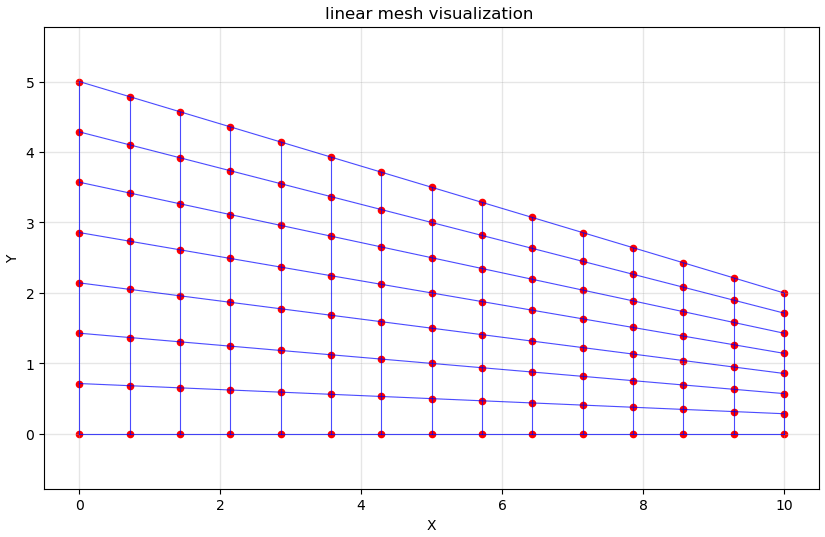
In [4]:
```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
import scipy.sparse as sp
from scipy.sparse.linalg import spsolve
```

In [7]:
```
class Coordinate2D():
    def __init__(self, x, y):
        self.x = x
        self.y = y

def calculate_area(ul, bl, br, ur):
    # calculate the area of the cell
    # ul (upper left), bl (bottom left), br (bottom right), ur (upper right) are the coordinates of the four vertices of the cell
    # apply Gaussian trapezoidal formula to calculate the areas

    x = [ul.x, bl.x, br.x, ur.x]
    y = [ul.y, bl.y, br.y, ur.y]

    area = 0.5 * abs(
        x[0]*y[1] + x[1]*y[2] + x[2]*y[3] + x[3]*y[0] - y[0]*x[1] - y[1]*x[2] - y[2]*x[3] - y[3]*x[0]
        )
    return area

def dy(a, b):
    # Calculate distance between 'a' and 'b' along the y axis
    return (b.y - a.y)

def dx(a, b):
    # Calculate distance between 'a' and 'b' along the x axis
    return (b.x - a.x)

def dist(a, b):
    # Calculate the euclidean distance between 'a' and 'b'
    return np.sqrt((b.x - a.x)**2 + (b.y - a.y)**2)
```

In [8]:
```
def formfunction(x, shape):
    """
    Defines the shape of north boundary
    takes an array and shape
    returns an array
    """
    h1 = x[-1]           # west boundary height
    h2 = x[-1] / 10 * 4  # east boundary height
    l  = x[-1]           # domain length
    if shape == 'linear':
        m = (h2 - h1) / (2 * l)
        b = h1 / 2
        return m * x + b

    elif shape == 'rectangular':
        return l * np.ones((x.size, 1))

    elif shape == 'quadratic':
        k = 2 * l**2 / (h1 - h2)
        return (x - l)**2 / k + h2 / 2

    elif shape == 'crazy':
        return h1/2 + (h2/2 - h1/2) * x / l + 0.25*(-h1 + h2/2) * np.sin(np.pi*x/l)**2

    else:
        raise ValueError('Unknown shape: %s' % shape)

def setUpMesh(nodes_x, nodes_y, length, formfunction, shape):
    # 1D x coordinates
    X = np.linspace(0, length, nodes_x)

    # Compute half-height profile along X
    y_profile = formfunction(X, shape).flatten()

    # Initialize Y mesh
    Y = np.zeros((nodes_y, nodes_x))

    for j in range(nodes_x):
        # Local half height at this x-position
        local_h = y_profile[j]

        # Linear distribution from local height to 0 along Y-direction
        Y[:, j] = np.linspace(local_h, 0, nodes_y)

    # Repeat X along rows to create 2D mesh
    X = np.array([X[:] for i in range(nodes_y)])

    return X, Y
# Parameters for mesh generation
nodes_x = 15
nodes_y = 8
length = 10

# Change this variable to plot different shapes
selected_shape = 'linear'  # Options: 'linear', 'rectangular', 'quadratic', 'crazy'

# Generate and visualize mesh for the selected shape
X, Y = setUpMesh(nodes_x, nodes_y, length, formfunction, selected_shape)

plt.figure(figsize=(10, 6))
plt.plot(X, Y, 'b-', linewidth=0.8, alpha=0.7)
plt.plot(X.T, Y.T, 'b-', linewidth=0.8, alpha=0.7)
plt.scatter(X, Y, c='red', s=20)
plt.xlabel('X')
plt.ylabel('Y')
plt.title(f'{selected_shape} mesh visualization ')
plt.grid(True, alpha=0.3)
plt.axis('equal')
plt.show()
```



Original Solver

In [10]:

```python
class SteadyHeat2D_FVM:
    """
    2D steady-state heat conduction solver using Finite Volume Method.
    Handles arbitrary quadrilateral meshes and mixed boundary conditions.
    """

    def __init__(self, X, Y, boundary=[], TD=[], q=0.0, alpha=0.0, Tinf=0.0):
        """
        Initialize FVM solver.

        Args:
            X, Y: mesh coordinate arrays (m × n)
            boundary: BC types [North, South, West, East] ('D', 'N', or 'R')
            TD: Dirichlet temperatures [North, South, West, East]
            q: heat flux for Neumann BC (W/m²)
            alpha: heat transfer coefficient for Robin BC (W/m²K)
            Tinf: ambient temperature for Robin BC (K)
        """
        self.X = X
        self.Y = Y
        self.boundary = boundary
        self.TD = TD
        self.q = q
        self.alpha = alpha
        self.Tinf = Tinf

        # Grid dimensions: m = rows (y-direction), n = columns (x-direction)
        self.m, self.n = X.shape

        # Initialize sparse linear system
        total_nodes = self.m * self.n
        self.A = sp.lil_matrix((total_nodes, total_nodes))
        self.B = np.zeros(total_nodes)

    def index(self, i, j):
        """Map 2D grid index (i,j) to 1D array index"""
        return i * self.n + j

    def stable_area(self, *args):
        """Wrapper for area calculation with stability check"""
        area = calculate_area(*args)
        return area if area > 1e-12 else 1e-12

    def set_stencil(self, i, j):
        """
        Determine node type and build appropriate stencil.
        Grid indexing: i=0 is North (top), i=m-1 is South (bottom)
                       j=0 is West (left), j=n-1 is East (right)
        """
        # Check corners first
        if i == 0 and j == 0:
            return self.build_NW(i, j)
        elif i == 0 and j == self.n-1:
            return self.build_NE(i, j)
        elif i == self.m-1 and j == 0:
            return self.build_SW(i, j)
        elif i == self.m-1 and j == self.n-1:
            return self.build_SE(i, j)

        # Check boundaries
        elif i == 0:
            return self.build_north(i, j)
        elif i == self.m-1:
            return self.build_south(i, j)
        elif j == 0:
            return self.build_west(i, j)
        elif j == self.n-1:
            return self.build_east(i, j)

        # Interior node
        else:
            return self.build_inner(i, j)

    def build_inner(self, i, j):
        """
```

```
    Build 9-point stencil for interior nodes using Green's theorem.

    Node nomenclature:
        NW   Nw    N    Ne   NE
              |         |
        nW   nw    n    ne   nE
         |         |         |
        W    w     P    e    E
         |         |         |
        sW   sw    s    se   sE
              |         |
        SW   Sw    S    Se   SE
    """
    stencil = np.zeros(self.n * self.m)
    b = np.zeros(1)

    # Principal nodes
    P = Coordinate2D(self.X[i, j], self.Y[i, j])
    N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
    S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
    W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
    E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
    NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])
    NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])
    SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
    SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

    # Auxiliary nodes (midpoints)
    Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
    Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
    Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
    Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
    nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
    nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)
    sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
    sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)

    # Face centers
    n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
    s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
    w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
    e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

    # Control volume corners
    se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)
    sw = Coordinate2D((Sw.x + w.x)/2, (Sw.y + w.y)/2)
    ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
    nw = Coordinate2D((Nw.x + w.x)/2, (Nw.y + w.y)/2)

    # Control volume areas
    S_P = calculate_area(ne, se, sw, nw)  # Main cell
    S_n = calculate_area(Ne, e, w, Nw)    # North face
    S_s = calculate_area(e, Se, Sw, w)    # South face
    S_w = calculate_area(n, s, sW, nW)    # West face
    S_e = calculate_area(nE, sE, s, n)    # East face

    # Coefficients from Green's theorem application
    # East neighbor
    D3 = ((dx(se, ne) * (dx(nE, n)/4 + dx(s, sE)/4 + dx(sE, nE))) / S_e +
         (dy(se, ne) * (dy(nE, n)/4 + dy(s, sE)/4 + dy(sE, nE))) / S_e +
         (dx(e, Ne) * dx(ne, nw)) / (4*S_n) + (dx(Se,e) * dx(sw,se)) / (4*S_s) +
         (dy(e, Ne) * dy(ne, nw)) / (4*S_n) + (dy(Se,e) * dy(sw,se)) / (4*S_s)) / S_P

    # West neighbor
    D_3 = ((dx(nw, sw) * (dx(n, nW) / 4 + dx(sW, s) / 4 + dx(nW, sW))) / S_w +
          (dy(nw, sw) * (dy(n, nW) / 4 + dy(sW, s) / 4 + dy(nW, sW))) / S_w +
          (dx(Nw, w) * dx(ne, nw)) / (4 * S_n) +
          (dx(w, Sw) * dx(sw, se)) / (4 * S_s) +
          (dy(Nw, w) * dy(ne, nw)) / (4 * S_n) +
          (dy(w, Sw) * dy(sw, se)) / (4 * S_s)) / S_P

    # South neighbor
    D1 = ((dx(sw, se) * (dx(Se, e) / 4 + dx(w, Sw) / 4 + dx(Sw, Se))) / S_s +
        (dy(sw, se) * (dy(Se, e) / 4 + dy(w, Sw) / 4 + dy(Sw, Se))) / S_s +
        (dx(s, sE) * dx(se, ne)) / (4 * S_e) +
        (dx(sW, s) * dx(nw, sw)) / (4 * S_w) +
        (dy(s, sE) * dy(se, ne)) / (4 * S_e) +
        (dy(sW, s) * dy(nw, sw)) / (4 * S_w)) / S_P

    # North neighbor
    D_1 = ((dx(ne, nw) * (dx(e, Ne) / 4 + dx(Nw, w) / 4 + dx(Ne, Nw))) / S_n +
        (dy(ne, nw) * (dy(e, Ne) / 4 + dy(Nw, w) / 4 + dy(Ne, Nw))) / S_n +
        (dx(nE, n) * dx(se, ne)) / (4 * S_e) +
        (dx(n, nW) * dx(nw, sw)) / (4 * S_w) +
        (dy(nE, n) * dy(se, ne)) / (4 * S_e) +
        (dy(n, nW) * dy(nw, sw)) / (4 * S_w)) / S_P

    # NW diagonal
    D_4 = ((dx(Nw, w) * dx(ne, nw)) / (4 * S_n) +
        (dx(n, nW) * dx(nw, sw)) / (4 * S_w) +
        (dy(Nw, w) * dy(ne, nw)) / (4 * S_n) +
        (dy(n, nW) * dy(nw, sw)) / (4 * S_w)) / S_P

    # NE diagonal
    D2 = ((dx(nE, n) * dx(se, ne)) / (4 * S_e) +
        (dx(e, Ne) * dx(ne, nw)) / (4 * S_n) +
        (dy(nE, n) * dy(se, ne)) / (4 * S_e) +
        (dy(e, Ne) * dy(ne, nw)) / (4 * S_n)) / S_P

    # SW diagonal
    D_2 = ((dx(w, Sw) * dx(sw, se)) / (4 * S_s) +
        (dx(sW, s) * dx(nw, sw)) / (4 * S_w) +
        (dy(w, Sw) * dy(sw, se)) / (4 * S_s) +
        (dy(sW, s) * dy(nw, sw)) / (4 * S_w)) / S_P

    # SE diagonal
    D4 = ((dx(s, sE) * dx(se, ne)) / (4 * S_e) +
        (dx(Se, e) * dx(sw, se)) / (4 * S_s) +
        (dy(s, sE) * dy(se, ne)) / (4 * S_e) +
        (dy(Se, e) * dy(sw, se)) / (4 * S_s)) / S_P

    # Center (P)
```

```
    D0 = ((dx(se, ne) * (dx(n, s) + dx(nE, n) / 4 + dx(s, sE) / 4)) / S_e +
        (dx(ne, nw) * (dx(w, e) + dx(e, Ne) / 4 + dx(Nw, w) / 4)) / S_n +
        (dx(sw, se) * (dx(e, w) + dx(Se, e) / 4 + dx(w, Sw) / 4)) / S_s +
        (dx(nw, sw) * (dx(s, n) + dx(n, nW) / 4 + dx(sW, s) / 4)) / S_w +
        (dy(se, ne) * (dy(n, s) + dy(nE, n) / 4 + dy(s, sE) / 4)) / S_e +
        (dy(ne, nw) * (dy(w, e) + dy(e, Ne) / 4 + dy(Nw, w) / 4)) / S_n +
        (dy(sw, se) * (dy(e, w) + dy(Se, e) / 4 + dy(w, Sw) / 4)) / S_s +
        (dy(nw, sw) * (dy(s, n) + dy(n, nW) / 4 + dy(sW, s) / 4)) / S_w) / S_P

    # Assemble stencil
    stencil[self.index(i, j)] = D0
    stencil[self.index(i-1, j)] = D_1
    stencil[self.index(i+1, j)] = D1
    stencil[self.index(i, j-1)] = D_3
    stencil[self.index(i, j+1)] = D3
    stencil[self.index(i-1, j-1)] = D_4
    stencil[self.index(i-1, j+1)] = D2
    stencil[self.index(i+1, j-1)] = D_2
    stencil[self.index(i+1, j+1)] = D4

    return stencil, b[0]


def build_north(self, i, j):
    """Build stencil for north boundary (i=0)"""
    stencil = np.zeros(self.n * self.m)
    b = np.zeros(1)

    # Dirichlet: impose temperature directly
    if self.boundary[0] == 'D':
        stencil[self.index(i, j)] = 1.0
        b[0] = self.TD[0]
        return stencil, b[0]

    # Neumann or Robin: apply flux balance
    # Get coordinates (no north neighbor available)
    P = Coordinate2D(self.X[i, j], self.Y[i, j])
    S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
    W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
    E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
    SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
    SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

    # Auxiliary nodes
    Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
    Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
    sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
    sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)

    s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
    w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
    e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

    se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)
    sw = Coordinate2D((Sw.x + w.x)/2, (Sw.y + w.y)/2)

    # Control volume areas
    S_ss = calculate_area(e, se, sw, w)
    S_s = calculate_area(e, Se, Sw, w)
    S_ssw = calculate_area(P, s, sW, W)
    S_sse = calculate_area(E, sE, s, P)

    # Neighbor coefficients
    D3 = (dy(sw, se) * (dy(Se, e) / 4) / S_s + dx(sw, se) * (dx(Se, e) / 4) / S_s +
        dy(se, e) * (dy(s, sE) / 4 + 3 * dy(sE, E) / 4 + dy(E, P) / 2) / S_sse +
        dx(se, e) * (dx(s, sE) / 4 + 3 * dx(sE, E) / 4 + dx(E, P) / 2) / S_sse) / S_ss

    D_3 = (dy(w, sw) * (3 * dy(W, sW) / 4 + dy(sW, s) / 4 + dy(P, W) / 2) / S_ssw +
        dx(w, sw) * (3 * dx(W, sW) / 4 + dx(sW, s) / 4 + dx(P, W) / 2) / S_ssw +
        dy(sw, se) * (dy(w, Sw) / 4) / S_s + dx(sw, se) * (dx(w, Sw) / 4) / S_s) / S_ss

    D1 = (dy(w, sw) * (dy(sW, s) / 4 + dy(s, P) / 4) / S_ssw +
        dx(w, sw) * (dx(sW, s) / 4 + dx(s, P) / 4) / S_ssw +
        dy(sw, se) * (dy(w, Sw) / 4 + dy(Sw, Se) + dy(Se, e) / 4) / S_s +
        dx(sw, se) * (dx(w, Sw) / 4 + dx(Sw, Se) + dx(Se, e) / 4) / S_s +
        dy(se, e) * (dy(P, s) / 4 + dy(s, sE) / 4) / S_sse +
        dx(se, e) * (dx(P, s) / 4 + dx(s, sE) / 4) / S_sse) / S_ss

    D_2 = (dy(w, sw) * (dy(W, sW) / 4 + dy(sW, s) / 4) / S_ssw +
        dx(w, sw) * (dx(W, sW) / 4 + dx(sW, s) / 4) / S_ssw +
        dy(sw, se) * (dy(w, Sw) / 4) / S_s + dx(sw, se) * (dx(w, Sw) / 4) / S_s) / S_ss

    D4 = (dy(sw, se) * (dy(Se, e) / 4) / S_s + dx(sw, se) * (dx(Se, e) / 4) / S_s +
        dy(se, e) * (dy(s, sE) / 4 + dy(sE, E) / 4) / S_sse +
        dx(se, e) * (dx(s, sE) / 4 + dx(sE, E) / 4) / S_sse) / S_ss

    # Boundary condition coefficient and source term
    coefficient = 0.0
    if self.boundary[0] == 'N':
        coefficient = 0.0
        b[0] = self.q * dist(e, w) / S_ss
    elif self.boundary[0] == 'R':
        coefficient = -self.alpha
        b[0] = -self.alpha * self.Tinf * dist(e, w) / S_ss
    else:
        raise ValueError(f'Unknown boundary type: {self.boundary[0]}')

    # Center coefficient
    D0 = (coefficient * dist(e, w) +
        dy(w, sw) * (dy(sW, s) / 4 + 3 * dy(s, P) / 4 + dy(P, W) / 2) / S_ssw +
        dx(w, sw) * (dx(sW, s) / 4 + 3 * dx(s, P) / 4 + dx(P, W) / 2) / S_ssw +
        dy(sw, se) * (dy(w, Sw) / 4 + dy(Se, e) / 4 + dy(e, w)) / S_s +
        dx(sw, se) * (dx(w, Sw) / 4 + dx(Se, e) / 4 + dx(e, w)) / S_s +
        dy(se, e) * (3 * dy(P, s) / 4 + dy(s, sE) / 4 + dy(E, P) / 2) / S_sse +
        dx(se, e) * (3 * dx(P, s) / 4 + dx(s, sE) / 4 + dx(E, P) / 2) / S_sse) / S_ss

    stencil[self.index(i, j)] = D0
    stencil[self.index(i+1, j)] = D1
    stencil[self.index(i, j-1)] = D_3
    stencil[self.index(i, j+1)] = D3
    stencil[self.index(i+1, j-1)] = D_2
    stencil[self.index(i+1, j+1)] = D4
```

```
    return stencil, b[0]

def build_south(self, i, j):
    """Build stencil for south boundary (i=m-1)"""
    stencil = np.zeros(self.m * self.n)
    b = np.zeros(1)

    if self.boundary[1] == 'D':
        stencil[self.index(i, j)] = 1.0
        b[0] = self.TD[1]
        return stencil, b[0]

    # Get coordinates (no south neighbor)
    P  = Coordinate2D(self.X[i, j], self.Y[i, j])
    N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
    W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
    E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
    NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])
    NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

    Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
    Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
    nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
    nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)

    n  = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
    w  = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
    e  = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

    ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
    nw = Coordinate2D((Nw.x + w.x)/2, (Nw.y + w.y)/2)

    S_nn  = self.stable_area(e, ne, nw, w)
    S_n   = self.stable_area(e, Ne, Nw, w)
    S_nnw = self.stable_area(P, n, nW, W)
    S_nne = self.stable_area(E, nE, n, P)

    # Neighbor coefficients

    D3 = (dy(nw, ne) * dy(Ne, e) / 4 / S_n + dx(nw, ne) * dx(Ne, e) / 4 / S_n +
        dy(ne, e) * (dy(n, nE)/4 + 3*dy(nE, E)/4 + dy(E, P)/2) / S_nne +
        dx(ne, e) * (dx(n, nE)/4 + 3*dx(nE, E)/4 + dx(E, P)/2) / S_nne) / S_nn

    D_3 = (dy(w, nw) * (3*dy(W, nW)/4 + dy(nW, n)/4 + dy(P, W)/2) / S_nnw +
        dx(w, nw) * (3*dx(W, nW)/4 + dx(nW, n)/4 + dx(P, W)/2) / S_nnw +
        dy(nw, ne) * dy(w, Nw)/4 / S_n + dx(nw, ne) * dx(w, Nw)/4 / S_n) / S_nn

    D_1 = (dy(w, nw) * (dy(nW, n)/4 + dy(n, P)/4) / S_nnw +
        dx(w, nw) * (dx(nW, n)/4 + dx(n, P)/4) / S_nnw +
        dy(nw, ne) * (dy(w, Nw)/4 + dy(Nw, Ne) + dy(Ne, e)/4) / S_n +
        dx(nw, ne) * (dx(w, Nw)/4 + dx(Nw, Ne) + dx(Ne, e)/4) / S_n +
        dy(ne, e) * (dy(P, n)/4 + dy(n, nE)/4) / S_nne +
        dx(ne, e) * (dx(P, n)/4 + dx(n, nE)/4) / S_nne) / S_nn

    D4 = (dy(w, nw) * (dy(W, nW)/4 + dy(nW, n)/4) / S_nnw +
        dx(w, nw) * (dx(W, nW)/4 + dx(nW, n)/4) / S_nnw +
        dy(nw, ne) * dy(w, Nw)/4 / S_n + dx(nw, ne) * dx(w, Nw)/4 / S_n) / S_nn

    D2 = (dy(nw, ne) * dy(Ne, e)/4 / S_n + dx(nw, ne) * dx(Ne, e)/4 / S_n +
        dy(ne, e) * (dy(n, nE)/4 + dy(nE, E)/4) / S_nne +
        dx(ne, e) * (dx(n, nE)/4 + dx(nE, E)/4) / S_nne) / S_nn

    if self.boundary[1] == 'N':
        coefficient = 0.0
        b[0] = self.q * dist(e, w) / S_nn
    elif self.boundary[1] == 'R':
        coefficient = -self.alpha
        b[0] = -self.alpha * self.Tinf * dist(e, w) / S_nn
    else:
        raise ValueError(f'Unknown boundary type: {self.boundary[1]}')

    D0 = (coefficient * dist(e, w) +
        dy(w, nw) * (dy(nW, n)/4 + 3*dy(n, P)/4 + dy(P, W)/2) / S_nnw +
        dx(w, nw) * (dx(nW, n)/4 + 3*dx(n, P)/4 + dx(P, W)/2) / S_nnw +
        dy(nw, ne) * (dy(w, Nw)/4 + dy(Ne, e)/4 + dy(e, w)) / S_n +
        dx(nw, ne) * (dx(w, Nw)/4 + dx(Ne, e)/4 + dx(e, w)) / S_n +
        dy(ne, e) * (3*dy(P, n)/4 + dy(n, nE)/4 + dy(E, P)/2) / S_nne +
        dx(ne, e) * (3*dx(P, n)/4 + dx(n, nE)/4 + dx(E, P)/2) / S_nne) / S_nn

    stencil[self.index(i, j)]     = D0
    stencil[self.index(i-1, j)]   = D_1
    stencil[self.index(i, j-1)]   = D_3
    stencil[self.index(i, j+1)]   = D3
    stencil[self.index(i-1, j-1)] = D4
    stencil[self.index(i-1, j+1)] = D2

    return stencil, b[0]

def build_west(self, i, j):
    """Build stencil for west boundary"""
    stencil = np.zeros(self.m * self.n)
    b = np.zeros(1)

    if self.boundary[2] == 'D':
        stencil[self.index(i, j)] = 1.0
        b[0] = self.TD[2]
        return stencil, b[0]

    P = Coordinate2D(self.X[i, j], self.Y[i, j])
    S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
    N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
    E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
    SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])
    NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

    Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
    Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
    sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)
    nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)

    s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
```

```
        n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
        se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)

        S_ww = self.stable_area(s, n, ne, se)
        S_w = self.stable_area(s, n, nE, sE)
        S_wws = self.stable_area(S, P, e, Se)
        S_wwn = self.stable_area(P, N, Ne, e)

        D_1 = -(dy(se, ne) * (dy(nE, n) / 4) / S_w + dx(se, ne) * (dx(nE, n) / 4) / S_w +
                dy(ne, n) * (dy(e, Ne) / 4 + 3 * dy(Ne, N) / 4 + dy(N, P) / 2) / S_wwn +
                dx(ne, n) * (dx(e, Ne) / 4 + 3 * dx(Ne, N) / 4 + dx(N, P) / 2) / S_wwn) / S_ww

        D1 = -(dy(se, ne) * (dy(s, sE) / 4) / S_w + dx(se, ne) * (dx(s, sE) / 4) / S_w +
                dy(se, s) * (dy(sE, S) / 4 + 3 * dy(Se, S) / 4 + dy(S, P) / 2) / S_wws +
                dx(se, s) * (dx(sE, S) / 4 + 3 * dx(Se, S) / 4 + dx(S, P) / 2) / S_wws) / S_ww

        D3 = -(dy(s, se) * (dy(Se, e) / 4 + dy(e, P) / 4) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + dx(e, P) / 4) / S_wws +
                dy(se, ne) * (dy(s, sE) / 4 + dy(sE, nE) + dy(nE, n) / 4) / S_w +
                dx(se, ne) * (dx(s, sE) / 4 + dx(sE, nE) + dx(nE, n) / 4) / S_w +
                dy(ne, n) * (dy(P, e) / 4 + dy(e, Ne) / 4) / S_wwn +
                dx(ne, n) * (dx(P, e) / 4 + dx(e, Ne) / 4) / S_wwn) / S_ww

        D2 = -(dy(ne, n) * (dy(e, Ne) / 4 + dy(Ne, N) / 4) / S_wwn +
                dx(ne, n) * (dx(e, Ne) / 4 + dx(Ne, N) / 4) / S_wwn +
                dy(se, ne) * (dy(nE, n) / 4) / S_w + dx(se, ne) * (dx(nE, n) / 4) / S_w) / S_ww

        D4 = -(dy(se, ne) * (dy(s, sE) / 4) / S_w + dx(se, ne) * (dx(Se, s) / 4) / S_w +
                dy(s, se) * (dy(Se, e) / 4 + dy(Se, S) / 4) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + dx(Se, S) / 4) / S_wws) / S_ww

        if self.boundary[2] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(n, s) / S_ww
        elif self.boundary[2] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(n, s) / S_ww
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[2]}')

        D0 = (coefficient * dist(ne, se) +
                dy(s, se) * (dy(Se, e) / 4 + 3 * dy(e, P) / 4 + dy(P, S) / 2) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + 3 * dx(e, P) / 4 + dx(P, S) / 2) / S_wws +
                dy(se, ne) * (dy(s, sE) / 4 + dy(nE, n) / 4 + dy(n, s)) / S_w +
                dx(se, ne) * (dx(s, sE) / 4 + dx(nE, n) / 4 + dx(n, s)) / S_w +
                dy(ne, n) * (3 * dy(P, e) / 4 + dy(e, Ne) / 4 + dy(N, P) / 2) / S_wwn +
                dx(ne, n) * (3 * dx(P, e) / 4 + dx(e, Ne) / 4 + dx(N, P) / 2) / S_wwn) / S_ww

        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j+1)] = D3
        stencil[self.index(i-1, j+1)] = D2
        stencil[self.index(i+1, j+1)] = D4

        return stencil, b[0]

    def build_east(self, i, j):
        """Build stencil for east boundary"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])

        Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)

        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)

        nw = Coordinate2D((nW.x + n.x)/2, (nW.y + n.y)/2)
        sw = Coordinate2D((sW.x + s.x)/2, (sW.y + s.y)/2)

        S_ee = self.stable_area(s, sw, nw, n)
        S_e = self.stable_area(n, s, sW, nW)
        S_ees = self.stable_area(P, S, Sw, w)
        S_een = self.stable_area(N, P, w, Nw)

        D_1 = (dy(nw, sw) * (dy(n, nW) / 4) / S_e + dx(nw, sw) * (dx(n, nW) / 4) / S_e +
                dy(n, nw) * (dy(Nw, w) / 4 + 3 * dy(N, Nw) / 4 + dy(P, N) / 2) / S_een +
                dx(n, nw) * (dx(Nw, w) / 4 + 3 * dx(N, Nw) / 4 + dx(P, N) / 2) / S_een) / S_ee

        D1 = (dy(nw, sw) * (dy(sW, s) / 4) / S_e + dx(nw, sw) * (dx(sW, s) / 4) / S_e +
                dy(sw, s) * (dy(w, Sw) / 4 + 3 * dy(Sw, S) / 4 + dy(S, P) / 2) / S_ees +
                dx(sw, s) * (dx(w, Sw) / 4 + 3 * dx(Sw, S) / 4 + dx(S, P) / 2) / S_ees) / S_ee

        D_3 = (dy(sw, s) * (dy(w, Sw) / 4 + dy(P, w) / 4) / S_ees +
                dx(sw, s) * (dx(w, Sw) / 4 + dx(P, w) / 4) / S_ees +
                dy(nw, sw) * (dy(sW, s) / 4 + dy(nW, sW) + dy(n, nW) / 4) / S_e +
                dx(nw, sw) * (dx(sW, s) / 4 + dx(nW, sW) + dx(n, nW) / 4) / S_e +
                dy(n, nw) * (dy(w, P) / 4 + dy(Nw, w) / 4) / S_een +
                dx(n, nw) * (dx(w, P) / 4 + dx(Nw, w) / 4) / S_een) / S_ee

        D_4 = (dy(n, nw) * (dy(N, Nw) / 4 + dy(Nw, w) / 4) / S_een +
                dx(n, nw) * (dx(N, Nw) / 4 + dx(Nw, w) / 4) / S_een +
                dy(nw, sw) * (dy(n, nW) / 4) / S_e + dx(nw, sw) * (dx(n, nW) / 4) / S_e) / S_ee
```

```
        D_2 = (dy(nw, sw) * (dy(sW, s) / 4) / S_e + dx(nw, sw) * (dx(sW, s) / 4) / S_e +
               dy(sw, s) * (dy(Sw, S) / 4 + dy(w, Sw) / 4) / S_ees +
               dx(sw, s) * (dx(Sw, S) / 4 + dx(w, Sw) / 4) / S_ees) / S_ee

        if self.boundary[3] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(n, s) / S_ee
        elif self.boundary[3] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(n, s) / S_ee
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[3]}')

        D0 = (coefficient * dist(nw, sw) +
               dy(sw, s) * (dy(w, Sw) / 4 + 3 * dy(P, w) / 4 + dy(S, P) / 2) / S_ees +
               dx(sw, s) * (dx(w, Sw) / 4 + 3 * dx(P, w) / 4 + dx(S, P) / 2) / S_ees +
               dy(nw, sw) * (dy(sW, s) / 4 + dy(n, nW) / 4 + dy(s, n)) / S_e +
               dx(nw, sw) * (dx(sW, s) / 4 + dx(n, nW) / 4 + dx(s, n)) / S_e +
               dy(n, nw) * (3 * dy(w, P) / 4 + dy(Nw, w) / 4 + dy(P, N) / 2) / S_een +
               dx(n, nw) * (3 * dx(w, P) / 4 + dx(Nw, w) / 4 + dx(P, N) / 2) / S_een) / S_ee

        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j-1)] = D_3
        stencil[self.index(i-1, j-1)] = D_4
        stencil[self.index(i+1, j-1)] = D_2

        return stencil, b[0]

    def build_NW(self, i, j):
        """Build stencil for North-West corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        S  = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

        s   = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        e   = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)
        Se  = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        sE  = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)
        se  = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)

        S_nw  = self.stable_area(P, s, se, e)
        S_nws = self.stable_area(P, S, Se, e)
        S_nwe = self.stable_area(P, s, sE, E)

        coeff_N = coeff_W = 0.0
        b[0] = 0.0

        if self.boundary[0] == 'N':
            b[0] += self.q * dist(e, P) / S_nw
        elif self.boundary[0] == 'R':
            coeff_N = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(e, P) / S_nw

        if self.boundary[2] == 'N':
            b[0] += self.q * dist(P, s) / S_nw
        elif self.boundary[2] == 'R':
            coeff_W = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, s) / S_nw

        D0 = (
            coeff_N * dist(e, P) +
            coeff_W * dist(s, P) +
            dy(s, se) * (dy(Se, e) / 4 + 3 * dy(e, P) / 4 + dy(P, S) / 2) / S_nws +
            dx(s, se) * (dx(Se, e) / 4 + 3 * dx(e, P) / 4 + dx(P, S) / 2) / S_nws +
            dy(se, e) * (3 * dy(s, P)/4 + dy(sE,s)/4 + dy(P, E)/2) / S_nwe +
            dx(se, e) * (3 * dx(s, P)/4 + dx(sE,s)/4 + dx(P, E)/2) / S_nwe
        ) / S_nw

        D1 = (
            dy(s, se) * (dy(e, Se) / 4 + 3 * dy(Se, S) / 4 + dy(S, P) / 2) / S_nws +
            dx(s, se) * (dx(e, Se) / 4 + 3 * dx(Se, S) / 4 + dx(S, P) / 2) / S_nws +
            dy(se, e) * (dy(s, sE)/4 + dy(sE, E)/4) / S_nwe +
            dx(se, e) * (dx(s, sE)/4 + dx(sE, E)/4) / S_nwe
        ) / S_nw

        D3 = (
            dy(se, e) * (dy(s, sE) / 4 + 3 * dy(sE, E) / 4 + dy(E, P) / 2) / S_nwe +
            dx(se, e) * (dx(s, sE) / 4 + 3 * dx(sE, E) / 4 + dx(E, P) / 2) / S_nwe +
            dy(s, se) * (dy(e, Se)) /4 / S_nws +
            dx(s, se) * (dx(e, Se)) /4 / S_nws
        ) / S_nw

        D4 = (
            dy(s, se) * (dy(Se, S) / 4 + dy(e, Se) / 4) / S_nws +
            dx(s, se) * (dx(Se, S) / 4 + dx(e, Se) / 4) / S_nws +
            dy(se, e) * (dy(s, sE) / 4 + dy(sE, E) / 4) / S_nwe +
            dx(se, e) * (dx(s, sE) / 4 + dx(sE, E) / 4) / S_nwe
        ) / S_nw

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i+1, j)]     = D1
        stencil[self.index(i, j+1)]     = D3
        stencil[self.index(i+1, j+1)]   = D4

        return stencil, b[0]
```

```
    def build_NE(self, i, j):
        """Build stencil for North-East corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        S  = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])

        s   = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w   = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        Sw  = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        sW  = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        sw  = Coordinate2D((sW.x + s.x)/2, (sW.y + s.y)/2)

        S_ne  = self.stable_area(w, sw, s, P)
        S_nes = self.stable_area(P, w, Sw, S)
        S_new = self.stable_area(P, W, sW, s)

        coeff_N = coeff_E = 0.0
        b[0] = 0.0

        if self.boundary[0] == 'N':
            b[0] += self.q * dist(w, P) / S_ne
        elif self.boundary[0] == 'R':
            coeff_N = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(w, P) / S_ne
        if self.boundary[3] == 'N':
            b[0] += self.q * dist(P, s) / S_ne
        elif self.boundary[3] == 'R':
            coeff_E = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, s) / S_ne
        D0 = (
            coeff_N * dist(w, P) +
            coeff_E * dist(P, s) +
            dy(sw, s) * (3 * dy(P, w) / 4 + dy(w, Sw) / 4 +  dy(S, P) / 2) / S_nes +
            dx(sw, s) * (3 * dx(P, w) / 4 + dx(w, Sw) / 4  + dx(S, P) / 2) / S_nes +
            dy(w, sw) * (3 * dy(s, P) / 4 + dy(sW, s) / 4  + dy(P, W) / 2) / S_new +
            dx(w, sw) * (3 * dx(s, P) / 4 + dx(sW, s) / 4 +  dx(P, W) / 2) / S_new
        ) / S_ne

        D1 = (
            dy(sw, s) * (3 * dy(Sw, S) / 4 + dy(w, Sw) / 4  + dy(S, P) / 2) / S_nes +
            dx(sw, s) * (3 * dx(Sw, S) / 4 + dx(w, Sw) / 4 +  dx(S, P) / 2) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + dy(s, P) / 4) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + dx(s, P) / 4) / S_new
        ) / S_ne

        D_3 = (
            dy(sw, s) * (dy(w, Sw) / 4 + dy(P, w) / 4) / S_nes +
            dx(sw, s) * (dx(w, Sw) / 4 + dx(P, w) / 4) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + 3 * dy(W, sW) / 4 + dy(P, W) / 2) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + 3 * dx(W, sW) / 4 + dx(P, W) / 2) / S_new
        ) / S_ne

        D_2 = (
            dy(sw, s) * (dy(Sw, S) / 4 + dy(w, Sw) / 4) / S_nes +
            dx(sw, s) * (dx(Sw, S) / 4 + dx(w, Sw) / 4) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + dy(W, sW) / 4) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + dx(W, sW) / 4) / S_new
        ) / S_ne

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i+1, j)]     = D1
        stencil[self.index(i, j-1)]     = D_3
        stencil[self.index(i+1, j-1)]   = D_2

        return stencil, b[0]

    def build_SW(self, i, j):
        """Build stencil for South-West corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

        n   = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        e   = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)
        Ne  = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        nE  = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)
        ne  = Coordinate2D((nE.x + n.x)/2, (nE.y + n.y)/2)

        S_sw  = self.stable_area(e, ne, n, P)
        S_swn = self.stable_area(P, e, Ne, N)
        S_swe = self.stable_area(P, E, nE, n)

        coeff_S = coeff_W = 0.0
```

```
        b[0] = 0.0

        if self.boundary[1] == 'N':
            b[0] += self.q * dist(e, P) / S_sw
        elif self.boundary[1] == 'R':
            coeff_S = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(e, P) / S_sw

        if self.boundary[2] == 'N':
            b[0] += self.q * dist(P, n) / S_sw
        elif self.boundary[2] == 'R':
            coeff_W = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, n) / S_sw
        D0 = (
            coeff_S * dist(e, P) +
            coeff_W * dist(P, n) +
            dy(n, ne) * (dy(Ne, e) / 4 + 3 * dy(e, P) / 4 + dy(P, N) / 2) / S_swn +
            dx(n, ne) * (dx(Ne, e) / 4 + 3 * dx(e, P) / 4 + dx(P, N) / 2) / S_swn +
            dy(ne, e) * (dy(nE, n) / 4 + 3 * dy(n, P) / 4 + dy(P, E) / 2) / S_swe +
            dx(ne, e) * (dx(nE, n) / 4 + 3 * dx(n, P) / 4 + dx(P, E) / 2) / S_swe
        ) / S_sw

        D_1 = ( dy(n, ne) * (dy(e, Ne) / 4 + 3 * dy(Ne, N) / 4 + dy(N, P) / 2) / S_swn +
                dx(n, ne) * (dx(e, Ne) / 4 + 3 * dx(Ne, N) / 4 + dx(N, P) / 2) / S_swn +
                dy(ne, e) * (dy(n, nE)/4 ) / S_swe +
                dx(ne, e) * (dx(n, nE)/4  / S_swe )
        )/ S_sw

        D3 = ( dy(n, ne) * (dy(e, Ne) / 4) / S_swn +
                dx(n, ne) * (dx(e, Ne) / 4) / S_swn +
                dy(ne, e) * (dy(n, nE) / 4 + 3 * dy(nE, E) / 4 +  dy(E, P) / 2) / S_swe +
                dx(ne, e) * (dx(n, nE) / 4 + 3 * dx(nE, E) / 4 + dx(E, P) / 2) / S_swe
            ) / S_sw

        D2 = (  dy(n, ne) * (dy(Ne, N) / 4 + dy(e, Ne) / 4) / S_swn +
                dx(n, ne) * (dx(Ne, N) / 4 + dx(e, Ne) / 4) / S_swn +
                dy(ne, e) * (dy(n, nE) / 4 + dy(nE, E) / 4) / S_swe +
                dx(ne, e) * (dx(n, nE) / 4 + dx(nE, E) / 4) / S_swe
            ) / S_sw

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i-1, j)]     = D_1
        stencil[self.index(i, j+1)]     = D3
        stencil[self.index(i-1, j+1)]   = D2

        return stencil, b[0]

    def build_SE(self, i, j):
        """Build stencil for South-East corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])

        n   = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        w   = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        Nw  = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        nW  = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
        nw  = Coordinate2D((nW.x + n.x)/2, (nW.y + n.y)/2)

        S_se  = self.stable_area(P, n, nw, w)
        S_sen = self.stable_area(P, N, Nw, w)
        S_sew = self.stable_area(P, n, nW, W)

        coeff_S = coeff_E = 0.0
        b[0] = 0.0


        if self.boundary[1] == 'N':
            b[0] += self.q * dist(w, P) / S_se
        elif self.boundary[1] == 'R':
            coeff_S = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(w, P) / S_se

        if self.boundary[3] == 'N':
            b[0] += self.q * dist(P, n) / S_se
        elif self.boundary[3] == 'R':
            coeff_E = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, n) / S_se

        D0 = (
            coeff_S * dist(w, P) +
            coeff_E * dist(P, n) +
            dy(nw, w) * (dy(W, P) / 2 + dy(n, nW) / 4 + 3 * dy(P, n) / 4) / S_sew +
            dx(nw, w) * (dx(W, P) / 2 + dx(n, nW) / 4 + 3 * dx(P, n) / 4) / S_sew +
            dy(n, nw) * (3 * dy(w, P) / 4 + dy(Nw, w) / 4 + dy(P, N) / 2) / S_sen +
            dx(n, nw) * (3 * dx(w, P) / 4 + dx(Nw, w) / 4 + dx(P, N) / 2) / S_sen
        ) / S_se

        D_1 = (
            dy(nw, w) * (dy(n, nW) / 4 + dy(P, n) / 4) / S_sew +
            dx(nw, w) * (dx(n, nW) / 4 + dx(P, n) / 4) / S_sew +
            dy(n, nw) * (dy(Nw, w) / 4 + 3 * dy(N, Nw) / 4 + dy(P, N) / 2) / S_sen +
            dx(n, nw) * (dx(Nw, w) / 4 + 3 * dx(N, Nw) / 4 + dx(P, N) / 2) / S_sen
        ) / S_se

        D_3 = (
            dy(nw, w) * (dy(W, P) / 2 + 3 * dy(nW, W) / 4 + dy(n, nW) / 4) / S_sew +
            dx(nw, w) * (dx(W, P) / 2 + 3 * dx(nW, W) / 4 + dx(n, nW) / 4) / S_sew +
```

```
            dy(n, nw) * (dy(w, P) / 4 + dy(Nw, w) / 4) / S_sen +
            dx(n, nw) * (dx(w, P) / 4 + dx(Nw, w) / 4) / S_sen
        ) / S_se

        D_4 = (
            dy(n, nw) * (dy(N, Nw) / 4 + dy(Nw, w) / 4) / S_sen +
            dx(n, nw) * (dx(N, Nw) / 4 + dx(Nw, w) / 4) / S_sen +
            dy(nw, w) * (dy(n, nW) / 4 + dy(nW, W) / 4) / S_sew +
            dx(nw, w) * (dx(n, nW) / 4 + dx(nW, W) / 4) / S_sew
        ) / S_se

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i-1, j)]     = D_1
        stencil[self.index(i, j-1)]     = D_3
        stencil[self.index(i-1, j-1)]   = D_4

        return stencil, b[0]

    def solve(self, state="steady", t_end=None, dt=None, T0=None):
        """Solve the linear system A*T = B """
        print("Assembling linear system...")

        if state == "steady":

            # Assemble the linear system
            for i in range(self.m):
                for j in range(self.n):
                    idx = self.index(i, j)
                    stencil, b_val = self.set_stencil(i, j)

                    # Set matrix coefficients
                    for k, coeff in enumerate(stencil):
                        if abs(coeff) > 1e-15:
                            self.A[idx, k] = coeff

                    # Set source term
                    self.B[idx] = b_val

            print(f"Matrix size: {self.A.shape}, Non-zero elements: {self.A.nnz}")

            # Solve linear system with regularization
            A_csr = self.A.tocsr() + sp.eye(self.A.shape[0]) * 1e-12

            print("Solving linear system...")
            try:
                T_flat = spsolve(A_csr, self.B)
                T = T_flat.reshape(self.m, self.n)
                print("Solution completed successfully.")
                return T
            except Exception as e:
                print(f"Error solving linear system: {e}")
                return np.zeros((self.m, self.n))

        elif state == "unsteady":
            #T_n+1 = (-b + A * T_n)
                    # 1) check time parameters
            if t_end is None or dt is None:
                raise ValueError("For unsteady, please provide t_end and dt")

            # 2) assemble spatial operator once (reuse your steady assembly)
            N = self.m * self.n
            self.A = sp.lil_matrix((N, N))
            self.B = np.zeros(N)
            for i in range(self.m):
                for j in range(self.n):
                    idx = self.index(i, j)
                    stencil, b_val = self.set_stencil(i, j)
                    for k, coeff in enumerate(stencil):
                        if abs(coeff) > 1e-15:
                            self.A[idx, k] = coeff
                    self.B[idx] = b_val

            # interpret A,B as RHS: dT/dt = A*T + B  (or with an alpha if you have it)
            L = self.A.tocsr()
            f = self.B.copy()

            # 3) initial condition
            if T0 is None:
                T = np.zeros(N)
            else:
                T = np.asarray(T0).reshape(-1).copy()

            # 4) explicit time stepping
            n_steps = int(np.ceil(t_end / dt))
            for _ in range(n_steps):
                RHS = L.dot(T) + f          # dT/dt
                T = T + dt * RHS           # T^{n+1} = T^n + dt * RHS

            return T.reshape(self.m, self.n)


        else:
            raise ValueError ("the state must be steady or unsteady state")


    def plot_Result(self, T, plot_type="2D"):
      # Create 2D and 3D plots
     X, Y = self.X, self.Y
     T_plot = np.array(T).reshape(X.shape)

     # Use actual temperature range
     vmin = np.min(T_plot)
     vmax = np.max(T_plot)

     if plot_type == "2D":
         plt.figure(figsize=(8, 6))

         # Create symmetric mesh by combining original and flipped
         X_combined = np.vstack([X, X])
         Y_combined = np.vstack([Y, -Y])
         T_combined = np.vstack([T_plot, T_plot])
```

```
        pcm = plt.pcolormesh(X_combined, Y_combined, T_combined,
                             shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
        plt.colorbar(pcm, label="Temperature")
        plt.xlabel("X-axis")
        plt.ylabel("Y-axis")
        plt.title("Temperature Distribution")
        plt.axis('equal')
        plt.tight_layout()
        plt.show()

    elif plot_type == "3D":
        fig = plt.figure(figsize=(10, 7))
        ax = fig.add_subplot(111, projection='3d')

        # Plot original surface
        surf1 = ax.plot_surface(X, Y, T_plot, cmap='jet',
                                edgecolor='none', antialiased=True,
                                vmin=vmin, vmax=vmax)

        # Plot symmetric surface (y-flipped)
        surf2 = ax.plot_surface(X, -Y, T_plot, cmap='jet',
                                edgecolor='none', antialiased=True,
                                vmin=vmin, vmax=vmax)

        ax.set_xlabel("X-axis")
        ax.set_ylabel("Y-axis")
        ax.set_zlabel("Temperature")
        ax.set_title("3D Temperature Distribution")
        ax.view_init(elev=45, azim=300)
        plt.colorbar(surf1, label="Temperature")
        plt.tight_layout()
        plt.show()

    elif plot_type == "both":
        # Create a figure with two subplots
        fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(16, 6))

        # 2D plot on left subplot
        X_combined = np.vstack([X, X])
        Y_combined = np.vstack([Y, -Y])
        T_combined = np.vstack([T_plot, T_plot])

        pcm = ax1.pcolormesh(X_combined, Y_combined, T_combined,
                             shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
        plt.colorbar(pcm, ax=ax1, label="Temperature")
        ax1.set_xlabel("X-axis")
        ax1.set_ylabel("Y-axis")
        ax1.set_title("2D Temperature Distribution")
        ax1.axis('equal')

        # 3D plot on right subplot
        ax2 = fig.add_subplot(122, projection='3d')
        surf1 = ax2.plot_surface(X, Y, T_plot, cmap='jet',
                                 edgecolor='none', antialiased=True,
                                 vmin=vmin, vmax=vmax)
        surf2 = ax2.plot_surface(X, -Y, T_plot, cmap='jet',
                                 edgecolor='none', antialiased=True,
                                 vmin=vmin, vmax=vmax)
        ax2.set_xlabel("X-axis")
        ax2.set_ylabel("Y-axis")
        ax2.set_zlabel("Temperature")
        ax2.set_title("3D Temperature Distribution")
        ax2.view_init(elev=45, azim=300)
        plt.colorbar(surf1, ax=ax2, label="Temperature")

        plt.tight_layout()
        plt.show()

    else:
       raise ValueError("plot_type must be '2D', '3D', or 'both'")
```

Solver for Unsteady state

In [13]:

```
class UnsteadyHeat2D_FVM:
    """
    2Dcl steady-state heat conduction solver using Finite Volume Method.
    Handles arbitrary quadrilateral meshes and mixed boundary conditions.
    """

    def __init__(self, X, Y, boundary=[], TD=[], q=0.0, alpha=0.0, Tinf=0.0, lambda_conductivity=1.0):
        """
        Initialize FVM solver.

        Args:
            X, Y: mesh coordinate arrays (m × n)
            boundary: BC types [North, South, West, East] ('D', 'N', or 'R')
            TD: Dirichlet temperatures [North, South, West, East]
            q: heat flux for Neumann BC (W/m²)
            alpha: heat transfer coefficient for Robin BC (W/m²K)
            Tinf: ambient temperature for Robin BC (K)
            lambda_conductivity: thermal conductivity (W/mK)
        """
        self.X = X
        self.Y = Y
        self.boundary = boundary
        self.TD = TD
        self.q = q
        self.alpha = alpha
        self.Tinf = Tinf
        self.lambda_conductivity = lambda_conductivity

        # Grid dimensions
        self.m, self.n = X.shape
        self.total_nodes = self.m * self.n

        # Initialize sparse linear system
        self.A = sp.lil_matrix((self.total_nodes, self.total_nodes))
        self.B = np.zeros(self.total_nodes)
```

```
        # For unsteady problems
        self.Matrix = None
        self.RHS = None

    def index(self, i, j):
        """Map 2D grid index (i,j) to 1D array index"""
        return i * self.n + j

    def stable_area(self, *args):
        """Wrapper for area calculation with stability check"""
        area = calculate_area(*args)
        return area if area > 1e-12 else 1e-12

    def set_stencil(self, i, j):
        """
        Determine node type and build appropriate stencil.
        Grid indexing: i=0 is North (top), i=m-1 is South (bottom)
                       j=0 is West (left), j=n-1 is East (right)
        """
        # Check corners first
        if i == 0 and j == 0:
            return self.build_NW(i, j)
        elif i == 0 and j == self.n-1:
            return self.build_NE(i, j)
        elif i == self.m-1 and j == 0:
            return self.build_SW(i, j)
        elif i == self.m-1 and j == self.n-1:
            return self.build_SE(i, j)

        # Check boundaries
        elif i == 0:
            return self.build_north(i, j)
        elif i == self.m-1:
            return self.build_south(i, j)
        elif j == 0:
            return self.build_west(i, j)
        elif j == self.n-1:
            return self.build_east(i, j)

        # Interior node
        else:
            return self.build_inner(i, j)

    def build_inner(self, i, j):
        """Build 9-point stencil for interior nodes"""
        stencil = np.zeros(self.n * self.m)
        b = np.zeros(1)

        # Principal nodes
        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

        # Auxiliary nodes (midpoints)
        Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
        nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)
        sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)

        # Face centers
        n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        # Control volume corners
        se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)
        sw = Coordinate2D((Sw.x + w.x)/2, (Sw.y + w.y)/2)
        ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
        nw = Coordinate2D((Nw.x + w.x)/2, (Nw.y + w.y)/2)

        # Control volume areas
        S_P = calculate_area(ne, se, sw, nw)
        S_n = calculate_area(Ne, e, w, Nw)
        S_s = calculate_area(e, Se, Sw, w)
        S_w = calculate_area(n, s, sW, nW)
        S_e = calculate_area(nE, sE, s, n)

        # Coefficients (same as working solver)
        D3 = ((dx(se, ne) * (dx(nE, n)/4 + dx(s, sE)/4 + dx(sE, nE))) / S_e +
              (dy(se, ne) * (dy(nE, n)/4 + dy(s, sE)/4 + dy(sE, nE))) / S_e +
              (dx(e, Ne) * dx(ne, nw)) / (4*S_n) + (dx(Se,e) * dx(sw,se)) / (4*S_s) +
              (dy(e, Ne) * dy(ne, nw)) / (4*S_n) + (dy(Se,e) * dy(sw,se)) / (4*S_s)) / S_P

        D_3 = ((dx(nw, sw) * (dx(n, nW) / 4 + dx(sW, s) / 4 + dx(nW, sW))) / S_w +
               (dy(nw, sw) * (dy(n, nW) / 4 + dy(sW, s) / 4 + dy(nW, sW))) / S_w +
               (dx(Nw, w) * dx(ne, nw)) / (4 * S_n) +
               (dx(w, Sw) * dx(sw, se)) / (4 * S_s) +
               (dy(Nw, w) * dy(ne, nw)) / (4 * S_n) +
               (dy(w, Sw) * dy(sw, se)) / (4 * S_s)) / S_P

        D1 = ((dx(sw, se) * (dx(Se, e) / 4 + dx(w, Sw) / 4 + dx(Sw, Se))) / S_s +
              (dy(sw, se) * (dy(Se, e) / 4 + dy(w, Sw) / 4 + dy(Sw, Se))) / S_s +
              (dx(s, sE) * dx(se, ne)) / (4 * S_e) +
              (dx(sW, s) * dx(nw, sw)) / (4 * S_w) +
              (dy(s, sE) * dy(se, ne)) / (4 * S_e) +
              (dy(sW, s) * dy(nw, sw)) / (4 * S_w)) / S_P

        D_1 = ((dx(ne, nw) * (dx(e, Ne) / 4 + dx(Nw, w) / 4 + dx(Ne, Nw))) / S_n +
              (dy(ne, nw) * (dy(e, Ne) / 4 + dy(Nw, w) / 4 + dy(Ne, Nw))) / S_n +
              (dx(nE, n) * dx(se, ne)) / (4 * S_e) +
              (dx(n, nW) * dx(nw, sw)) / (4 * S_w) +
```

```
            (dy(nE, n) * dy(se, ne)) / (4 * S_e) +
            (dy(n, nW) * dy(nw, sw)) / (4 * S_w)) / S_P

        D_4 = ((dx(Nw, w) * dx(ne, nw)) / (4 * S_n) +
            (dx(n, nW) * dx(nw, sw)) / (4 * S_w) +
            (dy(Nw, w) * dy(ne, nw)) / (4 * S_n) +
            (dy(n, nW) * dy(nw, sw)) / (4 * S_w)) / S_P

        D2 = ((dx(nE, n) * dx(se, ne)) / (4 * S_e) +
            (dx(e, Ne) * dx(ne, nw)) / (4 * S_n) +
            (dy(nE, n) * dy(se, ne)) / (4 * S_e) +
            (dy(e, Ne) * dy(ne, nw)) / (4 * S_n)) / S_P

        D_2 = ((dx(w, Sw) * dx(sw, se)) / (4 * S_s) +
            (dx(sW, s) * dx(nw, sw)) / (4 * S_w) +
            (dy(w, Sw) * dy(sw, se)) / (4 * S_s) +
            (dy(sW, s) * dy(nw, sw)) / (4 * S_w)) / S_P

        D4 = ((dx(s, sE) * dx(se, ne)) / (4 * S_e) +
            (dx(Se, e) * dx(sw, se)) / (4 * S_s) +
            (dy(s, sE) * dy(se, ne)) / (4 * S_e) +
            (dy(Se, e) * dy(sw, se)) / (4 * S_s)) / S_P

        D0 = ((dx(se, ne) * (dx(n, s) + dx(nE, n) / 4 + dx(s, sE) / 4)) / S_e +
            (dx(ne, nw) * (dx(w, e) + dx(e, Ne) / 4 + dx(Nw, w) / 4)) / S_n +
            (dx(sw, se) * (dx(e, w) + dx(Se, e) / 4 + dx(w, Sw) / 4)) / S_s +
            (dx(nw, sw) * (dx(s, n) + dx(n, nW) / 4 + dx(sW, s) / 4)) / S_w +
            (dy(se, ne) * (dy(n, s) + dy(nE, n) / 4 + dy(s, sE) / 4)) / S_e +
            (dy(ne, nw) * (dy(w, e) + dy(e, Ne) / 4 + dy(Nw, w) / 4)) / S_n +
            (dy(sw, se) * (dy(e, w) + dy(Se, e) / 4 + dy(w, Sw) / 4)) / S_s +
            (dy(nw, sw) * (dy(s, n) + dy(n, nW) / 4 + dy(sW, s) / 4)) / S_w) / S_P

        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j-1)] = D_3
        stencil[self.index(i, j+1)] = D3
        stencil[self.index(i-1, j-1)] = D_4
        stencil[self.index(i-1, j+1)] = D2
        stencil[self.index(i+1, j-1)] = D_2
        stencil[self.index(i+1, j+1)] = D4

        return stencil, b[0]

    def build_north(self, i, j):
        """Build stencil for north boundary (i=0)"""
        stencil = np.zeros(self.n * self.m)
        b = np.zeros(1)

        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        # Same as working solver
        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

        Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)

        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)
        sw = Coordinate2D((Sw.x + w.x)/2, (Sw.y + w.y)/2)

        S_ss = calculate_area(e, se, sw, w)
        S_s = calculate_area(e, Se, Sw, w)
        S_ssw = calculate_area(P, s, sW, W)
        S_sse = calculate_area(E, sE, s, P)

        D3 = (dy(sw, se) * (dy(Se, e) / 4) / S_s + dx(sw, se) * (dx(Se, e) / 4) / S_s +
            dy(se, e) * (dy(s, sE) / 4 + 3 * dy(sE, E) / 4 + dy(E, P) / 2) / S_sse +
            dx(se, e) * (dx(s, sE) / 4 + 3 * dx(sE, E) / 4 + dx(E, P) / 2) / S_sse) / S_ss

        D_3 = (dy(w, sw) * (3 * dy(W, sW) / 4 + dy(sW, s) / 4 + dy(P, W) / 2) / S_ssw +
            dx(w, sw) * (3 * dx(W, sW) / 4 + dx(sW, s) / 4 + dx(P, W) / 2) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4) / S_s + dx(sw, se) * (dx(w, Sw) / 4) / S_s) / S_ss

        D1 = (dy(w, sw) * (dy(sW, s) / 4 + dy(s, P) / 4) / S_ssw +
            dx(w, sw) * (dx(sW, s) / 4 + dx(s, P) / 4) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4 + dy(Sw, Se) + dy(Se, e) / 4) / S_s +
            dx(sw, se) * (dx(w, Sw) / 4 + dx(Sw, Se) + dx(Se, e) / 4) / S_s +
            dy(se, e) * (dy(P, s) / 4 + dy(s, sE) / 4) / S_sse +
            dx(se, e) * (dx(P, s) / 4 + dx(s, sE) / 4) / S_sse) / S_ss

        D_2 = (dy(w, sw) * (dy(W, sW) / 4 + dy(sW, s) / 4) / S_ssw +
            dx(w, sw) * (dx(W, sW) / 4 + dx(sW, s) / 4) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4) / S_s + dx(sw, se) * (dx(w, Sw) / 4) / S_s) / S_ss

        D4 = (dy(sw, se) * (dy(Se, e) / 4) / S_s + dx(sw, se) * (dx(Se, e) / 4) / S_s +
            dy(se, e) * (dy(s, sE) / 4 + dy(sE, E) / 4) / S_sse +
            dx(se, e) * (dx(s, sE) / 4 + dx(sE, E) / 4) / S_sse) / S_ss

        coefficient = 0.0
        if self.boundary[0] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(e, w) / S_ss
        elif self.boundary[0] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(e, w) / S_ss
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[0]}')
```

```
        D0 = (coefficient * dist(e, w) +
            dy(w, sw) * (dy(sW, s) / 4 + 3 * dy(s, P) / 4 + dy(P, W) / 2) / S_ssw +
            dx(w, sw) * (dx(sW, s) / 4 + 3 * dx(s, P) / 4 + dx(P, W) / 2) / S_ssw +
            dy(sw, se) * (dy(w, Sw) / 4 + dy(Se, e) / 4 + dy(e, w)) / S_s +
            dx(sw, se) * (dx(w, Sw) / 4 + dx(Se, e) / 4 + dx(e, w)) / S_s +
            dy(se, e) * (3 * dy(P, s) / 4 + dy(s, sE) / 4 + dy(E, P) / 2) / S_sse +
            dx(se, e) * (3 * dx(P, s) / 4 + dx(s, sE) / 4 + dx(E, P) / 2) / S_sse) / S_ss

        stencil[self.index(i, j)] = D0
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j-1)] = D_3
        stencil[self.index(i, j+1)] = D3
        stencil[self.index(i+1, j-1)] = D_2
        stencil[self.index(i+1, j+1)] = D4

        return stencil, b[0]

    def build_south(self, i, j):
        """Build stencil for south boundary (i=m-1)"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

        Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
        nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)

        n  = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        w  = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        e  = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
        nw = Coordinate2D((Nw.x + w.x)/2, (Nw.y + w.y)/2)

        S_nn  = self.stable_area(e, ne, nw, w)
        S_n   = self.stable_area(e, Ne, Nw, w)
        S_nnw = self.stable_area(P, n, nW, W)
        S_nne = self.stable_area(E, nE, n, P)

        D3 = (dy(nw, ne) * dy(Ne, e) / 4 / S_n + dx(nw, ne) * dx(Ne, e) / 4 / S_n +
            dy(ne, e) * (dy(n, nE)/4 + 3*dy(nE, E)/4 + dy(E, P)/2) / S_nne +
            dx(ne, e) * (dx(n, nE)/4 + 3*dx(nE, E)/4 + dx(E, P)/2) / S_nne) / S_nn

        D_3 = (dy(w, nw) * (3*dy(W, nW)/4 + dy(nW, n)/4 + dy(P, W)/2) / S_nnw +
            dx(w, nw) * (3*dx(W, nW)/4 + dx(nW, n)/4 + dx(P, W)/2) / S_nnw +
            dy(nw, ne) * dy(w, Nw)/4 / S_n + dx(nw, ne) * dx(w, Nw)/4 / S_n) / S_nn

        D_1 = (dy(w, nw) * (dy(nW, n)/4 + dy(n, P)/4) / S_nnw +
            dx(w, nw) * (dx(nW, n)/4 + dx(n, P)/4) / S_nnw +
            dy(nw, ne) * (dy(w, Nw)/4 + dy(Nw, Ne) + dy(Ne, e)/4) / S_n +
            dx(nw, ne) * (dx(w, Nw)/4 + dx(Nw, Ne) + dx(Ne, e)/4) / S_n +
            dy(ne, e) * (dy(P, n)/4 + dy(n, nE)/4) / S_nne +
            dx(ne, e) * (dx(P, n)/4 + dx(n, nE)/4) / S_nne) / S_nn

        D4 = (dy(w, nw) * (dy(W, nW)/4 + dy(nW, n)/4) / S_nnw +
            dx(w, nw) * (dx(W, nW)/4 + dx(nW, n)/4) / S_nnw +
            dy(nw, ne) * dy(w, Nw)/4 / S_n + dx(nw, ne) * dx(w, Nw)/4 / S_n) / S_nn

        D2 = (dy(nw, ne) * dy(Ne, e)/4 / S_n + dx(nw, ne) * dx(Ne, e)/4 / S_n +
            dy(ne, e) * (dy(n, nE)/4 + dy(nE, E)/4) / S_nne +
            dx(ne, e) * (dx(n, nE)/4 + dx(nE, E)/4) / S_nne) / S_nn

        if self.boundary[1] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(e, w) / S_nn
        elif self.boundary[1] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(e, w) / S_nn
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[1]}')

        D0 = (coefficient * dist(e, w) +
            dy(w, nw) * (dy(nW, n)/4 + 3*dy(n, P)/4 + dy(P, W)/2) / S_nnw +
            dx(w, nw) * (dx(nW, n)/4 + 3*dx(n, P)/4 + dx(P, W)/2) / S_nnw +
            dy(nw, ne) * (dy(w, Nw)/4 + dy(Ne, e)/4 + dy(e, w)) / S_n +
            dx(nw, ne) * (dx(w, Nw)/4 + dx(Ne, e)/4 + dx(e, w)) / S_n +
            dy(ne, e) * (3*dy(P, n)/4 + dy(n, nE)/4 + dy(E, P)/2) / S_nne +
            dx(ne, e) * (3*dx(P, n)/4 + dx(n, nE)/4 + dx(E, P)/2) / S_nne) / S_nn

        stencil[self.index(i, j)]     = D0
        stencil[self.index(i-1, j)]   = D_1
        stencil[self.index(i, j-1)]   = D_3
        stencil[self.index(i, j+1)]   = D3
        stencil[self.index(i-1, j-1)] = D4
        stencil[self.index(i-1, j+1)] = D2

        return stencil, b[0]

    def build_west(self, i, j):
        """Build stencil for west boundary"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
```

```
        N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        E = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])

        Se = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        Ne = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        sE = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)
        nE = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)

        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        e = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)

        ne = Coordinate2D((Ne.x + e.x)/2, (Ne.y + e.y)/2)
        se = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)

        S_ww = self.stable_area(s, n, ne, se)
        S_w = self.stable_area(s, n, nE, sE)
        S_wws = self.stable_area(S, P, e, Se)
        S_wwn = self.stable_area(P, N, Ne, e)

        D_1 = -(dy(se, ne) * (dy(nE, n) / 4) / S_w + dx(se, ne) * (dx(nE, n) / 4) / S_w +
                dy(ne, n) * (dy(e, Ne) / 4 + 3 * dy(Ne, N) / 4 + dy(N, P) / 2) / S_wwn +
                dx(ne, n) * (dx(e, Ne) / 4 + 3 * dx(Ne, N) / 4 + dx(N, P) / 2) / S_wwn) / S_ww

        D1 = -(dy(se, ne) * (dy(s, sE) / 4) / S_w + dx(se, ne) * (dx(s, sE) / 4) / S_w +
                dy(se, s) * (dy(sE, S) / 4 + 3 * dy(Se, S) / 4 + dy(S, P) / 2) / S_wws +
                dx(se, s) * (dx(sE, S) / 4 + 3 * dx(Se, S) / 4 + dx(S, P) / 2) / S_wws) / S_ww

        D3 = -(dy(s, se) * (dy(Se, e) / 4 + dy(e, P) / 4) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + dx(e, P) / 4) / S_wws +
                dy(se, ne) * (dy(s, sE) / 4 + dy(sE, nE) + dy(nE, n) / 4) / S_w +
                dx(se, ne) * (dx(s, sE) / 4 + dx(sE, nE) + dx(nE, n) / 4) / S_w +
                dy(ne, n) * (dy(P, e) / 4 + dy(e, Ne) / 4) / S_wwn +
                dx(ne, n) * (dx(P, e) / 4 + dx(e, Ne) / 4) / S_wwn) / S_ww

        D2 = -(dy(ne, n) * (dy(e, Ne) / 4 + dy(Ne, N) / 4) / S_wwn +
                dx(ne, n) * (dx(e, Ne) / 4 + dx(Ne, N) / 4) / S_wwn +
                dy(se, ne) * (dy(nE, n) / 4) / S_w + dx(se, ne) * (dx(nE, n) / 4) / S_w) / S_ww

        D4 = -(dy(se, ne) * (dy(s, sE) / 4) / S_w + dx(se, ne) * (dx(Se, s) / 4) / S_w +
                dy(s, se) * (dy(Se, e) / 4 + dy(Se, S) / 4) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + dx(Se, S) / 4) / S_wws) / S_ww

        if self.boundary[2] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(n, s) / S_ww
        elif self.boundary[2] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(n, s) / S_ww
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[2]}')

        D0 = (coefficient * dist(ne, se) +
                dy(s, se) * (dy(Se, e) / 4 + 3 * dy(e, P) / 4 + dy(P, S) / 2) / S_wws +
                dx(s, se) * (dx(Se, e) / 4 + 3 * dx(e, P) / 4 + dx(P, S) / 2) / S_wws +
                dy(se, ne) * (dy(s, sE) / 4 + dy(nE, n) / 4 + dy(n, s)) / S_w +
                dx(se, ne) * (dx(s, sE) / 4 + dx(nE, n) / 4 + dx(n, s)) / S_w +
                dy(ne, n) * (3 * dy(P, e) / 4 + dy(e, Ne) / 4 + dy(N, P) / 2) / S_wwn +
                dx(ne, n) * (3 * dx(P, e) / 4 + dx(e, Ne) / 4 + dx(N, P) / 2) / S_wwn) / S_ww

        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j+1)] = D3
        stencil[self.index(i-1, j+1)] = D2
        stencil[self.index(i+1, j+1)] = D4

        return stencil, b[0]

    def build_east(self, i, j):
        """Build stencil for east boundary"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P = Coordinate2D(self.X[i, j], self.Y[i, j])
        S = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        N = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])

        Sw = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        Nw = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        sW = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        nW = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)

        s = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        n = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)

        nw = Coordinate2D((nW.x + n.x)/2, (nW.y + n.y)/2)
        sw = Coordinate2D((sW.x + s.x)/2, (sW.y + s.y)/2)

        S_ee = self.stable_area(s, sw, nw, n)
        S_e = self.stable_area(n, s, sW, nW)
        S_ees = self.stable_area(P, S, Sw, w)
        S_een = self.stable_area(N, P, w, Nw)

        D_1 = (dy(nw, sw) * (dy(n, nW) / 4) / S_e + dx(nw, sw) * (dx(n, nW) / 4) / S_e +
                dy(n, nw) * (dy(Nw, w) / 4 + 3 * dy(N, Nw) / 4 + dy(P, N) / 2) / S_een +
                dx(n, nw) * (dx(Nw, w) / 4 + 3 * dx(N, Nw) / 4 + dx(P, N) / 2) / S_een) / S_ee

        D1 = (dy(nw, sw) * (dy(sW, s) / 4) / S_e + dx(nw, sw) * (dx(sW, s) / 4) / S_e +
                dy(sw, s) * (dy(w, Sw) / 4 + 3 * dy(Sw, S) / 4 + dy(S, P) / 2) / S_ees +
                dx(sw, s) * (dx(w, Sw) / 4 + 3 * dx(Sw, S) / 4 + dx(S, P) / 2) / S_ees) / S_ee
```

```
        D_3 = (dy(sw, s) * (dy(w, Sw) / 4 + dy(P, w) / 4) / S_ees +
               dx(sw, s) * (dx(w, Sw) / 4 + dx(P, w) / 4) / S_ees +
               dy(nw, sw) * (dy(sW, s) / 4 + dy(nW, sW) + dy(n, nW) / 4) / S_e +
               dx(nw, sw) * (dx(sW, s) / 4 + dx(nW, sW) + dx(n, nW) / 4) / S_e +
               dy(n, nw) * (dy(w, P) / 4 + dy(Nw, w) / 4) / S_een +
               dx(n, nw) * (dx(w, P) / 4 + dx(Nw, w) / 4) / S_een) / S_ee

        D_4 = (dy(n, nw) * (dy(N, Nw) / 4 + dy(Nw, w) / 4) / S_een +
               dx(n, nw) * (dx(N, Nw) / 4 + dx(Nw, w) / 4) / S_een +
               dy(nw, sw) * (dy(n, nW) / 4) / S_e + dx(nw, sw) * (dx(n, nW) / 4) / S_e) / S_ee

        D_2 = (dy(nw, sw) * (dy(sW, s) / 4) / S_e + dx(nw, sw) * (dx(sW, s) / 4) / S_e +
               dy(sw, s) * (dy(Sw, S) / 4 + dy(w, Sw) / 4) / S_ees +
               dx(sw, s) * (dx(Sw, S) / 4 + dx(w, Sw) / 4) / S_ees) / S_ee

        if self.boundary[3] == 'N':
            coefficient = 0.0
            b[0] = self.q * dist(n, s) / S_ee
        elif self.boundary[3] == 'R':
            coefficient = -self.alpha
            b[0] = -self.alpha * self.Tinf * dist(n, s) / S_ee
        else:
            raise ValueError(f'Unknown boundary type: {self.boundary[3]}')

        D0 = (coefficient * dist(nw, sw) +
               dy(sw, s) * (dy(w, Sw) / 4 + 3 * dy(P, w) / 4 + dy(S, P) / 2) / S_ees +
               dx(sw, s) * (dx(w, Sw) / 4 + 3 * dx(P, w) / 4 + dx(S, P) / 2) / S_ees +
               dy(nw, sw) * (dy(sW, s) / 4 + dy(n, nW) / 4 + dy(s, n)) / S_e +
               dx(nw, sw) * (dx(sW, s) / 4 + dx(n, nW) / 4 + dx(s, n)) / S_e +
               dy(n, nw) * (3 * dy(w, P) / 4 + dy(Nw, w) / 4 + dy(P, N) / 2) / S_een +
               dx(n, nw) * (3 * dx(w, P) / 4 + dx(Nw, w) / 4 + dx(P, N) / 2) / S_een) / S_ee

        stencil[self.index(i, j)] = D0
        stencil[self.index(i-1, j)] = D_1
        stencil[self.index(i+1, j)] = D1
        stencil[self.index(i, j-1)] = D_3
        stencil[self.index(i-1, j-1)] = D_4
        stencil[self.index(i+1, j-1)] = D_2

        return stencil, b[0]

    def build_NW(self, i, j):
        """Build stencil for North-West corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        S  = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        SE = Coordinate2D(self.X[i+1, j+1], self.Y[i+1, j+1])

        s   = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        e   = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)
        Se  = Coordinate2D((S.x + SE.x)/2, (S.y + SE.y)/2)
        sE  = Coordinate2D((E.x + SE.x)/2, (E.y + SE.y)/2)
        se  = Coordinate2D((Se.x + e.x)/2, (Se.y + e.y)/2)

        S_nw  = self.stable_area(P, s, se, e)
        S_nws = self.stable_area(P, S, Se, e)
        S_nwe = self.stable_area(P, s, sE, E)

        coeff_N = coeff_W = 0.0
        b[0] = 0.0

        if self.boundary[0] == 'N':
            b[0] += self.q * dist(e, P) / S_nw
        elif self.boundary[0] == 'R':
            coeff_N = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(e, P) / S_nw

        if self.boundary[2] == 'N':
            b[0] += self.q * dist(P, s) / S_nw
        elif self.boundary[2] == 'R':
            coeff_W = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, s) / S_nw

        D0 = (
            coeff_N * dist(e, P) +
            coeff_W * dist(s, P) +
            dy(s, se) * (dy(Se, e) / 4 + 3 * dy(e, P) / 4 + dy(P, S) / 2) / S_nws +
            dx(s, se) * (dx(Se, e) / 4 + 3 * dx(e, P) / 4 + dx(P, S) / 2) / S_nws +
            dy(se, e) * (3 * dy(s, P)/4 + dy(sE,s)/4 + dy(P, E)/2) / S_nwe +
            dx(se, e) * (3 * dx(s, P)/4 + dx(sE,s)/4 + dx(P, E)/2) / S_nwe
        ) / S_nw

        D1 = (
            dy(s, se) * (dy(e, Se) / 4 + 3 * dy(Se, S) / 4 + dy(S, P) / 2) / S_nws +
            dx(s, se) * (dx(e, Se) / 4 + 3 * dx(Se, S) / 4 + dx(S, P) / 2) / S_nws +
            dy(se, e) * (dy(s, sE)/4 + dy(sE, E)/4) / S_nwe +
            dx(se, e) * (dx(s, sE)/4 + dx(sE, E)/4) / S_nwe
        ) / S_nw

        D3 = (
            dy(se, e) * (dy(s, sE) / 4 + 3 * dy(sE, E) / 4 + dy(E, P) / 2) / S_nwe +
            dx(se, e) * (dx(s, sE) / 4 + 3 * dx(sE, E) / 4 + dx(E, P) / 2) / S_nwe +
            dy(s, se) * (dy(e, Se)) /4 / S_nws +
            dx(s, se) * (dx(e, Se)) /4 / S_nws
        ) / S_nw

        D4 = (
            dy(s, se) * (dy(Se, S) / 4 + dy(e, Se) / 4) / S_nws +
```

```
            dx(s, se) * (dx(Se, S) / 4 + dx(e, Se) / 4) / S_nws +
            dy(se, e) * (dy(s, sE) / 4 + dy(sE, E) / 4) / S_nwe +
            dx(se, e) * (dx(s, sE) / 4 + dx(sE, E) / 4) / S_nwe
        ) / S_nw

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i+1, j)]     = D1
        stencil[self.index(i, j+1)]     = D3
        stencil[self.index(i+1, j+1)]   = D4

        return stencil, b[0]

    def build_NE(self, i, j):
        """Build stencil for North-East corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[0] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[0]
            return stencil, b[0]

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        S  = Coordinate2D(self.X[i+1, j], self.Y[i+1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        SW = Coordinate2D(self.X[i+1, j-1], self.Y[i+1, j-1])

        s   = Coordinate2D((S.x + P.x)/2, (S.y + P.y)/2)
        w   = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        Sw  = Coordinate2D((S.x + SW.x)/2, (S.y + SW.y)/2)
        sW  = Coordinate2D((W.x + SW.x)/2, (W.y + SW.y)/2)
        sw  = Coordinate2D((sW.x + s.x)/2, (sW.y + s.y)/2)

        S_ne  = self.stable_area(w, sw, s, P)
        S_nes = self.stable_area(P, w, Sw, S)
        S_new = self.stable_area(P, W, sW, s)

        coeff_N = coeff_E = 0.0
        b[0] = 0.0

        if self.boundary[0] == 'N':
            b[0] += self.q * dist(w, P) / S_ne
        elif self.boundary[0] == 'R':
            coeff_N = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(w, P) / S_ne
        if self.boundary[3] == 'N':
            b[0] += self.q * dist(P, s) / S_ne
        elif self.boundary[3] == 'R':
            coeff_E = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, s) / S_ne
        D0 = (
            coeff_N * dist(w, P) +
            coeff_E * dist(P, s) +
            dy(sw, s) * (3 * dy(P, w) / 4 + dy(w, Sw) / 4 +  dy(S, P) / 2) / S_nes +
            dx(sw, s) * (3 * dx(P, w) / 4 + dx(w, Sw) / 4  + dx(S, P) / 2) / S_nes +
            dy(w, sw) * (3 * dy(s, P) / 4 + dy(sW, s) / 4  + dy(P, W) / 2) / S_new +
            dx(w, sw) * (3 * dx(s, P) / 4 + dx(sW, s) / 4 +  dx(P, W) / 2) / S_new
        ) / S_ne

        D1 = (
            dy(sw, s) * (3 * dy(Sw, S) / 4 + dy(w, Sw) / 4  + dy(S, P) / 2) / S_nes +
            dx(sw, s) * (3 * dx(Sw, S) / 4 + dx(w, Sw) / 4 +  dx(S, P) / 2) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + dy(s, P) / 4) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + dx(s, P) / 4) / S_new
        ) / S_ne

        D_3 = (
            dy(sw, s) * (dy(w, Sw) / 4 + dy(P, w) / 4) / S_nes +
            dx(sw, s) * (dx(w, Sw) / 4 + dx(P, w) / 4) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + 3 * dy(W, sW) / 4 + dy(P, W) / 2) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + 3 * dx(W, sW) / 4 + dx(P, W) / 2) / S_new
        ) / S_ne

        D_2 = (
            dy(sw, s) * (dy(Sw, S) / 4 + dy(w, Sw) / 4) / S_nes +
            dx(sw, s) * (dx(Sw, S) / 4 + dx(w, Sw) / 4) / S_nes +
            dy(w, sw) * (dy(sW, s) / 4 + dy(W, sW) / 4) / S_new +
            dx(w, sw) * (dx(sW, s) / 4 + dx(W, sW) / 4) / S_new
        ) / S_ne

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i+1, j)]     = D1
        stencil[self.index(i, j-1)]     = D_3
        stencil[self.index(i+1, j-1)]   = D_2

        return stencil, b[0]

    def build_SW(self, i, j):
        """Build stencil for South-West corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        if self.boundary[2] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[2]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        E  = Coordinate2D(self.X[i, j+1], self.Y[i, j+1])
        NE = Coordinate2D(self.X[i-1, j+1], self.Y[i-1, j+1])
```

```
        n   = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        e   = Coordinate2D((E.x + P.x)/2, (E.y + P.y)/2)
        Ne  = Coordinate2D((N.x + NE.x)/2, (N.y + NE.y)/2)
        nE  = Coordinate2D((E.x + NE.x)/2, (E.y + NE.y)/2)
        ne  = Coordinate2D((nE.x + n.x)/2, (nE.y + n.y)/2)

        S_sw  = self.stable_area(e, ne, n, P)
        S_swn = self.stable_area(P, e, Ne, N)
        S_swe = self.stable_area(P, E, nE, n)

        coeff_S = coeff_W = 0.0
        b[0] = 0.0

        if self.boundary[1] == 'N':
            b[0] += self.q * dist(e, P) / S_sw
        elif self.boundary[1] == 'R':
            coeff_S = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(e, P) / S_sw

        if self.boundary[2] == 'N':
            b[0] += self.q * dist(P, n) / S_sw
        elif self.boundary[2] == 'R':
            coeff_W = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, n) / S_sw
        D0 = (
            coeff_S * dist(e, P) +
            coeff_W * dist(P, n) +
            dy(n, ne) * (dy(Ne, e) / 4 + 3 * dy(e, P) / 4 + dy(P, N) / 2) / S_swn +
            dx(n, ne) * (dx(Ne, e) / 4 + 3 * dx(e, P) / 4 + dx(P, N) / 2) / S_swn +
            dy(ne, e) * (dy(nE, n) / 4 + 3 * dy(n, P) / 4 + dy(P, E) / 2) / S_swe +
            dx(ne, e) * (dx(nE, n) / 4 + 3 * dx(n, P) / 4 + dx(P, E) / 2) / S_swe
        ) / S_sw

        D_1 = ( dy(n, ne) * (dy(e, Ne) / 4 + 3 * dy(Ne, N) / 4 + dy(N, P) / 2) / S_swn +
                dx(n, ne) * (dx(e, Ne) / 4 + 3 * dx(Ne, N) / 4 + dx(N, P) / 2) / S_swn +
                dy(ne, e) * (dy(n, nE)/4 ) / S_swe +
                dx(ne, e) * (dx(n, nE)/4  / S_swe )
        )/ S_sw

        D3 = ( dy(n, ne) * (dy(e, Ne) / 4) / S_swn +
                dx(n, ne) * (dx(e, Ne) / 4) / S_swn +
                dy(ne, e) * (dy(n, nE) / 4 + 3 * dy(nE, E) / 4 +  dy(E, P) / 2) / S_swe +
                dx(ne, e) * (dx(n, nE) / 4 + 3 * dx(nE, E) / 4 + dx(E, P) / 2) / S_swe
            ) / S_sw

        D2 = (  dy(n, ne) * (dy(Ne, N) / 4 + dy(e, Ne) / 4) / S_swn +
                dx(n, ne) * (dx(Ne, N) / 4 + dx(e, Ne) / 4) / S_swn +
                dy(ne, e) * (dy(n, nE) / 4 + dy(nE, E) / 4) / S_swe +
                dx(ne, e) * (dx(n, nE) / 4 + dx(nE, E) / 4) / S_swe
            ) / S_sw

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i-1, j)]     = D_1
        stencil[self.index(i, j+1)]     = D3
        stencil[self.index(i-1, j+1)]   = D2

        return stencil, b[0]

    def build_SE(self, i, j):
        """Build stencil for South-East corner"""
        stencil = np.zeros(self.m * self.n)
        b = np.zeros(1)

        if self.boundary[1] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[1]
            return stencil, b[0]

        if self.boundary[3] == 'D':
            stencil[self.index(i, j)] = 1.0
            b[0] = self.TD[3]
            return stencil, b[0]

        P  = Coordinate2D(self.X[i, j], self.Y[i, j])
        N  = Coordinate2D(self.X[i-1, j], self.Y[i-1, j])
        W  = Coordinate2D(self.X[i, j-1], self.Y[i, j-1])
        NW = Coordinate2D(self.X[i-1, j-1], self.Y[i-1, j-1])

        n   = Coordinate2D((N.x + P.x)/2, (N.y + P.y)/2)
        w   = Coordinate2D((W.x + P.x)/2, (W.y + P.y)/2)
        Nw  = Coordinate2D((N.x + NW.x)/2, (N.y + NW.y)/2)
        nW  = Coordinate2D((W.x + NW.x)/2, (W.y + NW.y)/2)
        nw  = Coordinate2D((nW.x + n.x)/2, (nW.y + n.y)/2)

        S_se  = self.stable_area(P, n, nw, w)
        S_sen = self.stable_area(P, N, Nw, w)
        S_sew = self.stable_area(P, n, nW, W)

        coeff_S = coeff_E = 0.0
        b[0] = 0.0


        if self.boundary[1] == 'N':
            b[0] += self.q * dist(w, P) / S_se
        elif self.boundary[1] == 'R':
            coeff_S = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(w, P) / S_se

        if self.boundary[3] == 'N':
            b[0] += self.q * dist(P, n) / S_se
        elif self.boundary[3] == 'R':
            coeff_E = -self.alpha
            b[0] += -self.alpha * self.Tinf * dist(P, n) / S_se

        D0 = (
            coeff_S * dist(w, P) +
            coeff_E * dist(P, n) +
            dy(nw, w) * (dy(W, P) / 2 + dy(n, nW) / 4 + 3 * dy(P, n) / 4) / S_sew +
            dx(nw, w) * (dx(W, P) / 2 + dx(n, nW) / 4 + 3 * dx(P, n) / 4) / S_sew +
            dy(n, nw) * (3 * dy(w, P) / 4 + dy(Nw, w) / 4 + dy(P, N) / 2) / S_sen +
            dx(n, nw) * (3 * dx(w, P) / 4 + dx(Nw, w) / 4 + dx(P, N) / 2) / S_sen
        ) / S_se
```

```
        D_1 = (
            dy(nw, w) * (dy(n, nW) / 4 + dy(P, n) / 4) / S_sew +
            dx(nw, w) * (dx(n, nW) / 4 + dx(P, n) / 4) / S_sew +
            dy(n, nw) * (dy(Nw, w) / 4 + 3 * dy(N, Nw) / 4 + dy(P, N) / 2) / S_sen +
            dx(n, nw) * (dx(Nw, w) / 4 + 3 * dx(N, Nw) / 4 + dx(P, N) / 2) / S_sen
        ) / S_se

        D_3 = (
            dy(nw, w) * (dy(W, P) / 2 + 3 * dy(nW, W) / 4 + dy(n, nW) / 4) / S_sew +
            dx(nw, w) * (dx(W, P) / 2 + 3 * dx(nW, W) / 4 + dx(n, nW) / 4) / S_sew +
            dy(n, nw) * (dy(w, P) / 4 + dy(Nw, w) / 4) / S_sen +
            dx(n, nw) * (dx(w, P) / 4 + dx(Nw, w) / 4) / S_sen
        ) / S_se

        D_4 = (
            dy(n, nw) * (dy(N, Nw) / 4 + dy(Nw, w) / 4) / S_sen +
            dx(n, nw) * (dx(N, Nw) / 4 + dx(Nw, w) / 4) / S_sen +
            dy(nw, w) * (dy(n, nW) / 4 + dy(nW, W) / 4) / S_sew +
            dx(nw, w) * (dx(n, nW) / 4 + dx(nW, W) / 4) / S_sew
        ) / S_se

        stencil[self.index(i, j)]       = D0
        stencil[self.index(i-1, j)]     = D_1
        stencil[self.index(i, j-1)]     = D_3
        stencil[self.index(i-1, j-1)]   = D_4

        return stencil, b[0]

    def assemble_spatial_operator(self):
        N = self.total_nodes
        self.Matrix = sp.lil_matrix((N, N))
        self.RHS = np.zeros(N)

        for i in range(self.m):
            for j in range(self.n):
                idx = self.index(i, j)
                stencil, b_val = self.set_stencil(i, j)

                # Identify Dirichlet from stencil pattern
                is_dirichlet = abs(stencil[idx] - 1.0) < 1e-10

                if is_dirichlet:
                    # Dirichlet: dT/dt = 0 (row of zeros in L, zero in f)
                    # Don't add any matrix entries for this row
                    self.RHS[idx] = 0.0
                else:
                    # Apply physical conductivity to geometric stencil
                    for k, coeff in enumerate(stencil):
                        if abs(coeff) > 1e-15:
                            self.Matrix[idx, k] = self.lambda_conductivity * coeff
                    self.RHS[idx] = - self.lambda_conductivity * b_val

        self.Matrix = self.Matrix.tocsr()

        # Convert to CSR for efficient operations
        self.Matrix = self.Matrix.tocsr()

    def enforce_boundary_values(self, T_flat):
        """Forcefully set Dirichlet values to prevent drift"""
        if self.boundary[0] == 'D':  # North
            for j in range(self.n):
                T_flat[self.index(0, j)] = self.TD[0]
        if self.boundary[1] == 'D':  # South
            for j in range(self.n):
                T_flat[self.index(self.m-1, j)] = self.TD[1]
        if self.boundary[2] == 'D':  # West
            for i in range(self.m):
                T_flat[self.index(i, 0)] = self.TD[2]
        if self.boundary[3] == 'D':  # East
            for i in range(self.m):
                T_flat[self.index(i, self.n-1)] = self.TD[3]

    def solve(self, state="steady", t_end=None, dt=None, T0=None, time_scheme="euler", output_times=None):
        """
        Solve the heat equation.

        Args:
            state: "steady" or "unsteady"
            t_end: end time for unsteady (required if state="unsteady")
            dt: time step (required if state="unsteady")
            T0: initial condition for unsteady (m×n array or None for zeros)
            time_scheme: "euler", "rk4", or "crank-nicolson"/"cn"
            output_times: list of times to record snapshots (optional)

        Returns:
            steady: T (m×n array)
            unsteady without output_times: T_final (m×n array)
            unsteady with output_times: list of (time, T) tuples
        """
        print("Assembling linear system...")

        # ========== STEADY STATE ==========
        if state == "steady":
            for i in range(self.m):
                for j in range(self.n):
                    idx = self.index(i, j)
                    stencil, b_val = self.set_stencil(i, j)

                    for k, coeff in enumerate(stencil):
                        if abs(coeff) > 1e-15:
                            self.A[idx, k] = coeff

                    self.B[idx] = b_val

            print(f"Matrix size: {self.A.shape}, Non-zero elements: {self.A.nnz}")

            # Solve with regularization
            A_csr = self.A.tocsr() + sp.eye(self.A.shape[0]) * 1e-12

            print("Solving linear system (steady)...")
            try:
```

```
            T_flat = spsolve(A_csr, self.B)
            T = T_flat.reshape(self.m, self.n)
            print("Steady solution completed successfully.")
            return T
        except Exception as e:
            print(f"Error solving linear system: {e}")
            return np.zeros((self.m, self.n))

    # ========== UNSTEADY ==========
    elif state == "unsteady":
        if t_end is None or dt is None:
            raise ValueError("Provide t_end and dt for unsteady solve")


        if self.Matrix is None or self.RHS is None:
            self.assemble_spatial_operator()

        # Check stability for explicit schemes
        if time_scheme in ["euler", "rk4"]:
            dx_min = np.min(np.abs(np.diff(self.X[0, :])))
            dy_min = np.min(np.abs(np.diff(self.Y[:, 0])))
            dx_grid = min(dx_min, dy_min)
            # Assuming rho*cp = 1.0, so diffusivity = lambda
            dt_max = dx_grid**2 / (4 * self.lambda_conductivity)

            print(f"Grid spacing: {dx_grid:.6e}")
            print(f"Max stable dt: {dt_max:.6e}")
            print(f"Your dt: {dt:.6e}")

            if dt > dt_max:
                print(f"⚠️  WARNING: dt exceeds stability limit!")
                print(f"   Solution may become unstable/diverge")

        # Initial Condition
        T_n = np.zeros(self.total_nodes) if T0 is None else np.array(T0).flatten()
        self.enforce_boundary_values(T_n)

        # Setup History Recording
        history = []
        current_time = 0.0
        output_times_remaining = None

        if output_times is not None:
            output_times_remaining = sorted(output_times)
            # Save t=0 if requested
            if output_times_remaining and abs(output_times_remaining[0]) < 1e-9:
                history.append((0.0, T_n.reshape(self.m, self.n).copy()))
                output_times_remaining.pop(0)

        # Time Stepping
        num_steps = int(np.ceil(t_end / dt))
        L = self.Matrix
        f = self.RHS

        # Pre-build identity for Crank-Nicolson
        if time_scheme in ["cn", "crank-nicolson"]:
            I = sp.eye(self.total_nodes, format="csr")
            A_imp = I - 0.5 * dt * L

        print(f"Time-stepping ({time_scheme}): {num_steps} steps, dt={dt:.6e}")

        for step in range(num_steps):
            current_time += dt

            # ===== Time integration =====
            if time_scheme == "euler":
                dTdt = L.dot(T_n) + f
                T_n = T_n + dt * dTdt

            elif time_scheme == "rk4":
                k1 = L.dot(T_n) + f
                k2 = L.dot(T_n + 0.5 * dt * k1) + f
                k3 = L.dot(T_n + 0.5 * dt * k2) + f
                k4 = L.dot(T_n + dt * k3) + f
                T_n = T_n + (dt / 6.0) * (k1 + 2*k2 + 2*k3 + k4)

            elif time_scheme in ["cn", "crank-nicolson"]:
                rhs = (I + 0.5 * dt * L).dot(T_n) + dt * f
                T_n = spsolve(A_imp, rhs)

            else:
                raise ValueError(f"Unknown time_scheme: {time_scheme}")

            # Re-enforce Dirichlet BCs (prevents numerical drift)
            self.enforce_boundary_values(T_n)

            # Record snapshots at requested times
            if output_times_remaining:
                # Check if we just passed a target time (or are very close)
                while output_times_remaining and current_time >= output_times_remaining[0] - dt*1e-6:
                    t_out = output_times_remaining.pop(0)
                    history.append((t_out, T_n.reshape(self.m, self.n).copy()))

            # Progress indicator
            if (step + 1) % max(1, num_steps // 10) == 0:
                print(f"  Step {step+1}/{num_steps} (t={current_time:.4f}) - "
                      f"T_range=[{T_n.min():.2f}, {T_n.max():.2f}]")

        print("Unsteady solution completed successfully.")

        if output_times is not None:
            return history
        else:
            return T_n.reshape(self.m, self.n)

    else:
        raise ValueError("state must be 'steady' or 'unsteady'")

def plot_Result(self, T, plot_type="2D"):
  # Create 2D and 3D plots
 X, Y = self.X, self.Y
 T_plot = np.array(T).reshape(X.shape)
```

```
    # Use actual temperature range
    vmin = np.min(T_plot)
    vmax = np.max(T_plot)

    if plot_type == "2D":
        plt.figure(figsize=(8, 6))

        # Create symmetric mesh by combining original and flipped
        X_combined = np.vstack([X, X])
        Y_combined = np.vstack([Y, -Y])
        T_combined = np.vstack([T_plot, T_plot])

        pcm = plt.pcolormesh(X_combined, Y_combined, T_combined,
                             shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
        plt.colorbar(pcm, label="Temperature")
        plt.xlabel("X-axis")
        plt.ylabel("Y-axis")
        plt.title("Temperature Distribution")
        plt.axis('equal')
        plt.tight_layout()
        plt.show()

    elif plot_type == "3D":
        fig = plt.figure(figsize=(10, 7))
        ax = fig.add_subplot(111, projection='3d')

        # Plot original surface
        surf1 = ax.plot_surface(X, Y, T_plot, cmap='jet',
                                edgecolor='none', antialiased=True,
                                vmin=vmin, vmax=vmax)

        # Plot symmetric surface (y-flipped)
        surf2 = ax.plot_surface(X, -Y, T_plot, cmap='jet',
                                edgecolor='none', antialiased=True,
                                vmin=vmin, vmax=vmax)

        ax.set_xlabel("X-axis")
        ax.set_ylabel("Y-axis")
        ax.set_zlabel("Temperature")
        ax.set_title("3D Temperature Distribution")
        ax.view_init(elev=45, azim=300)
        plt.colorbar(surf1, label="Temperature")
        plt.tight_layout()
        plt.show()

    elif plot_type == "both":
        # Create a figure with two subplots
        fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(16, 6))

        # 2D plot on left subplot
        X_combined = np.vstack([X, X])
        Y_combined = np.vstack([Y, -Y])
        T_combined = np.vstack([T_plot, T_plot])

        pcm = ax1.pcolormesh(X_combined, Y_combined, T_combined,
                             shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
        plt.colorbar(pcm, ax=ax1, label="Temperature")
        ax1.set_xlabel("X-axis")
        ax1.set_ylabel("Y-axis")
        ax1.set_title("2D Temperature Distribution")
        ax1.axis('equal')

        # 3D plot on right subplot
        ax2 = fig.add_subplot(122, projection='3d')
        surf1 = ax2.plot_surface(X, Y, T_plot, cmap='jet',
                                 edgecolor='none', antialiased=True,
                                 vmin=vmin, vmax=vmax)
        surf2 = ax2.plot_surface(X, -Y, T_plot, cmap='jet',
                                 edgecolor='none', antialiased=True,
                                 vmin=vmin, vmax=vmax)
        ax2.set_xlabel("X-axis")
        ax2.set_ylabel("Y-axis")
        ax2.set_zlabel("Temperature")
        ax2.set_title("3D Temperature Distribution")
        ax2.view_init(elev=45, azim=300)
        plt.colorbar(surf1, ax=ax2, label="Temperature")

        plt.tight_layout()
        plt.show()

    else:
       raise ValueError("plot_type must be '2D', '3D', or 'both'")
```

Test Case (Unsteady)

In [15]:

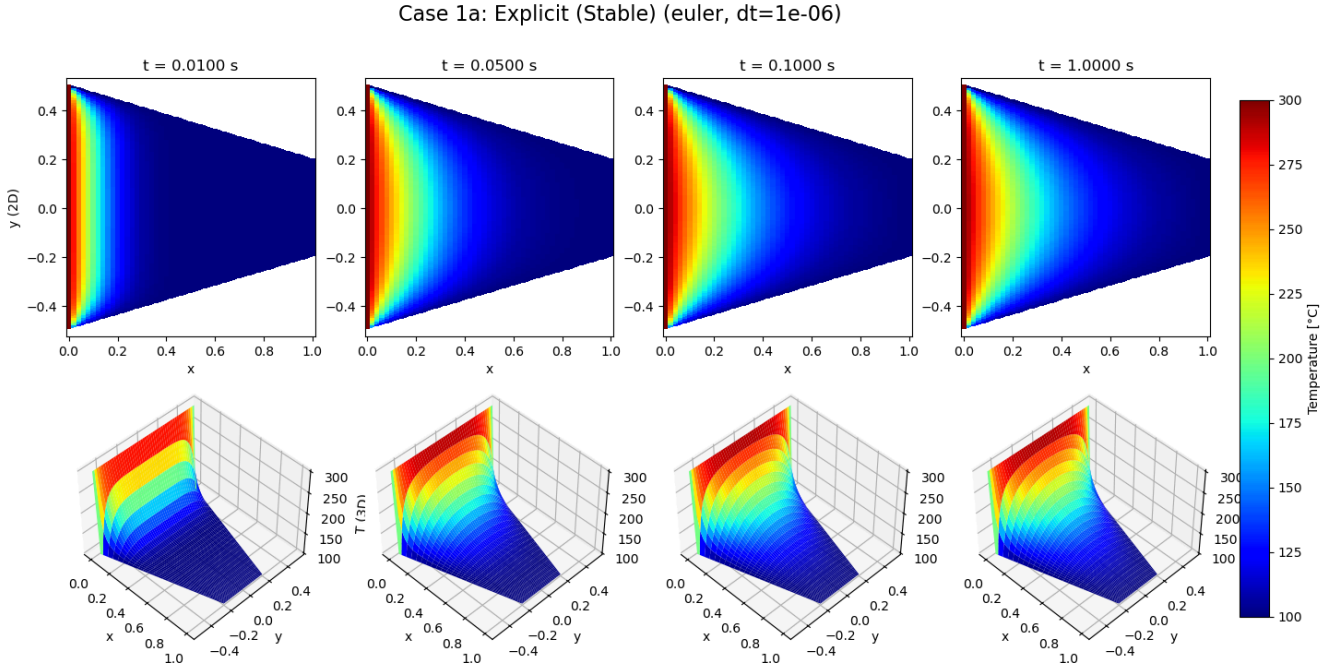
```
def run_and_plot_case(case_name, shape, l, dimX, dimY, boundary, TD, q, alpha, Tinf,
                      t_end, dt, time_scheme, plot_times, unstable=False):
    print(f"\n{'='*60}")
    print(f"Running {case_name}")
    print(f"{'='*60}")

    # 1. Generate Mesh
    X, Y = setUpMesh(dimX, dimY, l, formfunction, shape)

    # 2. Initialize Solver
    heat = UnsteadyHeat2D_FVM(X, Y, lambda_conductivity=1.0,
                              boundary=boundary, TD=TD, q=q, alpha=alpha, Tinf=Tinf)

    # 3. Initial Condition (T0)
    T0 = np.full((dimY, dimX), 100.0)

    # Enforce Dirichlet BCs on T0
    if boundary[0] == 'D': T0[0, :]  = TD[0]
    if boundary[1] == 'D': T0[-1, :] = TD[1]
    if boundary[2] == 'D': T0[:, 0]  = TD[2]
    if boundary[3] == 'D': T0[:, -1] = TD[3]

    print(f"Mesh: {dimX}x{dimY}")
    print(f"Time Scheme: {time_scheme}, dt: {dt}")
```

```
    # 4. Run Solver
    try:
        history = heat.solve(
            state="unsteady",
            t_end=t_end,
            dt=dt,
            T0=T0.flatten(),
            time_scheme=time_scheme,
            output_times=plot_times
        )

        # 5. Visualization (2D Top, 3D Bottom)
        num_cols = len(history)
        fig = plt.figure(figsize=(4 * num_cols, 8)) # Taller figure for 2 rows
        fig.suptitle(f"{case_name} ({time_scheme}, dt={dt})", fontsize=16)

        # Find global min/max for consistent colorbar
        all_T = np.concatenate([snap[1] for snap in history])
        if unstable:
            vmin, vmax = -1000, 1000
            print("Note: Colorbar clamped for unstable plot")
        else:
            vmin, vmax = np.min(all_T), np.max(all_T)

        for idx, (time, T_flat) in enumerate(history):
            T_grid = T_flat.reshape(dimY, dimX)

            # Prepare Symmetric Data
            X_comb = np.vstack([X, X])
            Y_comb = np.vstack([Y, -Y])
            T_comb = np.vstack([T_grid, T_grid])

            # --- Row 1: 2D Contour Plots ---
            ax2d = fig.add_subplot(2, num_cols, idx + 1)
            pcm = ax2d.pcolormesh(X_comb, Y_comb, T_comb, shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
            ax2d.set_title(f"t = {time:.4f} s")
            ax2d.axis('equal')
            ax2d.set_xlabel('x')
            if idx == 0: ax2d.set_ylabel('y (2D)')

            # --- Row 2: 3D Surface Plots ---
            ax3d = fig.add_subplot(2, num_cols, num_cols + idx + 1, projection='3d')
            ax3d.plot_surface(X, Y, T_grid, cmap='jet', vmin=vmin, vmax=vmax, rstride=2, cstride=2)
            ax3d.plot_surface(X, -Y, T_grid, cmap='jet', vmin=vmin, vmax=vmax, rstride=2, cstride=2)
            ax3d.set_xlabel('x')
            ax3d.set_ylabel('y')
            if idx == 0: ax3d.set_zlabel('T (3D)')
            ax3d.view_init(elev=45, azim=-45) # Angle from your reference image

        # Add a single colorbar
        fig.subplots_adjust(right=0.9)
        cbar_ax = fig.add_axes([0.92, 0.15, 0.02, 0.7])
        fig.colorbar(pcm, cax=cbar_ax, label='Temperature [°C]')

        plt.show()

    except Exception as e:
        print(f"Solver failed: {e}")
        import traceback
        traceback.print_exc()

# ========================================
# CASE DEFINITIONS
# ========================================

if __name__ == "__main__":

    # Case 1a: Explicit scheme (Stable)
    run_and_plot_case(
        case_name="Case 1a: Explicit (Stable)",
        shape='linear', l=1, dimX=51, dimY=51,
        boundary=['D', 'N', 'D', 'D'],
        TD=[100, 100, 300, 100], q=0, alpha=20, Tinf=90,
        t_end=1.0,
        dt=1e-6,
        time_scheme="euler",
        plot_times=[0.01, 0.05, 0.1, 1.0]
    )

    # Case 1b: Explicit scheme (Unstable)
    run_and_plot_case(
        case_name="Case 1b: Explicit (Unstable)",
        shape='linear', l=1, dimX=51, dimY=51,
        boundary=['D', 'N', 'D', 'D'],
        TD=[100, 100, 300, 100], q=0, alpha=20, Tinf=90,
        t_end=0.05,
        dt=2e-6,
        time_scheme="euler",
        plot_times=[0.001, 0.01, 0.03, 0.05],
        unstable=True
    )

    # Case 2a: Explicit, Mixed BCs
    run_and_plot_case(
        case_name="Case 2a: Explicit, Mixed BCs",
        shape='linear', l=1, dimX=51, dimY=51,
        boundary=['R', 'N', 'D', 'R'],
        TD=[100, 100, 100, 100],
        q=0, alpha=20, Tinf=90,
        t_end=1.0,
        dt=1.5e-6,
        time_scheme="euler",
        plot_times=[0.0, 0.05, 0.075, 0.11]
    )

    # Case 3: Crank-Nicolson (Implicit)
    run_and_plot_case(
        case_name="Case 3: Crank-Nicolson (Implicit)",
        shape='linear', l=1, dimX=51, dimY=51,
        boundary=['R', 'N', 'D', 'R'],
        TD=[100, 100, 100, 100],
```

```
        q=0, alpha=20, Tinf=90,
        t_end=1.0,
        dt=1e-5,
        time_scheme="crank-nicolson",
        plot_times=[0.0, 0.05, 0.075, 0.11]
    )

    # Case 4: RK4 Scheme
    run_and_plot_case(
        case_name="Case 4: RK4 Scheme",
        shape='linear', l=1, dimX=51, dimY=51,
        boundary=['R', 'N', 'D', 'R'],
        TD=[100, 100, 100, 100],
        q=0, alpha=20, Tinf=90,
        t_end=1.0,
        dt=1e-5,
        time_scheme="rk4",
        plot_times=[0.0, 0.05, 0.5, 1.0]
    )
```

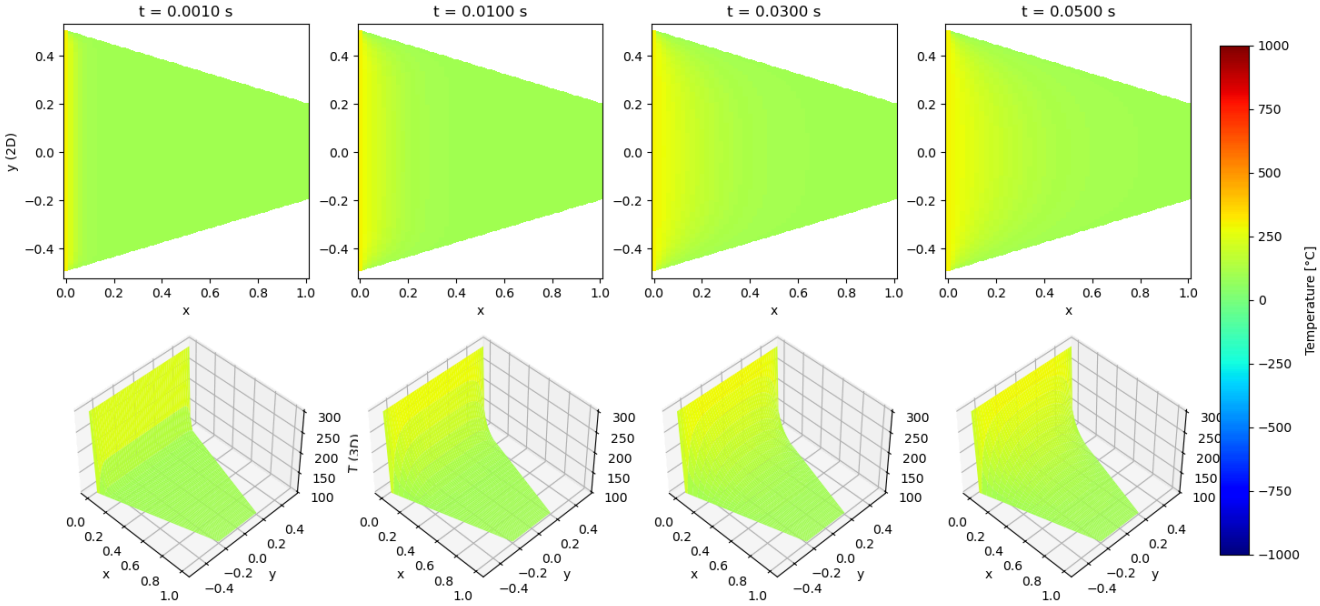
```
============================================================
Running Case 1a: Explicit (Stable)
============================================================
Mesh: 51x51
Time Scheme: euler, dt: 1e-06
Assembling linear system...
Grid spacing: 1.000000e-02
Max stable dt: 2.500000e-05
Your dt: 1.000000e-06
Time-stepping (euler): 1000000 steps, dt=1.000000e-06
  Step 100000/1000000 (t=0.1000) - T_range=[100.00, 300.00]
  Step 200000/1000000 (t=0.2000) - T_range=[100.00, 300.00]
  Step 300000/1000000 (t=0.3000) - T_range=[100.00, 300.00]
  Step 400000/1000000 (t=0.4000) - T_range=[100.00, 300.00]
  Step 500000/1000000 (t=0.5000) - T_range=[100.00, 300.00]
  Step 600000/1000000 (t=0.6000) - T_range=[100.00, 300.00]
  Step 700000/1000000 (t=0.7000) - T_range=[100.00, 300.00]
  Step 800000/1000000 (t=0.8000) - T_range=[100.00, 300.00]
  Step 900000/1000000 (t=0.9000) - T_range=[100.00, 300.00]
  Step 1000000/1000000 (t=1.0000) - T_range=[100.00, 300.00]
Unsteady solution completed successfully.
```

```
C:\Users\thomd\AppData\Local\Temp\ipykernel_17640\1647250858.py:60: UserWarning: The input coordinates to pcolormesh are interpreted as cell centers, but are not monotonically increasing or decreasing. This may lead to incorrectly calculated cell edges, in which case, please supply explicit cell edges to pcolormesh.
  pcm = ax2d.pcolormesh(X_comb, Y_comb, T_comb, shading='auto', cmap='jet', vmin=vmin, vmax=vmax)
```



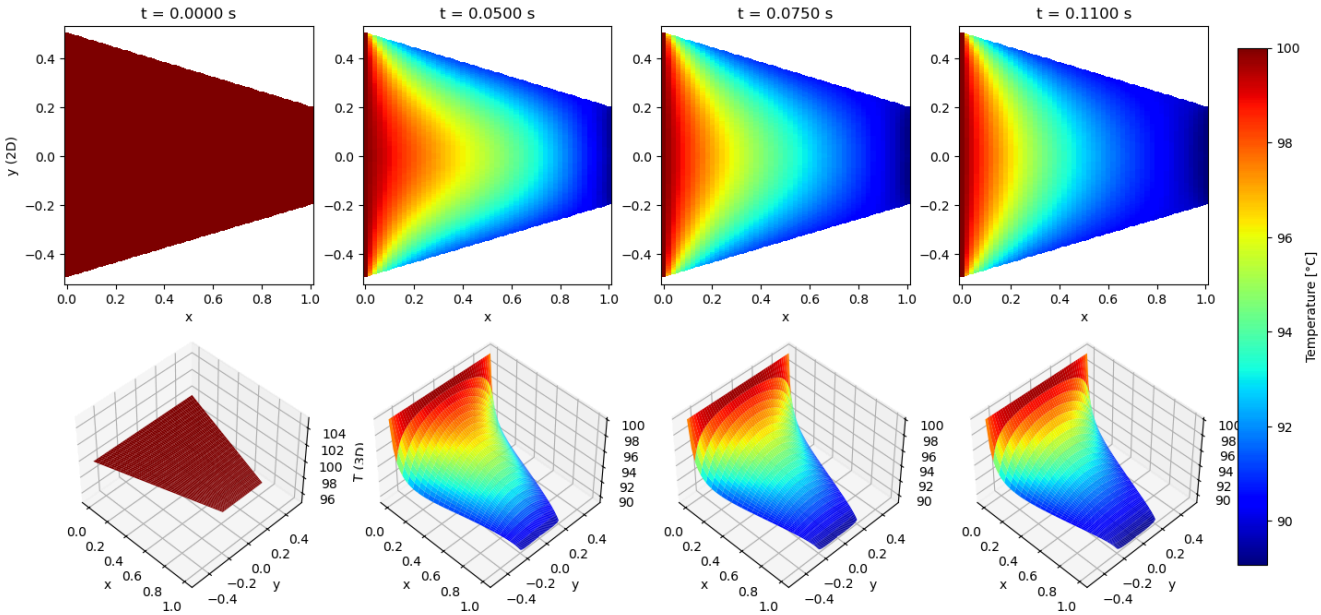

```
============================================================
Running Case 1b: Explicit (Unstable)
============================================================
Mesh: 51x51
Time Scheme: euler, dt: 2e-06
Assembling linear system...
Grid spacing: 1.000000e-02
Max stable dt: 2.500000e-05
Your dt: 2.000000e-06
Time-stepping (euler): 25001 steps, dt=2.000000e-06
  Step 2500/25001 (t=0.0050) - T_range=[100.00, 300.00]
  Step 5000/25001 (t=0.0100) - T_range=[100.00, 300.00]
  Step 7500/25001 (t=0.0150) - T_range=[100.00, 300.00]
  Step 10000/25001 (t=0.0200) - T_range=[100.00, 300.00]
  Step 12500/25001 (t=0.0250) - T_range=[100.00, 300.00]
  Step 15000/25001 (t=0.0300) - T_range=[100.00, 300.00]
  Step 17500/25001 (t=0.0350) - T_range=[100.00, 300.00]
  Step 20000/25001 (t=0.0400) - T_range=[100.00, 300.00]
  Step 22500/25001 (t=0.0450) - T_range=[100.00, 300.00]
  Step 25000/25001 (t=0.0500) - T_range=[100.00, 300.00]
Unsteady solution completed successfully.
Note: Colorbar clamped for unstable plot
```

Case 1b: Explicit (Unstable) (euler, dt=2e-06)



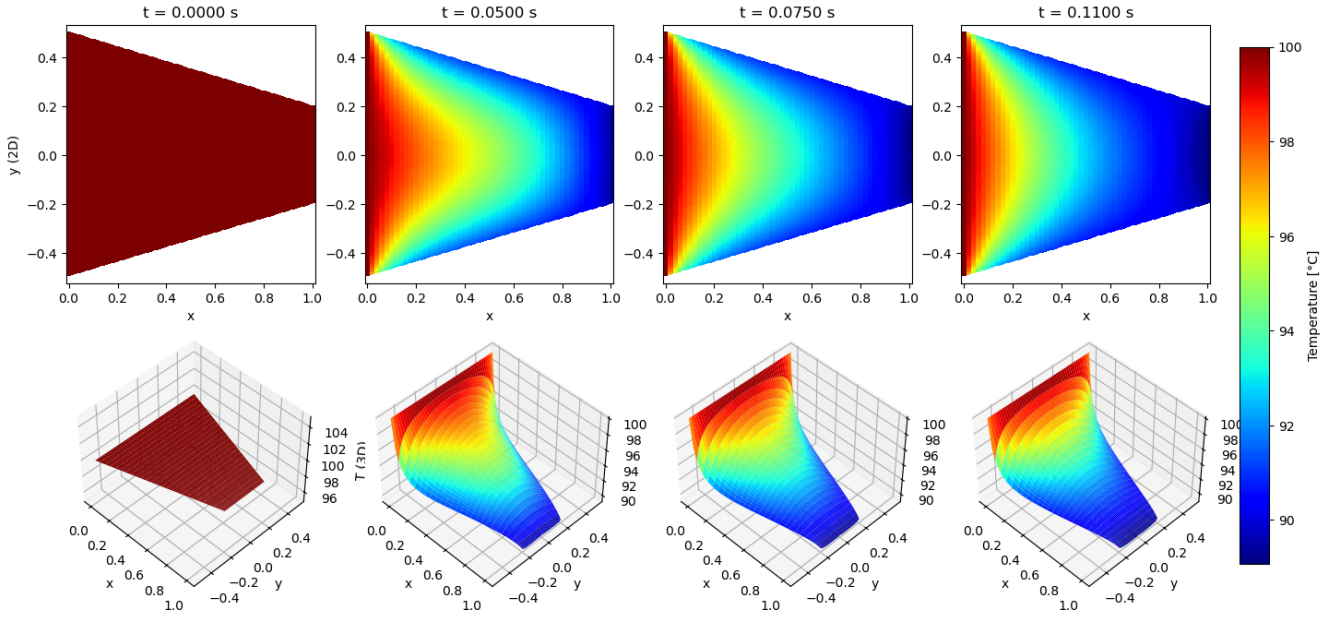

```
============================================================
Running Case 2a: Explicit, Mixed BCs
============================================================
Mesh: 51x51
Time Scheme: euler, dt: 1.5e-06
Assembling linear system...
Grid spacing: 1.000000e-02
Max stable dt: 2.500000e-05
Your dt: 1.500000e-06
Time-stepping (euler): 666667 steps, dt=1.500000e-06
  Step 66666/666667 (t=0.1000) - T_range=[89.10, 100.00]
  Step 133332/666667 (t=0.2000) - T_range=[89.01, 100.00]
  Step 199998/666667 (t=0.3000) - T_range=[89.01, 100.00]
  Step 266664/666667 (t=0.4000) - T_range=[89.01, 100.00]
  Step 333330/666667 (t=0.5000) - T_range=[89.01, 100.00]
  Step 399996/666667 (t=0.6000) - T_range=[89.01, 100.00]
  Step 466662/666667 (t=0.7000) - T_range=[89.01, 100.00]
  Step 533328/666667 (t=0.8000) - T_range=[89.01, 100.00]
  Step 599994/666667 (t=0.9000) - T_range=[89.01, 100.00]
  Step 666660/666667 (t=1.0000) - T_range=[89.01, 100.00]
Unsteady solution completed successfully.
```

Case 2a: Explicit, Mixed BCs (euler, dt=1.5e-06)



```
============================================================
Running Case 3: Crank-Nicolson (Implicit)
============================================================
Mesh: 51x51
Time Scheme: crank-nicolson, dt: 1e-05
Assembling linear system...
Time-stepping (crank-nicolson): 100000 steps, dt=1.000000e-05
  Step 10000/100000 (t=0.1000) - T_range=[89.10, 100.00]
  Step 20000/100000 (t=0.2000) - T_range=[89.01, 100.00]
  Step 30000/100000 (t=0.3000) - T_range=[89.01, 100.00]
  Step 40000/100000 (t=0.4000) - T_range=[89.01, 100.00]
  Step 50000/100000 (t=0.5000) - T_range=[89.01, 100.00]
  Step 60000/100000 (t=0.6000) - T_range=[89.01, 100.00]
  Step 70000/100000 (t=0.7000) - T_range=[89.01, 100.00]
  Step 80000/100000 (t=0.8000) - T_range=[89.01, 100.00]
  Step 90000/100000 (t=0.9000) - T_range=[89.01, 100.00]
  Step 100000/100000 (t=1.0000) - T_range=[89.01, 100.00]
Unsteady solution completed successfully.
```

Case 3: Crank-Nicolson (Implicit) (crank-nicolson, dt=1e-05)




```
============================================================
Running Case 4: RK4 Scheme
============================================================
Mesh: 51x51
Time Scheme: rk4, dt: 1e-05
Assembling linear system...
Grid spacing: 1.000000e-02
Max stable dt: 2.500000e-05
Your dt: 1.000000e-05
Time-stepping (rk4): 100000 steps, dt=1.000000e-05
  Step 10000/100000 (t=0.1000) - T_range=[89.10, 100.00]
  Step 20000/100000 (t=0.2000) - T_range=[89.01, 100.00]
  Step 30000/100000 (t=0.3000) - T_range=[89.01, 100.00]
  Step 40000/100000 (t=0.4000) - T_range=[89.01, 100.00]
  Step 50000/100000 (t=0.5000) - T_range=[89.01, 100.00]
  Step 60000/100000 (t=0.6000) - T_range=[89.01, 100.00]
  Step 70000/100000 (t=0.7000) - T_range=[89.01, 100.00]
  Step 80000/100000 (t=0.8000) - T_range=[89.01, 100.00]
  Step 90000/100000 (t=0.9000) - T_range=[89.01, 100.00]
  Step 100000/100000 (t=1.0000) - T_range=[89.01, 100.00]
Unsteady solution completed successfully.
```

Case 4: RK4 Scheme (rk4, dt=1e-05)