Lecture-1

Functions as Vectors – Positive semidefiniteness

Q) What are we gonna we study here?

Ans The subject is statistical signal processing it gonna invoke lots of Math and Statistics

→ We will use a lot of Math which you already know specially 1) Linear Algebra
2) Probability and statistics

→ There will be Derivations & proofs but they will be more from Engineering perspective.

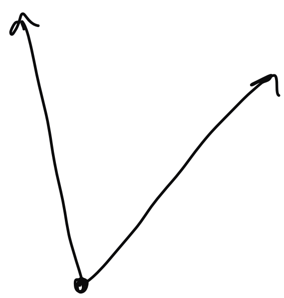
→ Signals are the carriers of Information.

We are interested in designing Devices or piece of software that process information and delivers it to the End user!
We are more interested in the information carrier and information Payers!
Our Goal is to learn what are the methods, we need to know to process information effectively to earn a lot of Money! There is a Big economical push for Doing So.

→ Regrettably (or Not Regrettably) we need a lot of Sophisticated Math and Analytic Geometry to do Statistical Signal Processing.

The Methods we device are extremely effective and do wonderful things in the end.

→ Since we are gonna use Geometry there will be Vector Spaces. They are Arrows in Space



$$x = \begin{bmatrix} x_0 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Vectors are columns filled of Real or Complex Numbers.

But, These are introductory definitions of vectors. in reality, vectors are abstract objects and the only features that characterises vectors is that they have certain Operations between them. This is extremely important.

→ Because when we say we want to process signals, we change the models we give to signals. Because we think signal as a time varying voltage (or a

- time varying current (or) • Amount of charge stored in a capacitor. etc.

The above description is too limited. Models, Signals are different, Signals are many things the only way to capture what Signals can be is a Mathematical Model.

→ Since we want to process signals we can't avoid Statistics and probability. Because there is no point in transmitting something that is already known at the Receiver

ex: There is no point in listening to a Song which makes only One tone.

→ So too make sense of uncertain world, we need to make use of Statistics to model signals.
What we would be doing is Manipulating probabilities.

→ The ultimate goal of signal processor both as a Machine and as a guide that designs algorithm is to deal with probabilities.

→ when we design a Compress algorithm for Music we don't know the Track in advance.

→ we are not dealing with methods to compute Finite

Sinusoids for which Amplitude and phase is known.
So forget about sinusoids when the Amplitude and phase are not known and ~~we~~ have to Model them.

→ Therefore when you do something about the Linear Filters it won't be Resistor, Capacitor and ...
it will be quite different.

→ It is very important to immerse ourselves in Statistics and Geometry and first thing to do is Raise the abstraction level.

A vectors are the only objects that satisfy certain type of properties.

→ Now, we should know these properties because they are the axioms defining the Vector Spaces.

→ Basically, vector space is a set of objects that can be summed together and the resultant is still in that vector space and there is a companion set of scalars when scalar multiply those vectors the resultant is still in that vector space.

→ The Two operations summing two vectors and Multiplication by a Scalar have a certain no. of properties that concern their possible combinations.

Axioms of vector Space:

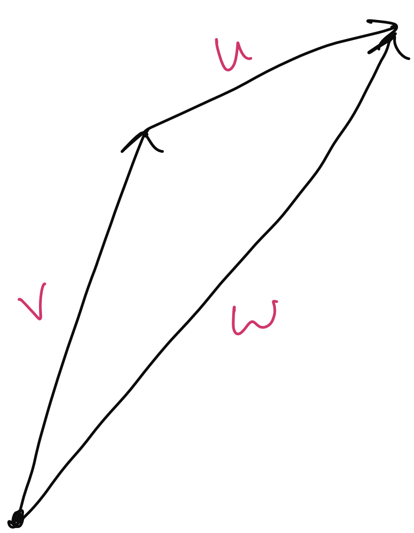
1) $\alpha V = U$

scaler vector vector

2) $U + V = W$

vector vector vector

• Here the definition of sum is completely different If we have to sum two Arrows in a plane we have to put the End of 2nd Arrow on the tip of first Arrow and join the End of first Arrow and tip of 2nd Arrow for the Resultant vector!



$\vec{V} + \vec{u} = \vec{w}$

In the case of vectors as a column of values.
The Resultant is by adding the respective values of the columns of same shape

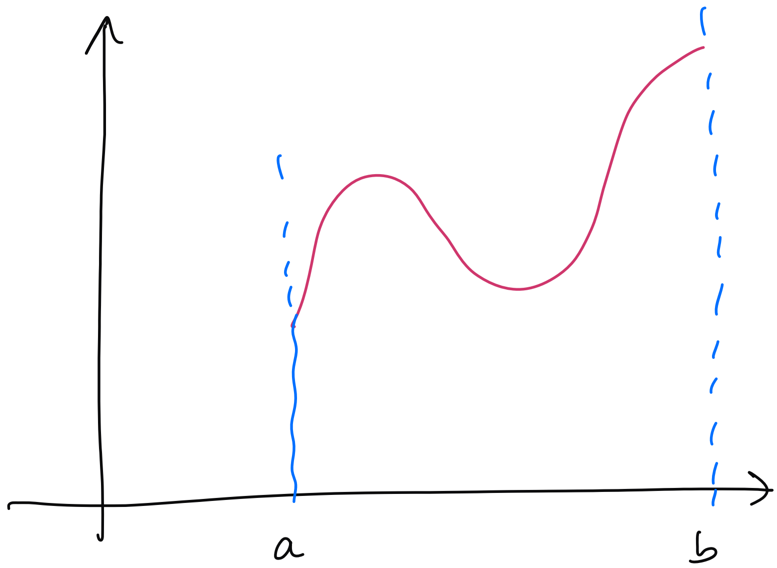
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

* In both the cases the definition of Sum is different. In principle they are two different operations but they are still Sum of two vectors!

→ Interesting point is there are many other useful vectors and the first kind of vector we will see is not arrows or arrays of numbers instead a set of functions.

Function are vectors. Because they do satisfy the Axioms of vector space. Most elementary example would be

ex: A set of Functions that have a certain Domain.



$f : [a, b] \to \mathbb{R}$
$g : [a, b] \to \mathbb{R}$

Two function of same Domain.

And we can sum these two functions to get a third function. i.e

$h = f + g$ (summing Two functions)

αf (multiplication by a scalar)

→ So, vectors can be almost anything, we will see that Functions will be vectors we will use also Random variables as also Vectors. we will make a lot of Geometry about these vectors spaces.

The Geometry is what we mainly call the Euclidean Geometry. Vectors alone donot have the concept of angle b/w them not intrinsically,
we know that if two vectors are arrows in a plane then the angle is naturally defined.

Q) But if two vectors are defined as Arrays. what is the angle b/w them ?

And all the Geometry we stay in in i.e Euclidean Geometry depends on the definition of the further operation b/w two pair of vectors. which is called scalar product

Axioms of scalar product

$$\underbrace{\langle u}_{\text{vector}}, \underbrace{v}_{\text{vector}} \rangle = \underbrace{\alpha}_{\text{scalar}}$$

So it takes two objects in a set of vectors and gives an object in the companion set of scalars. So, it stays in the same structure of Vector space. But it does this Mapping.

Not every function taking a pair of vectors and resulting in a scalar can be a scalar product. It must have certain no. of requirements satisfied.

- one of the most important property of scalar product is it distributes w.r.t. sum.

$$\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle$$

it is the generalization of product b/w two real numbers.

→ Scalar product for definition of vectors as arrays of Numbers.

$$x = \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} \qquad y = \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

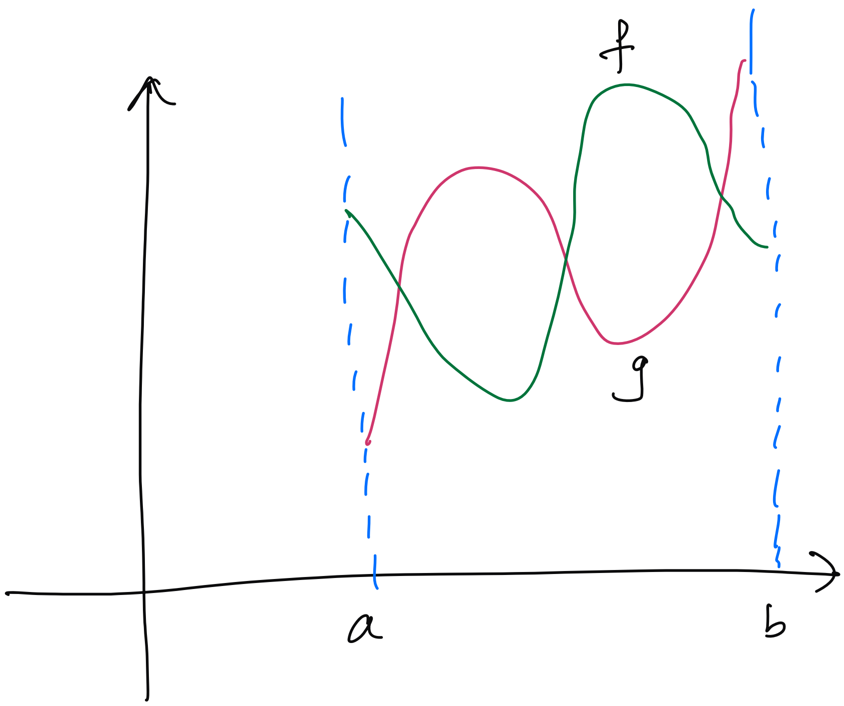
$$\boxed{\langle x, y \rangle = \sum_{j=0}^{n-1} x_j y_j = \underbrace{x^T y}_{\text{Dot product / row by column}}}$$

vector Transpose

* This is a legitimate scalar product. It can be also defined in Matrix form. So exploiting operations which we usually define b/w vectors and matrices.

→ d) How can we extent the above definition to vectors that are no longer Arrays of Numbers but are functions?

Assume the functions are defined on the domain $[a, b]$ as we discussed earlier.



$$f : [a, b] \rightarrow \mathbb{R}$$
$$g : [a, b] \rightarrow \mathbb{R}$$

Now we want to do scalar product b/w these two things. we are free to choose whatever definition

$$\boxed{\langle f, g \rangle = \int_a^b f(\xi)\, g(\xi)\, d\xi}$$

we want and we choose the definition as above.

Here the scalar product is Integral b/w their common domain of the product of functions in that domain. This definition satisfies all axioms about the scalar product. importantly the Distribution over sum.

$$\langle f, g+h \rangle = \int_a^b f(\xi)\left[g(\xi)+h(\xi)\right] d\xi$$

$$\langle f, g \rangle + \langle f, h \rangle \Rightarrow \int_a^b f(\xi)\, g(\xi)\, d\xi + \int_a^b f(\xi)\, h(\xi)\, d\xi$$

→ Now we will see the parallels b/w the

Scalar product of vectors as Rows as columns

& Scalar product of vectors that are functions.

$$\langle x, y \rangle = \sum_{j=0}^{n-1} x_j y_j = 0$$

$$\langle f, g \rangle = \int_a^b f(\xi)\, g(\xi)\, d\xi = 0$$

When the scalar product is Zero then we have the concept of Orthogonality

$$\langle u, v \rangle = 0 \quad (\text{Orthogonality})$$

Having a scalar product is fundamental to having a Geometry because if you have a Right angle then you can define angles using the trigonometric functions.

→ There are many other things available as operations b/w vectors.

We not only have vectors, we usually deal with Matrices.

→ You may know Matrix as a Rectangular Array of Numbers. But we should know that
** Matrix is a representation of Linear Transformation. This is The true Nature of a Matrix.
It is a representation of linear transformation.
So, it is a mapping b/w two Vector Spaces and the Matrix is one way of representing a Mapping when this Mapping is Linear.

Nevertheless, we are used to see Matrices as Array of Numbers and we are used to see the application of Mapping to a certain vector as the Multiplication of The Matrix by the vector.

i.e $\boxed{Y = AX}$

X, Y are vectors   'A' is a Matrix

we are applying Transformation 'A' to vector 'X' and obtaining another vector 'Y'. The transformation is Linear one That is why it is a Matrix Multiplication.

ex:

$$A_{JK} \qquad \begin{matrix} j = 0, \dots, n-1 \\ k = 0, \dots, n-1 \end{matrix}$$

$$\boxed{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & \cdots & \\ A_{10} & A_{11} & \cdots & \\ & A_{JK} & & \\ & & & A_{n-1\,n-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}}$$

$$\boxed{y_J = \sum_{k=0}^{n-1} A_{JK} x_k}$$

→ Let's try to Generalize this concept to Functions that are vectors.

In the Discrete and Finite Domain we live in we have Tensors for everything.

ex:

$$X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

let's generalize for Functions

$$x_k \longrightarrow f(\beta)$$

$$A_{jk} \longrightarrow H(\alpha, \beta)$$

$$y_k \longrightarrow g(\alpha)$$

Then

$$\boxed{g(\alpha) = \int_a^b H(\alpha, \beta) f(\beta) \, d\beta}$$

Generalization is straight forward and it's solid.

$H(\alpha, \beta)$ as linear operation on function $f(\beta)$.

→ What we have a linear mapping from one function to another function.

Here $H(\alpha, \beta)$ is a two argument function.

→ Quadratic form in Finite, Discrete domain

$$x^T A x$$

Quadratic form of vector 'x' defined by the Matrix 'A'

$$x \in \mathbb{R}^n \qquad x = \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$A \in \mathbb{R}^{n \times n}$$

$$\underbrace{x^T}_{\text{vector}} \underbrace{A}_{\text{Matrix}} \underbrace{x}_{\text{vector}}$$

- It is completely defined by the Row by column product
- We know Row by column product in vectors is not commutative but Associative.

$$\underbrace{x^T}_{\text{Row vector}} \underbrace{Ax}_{\text{Column vector}}$$

Q) Why is the Quadratic form very popular?

Ans It is a way of writing 2nd order polynomial in the components of X

And 2nd order polynomical are very usual for us we will see why!

The reason why is that if write row by column product that we perform first Ax and then we perform.
$X^T A X$ if we write down all the sums that are implicit in this multiplication we come up with

$$X^T A X = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} A_{JK} x_J x_K$$

Inside this Double sum we have all the possible combinations of the pair $X_J X_K$.

If we consider X_J, X_K are the unknowns of a polynomical and it includes all the possible monomials of 2nd degree like X_J^2 i.e $(J=K)$

Note: Now if we are changing A_{JK} we are simply changing the coefficients of generic second order polynomials in the components of X.

* When we define a Quadratic form we absolutely don't care whether 'A' is symmetric or Not symmetric. The reason is we can substitute with 'A' any symmetric matrix. such that the result of the computation is absolutely the same.

Reasoning: Assume $A^T \neq A$

$$2\left(\underline{x^T A x}\right) = x^T A x + \underline{\left(x^T A x\right)^T}$$

The result is a scalar so we can do this Transpose

The result of this thing is a scalar

$$\Rightarrow x^T A x + (Ax)^T (x^T)^T$$

$$\Rightarrow x^T A x + x^T A^T x$$

$$\Rightarrow x^T (Ax + A^T x)$$

$$\Rightarrow x^T (A + A^T) x$$

We have,

$$2x^TAx = x^T(A+A^T)x$$

$$\boxed{x^TAx = x^T \underline{\frac{(A+A^T)}{2}} x}$$

This is a symmetric matrix

→ So, the idea is when you look at a Quadratic form you always assume Symmetric Matrix

That's wonderful because we know a lot of things about Symmetric Matrices.

Q) Why Quadratic forms are Important?

The reasons is 2^{nd} order polynomial in Quantities are fundamental because they are Energies.

Ex: Kinetic Energy $= \frac{1}{2}mv^2 \geq 0$

Energy in a Capacitor $= \frac{1}{2}cv^2 \geq 0$

Q) How are properly designed Quadratic forms That can be used as Energies ?

If we want to use Quadratic forms as more sophesticated form of Energy. I must be sure That the result is not less than zero (positive)

But, we can't always be sure of The expression

$$x^T A x \geq 0 \ ?$$

Q) Under what conditions This is greater than zero ?

Any

$$\forall x \quad x^T A x \geq 0$$ Positive Semi definite

$$\forall x \neq 0 \quad x^T A x > 0$$ positive Definite

everything now is dependent on the

Eigen Values & Eigen vectors of Matrix 'A'

$v \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$

$$\boxed{\lambda v = A v} \qquad v \neq 0$$

Scalar (eigen value)

eigen vector

Q) What kind of property that comes out when we consider matrices that are symmetric?

unknown

$$\det(A - \lambda I) = 0$$ to calculate Eigen values

$\lambda_0, \lambda_1, \ldots \ldots \lambda_n$ solutions

even though there can be repeated Eigen values!

→ But in our case i.e symmetric matrices there is a beautiful theorem that shows a lot of things about Eigen values and Eigen vectors of our Matrix.

It is called Spectral Decomposition Theorem.
and it applies to Symmetric matrices!

Then we are sure that we have 'n'
Eigen values and corresponding Eigenvectors

$$\lambda_j, \; u_j \quad \text{for } j = 0, 1, \cdots n-1$$

$$D = \begin{bmatrix} \lambda_0 & & & & \\ & \lambda_1 & & 0 & \\ & & \lambda_2 & & \\ & 0 & & \ddots & \\ & & & & \lambda_{n-1} \end{bmatrix}$$

$$U = \begin{bmatrix} | & | & | & & | \\ u_0 & u_1 & u_2 & \cdots & u_{n-1} \\ | & | & | & & | \end{bmatrix}$$

Then $\boxed{A = U D U^T}$

'A' can be decomposed in terms of eigen values & Eigen vector matrices.

& $U^T U = I$

i.e 'U' is orthonormal

$$\begin{bmatrix} u_0^t \\ u_1^t \\ \vdots \\ u_{n-1}^t \end{bmatrix} \begin{bmatrix} u_0 & u_1 & \cdots & u_{n-1} \end{bmatrix} = \begin{bmatrix} u_0^T u_0 & & & \\ u_1^T u_0 & u_1^T u_1 & & \\ \vdots & & u_2^T u_2 & \\ \vdots & & & \ddots \\ & & & u_{n-1}^T u_{n-1} \end{bmatrix}$$

$$\Downarrow$$

$$I_{n \times n}$$

→ When we have a Scalar product we define Norm

$$\langle \cdot , \cdot \rangle \rightarrow \| \cdot \| = \sqrt{\langle \cdot , \cdot \rangle}$$

$$\| U_J \| = \sqrt{\langle U_J , U_J \rangle}$$

$\rightarrow$ Check for non-negativity of

$$x^T A x \geq 0$$

$$\forall x$$

$\because$ 'A' is symmetric we substitute with UDU^T for 'A'

$$\underbrace{x^T U}_{y^T} D \underbrace{U^T x}_{y} \geq 0$$

$$\therefore \forall y \quad y^T D y = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} D_{JK} y_J y_K$$

$$\because D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_{n-1} \end{bmatrix} = \sum_{j=0}^{n-1} D_{JJ} y_J^2$$

$$\Rightarrow \sum_{j=0}^{n-1} \lambda_J y_J^2$$

$$\lambda_J \geq 0$$

$\therefore$ if Eigen values are +ve Then The Quadratic form is positive.

→ Extension to complex case

$z \in \mathbb{C}^n$ $\quad z = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix}$ $\quad \underset{\text{Hermetian Operator}}{z^{\dagger}} A z$

$A \in \mathbb{C}^{n \times n}$

$$z^{\dagger} = \underbrace{\begin{bmatrix} z_0^* & z_1^* & \cdots & z_{n-1}^* \end{bmatrix}}_{\text{complex conjugate}}$$

$\therefore$ for Spectral Decomposition

$A^{\dagger} = A$ (Hermitian Matrix)

$A = U D U^{\dagger}$ $\qquad U_j^{\dagger} U_j = 1$

$U \rightarrow$ orthonormal $\quad U^{\dagger} U = I$

$U \in \mathbb{C}^n, V \in \mathbb{C}^n$

$$\langle U, V \rangle = \sum_{j=0}^{n-1} U_j^* V_j$$ } Scalar Product for complex vectors

$$= U^\dagger V$$

we conjugate and Transpose the vector

→ **Extension to Function Vector Space**

Q) What's a Quadratic form for Functions?

Ans we can do it like this

Real $X^T A X = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} A_{jk} x_j x_k = \int_a^b \int_a^b A(\alpha, \beta) f(\alpha) f(\beta) d\alpha\, d\beta$

complex $z^\dagger A z = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} A_{jk} z_j^* z_k = \iint A(\alpha, \beta) f^*(\alpha) f(\beta) d\alpha\, d\beta$

Generalization for functional vector space

→ Now that we have generalized the concept of Quadratic form to functional vector space.

Q) Which parameter will determine the positive semidefiniteness in Functional Domain?

What is the parallel for Eigen Values in Functional vector space?

Ans We will use the concept of

Eigen Value & Eigen Function

In this case Regrettably we will not be exploiting Spectral Decomposition Theorem. The Reason is the Theorem hinges on the fact that I have Finite no. of Dimensions.

The Dimension for a Vector Space } = The Maximum no. of Linear independent vectors.

But, Vector Space of Functions is not finite Dimensional

→ $\forall f$ The Quadratic form in Functional space

$$\iint A(\alpha, \beta)\, f(\alpha)\, f(\beta)\, d\alpha\, d\beta \geq 0$$

Here The Matrix 'A' has been generalized into a function.

1) Symmetric $A(\alpha, \beta) = A(\beta, \alpha)$ [$A_{JK} = A_{KJ}$ for matrices]

2) Hermetian $A(\alpha, \beta) = A^*(\beta, \alpha)$ [$A_{JK} = A_{KJ}^*$ for matrices]

3) Töplitz $A(\alpha, \beta) = A(\alpha - \beta)$

This 3rd Assumption is necessary for a functional vector because dimension can go to Infinity

** $A(\alpha, \beta)$ has been generalized in such a way that it mimics symmetric Matrix. i.e it follows above three cases.

The Symmetric & Hermetian assumptions are not sufficient they are only for finite dimensions. Therefore we need another condition for Infinite dimension vector (functional space) i.e Töplitz

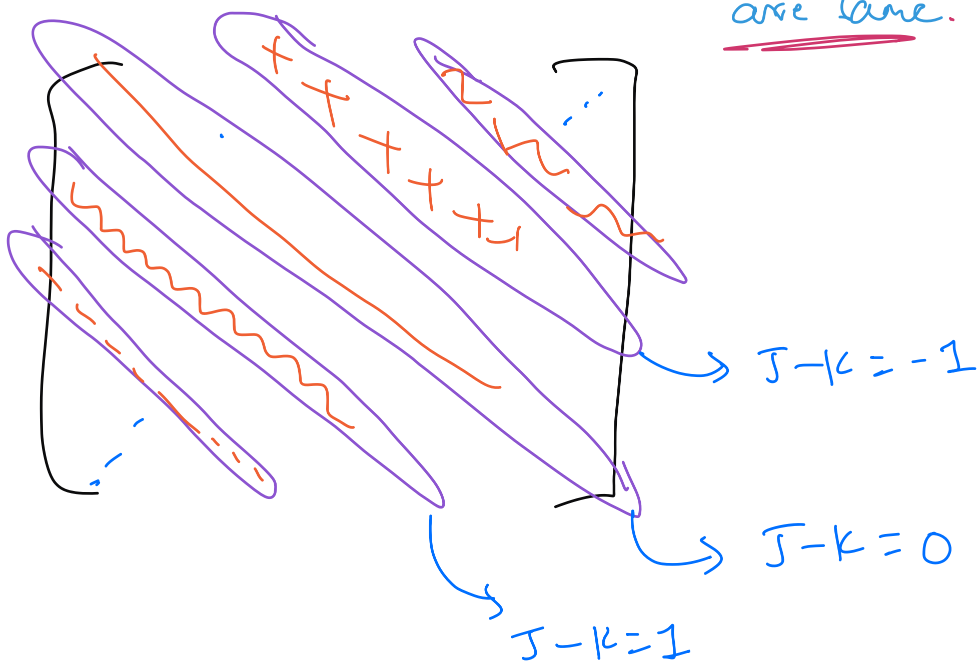
$$\boxed{\underbrace{A(\alpha, \beta)}_{\text{Two argument function}} = \underbrace{A(\alpha - \beta)}_{\text{But actual value is dependent on the difference of arguments.}}}$$

The equivalent of this assumption is not needed in finite domain, Because everything holds with either Symmetry (or) Hermetician assumption.

Q) How is Töplitz matrix made?

The value of the function depends on the difference b/w two arguments.

Therefore the Töplitz matrix would be a matrix where all the diagonal elements & super diagonal elements are same.



→ To exploit all the above facts we need to make a slight detour and define another function

'δ' → Dirac Delta function

It's true name is distribution not a function

It is defined as $\delta(\xi) = \begin{cases} 0 & \xi \neq 0 \\ ? & \xi = 0 \end{cases}$

It's a difficult mathematical entity with a particular definition.

* we wanna know how this delta function operates and what are it's properties that we can make use of.

$$\delta(\xi) = 0 \text{ for } \xi \neq 0$$

for $\xi = 0$ we will define another property which is the one we will use a lot. i.e to compute an Integral, inside which we have a $\delta(\xi)$ function.

$$\boxed{\int_{D} f(\xi)\, \delta(\xi - \xi_0)\, d\xi_0 = \begin{cases} f(\xi_0) & \text{if } \xi_0 \in D \\ 0 & \text{otherwise} \end{cases}}$$

You may take it as a Definition!

→ we can use this property to calculate the Fourier Transform of Delta function.

$$\boxed{\mathcal{F}[\delta(\xi)](\omega) = \int_{-\infty}^{+\infty} \delta(\xi)\, e^{-2\pi i \omega \xi}\, d\xi = 1}$$

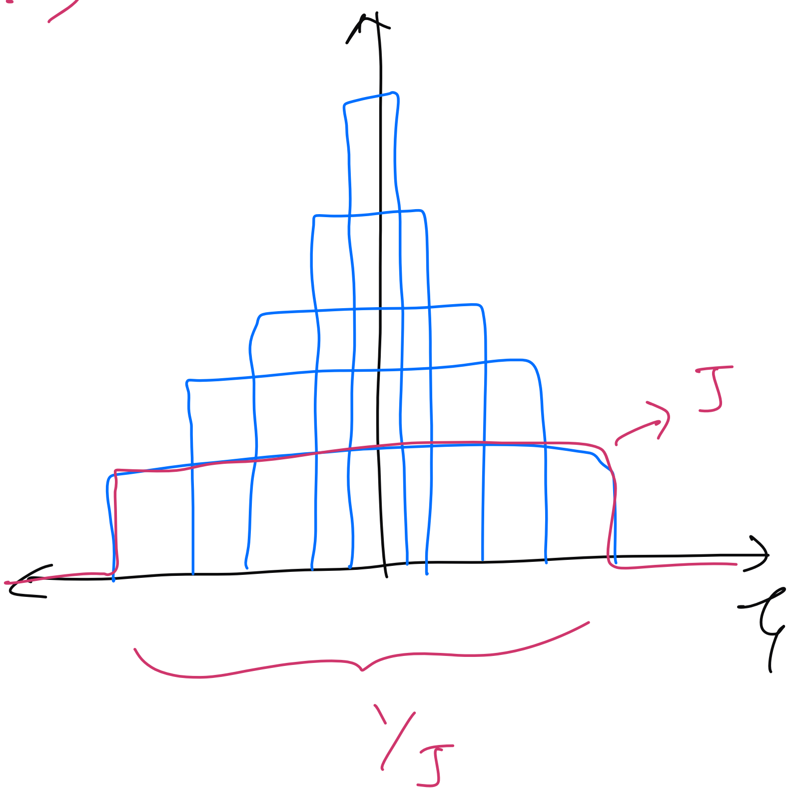
ξ & ω have Inverse dimensions!

→ By the above definition of calculate integral of a function multiplied by $\delta(\xi)$ we can say

$$\boxed{\mathcal{F}[\delta(\xi)](\omega) = \int_{-\infty}^{\infty} \delta(\xi)\, e^{-i2\pi\omega\xi}\, d\xi = e^{-i2\pi\omega(0)} = 1}$$

Which implies The inverse fourier transform of '1' is 'δ'

→



'δ' is the limit of the the functions.

$$\delta_J(\tau) = \begin{cases} 0 & \text{if } \tau < -\frac{1}{2J} \\ 0 & \text{if } \tau > \frac{1}{2J} \\ J & \text{otherwise} \end{cases}$$

$$\delta(\tau) = \lim_{J \to \infty} \delta_J(\tau)$$

intuitively too

$$\boxed{\int_{-\infty}^{\infty} 1\,\delta(q)\,dq = 1}$$

All the functions that have as a limit of Delta have integral equal to 1.

Note:- Delta function was invented by the physisit "Dirac" to make a coherent formulation of Maxwell equation going inside and outside different materials.

Now that we have Delta we can state the Necessary condition for Positive Semi Definite-ness

$$\boxed{\forall f \quad \iint A(\alpha, \beta)\, f(\alpha)\, f(\beta)\, d\alpha\, d\beta \geq 0 \rightarrow \text{Condition}}$$

To derive this condition let's start from the definition of Positive semidefinite.

we know our Quadratic form is positive semi definite so what ever 'f' we put into the quadratic form we get a value higher than zero!

let's choose an 'f'

$$\boxed{f(\alpha) = \delta(\alpha) \pm \delta(\alpha - \alpha_0)}$$

In fact it is two functions.

let's put this function in our Quadratic form and we know our Quadratic form should be positive semidefinite.

$$\iint A(\alpha, \beta) \Big(\underbrace{\delta(\alpha) \pm \delta(\alpha - \alpha_0)}_{f(\alpha)}\Big)\Big(\underbrace{\delta(\beta) \pm \delta(\beta - \alpha_0)}_{f(\beta)}\Big) \, d\alpha \, d\beta \geq 0$$

we also assume that $A(\alpha, \beta)$ is Töplitz

$$\Rightarrow \iint A(\alpha - \beta)\Big[\delta(\alpha)\delta(\beta) \pm \delta(\alpha)\delta(\beta - \alpha_0) \pm \delta(\alpha - \delta_0)\delta(\beta) + \delta(\alpha - \alpha_0)\delta(\beta - \alpha_0)\Big]$$

$$d\alpha\, d\beta \geq 0$$

$$\Downarrow$$

$$\Rightarrow \iint \underbrace{A(\alpha-\beta)\,\delta(\alpha)\,\delta(\beta)\,d\alpha\, d\beta}_{1} \pm \iint \underbrace{A(\alpha-\beta)\,\delta(\alpha)\,\delta(\beta-\alpha_0)\,d\alpha\, d\beta}_{2}$$

$$\pm \iint \underbrace{A(\alpha-\beta)\,\delta(\alpha-\alpha_0)\,\delta\beta}_{3} + \iint \underbrace{A(\alpha-\beta)\,\delta(\alpha-\alpha_0)\,\delta(\beta-\alpha_0)\,d\alpha\, d\beta}_{4} \geq 0$$

from the definition of

$$\boxed{\int f(\tau)\,\delta(\tau-\tau_0)\,d\tau = f(\tau_0)}$$

This is equal to the function where $\delta(\tau-\tau_0) = 1$

i.e $f(\tau_0)$

$\therefore$ 1 + 2 + 3 + 4

$$\boxed{A(0) \pm A(-\alpha_0) \pm A(\alpha_0) + A(0) \geq 0}$$

①

$$\iint A(\alpha-\beta)\,\delta(\alpha)\,\delta(\beta)\,d\alpha\,d\beta$$

for α

$$\iint A(0-\beta)\,\delta(\beta)\,d\beta \Rightarrow A(0-0) = A(0)$$

for $\beta=0$

②

$$\iint A(\alpha-\beta)\,\delta(\alpha)\,\delta(\beta-\alpha_0)\,d\alpha\,d\beta$$

$$\Rightarrow \int A(-\beta)\,\delta(\beta-\alpha_0)\,d\beta \Rightarrow A(-\alpha_0)$$

for $\beta=\alpha_0$

Since we have consider $A(\alpha,\beta)$ to be Symmetric & Töplitz

Symmetry $A(\alpha,\beta) = A(\beta,\alpha)$

Töplitz $A(\alpha-\beta) = A(\beta-\alpha)$

$\Rightarrow A(\eta) = A(-\eta)$ → 'A' Even function

$$\therefore \quad A(0) \pm A(-\alpha_0) \pm A(\alpha_0) + A(0) \geq 0$$

$$\Rightarrow 2A(0) \pm A(\alpha_0) \geq 0$$

$$\Rightarrow \quad A(0) \pm A(\alpha_0) \geq 0$$

$$\Rightarrow$$

$$A(0) \geq A(\alpha_0) \qquad\qquad A(0) \geq -A(\alpha_0)$$

ie If $A(0)$ is the largest value of the whole function Then it is

$$\boxed{A(0) \geq |A(\alpha_0)| \quad \forall \alpha_0}$$

This is called Diagonal Dominance

$\longrightarrow$ Necessary & Sufficient condition for positive Semi Definiteness

$$\boxed{\forall f \quad \iint A(\alpha, \beta)\, f(\alpha)\, f(\beta)\, d\alpha\, d\beta \geq 0}$$

$$\boxed{A(\alpha,\beta) = A(\beta-\alpha)} \quad \Updownarrow \quad \boxed{\mathcal{F}[A](\omega) \geq 0}$$

Fourier Transform of 'A' is Real & Greater than Zero

Note

we know that Fourier transform a function is called a Spectrum & also the collection of Eigen values of a Matrix is also called its Spectrum

That is why we called above the theory of Symmetric Matrix above as Spectral Decomposition Theorem.

Q) What do Eigen values have in common with Fourier Transform?

In next Class.