№1 $\boxed{14}$

Каноним. ур-е окр-ти. $x^2+y^2=r^2$

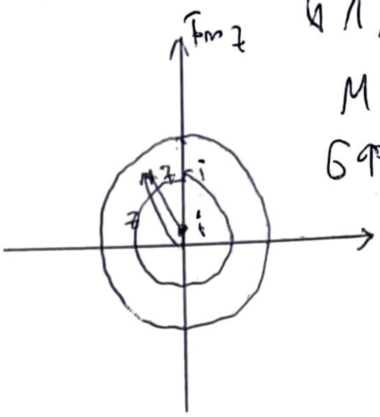
1) $2\leq|z-i|\leq 4$

$2\leq|x+i(y-1)|\leq 4$

$4\leq(x+i(y-1))(x-i(y-1))\leq 16$

$4\leq x^2+(y-1)^2\leq 16$

Каждетя и правда кольцо, $r=2$; $R=4$.



ГЛУМЕНАУ
М. И.
БФЗ191

$S=\pi(R^2-r^2)=\pi(16-4)=12\pi$

2) $|z-4i|+|z+4i|=10$ каноним. ур-е эллипса! $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$|x+i(y-4)|+|x+i(y+4)|=10$ (обе части в ^{2} и раскрыл модули)

$x^2+(y-4)^2+x^2+(y+4)^2+2\sqrt{(x+i(y-4))(x+i(y+4))(x-i(y-4))(x+i(y+4))}=100$

$2x^2+2y^2+32+2\sqrt{(x^2+y^2-8y+16)(x^2+y^2+8y+16)}=100$

$x^2+y^2+16+\sqrt{((x^2+y^2+16)-8y)((x^2+y^2+16)+8y)}=50$

$x^2+y^2+16+\sqrt{x^4+x^2y^2+16x^2+x^2y^2+y^4+16y^2+16x^2+16y^2+256-64y^2}=50$

$x^2+y^2+16+\sqrt{x^4+y^4+32x^2+2x^2y^2-32y^2+256}=50$

$x^2+y^2+16+\sqrt{(x^2+y^2)^2+32(x^2-y^2+8)}=50$

$34-x^2-y^2=\sqrt{(x^2+y^2)^2+32(x^2-y^2+8)}$

$34^2-68(x^2+y^2)+\cancel{(x^2+y^2)^2}=\cancel{(x^2+y^2)^2}+32(x^2-y^2+8)$

$34^2-100x^2-36y^2-256=0$

$100x^2+36y^2=900$

$\frac{x^2}{9}+\frac{y^2}{25}=1$

Большая полуось $b\Rightarrow b=\sqrt{25}=5$ $\boxed{1/9}$

3) $\operatorname{Im}\frac{1}{z}=1$ $R=\frac{1}{2}$

$\operatorname{Im}\frac{x-iy}{x^2+y^2}=1\Rightarrow$ $x^2+y^2=-y\Rightarrow x^2+(y^2+y+\frac{1}{4})=\frac{1}{4}\Rightarrow x^2+(y+\frac{1}{2})^2=\frac{1}{4}$

$$\sum_{i=0}^{n-1} i\varepsilon^i = \frac{n}{\varepsilon-1}$$

№ 2 $\boxed{1H}$

(везде у сумм $i=0$)

$S = \frac{b_1(1-q^n)}{1-q}$ $b_1 = \varepsilon$ $q = \varepsilon$

$$\sum_{0}^{n-1} i\varepsilon^i = \frac{d}{d\varepsilon}\sum_{0}^{n-1}\varepsilon^{i+1} = \frac{d}{d\varepsilon}\sum_{k=1}^{n}\varepsilon^k = \frac{d}{d\varepsilon}\frac{\varepsilon(1-\varepsilon^n)}{1-\varepsilon} = \frac{(1-(n+1)\varepsilon^n)(1-\varepsilon)-\varepsilon^{n+1}+\varepsilon}{(1-\varepsilon)^2}$$

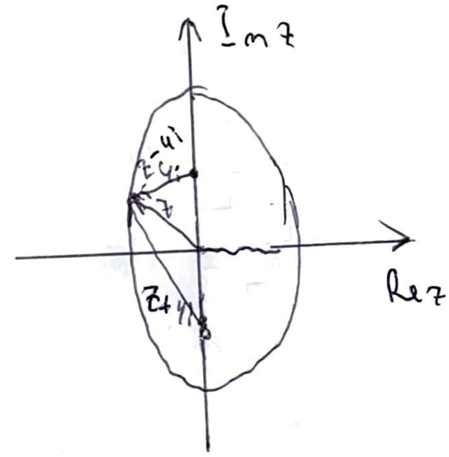
$$= \frac{1-\cancel{\varepsilon}-(n+1)\varepsilon^n+\varepsilon^{n+1}(n+1)-\varepsilon^{n+1}+\cancel{\varepsilon}}{(1-\varepsilon)^2} = \frac{(n+1)(\varepsilon-1)\varepsilon^n-\varepsilon^{n+1}+1}{(\varepsilon-1)^2} =$$

$$= \frac{((n+1)(\varepsilon-1)-\varepsilon)\varepsilon^n+1}{(\varepsilon-1)^2} = \frac{(n(\varepsilon-1)+\cancel{\varepsilon}-1-\cancel{\varepsilon})\varepsilon^n+1}{(\varepsilon-1)^2} = \frac{(n(\varepsilon-1)-1)\varepsilon^n+1}{(\varepsilon-1)^2} =>$$

при усл.:

$\varepsilon^n = 1$

$$=> \frac{n}{\varepsilon-1} = \frac{n(\varepsilon-1)-\cancel{1}+\cancel{1}}{(\varepsilon-1)^2} = \frac{n}{\varepsilon-1}$$



№ 3.

$\operatorname{Im} z = 1 \quad => \quad z = x + i \quad (y=1, \text{т.к. } \operatorname{Im} z = 1)$

$z \to w(z) = z^3 + 3z - i$

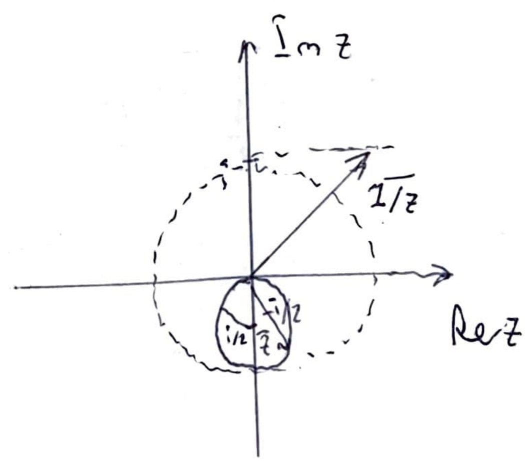
$$w(z) \to (x+i)^3 + 3(x+i) - i = x^3 + 3x^2 i - \cancel{3x} - i + \cancel{3x} + 3i - i =$$
$$= x^3 + 3x^2 i + i = x^3 + i(3x^2+1)$$

1) $\operatorname{Re}(w) = x^3$

$\operatorname{Im}(w) = 3x^2 + 1 = 3|\operatorname{Re}(w)|^{2/3} + 1$

2) $|z - i| = 1$

$z \to w(z) = \frac{1}{z-2i}$



$|z - i| = 1$

$|x + i(y-1)| = 1$ (в квадрат)

$x^2 + (y-1)^2 = 1$ $(*)$

$$w(z) \to \frac{1}{x+i(y-2)} = \frac{x-i(y-2)}{x^2+(y-2)^2} = \frac{x}{x^2+(y-2)^2} - \frac{y-2}{x^2+(y-2)^2}i$$

$$\operatorname{Im}(w) = \frac{2-y}{x^2+y^2-4y+4} = \frac{2-y}{\underbrace{(x^2-2y+y^2+1)}_{(y-1)^2}+3-2y} = \frac{2-y}{4-2y} = \frac{1}{2}$$

см. $*$ $\quad 1$

$\boxed{2/9}$

№4.

$\boxed{1M}$

$z = x + iy$

$$\boxed{\begin{aligned} &\mathrm{I} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ &\mathrm{II} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{aligned}}$$

а) $w(z) = x^2 + y^2$

$u = x^2 + y^2$

$v = 0$

I: $2x \neq 0 \Rightarrow$ нет

б) $w(z) = x^2 - y^2 + 2ixy$

$u = x^2 - y^2$

$v = 2xy$

I: $2x = 2x$

II: $-2y = -2y$ $\Rightarrow$ вып.

в) $w(z) = \frac{1}{x+iy}$

$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$u = \frac{x}{x^2+y^2}$

$v = -\frac{y}{x^2+y^2}$

I: $\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = -\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

II: $-\frac{2yx}{(x^2+y^2)^2} = -\left(-\left(-\frac{2xy}{(x^2+y^2)^2}\right)\right) = -\frac{2xy}{(x^2+y^2)^2}$

$\Downarrow$

вып.

№5.

$f(z = x + iy)$

$\cos 2\varphi = \frac{e^{2i\varphi} + e^{-2i\varphi}}{2}$

1) $|f| = e^{r^2 \cos 2\varphi}$ $\quad z = re^{i\varphi}$

$|f| = e^{r^2\left(\frac{e^{2i\varphi}+e^{-2i\varphi}}{2}\right)} = e^{\frac{z^2+z^{*2}}{2}}$

$w = \ln f = \underbrace{\ln|f|}_{u} + \underbrace{i\arg f}_{iv}$

$\frac{z^2+z^{*2}}{2} = \frac{x^2 + \cancel{2ixy} - y^2 + x^2 - \cancel{2ixy} - y^2}{2} = x^2 - y^2$

$u = x^2 - y^2$

I: $2x = \frac{\partial v}{\partial y} \Rightarrow v = 2xy + C(x)$

II: $-2y = -\frac{\partial v}{\partial x} = -2y - C'(x) \Rightarrow C'(x) = 0 \Rightarrow C(x) = \underset{\uparrow\,const}{C_0}$

$u = x^2 - y^2$

$v = 2xy + C_0$

$f = e^{x^2-y^2} e^{i(2xy+C_0)} = e^{z^2} e^{C_0}$, частное e^{z^2}

2) $\mathrm{Arg}(f) = xy$

$w = \underbrace{\ln|f|}_{u} + \underbrace{i\arg f}_{iv}$

$v = xy$

I: $x = \frac{\partial u}{\partial x} \Rightarrow u = \frac{x^2}{2} + C(y)$

II: $C'(y) = -\frac{\partial v}{\partial x} = -y \Rightarrow C(y) = -\frac{y^2}{2} + \underset{\uparrow\,const}{C_0}$

$f = e^{\frac{x^2-y^2}{2}} e^{C_0} e^{ixy} = e^{z^2/2} e^{C_0}$; частное $e^{z^2/2}$

№6. $\boxed{1H}$

3) $u = \varphi(x^2-y^2)$

$\partial_x u = \partial_y v$
$\partial_y u = -\partial_x v$ — усл. Коши–Римана

$\partial^2_{xy} v = \partial_x \partial_y v = \partial_x^2 u$
$\partial^2_{xy} v = \partial_y \partial_x v = -\partial_y^2 u$ — это и для 2 пункта

$\Delta u = 0$

$\partial_x\, 2x\, \varphi'(x^2-y^2) = 2\varphi'(x^2-y^2) + 4x^2\varphi''(x^2-y^2)$

$\partial_y\, -2y\, \varphi'(x^2-y^2) = -2\varphi'(x^2-y^2) + 4y^2\varphi''(x^2-y^2)$

$\Delta u = 4(x^2+y^2)\,\varphi''(x^2-y^2) = 0$

$\varphi''(x^2-y^2) = 0$

$\varphi''(t) = 0$

$\varphi(t) = C_0 t + C_1$

$u(x,y) = C_1 + C_0(x^2-y^2)$

$\left.\begin{aligned} \partial_x u &= 2x C_0 = \partial_y v \\ \partial_y u &= -C_0 2y = -\partial_x v \end{aligned}\right\} \Rightarrow$

$f(z) = u(x,y) + i v(x,y)$

$f(z) = iC_2 + C_1 + C_0(x^2 + 2ixy - y^2) = C_1 + C_0 z^2 + iC_2$ $\quad (C_1 + iC_2) = a$, $C_0 = b$

2) $\varphi\left(\frac{y}{x}\right) = u$

$\partial_x\left(-\frac{y}{x^2}\varphi'\left(\frac{y}{x}\right)\right) = \frac{2y}{x^3}\varphi'\left(\frac{y}{x}\right) + \frac{y^2}{x^4}\varphi''\left(\frac{y}{x}\right)$

$\partial_y\left(\frac{1}{x}\varphi'\left(\frac{y}{x}\right)\right) = \varphi''\left(\frac{y}{x}\right)\cdot\frac{1}{x^2}$

$\Rightarrow \varphi''\left(\frac{y}{x}\right)\left[\frac{y^2+x^2}{x^4}\right] + \frac{2y}{x^3}\varphi'\left(\frac{y}{x}\right) = 0$

$\varphi''\left(\frac{y}{x}\right)\left[\left(\frac{y}{x}\right)^2+1\right] = -\frac{2y}{x}\varphi'\left(\frac{y}{x}\right)$ $\boxed{1H}$

$\frac{y}{x} = t$

$\varphi''(t)\left[1+t^2\right] = -2t\,\varphi'(t)$

$\frac{d\varphi'}{dt} = \frac{-2t}{1+t^2}\varphi'$

$\frac{d\varphi'}{\varphi'} = -\frac{2t}{1+t^2}dt$

$\ln\varphi' = -\ln(1+t^2)+\tilde{C}_0$

$\varphi' = \frac{C_0}{1+t^2}$ $\qquad C_0 = e^{\tilde{C}_0}$

$d\varphi = \frac{C_0\,dt}{1+t^2}$

$\frac{1}{(1-it)(1+it)} = \left(\frac{1}{1-it}+\frac{1}{1+it}\right)\frac{1}{2}$

$\varphi = \frac{iC_0}{2}\left(\ln(1+it)+\ln(1-it)\right)+C_1$

$u = \frac{iC_0}{2}\left(\ln\left(\frac{1-i\,y/x}{1+i\,y/x}\right)\right)+C_1$

$\partial_x u = -C_0\frac{1}{1+y^2/x^2}\cdot\frac{y}{x^2} = \partial_y v \Rightarrow \partial_y v = -\frac{C_0 y}{x^2+y^2}$

$v = -C_0\int\frac{y\,dy}{x^2+y^2} = -\frac{C_0}{2}\ln(x^2+y^2)+\theta(x)$

$\partial_y u = \frac{C_0}{1+y^2/x^2}\cdot\frac{1}{x} = \frac{C_0 x}{x^2+y^2} = -\partial_x v \Rightarrow \frac{C_0 x}{x^2+y^2}+\theta'(x) = \frac{C_0 x}{x^2+y^2} \Rightarrow \theta(x) = \text{const}$

$v = -\frac{C_0\ln(x^2+y^2)}{2}+C_2$

$f = \underbrace{C_1+iC_2}_{\text{as } a} - \underbrace{\frac{iC_0}{2}}_{\text{as } b}\ln(x^2+y^2)+\frac{iC_0}{2}\left(\ln\left(\frac{x-iy}{x+iy}\right)\right) = a+\frac{ib}{2}\left(\frac{\cancel{x-iy}}{(x+iy)(x+iy)\cancel{(x-iy)}}\right) =$

$= a+\frac{ib}{2}\ln\left(\frac{1}{x^2+2xyi-y^2}\right) = a-\frac{ib}{2}\ln(z^2) = a+i\tilde{b}\ln(z)$

Contin [illegible], C+

№7.

1) $\int_c z\,dz = \int_c e^{i\varphi} e^{i\varphi} i\,d\varphi =$ $C: R=1$; center: $z=0$

$= \int_0^{2\pi} e^{2i\varphi} i\,d\varphi = \frac{\cancel{i} e^{2i\varphi}}{2\cancel{i}}\Big|_0^{2\pi} = \frac{e^{4\pi i}}{2} - \frac{1}{2} = 0$

2) $\int_c z^* dz = \int_c \cancel{e^{-i\varphi}}\, \cancel{e^{i\varphi}}\, i\,d\varphi = \int_0^{2\pi} i\,d\varphi = 2\pi i$

$\boxed{dz = R\,de^{i\varphi} = iRe^{i\varphi}d\varphi}$

$\boxed{z = Re^{i\varphi}}$

№8

$\int_c \frac{y\,dx - x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{\sin\varphi\, d(\cos\varphi) - \cos\varphi\, d(\sin\varphi)}{1} = \int_0^{2\pi} \frac{-(\sin^2\varphi + \cos^2\varphi)\,d\varphi}{1} = \int_0^{2\pi} -d\varphi = -2\pi$ (особая точка $(0;0)$)

$C: R=1$ center 1) $0;0$ 2) $2;0$

$z = (\cos\varphi + i\sin\varphi)R = x+iy$

2) В контуре нет особенностей $\Rightarrow 0$

№9

$p(n) = \frac{1}{2\pi i}\int_c dz\, z^{-1-n} \prod_{k=1}^{\infty} \frac{1}{1-z^k}$ $C: R<1$ $p(n)$ - пат?

$\prod_{k=1}^{\infty} \frac{1}{1-z^k} = \prod_{k=1}^{\infty} (1+z+z^2\ldots)(1+z^2+z^4\ldots)\ldots = \prod_{k=1}^{\infty}\sum_{m=0}^{\infty} z^{km}$

$p(n) = \frac{1}{2\pi i}\int_c \left(\prod_{k=1}^{\infty}\sum_{m=0}^{\infty} z^{km}\right) z^{-1-n}\,dz = \frac{\cancel{i}}{2\pi \cancel{i}}\int_0^{2\pi} (Re^{i\varphi})^{-1-n} \left(\prod_{k=1}^{\infty}\sum_{m=0}^{\infty}(Re^{i\varphi})^{km}\right) e^{i\varphi} R\,d\varphi =$

$\boxed{z = Re^{i\varphi}}$ $\boxed{dz = Ri\,d\varphi \cdot e^{i\varphi}}$

$= \frac{1}{2\pi}\int_0^{2\pi} (Re^{i\varphi})^{-n} \prod_{k=1}^{\infty}\sum_{m=0}^{\infty}(Re^{i\varphi})^{km}\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi}(Re^{i\varphi})^{-n} \cdot Q \cdot (Re^{i\varphi})^{l}\,d\varphi =$

$= 2\pi/2\pi\ \delta_{nl} \cdot Q$, где Q - кол-во способов получить $n=l$;

натуральное число

т.е. $0 \in N$

1Ч

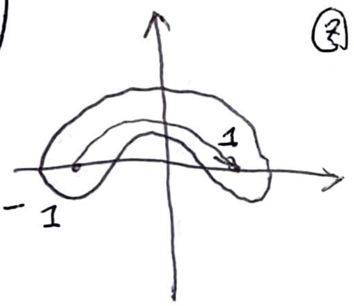
$P(1) = 1$

$P(4) = 5$

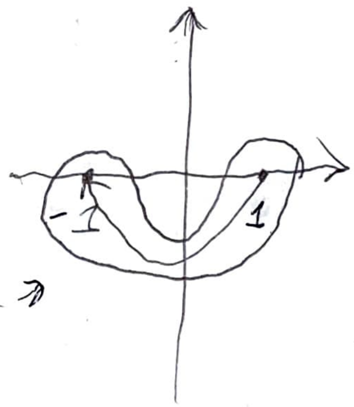
I	II	III	IV	I	II	III
1	1	1	z^4	1	z^4	1
z^2	z^2	1		z	1	z^3
z^4	1	1				

№10.

а)



б)



Ф

рассм. дугу $R=1$

$y(1) = 0$

$y'(z) = \frac{1}{2z}$

$$\int_{\Phi a} \frac{1}{2z} dz = \int_0^{\pi} \frac{1}{2} \frac{1}{\cancel{R}} \cancel{e^{-i\varphi}} \cdot i\cancel{R}\cancel{e^{i\varphi}} d\varphi = \int_0^{\pi} \frac{1}{2} i \, d\varphi = \frac{\pi i}{2}$$

$z = Re^{i\varphi}$

$dz = iRe^{i\varphi} d\varphi$

$$\int_{\Phi б} \frac{1}{2z} = \int_0^{-\pi} \frac{1}{2\cancel{R}} \cancel{e^{-i\varphi}} i\cancel{R}\cancel{e^{i\varphi}} d\varphi = \int_0^{-\pi} \frac{1}{2} i \, d\varphi = -\frac{\pi i}{2}$$

№11.

$\boxed{1H}$

$$\frac{1+2z^2}{z^3+z^5} \qquad z=0$$

не имеют смысла, т.к. 4 степень

$$\frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\left(\frac{1+2z^2}{(1+z^2)}\right) = \frac{1}{z^3}(1+2z^2)(1-z^2+\dots) = \frac{1}{z^3}+\frac{2}{z}-\frac{1}{z}+\dots =$$

знак не минус

$$= \frac{1}{z^3}+\frac{1}{z} = \frac{1}{z^3}+\frac{0}{z^2}+\frac{1}{z}$$

№12.

$$f(z)=\frac{1}{z(e^z-1)}$$

знак не минус

$$\frac{1}{z(e^z-1)} = \frac{1}{z}\frac{1}{(\cancel{1}+z+z^2/2+\dots-\cancel{1})} = \frac{1}{z^2}\frac{1}{1+z/2+\dots} = \frac{1}{z^2}\left(1-\frac{z}{2}+\dots\right) =$$

$$= \frac{1}{z^2}-\frac{1}{2z}+\dots$$

2 порядк., $-\frac{1}{2}$ коэф

№13

$$\frac{1}{z(z-1)}$$

i) $|z|\in(0,1)$:

$$\frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{(1-z)} = -\frac{1}{z}\left(1+z+\frac{2}{2!}z^2+\frac{(-1)(-1-1)(-1-1-1)}{3!}(-z)^3+\dots\right)$$

$$= -\sum_{n=0}^{\infty} z^{-1+n} = -\frac{1}{z}-\sum_{n=0}^{\infty} z^n$$

ii) $\frac{1}{z(z-1)}$ $|z|\in(1,\infty)$

$$\frac{1}{z(z-1)} = \{\text{поскольку раскилад. ф-ция должна быть малой}\} = \frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z}}\right) =$$

$$= \frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\dots\right) = \sum_{n=0}^{\infty} z^{-2-n} = \sum_{n=2}^{\infty} z^{-n}$$

№4. $\boxed{1Н}$

$$\frac{z}{z^2+1}$$

$z = i$

$$\frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)} = \frac{1}{2(z-i)} + \frac{1}{2(z+i)} = \frac{1}{2(z-i)} + \frac{1}{2(z-i)+2i} = \frac{1}{2(z-i)} +$$

$$+ \frac{1}{4i} \cdot \frac{1}{\frac{(z-i)}{2i}+1} = \frac{1}{2(z-i)} - \frac{i}{4} \cdot \left(1 + \frac{i(z-i)}{2} + \frac{(z-i)^2}{4} + \frac{i(z-i)^3}{8} + \ldots\right) =$$

$$= \frac{1}{2(z-i)} - \frac{i}{4}\left(\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n (z-i)^n\right)$$

т.к. расст. до ближ. точки сингулярности $(z=-i)$ от данной точки сост. 2, то

$R = 2$