№1

2H

$z = \pm\pi$

$$\frac{1}{\sin z} + \frac{2z}{z^2-\pi^2} = \frac{1}{\sin z} + \frac{2z}{(z-\pi)(z+\pi)} = \frac{1}{\sin z} + \frac{1}{z-\pi} + \frac{1}{z+\pi}$$

$\sin(x+\pi) = -\sin x \Rightarrow$

берём только нужные

$$\Rightarrow \frac{1}{\sin z} \approx \left(-(z\mp\pi) + \frac{(z\mp\pi)^3}{6} \ldots\right)^{-1} \approx \frac{1}{-(z\mp\pi)}\left(1 - \frac{(z\mp\pi)^2}{6}\ldots\right)^{-1} =$$

$$= -\frac{1}{z\mp\pi}\left(1 + \frac{(z\mp\pi)^2}{6}\ldots\right) \Rightarrow$$ гл. нужный?

$$-\frac{1}{z\mp\pi}$$

Блуменау М.И. БФЗ191

при π: $$-\cancel{\frac{1}{z-\pi}} + \cancel{\frac{1}{z-\pi}} + \frac{1}{z+\pi} = (2\pi)^{-1}$$

при $-\pi$: $$-\cancel{\frac{1}{z+\pi}} + \cancel{\frac{1}{z+\pi}} + \frac{1}{z-\pi} = -(2\pi)^{-1}$$

т.е. ост. только const $\Rightarrow$ removable sing.

№2.

1) $f(z) = \dfrac{\sin z}{1-\operatorname{tg} z}$

Особые точки:

$1 - \operatorname{tg} z = 0 \Rightarrow$

$\Rightarrow z = \underbrace{\frac{\pi}{4} + \pi n}_{\gamma_0}, n \in \mathbb{Z}$

$$\sin z \approx \pm\left(\frac{\sqrt{2}}{2} + \frac{z-\gamma_0}{\sqrt{2}}\right)$$

$$1 - \operatorname{tg} z \approx 1 - (1 + 2(z-\gamma_0))$$

Тогда:

выкинуть в силу незначительности

$$\frac{\sin z}{1-\operatorname{tg} z} = \frac{\pm\left(\frac{1}{\sqrt{2}} + \frac{z-\gamma_0}{\sqrt{2}}\right)}{-2(z-\gamma_0)} \approx \frac{\mp 1}{2\sqrt{2}(z-\gamma_0)}$$

$\frac{1}{z-z_0}\cdot\text{const}$, т.е. полюс

2) $f(z) = \dfrac{e^{c/(z-a)}}{e^{z/a}-1}$

$e^{z/a} = 1$

$z = 2\pi i n a, n \in \mathbb{Z}$ – simple poles

case $z = a$

этих инф.

$$f(z) \approx \frac{1 + \frac{c}{z-a} + \frac{c^2}{(z-a)^2 2} + \ldots}{e + \frac{z-a}{a}e + \frac{(z-a)^2}{2a^2}e \ldots -1}$$ – essential

case $z = 2\pi i n a$

$$f(z) \simeq \frac{e^{c/(z-a)}}{\cancel{1} + \frac{z-2\pi i n a}{a} + \frac{z-(2i\pi n a)^2}{2a^2} + \dots - \cancel{1}} =$$

$$= \frac{\overbrace{e^{c/(z-a)}}^{\gamma_2}\, a}{z - 2\pi i n a}\left(1 + \frac{z - 2i\pi n a}{2a}\right)^{-1} \simeq \frac{\gamma_2}{z - 2\pi i n a}\left(1 - \frac{z - 2\pi n a i}{2a}\right) =>$$

$=>$ сущ. член $\frac{\gamma_2}{z - 2\pi i n a}$ $=>$ s. pole

N 3

$f(z) = z\, e^{1/z}\, e^{-1/z^2}$ @ $z=0$

$$f(z) \simeq z\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots\right)\left(1 - \frac{1}{z^2} + \frac{1}{2z^4} - \dots\right) => \text{essential}$$

всего z^1

могут иметь с обратн. см.

могут посчитать

1) $\int_C \frac{ze^z}{\tan z^2} dz = 2\pi i$ $\quad C: \begin{cases} R = 1 \\ \text{center } z = 0 \end{cases}$

По теореме Коши:

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=a_k}{\operatorname{res}} f(z)$$

Очевидным образом т. сингл. будет $z=0$

Тогда

$$\int_C \frac{ze^z}{tg z^2} dz = 2\pi i \underset{z=0}{\operatorname{res}} \frac{ze^z}{tg z^2} = 2\pi i$$

$\frac{ze^z}{tg z^2}$:

$tg z^2 \approx \left(x^2 + \frac{x^6}{3} + \frac{2x^{10}}{15} + \dots\right)$ $\quad (tg z^2)^{-1} \approx \frac{1}{z^2 + \frac{z^6}{3}} \approx \frac{1}{z^2} \frac{1}{\left(1 + \frac{z^4}{3}\right)} \approx$

$ze^z \approx z\left(1 + z + \frac{z^2}{2} \dots\right)$ $\quad = \frac{1}{z^2}\left(1 - \frac{z^4}{3}\right)$

$\frac{z}{z^2}\left(1 - \frac{z^4}{3}\right)\left(1 + z + \frac{z^2}{2} \dots\right) \Rightarrow \underset{0}{\operatorname{res}} f(z) = 1$

2) $\int_C e^{-1/z} \sin\left(\frac{1}{z}\right) dz = 2\pi i$

Всё то же

$e^{-1/z} = \left(1 - \frac{1}{z} + \frac{1}{2z^2} \dots\right)$

$\sin\left(\frac{1}{z}\right) = \sin(t) = t - \frac{t^3}{6} \dots$
$t = \frac{1}{z}$

$$\left(1 - \frac{1}{z} + \frac{1}{2z^2}\right)\left(\frac{1}{z} - \frac{1}{6z^3} \dots\right)$$

Очевидно $\underset{0}{\operatorname{res}} f(z) = 1$

$\int_C e^{-1/z} \sin\left(\frac{1}{z}\right) dz = 2\pi i$

3) $n \in \mathbb{N}$

$\int_C \frac{e^z}{z^n} dz = \frac{2\pi i}{(n-1)!}$

$e^z \approx 1 + z + z^2/2 \dots$

т.к. нужно $\frac{1}{z}$, то понадобится $n-1$ степень

№5.

$\boxed{2H}$

1) $\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}dx$

Замкнём контур в верхней полуплоскости ($\varphi \in [0, \pi]$)

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}dx = \int_C \frac{z^4}{1+z^6}dz$$

$1+z^6=0$

$z^6=-1$

$e^{6i\varphi} = e^{-i\pi+2i\pi n}, n \in \mathbb{Z}$

$\varphi = -\frac{\pi}{6} + \frac{2\pi n}{6} = -\frac{\pi}{6} + \frac{\pi n}{3}$

$\varphi = \frac{\pi}{6}; \frac{\pi}{2}; \frac{5\pi}{6}$

$\int_C \frac{z^4}{1+z^6} = 2\pi i \sum_k \operatorname{res}_{z=a_k} f(z) \ominus$

$f(z) = \frac{g(z)}{h(z)}$ with $h'(a_k) \neq 0$, $h(a_k)=0$;

$\operatorname{res}_{a_k} f(z) = \frac{g(a_k)}{h'(a_k)}$

$\operatorname{res}_{a_k} \frac{z^4}{1+z^6} = \frac{z^4}{6z^5}\Big|_{z=a_k} = \frac{e^{-ia_k}}{6}$

$\ominus \frac{2\pi i}{6}\left(e^{-\frac{i\pi}{6}} + e^{-\frac{i\pi}{2}} + e^{-\frac{5\pi i}{6}}\right) = \frac{2\pi i}{6}\left(\cancel{\cos\frac{\pi}{6}} - i\sin\frac{\pi}{6} - i + \cancel{\cos\frac{5\pi}{6}} - i\sin\frac{5\pi}{6}\right) =$

$= -\frac{2\pi}{6}\left(-\frac{1}{2} - 1 - \frac{1}{2}\right) = \frac{2\pi}{3}$

2) $\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}d\theta = -\int_C \frac{i}{z} \frac{z^2+1/z^2}{2\left(2+\frac{z+1/z}{2}\right)}dz = -\int_C \frac{i}{z}\frac{z^2+1/z^2}{4+z+1/z}dz =$

$= -\int_C \frac{i(z^2+1/z^2)}{z^2+4z+1}dz = -\int_C \frac{1}{z^2}\frac{i(z^4+1)}{z^2+4z+1}dz \ominus$

$e^{i\theta} = \cos\theta + i\sin\theta$

$dz = iz\,d\theta \Rightarrow$

$\Rightarrow d\theta = \frac{dz}{iz}$

$C: R=1$

center: $z=0$

особые т.:

$z=0$

$z^2+4z+1=0$ $\quad D = 16-4=12$ $\quad \frac{-4\pm 2\sqrt{3}}{2}$

$z = -2\pm\sqrt{3}$

$-2-\sqrt{3} < -1$ – вне контура

$-1 < -2+\sqrt{3} < 1$ – внутри

$\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$

$\frac{e^{i2\theta}+e^{-i2\theta}}{2} = \cos 2\theta$

$\ominus -i\left(2\pi i\left(\operatorname{res}_{a_1=0} f(a_1) + \operatorname{res}_{a_2=-2+\sqrt{3}} f(a_2)\right)\right) \ominus$

$$\operatorname*{res}_{a_1=0} f(a_1) = -4$$

$$\frac{z^4+1}{z^2(z^2+4z+1)} = \{\text{уч., има интерес } y\ \tfrac{1}{z}\} \approx \frac{(z^4+1)}{z^2}(1+4z)^{-1} = z^{-2}(1+z^4)(1-4z) \Rightarrow$$

$$\Rightarrow \operatorname*{res}_{a_1=0} f(a_1) = -4$$

$$\operatorname*{res}_{a_2=-2+\sqrt{3}} f(a_2) = \frac{7\sqrt{3}}{3}$$

Исп.

$$\frac{g(c)}{h'(c)} \quad \text{т.к.}$$

$$h'(c) \neq 0$$
$$h(c) = 0$$

$$\frac{z^4+1}{z^2(\underbrace{z+2-\sqrt{3}}_{-\alpha})(\underbrace{z+2+\sqrt{3}}_{-\beta})} = \frac{z^4+1}{z^2(z-\alpha)(z-\beta)}$$

$$h' = 2z(z-\alpha)(z-\beta) + z^2(z-\beta) + z^2(z-\alpha)$$

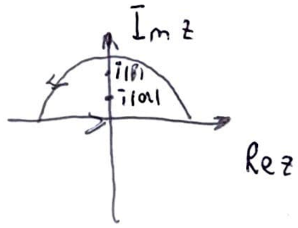
$$\operatorname*{res}_{a_1=-2+\sqrt{3}} f(z) = \frac{(-2+\sqrt{3})^4+1}{\cancel{2(-2+\sqrt{3})(\alpha-\alpha)(\alpha-\beta)} + (-2+\sqrt{3})^2(-2+\sqrt{3}+2+\sqrt{3}) + 0} =$$

$$= \frac{(4-4\sqrt{3}+3)^2+1}{(7-4\sqrt{3})\cdot 2\sqrt{3}} = \frac{49-56\sqrt{3}+48+1}{14\sqrt{3}-24} = \frac{98-56\sqrt{3}}{14\sqrt{3}-24} = \frac{\cancel{14}^{7}\cancel{(7-4\sqrt{3})}}{2\cancel{(7-4\sqrt{3})}\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

Итого:

$$\circledeq\ -i\left(2\pi i\left(-4+\frac{7\sqrt{3}}{3}\right)\right) = 2\pi\left(\frac{7\sqrt{3}}{3}-4\right)$$

3) $$\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)^2} \circledeq \quad a,b\in\mathbb{R}$$ assuming $|b|>|a|$



$$\circledeq \int_{-\infty}^{\infty}\frac{dx}{(x-i|a|)(x+i|a|)(x-i|b|)^2(x+i|b|)^2} = 2\pi i\sum_k \operatorname*{res}_{a_k} f(z)$$

$$\operatorname*{res}_{a_k=i|b|} f(x) = \lim_{x\to i|b|}\frac{d}{dx}\left[(x-i|b|)^2 f(x)\right) = \lim_{x\to i|b|}\frac{d}{dx}\left(\frac{1}{(x^2+a^2)(x+i|b|)^2}\right) =$$

$$= \lim_{x\to i|b|}\frac{-2(x^2+a^2)(x+i|b|)-2x(x+i|b|)^2}{((x^2+a^2)(x+i|b|)^2)^2} = -\frac{2i|b|^3+i|b|(a^2-b^2)}{4|b|^4(-a^2+b^2)^2} = \frac{i(-a^2+3b^2)}{4|b|^3(-a^2+b^2)^2}$$

$$\operatorname*{res}_{a_k = i|a|} f(x) = \left\{ f(x) = \frac{g(x)}{h(x)},\ \text{доп. усл. и т.д.} \right\} = \frac{1}{(x+i|a| + x - i|a|)(x+i|b|)(x-i|b|)^2 + 2(x^2+a^2)(x^2+b^2)}\Bigg|_{i|a|} = \frac{1}{2i|a|(b^2-a^2)^2}$$

$\boxed{2H}$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = 2\pi i \left(\frac{1}{2i|a|(-a^2+b^2)^2} + \frac{i(-a^2+3b^2)}{4|b|^3(-a^2+b^2)^2} \right) =$$

$$= 2\pi i \left(\frac{2|b|^3 - |a|(-a^2+3b^2)}{4i|a||b|^3(-a^2+b^2)^2} \right) = \cancel{2}\pi \cancel{i} \frac{2|b|^3 + |a|^3 - 3b^2|a|}{\cancel{4}_2 \cancel{i} |a||b|^3(b^2-a^2)^2} \overset{*}{=}$$

$$\overset{*}{=} \pi \frac{(|a|-|b|)(|a|(|a|+|b|) - 2b^2)}{2|a||b|^3(a^2-b^2)^2} = \frac{\pi}{2} \cdot \frac{(|a|-|b|)^2(|a|+2|b|)}{|a||b|^3(a^2-b^2)^2} =$$

$$= \underbrace{\frac{|a|+2|b|}{|a||b|^3(|a|+|b|)^2} \frac{\pi}{2}}_{\text{итог}}$$

$$*:\ 2|b|^3 + |a|^3 - 3b^2|a| = |a|^3 - |a|b^2 + 2|b|^3 - 2|a|b^2 =$$

$$= |a|(|a|-|b|)(|a|+|b|) - 2b^2(|a|-|b|) = (|a|-|b|)(|a|(|a|+|b|) - 2b^2)$$

№6.

2H

$\int_C \frac{z^5\,dz}{1+z^6} = 2\pi i$

C: $\begin{cases} \text{центр: } z=0 \\ R=2 \end{cases}$ $\varphi \in [-\pi; \pi]$

$z^6 = -1$

$e^{6i\varphi} = e^{-i\pi + 2i\pi n}$, $n \in \mathbb{Z}$

$\varphi = -\frac{5\pi}{6}; -\frac{\pi}{2}; -\frac{\pi}{6}; \frac{\pi}{6}; \frac{\pi}{2}; \frac{5\pi}{6}$

См. 5 номер, аналогично

$f(z) = \frac{g(z)}{h(z)}$ и т.д. ...

$\operatorname{res}_{a_k} \frac{z^5}{1+z^6} = \frac{1}{6}\Big|_{z=a_k}$

$\int_C \frac{z^5\,dz}{1+z^6} = 2\pi i = 2\pi i\left(\frac16 + \frac16 + \frac16 + \frac16 + \frac16 + \frac16\right)$

№7.

$\int_{-\infty}^{\infty} \frac{\sin^2 x\,dx}{x^2(1+x^2)} = -\frac14 \int_{-\infty}^{\infty} \frac{(e^{ix}-e^{-ix})^2}{x^2(1+x^2)}\,dx = -\int_{-\infty}^{\infty} \frac14 \left(\frac{(e^{2ix} - 2 + e^{-2ix})}{x^2(1+x^2)}\right) dx \Rightarrow$

$\Rightarrow -\int_C \frac14 \frac{e^{2iz}\,dz}{z^2(1+z^2)} - \int_{C_2} \frac14 \frac{e^{-2iz}}{z^2(1+z^2)} + \int_C \frac12 \frac{1 \cdot dz}{z^2(1+z^2)}$

C – против часовой, сверху

C_2 – по часовой, снизу

C – верхняя полукр.

Особые точки в контуре: $z=i$; $z=0$ (на гр.)

Далее по порядку рассм. кажд. точку

$z = i$: $\operatorname{res}_{a_k=i} f(z) = \frac{e^{2iz}}{2(2z^3+z)}\Big|_i = \frac{i}{2e^2}$

$z=-i$: $\operatorname{res}_{a_k=-i} f(z) = \frac{e^{-2iz}}{2(2z^3+z)}\Big|_{-i} = -\frac{i}{2e^2}$

(т.к. в exp −)

$\operatorname{res}_{a_k=i} f(z) = \frac{1}{2(2z^3+z)}\Big|_i = \frac{1}{2(-2i+i)} = \frac{i}{2}$

$z=0$? $\frac{e^{2iz}}{z^2(1+z^2)} = (1+2iz)\left(\frac{1}{z^2}\right)(1-z^2) \Rightarrow \operatorname{res}_{z=0} f(z) = 2i$

т.к. изм только exp: $\operatorname{res}_{z=0} f(z) = -2i$

По аналогии для 3 части $\operatorname{res} f(z) = 0$

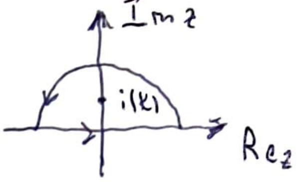
0 на границе

Итого с учётом мал. контуров и прямого (полукруга): $\boxed{2H}$

$$-\frac{1}{4}\left(\cancel{2}\pi i\cdot\frac{i}{\cancel{2}e^2}+2i\cdot\pi i\right)-\frac{1}{4}\left(-\cancel{2}\pi i\left(-\frac{i}{\cancel{2}e^2}\right)+(-2i)(-\pi i)\right)+\frac{1}{\cancel{2}}i/2\cdot\cancel{2}\pi i=$$

$$=\frac{\pi}{4e^2}+\frac{2\pi}{4}+\frac{\pi}{4e^2}+\frac{2\pi}{4}-\frac{\pi}{2}=\frac{\pi}{2}\left(1+\frac{1}{e^2}\right)$$

№8. $\boxed{\text{ДЗ}}$



$$\int_0^\infty \frac{x\sin(ax)}{x^2+k^2}dx$$

Чтобы замкнуть контур сверху $a=|a|\,\mathrm{sign}(a)$

$$\mathrm{sign}(a)\int_0^\infty \frac{x\sin(|a|x)}{x^2+k^2}dx \underset{\text{чётная}}{=} \frac{\mathrm{sign}(a)}{2}\int_{-\infty}^{\infty}\frac{x\sin(|a|x)}{x^2+k^2}dx = \frac{\mathrm{sign}(a)}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{xe^{i|a|x}}{x^2+k^2}dx =$$

$$\Rightarrow \mathrm{Im}\int_C \frac{\mathrm{sign}(a)}{2}\,\frac{ze^{i|a|z}}{z^2+k^2} =$$

Особые точки внутри контура: $i|k|$

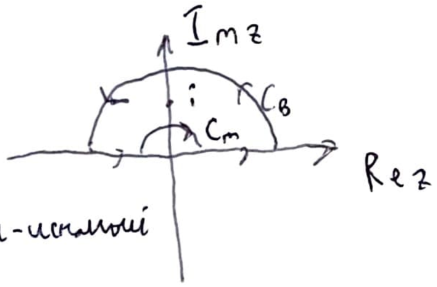
C: верхн. полукр.

$$\underset{a_0=ik}{\mathrm{res}}\, f(z) = \left.\frac{\cancel{z}e^{i|a|z}}{2\cancel{z}}\right|_{i|k|} = \left.\frac{e^{i|a|z}}{2}\right|_{i|k|} = \frac{e^{-|a||k|}}{2}$$

$$= \mathrm{Im}\,\cancel{2}\pi i\cdot\frac{e^{-|a||k|}}{\cancel{2}}\cdot\frac{\mathrm{sign}(a)}{2} = \frac{\pi\,\mathrm{sign}(a)}{2}e^{-|a||k|}$$

№9.

$$\int_{-\infty}^{\infty}\frac{\cos(x-\frac{1}{x})}{1+x^2}dx = \mathrm{Re}\int_{-\infty}^{\infty}\frac{\exp(iz-i/z)}{1+z^2}dz = \mathrm{Re}\int_{-\infty}^{\infty}\frac{\exp iz\,\exp -i/z}{1+z^2}dz$$



u-искомый

Особые точки (интересующие): $z=0$; $z=i$

$$\oint = \int_u + \int_{C_B} + \int_{C_m}$$

Рассм. каждую из дуг:

$\int_{C_m}$: $z=\varepsilon e^{i\varphi}$, $dz=\varepsilon\, de^{i\varphi}$ Оценим по модулю:

по пределам $\pi\ldots0$

$$\left|\int_{C_m}\frac{\exp i\varepsilon e^{i\varphi}\,\exp -\frac{i}{\varepsilon}e^{-i\varphi}}{1+\varepsilon^2e^{2i\varphi}}\cdot i\varepsilon e^{i\varphi}d\varphi\right| \leqslant$$

$$\leqslant \varepsilon\pi\,\frac{|e^{i\varphi}||e^{i\varepsilon e^{i\varphi}}||e^{-\frac{i}{\varepsilon}e^{-i\varphi}}|}{|1+\varepsilon^2e^{2i\varphi}|} = \frac{\varepsilon\pi}{|1+\varepsilon^2e^{2i\varphi}|}\,|e^{i\varphi}||e^{i\varepsilon\cos\varphi}||e^{-\frac{i}{\varepsilon}\cos\varphi}||e^{-\varepsilon\sin\varphi}|\cdot$$

$\cdot|e^{-\frac{1}{\varepsilon}\sin\varphi}|$, очевидно, что каждая $\leqslant 1$ из exp.

$\Downarrow$

$\leqslant \mathrm{const}\cdot\pi\varepsilon$, при $\varepsilon\to0$, инт. $\to 0$

(знаменатель $\geqslant 1/2$)

По аналогии для C_B

$\int$ $z = re^{i\varphi}$

$dz = r\,de^{i\varphi}$, т.е. просто $\varepsilon \to r$

$\left|\int_{C_B} f(z)\,dz\right| \leqslant \frac{\pi r}{1 + r e^{2i\varphi}}$; $r \to \infty \Rightarrow \leqslant \frac{\pi}{r}$ и $\to 0$ при $r \to \infty$

Т.к. $\int_{C_m}$ и $\int_{C_B} = 0$:

$\oint = \int_u$

используя $f(z) = \frac{g(z)}{h(z)}$ и т.д.

$2\pi i \operatorname{res}_{a_k = i}\left(\frac{\exp(iz - i/z)}{1+z^2}\right) = 2\pi i \frac{\exp(i^2 - 1)}{2i} = \pi e^{-2}$

$\int_u = \pi e^{-2} = \int_{-\infty}^{\infty} \frac{\cos(x - 1/x)}{1+x^2}\,dx$

№10.

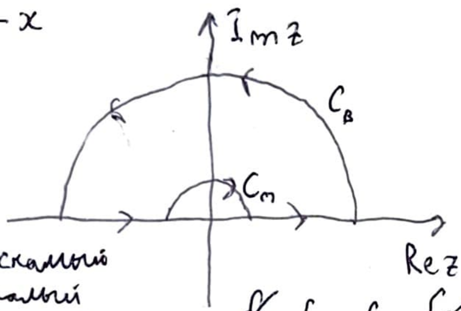
1) $\int_0^\infty \frac{x-\sin x}{x^3}dx = \int_0^\infty \frac{x-\frac{(e^{ix}-e^{-ix})}{2i}}{x^3} = \frac{1}{2}\int_0^\infty \frac{2xi - e^{ix}+e^{-ix}}{ix^3}dx =$

$= \int_0^\infty \frac{2ix - e^{ix}+e^{-ix}}{2ix^3}dx = \int_\varepsilon^\infty \frac{ix-e^{ix}}{2ix^3}dx + \int_\varepsilon^\infty \frac{ix+e^{-ix}}{2ix^3}dx = \frac{1}{2}\Big(\int_\varepsilon^\infty \frac{ix-e^{ix}}{x^3 i}dx +$

$+ \int_{-\infty}^{-\varepsilon} \frac{ix-e^{ix}}{ix^3}\Big) = 0,5\ \text{p.V.}\int_{-\infty}^{\infty} \frac{ix-e^{ix}}{ix^3}dx$

замена $x \to -x$

без $-\varepsilon \dots \varepsilon$



И - искомый
m - малый
B - большой

$\frac{1}{2}\int_{C_m} \frac{iz - e^{iz}}{iz^3} \ominus$

$iz - e^{iz} \simeq \cancel{iz} - 1 - \cancel{iz} - \frac{(iz)^2}{2} \dots = -1 - \frac{(iz)^2}{2} =$

$= -1 + \frac{z^2}{2}$

$\oint = \int_И + \int_{C_m} + \cancel{\int_{C_B}} \Rightarrow \int_И = -\int_{C_m}$

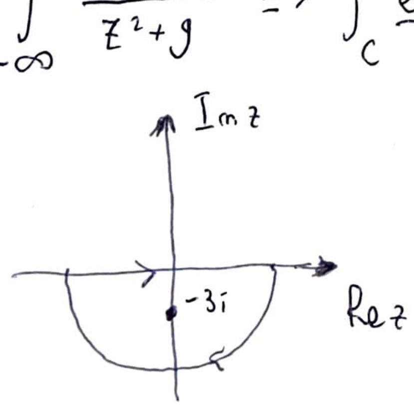
$\ominus \frac{1}{2}\int_{C_m}\left[\frac{i}{z^3} - \frac{i}{2z}\right]dz = \left\{\begin{matrix} z = \varepsilon e^{i\varphi} \\ dz = \varepsilon d e^{i\varphi}\end{matrix}\right\} = \frac{1}{2}\int_\pi^0 \left[\frac{i}{\varepsilon^3 e^{i3\varphi}} - \frac{i}{2\varepsilon e^{i\varphi}}\right]\varepsilon d e^{i\varphi} = \frac{1}{2}\int_\pi^0 \left[-\frac{e^{-2i\varphi}}{\varepsilon^2} + \frac{1}{2}\right]d\varphi$

$= \frac{1}{2}\left[\frac{e^{-2i\varphi}}{2i\varepsilon^2}\Big|_\pi^0 + \frac{\varphi}{2}\Big|_\pi^0\right] = \frac{1}{2}\left[\cancel{\frac{1}{2i\varepsilon^2}} - \cancel{\frac{1}{2i\varepsilon^2}} + 0 - \frac{\pi}{2}\right] = -\frac{\pi}{4}$

$\int_0^\infty \frac{x-\sin x}{x^3}dx = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

2) $\int_{-\infty}^{\infty} \frac{e^{-iz}dz}{z^2+9} \Rightarrow \int_C \frac{e^{-iz}dz}{z^2+9} = -2\pi i \sum_k \underset{a_k}{\text{res}} f(z) \ominus$

по часовой



Внутри только $z=-3i$ из особых

Используя $f(z) = \frac{g(z)}{h(z)}$: $\underset{a_k=-3i}{\text{res}} \frac{e^{-iz}}{z^2+9} = \frac{e^{-iz}}{2z}\Big|_{-3i} = \frac{ie^{-3}}{6}$

$\ominus \frac{-\cancel{2}\pi (i)^2 e^{-3}}{\cancel{6}_3} = \frac{\pi}{3e^3}$

№11.

$f(z) = z^3 \cos \frac{1}{z-2}$

$z = \infty$

Рассм.: (ряд Лорана для $z_0 = 2$, далее сумма выч. на инф. и обычных $= 0$)

$$\cos\frac{1}{z-2} \simeq 1 - \frac{1}{2(z-2)^2} + \frac{1}{24(z-2)^4} \ldots \simeq 1 - \frac{1}{2z^2}\left(1-\frac{2}{z}\right)^{-2} + \frac{1}{24z^4}\left(1-\frac{2}{z}\right)^{-4}$$

$$= 1 - \frac{1}{2z^2}\left(1 + \frac{4}{z} + \frac{2^{\cancel{2}}}{\cancel{2}z^2}(-2)(-2-1)\ldots\right) + \frac{1}{24z^4}\left(1 + \frac{8}{z}\ldots\right) \ominus$$

т.к. с учётом z^3 перед $\cos$ интересны только члены с $\frac{1}{z^4} \cdot \frac{a}{b}$ const

$\ominus$ {не пишу ничто, кроме } $\simeq -\frac{6}{z^4} + \frac{1}{24z^2} = -\frac{143}{24} z^{-4}$

Ответ: $\frac{143}{24}$

№12

$f(z) = \frac{1}{z^3 - z^5}$

Sum of residues is 0.

$z = -1$

$z = 0$

$z = 1$

$z = \infty$

$z = -1$: Using $\mathrm{Res}(f, c) = \frac{g(c)}{h'(c)}$, where $f(z) = \frac{g(z)}{h(z)}$ with $h'(c) \neq 0$, $h(c) = 0$

$\mathrm{Res}(f; -1) = \frac{1}{3-5} = -\frac{1}{2}$

$z = 1$:

$\mathrm{Res}(f; 1) = \frac{1}{3-5} = -\frac{1}{2}$

$z = 0$:

$$\frac{1}{z^3 - z^5} = \frac{1}{z^3(1-z^2)} = \frac{1}{z^3}(1-z^2)^{-1} = \frac{1}{z^3}(1+z^2) = \frac{1}{z^3} + \frac{1}{z} + \ldots$$

$\mathrm{Res}(f, 0) = 1$

$z = \infty$:

$$\frac{1}{z^3 - z^5} = -\frac{1}{z^5\left(1 - \frac{1}{z^2}\right)} = -\frac{1}{z^5}\left(1 - \frac{1}{z^2}\right)^{-1} = -\frac{1}{z^5}\left(1 + \frac{1}{z^2}\ldots\right) \Rightarrow \mathrm{Res}(f, \infty) = 0$$

№13. 2H

1) $f(z)=\dfrac{\sin\frac{1}{z}}{1-z}$

$z=0$

$(1-z)^{-1} = 1+z+z^2\ldots$

$\sin\left(\frac{1}{z}\right) = \sin(t) = t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots$

$t=\frac{1}{z}$

$\Downarrow$

$\sin\left(\frac{1}{z}\right) = \frac{1}{z}-\frac{1}{6z^3}+\frac{1}{5!z^5}\ldots$

$(1+z\ldots)\left(\frac{1}{z}-\frac{1}{6z^3}\ldots\right)$

$\sum\limits_{n=0}^{\infty} z^n \sum\limits_{k=0}^{\infty} \frac{z^{-(2k+1)}(-1)^k}{(2k+1)!} \Rightarrow$ коэфф. перед z^{-1};

$\sum\limits_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!} = \sin(1)$

$f(z) = \exp(-\exp(\frac{1}{z}))$

$t=\frac{1}{z}$

$\exp(-\exp t) = \exp(-1-t-\frac{t^2}{2}\ldots) =$

$= 1+(-1-t-\frac{t^2}{2}\ldots)+\frac{(-1-t-\frac{t^2}{2}\ldots)^2}{2}\ldots$

$t=\frac{1}{z} \Rightarrow 1+(-1-\frac{1}{z}-\frac{1}{2z^2}\ldots)+$

$+\frac{1}{2}(-1-\frac{1}{z}-\frac{1}{2z}\ldots)^2+\ldots$

Перед $\frac{1}{z}$:

$-1+\frac{1}{2}\cdot 2\cdot 1-\frac{1}{6}\cdot 3\cdot 1\ldots =$

$=\sum\limits_{k=1}^{\infty}\frac{(-1)^k k}{k!} = -\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}k}{k!} =$

$=-e^{-1}$

№14

1) $\lim_{R\to\infty}\int_{C_R} e^{iz}dz = \lim_{R\to\infty} \left.\frac{e^{iz}}{i}\right|_{-R}^{R} = \lim_{R\to\infty}\frac{e^{iR}-e^{-iR}}{i} = \lim_{R\to\infty} 2\sin(R)$ – не сущ.

$C_R: |z|=R;\ \varphi\in[0,\pi]$

2) $\lim_{R\to\infty}\int_{C_R} e^{iz^2}dz = \lim_{R\to\infty}\int_0^{\pi/4} e^{iR^2(\cos 2\varphi+i\sin 2\varphi)}\, iRe^{i\varphi}d\varphi$

$\boxed{C_R: z=Re^{i\varphi}; 0\le\varphi\le\frac{\pi}{4}}$

Сделаем оценку:

$I_R = \left|\int_{C_R} e^{iz^2}dz\right| \le R\int_0^{\pi/4}|ie^{i\varphi}e^{R^2(i\cos 2\varphi-\sin 2\varphi)}|d\varphi \le$

$\le \int_0^{\pi/4} Re^{-R^2\sin 2\varphi}d\varphi \le \int_0^{\pi/4} Re^{-\frac{R^2 4\varphi}{\pi}}d\varphi = -\frac{\pi}{4R}\left(\left.e^{-\frac{R^2 4\varphi}{\pi}}\right|_0^{\pi/4}\right) = \frac{\pi}{4R}(1-e^{-R^2}) \to 0$ case $R\to\infty$

$\sin\varphi \ge \frac{2\varphi}{\pi} \Rightarrow \sin 2\varphi \ge \frac{4\varphi}{\pi}$

для $\varphi\in[0;\frac{\pi}{2}]$ для $\varphi\in[0;\frac{\pi}{4}]$

$\Downarrow$

$\lim_{R\to\infty}\int_{C_R} e^{iz^2}dz = 0$