
$p_0\ p_1$ have zero covariance
$\} \rightarrow p_0, p_1$ are independent