1)

Proof. To begin, we will prove some lemmas that will be useful later.

Lemma 0.1. *If G is a group with $H \triangleleft G$, $K \triangleleft G$ such that $H \cap K = \{e\}$. then $hk = kh \ \forall \ h \in H$ and $k \in K$.*

Proof. Let $h \in H$ and $k \in K$. So, $h^{-1} \in H$. Now, since both are normal subgroups, we have that $kh^{-1}k^{-1} \in H$ and $hkh^{-1} \in K$. Since $kh^{-1}k^{-1} \in H$ and $h \in H$, then $hkh^{-1}k^{-1} \in H$. Similarly, since $hkh^{-1} \in K$ and $k^{-1} \in K$, then $hkh^{-1}k^{-1} \in K$. Therefore, $hkh^{-1}k^{-1} \in H \cap K$. So, $hkh^{-1}k^{-1} = e$ and hence, $hk = kh$. So, since we have shown that $kh = hk \ \forall \ k \in K$ and $\forall \ h \in H$, we have that $HK = KH$. □

Lemma 0.2. $G = HK$

Proof. Part one: we know the subset $HK \subseteq G$ has size

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

So, $|H : H \cap K| = \dfrac{|HK|}{|K|} \leq \dfrac{|G|}{|K|} = [G : K]$ with equality if and only if $|HK| = |G|$, i.e. $HK = G$.

Part two: we know that $|G : K|$ divides $|G : H \cap K| = |G : H||H : H \cap K|$. Since $|G : K|$ and $|G : H|$ are co-prime, $|G : H|$ divides $|H : H \cap K|$. In particular, $|G : K| \leq |H : H \cap K|$. We always have the reverse inequality by part one. So, we get equality and again, by part one, we conclude $HK = G$. □

Lemma 0.3. *The direct product $G_1 \times G_2$ of two groups is abelian if and only if both G_1 and G_2 are abelian.*

Proof. Suppose that $G_1 \times G_2$ is abelian. Let $a, b \in G$ and let $e_2 \in G_2$ be the identity element of G_2. Then,

$$(ab, e_2) = (a, e_2) \cdot (b, e_2) = (b, e_2) \cdot (a, e_2) = (ba, e_2)$$

so, $ab = ba$. Next, let $c, d \in G_2$ and let $e_1 \in G_1$ be the identity element of G_1. Then,

$$(e_1, cd) = (e_1, c) \cdot (e_1, d) = (e_1, d) \cdot (e_1, c) = (e_1, dc)$$

so, $cd = dc$. Thus, G_1 and G_2 are both abelian.

□

Now, that we have gotten that out of the way, on to the problem! So, suppose that $g_1 \in G$ and $g_2 \in G$. Then, $g_1 \in G/H$, $g_1 \in G/K$, $g_2 \in G/H$, and $g_2 \in G/K$. Then, we have

$$\begin{gathered} g_1 H g_2 H g_1 K g_2 K \\ = (g_1 g_2) H \ (g_1 g_2) K \\ = (g_1 g_2) HK \\ = g_2 g_1 KH \end{gathered}$$

Since the product of two abelian groups is abelian by our lemma. Now, if we look more closely at

$$g_1 g_2 HK = g_2 g_1 KH$$

and divide out by $g_1 g_2$, we have that $HK = KH$. Now, by our lemma, we showed that $hk = kh \ \forall \ k \in K$ and $h \in H$, and so we have $HK = KH$. And, since we already have $KH = HK$ after dividing $g_1 g_2) HK = g_2 g_1 KH$ by $g_1 g_2$, we have that $hk = kh$. Next, by our lemma 2, we have that $G = KH$. Also, by definition of the problem, we have $H \cap K = \{e_G\}$. So, at this point, we have satisfied all three of the necessary conditions for G to be an internal direct product of H and K. Now, we have shown that $HK = KH$, or that the internal direct product of subgroups H and K are abelian. Then, since we have show G to be an internal direct product of H and K, then by *Theorem 9.27* from the chapter on Isomorphisms, G is isomorphic to $H \times K$. Finally, since $H \times K$ is abelian and G is isomorphic to $H \times K$, then we have that G is abelian.

□

3)

Proof. Let $x + H$ have finite order in G/H. Then, there is some integer $n > 0$ such that $nx + H = H$. So, nx is in H. It then follows that nx has finite order in G. So, there is some integer $m > 0$ such that $m(nx) = 0$. Next, note that $m(nx) = (mn)x$. Using this, we arrive at $(mn)x = 0$. Therefore, x has finite order in G, so x is in H, and $x + H = H$ was the identity in G/H.

□

4)

Proof. Let $G = A_4$, $H = \{(12)(34), (13)(24), (14)(23), e\}$, $K = \{e, (12)(34)\}$. Then, H is normal in G and K is normal in H. But, K is not normal in G. $\square$