Stage 1 – Boundary Derivation

For the **South–West corner node**, This is the first general finite volume equation , including all surface contributions.

The **Neumann condition** is imposed on the South boundary and the **Dirichlet condition** on the West boundary.
For the remaining faces, apply Green's theorem as shown in *Appendix A.2.1*. This is the resulting equation

South-West Corner Node

General FVM equation:

$$\nabla^2 T\big|_P \approx \frac{\lambda}{S_P}\left[\Delta y_{sw}^{se}\frac{\partial T}{\partial x}\bigg|_s - \Delta x_{sw}^{se}\frac{\partial T}{\partial y}\bigg|_s + \Delta y_{se}^{ne}\frac{\partial T}{\partial x}\bigg|_e - \Delta x_{se}^{ne}\frac{\partial T}{\partial y}\bigg|_e + \Delta y_{ne}^{nw}\frac{\partial T}{\partial x}\bigg|_n - \Delta x_{ne}^{nw}\frac{\partial T}{\partial y}\bigg|_n + \Delta y_{nw}^{sw}\frac{\partial T}{\partial x}\bigg|_w - \Delta x_{nw}^{sw}\frac{\partial T}{\partial y}\bigg|_w\right] = 0$$

East face (Green's theorem)

$$\begin{aligned}
\frac{\partial T}{\partial x}\bigg|_e &\approx \frac{1}{S_e}\left(\Delta y_s^{sE} T_{se} + \Delta y_{sE}^{nE} T_E + \Delta y_{nE}^{n} T_{ne} + \Delta y_n^{s} T_P\right), \\
\frac{\partial T}{\partial y}\bigg|_e &\approx -\frac{1}{S_e}\left(\Delta x_s^{sE} T_{se} + \Delta x_{sE}^{nE} T_E + \Delta x_{nE}^{n} T_{ne} + \Delta x_n^{s} T_P\right).
\end{aligned}$$

North face (Green's theorem)

$$\begin{aligned}
\frac{\partial T}{\partial x}\bigg|_n &\approx \frac{1}{S_n}\left(\Delta y_w^{e} T_P + \Delta y_e^{Ne} T_{ne} + \Delta y_{Ne}^{Nw} T_N + \Delta y_{Nw}^{w} T_{nw}\right), \\
\frac{\partial T}{\partial y}\bigg|_n &\approx -\frac{1}{S_n}\left(\Delta x_w^{e} T_P + \Delta x_e^{Ne} T_{ne} + \Delta x_{Ne}^{Nw} T_N + \Delta x_{Nw}^{w} T_{nw}\right).
\end{aligned}$$

South face (Neumann BC)

$$\frac{\partial T}{\partial y}\bigg|_s = -\frac{q_s}{\lambda}$$

$$\frac{\partial T}{\partial x}\bigg|_s \approx \frac{1}{S_s}\left(\Delta y_{Sw}^{Se} T_S + \Delta y_{Se}^{e} T_{se} + \Delta y_e^{w} T_P + \Delta y_w^{Sw} T_{sw}\right)$$

West face (Dirichlet BC)

$$\frac{\partial T}{\partial x}\bigg|_w \approx \frac{1}{S_w}\left(\Delta y_s^{n} T_P + \Delta y_n^{Nw} T_{nw} + \Delta y_{Nw}^{Sw} T_W^D + \Delta y_{Sw}^{s} T_{sw}\right)$$

$$\frac{\partial T}{\partial y}\bigg|_w \approx -\frac{1}{S_w}\left(\Delta x_s^{n} T_P + \Delta x_n^{Nw} T_{nw} + \Delta x_{Nw}^{Sw} T_W^D + \Delta x_{Sw}^{s} T_{sw}\right)$$

Corner interpolations

$$\begin{aligned}
T_{se} &= \frac{T_S + T_{SE} + T_E + T_P}{4}, \\
T_{ne} &= \frac{T_P + T_E + T_{NE} + T_N}{4}, \\
T_{nw} &= \frac{T_W^D + T_P + T_N + T_{NW}}{4}, \\
T_{sw} &= \frac{T_{SW} + T_S + T_P + T_W^D}{4}.
\end{aligned}$$

Stage 2 – Solver Implementation

In this stage, you will implement a **2D steady-state heat conduction solver** using the Finite Volume Method (FVM) with all boundary condition types.

The **form function** defining the geometry is already provided — do **not** modify it.
Your task starts from the **mesh generation** using `setupMesh()`.
First, **visualize the mesh** to verify that the geometry and node layout are correct.

Next, complete the helper class and functions:

- `Coordinate2D` class
- `calculate_area`, `dx`, `dy`, `dist`, and `index` functions

Refer to the provided Skriptum for the necessary geometric and differential relations.

In the **main solver class**:

1. Initialize all required parameters similarly to the Finite Difference (FD) exercise.
2. In the `set_stencil()` method, identify whether each node belongs to the interior, boundary, or corner region, and call the respective `build_*()` function.
   - `build_inner()` and `build_north()` are already implemented as examples.
   - You must implement the remaining boundary and corner functions yourself.

Inside the `solve()` method:

- Loop through each node in the computational domain.
- Call `set_stencil()` for each node to assemble the global coefficient matrix **A** and source vector **B**.
- After assembly, solve the linear system and reshape the resulting temperature field into 2D form.

Finally, visualize the computed temperature field using the provided plotting routine.
Note that you are solving only the **upper half** of the geometry.
Reconstruct the full domain for plotting using `np.flipud()` and `np.vstack()` to mirror the field across the symmetry line.

In [1]:

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
import scipy.sparse as sp
from scipy.sparse.linalg import spsolve
```

Form function to change the shape of domain

Domain see Figure 3.4

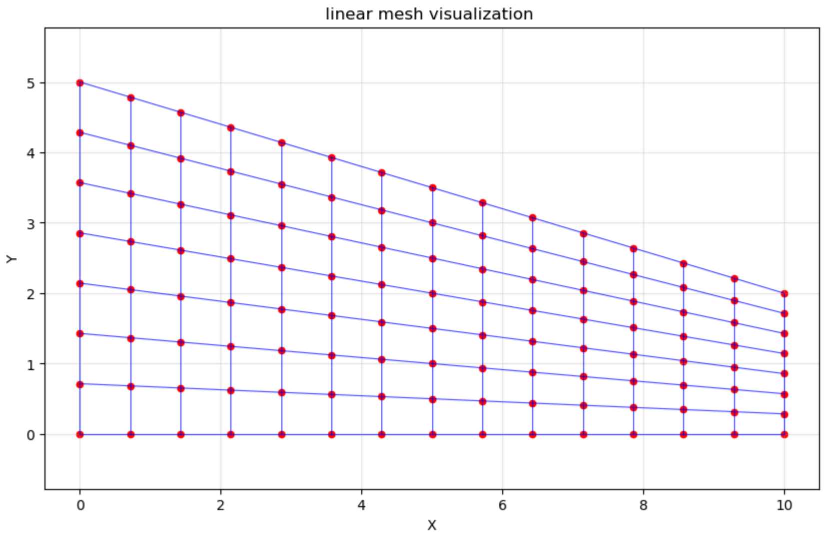
In [3]:

```python
class Coordinate2D():
    def __init__(self, x, y):
        self.x = x
        self.y = y

def calculate_area(ul, bl, br, ur):
    # calculate the area of the cell
    # ul (upper left), bl (bottom left), br (bottom right), ur (upper right) are the coordinates of the four vertices of the cell
    # apply Gaussian trapezoidal formula to calculate the areas

    x = [ul.x, bl.x, br.x, ur.x]
    y = [ul.y, bl.y, br.y, ur.y]

    area = 0.5 * abs(
        x[0]*y[1] + x[1]*y[2] + x[2]*y[3] + x[3]*y[0] - y[0]*x[1] - y[1]*x[2] - y[2]*x[3] - y[3]*x[0]
        )
    return area

def dy(a, b):
    # Calculate distance between 'a' and 'b' along the y axis
    return (b.y - a.y)

def dx(a, b):
    # Calculate distance between 'a' and 'b' along the x axis
    return (b.x - a.x)

def dist(a, b):
    # Calculate the euclidean distance between 'a' and 'b'
    return np.sqrt((b.x - a.x)**2 + (b.y - a.y)**2)
```

In [4]:

```python
def formfunction(x, shape):
    """
    Defines the shape of north boundary
    takes an array and shape
    returns an array
    """
    h1 = x[-1]           # west boundary height
    h2 = x[-1] / 10 * 4  # east boundary height
    l  = x[-1]           # domain length
    if shape == 'linear':
        m = (h2 - h1) / (2 * l)
        b = h1 / 2
        return m * x + b

    elif shape == 'rectangular':
        return l * np.ones((x.size, 1))

    elif shape == 'quadratic':
        k = 2 * l**2 / (h1 - h2)
        return (x - l)**2 / k + h2 / 2

    elif shape == 'crazy':
        return h1/2 + (h2/2 - h1/2) * x / l + 0.25*(-h1 + h2/2) * np.sin(np.pi*x/l)**2

    else:
        raise ValueError('Unknown shape: %s' % shape)

def setUpMesh(nodes_x, nodes_y, length, formfunction, shape):
    # 1D x coordinates
    X = np.linspace(0, length, nodes_x)

    # Compute half-height profile along X
    y_profile = formfunction(X, shape).flatten()

    # Initialize Y mesh
    Y = np.zeros((nodes_y, nodes_x))

    for j in range(nodes_x):
        # Local half height at this x-position
        local_h = y_profile[j]

        # Linear distribution from local height to 0 along Y-direction
        Y[:, j] = np.linspace(local_h, 0, nodes_y)

    # Repeat X along rows to create 2D mesh
    X = np.array([X[:] for i in range(nodes_y)])

    return X, Y
# Parameters for mesh generation
nodes_x = 15
nodes_y = 8
length = 10

# Change this variable to plot different shapes
selected_shape = 'linear'  # Options: 'linear', 'rectangular', 'quadratic', 'crazy'

# Generate and visualize mesh for the selected shape
X, Y = setUpMesh(nodes_x, nodes_y, length, formfunction, selected_shape)

plt.figure(figsize=(10, 6))
plt.plot(X, Y, 'b-', linewidth=0.8, alpha=0.7)
plt.plot(X.T, Y.T, 'b-', linewidth=0.8, alpha=0.7)
plt.scatter(X, Y, c='red', s=20)
plt.xlabel('X')
plt.ylabel('Y')
plt.title(f'{selected_shape} mesh visualization ')
plt.grid(True, alpha=0.3)
```

```python
plt.axis('equal')
plt.show()
```



In [5]:
```python
class SteadyHeat2D_FVM:
    """
    2D steady-state heat conduction solver using Finite Volume Method.
    Handles arbitrary quadrilateral meshes and mixed boundary conditions.
    """

    def __init__(self, X, Y, boundary=[], TD=[], q=0.0, alpha=0.0, Tinf=0.0):
        """
        Initialize FVM solver.

        Args:
            X, Y: mesh coordinate arrays (m × n)
            boundary: BC types [North, South, West, East] ('D', 'N', or 'R')
            TD: Dirichlet temperatures [North, South, West, East]
            q: heat flux for Neumann BC (W/m²)
            alpha: heat transfer coefficient for Robin BC (W/m²K)
            Tinf: ambient temperature for Robin BC (K)
        """
        self.X = X
        self.Y = Y
        self.boundary = boundary
        self.TD = TD
        self.q = q
        self.alpha = alpha
        self.Tinf = Tinf

        # Grid dimensions: m = rows (y-direction), n = columns (x-direction)
        self.m, self.n = X.shape

        # Initialize sparse linear system
        total_nodes = self.m * self.n
        self.A = sp.lil_matrix((total_nodes, total_nodes))
        self.B = np.zeros(total_nodes)

    def index(self, i, j):
        """Map 2D grid index (i,j) to 1D array index"""
        return i * self.n + j

    def stable_area(self, *args):
        """Wrapper for area calculation with stability check"""
        area = calculate_area(*args)
        return area if area > 1e-12 else 1e-12

    def set_stencil(self, i, j):
        """
        Determine node type and build appropriate stencil.
        Grid indexing: i=0 is North (top), i=m-1 is South (bottom)
                       j=0 is West (left), j=n-1 is East (right)
        """
        # Check corners first
        if i == 0 and j == 0:
            return self.build_NW(i, j)
        elif i == 0 and j == self.n-1:
            return self.build_NE(i, j)
        elif i == self.m-1 and j == 0:
            return self.build_SW(i, j)
        elif i == self.m-1 and j == self.n-1:
            return self.build_SE(i, j)

        # Check boundaries
        elif i == 0:
            return self.build_north(i, j)
        elif i == self.m-1:
            return self.build_south(i, j)
        elif j == 0:
            return self.build_west(i, j)
        elif j == self.n-1:
            return self.build_east(i, j)

        # Interior node
        else:
            return self.build_inner(i, j)

    def build_inner(self, i, j):
        """
```

```
    Build 9-point stencil for interior nodes using Green's theorem.

    Node nomenclature:
        NW   Nw    N    Ne   NE
              |         |
        nW   nw    n    ne   nE
         |         |         |
        W    w     P    e    E
         |         |         |
        sW   sw    s    se   sE
              |         |
        SW   Sw    S    Se   SE
    """
    stencil = np.zeros(self.n * self.m)
    b = np.zeros(1)

    # Principal nodes
    P = Coordinate2D(self.X[i, j], self.Y[i, j])
    N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
    S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
    W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
    E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
    NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])
    NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])
    SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
    SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

    # Auxiliary nodes (midpoints)
    Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
    Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
    Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
    Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
    nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
    nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)
    sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
    sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)

    # Face centers
    n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
    s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
    w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
    e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

    # Control volume corners
    se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)
    sw = Coordinate2D((Sw.x + w.x)/2, (Sw.y + w.y)/2)
    ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
    nw = Coordinate2D((Nw.x + w.x)/2, (Nw.y + w.y)/2)

    # Control volume areas
    S_P = calculate_area(ne, se, sw, nw)  # Main cell
    S_n = calculate_area(Ne, e, w, Nw)    # North face
    S_s = calculate_area(e, Se, Sw, w)    # South face
    S_w = calculate_area(n, s, sW, nW)    # West face
    S_e = calculate_area(nE, sE, s, n)    # East face

    # Coefficients from Green's theorem application
    # East neighbor
    D3 = ((dx(se, ne) * (dx(nE, n)/4 + dx(s, sE)/4 + dx(sE, nE))) / S_e +
         (dy(se, ne) * (dy(nE, n)/4 + dy(s, sE)/4 + dy(sE, nE))) / S_e +
         (dx(e, Ne) * dx(ne, nw)) / (4*S_n) + (dx(Se,e) * dx(sw,se)) / (4*S_s) +
         (dy(e, Ne) * dy(ne, nw)) / (4*S_n) + (dy(Se,e) * dy(sw,se)) / (4*S_s)) / S_P

    # West neighbor
    D_3 = ((dx(nw, sw) * (dx(n, nW) / 4 + dx(sW, s) / 4 + dx(nW, sW))) / S_w +
          (dy(nw, sw) * (dy(n, nW) / 4 + dy(sW, s) / 4 + dy(nW, sW))) / S_w +
          (dx(Nw, w) * dx(ne, nw)) / (4 * S_n) +
          (dx(w, Sw) * dx(sw, se)) / (4 * S_s) +
          (dy(Nw, w) * dy(ne, nw)) / (4 * S_n) +
          (dy(w, Sw) * dy(sw, se)) / (4 * S_s)) / S_P

    # South neighbor
    D1 = ((dx(sw, se) * (dx(Se, e) / 4 + dx(w, Sw) / 4 + dx(Sw, Se))) / S_s +
        (dy(sw, se) * (dy(Se, e) / 4 + dy(w, Sw) / 4 + dy(Sw, Se))) / S_s +
        (dx(s, sE) * dx(se, ne)) / (4 * S_e) +
        (dx(sW, s) * dx(nw, sw)) / (4 * S_w) +
        (dy(s, sE) * dy(se, ne)) / (4 * S_e) +
        (dy(sW, s) * dy(nw, sw)) / (4 * S_w)) / S_P

    # North neighbor
    D_1 = ((dx(ne, nw) * (dx(e, Ne) / 4 + dx(Nw, w) / 4 + dx(Ne, Nw))) / S_n +
        (dy(ne, nw) * (dy(e, Ne) / 4 + dy(Nw, w) / 4 + dy(Ne, Nw))) / S_n +
        (dx(nE, n) * dx(se, ne)) / (4 * S_e) +
        (dx(n, nW) * dx(nw, sw)) / (4 * S_w) +
        (dy(nE, n) * dy(se, ne)) / (4 * S_e) +
        (dy(n, nW) * dy(nw, sw)) / (4 * S_w)) / S_P

    # NW diagonal
    D_4 = ((dx(Nw, w) * dx(ne, nw)) / (4 * S_n) +
        (dx(n, nW) * dx(nw, sw)) / (4 * S_w) +
        (dy(Nw, w) * dy(ne, nw)) / (4 * S_n) +
        (dy(n, nW) * dy(nw, sw)) / (4 * S_w)) / S_P

    # NE diagonal
    D2 = ((dx(nE, n) * dx(se, ne)) / (4 * S_e) +
        (dx(e, Ne) * dx(ne, nw)) / (4 * S_n) +
        (dy(nE, n) * dy(se, ne)) / (4 * S_e) +
        (dy(e, Ne) * dy(ne, nw)) / (4 * S_n)) / S_P

    # SW diagonal
    D_2 = ((dx(w, Sw) * dx(sw, se)) / (4 * S_s) +
        (dx(sW, s) * dx(nw, sw)) / (4 * S_w) +
        (dy(w, Sw) * dy(sw, se)) / (4 * S_s) +
        (dy(sW, s) * dy(nw, sw)) / (4 * S_w)) / S_P

    # SE diagonal
    D4 = ((dx(s, sE) * dx(se, ne)) / (4 * S_e) +
        (dx(Se, e) * dx(sw, se)) / (4 * S_s) +
        (dy(s, sE) * dy(se, ne)) / (4 * S_e) +
        (dy(Se, e) * dy(sw, se)) / (4 * S_s)) / S_P

    # Center (P)
```

```
        D0 = ((dx(se, ne) * (dx(n, s) + dx(nE, n) / 4 + dx(s, sE) / 4)) / S_e +
            (dx(ne, nw) * (dx(w, e) + dx(e, Ne) / 4 + dx(Nw, w) / 4)) / S_n +
            (dx(sw, se) * (dx(e, w) + dx(Se, e) / 4 + dx(w, Sw) / 4)) / S_s +
            (dx(nw, sw) * (dx(s, n) + dx(n, nW) / 4 + dx(sW, s) / 4)) / S_w +
            (dy(se, ne) * (dy(n, s) + dy(nE, n) / 4 + dy(s, sE) / 4)) / S_e +
            (dy(ne, nw) * (dy(w, e) + dy(e, Ne) / 4 + dy(Nw, w) / 4)) / S_n +
            (dy(sw, se) * (dy(e, w) + dy(Se, e) / 4 + dy(w, Sw) / 4)) / S_s +
            (dy(nw, sw) * (dy(s, n) + dy(n, nW) / 4 + dy(sW, s) / 4)) / S_w) / S_P

        # Assemble stencil
        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j-1)] = D_3
        stencil[self.index(i, j+1)] = D3
        stencil[self.index(i-1, j-1)] = D_4
        stencil[self.index(i-1, j+1)] = D2
        stencil[self.index(i+1, j-1)] = D_2
        stencil[self.index(i+1, j+1)] = D4

        return stencil, b[0]


    def build_north(self, i, j):
        """Build stencil for north boundary (i=0)"""
        stencil = np.zeros(self.n * self.m)
        b = np.zeros(1)

        # Dirichlet: impose temperature directly
        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        # Neumann or Robin: apply flux balance
        # Get coordinates (no north neighbor available)
        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

        # Auxiliary nodes
        Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)

        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)
        sw = Coordinate2D((Sw.x + w.x)/2, (Sw.y + w.y)/2)

        # Control volume areas
        S_ss = calculate_area(e, se, sw, w)
        S_s = calculate_area(e, Se, Sw, w)
        S_ssw = calculate_area(P, s, sW, W)
        S_sse = calculate_area(E, sE, s, P)

        # Neighbor coefficients
        D3 = (dy(sw, se) * (dy(Se, e) / 4) / S_s + dx(sw, se) * (dx(Se, e) / 4) / S_s +
            dy(se, e) * (dy(s, sE) / 4 + 3 * dy(sE, E) / 4 + dy(E, P) / 2) / S_sse +
            dx(se, e) * (dx(s, sE) / 4 + 3 * dx(sE, E) / 4 + dx(E, P) / 2) / S_sse) / S_ss

        D_3 = (dy(w, sw) * (3 * dy(W, sW) / 4 + dy(sW, s) / 4 + dy(P, W) / 2) / S_ssw +
            dx(w, sw) * (3 * dx(W, sW) / 4 + dx(sW, s) / 4 + dx(P, W) / 2) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4) / S_s + dx(sw, se) * (dx(w, Sw) / 4) / S_s) / S_ss

        D1 = (dy(w, sw) * (dy(sW, s) / 4 + dy(s, P) / 4) / S_ssw +
            dx(w, sw) * (dx(sW, s) / 4 + dx(s, P) / 4) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4 + dy(Sw, Se) + dy(Se, e) / 4) / S_s +
            dx(sw, se) * (dx(w, Sw) / 4 + dx(Sw, Se) + dx(Se, e) / 4) / S_s +
            dy(se, e) * (dy(P, s) / 4 + dy(s, sE) / 4) / S_sse +
            dx(se, e) * (dx(P, s) / 4 + dx(s, sE) / 4) / S_sse) / S_ss

        D_2 = (dy(w, sw) * (dy(W, sW) / 4 + dy(sW, s) / 4) / S_ssw +
            dx(w, sw) * (dx(W, sW) / 4 + dx(sW, s) / 4) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4) / S_s + dx(sw, se) * (dx(w, Sw) / 4) / S_s) / S_ss

        D4 = (dy(sw, se) * (dy(Se, e) / 4) / S_s + dx(sw, se) * (dx(Se, e) / 4) / S_s +
            dy(se, e) * (dy(s, sE) / 4 + dy(sE, E) / 4) / S_sse +
            dx(se, e) * (dx(s, sE) / 4 + dx(sE, E) / 4) / S_sse) / S_ss

        # Boundary condition coefficient and source term
        coefficient = 0.0
        if self.boundary[0] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(e, w) / S_ss
        elif self.boundary[0] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(e, w) / S_ss
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[0]}')

        # Center coefficient
        D0 = (coefficient * dist(e, w) +
            dy(w, sw) * (dy(sW, s) / 4 + 3 * dy(s, P) / 4 + dy(P, W) / 2) / S_ssw +
            dx(w, sw) * (dx(sW, s) / 4 + 3 * dx(s, P) / 4 + dx(P, W) / 2) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4 + dy(Se, e) / 4 + dy(e, w)) / S_s +
            dx(sw, se) * (dx(w, Sw) / 4 + dx(Se, e) / 4 + dx(e, w)) / S_s +
            dy(se, e) * (3 * dy(P, s) / 4 + dy(s, sE) / 4 + dy(E, P) / 2) / S_sse +
            dx(se, e) * (3 * dx(P, s) / 4 + dx(s, sE) / 4 + dx(E, P) / 2) / S_sse) / S_ss

        stencil[self.index(i, j)] = D0
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j-1)] = D_3
        stencil[self.index(i, j+1)] = D3
        stencil[self.index(i+1, j-1)] = D_2
        stencil[self.index(i+1, j+1)] = D4
```

```
        return stencil, b[0]

    def build_south(self, i, j):
        """Build stencil for south boundary (i=m-1)"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        # Get coordinates (no south neighbor)
        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

        Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
        nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)

        n  = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        w  = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        e  = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
        nw = Coordinate2D((Nw.x + w.x)/2, (Nw.y + w.y)/2)

        S_nn  = self.stable_area(e, ne, nw, w)
        S_n   = self.stable_area(e, Ne, Nw, w)
        S_nnw = self.stable_area(P, n, nW, W)
        S_nne = self.stable_area(E, nE, n, P)

        # Neighbor coefficients

        D3 = (dy(nw, ne) * dy(Ne, e) / 4 / S_n + dx(nw, ne) * dx(Ne, e) / 4 / S_n +
            dy(ne, e) * (dy(n, nE)/4 + 3*dy(nE, E)/4 + dy(E, P)/2) / S_nne +
            dx(ne, e) * (dx(n, nE)/4 + 3*dx(nE, E)/4 + dx(E, P)/2) / S_nne) / S_nn

        D_3 = (dy(w, nw) * (3*dy(W, nW)/4 + dy(nW, n)/4 + dy(P, W)/2) / S_nnw +
            dx(w, nw) * (3*dx(W, nW)/4 + dx(nW, n)/4 + dx(P, W)/2) / S_nnw +
            dy(nw, ne) * dy(w, Nw)/4 / S_n + dx(nw, ne) * dx(w, Nw)/4 / S_n) / S_nn

        D_1 = (dy(w, nw) * (dy(nW, n)/4 + dy(n, P)/4) / S_nnw +
            dx(w, nw) * (dx(nW, n)/4 + dx(n, P)/4) / S_nnw +
            dy(nw, ne) * (dy(w, Nw)/4 + dy(Nw, Ne) + dy(Ne, e)/4) / S_n +
            dx(nw, ne) * (dx(w, Nw)/4 + dx(Nw, Ne) + dx(Ne, e)/4) / S_n +
            dy(ne, e) * (dy(P, n)/4 + dy(n, nE)/4) / S_nne +
            dx(ne, e) * (dx(P, n)/4 + dx(n, nE)/4) / S_nne) / S_nn

        D4 = (dy(w, nw) * (dy(W, nW)/4 + dy(nW, n)/4) / S_nnw +
            dx(w, nw) * (dx(W, nW)/4 + dx(nW, n)/4) / S_nnw +
            dy(nw, ne) * dy(w, Nw)/4 / S_n + dx(nw, ne) * dx(w, Nw)/4 / S_n) / S_nn

        D2 = (dy(nw, ne) * dy(Ne, e)/4 / S_n + dx(nw, ne) * dx(Ne, e)/4 / S_n +
            dy(ne, e) * (dy(n, nE)/4 + dy(nE, E)/4) / S_nne +
            dx(ne, e) * (dx(n, nE)/4 + dx(nE, E)/4) / S_nne) / S_nn

        if self.boundary[1] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(e, w) / S_nn
        elif self.boundary[1] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(e, w) / S_nn
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[1]}')

        D0 = (coefficient * dist(e, w) +
            dy(w, nw) * (dy(nW, n)/4 + 3*dy(n, P)/4 + dy(P, W)/2) / S_nnw +
            dx(w, nw) * (dx(nW, n)/4 + 3*dx(n, P)/4 + dx(P, W)/2) / S_nnw +
            dy(nw, ne) * (dy(w, Nw)/4 + dy(Ne, e)/4 + dy(e, w)) / S_n +
            dx(nw, ne) * (dx(w, Nw)/4 + dx(Ne, e)/4 + dx(e, w)) / S_n +
            dy(ne, e) * (3*dy(P, n)/4 + dy(n, nE)/4 + dy(E, P)/2) / S_nne +
            dx(ne, e) * (3*dx(P, n)/4 + dx(n, nE)/4 + dx(E, P)/2) / S_nne) / S_nn

        stencil[self.index(i, j)]     = D0
        stencil[self.index(i-1, j)]   = D_1
        stencil[self.index(i, j-1)]   = D_3
        stencil[self.index(i, j+1)]   = D3
        stencil[self.index(i-1, j-1)] = D4
        stencil[self.index(i-1, j+1)] = D2

        return stencil, b[0]

    def build_west(self, i, j):
        """Build stencil for west boundary"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

        Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)
        nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)

        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
```

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        n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
        se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)

        S_ww = self.stable_area(s, n, ne, se)
        S_w = self.stable_area(s, n, nE, sE)
        S_wws = self.stable_area(S, P, e, Se)
        S_wwn = self.stable_area(P, N, Ne, e)

        D_1 = -(dy(se, ne) * (dy(nE, n) / 4) / S_w + dx(se, ne) * (dx(nE, n) / 4) / S_w +
                dy(ne, n) * (dy(e, Ne) / 4 + 3 * dy(Ne, N) / 4 + dy(N, P) / 2) / S_wwn +
                dx(ne, n) * (dx(e, Ne) / 4 + 3 * dx(Ne, N) / 4 + dx(N, P) / 2) / S_wwn) / S_ww

        D1 = -(dy(se, ne) * (dy(s, sE) / 4) / S_w + dx(se, ne) * (dx(s, sE) / 4) / S_w +
                dy(se, s) * (dy(sE, S) / 4 + 3 * dy(Se, S) / 4 + dy(S, P) / 2) / S_wws +
                dx(se, s) * (dx(sE, S) / 4 + 3 * dx(Se, S) / 4 + dx(S, P) / 2) / S_wws) / S_ww

        D3 = -(dy(s, se) * (dy(Se, e) / 4 + dy(e, P) / 4) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + dx(e, P) / 4) / S_wws +
                dy(se, ne) * (dy(s, sE) / 4 + dy(sE, nE) + dy(nE, n) / 4) / S_w +
                dx(se, ne) * (dx(s, sE) / 4 + dx(sE, nE) + dx(nE, n) / 4) / S_w +
                dy(ne, n) * (dy(P, e) / 4 + dy(e, Ne) / 4) / S_wwn +
                dx(ne, n) * (dx(P, e) / 4 + dx(e, Ne) / 4) / S_wwn) / S_ww

        D2 = -(dy(ne, n) * (dy(e, Ne) / 4 + dy(Ne, N) / 4) / S_wwn +
                dx(ne, n) * (dx(e, Ne) / 4 + dx(Ne, N) / 4) / S_wwn +
                dy(se, ne) * (dy(nE, n) / 4) / S_w + dx(se, ne) * (dx(nE, n) / 4) / S_w) / S_ww

        D4 = -(dy(se, ne) * (dy(s, sE) / 4) / S_w + dx(se, ne) * (dx(Se, s) / 4) / S_w +
                dy(s, se) * (dy(Se, e) / 4 + dy(Se, S) / 4) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + dx(Se, S) / 4) / S_wws) / S_ww

        if self.boundary[2] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(n, s) / S_ww
        elif self.boundary[2] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(n, s) / S_ww
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[2]}')

        D0 = (coefficient * dist(ne, se) +
                dy(s, se) * (dy(Se, e) / 4 + 3 * dy(e, P) / 4 + dy(P, S) / 2) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + 3 * dx(e, P) / 4 + dx(P, S) / 2) / S_wws +
                dy(se, ne) * (dy(s, sE) / 4 + dy(nE, n) / 4 + dy(n, s)) / S_w +
                dx(se, ne) * (dx(s, sE) / 4 + dx(nE, n) / 4 + dx(n, s)) / S_w +
                dy(ne, n) * (3 * dy(P, e) / 4 + dy(e, Ne) / 4 + dy(N, P) / 2) / S_wwn +
                dx(ne, n) * (3 * dx(P, e) / 4 + dx(e, Ne) / 4 + dx(N, P) / 2) / S_wwn) / S_ww

        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j+1)] = D3
        stencil[self.index(i-1, j+1)] = D2
        stencil[self.index(i+1, j+1)] = D4

        return stencil, b[0]

    def build_east(self, i, j):
        """Build stencil for east boundary"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])

        Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)

        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)

        nw = Coordinate2D((nW.x + n.x)/2, (nW.y + n.y)/2)
        sw = Coordinate2D((sW.x + s.x)/2, (sW.y + s.y)/2)

        S_ee = self.stable_area(s, sw, nw, n)
        S_e = self.stable_area(n, s, sW, nW)
        S_ees = self.stable_area(P, S, Sw, w)
        S_een = self.stable_area(N, P, w, Nw)

        D_1 = (dy(nw, sw) * (dy(n, nW) / 4) / S_e + dx(nw, sw) * (dx(n, nW) / 4) / S_e +
                dy(n, nw) * (dy(Nw, w) / 4 + 3 * dy(N, Nw) / 4 + dy(P, N) / 2) / S_een +
                dx(n, nw) * (dx(Nw, w) / 4 + 3 * dx(N, Nw) / 4 + dx(P, N) / 2) / S_een) / S_ee

        D1 = (dy(nw, sw) * (dy(sW, s) / 4) / S_e + dx(nw, sw) * (dx(sW, s) / 4) / S_e +
                dy(sw, s) * (dy(w, Sw) / 4 + 3 * dy(Sw, S) / 4 + dy(S, P) / 2) / S_ees +
                dx(sw, s) * (dx(w, Sw) / 4 + 3 * dx(Sw, S) / 4 + dx(S, P) / 2) / S_ees) / S_ee

        D_3 = (dy(sw, s) * (dy(w, Sw) / 4 + dy(P, w) / 4) / S_ees +
                dx(sw, s) * (dx(w, Sw) / 4 + dx(P, w) / 4) / S_ees +
                dy(nw, sw) * (dy(sW, s) / 4 + dy(nW, sW) + dy(n, nW) / 4) / S_e +
                dx(nw, sw) * (dx(sW, s) / 4 + dx(nW, sW) + dx(n, nW) / 4) / S_e +
                dy(n, nw) * (dy(w, P) / 4 + dy(Nw, w) / 4) / S_een +
                dx(n, nw) * (dx(w, P) / 4 + dx(Nw, w) / 4) / S_een) / S_ee

        D_4 = (dy(n, nw) * (dy(N, Nw) / 4 + dy(Nw, w) / 4) / S_een +
                dx(n, nw) * (dx(N, Nw) / 4 + dx(Nw, w) / 4) / S_een +
                dy(nw, sw) * (dy(n, nW) / 4) / S_e + dx(nw, sw) * (dx(n, nW) / 4) / S_e) / S_ee
```

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        D_2 = (dy(nw, sw) * (dy(sW, s) / 4) / S_e + dx(nw, sw) * (dx(sW, s) / 4) / S_e +
                dy(sw, s) * (dy(Sw, S) / 4 + dy(w, Sw) / 4) / S_ees +
                dx(sw, s) * (dx(Sw, S) / 4 + dx(w, Sw) / 4) / S_ees) / S_ee

        if self.boundary[3] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(n, s) / S_ee
        elif self.boundary[3] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(n, s) / S_ee
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[3]}')

        D0 = (coefficient * dist(nw, sw) +
                dy(sw, s) * (dy(w, Sw) / 4 + 3 * dy(P, w) / 4 + dy(S, P) / 2) / S_ees +
                dx(sw, s) * (dx(w, Sw) / 4 + 3 * dx(P, w) / 4 + dx(S, P) / 2) / S_ees +
                dy(nw, sw) * (dy(sW, s) / 4 + dy(n, nW) / 4 + dy(s, n)) / S_e +
                dx(nw, sw) * (dx(sW, s) / 4 + dx(n, nW) / 4 + dx(s, n)) / S_e +
                dy(n, nw) * (3 * dy(w, P) / 4 + dy(Nw, w) / 4 + dy(P, N) / 2) / S_een +
                dx(n, nw) * (3 * dx(w, P) / 4 + dx(Nw, w) / 4 + dx(P, N) / 2) / S_een) / S_ee

        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j-1)] = D_3
        stencil[self.index(i-1, j-1)] = D_4
        stencil[self.index(i+1, j-1)] = D_2

        return stencil, b[0]

    def build_NW(self, i, j):
        """Build stencil for North-West corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        S  = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

        s   = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        e   = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)
        Se  = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        sE  = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)
        se  = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)

        S_nw  = self.stable_area(P, s, se, e)
        S_nws = self.stable_area(P, S, Se, e)
        S_nwe = self.stable_area(P, s, sE, E)

        coeff_N = coeff_W = 0.0
        b[0] = 0.0

        if self.boundary[0] == 'N':
            b[0] += self.q * dist(e, P) / S_nw
        elif self.boundary[0] == 'R':
            coeff_N = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(e, P) / S_nw

        if self.boundary[2] == 'N':
            b[0] += self.q * dist(P, s) / S_nw
        elif self.boundary[2] == 'R':
            coeff_W = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, s) / S_nw

        D0 = (
            coeff_N * dist(e, P) +
            coeff_W * dist(s, P) +
            dy(s, se) * (dy(Se, e) / 4 + 3 * dy(e, P) / 4 + dy(P, S) / 2) / S_nws +
            dx(s, se) * (dx(Se, e) / 4 + 3 * dx(e, P) / 4 + dx(P, S) / 2) / S_nws +
            dy(se, e) * (3 * dy(s, P)/4 + dy(sE,s)/4 + dy(P, E)/2) / S_nwe +
            dx(se, e) * (3 * dx(s, P)/4 + dx(sE,s)/4 + dx(P, E)/2) / S_nwe
        ) / S_nw

        D1 = (
            dy(s, se) * (dy(e, Se) / 4 + 3 * dy(Se, S) / 4 + dy(S, P) / 2) / S_nws +
            dx(s, se) * (dx(e, Se) / 4 + 3 * dx(Se, S) / 4 + dx(S, P) / 2) / S_nws +
            dy(se, e) * (dy(s, sE)/4 + dy(sE, E)/4) / S_nwe +
            dx(se, e) * (dx(s, sE)/4 + dx(sE, E)/4) / S_nwe
        ) / S_nw

        D3 = (
            dy(se, e) * (dy(s, sE) / 4 + 3 * dy(sE, E) / 4 + dy(E, P) / 2) / S_nwe +
            dx(se, e) * (dx(s, sE) / 4 + 3 * dx(sE, E) / 4 + dx(E, P) / 2) / S_nwe +
            dy(s, se) * (dy(e, Se)) /4 / S_nws +
            dx(s, se) * (dx(e, Se)) /4 / S_nws
        ) / S_nw

        D4 = (
            dy(s, se) * (dy(Se, S) / 4 + dy(e, Se) / 4) / S_nws +
            dx(s, se) * (dx(Se, S) / 4 + dx(e, Se) / 4) / S_nws +
            dy(se, e) * (dy(s, sE) / 4 + dy(sE, E) / 4) / S_nwe +
            dx(se, e) * (dx(s, sE) / 4 + dx(sE, E) / 4) / S_nwe
        ) / S_nw

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i+1, j)]     = D1
        stencil[self.index(i, j+1)]     = D3
        stencil[self.index(i+1, j+1)]   = D4

        return stencil, b[0]
```

```python
    def build_NE(self, i, j):
        """Build stencil for North-East corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        S  = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])

        s   = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w   = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        Sw  = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        sW  = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        sw  = Coordinate2D((sW.x + s.x)/2, (sW.y + s.y)/2)

        S_ne  = self.stable_area(w, sw, s, P)
        S_nes = self.stable_area(P, w, Sw, S)
        S_new = self.stable_area(P, W, sW, s)

        coeff_N = coeff_E = 0.0
        b[0] = 0.0

        if self.boundary[0] == 'N':
            b[0] += self.q * dist(w, P) / S_ne
        elif self.boundary[0] == 'R':
            coeff_N = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(w, P) / S_ne
        if self.boundary[3] == 'N':
            b[0] += self.q * dist(P, s) / S_ne
        elif self.boundary[3] == 'R':
            coeff_E = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, s) / S_ne
        D0 = (
            coeff_N * dist(w, P) +
            coeff_E * dist(P, s) +
            dy(sw, s) * (3 * dy(P, w) / 4 + dy(w, Sw) / 4 +  dy(S, P) / 2) / S_nes +
            dx(sw, s) * (3 * dx(P, w) / 4 + dx(w, Sw) / 4  + dx(S, P) / 2) / S_nes +
            dy(w, sw) * (3 * dy(s, P) / 4 + dy(sW, s) / 4  + dy(P, W) / 2) / S_new +
            dx(w, sw) * (3 * dx(s, P) / 4 + dx(sW, s) / 4 +  dx(P, W) / 2) / S_new
        ) / S_ne

        D1 = (
            dy(sw, s) * (3 * dy(Sw, S) / 4 + dy(w, Sw) / 4  + dy(S, P) / 2) / S_nes +
            dx(sw, s) * (3 * dx(Sw, S) / 4 + dx(w, Sw) / 4 +  dx(S, P) / 2) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + dy(s, P) / 4) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + dx(s, P) / 4) / S_new
        ) / S_ne

        D_3 = (
            dy(sw, s) * (dy(w, Sw) / 4 + dy(P, w) / 4) / S_nes +
            dx(sw, s) * (dx(w, Sw) / 4 + dx(P, w) / 4) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + 3 * dy(W, sW) / 4 + dy(P, W) / 2) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + 3 * dx(W, sW) / 4 + dx(P, W) / 2) / S_new
        ) / S_ne

        D_2 = (
            dy(sw, s) * (dy(Sw, S) / 4 + dy(w, Sw) / 4) / S_nes +
            dx(sw, s) * (dx(Sw, S) / 4 + dx(w, Sw) / 4) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + dy(W, sW) / 4) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + dx(W, sW) / 4) / S_new
        ) / S_ne

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i+1, j)]     = D1
        stencil[self.index(i, j-1)]     = D_3
        stencil[self.index(i+1, j-1)]   = D_2

        return stencil, b[0]

    def build_SW(self, i, j):
        """Build stencil for South-West corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

        n   = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        e   = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)
        Ne  = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        nE  = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)
        ne  = Coordinate2D((nE.x + n.x)/2, (nE.y + n.y)/2)

        S_sw  = self.stable_area(e, ne, n, P)
        S_swn = self.stable_area(P, e, Ne, N)
        S_swe = self.stable_area(P, E, nE, n)

        coeff_S = coeff_W = 0.0
```

```
        b[0] = 0.0

        if self.boundary[1] == 'N':
            b[0] += self.q * dist(e, P) / S_sw
        elif self.boundary[1] == 'R':
            coeff_S = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(e, P) / S_sw

        if self.boundary[2] == 'N':
            b[0] += self.q * dist(P, n) / S_sw
        elif self.boundary[2] == 'R':
            coeff_W = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, n) / S_sw
        D0 = (
            coeff_S * dist(e, P) +
            coeff_W * dist(P, n) +
            dy(n, ne) * (dy(Ne, e) / 4 + 3 * dy(e, P) / 4 + dy(P, N) / 2) / S_swn +
            dx(n, ne) * (dx(Ne, e) / 4 + 3 * dx(e, P) / 4 + dx(P, N) / 2) / S_swn +
            dy(ne, e) * (dy(nE, n) / 4 + 3 * dy(n, P) / 4 + dy(P, E) / 2) / S_swe +
            dx(ne, e) * (dx(nE, n) / 4 + 3 * dx(n, P) / 4 + dx(P, E) / 2) / S_swe
        ) / S_sw

        D_1 = ( dy(n, ne) * (dy(e, Ne) / 4 + 3 * dy(Ne, N) / 4 + dy(N, P) / 2) / S_swn +
                dx(n, ne) * (dx(e, Ne) / 4 + 3 * dx(Ne, N) / 4 + dx(N, P) / 2) / S_swn +
                dy(ne, e) * (dy(n, nE)/4 ) / S_swe +
                dx(ne, e) * (dx(n, nE)/4  / S_swe )
        )/ S_sw

        D3 = ( dy(n, ne) * (dy(e, Ne) / 4) / S_swn +
                dx(n, ne) * (dx(e, Ne) / 4) / S_swn +
                dy(ne, e) * (dy(n, nE) / 4 + 3 * dy(nE, E) / 4 +  dy(E, P) / 2) / S_swe +
                dx(ne, e) * (dx(n, nE) / 4 + 3 * dx(nE, E) / 4 + dx(E, P) / 2) / S_swe
            ) / S_sw

        D2 = (  dy(n, ne) * (dy(Ne, N) / 4 + dy(e, Ne) / 4) / S_swn +
                dx(n, ne) * (dx(Ne, N) / 4 + dx(e, Ne) / 4) / S_swn +
                dy(ne, e) * (dy(n, nE) / 4 + dy(nE, E) / 4) / S_swe +
                dx(ne, e) * (dx(n, nE) / 4 + dx(nE, E) / 4) / S_swe
            ) / S_sw

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i-1, j)]     = D_1
        stencil[self.index(i, j+1)]     = D3
        stencil[self.index(i-1, j+1)]   = D2

        return stencil, b[0]

    def build_SE(self, i, j):
        """Build stencil for South-East corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])

        n   = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        w   = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        Nw  = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        nW  = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
        nw  = Coordinate2D((nW.x + n.x)/2, (nW.y + n.y)/2)

        S_se  = self.stable_area(P, n, nw, w)
        S_sen = self.stable_area(P, N, Nw, w)
        S_sew = self.stable_area(P, n, nW, W)

        coeff_S = coeff_E = 0.0
        b[0] = 0.0


        if self.boundary[1] == 'N':
            b[0] += self.q * dist(w, P) / S_se
        elif self.boundary[1] == 'R':
            coeff_S = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(w, P) / S_se

        if self.boundary[3] == 'N':
            b[0] += self.q * dist(P, n) / S_se
        elif self.boundary[3] == 'R':
            coeff_E = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, n) / S_se

        D0 = (
            coeff_S * dist(w, P) +
            coeff_E * dist(P, n) +
            dy(nw, w) * (dy(W, P) / 2 + dy(n, nW) / 4 + 3 * dy(P, n) / 4) / S_sew +
            dx(nw, w) * (dx(W, P) / 2 + dx(n, nW) / 4 + 3 * dx(P, n) / 4) / S_sew +
            dy(n, nw) * (3 * dy(w, P) / 4 + dy(Nw, w) / 4 + dy(P, N) / 2) / S_sen +
            dx(n, nw) * (3 * dx(w, P) / 4 + dx(Nw, w) / 4 + dx(P, N) / 2) / S_sen
        ) / S_se

        D_1 = (
            dy(nw, w) * (dy(n, nW) / 4 + dy(P, n) / 4) / S_sew +
            dx(nw, w) * (dx(n, nW) / 4 + dx(P, n) / 4) / S_sew +
            dy(n, nw) * (dy(Nw, w) / 4 + 3 * dy(N, Nw) / 4 + dy(P, N) / 2) / S_sen +
            dx(n, nw) * (dx(Nw, w) / 4 + 3 * dx(N, Nw) / 4 + dx(P, N) / 2) / S_sen
        ) / S_se

        D_3 = (
            dy(nw, w) * (dy(W, P) / 2 + 3 * dy(nW, W) / 4 + dy(n, nW) / 4) / S_sew +
            dx(nw, w) * (dx(W, P) / 2 + 3 * dx(nW, W) / 4 + dx(n, nW) / 4) / S_sew +
```

```python
            dy(n, nw) * (dy(w, P) / 4 + dy(Nw, w) / 4) / S_sen +
            dx(n, nw) * (dx(w, P) / 4 + dx(Nw, w) / 4) / S_sen
        ) / S_se

        D_4 = (
            dy(n, nw) * (dy(N, Nw) / 4 + dy(Nw, w) / 4) / S_sen +
            dx(n, nw) * (dx(N, Nw) / 4 + dx(Nw, w) / 4) / S_sen +
            dy(nw, w) * (dy(n, nW) / 4 + dy(nW, W) / 4) / S_sew +
            dx(nw, w) * (dx(n, nW) / 4 + dx(nW, W) / 4) / S_sew
        ) / S_se

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i-1, j)]     = D_1
        stencil[self.index(i, j-1)]     = D_3
        stencil[self.index(i-1, j-1)]   = D_4

        return stencil, b[0]

    def solve(self):
        """Solve the linear system A*T = B """
        print("Assembling linear system...")

        # Assemble the linear system
        for i in range(self.m):
            for j in range(self.n):
                idx = self.index(i, j)
                stencil, b_val = self.set_stencil(i, j)

                # Set matrix coefficients
                for k, coeff in enumerate(stencil):
                    if abs(coeff) > 1e-15:
                        self.A[idx, k] = coeff

                # Set source term
                self.B[idx] = b_val

        print(f"Matrix size: {self.A.shape}, Non-zero elements: {self.A.nnz}")

        # Solve linear system with regularization
        A_csr = self.A.tocsr() + sp.eye(self.A.shape[0]) * 1e-12

        print("Solving linear system...")
        try:
            T_flat = spsolve(A_csr, self.B)
            T = T_flat.reshape(self.m, self.n)
            print("Solution completed successfully.")
            return T
        except Exception as e:
            print(f"Error solving linear system: {e}")
            return np.zeros((self.m, self.n))

    def plot_Result(self, T, plot_type="2D"):
      # Create 2D and 3D plots
     X, Y = self.X, self.Y
     T_plot = np.array(T).reshape(X.shape)

     # Use actual temperature range
     vmin = np.min(T_plot)
     vmax = np.max(T_plot)

     if plot_type == "2D":
         plt.figure(figsize=(8, 6))

         # Create symmetric mesh by combining original and flipped
         X_combined = np.vstack([X, X])
         Y_combined = np.vstack([Y, -Y])
         T_combined = np.vstack([T_plot, T_plot])

         pcm = plt.pcolormesh(X_combined, Y_combined, T_combined,
                             shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
         plt.colorbar(pcm, label="Temperature")
         plt.xlabel("X-axis")
         plt.ylabel("Y-axis")
         plt.title("Temperature Distribution")
         plt.axis('equal')
         plt.tight_layout()
         plt.show()

     elif plot_type == "3D":
         fig = plt.figure(figsize=(10, 7))
         ax = fig.add_subplot(111, projection='3d')

         # Plot original surface
         surf1 = ax.plot_surface(X, Y, T_plot, cmap='jet',
                                edgecolor='none', antialiased=True,
                                vmin=vmin, vmax=vmax)

         # Plot symmetric surface (y-flipped)
         surf2 = ax.plot_surface(X, -Y, T_plot, cmap='jet',
                                edgecolor='none', antialiased=True,
                                vmin=vmin, vmax=vmax)

         ax.set_xlabel("X-axis")
         ax.set_ylabel("Y-axis")
         ax.set_zlabel("Temperature")
         ax.set_title("3D Temperature Distribution")
         ax.view_init(elev=45, azim=300)
         plt.colorbar(surf1, label="Temperature")
         plt.tight_layout()
         plt.show()

     elif plot_type == "both":
         # Create a figure with two subplots
         fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(16, 6))

         # 2D plot on left subplot
         X_combined = np.vstack([X, X])
         Y_combined = np.vstack([Y, -Y])
         T_combined = np.vstack([T_plot, T_plot])

         pcm = ax1.pcolormesh(X_combined, Y_combined, T_combined,
                             shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
```

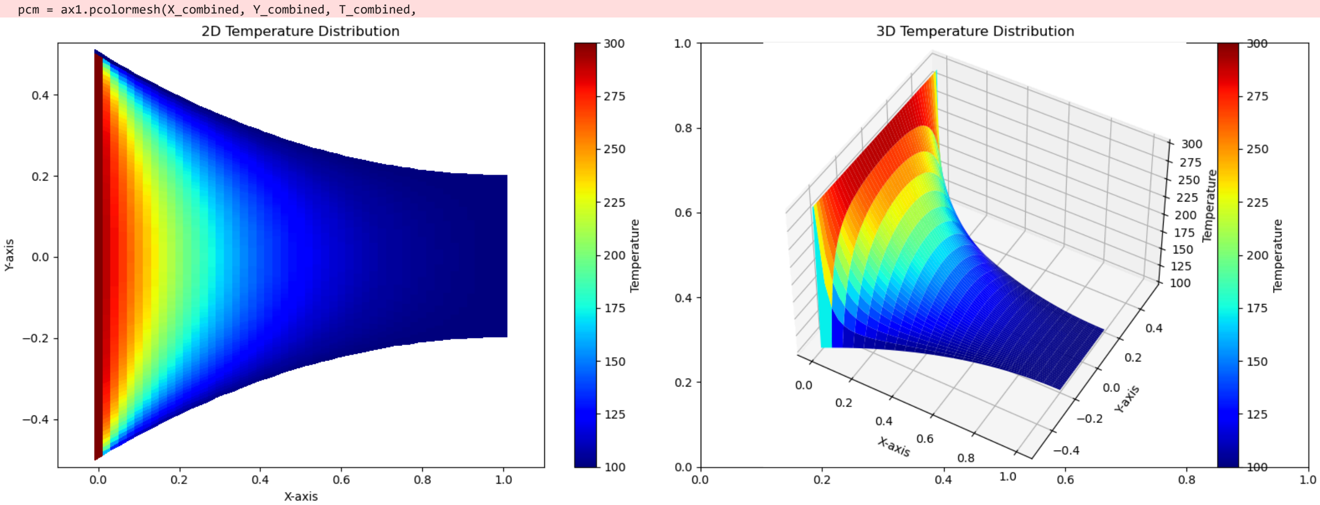
```python
        plt.colorbar(pcm, ax=ax1, label="Temperature")
        ax1.set_xlabel("X-axis")
        ax1.set_ylabel("Y-axis")
        ax1.set_title("2D Temperature Distribution")
        ax1.axis('equal')

        # 3D plot on right subplot
        ax2 = fig.add_subplot(122, projection='3d')
        surf1 = ax2.plot_surface(X, Y, T_plot, cmap='jet',
                                 edgecolor='none', antialiased=True,
                                 vmin=vmin, vmax=vmax)
        surf2 = ax2.plot_surface(X, -Y, T_plot, cmap='jet',
                                 edgecolor='none', antialiased=True,
                                 vmin=vmin, vmax=vmax)
        ax2.set_xlabel("X-axis")
        ax2.set_ylabel("Y-axis")
        ax2.set_zlabel("Temperature")
        ax2.set_title("3D Temperature Distribution")
        ax2.view_init(elev=45, azim=300)
        plt.colorbar(surf1, ax=ax2, label="Temperature")

        plt.tight_layout()
        plt.show()

    else:
       raise ValueError("plot_type must be '2D', '3D', or 'both'")
```

Stage 3 - Test Cases

In [ ]:
```python
shape =  'quadratic'    # 'rectangular', 'linear',  'quadratic', 'crazy'

l = 1
dimX = 51
dimY = 51

boundary =   ['D', 'N', 'D', 'D'] # [N,S,W,E] : D : Dirichlet, N : Neumann, R : Robin
TD =  [100 , 100 , 300 , 100] # [N,S,W,E]
alpha = 20
Tinf = 90
q = 0

X, Y = setUpMesh(dimX, dimY, l, formfunction, shape)
heat = SteadyHeat2D_FVM(X, Y, boundary, TD, q, alpha, Tinf)
T = heat.solve()
heat.plot_Result(T, "both")
```

```
Assembling linear system...
Matrix size: (2601, 2601), Non-zero elements: 22054
Solving linear system...
Solution completed successfully.
```

```
C:\Users\thomd\AppData\Local\Temp\ipykernel_3368\3703730366.py:970: UserWarning: The input coordinates to pcolormesh are interpreted as cell centers, but are not monotonically increasin
g or decreasing. This may lead to incorrectly calculated cell edges, in which case, please supply explicit cell edges to pcolormesh.
  pcm = ax1.pcolormesh(X_combined, Y_combined, T_combined,
```

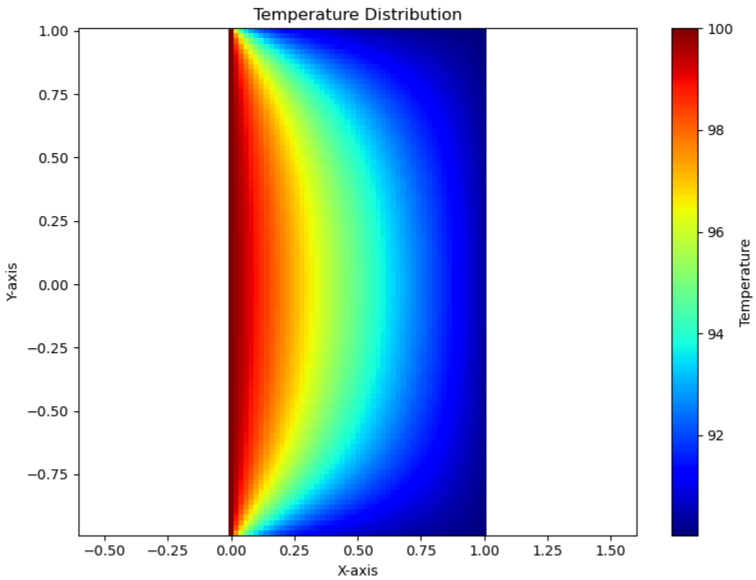


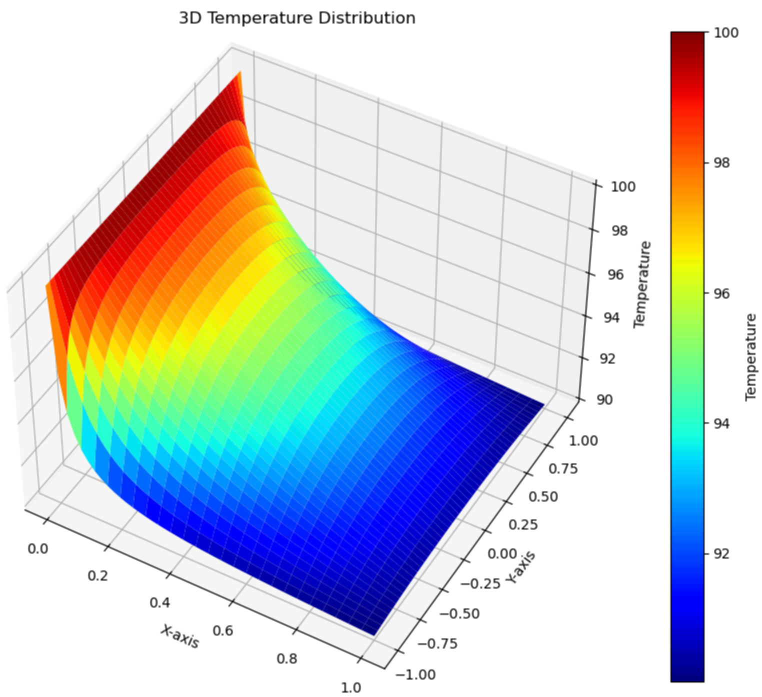

```
The Kernel crashed while executing code in the current cell or a previous cell.

Please review the code in the cell(s) to identify a possible cause of the failure.

Click <a href='https://aka.ms/vscodeJupyterKernelCrash'>here</a> for more info.

View Jupyter <a href='command:jupyter.viewOutput'>log</a> for further details.
```

In [ ]:
```python
shape =  'rectangular'    # 'rectangular', 'linear',  'quadratic', 'crazy'

l = 1
dimX = 51
dimY = 51

boundary =   ['R', 'N', 'D', 'R'] # [N,S,W,E] : D : Dirichlet, N : Neumann, R : Robin
TD =  [100 , 100 , 100 , 100] # [N,S,W,E]
alpha = 20
Tinf = 90
q = 0

X, Y = setUpMesh(dimX, dimY, l, formfunction, shape)
heat = SteadyHeat2D_FVM(X, Y, boundary, TD, q, alpha, Tinf)
T = heat.solve()
heat.plot_Result(T, "2D")
heat.plot_Result(T, "3D")
```

```
Assembling linear system...
Matrix size: (2601, 2601), Non-zero elements: 12946
Solving linear system...
Solution completed successfully.
```

```
C:\Users\thomd\AppData\Local\Temp\ipykernel_3368\3703730366.py:928: UserWarning: The input coordinates to pcolormesh are interpreted as cell centers, but are not monotonically increasin
g or decreasing. This may lead to incorrectly calculated cell edges, in which case, please supply explicit cell edges to pcolormesh.
  pcm = plt.pcolormesh(X_combined, Y_combined, T_combined,
```

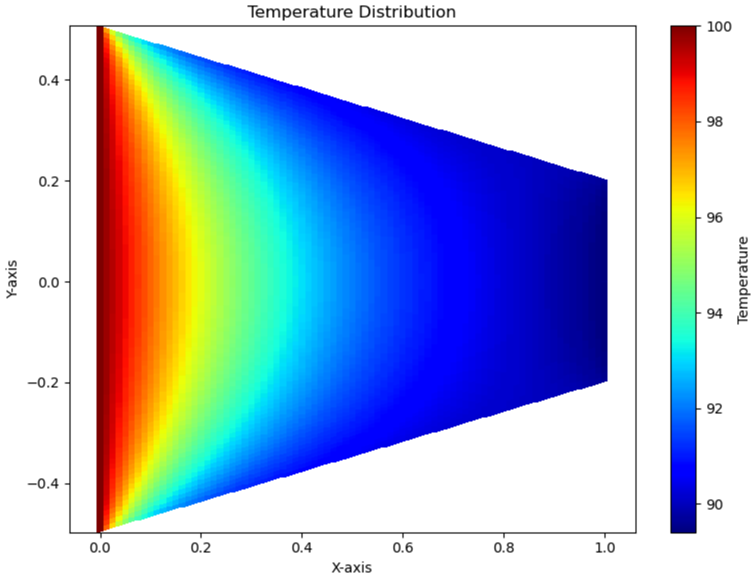


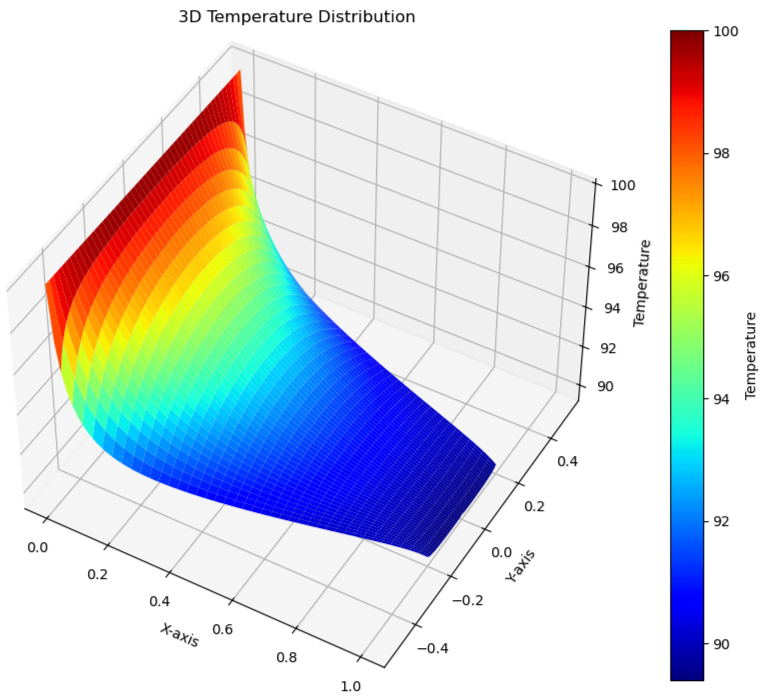


In [8]:
```python
# if you fail to implement linear, go with rectangular case!
shape =  'linear'    # 'rectangular', 'linear',  'quadratic', 'crazy'

l = 1
dimX = 81
dimY = 51

boundary =   ['R', 'N', 'D', 'R'] # [N,S,W,E] : D : Dirichlet, N : Neumann, R : Robin
TD =  [100 , 100 , 100 , 100] # [N,S,W,E]
alpha = 20
Tinf = 90
q = 0

X, Y = setUpMesh(dimX, dimY, l, formfunction, shape)
heat = SteadyHeat2D_FVM(X, Y, boundary, TD, q, alpha, Tinf)
T = heat.solve()
heat.plot_Result(T, "2D")
heat.plot_Result(T, "3D")
```

```
Assembling linear system...
Matrix size: (4131, 4131), Non-zero elements: 36140
Solving linear system...
Solution completed successfully.
```

```
C:\Users\thomd\AppData\Local\Temp\ipykernel_3368\3703730366.py:928: UserWarning: The input coordinates to pcolormesh are interpreted as cell centers, but are not monotonically increasin
g or decreasing. This may lead to incorrectly calculated cell edges, in which case, please supply explicit cell edges to pcolormesh.
  pcm = plt.pcolormesh(X_combined, Y_combined, T_combined,
```


Temperature Distribution
0.4
0.2
0.0
−0.2
−0.4
Y-axis
0.0
0.2
0.4
0.6
0.8
1.0
X-axis
100
98
96
94
92
90
Temperature


3D Temperature Distribution
100
98
96
94
92
90
Temperature
0.0
0.2
0.4
0.6
0.8
1.0
X-axis
0.4
0.2
0.0
−0.2
−0.4
Y-axis
100
98
96
94
92
90
Temperature

In [ ]: