MATH
600.475 MACHINE LEARNING

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We will define our energy function E. Suppose that we have a game with blocks $b_1, \dots, b_n$, and empty territories $e_1, \dots, e_t$ at the end of the game. and our network's hidden layer has m nodes. Our learning algorithm uses a hidden layer of size m, a matrix of input layer weights W of size $m \times n$, a vector of hidden layer weights $\mathbf{v} \in \mathbb{R}^m$, and two vector biases $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. Furthermore, for the second half of the algorithm, we use a matrix of weights V of size 4×5, a vector of hidden weights $\mathbf{g} \in \mathbb{R}^4$, and two bias vectors $\mathbf{c} \in \mathbb{R}^5$ and $\mathbf{d} \in \mathbb{R}^4$.

Suppose we have a game G for which we would like to determine $E(G)$. Let k be the komi, p_b and p_w the prisoners held by black and white, respectively, and the true final score S. Define $\text{touches}_e(b)$ to be the indicator function taking the value 1 when the territory e touches the block b. Define $\text{color}(b)$ to be the indicator function taking the value 1 when the block b is white. Let $\text{size}(e)$ return the size of the territory e. Let $G(x)$ and $L(x)$ be the indicator functions that return 1 when x is greater than and less than x. Note that when $x \neq 0$, $L(x) = (|x| - x)/2x$ and $G(x) = (|x| + x)/2x$. The energy function runs algorithmically as follows:

(1) For each block b_i calculate a feature vector $\mathbf{x}_i \in \mathbb{R}^n$.
(2) Calculate $r_i = \mathbf{v}[W(\mathbf{x}_i + \mathbf{a}) + \mathbf{b}]$ for each i.
(3) For each territory e_i compute

$$N_b^{(i)} = \sum_{j=1}^{n} (1 - \text{color}(b_j))\text{touches}_{e_i}(b_j) \qquad N_w^{(i)} = \sum_{j=1}^{n} \text{color}(b_j)\text{touches}_{e_i}(b_j)$$

$$R_b^{(i)} = \sum_{j=1}^{n} (1 - \text{color}(b_j))\text{touches}_{e_i}(b_j) r_j \qquad R_w^{(i)} = \sum_{j=1}^{n} \text{color}(b_j)\text{touches}_{e_i}(b_j) r_j$$

(4) Construct the vector $\mathbf{r}_i = \{N_b^{(i)}, N_w^{(i)}, R_b^{(i)}, R_w^{(i)}, \text{size}(e_i)\}$.
(5) Compute $s_i = \mathbf{g}[V(\mathbf{r}_i + \mathbf{c}) + \mathbf{d}]$.
(6) Compute $S^* = k - c_b + c_w + \sum_{i=1}^{t} s_i$.
(7) We will assume that $r_i \neq 0$. Compute

$$D_b = 2 \cdot \sum_{i=1}^{n} (1 - \text{color}(b_i))\text{size}(b_i) L(r_i) \qquad D_w = 2 \cdot \sum_{i=1}^{n} \text{color}(b_i)\text{size}(b_i) G(r_i)$$

(8) Output the energy $|S - (S^* + D_b - D_w)|$.

We can write it out algebraically as

$$\begin{aligned}
E(G) &= |E^*(G)| = |S - (S^* + D_b - D_w)| \\
&= \left| S - \left(k - c_b + c_w + \sum_{i=1}^{t} s_i + 2 \cdot \sum_{i=1}^{n} (1 - \text{color}(b_i))\text{size}(b_i) L(r_i) - 2 \cdot \sum_{i=1}^{n} \text{color}(b_i)\text{size}(b_i) G(r_i) \right) \right| \\
&= \left| S - k + c_b - c_w - \sum_{i=1}^{t} s_i + 2 \cdot \sum_{i=1}^{n} \text{size}(b_i) \left(\text{color}(b_i) G(r_i) - (1 - \text{color}(b_i)) L(r_i) \right) \right| \\
&= \left| S - k + c_b - c_w - \sum_{i=1}^{t} \mathbf{g}[V(\mathbf{r}_i + \mathbf{c}) + \mathbf{d}] + 2 \cdot \sum_{i=1}^{n} \text{size}(b_i) \left(\text{color}(b_i) G(r_i) - (1 - \text{color}(b_i)) L(r_i) \right) \right|
\end{aligned}$$

Therefore,

$$\frac{\partial E^*(G)}{\partial \mathbf{g}_i} = -\sum_{i=1}^{t} [V(\mathbf{r}_i + \mathbf{c}) + \mathbf{d}] \frac{\partial}{\partial \mathbf{g}_i} \mathbf{g}$$

$$\frac{\partial E^*(G)}{\partial V_{ij}} = -\sum_{i=1}^{t} \mathbf{g} \left(\frac{\partial}{\partial V_{ij}} V \right) (\mathbf{r}_i + \mathbf{c})$$

$$\frac{\partial E^*(G)}{\partial \mathbf{c}_i} = -\sum_{i=1}^{t} \mathbf{g} V \frac{\partial}{\partial \mathbf{c}_i} \mathbf{c}$$

$$\frac{\partial E^*(G)}{\partial \mathbf{d}_i} = \mathbf{g} \frac{\partial}{\partial \mathbf{d}_i} \mathbf{d}$$

Furthermore, since

$$\frac{dL(x)}{dx} = 0 \qquad\qquad \frac{dG(x)}{dx} = 0$$

then when $r_i \neq 0$

$$\frac{\partial E^*(G)}{\partial r_i} = \sum_{i=1}^{t} \mathbf{g} V \frac{\partial}{\partial r_i} \mathbf{r}_i = \sum_{i=1}^{t} \mathbf{g} V \left\{ 0, 0, \frac{\partial}{\partial r_i} R_b^{(i)}, \frac{\partial}{\partial r_i} R_w^{(i)}, 0 \right\}$$

Therefore,

$$\frac{\partial E^*(G)}{\partial \mathbf{v}_i} = \sum_{i=1}^{t} \mathbf{g} V \left\{ 0, 0, \sum_{j=1}^{n} (1 - \mathrm{color}(b_j)) \mathrm{touches}_{e_i}(b_j) [W(\mathbf{x}_j + \mathbf{a}) + \mathbf{b}] \frac{\partial}{\partial \mathbf{v}_i} \mathbf{v}, \right.$$
$$\left. \sum_{j=1}^{n} \mathrm{color}(b_j) \mathrm{touches}_{e_i}(b_j) [W(\mathbf{x}_j + \mathbf{a}) + \mathbf{b}] \frac{\partial}{\partial \mathbf{v}_i} \mathbf{v}, 0 \right\}$$

$$\frac{\partial E^*(G)}{\partial W_{ij}} = \sum_{i=1}^{t} \mathbf{g} V \left\{ 0, 0, \sum_{j=1}^{n} (1 - \mathrm{color}(b_j)) \mathrm{touches}_{e_i}(b_j) \mathbf{v} \left(\frac{\partial}{\partial W_{ij}} W \right) (\mathbf{x}_j + \mathbf{a}), \right.$$
$$\left. \sum_{j=1}^{n} \mathrm{color}(b_j) \mathrm{touches}_{e_i}(b_j) \mathbf{v} \left(\frac{\partial}{\partial W_{ij}} W \right) (\mathbf{x}_j + \mathbf{a}), 0 \right\}$$

$$\frac{\partial E^*(G)}{\partial \mathbf{a}_i} = \sum_{i=1}^{t} \mathbf{g} V \left\{ 0, 0, \sum_{j=1}^{n} (1 - \mathrm{color}(b_j)) \mathrm{touches}_{e_i}(b_j) \mathbf{v} W \frac{\partial}{\partial \mathbf{a}_i} \mathbf{a}, \right.$$
$$\left. \sum_{j=1}^{n} \mathrm{color}(b_j) \mathrm{touches}_{e_i}(b_j) \mathbf{v} W \frac{\partial}{\partial \mathbf{a}_i} \mathbf{a}, 0 \right\}$$

$$\frac{\partial E^*(G)}{\partial \mathbf{b}_i} = \sum_{i=1}^{t} \mathbf{g} V \left\{ 0, 0, \sum_{j=1}^{n} (1 - \mathrm{color}(b_j)) \mathrm{touches}_{e_i}(b_j) \mathbf{v} \frac{\partial}{\partial \mathbf{b}_i} \mathbf{b}, \right.$$
$$\left. \sum_{j=1}^{n} \mathrm{color}(b_j) \mathrm{touches}_{e_i}(b_j) \mathbf{v} \frac{\partial}{\partial \mathbf{b}_i} \mathbf{b}, 0 \right\}$$

Therefore, we can easily compute the same for $E(G)$ knowing that $d|x|/dx = x/|x|$.