Group Quiz 4

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1) We want to show that $u_1(x,t) \leq u_2(x,t) \; \forall \; (x,t) \in \Omega_{LT}$

So, we begin by letting $\left(\hat{x},\hat{t}\right)$ be a maximizer of $u_1(x,t)$ on the "edges", and $u_2(x,t)$ be the maximum value on the "edges." So, $u_1(x,t) \leq u_2(x,t) \; \forall \; (x,t)$ on the "edges." Our goal is to show that the above claim holds $\forall \; (x,t) \in \Omega_{LT}$. As with many proofs, let $\epsilon > 0$ be a fixed constant. Then, define $v(x,t) = u_1(x,t) + \epsilon x^2$ on Ω_{LT}. Next, we want to establish some derivative relationships. Taking derivatives with respect to x and t for our $v(x,t)$ function, we have

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + 2\epsilon$$

Then, if we substitute correctly the above derivatives into our diffusion equation, we have

$$\frac{\partial v}{\partial t} - D\frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} - D\left[\frac{\partial^2 u}{\partial x^2} + 2\epsilon\right]$$

$$\frac{\partial v}{\partial t} - D\frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} - 2D\epsilon$$

$$\frac{\partial v}{\partial t} - D\frac{\partial^2 v}{\partial x^2} = -2D\epsilon$$

So, we now know that $\dfrac{\partial v}{\partial t} - D\dfrac{\partial^2 v}{\partial x^2} < 0 \; \forall (x,t) \in \Omega_{LT}$.

Now, suppose $v(x,t)$ attains a maximum at an interior point (x_0,t_0) of Ω_{LT} where $0 < x_0 < L$, $0 < x_0 < L$. If we apply Fermat's theorem, we get

$$\frac{\partial v}{\partial t}\left(x_0,t_0\right) = 0 \quad \text{and} \quad \frac{\partial v}{\partial x}\left(x_0,t_0\right)$$

We also know that $\dfrac{\partial^2 v}{\partial x^2}\left(x_0,t_0\right) \leq 0$, so our function will be concave down. Next, plugging in what we have just discovered back into our diffusion equation, we arrive at

$$\frac{\partial v}{\partial t}\left(x_0,t_0\right) - D\frac{\partial^2 v}{\partial x^2}\left(x_0,t_0\right) \geq 0$$

Which contradicts the earlier diffusion inequality we established. So, $v(x,t)$ cannot attain a maximum at an interior point.

Next, consider the case where a maximizer, call it $\left(x_0,t_)\right)$, of $\left(x,t)\right)$ lies on the "top edge", where $0 < x_0 < L$ and $t = t_0$. Again, if we apply Fermat's theorem, we have

$$\frac{\partial v}{\partial x}\left(x_0,t_0\right) = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}\left(x_0,t_0\right) \leq 0 \quad \text{and} \quad \frac{\partial v}{\partial t}\left(x_0,t_0\right) \geq 0$$

Plugging in what we have just derived into our diffusion equation, we get

$$\frac{\partial v}{\partial x}\Big(x_0, t)0\Big) - D\frac{\partial^2 v}{\partial x^2}\Big(x_0, t_0\Big) \geq 0,$$

which is yet another contradiction to our diffusion inequality established at the beginning of the problem. So, $v(x,t)$ cannot attain a maximum at the "top edge" either.

Therefore, the maximizer $\Big(x_0, t_0\Big)$ of $v(x,t)$ on the domain Ω_{LT} must be on the remaining "edges." So, $v(x,t) \leq v\Big(x_0, t_0\Big)\ \forall\ (x,t) \in \Omega_{LT}$. So, $u(x,t) \leq v(x,t) \leq v\Big(x_0, t_0\Big) \leq u_2(x,t) + \epsilon(L)^2\ \forall\ (x,t) \in \Omega_{LT}$. Thus, $u_1(x,t) \leq u_2(x,t) + \epsilon(L)^2\ \forall\ (x,t) \in \Omega_{LT}$. Finally, if you consider $\epsilon \to 0$, then $u_1(x,t) \leq u_2(x,t)\ \forall\ (x,t) \in \Omega_{LT}$.

2) For $u(x,0)$ and $u_2(x,0)$, since time is zero in both of these cases, we are only looking at values along the $x-axis$. However, from our work in question number one, we know that the $x-axis$ is one of the "edges" where $u_1(x,t) \leq u_2(x,t)\ \forall (x,t) \in$ the domain Ω_{LT}. So, using the notation $\phi_1(x)$ and $\phi_2(x)$, we can write that $\phi_1(x) \leq \phi_2(x) \in$ our domain Ω_{LT}.

Next, if we consider the first boundary condition(s)

$$\begin{cases} u_1(0,t) = g_1(t) \\ u_1(L,t) = h_1(t) \end{cases}$$

We see that $u_1(0,t) = g_1(t)$ represents the concentration of the kool aid along the "left" boundary (of the $t-axis$) as time progresses, and $u_1(L,t) = h_1(t)$ represents the concentration of kool aid along the "right" boundary in our x, t plane as time progresses.

If we then look at the secondary boundary condition(s) with the same lens as for the first boundary condition(s)

$$\begin{cases} u_2(0,t) = g_2(t) \\ u_2(L,t) = h_2(t) \end{cases}$$

$u_2(0,t) = g_2(t)$ represents the concentration of the kool aid along the "left" boundary (or the $t-axis$) as time progresses, and $u_2(L,t) = h_2(t)$ represents the concentration of kool aid along the "right" boundary in our x, t plane as time increases.

If we recall the conclusion we came to in question number one, that $u_1(x,t) \leq u_2(x,t)$, and think about it in the context of the initial conditions, then as time progresses, we know that $g_1(t)$ and $h_1(t)$ will increase at a rate less than or equal to $g_2(t)$ and $h_2(t)$. Physically, this corresponds to the concentration of kool aid in our experiment one increasing at a rate within the tube less than or equal to that of our experiment two along the left and right "edges" in our x, t plane over our domain Ω_{LT}.