№1

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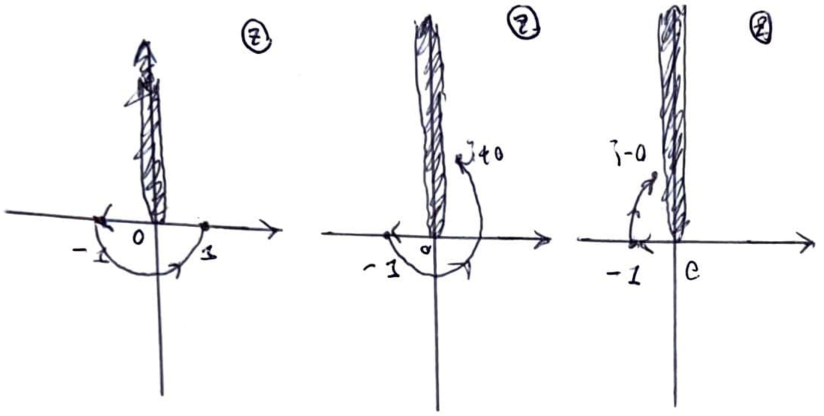
I. $\varphi(1)$; $\varphi(i+0)$; $\varphi(i-0)$ – ?

$\varphi(z)=\sqrt[3]{z}$ $\varphi(-1)=e^{i\pi/3}$

$z\in[0;\ i\infty]$



1) $\Delta\arg z=\pi$

$\varphi(1)=\left|\frac{g(1)}{g(-1)}\right|e^{i\Delta\arg g}\varphi(-1)=e^{2i\pi/3}\left|\frac{\sqrt[3]{1}}{e^{i\pi/3}}\right|=e^{2i\pi/3}$

2) $\Delta\arg z=\frac{3\pi}{2}\Rightarrow\Delta\arg g=\frac{\pi}{2}$

$\varphi(i+0)=\left|\frac{\sqrt[3]{i+0}}{e^{i\pi/3}}\right|e^{i\pi\left(\frac{1}{3}+\frac{1}{2}\right)}=e^{\frac{5\pi i}{6}}$

3) $\Delta\arg z=-\frac{\pi}{2}\Rightarrow\Delta\arg g=-\frac{\pi}{6}$

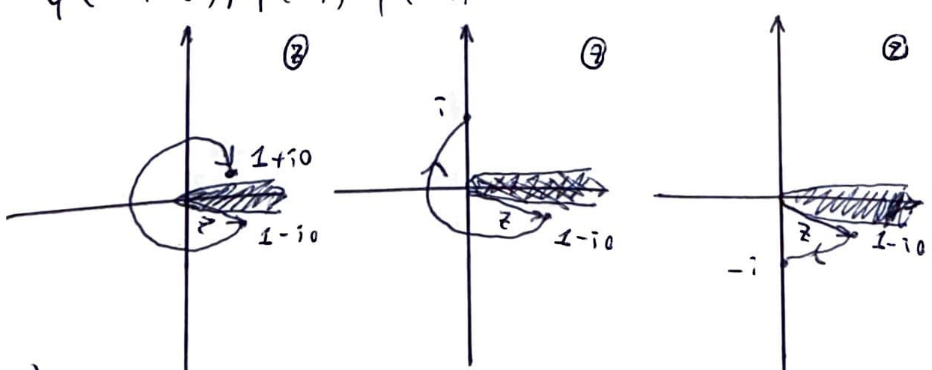
$\varphi(i-0)=\left|\frac{\sqrt[3]{i-0}}{e^{i\pi/3}}\right|e^{-i\pi/6}e^{i\pi/3}=e^{\pi i/6}$

II. $\varphi(z)=\ln z$

$z\in[0;\infty]$

$\varphi(1-i0)=0$

$\varphi(1+i0)$, $\varphi(i)$, $\varphi(-i)$ – ?



2) аналогично, но $\Delta\arg z=-\frac{3\pi}{2}\Rightarrow\varphi(i)=-\frac{3\pi i}{2}$

3) $\Delta\arg z=-\frac{\pi}{2}\Rightarrow\varphi(-i)=-\frac{\pi i}{2}$

1) $\Delta\arg z=-2\pi=\Delta\arg g$

$\varphi(1+i0)=\ln\left|\frac{g(1+i0)}{g(1-i0)}\right|+\varphi(1-i0)+i\Delta\arg g=0+0-2\pi i$

N 2.

$\varphi_1(z) = \sqrt{z - e^{-i\alpha}}$

$\varphi_2(z) = \ln(z - e^{-i\alpha})$

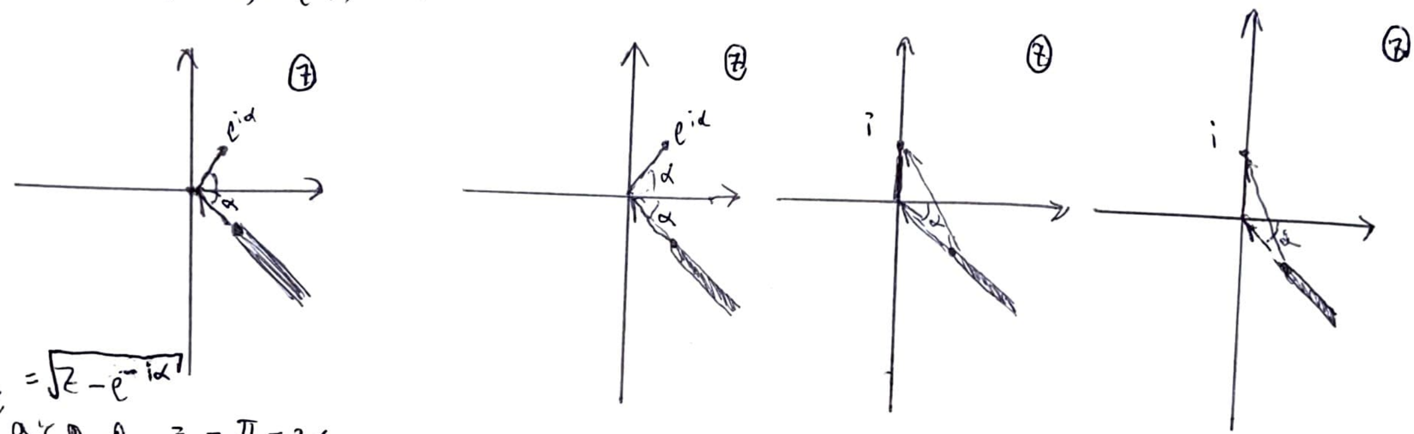
$\alpha \in (0; \pi/2)$

$\varphi_1(0) = i e^{-i\alpha/2}$

$\varphi_2(0) = -\pi i - i\alpha$

$\varphi_{1,2}(e^{i\alpha}),\ \varphi_{1,2}(i)$ — ?

1)



$g_1 = \sqrt{z - e^{-i\alpha}}$

$\Delta \arg g_1 = -\frac{\pi - 2\alpha}{4} = -\frac{\pi}{4} + \frac{\alpha}{2}$

равнобед. треуг.

$\varphi_1(e^{i\alpha}) = \left|\frac{\sqrt{e^{i\alpha} - e^{-i\alpha}}}{\sqrt{e^{-i\alpha}}}\right| i \cancel{e^{-i\alpha/2}} e^{-i(\frac{\pi}{4} - \cancel{\frac{\alpha}{2}})} =$

$= \sqrt{2\sin\alpha}\ e^{-i\pi/4}\ i = e^{\pi/4}\sqrt{2\sin\alpha}$

$\varphi_2(e^{i\alpha}) = \ln\left|\frac{e^{i\alpha} - e^{-i\alpha}}{-e^{-i\alpha}}\right| - i\pi - i\cancel{\alpha} - \frac{i\pi}{2} + i\cancel{\alpha} = \ln(2\sin\alpha) - \frac{3i\pi}{2}$

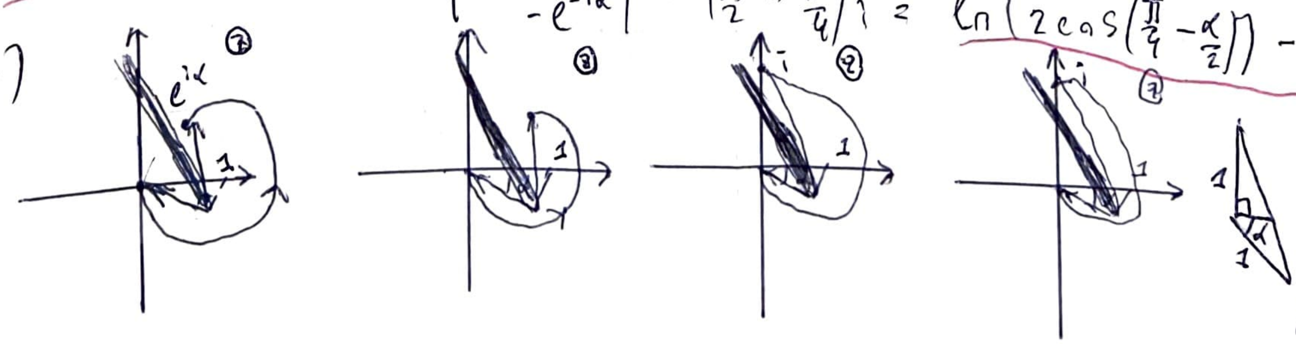
$\Delta \arg(z - e^{-i\alpha}) = -\frac{(\pi - (\frac{\pi}{2} + \alpha))}{2} = -\frac{\pi}{4} + \frac{\alpha}{2}$

$\varphi_1(i) = \left|\frac{\sqrt{i - e^{-i\alpha}}}{\sqrt{-e^{-i\alpha}}}\right| i e^{-i\alpha/2} e^{-i(\pi/8 - \alpha/4)} = i e^{-\alpha i/4 - \pi i/8} \sqrt{2\sin(\frac{\pi}{4} + \frac{\alpha}{2})} =$

$= e^{3\pi i/8 - \alpha i/4} \sqrt{2\cos(\frac{\pi}{4} - \frac{\alpha}{2})}$

$\varphi_2(i) = -i\pi - i\alpha + \ln\left|\frac{i - e^{-i\alpha}}{-e^{-i\alpha}}\right| + (\frac{\alpha}{2} - \frac{\pi}{4})i = \ln(2\cos(\frac{\pi}{4} - \frac{\alpha}{2})) - \frac{5\pi i}{4} - \frac{i\alpha}{2}$

2)



г) $\Delta \arg (z - e^{-i\alpha}) = 2\pi - \frac{\pi}{2} + \alpha = \frac{3\pi}{2} + \alpha$

$\varphi_1(e^{i\alpha}) = i\cancel{e^{-i\alpha/2}} \left| \frac{\sqrt{e^{i\alpha} - e^{-i\alpha}}}{\underbrace{\sqrt{-e^{-i\alpha}}}_{1}} \right| e^{i(\frac{3\pi}{2} + \cancel{\alpha})/2} = \sqrt{2\sin\alpha}\, e^{(5\pi/4)i}$

$\varphi_2(e^{i\alpha}) = -i\pi - i\alpha + i(\frac{3\pi}{2} + \alpha) + \ln \left| \frac{e^{i\alpha} - e^{-i\alpha}}{-e^{-i\alpha}} \right| = \frac{\pi i}{2} + \ln(2\sin a)$

$\Delta \arg (z - e^{-i\alpha}) = 2\pi - \frac{\pi}{4} + \frac{\alpha}{2} = \frac{7\pi}{4} + \frac{\alpha}{2}$

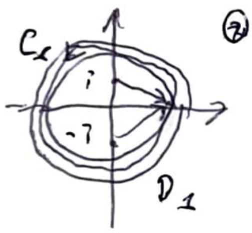
$\varphi_1(i) = i e^{-i\alpha/2} e^{i\alpha/4 + \frac{7\pi i}{8}} \left| \frac{\sqrt{i - e^{-i\alpha}}}{\sqrt{-e^{-i\alpha}}} \right| = e^{-\frac{5\pi i}{8} - \frac{i\alpha}{4}} \sqrt{2\cos(\frac{\pi}{4} - \frac{\alpha}{2})}$ см. п. 1

$\varphi_2(i) = -i\pi - i a + i(\frac{7\pi}{4} + \frac{\alpha}{2}) + \ln \left| \frac{i - e^{-i\alpha}}{-e^{-i\alpha}} \right| = \frac{3\pi i}{4} - \frac{i\alpha}{2} + \ln\left(2\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right)$

№3.

$f(z)$ − аналит. прод. $\sqrt{1+z^2}$ из $z>0$ вещ. в D: $\sqrt{1+z^2} = \sqrt{(z-i)(z+i)}$

1)



Рассм. оборот:
$\Delta\arg g = 2\pi \cdot 2/2 = 2\pi$

$f(z)\big|_{z=z_0} = f(z_0) \left| \frac{f(z)}{f(z_0)} \right| e^{2\pi i} = \sqrt{1+z_0^2} \left| \frac{\sqrt{1+z_0^2}}{\sqrt{1+z_0^2}} \right| = \sqrt{1+z_0^2} = f(z_0)$ => однозначная

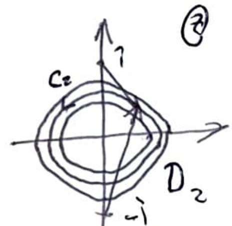
$I = \oint_{C_1} \sqrt{1+z^2}\,dz = -2\pi i \operatorname{res}_{z=\infty} f(z)$

При $z \to \infty$: $z\sqrt{1 + \frac{1}{z^2}} \approx z\left(1 + \frac{1}{2z^2} + \ldots\right)$

$1/2 \Rightarrow \operatorname{res}_{z=\infty} = -\frac{1}{2}$

$I = \pi i$

2)

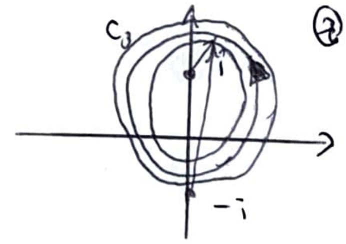


Рассм. оборот:
$\Delta\arg g = 0$

$f(z) = \{$то же самое$\}$ => но $\Delta\arg g = 0$
=> однозначна

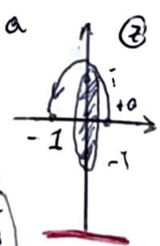
Поскольку внутри контура нет никаких особенностей
$I = 0$

3)

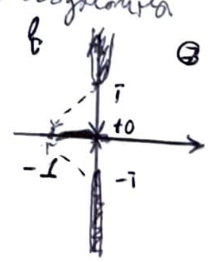


$\Delta\arg g = (2\pi + 0)/2 = \pi$
Совершая полный оборот:
$f(z_0) = f(z_0)|1| e^{\pi i} = -f(z_0)$ => ф-ция многозначна

а



б



а: $\Delta\arg(z - i) = \frac{7\pi}{4}$ $(2\pi - \frac{\pi}{4})$

$\Delta\arg(z + i) = \frac{\pi}{4}$

$\Delta\arg f = 2\pi/2 = \pi$

⇓

$f(-1) = f(+0) \left| \frac{f(-1)}{f(+0)} \right| e^{i\pi} = -1 \cdot \sqrt{2} = -\sqrt{2}$

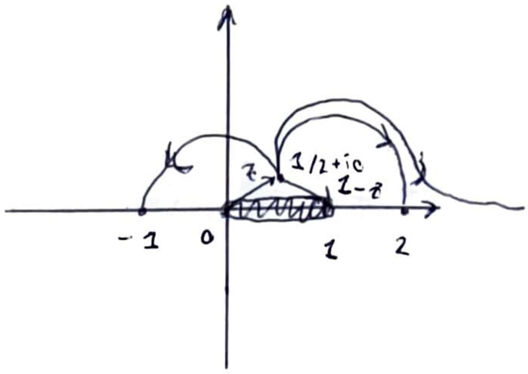
б) $\Delta\arg f = 0$; $f(-1) = \sqrt{2} \cdot e^0 = \sqrt{2}$

№4.

$\varphi(z) = z^{\mu}(1-z)^{1-\mu}$

$\varphi(\frac{1}{2}+i0) = \frac{1}{2}$ $\quad z \in [0,1]$

$\varphi(2); \varphi(-1); \lim_{z\to\infty} \frac{\varphi(z)}{z}$



$\varphi(2)$:

$\Delta \arg z = 0$

$\Delta \arg (1-z) = -\pi$

$\Delta \arg \varphi = \pi(\mu-1)$

$\varphi(2) = \cancel{\frac{1}{2}} \left| \frac{2^{\mu}(1-2)^{1-\mu}}{\cancel{\varphi(1/2+i0)}} \right| e^{i\pi(\mu-1)} =$

$= |2^{\mu}\underbrace{(1-2)}_{-1}^{1-\mu}| e^{i\pi(\mu-1)} =$

$= 2^{\mu} e^{i\pi(\mu-1)}$

$\varphi(-1)$:

$\Delta \arg z = \pi$

$\Delta \arg (1-z) = 0$

$\Delta \arg \varphi = \pi\mu$

$\varphi(-1) = \left| \frac{2^{1-\mu}(-1)^{\mu}}{\varphi(\frac{1}{2}+i0)} \right| \cancel{\varphi(\frac{1}{2}+i0)} e^{i\pi\mu} =$

$= 2^{1-\mu} e^{i\pi\mu}$

$\Delta \arg \varphi = \pi(\mu-1)$

$\lim_{z\to\infty} \frac{\varphi(z)}{z} = \lim_{z\to\infty} \cancel{\varphi(\frac{1}{2}+i0)} \left| \frac{z^{\mu}(1-z)^{1-\mu}}{\cancel{\varphi(1/2+i0)}} \right| e^{i\pi(\mu-1)} \frac{1}{z} =$

$= \lim_{z\to\infty} \left| \frac{z^{\mu-1}}{\cancel{(1-z)}^{\mu-1}} \right| e^{i\pi(\mu-1)} = e^{i\pi(\mu-1)}$

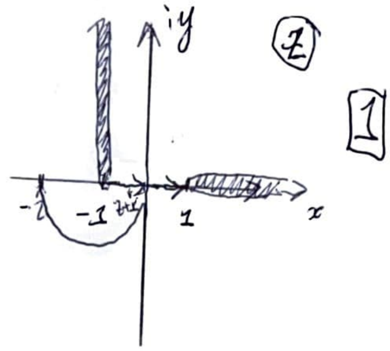
$\cancel{-1}$ 1

№5.

$D: [-1, -1+i\infty]$; $[1, +\infty]$ - ветви

$\varphi(z) = \ln(1-z^2)$ $\quad \varphi(a) = -2\pi i$

$g(z) = 1 - z^2$

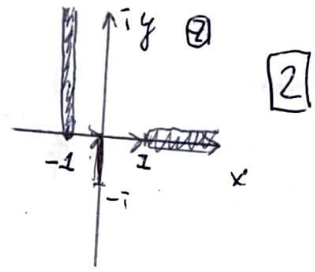


1) $\varphi(-2)$

$\Delta \arg(z+1) = -\pi$

$\Delta \arg(1-z) = 0$

$\Delta \arg g = -\pi$

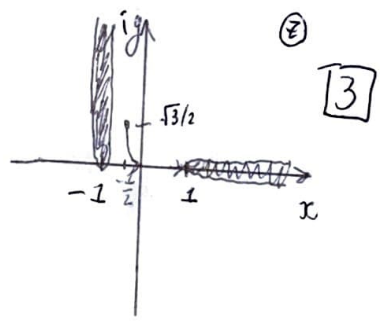


$\varphi(-2) = \ln\left[\left|\frac{g(z)}{g(z_0)}\right|\right] + \varphi(z_0) + i\Delta\arg g =$

$= \ln 3 + (-2\pi i) + i\cdot(-\pi) = \ln 3 - 3\pi i$

$\underline{\varphi(-2) = \ln 3 - 3\pi i}$

$\left|\frac{g(z)}{g(z_0)}\right| = \left|\frac{-3}{1}\right| = 3$



2) $\varphi(-i)$

$\Delta \arg(z+1) = -\pi/4$

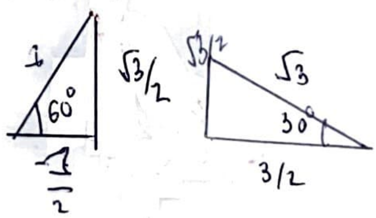
$\Delta \arg(1-z) = \pi/4$

$\Delta \arg g = 0$

$\varphi(-i) = \ln\left[\left|\frac{g(z)}{g(z_0)}\right|\right] + \varphi(z_0) + i\cancel{\Delta\arg}\overset{0}{} g =$

$\underline{= \ln 2 - 2\pi i}$

$\left|\frac{2}{1}\right| = 2 \qquad \varphi(-i) = \ln 2 - 2\pi i$



3) $\varphi\left(\frac{-1+\sqrt{3}i}{2}\right)$

$\Delta \arg(z+1) = \frac{\pi}{3}$

$\Delta \arg(1-z) = -\frac{\pi}{6}$

$\Delta \arg g = \frac{\pi}{6}$

$\left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{-\cancel{2}(1+\sqrt{3}i)}{\cancel{4}_2} \Rightarrow 1 - \ldots =$

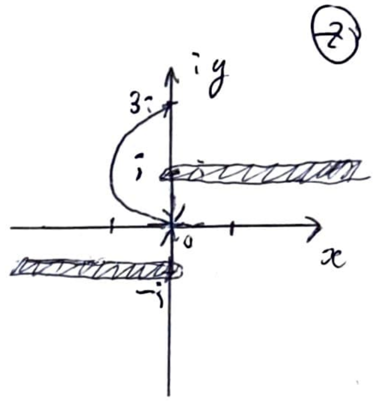
$= \frac{3+\sqrt{3}i}{2} \Rightarrow \left|\frac{3+\sqrt{3}i}{2}\right| = \sqrt{3}$

$\underline{\varphi(\ldots) = \ln(\sqrt{3}) - \frac{11\pi}{6}i}$

№6.

$\varphi(z) = \sqrt[3]{1+z^2}$; $\varphi(0)=1$ $\qquad \varphi(3i)-?$

1) ветви $[-i, -i-\infty]$ и $[i, i+\infty]$



$g(z) = (1+z^2) = (1+iz)(1-iz) = -(-i+z)(-i-z) =$
$= (z+i)(z-i)$

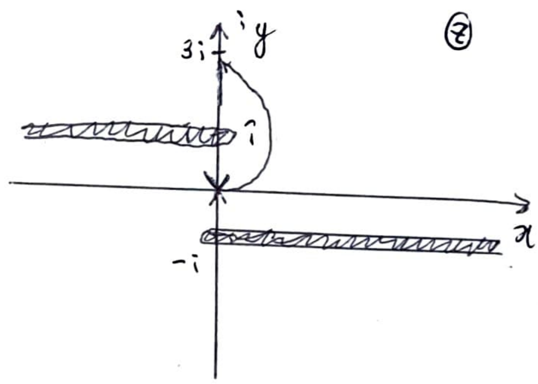
$\Delta \arg (z-i) = -\pi$

$\Delta \arg (z+i) = 0$

$\Delta \arg g = -\pi$

$\varphi(3i) = \sqrt[3]{\left|\frac{-8}{1}\right|}\, \varphi(0)\, e^{-i\pi/3} = 2e^{-i\pi/3}$

2) ветви $[-i, -i+\infty]$ $[i, i-\infty]$



$g(z)$ т. самое

$\Delta \arg (z-i) = \pi$

$\Delta \arg (z+i) = 0$

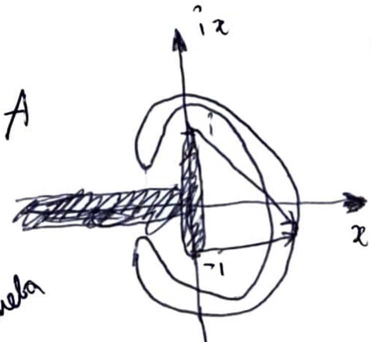
$\varphi(3i) = \sqrt[3]{\left|-\frac{8}{1}\right|}\, \varphi(0)\, e^{i\pi/3} = 2e^{i\pi/3}$

№7.

$f(x) = \ln[(x^2+1)^{1/2}]$

$\ln([(x+i)(x-i)]^{1/2})$

A



1) $\lim\limits_{\varepsilon\to 0} F(\varepsilon) = \lim\limits_{\varepsilon\to 0} \ln\sqrt{1+\varepsilon^2} = 0$

2) $\lim\limits_{\varepsilon\to 0} F(\varepsilon e^{3\pi i/4})$

$\Delta \arg (z+i) = 0$

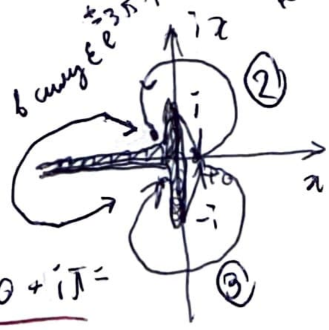
$\Delta \arg (z-i) = 2\pi$

$\Delta \arg g = \pi$

$g = \sqrt{1+z^2}$

$\lim\limits_{\varepsilon\to 0} F(\varepsilon e^{3\pi i/4}) = C + 0 + i\pi =$
$= i\pi$

в силу $\varepsilon e^{\pm 3\pi i/4}$ я как бы подхожу к нулю слева



3) $\lim\limits_{\varepsilon\to 0} F(\varepsilon e^{-3\pi i/4})$

$\Delta \arg (z+i) = -2\pi$

$\Delta \arg (z-i) = 0$

$\Downarrow$

$\lim\limits_{\varepsilon\to 0} F(\varepsilon e^{-3\pi i/4}) = -\pi i$

пусть есть аналит. продолжение $F(z)$. в A: $F(z) = f(x) = F(x) = \ln\sqrt{1+x^2}$

При движении по контуру (замк.) $\Delta \arg \sqrt{(x-i)(x+i)} = 0$, значит ф-ия $F(z) = \ln\sqrt{1+z^2}$ однозн. в A $(F(x) = \ln\sqrt{1+x^2})$

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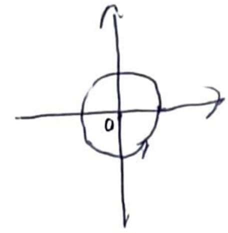
№8.

branch point @ $z=\infty$?

$f(z) = \ln z \to \ln \frac{1}{\xi}$ $\Delta \arg \xi = 2\pi$ $\Delta \arg f = -2\pi$



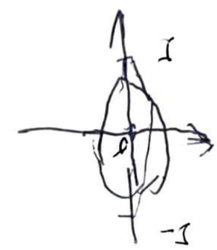
Полный оборот:

$f(\xi)_{\xi \to \xi_0} = f(\xi_0) + \ln \left| \frac{1/\xi_0}{1/\xi_0} \right| - 2\pi i =$

$= -2\pi i + f(\xi_0)$ => многозначная ф-ция

и $z = \infty$ - br. point

$f(z) = \ln \frac{z-1}{z+1} \to \ln \frac{1/\xi - 1}{1/\xi + 1} = \ln \frac{1-\xi}{1+\xi}$



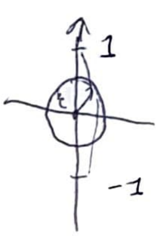
$\Delta \arg f = 0$ $\frac{1-\xi}{1+\xi} = 1$ при $\xi = 0$

$f(\xi)_{\xi \to \xi_0} = f(\xi_0) + \ln |1| + i \cdot 0 = f(\xi_0) \Rightarrow z = \infty$ - not br. point

3) $f(z) = \ln(z^2 - 1) \to \ln \left[\left(\frac{1}{\xi} - 1\right)\left(\frac{1}{\xi} + 1\right) \right] = \ln \left[\frac{(1-\xi)(1+\xi)}{\xi^2} \right]$

$f(\xi)_{\xi \to \xi_0} = f(\xi_0) + \ln \left| \frac{g(\xi)}{g(\xi_0)} \right| + i(-4\pi + 0 + 0) = -4\pi i + f(\xi_0)$; $\left[\ldots\right] = g$

от $\frac{1}{\xi^2}$



$\Downarrow$

$z = \infty$ - br. point

4) $f(z) = \sqrt{z^2 - 1} \to \frac{\sqrt{(1-\xi)(1+\xi)}}{\xi}$ $\Delta \arg g = -2\pi$ (контур тот же) вклад только от ξ в знам

$f(\xi)_{\xi \to \xi_0} = f(\xi_0) \left| \frac{g(\xi)}{g(\xi_0)} \right| e^{i \Delta \arg g} = e^{-2\pi i} f(\xi_0) = f(\xi_0) \Rightarrow z = \infty$ - not br. point

№9.

$f(z) = z^a (z-1)^b$

Исслед. ф-ции.

Кандидаты в т. ветвл.: 0, 1, ∞.

$z = 0$:

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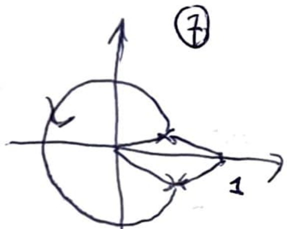

$\Delta \arg z = \pi \cdot 2$

$\Delta \arg z - 1 = 0$

Полный обход:

$$\frac{f(z+i0)}{f(z-i0)} = \underbrace{\left|\cancel{\frac{f(z+i0)}{f(z-i0)}}\right|}_{1} e^{i2\pi a}$$

$\Downarrow$

Если $a \in \mathbb{Z}$ - не т. ветвл.

$z = 1$: Полный обход:

$\Delta \arg z = 0$

$\Delta \arg z - 1 = 2\pi$

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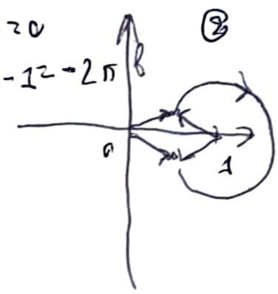

$$\frac{f(z+i0)}{f(z-i0)} = \overset{\text{см. пред. пункт}}{1} \cdot e^{i2\pi b}$$

$\Downarrow$

не т. ветв. при $b \in \mathbb{Z}$

$z = \infty$:

$$f(z) \to \left(\frac{1}{\xi}\right)^a \left(\frac{1}{\xi} - 1\right)^b = \xi^{-a-b}(1-\xi)^b$$

см. в т. 0

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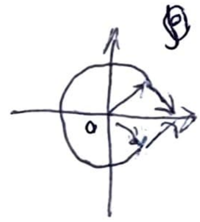


$\Delta \arg 1 - \xi$

$\Delta \arg \xi = 2\pi$

$\Downarrow$

$\Delta \arg g = 2\pi(-a-b)$

$$\frac{f(\xi_0+i0)}{f(\xi_0-i0)} = 1 \cdot e^{i(-a-b)2\pi}$$

$\Downarrow$

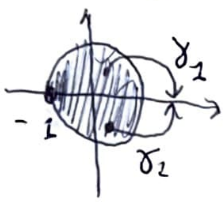
Если $a + b \in \mathbb{Z}$ - не т. ветв.

N		К т. ветв.
1	1	0
1	$\frac{1}{2}$	2
$\frac{1}{2}$	$\frac{1}{3}$	3
$\frac{2}{3}$	$\frac{1}{3}$	2

№10.

$f(z) \quad 1 > |z|$

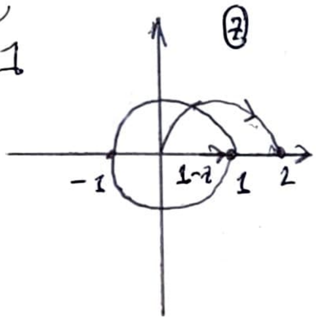
$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} z^n$$



$$f(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n+1} - \sum_{n=0}^{\infty} \frac{z^n}{n+2} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} - \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{n+2}}{n+2} =$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} - \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} + \frac{1}{z} = \frac{1}{z^2}\ln(1-z) - \frac{1}{z}\ln(1-z) + \frac{1}{z}$$

γ_1



$\psi(z) = \ln(1-z)$

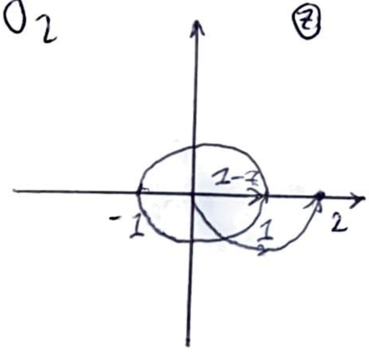
$g(z) = 1-z$

$\Delta \arg g = -\pi$

$$\psi(2) = \psi(0) + \ln\left|\frac{g(2)}{g(0)}\right| + i\Delta \arg g = 0 + \ln 1 - \pi i = -\pi i$$

$$f(2) = -\frac{1}{4}\pi i + \frac{\pi i}{2} + \frac{1}{2} = \frac{\pi i}{4} + \frac{1}{2}$$

γ_2



То же самое, но $\Delta \arg g = \pi \Rightarrow$

$\Rightarrow \psi(2) = \pi i$

$$f(2) = -\frac{\pi i}{4} + \frac{1}{2}$$

Ответ: красным.

№11

$z_n(t)$ $[n=0,1,2]$

$z^3-3z^2+t=0$

$z_n(t>>1)\approx t^{1/3}e^{\pi i+2\pi i n/3}$

1) $t_{0,1}$ — ?

$z^2(z-3)=-t$

Если $t=0$:

$z^2=0$, $z-3=0$ $\Rightarrow$ $z_1=0$, $z_2=0$, $z_3=3$ $\Rightarrow$ слились

$z^3-2z^2=z^2-t$

$z^2(z-2)=z^2-t$; $t=4$; $\sqrt{t}=2$

$z^2(z-2)=(z-2)(z+2)$

$z^2-z-2=0$

$\underline{t_{0,1}=0,4}$

$z_1=2$

$z_2=-1$ $\Rightarrow$ слились

$z_3=2$

2) $z_0(t)$ $t\approx t_0$ $t>t_0$ $f_0(t_0+\rho e^{i\varphi})$ ρ — мало, $\rho>0$

$z_0(t>>1)\approx t^{1/3}e^{\pi i}$ (<0)

$|f(t_0+\rho e^{i\varphi})|=\left|-\sqrt{\frac{0+\rho e^{i\varphi}}{3}}\right|=\sqrt{\frac{\rho}{3}}$

$|f(t_0+0)|=\left|-\sqrt{\frac{t_0+0}{3}}\right|\Rightarrow$

$\Rightarrow |f|=-f$

$\Downarrow$

$\underline{f(t_0+\rho e^{i\varphi})=e^{i\varphi/2}(-1)\cdot\sqrt{\frac{\rho}{3}}=}$

$\underline{=-\sqrt{\frac{\rho}{3}}e^{i\varphi/2}}$

Формулы Кардано:

$p=\frac{c}{a}-\frac{b^2}{3a^2}=-3$

$q=\frac{2b^3}{27a^3}-\frac{bc}{3a^2}+\frac{d}{a}=t-2$

$x=z-\frac{b}{3a}=z-1$

$Q=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2=-1+\left(\frac{t-2}{2}\right)^2$

$\alpha=\sqrt[3]{-\frac{q}{2}+\sqrt{Q}}=\sqrt[3]{\frac{2-t}{2}+\sqrt{\frac{(t-2)^2}{4}-1}}=\sqrt[3]{1-\frac{t}{2}+\sqrt{\frac{t^2}{4}-1}}\approx\sqrt[3]{1-\frac{t}{2}+i\sqrt{t}\sqrt{1-\frac{t}{4}}}\approx$

$\approx 1+i\frac{\sqrt{t}}{3}$

$\beta=\sqrt[3]{-\frac{q}{2}-\sqrt{Q}}=\sqrt[3]{1-\frac{t}{2}-\sqrt{t^2/4-1}}\approx 1-i\frac{\sqrt{t}}{3}$

$\frac{\alpha+\beta}{2}=1$; $\frac{\alpha-\beta}{2}\sqrt{3}=\frac{i\sqrt{t}}{\sqrt{3}}$

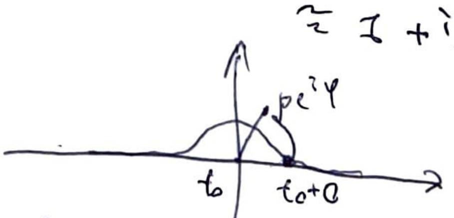
Ищу отрицат. корень:

$x=z_0-1=-1-\frac{\sqrt{t}}{\sqrt{3}}$

$\Downarrow$

$z_0=-\sqrt{\frac{t}{3}}$

(Можно проверить другие корни, но и так очевидно — знак min)



$f(t_0+\rho e^{i\varphi})=f(t_0+0)e^{i\Delta\arg f}\left|\frac{f(t_0+\rho e^{i\varphi})}{f(t_0+0)}\right|$

$\Delta\arg f=\frac{\varphi}{2}$

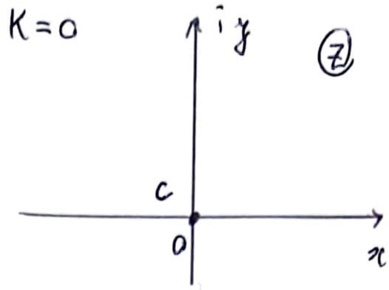
$$\frac{e^{iz}}{\cos z - 1} = \varphi(z)$$

$$\sum_{n=-\infty}^{\infty} c_n z^n \qquad |z| \in (2\pi k, 2\pi(k+1))$$

C_{-3} for $k=0,1$ - ?

$$C_{-3} = \frac{1}{2\pi i}\int_C z^2 \varphi(z)\, dz$$

$k=0$



Очевидно особ. т. $z=0$:

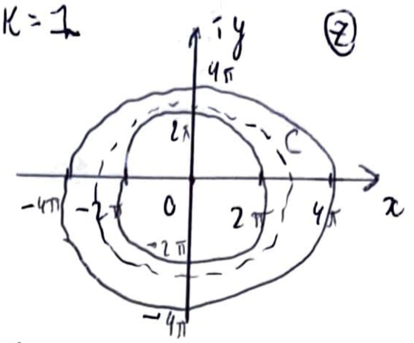
$$\int_C \frac{z^2 e^{iz}}{\cos z - 1} = 2\pi i \operatorname{res}_{z=0} (\varphi(z) z^2)$$

$$\lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{z^4 e^{iz}}{\cos z - 1}\right) = \lim_{z\to 0} \frac{e^{iz} z^3(-iz + z\sin x + (4+iz)\cos z - 4)}{(\cos z - 1)^2} =$$

$= 0$

(по Лопиталю)

$$C_{-3} = 0$$



$k=1$:

$$\int_C \frac{z^2 e^{iz}}{\cos z - 1} = 2\pi i \operatorname{res}_{z=0,\,-2\pi,\,2\pi} (\varphi(z) z^2)$$

При $z=0$: 0

При $z=2\pi$:

$$\xi = z - 2\pi \Rightarrow \frac{(\xi+2\pi)^2 e^{i\xi}}{\cos\xi - 1} \approx \frac{(\xi^2 + 4\pi\xi + 4\pi^2)(1+i\xi)}{\cancel{1} - \xi^2/2 - \cancel{1}} \approx$$

$$\approx -\frac{8\pi}{\xi} - \frac{8\pi^2 i}{\xi}$$

$$\operatorname{res}_{z=2\pi} \ldots = -(8\pi + 8\pi^2 i)$$

$z = -2\pi$:

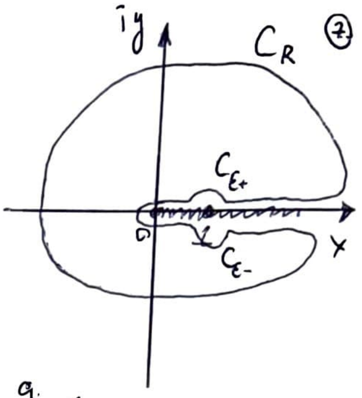
$\xi = z + 2\pi$

$$\frac{(\xi^2 - 4\pi\xi + 4\pi^2)(1+i\xi)}{-\xi^2/2} \approx \frac{8\pi}{\xi} - \frac{8\pi^2 i}{\xi}$$

$$\Rightarrow \int_C \ldots = -16\pi^2 i$$

№13

$$PV\int_0^\infty \frac{x^{a-1}}{1-x^b}dx \quad b>a>0$$



$$p.v.\int_0^\infty \frac{x^{a-1}}{1-x^b}dx = p.v.\int_0^\infty \frac{z^{\cancel{(a-1)/b}}\, z^{\frac{1}{b}-1}}{(1-z)b}dz =$$

$z=x^b$

$z^{1/b}=x$

$dx=\frac{1}{b}z^{1/b-1}dz$

$$= p.v.\int_0^\infty \frac{z^{a/b-1}}{b(1-z)}dz = I\cdot b$$

$$\oint = I + \int_{C_{\varepsilon+}} + \int_{C_{\varepsilon-}} + \cancel{\int_{C_R}}^{0,\text{л.ж.}} - e^{2\pi i\frac{a}{b}} I = 0$$

$$\left[\begin{array}{l} \cap: \frac{x^{\frac{a}{b}-1}}{1-x} \simeq \{x=1+\varepsilon e^{i\varphi}\} = -\frac{1}{\varepsilon}e^{-i\varphi} \\ \cup: \frac{x^{a/b-1}}{1-x} = -\frac{1}{\varepsilon}e^{2\pi i\overbrace{(a/b-1)}^{\gamma}-i\varphi} \end{array}\right.$$

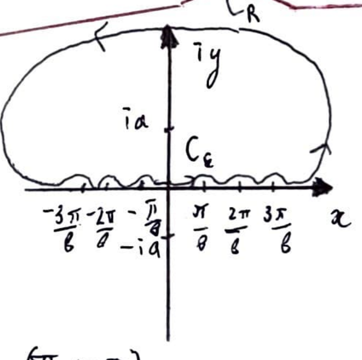
$$\int_{C_{\varepsilon+}} = \int_\pi^0 \cancel{\varepsilon} i \cancel{e^{i\varphi}}\left(-\frac{1}{\cancel{\varepsilon}}\cancel{e^{-i\varphi}}\right)d\varphi = \pi i$$

$$\int_{C_{\varepsilon-}} = \int_0^{-\pi} \cancel{\varepsilon} i \cancel{e^{i\varphi}}\left(-\frac{1}{\cancel{\varepsilon}}e^{2\pi i\gamma}\cancel{e^{-i\varphi}}\right)d\varphi = i\pi e^{2\pi i\gamma}$$

$$I = -\frac{\pi i(1+e^{2\pi i\gamma})}{b(1-e^{2\pi i\gamma})} = -\frac{\pi i}{b}\left(\frac{\cos(\pi\gamma)\cdot 2i}{i(1-e^{2\pi i\gamma})e^{-\pi i\gamma}}\right) = \frac{\pi}{b}ctg\left(\frac{\pi a}{b}-\pi\right) = \pi/b\; ctg\left(\frac{\pi a}{b}\right)$$

$$\sin x = \frac{e^{2ix}-1}{2i}e^{-ix}$$

$$\cos x = \frac{e^{2ix}+1}{2}e^{-ix}$$



$$p.v.\int_0^\infty \frac{x\,dx}{(x^2+a^2)\sin bx} = \frac{1}{2}\underbrace{p.v.\int_{-\infty}^\infty \frac{x\,dx}{(x^2+a^2)\sin bx}}_{I} \quad a>0, b>0$$

ф-ия чётная

Можно заметить, что особенности находятся в точках $\frac{(\pi+n\pi)}{b}$ и $-\frac{(\pi+n\pi)}{b}$, $n\in N$

$$\oint = p.v.\int_{-a}^a \frac{x\,dx}{(x^2+a^2)\sin bx} + \cancel{\int_{C_\varepsilon}}^{0,\text{ см. ниже}} + \cancel{\int_{C_R}}^{0,\text{ по л. Жордана}} = 2\pi i \operatorname*{res}_{z=ia} f(z) = \frac{\pi}{sh(ab)}$$

$a=\pi n+\frac{\pi}{2}, n\to\infty$
$\Rightarrow a\to\infty$

$$\int_{C_\varepsilon}\frac{z\,dz}{(z^2+a^2)\sin bz} = \left\{\begin{array}{l} z=z_0+\varepsilon e^{i\varphi} \\ z_0=\pm\left(\frac{\pi+n}{b}\right), n\in N \\ \varepsilon\to 0 \\ dz=\varepsilon e^{i\varphi}\cdot i\,d\varphi \end{array}\right\} = \int_\pi^0 \frac{(z_0+\varepsilon e^{i\varphi})\varepsilon e^{i\varphi}i\,d\varphi}{((z_0+\varepsilon e^{i\varphi})^2+a^2)\sin(b(z_0+e^{i\varphi}\varepsilon))} \simeq$$

$$\simeq \int_\pi^0 \frac{z_0 \cancel{\varepsilon e^{i\varphi}} i\,d\varphi}{(z_0^2+a^2)(-1)^n \cancel{\varepsilon} b \cancel{e^{i\varphi}}} = \int_\pi^0 \frac{z_0 i\,d\varphi}{(z_0^2+a^2)(-1)^n b}$$

, очевидным образом при совпадающих n, но $-\pi$ и π делают разный знак и сокращаются $\Rightarrow \int_{C_\varepsilon}=0$

$$\operatorname*{res}_{z=ia} f(z) = \frac{g(z)}{h'(z)}\Bigg|_{z=ia} = \frac{ia}{b(\underbrace{\cancel{a^2-a^2}}_{0})\cos(iab)+2ia\sin(iab)} =$$

$$= -\frac{i\cancel{a}}{2\cancel{a}\,sh(ab)} = -\frac{i}{2sh(ab)}$$

Ответ: $I=\frac{1}{2}\oint = \frac{\pi}{2sh(ab)}$

$\boxed{12/12}$