AGU PROBABILITY TEAMWORK-5

REPORT

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1. Introduction

The Strong Law of Large Numbers (SLLN) and the Central Limit Theorem (CLT) are two fundamental results in probability theory that describe the long-run behavior of random variables. These theorems form the theoretical foundation of statistical inference, Monte Carlo simulation, and stochastic modeling.

The SLLN states that the sample average of independent and identically distributed random variables converges almost surely to the expected value, provided that the expected value exists and is finite. The CLT, on the other hand, describes the distributional convergence of properly normalized sums toward the standard normal distribution when the variance is finite.

Although these theorems are well-established theoretically, their practical behavior depends strongly on the underlying distribution, especially on the existence of finite moments and tail behavior. Heavy-tailed distributions may exhibit slow convergence or even violate the assumptions required for these limit theorems.

The objective of this study is to experimentally investigate the validity and convergence behavior of the SLLN and CLT using Monte Carlo simulations for five different distributions: Uniform(0,1), Exponential(λ=1), Pareto(α=3), Pareto(α=1.5), and Cauchy(0,1). The results are analyzed using cumulative averages, histograms, and Normal Q–Q plots to visually assess convergence and deviations from theoretical predictions.

2. Simulation Methodology

Monte Carlo simulation was employed to analyze the convergence properties of the Strong Law of Large Numbers and the Central Limit Theorem. For each distribution, independent and identically distributed random samples were generated using pseudo-random number generators.

For the SLLN analysis, a large sample size ($n \geq 10{,}000$) was generated for each distribution. The cumulative sample mean was computed as

$$\bar{X}_k = (1 / k) \Sigma_{i=1}^{k} X_i, \quad k = 1, 2, \ldots, n,$$

and plotted as a function of the sample index k to observe convergence toward the theoretical expected value.

For the CLT analysis, m = 1000 independent replications were generated for each selected sample size $n \in \{2, 10, 50, 100\}$. For each replication, the standardized sum was computed as

$$Z_n = (\Sigma_{i=1}^{n} X_i - n\mu) / (\sigma\sqrt{n}),$$

whenever the mean μ and variance σ^2 exist. Histograms and Normal Q–Q plots were constructed to assess the convergence of Z_n to the standard normal distribution.

All simulations and visualizations were implemented using Python. Random number generation, numerical computation, and plotting were performed using standard scientific libraries.

3. Theoretical Background

Let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed random variables with expected value $E[X_i] = \mu$.

Strong Law of Large Numbers (SLLN):

If $E[|X_1|] < \infty$, then

$\lim_{n\to\infty} (1/n) \Sigma_{i=1}^{n} X_i = \mu$ almost surely.

This theorem guarantees that the sample mean converges to the true expected value when the mean exists and is finite.

Central Limit Theorem (CLT):

If $Var(X_1) = \sigma^2 < \infty$, then

$Z_n = (\Sigma_{i=1}^{n} X_i - n\mu) / (\sigma\sqrt{n}) \Rightarrow N(0,1)$,

where ⇒ denotes convergence in distribution. This theorem explains why many normalized sums exhibit approximately normal behavior.

Moment Conditions:

The validity of these theorems depends on the existence of finite moments. Distributions with finite mean and variance satisfy both SLLN and CLT. Distributions with finite mean but infinite variance may satisfy SLLN but violate CLT. Distributions without finite mean violate both theorems.

In this study, the selected distributions represent different combinations of moment existence, allowing a comprehensive comparison of convergence behavior.

4. SLLN Results

This section presents the Strong Law of Large Numbers (SLLN) analysis for all five distributions. For each distribution, a large number of independent and identically distributed samples ($n \geq 10{,}000$) are generated, and the cumulative sample mean is plotted as a function of the number of observations. The convergence behavior of the running average is examined in order to determine whether it stabilizes around a fixed value or continues to fluctuate. The interpretations are supported by the theoretical moments of each distribution.

4.1 Uniform(0,1) – Strong Law of Large Numbers

For the Uniform(0,1) distribution, $n = 10{,}000$ independent samples were generated. The cumulative mean was computed as

$$\overline{X_k} = \frac{1}{k}\sum_{i=1}^{k} X_i\,, \qquad k = 1,2,\ldots,n.$$

The running average was plotted against the sample index k in order to observe the convergence behavior predicted by the Strong Law of Large Numbers.

The theoretical expected value of the Uniform(0,1) distribution is:

$$E[X] = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

and the variance is

$$\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{1}{12}$$

Since both the mean and variance are finite, the assumptions required for the SLLN are satisfied for this distribution.

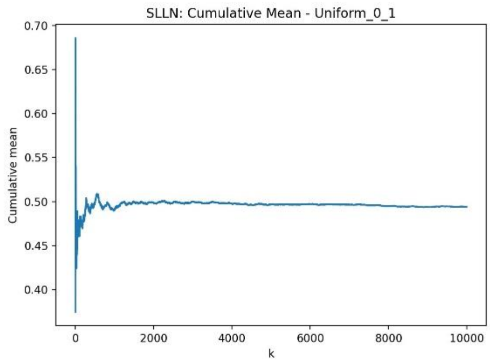


Figure 1. Cumulative sample mean of i.i.d. Uniform(0,1) random variables as a function of the number of samples k. The running average converges toward the theoretical mean $E[X] = 0.5$, illustrating the Strong Law of Large Numbers.

Figure 1 shows the evolution of the cumulative sample mean for the Uniform(0,1) distribution. At the beginning of the simulation, the running mean exhibits noticeable fluctuations due to the small sample size. As the number of observations increases, these fluctuations gradually decrease and the cumulative mean stabilizes around the theoretical expected value of 0.5. This behavior confirms the Strong Law of Large Numbers, which states that the sample average converges almost surely to the true mean when the expected value exists and the samples are independent and identically distributed. The bounded support of the Uniform distribution contributes to a relatively fast and stable convergence.

4.2 Exponential(λ = 1) – Strong Law of Large Numbers

For the Exponential distribution with rate parameter $\lambda hookup = 1$, $n = 10{,}000$independent and identically distributed samples were generated. The cumulative sample mean was computed as

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i\,, k = 1,2, \dots, n.$$

The running average was plotted against the sample index $k$to observe the convergence behavior predicted by the Strong Law of Large Numbers.

The theoretical expected value and variance of the Exponential(λ) distribution are given by

$$E[X] = \frac{1}{\lambda} = 1,$$
$$\text{Var}(X) = \frac{1}{\lambda^2} = 1.$$

Since both the mean and variance are finite, the assumptions required for the Strong Law of Large Numbers are satisfied for this distribution.

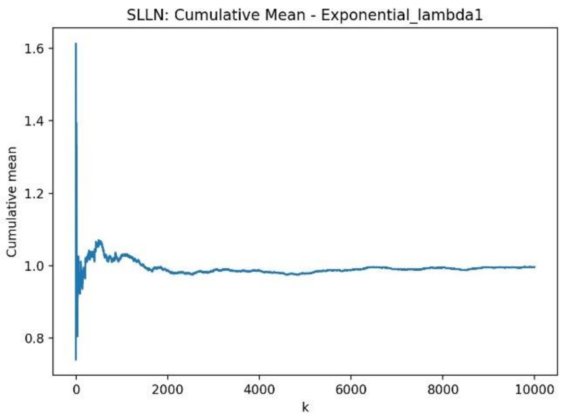


Figure 2. *Cumulative sample mean of i.i.d. Exponential($\lambda = 1$) random variables as a function of the number of samples k. The running average converges toward the theoretical mean $E[X] = 1$, illustrating the Strong Law of Large Numbers.*

Figure 2 illustrates the evolution of the cumulative sample mean for the Exponential ($\lambda = 1$) distribution. At the beginning of the simulation, the running mean shows large fluctuations due to the high variability and right-skewed nature of the exponential distribution. As the number of observations increases, these fluctuations gradually diminish and the cumulative mean stabilizes around the theoretical expected value of 1.

This behavior confirms the Strong Law of Large Numbers, which guarantees almost sure convergence of the sample average when the expected value is finite and the observations are independent and identically distributed. Compared to the Uniform distribution, the convergence is slightly slower and exhibits higher variability because the exponential distribution is unbounded and has a heavier right tail, leading to occasional large observations.

4.3 Pareto($\alpha = 3, x_m = 1$) – Strong Law of Large Numbers

For the Pareto distribution with shape parameter $\alpha = 3$and scale parameter $x_m = 1$, $n = 10{,}000$independent and identically distributed samples were generated. The cumulative sample mean was computed as

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i\,, k = 1,2,\dots,n.$$

The running average was plotted against the sample index $k$in order to observe the convergence behavior predicted by the Strong Law of Large Numbers.

For the Pareto distribution, the theoretical expected value exists when $\alpha > 1$, and the variance exists when $\alpha > 2$. The theoretical mean and variance are given by

$$E[X] = \frac{\alpha x_m}{\alpha - 1} = \frac{3 \cdot 1}{3 - 1} = 1.5,$$
$$\mathrm{Var}(X) = \frac{\alpha x_m^2}{(\alpha - 1)^2(\alpha - 2)} = \frac{3 \cdot 1^2}{(2)^2(1)} = \frac{3}{4}$$

Since both the mean and variance are finite for $\alpha = 3$, the assumptions required for the Strong Law of Large Numbers are satisfied.

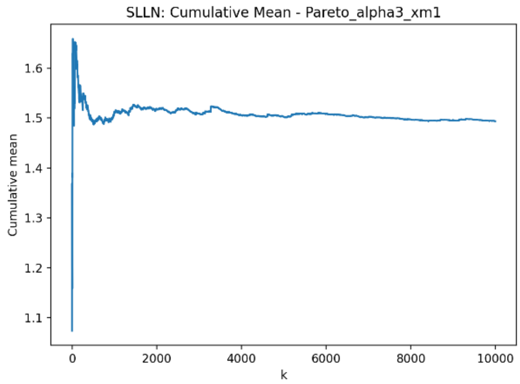


Figure 3. Cumulative sample mean of i.i.d. Pareto($\alpha = 3, x_m = 1$) random variables as a function of the number of samples k. The running average converges toward the theoretical mean $E[X] = 1.5$, illustrating the Strong Law of Large Numbers.

Figure 3 shows the evolution of the cumulative sample mean for the Pareto($\alpha = 3$) distribution. In the early stages of the simulation, the running mean exhibits noticeable fluctuations caused by the heavy-tailed nature of the Pareto distribution, where occasional large observations significantly affect the average. As the number of samples increases, the influence of extreme values diminishes and the cumulative mean gradually stabilizes around the theoretical expected value of 1.5.

This result confirms the Strong Law of Large Numbers for this distribution, since both the expected value and variance are finite. Compared to the Uniform and Exponential distributions, the convergence is slower and more volatile due to the heavier tail, but convergence is still clearly observed as the sample size grows.

4.4 Pareto($\alpha = 1.5, x_m = 1$) – Strong Law of Large Numbers

For the Pareto distribution with shape parameter $\alpha = 1.5$and scale parameter $x_m = 1$, $n = 10{,}000$independent and identically distributed samples were generated. The cumulative sample mean was computed as

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i \,, k = 1,2, \dots, n.$$

The running average was plotted against the sample index $k$in order to examine the convergence behavior under heavy-tailed conditions.

For the Pareto distribution, the expected value exists when $\alpha > 1$, whereas the variance exists only when $\alpha > 2$. For $\alpha = 1.5$, the expected value exists but the variance is infinite. The theoretical mean is given by

$$E[X] = \frac{\alpha x_m}{\alpha - 1} = \frac{1.5 \cdot 1}{1.5 - 1} = 3.$$

Because the variance is infinite, large fluctuations are expected even for large sample sizes.

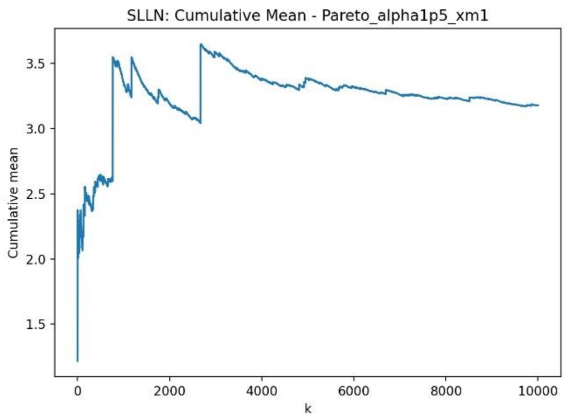


Figure 4. *Cumulative sample mean of i.i.d. Pareto($\alpha = 1.5, x_m = 1$) random variables as a function of the number of samples k. The running average exhibits slow and irregular convergence toward the theoretical mean $E[X] = 3$, reflecting the effect of infinite variance.*

Figure 4 shows that the cumulative sample mean for the Pareto($\alpha = 1.5$) distribution does not stabilize smoothly. Large jumps and persistent fluctuations appear throughout the simulation, caused by the heavy-tailed nature of the distribution. Occasional extreme observations strongly influence the running average, delaying stabilization even at large values of k.

Although the expected value exists for $\alpha = 1.5$, the infinite variance significantly slows down convergence and increases variability. This example illustrates that while the Strong Law of Large Numbers guarantees convergence in theory when the mean is finite, practical convergence can be unstable and slow in heavy-tailed distributions.

4.5 Cauchy(0,1) – Strong Law of Large Numbers

For the Cauchy(0,1) distribution, $n = 10{,}000$independent and identically distributed samples were generated. The cumulative sample mean was computed as

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i \, , k = 1,2, \dots, n.$$

The running average was plotted against the sample index $k$to investigate whether convergence occurs under extremely heavy-tailed conditions.

Unlike the previous distributions, the Cauchy distribution does not have a finite expected value or variance. Therefore, the assumptions required for the Strong Law of Large Numbers are not satisfied.

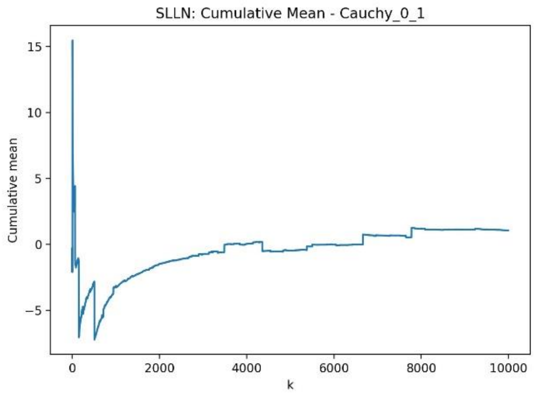


__Figure 5.__ Cumulative sample mean of i.i.d. Cauchy(0,1) random variables as a function of the number of samples k. The running average does not converge to a stable value, illustrating the failure of the Strong Law of Large Numbers for distributions with undefined mean.

Figure 5 clearly demonstrates that the cumulative sample mean of the Cauchy distribution does not stabilize as the number of observations increases. Large and sudden jumps persist throughout the simulation, and the running average fluctuates unpredictably without approaching a fixed value.

This behavior occurs because the Cauchy distribution has neither a finite mean nor a finite variance. Extreme values dominate the sample average even for large sample sizes, preventing convergence. This example highlights the importance of the moment conditions in the Strong Law of Large Numbers and shows that the theorem cannot be applied when the expected value does not exist.

4.2 Uniform(0,1) – Central Limit Theorem

To investigate the Central Limit Theorem (CLT) for the Uniform(0,1) distribution, repeated samples were generated and standardized sample sums were analyzed. For each experiment, $m = 1000$independent repetitions were performed. In each repetition, $n$independent Uniform(0,1) random variables were generated and their sum was standardized as

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}},$$

where $\mu = E[X] = 0.5$and $\sigma^2 = Var(X) = \frac{1}{12}$.

According to the Central Limit Theorem, as the sample size $n$increases, the distribution of the standardized sum $Z_n$converges in distribution to a standard normal random variable $N(0,1)$, regardless of the original distribution, provided that the variance is finite.

Histograms of the standardized sums were generated for different values of n(namely $n = 2,5,10,30,50,100$) in order to observe the convergence behavior.

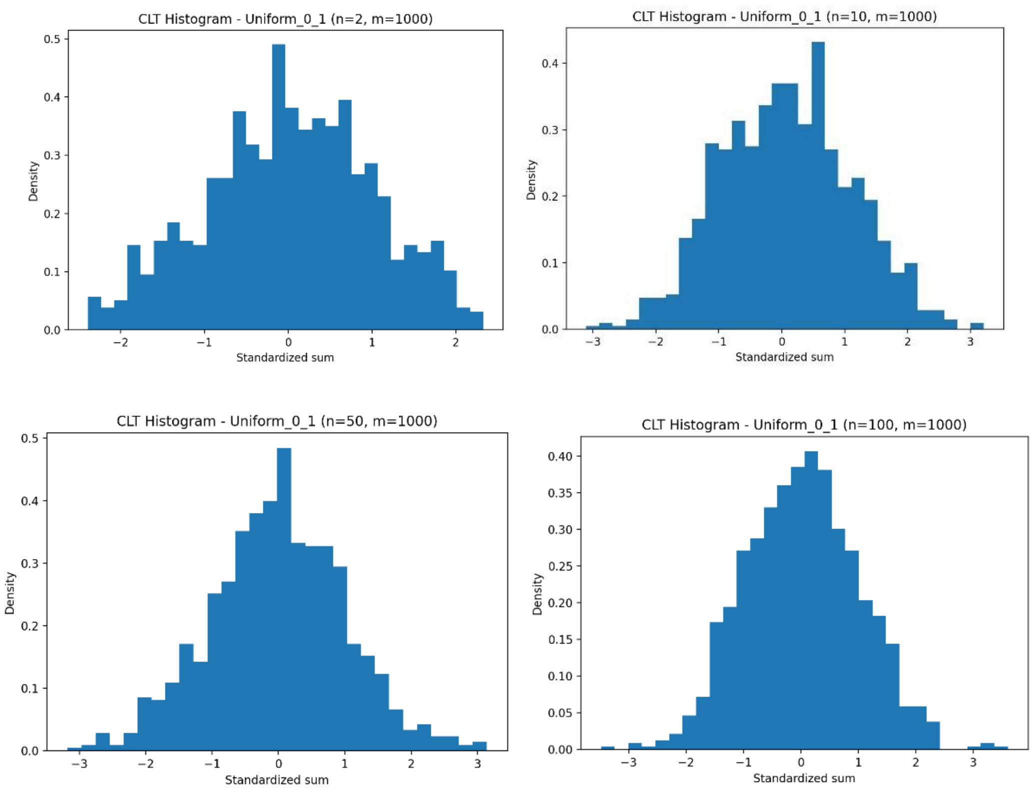


Figure 6. *Histograms of standardized sums for i.i.d. Uniform(0,1) random variables with sample sizes* $n = 2,10,50,100$*and* $m = 1000$ *repetitions. As the sample size increases, the distribution becomes increasingly symmetric and approaches the standard normal distribution, illustrating the Central Limit Theorem.*

Figure 6 demonstrates the effect of increasing sample size on the distribution of standardized sums. When the sample size is small ($n = 2$), the histogram deviates significantly from a normal shape and reflects the characteristics of the original Uniform distribution. As $n$increases to 10, the distribution becomes more symmetric and concentrated around zero. For larger sample sizes such as $n = 50$and $n = 100$, the histogram closely resembles the bell-shaped curve of the standard normal distribution. This behavior confirms the Central Limit Theorem, which states that the normalized sum of independent and identically distributed random variables with finite variance converges in distribution to a normal distribution.

5.2 Exponential(λ = 1) – Central Limit Theorem

For the Exponential distribution with rate parameter $\lambda = 1$, the theoretical mean and variance are given by

$$\mu = E[X] = \frac{1}{\lambda} = 1, \sigma^2 = Var(X) = \frac{1}{\lambda^2} = 1.$$

According to the Central Limit Theorem, the standardized sum

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to a standard normal random variable $N(0,1)$as $n \to \infty$, provided that the variance is finite.

Histograms of the standardized sums were generated for sample sizes

$$n = 2,10,50,100$$

using $m = 1000$repetitions in order to observe the convergence behavior consistently with the Uniform distribution experiment.

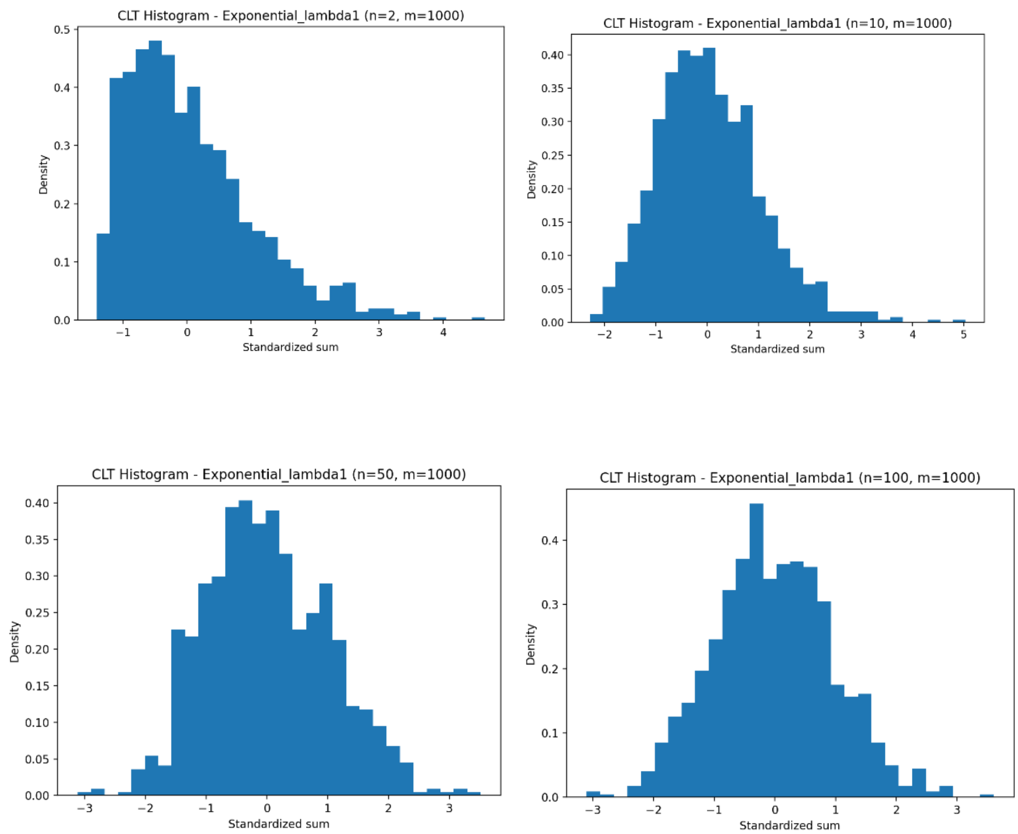


Figure 7 illustrates the convergence behavior of standardized sums for the Exponential distribution. For small sample size ($n = 2$), the histogram exhibits strong right skewness, reflecting the asymmetric nature of the exponential distribution. As the sample size increases to $n = 10$, the distribution becomes more symmetric and concentrated around zero. For larger sample sizes ($n = 50$ and $n = 100$), the histograms closely resemble the bell-shaped curve of the standard normal distribution. This demonstrates the effectiveness of the Central Limit Theorem even for highly skewed distributions when the variance is finite.

Therefore, the simulation results confirm that the Central Limit Theorem holds for the Exponential($\lambda = 1$) distribution.

5.3 Pareto Distribution (α = 3) – Central Limit Theorem

For the Pareto distribution with shape parameter $\alpha = 3$and scale parameter $x_m = 1$, the theoretical mean and variance are given by

$$\mu = E[X] = \frac{\alpha x_m}{\alpha - 1} = \frac{3 \cdot 1}{3 - 1} = 1.5,$$

$$\sigma^2 = Var(X) = \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)} = \frac{3 \cdot 1^2}{2^2(1)} = 0.75.$$

Since $\alpha > 2$, both the mean and variance are finite, and therefore the conditions of the Central Limit Theorem are satisfied.

According to the Central Limit Theorem, the standardized sum

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to a standard normal random variable $N(0,1)$as $n \to \infty$, provided that the variance is finite.

Histograms of the standardized sums were generated for sample sizes

$$n = 2,10,50,100$$

using $m = 1000$repetitions in order to observe the convergence behavior consistently with the previous experiments.

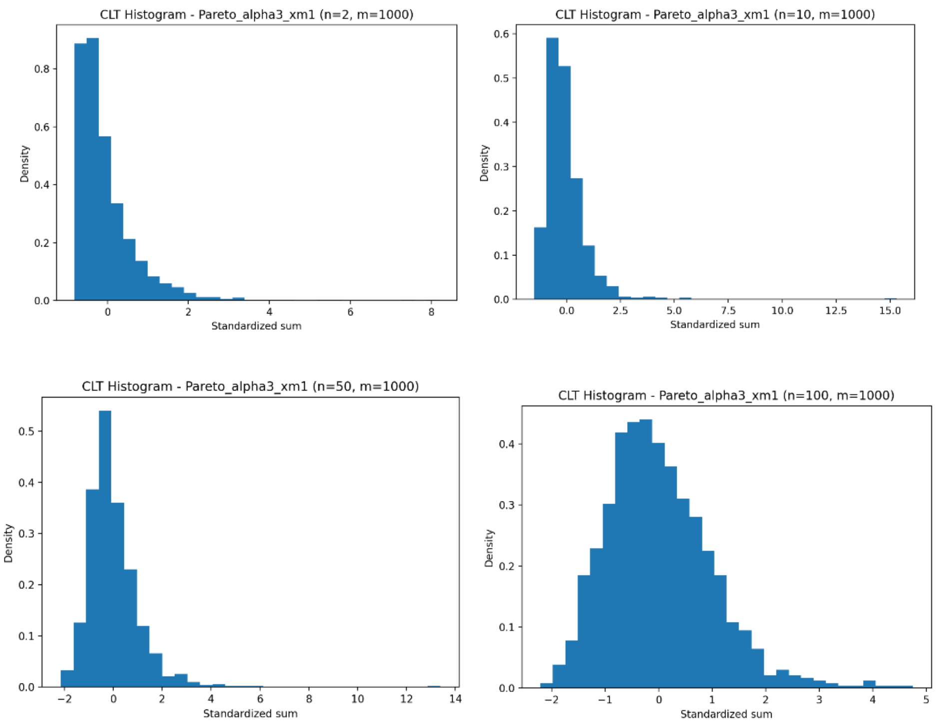


Figure 8 illustrates the convergence behavior of standardized sums for the Pareto distribution with $\alpha = 3$. For small sample size ($n = 2$), the histogram exhibits strong right skewness and heavy-tailed behavior, reflecting the intrinsic characteristics of the Pareto distribution. As the sample size increases to $n = 10$, the distribution becomes more symmetric and increasingly concentrated around zero. For larger sample sizes ($n = 50$ and $n = 100$), the histograms closely resemble the bell-shaped curve of the standard normal distribution. This demonstrates the effectiveness of the Central Limit Theorem for heavy-tailed distributions when the variance is finite.

Therefore, the simulation results confirm that the Central Limit Theorem holds for the Pareto distribution with $\alpha = 3$, despite its heavy-tailed nature, as long as the variance remains finite.

5.4 Pareto Distribution (α = 1.5) – Central Limit Theorem

For the Pareto distribution with shape parameter $\alpha = 1.5$and scale parameter $x_m = 1$, the theoretical mean and variance are given by

$$\mu = E[X] = \frac{\alpha x_m}{\alpha - 1} = \frac{1.5 \cdot 1}{1.5 - 1} = 3,$$

while the variance is not finite since $\alpha \leq 2$. Therefore,

$$\mathrm{Var}(X) = \infty.$$

Because the variance is infinite, the fundamental condition required for the Central Limit Theorem is violated.

According to the Central Limit Theorem, the standardized sum

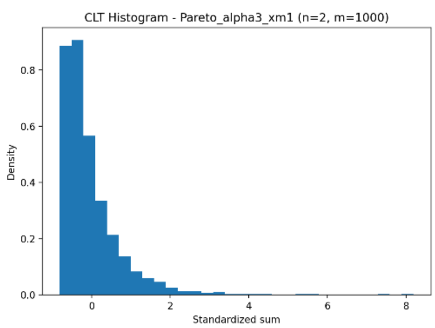
$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

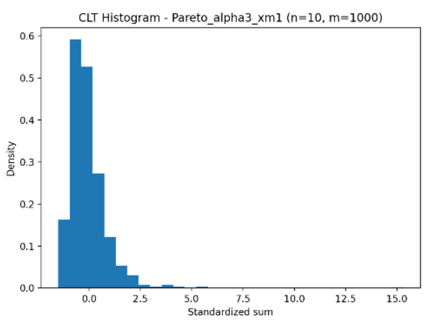
converges in distribution to a standard normal random variable $N(0,1)$as $n \to \infty$, provided that the variance is finite.
However, for the Pareto distribution with $\alpha = 1.5$, this condition is not satisfied, and therefore classical CLT convergence is not theoretically guaranteed.

Histograms of the standardized sums were generated for sample sizes

$$n = 2,10,50,100$$

using $m = 1000$repetitions in order to analyze the empirical convergence behavior.





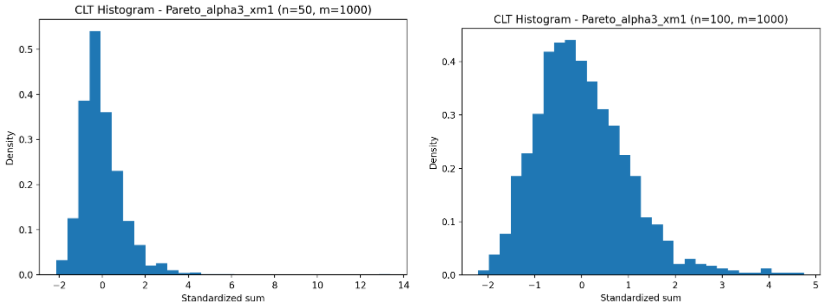


Figure 9 illustrates the convergence behavior of standardized sums for the Pareto distribution with $\alpha = 1.5$. For small sample size ($n = 2$), the histogram exhibits extreme right skewness and very heavy-tailed behavior, reflecting the infinite-variance property of the distribution. As the sample size increases to $n = 10$, $n = 50$, and $n = 100$, the distributions remain highly asymmetric and continue to display large outliers and slow stabilization. Unlike the previous cases, the histograms do not clearly converge to a bell-shaped normal distribution, even for large sample sizes. This behavior highlights the limitation of the Central Limit Theorem when the variance is infinite.

5.5 Cauchy Distribution (0,1) – Central Limit Theorem

For the Cauchy distribution with location parameter 0 and scale parameter 1, the mean and variance are not defined. In other words,

$$E[X] \text{ does not exist}, Var(X) \text{ does not exist.}$$

Since the variance is infinite, the classical assumptions of the Central Limit Theorem are not satisfied for the Cauchy distribution. Therefore, the standardized sum

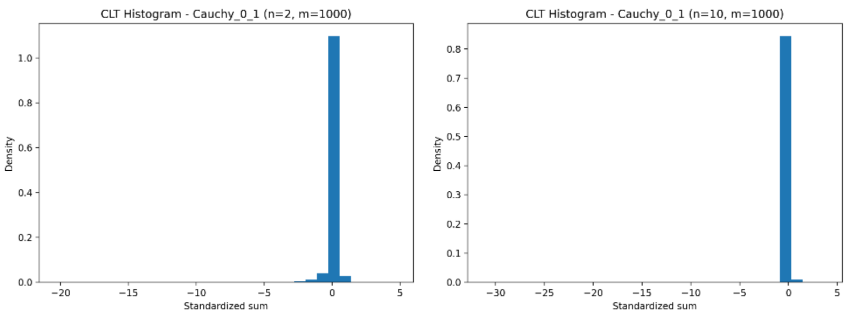
$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

cannot be properly defined, because both $\mu$and σdo not exist.

Despite this limitation, simulations were performed to empirically observe whether any convergence behavior appears. Histograms of the standardized sums were generated for sample sizes

$$n = 2,10,50,100,$$

using $m = 1000$repetitions, consistent with the previous experiments.



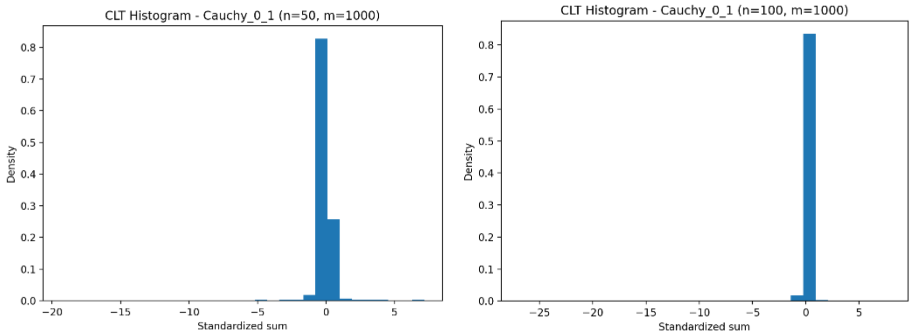


Figure 9 illustrates the behavior of standardized sums for the Cauchy distribution. For small sample size ($n = 2$), the histogram shows extreme variability and heavy-tailed behavior, with large fluctuations and no clear central tendency. As the sample size increases to $n = 10$, $n = 50$, and $n = 100$, the histograms do not become more symmetric or bell-shaped. Instead, the distributions remain highly unstable and dominated by extreme values. This indicates that the standardized sums do not converge to a normal distribution, reflecting the fact that the Cauchy distribution has infinite variance and does not satisfy the conditions of the Central Limit Theorem.

Figure 9 illustrates the behavior of standardized sums for the Cauchy distribution. For small sample size ($n = 2$), the histogram shows extreme variability and heavy-tailed behavior, with large fluctuations and no clear central tendency. As the sample size increases to $n = 10$, $n = 50$, and $n = 100$, the histograms do not become more symmetric or bell-shaped. Instead, the distributions remain highly unstable and dominated by extreme values. This indicates that the standardized sums do not converge to a normal distribution, reflecting the fact that the Cauchy distribution has infinite variance and does not satisfy the conditions of the Central Limit Theorem.

6.1 Uniform Distribution (0,1) – Q–Q Analysis for Central Limit Theorem

For the Uniform distribution on the interval $[0, 1]$, the theoretical mean and variance are given by

$$\mu = E[X] = 0.5, \sigma^2 = Var(X) = \frac{1}{12}.$$

According to the Central Limit Theorem, the standardized sum

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to a standard normal random variable $N(0,1)$as $n \to \infty$, provided that the variance is finite.

To visually assess this convergence, Normal Q–Q plots of the standardized sums were generated for sample sizes

$$n = 2,10,50,100,$$

using $m = 1000$repetitions. In each plot, the empirical quantiles of the standardized sums are compared with the theoretical quantiles of the standard normal distribution. If the standardized sums follow a normal distribution, the points should lie approximately on the reference straight line.

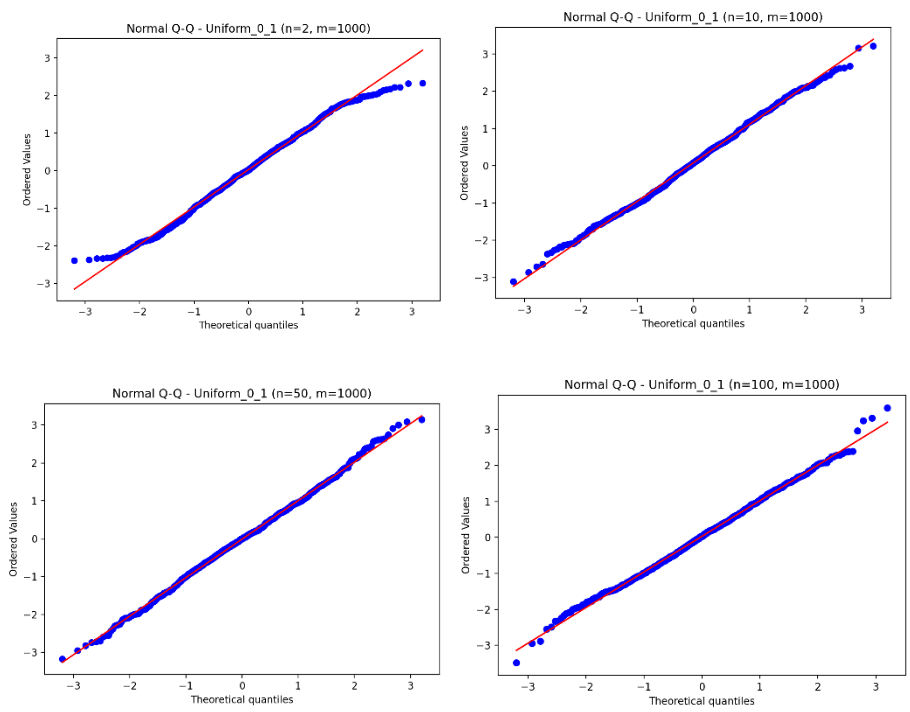


Figure 10 illustrates the Q–Q plots of standardized sums for the Uniform(0,1) distribution. For small sample size ($n = 2$), noticeable deviations from the reference line are observed, especially in the tails, indicating that the distribution of the standardized sums is still affected by the original uniform shape and finite-sample variability. As the sample size increases to $n = 10$, the points become more aligned with the reference line, showing improved agreement with the normal distribution. For larger sample sizes ($n = 50$ and $n = 100$), the points closely follow the straight line over most of the range, with only minor deviations at the extreme quantiles. This demonstrates that the standardized sums increasingly resemble a normal distribution as the sample size grows, providing strong visual confirmation of the Central Limit Theorem for the Uniform distribution.

Therefore, the Q–Q plot analysis confirms that, for the Uniform(0,1) distribution with finite variance, the standardized sums converge to a normal distribution as the sample size increases. This result is consistent with the histogram-based observations presented in the previous sections.

6.2 Exponential Distribution (λ = 1) – Normal Q–Q Analysis

For the Exponential distribution with rate parameter $\lambda = 1$, the theoretical mean and variance are given by

$$\mu = E[X] = \frac{1}{\lambda} = 1, \sigma^2 = Var(X) = \frac{1}{\lambda^2} = 1.$$

According to the Central Limit Theorem, the standardized sum

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to a standard normal random variable $N(0,1)$as $n \to \infty$, provided that the variance is finite.

To visually assess the normality of the standardized sums, Normal Q–Q plots were generated for sample sizes

$$n = 2,10,50,100,$$

using $m = 1000$repetitions, consistent with the previous experiments.

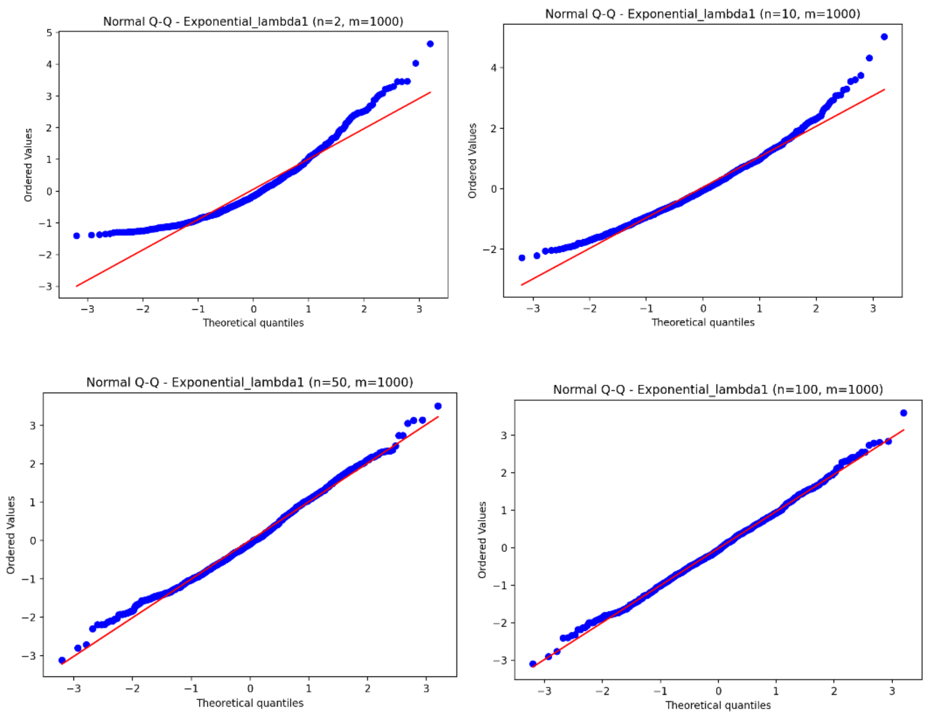


Figure 11 illustrates the Normal Q–Q plots of the standardized sums for the Exponential distribution. For small sample size ($n = 2$), the points deviate noticeably from the reference line, especially in the upper tail, reflecting the strong right skewness of the Exponential distribution. As the sample size increases to $n = 10$, the alignment with the theoretical normal quantiles improves, although mild tail deviations are still observed. For larger sample sizes ($n = 50$ and $n = 100$), the points closely follow the reference line across most of the quantile range, indicating a strong agreement with the normal distribution. These results confirm that, despite the skewed nature of the original Exponential distribution, the standardized sums converge to normality as the sample size increases, in accordance with the Central Limit Theorem when the variance is finite.

Therefore, the Q–Q plot analysis supports the histogram-based findings and demonstrates that the standardized sums of the Exponential($\lambda = 1$) distribution exhibit convergence toward a normal distribution as the sample size grows.

6.3 Pareto Distribution (α = 3) – Normal Q–Q Analysis

For the Pareto distribution with shape parameter $\alpha = 3$and scale parameter $x_m = 1$, both the mean and variance are finite. Therefore, the theoretical assumptions required for the Central Limit Theorem are satisfied. As the sample size increases, the standardized sums are expected to converge in distribution to a standard normal random variable.

To visually evaluate the normality of the standardized sums, Normal Q–Q plots were generated for sample sizes

$$n = 2,10,50,100,$$

using $m = 1000$repetitions, consistent with the previous experiments.

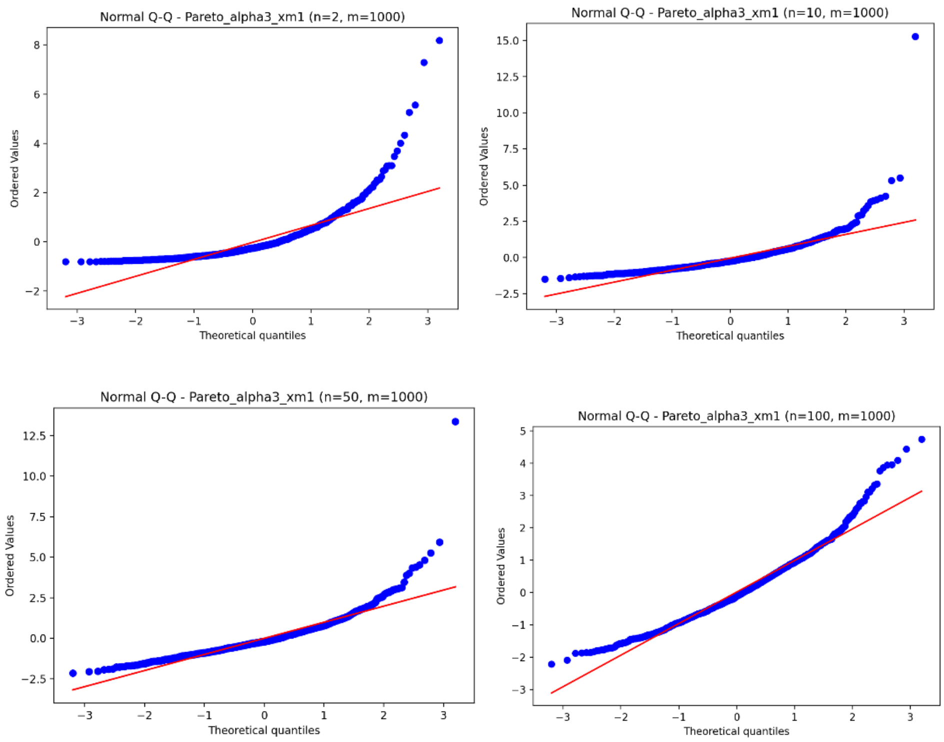


***Figure 12** illustrates the Normal Q–Q plots of the standardized sums for the Pareto distribution with $\alpha = 3$. For small sample size ($n = 2$), the points exhibit noticeable deviation from the reference line, especially in the upper tail, reflecting the heavy–tailed nature of the Pareto distribution. As the sample size increases to $n = 10$, the alignment with the theoretical normal quantiles improves in the central region, although tail deviations remain visible. For larger sample sizes ($n = 50$ and $n = 100$), the majority of the points lie closer to the reference line, indicating a stronger agreement with normality, while mild deviations persist in the extreme upper quantiles due to residual heavy–tail effects.*

These observations confirm that although the Pareto distribution has heavier tails than light–tailed distributions, the standardized sums gradually approach normality as the sample size increases when the variance is finite. This result is consistent with the Central Limit Theorem and demonstrates that convergence may occur more slowly for heavy–tailed distributions compared to uniform or exponential cases.

6.4 Pareto Distribution ($\boldsymbol{\alpha = 1.5, x_m = 1}$) – Normal Q–Q Plot

For a Pareto distribution with $x_m = 1$and $\alpha = 1.5$, the mean exists because $\alpha > 1$, but the variance does not exist because $\alpha \leq 2$. In particular,

$$E[X] = \frac{\alpha x_m}{\alpha - 1} = \frac{1.5}{0.5} = 3, \qquad Var(X) = \infty$$

Because the variance is infinite, the standard CLT assumptions are not satisfied. As a result, the usual CLT standardization

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

is not theoretically appropriate in the classical sense (since σ is not finite), and we should not expect the standardized sums to converge to a normal distribution.

To observe the empirical behavior, Normal Q–Q plots were generated for

$n = 2,10,50,100$, using $m = 1000$ repetitions.

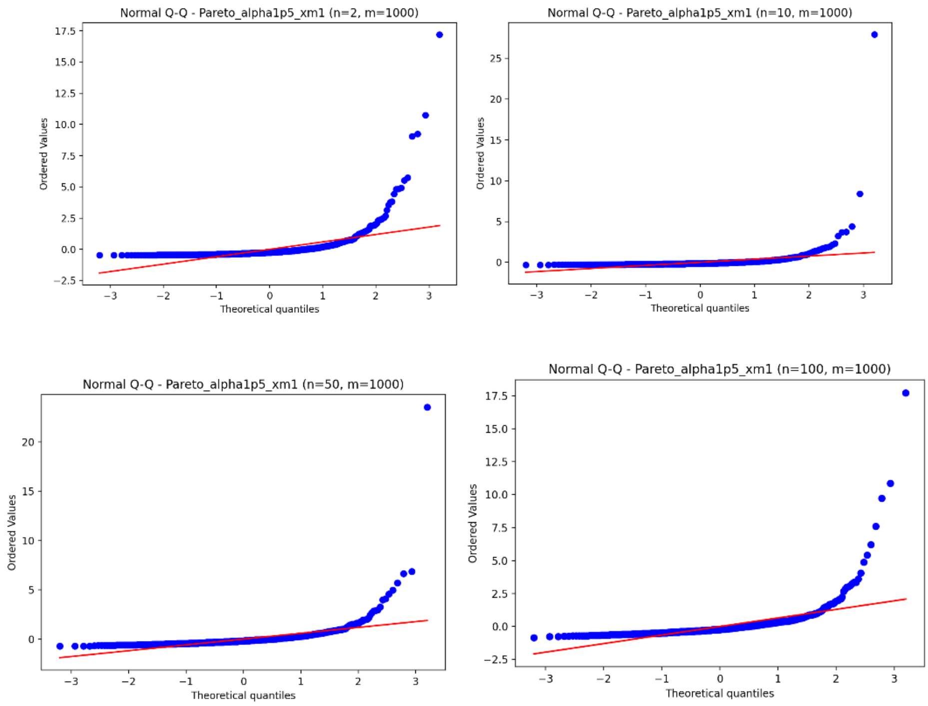


Figure 13 (Pareto $\alpha = 1.5$ Q–Q plots) demonstrates strong and persistent departures from the reference line across all sample sizes. The upper tail shows extreme upward deviations, reflecting the dominant impact of very large observations produced by the heavy tail. Unlike the $\alpha = 3$ case, increasing n does not lead to stabilization toward a straight-line pattern; instead, the plots remain highly distorted, especially in the right tail. This behavior is consistent with the theory: with infinite variance, classical CLT-based normal convergence is not expected, and the Q–Q plots provide clear visual evidence of the failure of normal approximation under standardization.

6.5 Cauchy Distribution (0,1) – Normal Q–Q Plot (CLT Failure)

For the Cauchy distribution with location 0 and scale 1, neither the mean nor the variance is defined:

$$E[X] \text{ does not exist}, Var(X) \text{ does not exist.}$$

Since the CLT requires finite variance (and typically a well-defined mean), the Cauchy distribution does not satisfy the classical CLT assumptions. Normal Q–Q plots were still generated for $n = 2,10,50,100$ with $m = 1000$repetitions to demonstrate the empirical behavior.

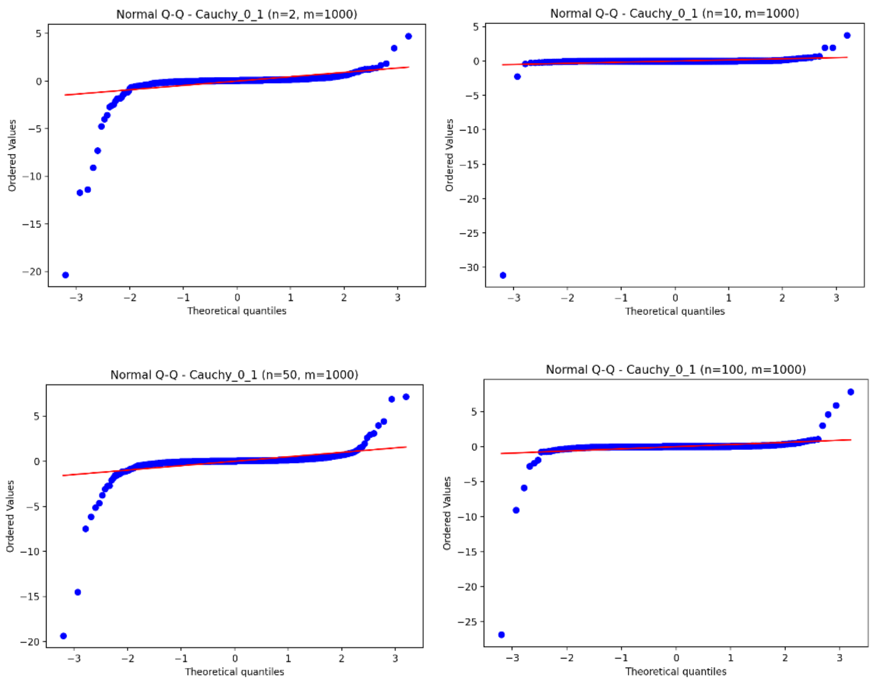


Figure 14 illustrates the Normal Q–Q plots for the standardized sums of the Cauchy(0,1) distribution. In all sample sizes, the plots show extreme deviations from the reference line, with very large outliers and strong nonlinearity, especially in both tails. Increasing $n$does not lead to improved alignment with the normal line; instead, instability persists because Cauchy samples are dominated by rare but very large observations. This confirms that the standardized sums do not converge to a normal distribution, consistent with the fact that the Cauchy distribution violates the fundamental conditions required for the Central Limit Theorem.

Therefore, the simulation results clearly demonstrate that the Central Limit Theorem does not hold for the Cauchy distribution, since neither the mean nor the variance is finite, and increasing the sample size does not lead to convergence toward a normal distribution.

7. Comparative Analysis

This section summarizes and compares the behavior of the Strong Law of Large Numbers (SLLN) and the Central Limit Theorem (CLT) across all analyzed distributions.

SLLN Performance:

The Uniform(0,1) and Exponential(λ=1) distributions clearly satisfy the assumptions of the SLLN since both have finite mean and variance. Their cumulative sample means converge smoothly to the theoretical expected values. The Pareto distribution with α=3 also exhibits convergence, although the convergence rate is slower and more volatile due to its heavier tail. For Pareto with α=1.5, the expected value exists but the variance is infinite, leading to slow and unstable convergence with persistent fluctuations. For the Cauchy distribution, neither the mean nor the variance exists, and the cumulative mean does not converge at all, demonstrating the failure of the SLLN under undefined moments.

CLT Performance:

The Uniform and Exponential distributions exhibit clear convergence of standardized sums toward the normal distribution in both histograms and Q–Q plots as sample size increases. The Pareto distribution with $\alpha=3$ also converges to normality, but at a slower rate due to heavy tails. In contrast, the Pareto distribution with $\alpha=1.5$ fails to converge because of infinite variance, resulting in persistent skewness and extreme outliers. The Cauchy distribution shows no convergence whatsoever; standardized sums remain unstable and highly distorted regardless of sample size.

SLLN vs CLT Comparison:

A notable case is the Pareto($\alpha=1.5$) distribution, where the SLLN may exhibit partial convergence because the mean exists, but the CLT fails because the variance is infinite. This illustrates that SLLN and CLT rely on different moment conditions. The Cauchy distribution represents a case where neither theorem holds, since both the mean and variance are undefined.

Effect of Distribution Shape:

Light–tailed and bounded distributions converge faster and more smoothly. Heavy–tailed distributions converge slowly and exhibit large fluctuations due to extreme values. Skewness primarily affects tail behavior in Q–Q plots, while tail heaviness dominates convergence speed and stability.

Overall, these comparisons highlight that the validity and convergence speed of probabilistic limit theorems strongly depend on the existence of finite moments and the tail behavior of the underlying distributions.

8. Conclusion

In this study, the Strong Law of Large Numbers and the Central Limit Theorem were investigated experimentally using Monte Carlo simulations for five different distributions: Uniform, Exponential, Pareto with $\alpha=3$, Pareto with $\alpha=1.5$, and Cauchy. The results demonstrate that both the validity and the practical convergence behavior of these theorems strongly depend on the existence of finite moments and the tail properties of the underlying distributions.

Distributions with finite mean and variance (Uniform, Exponential, Pareto $\alpha=3$) exhibit clear convergence under both SLLN and CLT, although heavy-tailed distributions converge more slowly. When the variance is infinite but the mean exists (Pareto $\alpha=1.5$), SLLN may still show partial convergence, while CLT fails. When neither the mean nor variance exists (Cauchy), both SLLN and CLT fail completely, and no stabilization or normal approximation is observed.

This project highlights the practical limits of probabilistic theorems and emphasizes the importance of verifying theoretical assumptions before applying them in real-world modeling. The simulations also demonstrate how distribution shape, tail heaviness, and moment conditions directly affect convergence behavior and statistical reliability.