$$I(x)=\int\limits_{\infty}^{x}\underbrace{\frac{e^{i\pi x}dx}{x^2-1}}_{f(z)},\quad x>1$$

№1.

Блуменау М.И.
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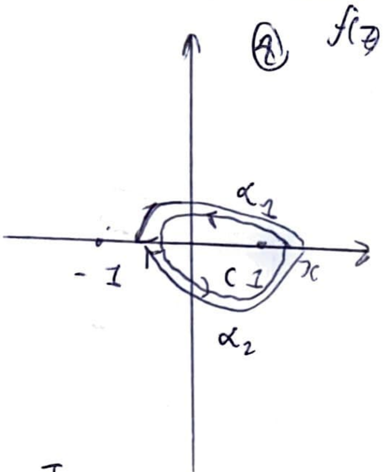
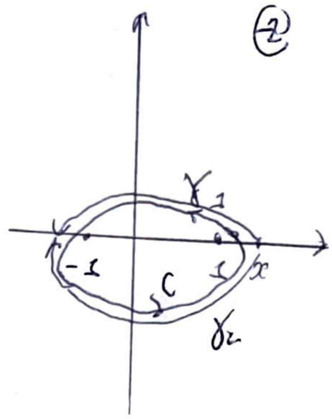




$$I_{\alpha_1}(x)-I_{\alpha_2}(x)=\int\limits_{\alpha_1}f(z)dz-\int\limits_{\alpha_2}f(z)dz=\oint\limits_C f(z)dz=$$

$$=2\pi i\,\underset{z=1}{\operatorname{res}}\,f(z)=2\pi i\,\underset{z=1}{\operatorname{res}}\,\frac{e^{i\pi z}}{z^2-1}=2\pi i\,\frac{e^{i\pi z}}{2z}\Big|_{z=1}=-\pi i$$

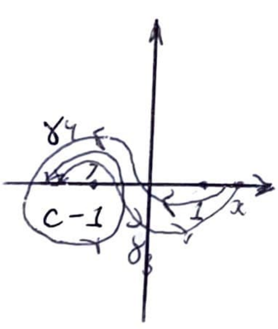
$$I_{\gamma_1}(x)-I_{\gamma_2}(x)=\int\limits_{\gamma_1}f(z)dz-\int\limits_{\gamma_2}f(z)dz=\oint\limits_C f(z)dz=\sum\operatorname{res}\ldots\cdot 2\pi i=0$$

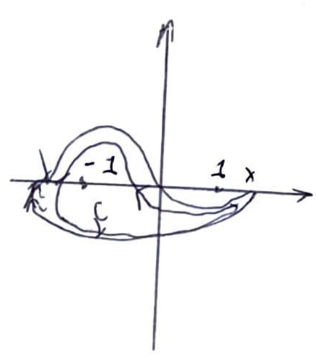
$z=-1$, $\underset{z=1}{\operatorname{res}}=-\frac{1}{2}$

$$\underset{z=-1}{\operatorname{res}}\,f(z)=\frac{e^{i\pi z}}{2z}\Big|_{z=-1}=\frac{e^{-i\pi}}{-2}=\frac{1}{2}$$

↓

$\frac{1}{2}-\frac{1}{2}=0$





$$I_{\gamma_3}(x)-I_{\gamma_2}(x)=\oint\limits_C f(z)dz=$$

$$=2\pi i\,\underset{z=-1}{\operatorname{res}}=\pi i$$

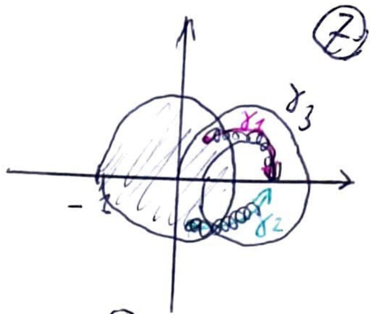
$$I_{\gamma_4}-I_{\gamma_3}=\oint\limits_C f(z)dz=$$

$$=2\pi i\,\underset{z=1}{\operatorname{res}}\,f(z)=i\pi$$

α - positive

$$F_\alpha(z)=\int_0^1 \overbrace{\frac{t^\alpha}{1-zt}(1-t)^\alpha}^{f(t)}\,dt$$

№2.



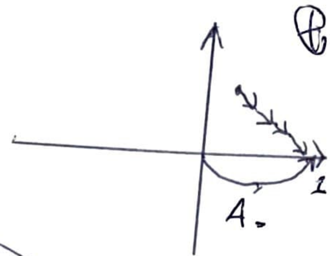
$|z|$

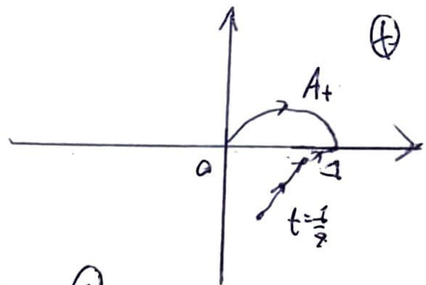
$z>1$ Будем в $\frac{1}{z}$

$\gamma_1,\gamma_2,\gamma_3$

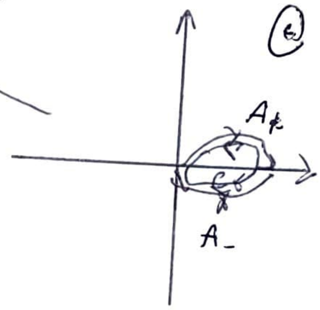
$$\Delta_{12}(x)=F_\alpha(x_1)-F_\alpha(x_2)=$$

$$=\oint_C f(t)\,dt=$$





$$\left\{\int_{A_+} f-\int_{A_-} f\right\}=-2\pi i \operatorname*{res}_{t=\frac1z} f(t)$$

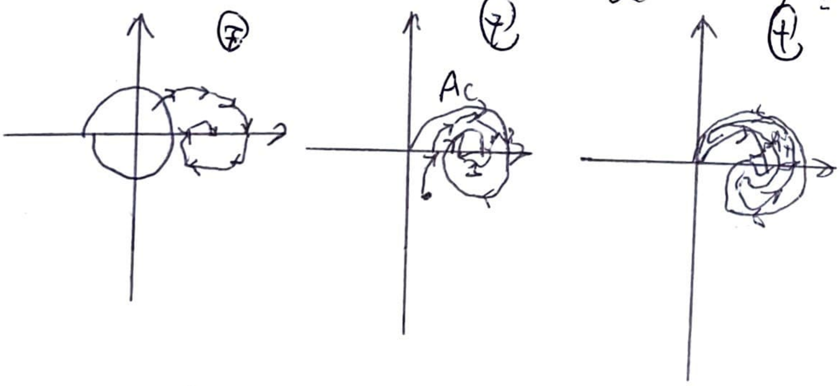


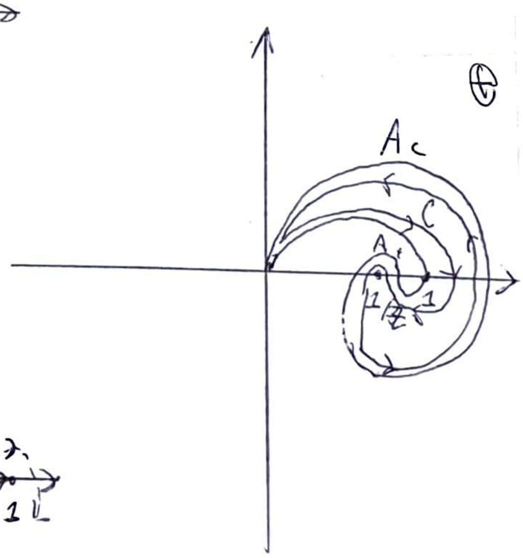
$$\operatorname*{res}_{t=\frac1z} f(t)=\operatorname*{res}_{t=\frac1z}\left[\frac{t^\alpha(1-t)^\alpha}{1-zt}\right]=$$

$$=\frac{t^\alpha(1-t)^\alpha}{-z}\Bigg|_{t=\frac1z}=\frac{(1/z)^\alpha(1-1/z)^\alpha}{-z}=-z^{-1-2\alpha}(z-1)^\alpha \Rightarrow$$

вне диска $\frac{1}{z}<1$

$$\Rightarrow \quad 0=-2\pi i\left(-(x-1)^\alpha x^{-1-2\alpha}\right)=2\pi i x^{-1-2\alpha}(x-1)^\alpha$$



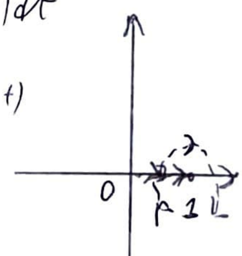


$$\Delta_{13}=\int_{A_+} f(t)\,dt-\int_{A_C} f(t)\,dt=\oint_C f(t)\,dt$$

$$\|$$

$$2\pi i \operatorname*{res}_{t=\frac1z} f(t)$$

$$\operatorname*{res}_{t=\frac1z} f(t)=-\frac{\left(\frac1z\right)^\alpha\left(1-\frac1z\right)^\alpha}{z}$$



$$g\left(\tfrac1z-i0\right)=\left|\frac{g(1/z-i0)}{g(1/z+i0)}\right| g\left(\tfrac1z+i0\right)e^{i\Delta\arg g}$$

Спиралька совершила оборот вокруг 1, которая т. ветвления, поэтому см. слева

$$\Delta\arg g=\alpha\Delta\arg t+\alpha\Delta\arg(1-t)=-2\pi\alpha$$

$$g\left(\tfrac1z-i0\right)=\left(\tfrac1z\right)^\alpha\left(1-\tfrac1z\right)^\alpha e^{-2\pi i\alpha}$$

$$\Delta_{13}=-e^{-2\pi i\alpha}z^{-1-2\alpha}(z-1)^\alpha\cdot 2\pi i=-2\pi i e^{-2\pi i\alpha}x^{-1-2\alpha}(x-1)^\alpha$$

№3.

$$I_0(z) = \int_0^\infty t^z e^{-t}\,dt$$

$\mathrm{Re}\, z \leq -1$

$\mathrm{Re}\, z > -1$

$$I_1(z) = \frac{1}{z+1}\int_0^\infty t^{z+1} e^{-t}\,dt$$

$\mathrm{Re}\, z > -2$

$I(z)$ $\quad z = -3$ $\quad \mathrm{res}$ - ?

$$I_2(z) = \frac{1}{z+1}\,\frac{1}{z+2}\int_0^\infty t^{z+2} e^{-t}\,dt$$

$\mathrm{Re} > -3$

$$I_3(z) = \frac{1}{(z+1)(z+2)(z+3)}\int_0^\infty t^{z+3} e^{-t}\,dt$$

$\mathrm{Re}\, z > -4$

$\mathrm{res}_{z=-3}\, I(z)$:

$$\mathrm{res}_{z=-3}\, I(z) = \underline{\int_0^\infty e^{-t}\,dt} = \underline{1}$$

$$\underset{z\to -3}{I(z)} = \frac{1}{(z+1)(z+2)(z+3)} = \frac{1}{z+3}\cdot\frac{1}{z+3-2}\cdot\frac{1}{z+3-1} =$$

$$= \frac{1}{z+3}\cdot\left(-\frac{1}{2\left(1-\frac{z+3}{2}\right)}\right)\left(-\frac{1}{1-(z+3)}\right) = \{\text{интересуют члены} \propto z^{-1}\} =$$

$$= \frac{1}{2}\,\frac{1}{z+3} \Rightarrow \mathrm{res}_{z=-3}\, I(z) = \frac{1}{2}$$

№4.

$$f(z) = \int_1^z \left(\frac{1}{w} + \frac{\alpha}{w^3}\right)\cos w\, dw \qquad \alpha - ?$$

$$f(z) = \underbrace{\int_1^z \frac{\cos w\, dw}{w}}_{I_0} + \underbrace{\int_1^z \frac{\alpha \cos w\, dw}{w^3}}_{I_1}$$

$$\int_1^z \frac{\alpha\cos w\, dw}{w^3} = -\frac{\alpha}{2}\int_1^z \cos w\; d(w^{-2}) = -\frac{\alpha}{2}\left(\frac{\cos w}{w^2}\Big|_1^z + \int_1^z \frac{\sin w}{w^2}dw\right) =$$

$$= -\frac{\alpha}{2}\left(\frac{\cos(z)}{z^2} - \cos(1) - \left(\frac{\sin w}{w}\Big|_1^z - \underbrace{\int_1^z \frac{\cos w}{w}dw}_{=I_0}\right)\right) =$$

$$= -\frac{\alpha}{2}\Big(\underbrace{\frac{\cos(z)}{z^2} - \cos(1) + \sin(1) - \frac{\sin z}{z}}_{\text{нет проблем с многозначностью}} + \underbrace{I_0}_{\text{а у этого есть}}\Big)$$

$$\Rightarrow I_0 + \left(-\frac{\alpha}{2} I_0\right) = 0 \quad \Rightarrow \alpha = 2$$

чтобы уничтожить проблемный член

№5

$$z u'' + (\gamma - 1 - iz)u'(z) + iu(z) = 0 \qquad \gamma > 0$$

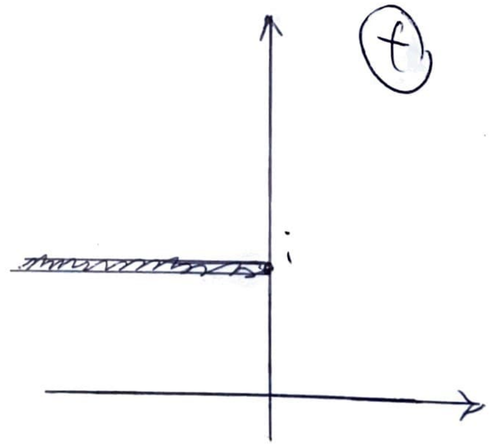
real pos. $z \to x$

$u_1(x) = \gamma - 1 - ix \qquad u_2(x) = \int_i^{-\infty} e^{xt}\,\frac{(t-i)^{\gamma-1}}{t^2}\,dt$

$g(t) = (t-i)^{\gamma-1}$

$g(i+\infty) > 0$

$u_2(x)$ for $x \to x e^{-2ix}$



Проверка:

u_1 просто подставляю?

$u'' = 0$

$u' = -i$

$$i(\gamma - 1 - iz) + i(\gamma - 1 - iz) = 0 \Rightarrow \text{удовл.}$$

u_2:

$$z\frac{d^2}{dz^2}\int_i^{-\infty} \underbrace{e^{zt}\frac{(t-i)^{\gamma-1}}{t^2}}_{f(t)}dt + (\gamma-1-iz)\frac{d}{dz}\int_i^{-\infty} e^{zt}\frac{(t-i)^{\gamma-1}}{t^2}dt + i\int_i^{\infty} e^{zt}\frac{(t-i)^{\gamma-1}}{t^2}dt = 0$$

$$\int_i^{-\infty} e^{zt}(t-i)^{\gamma-1}dt + (\gamma-1-iz)\int_i^{-\infty}\frac{e^{zt}(t-i)^{\gamma-1}}{t}dt + i\int_i^{-\infty}\frac{e^{zt}(t-i)^{\gamma-1}}{t^2}dt = 0$$

$$\int_i^{-\infty} e^{xt}(t-i)^{\gamma-1}dt = \frac{1}{x}\int_i^{-\infty}(t-i)^{\gamma-1}d(e^{xt}) = \frac1x\left(e^{xt}\cancel{(t-i)^{\gamma-1}}\Big|_i^{-\infty} - \int_i^{-\infty}(\gamma-1)(t-i)^{\gamma-2}\cdot e^{xt}dt\right) = -\frac1x\int_i^{-\infty}(\gamma-1)(t-i)^{\gamma-2}e^{xt}dt$$

$$\int_i^{-\infty}\frac{e^{xt}(t-i)^{\gamma-1}}{t^2}dt = -\left(\underbrace{\cancel{\frac{e^{xt}(t-i)^{\gamma-1}}{t}}}_{0}\Big|_i^{-\infty} - \int_i^{-\infty}\frac{xe^{xt}(t-i)^{\gamma-1}+(\gamma-1)(t-i)^{\gamma-2}e^{xt}}{t}dt\right) =$$

$$= x\int_i^{\infty}\frac{e^{xt}(t-i)^{\gamma-1}}{t}dt + \int_i^{-\infty}\frac{(\gamma-1)(t-i)^{\gamma-2}}{t}e^{xt}dt$$

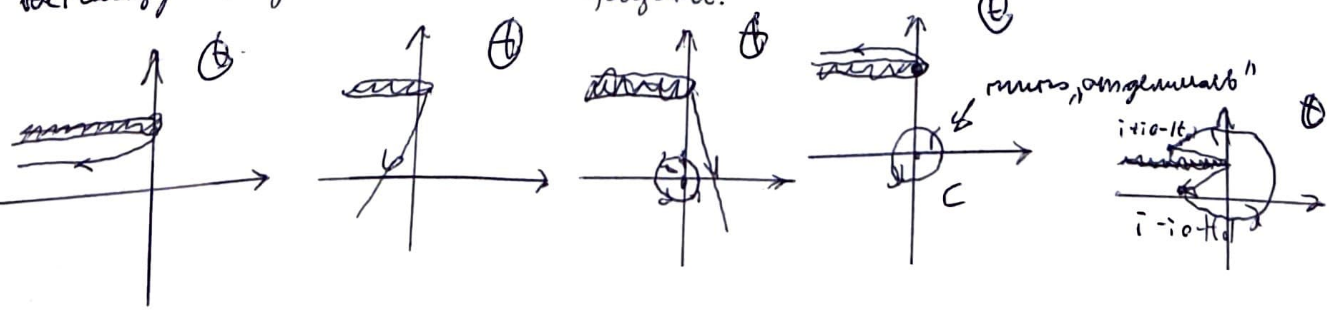
$$-(\gamma-1)\int_i^{-\infty}e^{xt}(t-i)^{\gamma-2}dt + (\gamma-1-\cancel{ix})\int_i^{-\infty}\frac{e^{xt}(t-i)^{\gamma-1}}{t} + \cancel{ix\int_i^{-\infty}\frac{e^{xt}(t-i)^{\gamma-1}}{t}dt} +$$
$$+ i\int_i^{-\infty}\frac{e^{xt}(\gamma-1)(t-i)^{\gamma-2}}{t}dt$$

$$\int_i^{-\infty}\frac{-(t-i)^{\gamma-2}e^{xt}+e^{xt}i(t-i)^{\gamma-2}}{t}dt = \int_i^{-\infty}e^{xt}(t-i)^{\gamma-2}\left(\frac{-t+i}{t}\right)dt = -\int_i^{-\infty}e^{xt}(t-i)^{\gamma-2}\frac{t-i}{t}dt =$$
$$= -\int_i^{\infty}\frac{e^{xt}(t-i)^{\gamma-1}}{t}dt$$

$$(\gamma-1)\left(-\int_i^{-\infty}\cancel{\frac{e^{xt}(t-i)^{\gamma-1}}{t}}dt + \int_i^{-\infty}\cancel{\frac{e^{xt}(t-i)^{\gamma-1}}{t}}dt\right) = 0$$

подходит.

Для сходимости интеграла e^{zt} — вещественна. Поэтому при вращении на полный оборот x надо крутить и t. Причём для компенсации будем крутить против часовой (как бы компенсируем „фазы"). В 0 также есть особенность, тогда воспользуемся значениями с картинки:



$g(i+i0-|t_0|) = 1 \cdot g(i-i0-|t_0|)e^{i\Delta \arg(g)} = e^{2\pi i \gamma} g(i-i0-|t_0|)$

$\Delta \arg(g) = (\gamma-1)2\pi$

$\oint f(t)\,dt = 2\pi i \operatorname{res}_{t=0} f(t)$

$\downarrow$

отсюда коэфф. перед $u_2(x)$

$\lim_{t\to 0} \frac{d}{dt}\left(e^{xt}(t-i)^{\gamma-1}\right) = \lim_{t\to 0}\left(xe^{xt}(t-i)^{\gamma-1} + (\gamma-1)(t-i)^{\gamma-2}e^{xt}\right) =$

$= \lim_{t\to 0} e^{xt}(t-i)^{\gamma-2}(\gamma-1+xt-ix)) = (\gamma-1-ix)(-i)^{\gamma-2} = (\gamma-1-ix)e^{-i\pi/2(\gamma-2)} =$

$= -(\gamma-1-ix)\, e^{-i\pi\gamma/2}$

$\underline{\quad} \; u_1 \quad e^{-i\pi\gamma/2}$

$\oint f(t)\,dt = -2\pi i\, u_1(x) e^{-i\pi\gamma/2}$

$u_2(|x| e^{-2\pi i}) = e^{2\pi i\gamma} u_2(x) - 2\pi i e^{-i\pi\gamma/2} u_1(x)$

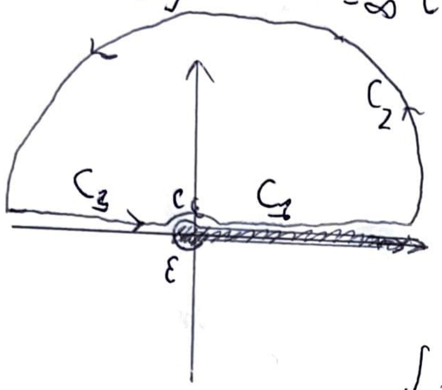
№ 6.

$$I = \int_0^\infty \frac{\ln(x)\,dx}{x^2+1}$$

0)

$$\begin{cases} x = e^t \\ dx = dt \cdot e^t \end{cases} \Rightarrow I = \int_{-\infty}^{\infty} \frac{t\, e^t dt}{e^{2t}+1} = \int_{-\infty}^{\infty} \frac{t\ \ dt}{2sh(t)} = 0$$

нечётная ф-ция по симм.



i) $\oint_1 = \int_{C_\varepsilon} + I_{1+2+3}$

$\int_{C_\varepsilon} : z = \varepsilon e^{i\varphi}, \varepsilon \to 0 : \lim_{\varepsilon \to 0} \int_\pi^0 \frac{\ln(\varepsilon e^{i\varphi})\, i\varepsilon e^{i\varphi}\, d\varphi}{\varepsilon^2 e^{2i\varphi}+1} =$

$$= \int_\pi^0 \lim_{\varepsilon \to 0} \left(\underbrace{\frac{i e^{i\varphi} \varepsilon (i\varphi)}{\varepsilon^2 e^{2i\varphi}+1}}_{0\ (\text{предел})} + \underbrace{\frac{i\varepsilon e^{i\varphi} \ln(\varepsilon)}{\varepsilon^2 e^{2i\varphi}+1}}_{0\ (\text{предел})} \right) d\varphi = \int_\pi^0 0\, d\varphi = 0$$

$\oint_1 = 2\pi i \operatorname{res} f(z) = 2\pi i \cdot \left.\frac{\ln z}{2z}\right|_{z=i} = \pi \ln i$

формулой с производной

$\ln i$:

$\Delta \arg = \pi/2$

$(1+0i)$

$\ln i = 0 + 0 + i\pi/2 = i\pi/2$

Ответ: $\frac{\pi^2 i}{2}$

iii) $\oint_1 = I_1 + I_3 =$

$(I_2 \to 0)$

$= 2I_1 + \int_0^\infty \frac{\pi i\, dx}{1+x^2}$

$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\pi i\, dx}{1+x^2} = \frac{1}{2}\oint = \frac{2\pi i}{2} \operatorname{res}_{z=i} \psi(z) =$

$= \frac{2(\pi i)^2}{2 \cdot 2i} = \frac{i\pi^2}{2}$

$2I_1 = \oint_1 - \frac{i\pi^2}{2} = 0 \Rightarrow I_1 = 0$

ii) I_3; $\ln(z) \to \ln(-z)$:

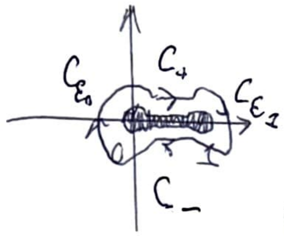
$\ln(-z_0) = \ln\left|\frac{-z_0}{z_0 + 0i}\right| + \ln(z_0 + i0) + i\Delta\arg z = \ln(z_0 + i0) + \pi i$

$$I_3 = I_1 + \int_0^\infty \frac{\pi i\, dx}{1+x^2}$$

№7

$\mathrm{Re}\,\alpha \in (-1; 3)$

$$I(\alpha) = \int_0^1 \frac{x^\alpha (1-x)^{2-\alpha}}{x+1}\,dx$$



$$\oint = \int_{C_+} + \int_{C_-} + \int_{C_{\varepsilon_1}} + \int_{C_{\varepsilon_0}} = I + \int_{C_-}$$

$\varphi(z) = x^\alpha (1-x)^{2-\alpha}$

$\varphi(x+i0) > 0, \ x \in [0;1]$

$$\varphi(x-i0) = \left|\cancel{\frac{\varphi(x-i0)}{\varphi(x+i0)}}\right| \varphi(x+i0) e^{i\Delta \arg \varphi}$$

$$\Delta \arg \varphi = \alpha \Delta \arg z + (2-\alpha)\underbrace{\Delta \arg (1-z)}_{=0} = 2\pi\alpha$$

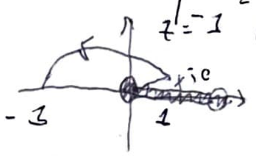
$\varphi(x-i0) = \varphi(x+i0)e^{i2\pi\alpha}$

$$\int_{C_-} = \int_1^0 \frac{\varphi(x+i0)e^{2i\pi\alpha}}{x+1}\,dx = -e^{2i\pi\alpha}\int_{C_+}$$

$\oint = I(1-e^{2i\pi\alpha}) = 2\pi i \sum \mathrm{res}\, f(z)$

$\underset{z=-1}{\mathrm{res}} f(z) = z^\alpha (1-z)^{2-\alpha}\Big|_{z=-1} = (-1)^\alpha 2^{2-\alpha}$

$(-1)^\alpha = e^{i\pi\alpha}$



$\underset{z=\infty}{\mathrm{res}} f(z)$

$$\frac{\cancel{z^\alpha}\left(\frac{1}{z}-1\right)^{2-\alpha} z^{\cancel{2-\alpha}}}{\cancel{z}\left(1+\frac{1}{z}\right)} = \frac{z\left(\frac{1}{z}-1\right)^{2-\alpha}}{1+\frac{1}{z}} = z\Big((-1)^{-\alpha} + (-1)^{-\alpha}(\alpha-2)\frac{1}{z} + \frac{(-1)^{-\alpha}}{2}(\alpha-2)\cdot$$

$= (-1)^{2-\alpha}$

$$\cdot(\alpha-1)\cdot\frac{1}{z^2}\Big)\left(1-\frac{1}{z}+\frac{1}{z^2}\right) \approx -\frac{(-1)^{-\alpha}\cancel{z}}{\cancel{z^2}}(\alpha-2) + \frac{1}{2}(-1)^{-\alpha}(\alpha-2)(\alpha-1)\frac{\cancel{z}}{z^2} +$$

$$\frac{\cancel{z}}{z^2}(-1)^{-\alpha} = (-1)^{-\alpha}(2-\alpha) + \frac{(-1)^{-\alpha}}{2} + \frac{1}{2}(-1)^{-\alpha}(\alpha-2)(\alpha-1) + (-1)^{-\alpha}\Big) =$$

$$= \frac{(-1)^{-\alpha}}{z}\left(1-\alpha+2+\frac{1}{2}(\alpha^2-3\alpha+2)\right) = \frac{1}{ze^{i\pi\alpha}}\left(\frac{\alpha^2}{2}-\frac{5\alpha}{2}+4\right)$$

$\underset{z=\infty}{\mathrm{res}} f(z) = -$

$$I = 2\pi i\left(-\frac{1}{e^{i\pi\alpha}}\left(\frac{\alpha^2}{2}-\frac{5\alpha}{2}+4\right) + e^{i\pi\alpha}2^{2-\alpha}\right) \Big/ \left(1-e^{2\pi i\alpha}\right) =$$

$$= \frac{1\cdot 2\pi i}{e^{i\pi\alpha}-e^{-i\pi\alpha}}\left(-2^{2-\alpha}+1\cdot\left(\frac{\alpha^2}{2}-\frac{5\alpha}{2}+4\right)\right) =$$

$$= \frac{\pi}{\sin(\pi\alpha)}\left(\frac{\alpha^2}{2}-\frac{5}{2}\alpha+4-2^{2-\alpha}\right)$$

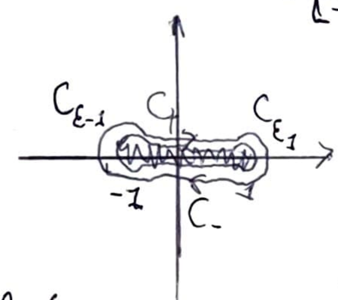
№8

$\mathrm{Re}\,\alpha \in (-1,2)$

$$I(\alpha) = \int_{-1}^{1} \frac{(1-x)^{a}(1+x)^{1-a}}{(x^2+1)}\,dx$$

$\varphi(z) = (1-z)^{a}(1+z)^{1-a}$

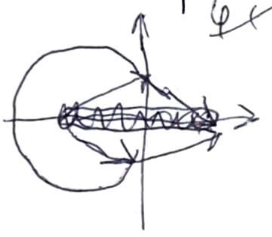
$\varphi(x+i0) > 0 \quad x \in [-1;1]$



$$\oint = I + \int_{C_-} + \cancel{\int_{C_{\varepsilon_{-1}}}} + \cancel{\int_{C_{\varepsilon_1}}}$$

$\int_{C_-}$:

$$\varphi(x-i0) = \left|\frac{\varphi(x+i0)}{\varphi(x-i0)}\right| \cancel{\varphi(x+i0)}\, e^{i\Delta\arg\varphi}$$



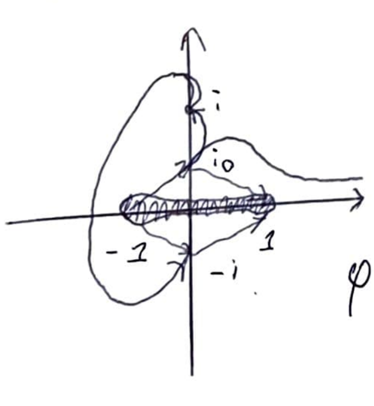
$\Delta\arg\varphi = (1-\alpha)2\pi$

$$\int_{C_-} = -e^{-2\pi i a}\int_{C_+} \Rightarrow \oint = I(1-e^{-2\pi i a})$$

$\oint = 2\pi i \sum \mathrm{res}\, f(z)$

$z=\pm i$:

$$\mathop{\mathrm{res}}_{z=\pm i} f(z) = \left.\frac{(1-z)^{a}(1+z)^{1-a}}{2z}\right|_{z=\pm i} = \frac{(1\mp i)^{a}(1\pm i)^{1-a}}{\pm 2i}$$



$$\varphi(i) = \left|\frac{\varphi(i)}{\varphi(i0)}\right| \cancel{\varphi(i0)}\, e^{i\Delta\arg\varphi} = |(1+i)^{1-a}(1-i)^{a}|\, e^{i(\pi/4 - \pi a/2)} =$$

$$\Delta\arg\varphi = a\left(-\frac{\pi}{4}\right) + (1-a)\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\pi a}{2}$$

$$= \sqrt{2}\exp\left(i\left(\frac{\pi}{4} - \frac{\pi a}{2}\right)\right)$$

$\varphi(-i) = \{$аналогично, но $\Delta\arg\varphi = a\cdot\frac{\pi}{4} + (1-a)\frac{7\pi}{4} = \frac{7\pi}{4} - \frac{3\pi a}{2}\} =$

$$= \sqrt{2}\exp\left(i\left(\frac{7\pi}{4} - \frac{3\pi a}{2}\right)\right)$$

$\mathop{\mathrm{res}}_{z=\infty} f(z)$:

$$\varphi(z) = \left|\frac{\varphi(z)}{\varphi(i0)}\right| \varphi(i0)\, e^{i\Delta\arg\varphi} = |\varphi(z)|\, e^{i\Delta\arg\varphi} = (z-1)^{a}(z+1)^{1-a}e^{-i\pi a}$$

$$f(z) = \frac{z^{a} z^{1-a}\left(1-\frac{1}{z}\right)^{a}\left(1+\frac{1}{z}\right)^{1-a}}{z^2\left(1+\frac{1}{z^2}\right)} e^{-i\pi a} = \left\{\propto \frac{1}{z}\right\} = \frac{e^{-i\pi a}}{z}$$

$$\mathop{\mathrm{res}}_{z=\infty} f(z) = -e^{-i\pi a}$$

$$\oint = 2\pi i\left(-\frac{e^{i(\pi/4 - \pi a/2)} + i e^{i(7\pi/4 - 3\pi a/2)}}{\sqrt{2}} - e^{-i\pi a}\right) =$$

$$= 2\pi i\left(-\frac{i e^{-i\pi a}}{\sqrt{2}}\left(\exp\left[i\left(\frac{\pi}{4} + \frac{\pi a}{2}\right)\right] - e^{i(-\pi/4 - \pi a/2)}\right) - e^{-i\pi a}\right) = 2\pi i e^{-i\pi a}\left(-\frac{i}{\sqrt{2}}\, 2i\sin\left(\frac{\pi}{4} + \frac{\pi a}{2}\right) - 1\right)$$

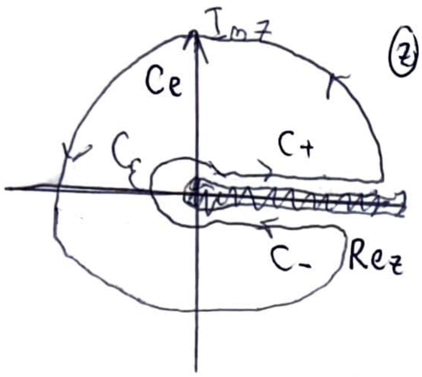
$$= 2\pi i e^{-i\pi a}\left(\sqrt{2}\left(\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi a}{2}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi a}{2}\right)\right) - 1\right) =$$

$$= 2\pi i e^{-i\pi a}\left(\sin\left(\frac{\pi a}{2}\right) + \cos\left(\frac{\pi a}{2}\right) - 1\right)$$

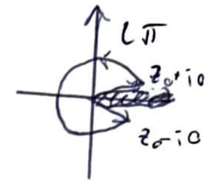
$$I = \frac{\oint}{1 - e^{-2\pi i a}} = \frac{2\pi i\left(\sin\left(\frac{\pi a}{2}\right) + \cos\left(\frac{\pi a}{2}\right) - 1\right)}{e^{i\pi a} - e^{-i\pi a}} = \frac{\pi}{\sin(\pi a)} \cdot \left(\sin\left(\frac{\pi a}{2}\right) + \cos\left(\frac{\pi a}{2}\right) - 1\right)$$

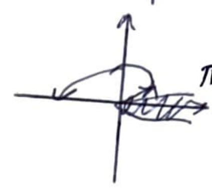
$\boxed{8/13}$

$$I = \int_0^{\infty} \frac{\ln x}{\sqrt[3]{x}\,(x+1)^2}\,dx$$



$$\oint = \int_{C_+} + \int_{C_-} + \int_{C_R}^{0} + \int_{C_\varepsilon}^{0}$$





$$\varphi_1(z) = z^{-1/3}$$

$$\varphi_2(z) = \ln(z)$$

$$\varphi_1(z-i0) = 1\cdot\varphi(z_0+i0)\,e^{i\Delta\arg\varphi_1} = \varphi_1(z_0+i0)\,e^{-2\pi i/3}$$

$$\varphi_2(z-i0) = \varphi_2(z_0+i0) + 2\pi i$$

$$\int_{C_-} = -\int_0^{\infty} \frac{(\varphi_2(z+i0)+2\pi i)\,\varphi_1(z_0+i0)\,e^{-2\pi i/3}}{(x+1)^2} = -e^{-2\pi i/3} I - e^{-2\pi i/3}\int_0^{\infty}\frac{2\pi i\,dx}{\sqrt[3]{x}\,(x+1)^2}$$

$$\Rightarrow \oint = 2\pi i \operatorname{res}_{z=-1} f(z) = 2\pi i \lim_{z\to -1}\frac{d}{dz}\left((z+1)^2 f(z)\right) = 2\pi i \lim_{z\to -1}\frac{d}{dz}\frac{\ln z}{\sqrt[3]{z}} =$$

$$= 2\pi i \lim_{z=-1}\left(\frac{\sqrt[3]{z}/z - \frac{1}{3}z^{-2/3}\ln(z)}{\sqrt[3]{z^2}}\right) = 2\pi i \lim_{z\to -1}\left(-\frac{\ln z}{3z\sqrt[3]{z}} + \frac{1}{z\sqrt[3]{z}}\right) = 2\pi i\frac{-1}{\sqrt[3]{-1}}\left(1 - \frac{\ln(-1)}{3}\right)$$

$$\varphi_1(-1) = \left|\cancel{\frac{\varphi_1(-1)}{\varphi_1(1)}}\right|\cancel{\varphi_1(1)}\,e^{i\Delta\arg\varphi_1} = e^{-i\pi/3}$$

$$\varphi_2(-1) = 0 + 0 + i\Delta\arg\varphi_2 = i\pi$$

$$\oint = 2\pi i\left(-e^{-i\pi/3}\left(1 - \frac{i\pi}{3}\right)\right) = -e^{-i\pi/3}\left(2\pi i + \frac{2\pi^2}{3}\right) = -\frac{2\pi^2}{3}e^{-i\pi/3} - 2\pi i e^{-i\pi/3}$$

$$I + \left(I + 2\pi i\int_0^{\infty}\frac{dx}{\sqrt[3]{x}\,(x+1)^2}\right)\left(-e^{-2\pi i/3}\right) =$$

$$I\left(e^{i\pi/3} - e^{-i\pi/3}\right) - 2\pi i e^{-i\pi/3}\int_0^{\infty}\frac{dx}{\sqrt[3]{x}\,(x+1)^2}$$

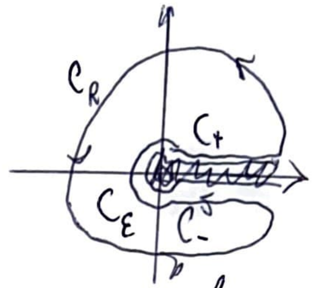
$$2i\sin\left(\frac{\pi}{3}\right)I - 2\pi i e^{-i\pi/3}\int_0^{\infty}\frac{dx}{\sqrt[3]{x}\,(x+1)^2} = -2\pi\left(\frac{\pi}{3} + i\right)$$

$$\frac{\sqrt{3}}{2}I - \pi e^{-i\pi/3}\int_0^{\infty}\frac{dx}{\sqrt[3]{x}\,(x+1)^2} = -2i\pi\left(1 - \frac{\pi i}{3}\right) \quad = -\pi\left(1 - \frac{\pi i}{3}\right)$$

$$\begin{cases} \frac{\sqrt{3}}{2}I - \pi\cos\left(-\frac{\pi}{3}\right)\int_0^{\infty}\frac{dx}{\sqrt[3]{x}\,(x+1)^2} = -\pi \\ -\pi\sin\left(-\frac{\pi}{3}\right)\int_0^{\infty}\frac{dx}{\sqrt[3]{x}\,(x+1)^2} = \frac{\pi^2}{3} \end{cases} \Rightarrow \quad = \frac{2\pi^{\cancel{2}}}{\sqrt{3}\,\cancel{\pi}\cdot 3} = \frac{2\pi}{3\sqrt{3}}$$

$$\underline{I = \frac{2}{\sqrt{3}}\left(-\pi + \frac{2\pi}{3\sqrt{3}}\cdot\frac{1}{2}\right) = \frac{2\pi^2}{9} - \frac{2\sqrt{3}\pi}{3}}$$

№10



$$I=\int_0^1 \ln\frac{1-x}{x}\,\frac{dx}{x^2+1}$$

$$x=\frac{1}{t}\quad dx=-\frac{1}{t^2}dt$$

$$I=\int_1^{\infty}\ln(t-1)\,\frac{dt}{t^2\left(1+\frac{1}{t^2}\right)}=\int_1^{\infty}\frac{\ln(t-1)}{t^2+1}dt=\{t=\eta+1\}=\int_0^{\infty}\frac{\ln(\eta)}{\eta^2+2\eta+2}d\eta$$

$\ln(x+i0)\in\mathbb{R}$
$x\in(0;+\infty)$

Посчитаем $\ln^2(z)$, генеральный ход с семинара

$$\tilde{I}=\int_0^{+\infty}\frac{d\eta\,\ln^2\eta}{\eta^2+2\eta+1}$$

$$\oint=\int_{C_-}+\int_{C_\varepsilon}+\int_{C_R}+\tilde{I}\qquad \int_{C_-}=-\int_0^{\infty}\frac{\ln^2\eta+4\pi i\ln\eta-4\pi^2}{\eta^2+2\eta+2}=\underbrace{-\int_0^{\infty}\frac{\ln^2\eta}{\eta^2+2\eta+2}}_{\tilde{I}}-4\pi i\underbrace{\int_0^{\infty}\frac{\ln\eta}{\eta^2+2\eta+2}}_{I}$$

$$+4\pi^2\int_0^{\infty}\frac{d\eta}{\eta^2+2\eta+2}$$

$$\oint=-4\pi i I+4\pi^2\int_0^{\infty}\frac{d\eta}{\eta^2+2\eta+2}$$

$$\oint=2\pi i\left(\operatorname*{res}_{z=i-1}f(z)+\operatorname*{res}_{z=-1-i}f(z)\right)$$

$$\operatorname*{res}_{z=-1+i}f(z)=\frac{\ln^2 z}{2z+2}\Big|_{-1+i}=\frac{\left(\ln\sqrt{2}+\frac{3\pi i}{4}\right)^2}{2i}$$

$$\ln z\Big|_{z=\sqrt{2}e^{3\pi i/4}}=\ln\sqrt{2}+\frac{3\pi i}{4}$$

В $z=-1-i$ аналогично: $\operatorname*{res}_{z=\ldots}f(z)=-\frac{\left(\ln\sqrt{2}+\frac{5\pi i}{4}\right)^2}{2i}$

$$\oint=\cancel{2}\pi\cancel{i}\,\frac{1}{\cancel{2}\cancel{i}}\left(\left(\ln\sqrt{2}+\frac{3\pi i}{4}\right)^2-\left(\ln\sqrt{2}+\frac{5\pi i}{4}\right)^2\right)=\pi\left(\ln^2\sqrt{2}+\frac{3\pi i}{2}\ln\sqrt{2}-\frac{9\pi^2}{16}-\ln^2\sqrt{2}-\frac{5\pi i}{2}\ln\sqrt{2}+\right.$$

$$\left.+\frac{25\pi^2}{16}\right)=\pi\left(-\pi i\ln\sqrt{2}+\pi^2\right)=\pi^3-\pi^2 i\ln\sqrt{2}$$

$$-4\pi i I+4\pi^2\int_0^{\infty}\frac{d\eta}{\eta^2+2\eta+2}=\pi^3-\pi^2 i\ln(\sqrt{2})$$

$$4\pi I=\pi^2\ln(\sqrt{2})$$

$$I=\frac{\pi\ln\sqrt{2}}{4}=\frac{\pi\ln(2)}{8}$$

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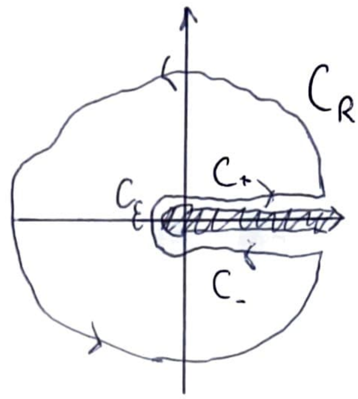
№11

$a>0\ ;\ \lambda\in(0,\pi)$

$$I(a,\lambda)=\int_0^\infty \frac{\ln x}{x^2+2ax\cos\lambda+a^2}$$

Типичный ход $\ln x \to \ln^2 x$



$$\tilde{I}=\int_0^\infty \frac{\ln^2 x}{x^2+2ax\cos\lambda+a^2}$$

$$\oint = I+\int_{C_-}+\cancel{\int_{C_R}}^{0}+\cancel{\int_{C_\varepsilon}}^{0}$$

$$\int_{C_-}=-\int_0^\infty \frac{\ln^2(x)+4\pi i\ln(x)-4\pi^2}{x^2+2ax\cos\lambda+a^2}=-\underbrace{\int_0^\infty\frac{\ln^2 x\,dx}{x^2+2ax\cos\lambda+a^2}}_{\tilde{I}}-4\pi i\underbrace{\int_0^\infty\frac{\ln(x)\,dx}{x^2+2ax\cos\lambda+a^2}}_{I}+$$

$$+4\pi^2\int_0^\infty\frac{dx}{x^2+2ax\cos\lambda+a^2}$$

$$\oint=-4\pi i I+4\pi^2\int_0^\infty\frac{dx}{x^2+2ax\cos\lambda+a^2}$$

$$\oint=2\pi i\sum_{z_i}\operatorname{res}f(z)$$

$$z^2+2az\cos\lambda+a^2=0$$

$$z=-a\cos\lambda\pm\sqrt{a^2\cos^2\lambda-a^2}=-a(\cos\lambda\pm i\sin\lambda)=-ae^{\pm i\lambda}$$

$$\operatorname*{res}_{z=-ae^{i\lambda}}f(z)=\frac{\ln^2(z)}{2z+2a\cos\lambda}\Big|_{z=-ae^{i\lambda}}=\frac{\ln^2(-ae^{i\lambda})}{-2ai\sin\lambda}=\frac{i(\ln(-a)+i\lambda)^2}{2a\sin\lambda}$$

$$\operatorname*{res}_{z=-ae^{-i\lambda}}f(z)=\{\text{аналогично}\}=-\frac{i(\ln(-a)-i\lambda)^2}{2a\sin\lambda}$$

$$\oint=2\pi i\,\frac{i}{2a\sin\lambda}\left((\ln(-a)+i\lambda)^2-(\ln(-a)-\lambda i)^2\right)=-\frac{\pi}{a\sin\lambda}\left(4\lambda i\ln(-a)-2\lambda^2\right)=$$

$$=-\frac{2\pi}{a\sin\lambda}\left(2i\lambda\ln(a)-2\pi\lambda-\lambda^2\right)$$

$$-4\pi i I+4\pi^2\int_0^\infty \text{добавка}=$$

$$\cancel{+4}\pi I=\cancel{+}\frac{2\pi}{a\sin\lambda}\cdot 2\lambda\ln a$$

$$I=\frac{\lambda\ln a}{a\sin\lambda}$$

$b > a > 0$

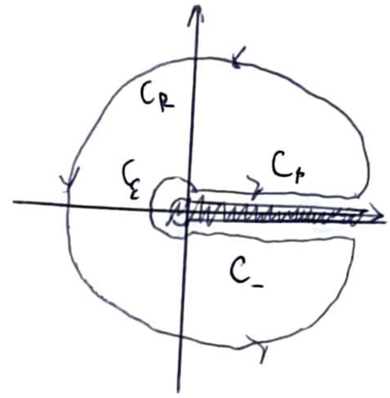
$I(a,b) = \int_0^\infty \frac{x^{a-1}}{1+x^b} dx$

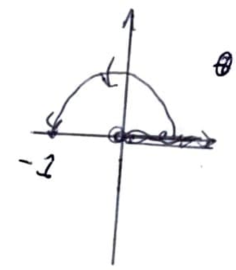
$x^b = t$

$I = \int\limits_0^\infty \frac{t^{(a-1)/b} \frac{1}{b} t^{1/b-1}}{1+t} dt = \frac{1}{b}\int\limits_0^\infty \frac{t^{a/b-1}}{1+t} dt$

$\tilde{I} = bI = \left\{ \frac{a}{b} = \gamma \right\} = \int\limits_0^\infty \frac{t^{\gamma-1}}{1+t} dt$

см. правило дз





$t^{\gamma-1} = \varphi(t)$

$\varphi(t_0 - i0) = \left|\frac{\varphi(t_0-i0)}{\varphi(t_0+i0)}\right| \varphi(t+i0)\, e^{i\Delta\arg\varphi} = \varphi(t_0+i0) e^{2\pi i\gamma}$

$\Delta\arg\varphi = (2\pi)(\gamma-1)$

$\int_{C_-} = -e^{2\pi i\gamma}\int_{C_+}$

$\oint = \tilde{I} + \int_{C_-} + \cancel{\int_{C_\varepsilon}}^{0} + \cancel{\int_{C_R}}^{0}$

$\oint = 2\pi i \operatorname*{res}_{z=-1} g(z)$

$g(z) = \frac{z^{\gamma-1}}{1+z}$; $\operatorname*{res}_{z=-1} g(z) = z^{\gamma-1}\Big|_{z=-1} = \varphi(-1)$

$\varphi(-1) = \left|\frac{\cancel{-1}^{1}}{\cancel{1+i0}}\right| \cancel{\varphi(1)}^{1}\, e^{i\Delta\arg z}$

$\Delta\arg\varphi = (\gamma-1)\pi$

$\text{res} = e^{i\pi(\gamma-1)}$

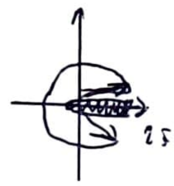
$\tilde{I}(1-e^{2\pi i\gamma}) = \oint$

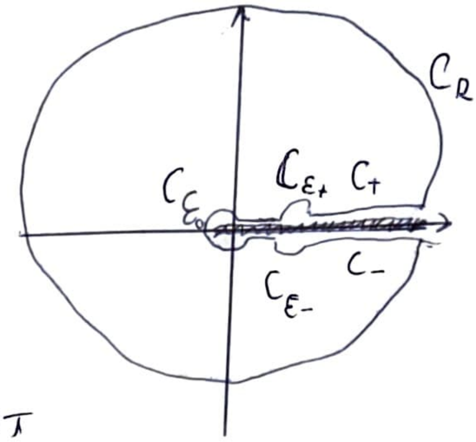
$\tilde{I} = \frac{2\pi i e^{i\pi(\gamma-1)}}{1-e^{2\pi i\gamma}} = \frac{2\pi i e^{-i\pi}}{e^{-\pi i\gamma} - e^{i\pi\gamma}} = \frac{\pi}{\sin(\pi\gamma)}$

$I = \frac{\pi}{b\sin(\pi\gamma)} = \frac{\pi}{b\sin\left(\pi\frac{a}{b}\right)}$

№13.

$$I = PV\int_0^\infty \frac{\sqrt{x}\,dx}{x^2-1}$$





$$\oint = I + \int_{C_-} + \int_{C_{\varepsilon+}} + \int_{C_{\varepsilon-}} + \cancel{\int_{C_R}} + \cancel{\int_{C_{\varepsilon_0}}}$$

$$\varphi(z) = \sqrt{z}$$

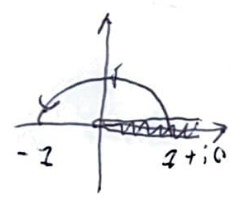
$$\varphi(z_0 - i0) = 1\cdot\varphi(z_0+i0)\,e^{i\Delta\arg\varphi} = \varphi(z_0+i0)\,e^{i\pi} = -\varphi(z_0+i0)$$

$$\int_{C_-} = \int_0^\infty \frac{\varphi(z_0+i0)}{z^2-1}\,dz = \int_{C_+}$$

$$\int_{C_{\varepsilon+}} : z = \varepsilon e^{i\varphi}+1$$

$$\int_{C_{\varepsilon+}} = \lim_{\varepsilon\to0}\int_\pi^0 \frac{\sqrt{1+\varepsilon e^{i\varphi}}\cdot e^{i\varphi}\cdot\varepsilon i\,d\varphi}{\varepsilon^2e^{2i\varphi}+2\varepsilon e^{i\varphi}} = \lim_{\varepsilon\to0}\int_\pi^0 \frac{i\sqrt{1+\varepsilon e^{i\varphi}}}{\varepsilon e^{i\varphi}+2}\,d\varphi = \int_\pi^0 \frac{i}{2}\,d\varphi = -\frac{i\pi}{2}$$

$$\int_{C_{\varepsilon-}} = \{\text{аналогично}\} = \frac{i\pi}{2}$$



$$\oint = 2\pi i \operatorname*{res}_{z=-1}\ldots = \cancel{2}\pi i\frac{\sqrt{z}}{\cancel{2}z}\Big|_{z=-1} = -i\pi\varphi(-1)$$

$$\varphi(-1) = 1\cdot1\cdot e^{i\Delta\arg\varphi} = e^{i\pi/2} = i$$

$$\oint = \pi$$

$$\oint = 2I + 0 \Rightarrow I = \frac{\pi}{2}$$