Homework 5

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1) Want to force σ to be in A_n, the alternating group on n letters.

So, we would need $\mu_1, ..., \mu_k$ to be made up of alternating elements. So, for any $\mu_i \in \sigma$, we would need $\mu_i = \mu_{i+2}$, $\mu_{i+1} = \mu_{i+3}$, etc.

2) For S_3, we have the element (123). For S_4, we have the element (1234). For S_5, we have the elements (12345) and (123)(45). For S_6, we have the elements (123456), (123)(456), and (123)(45). For S_7, we have the elements (1234567) and (1234)(567). For S_8, we have (123)(45678). For S_9, we have (12345)(678).

3a) We need to show symmetry, reflexivity, and transitivity

For Reflexivity

Proof. Let $x \in Cl(g)$. Then, by definition, we have $x = hxh^{-1}$ where $h \in G$. Since $h \in G$ and G is a group, we know that G has the identity element. Let h be the identity element. After substitution, we have $x = exe^{-1} = x$.

□

For transitivity

Proof. Let $a, b, c \in Cl$. Then, we have $Cl(a) = hah^{-1}, Cl(b) = hbh^{-1}$, and $Cl(c) = hch^{-1}$ for $h \in G$. Next, suppose $a \sim b$ and $b \sim c$. So we have $Cl(a) = Cl(b)$ or $hah^{-1} = hbh^{-1}$ and $Cl(b) = Cl(c)$ or $hbh^{-1} = hch^{-1}$. Since $h \in G$ and G is a group, we know the identity is in G. So let h be the identity element. After substitution, we have

$$eae^{-1} = ebe^{-1} \text{ and } ebe^{-1} = ece^{-1}$$

which simplifies to

$$a = b \text{ and } b = c$$

Substitute a for b and the result follows

□

For symmetry

Proof. Let $x, y \in Cl$. Then, we have $Cl(x) = hxh^{-1}$ and $Cl(y) = hyh^{-1}$ for $h \in G$. For xy, we have $xy = hxh^{-1} \cdot hyh^{-1}$. Now, since multiplication is associative, we can write $hxh^{-1} \cdot hyh^{-1}$ as $hyh^{-1} \cdot hxh^{-1}$. So, we have

$$hxh^{-1} \cdot hyh^{-1} = hyh^{-1} \cdot hxh^{-1}$$

Now, because $h \in G$ and G is a group, G contains the identity element. Let h be the identity element. Then, after substitution, we have

$$exe^{-1} \cdot eye^{-1} = eye^{-1} \cdot exe^{-1}$$
$$x \cdot y = y \cdot x, \text{ as desired}$$

□

3b)

Proof. Suppose that $g \in Z(G)$. Then, we know that g is an element that commutes with every element in G. Next, for $Cl(g)$, we have $Cl(g) = hgh^{-1}$ for $h \in G$. Now, since g communicates with all elements in G, we can write hgh^{-1} as $hh^{-1}g$. Now, since G is a group, we know that G contains its identity element. Let h be the identity of G. Then, we have
$hh^{-1}g = ee^{-1}g = g$, as desired

□

3c)

For S_3, there are three conjugate classes because 3 can be written as an ascending sum in three different ways: $3 + 0$, $1 + 2$, and $1 + 1 + 1$. The conjugate classes are $\{e\}, \{(12), (13)(23)\}, \{(123), (321)\}$

For S_4, there are five conjugate classes because four can be written as ascending sum in five different
ways: 4+0, 4+1, 1+3, 1+1+2, 2+2, and 1+1+1+1. The conjugate classes are
$\{e\}, \{(12), (13), (14), (23), (24), 34)\}\{(12)(34), (13)(24), (14)(23)\}\{(123), (132), (124), (142), (134), (14$

3d) Two elements of S_n are conjugate when they have the same cycle type

4a) Need to show one-to-one and onto

Proof. Suppose that $\lambda_g(h_1) = \lambda_g(h_2)$. Then, $gh_1 = gh_2$ and so $h_1 = g^{-1}gh_1 = g^{-1}gh_2 = h_2$. Thus, λ_g is one-to-one.

□

Proof. A function $\lambda_g(h) = gh$ is onto if every element of the codomain gh is the image of some element of h. Let $y \in gh$. We can show that $\exists$ a $x \in h$ such that $\lambda_g(x) = y$. Choose $x = \lambda_g^{-1}(y)$ and so $\lambda_g(\lambda_g^{-1}(y)) = y$. So, $\forall y \in gh$, $\exists$ a $x \in h$ such that $\lambda_g(x) = y$.

□

4b)

Proof. Suppose that $\lambda_g = \lambda_h$. Then, for some k, we have $\lambda_g(k) = kg$ and $\lambda_h(k) = kg$, where $g \in G$ and $h \in G$. We also have $kg = kh$, by supposition. Next, just divide through by k and, poof, $g = h$, as desired.

□

4c)

Proof. Let λ_{gh} be given. Then, for some k, we have have $\lambda_{gh}(k) = kgh$. Then, suppose we have $\lambda_g \circ \lambda_h$. By definition of function composition, $\lambda_g \circ \lambda_h = \lambda_g(\lambda_h(k))$ for some k. Next, doing some algebra yields the following:

$$\lambda_g(\lambda_h(k)) = \lambda_g(kh) = khg$$

Finally, since multiplication is associative, we can write khg as kgh. Thus, $\lambda_{gh} = \lambda_g \circ \lambda_h$, as desired.

□

5)

Proof. Let $\sigma \in S_n$ be a non-identity element, and suppose $\pi(i) = j$, for $j \neq i$. Then, since $n \geq 3$, $\exists$ a $k \neq i, j$. Let $\tau = (jk)$. Then,

$$\tau\sigma = \tau(j) = k \neq j = \sigma(i) = \sigma\tau(i)$$

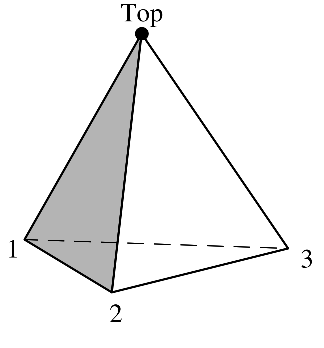
Hence, for every non-identity permutation in S_n, $\exists$ some element not commuting with it. Thus, $Z(S_n)$ must be trivial.

□

6) We want $(12)(34)h = h(12)(34)$ or $(12)(34)h((12)(34))^{-1} = h$. So, we have our centralizers:

$$\begin{aligned}
(12)(34) &\to (12)(34)(12)(34)(21)(43) = (12)(34)\\
(13)(24) &\to (12)(34)(13)(24)(21)(43) = (13)(24)\\
(14)(23) &\to (12)(34)(14)(23)(21)(43) = (14)(23)
\end{aligned}$$

7)



For each of the four faces, we have two non-identity rotations and one identity rotation. However, we don't want to count the identity rotation more times than necessary, so we will only count it once. So, up to this point, we have nine rotation symmetries. Now, if we consider the sides (the lines coming down from the top-vertex) of the tetrahedron, then, for each pair of sides together with the top-vertex, we can rotate over/around the top-vertex forwards or backwards. So, adding those three rotations to our collection of rotations, we have twelve rotations in total, which happens to be A_4.

8a)

8b) Let A_σ and A_τ be matrices in S_n. Then, according to the definition of matrix multiplication, we have

$$\left(A_\sigma A_\tau\right)_{ij} = \sum_{k=1}^{n} A_\sigma A_\tau$$

$$\left(A_\sigma A_\tau\right)_{ij} = \sum_{k=1}^{n} 1$$

Since in our definition of $\left(A_\sigma\right)_{jk}$ and $\left(A_\tau\right)_{kj}$, which is

$$\left(A_\sigma\right)_{ik} = \begin{cases} 1 & \text{if } \sigma(k) = i \\ 0 & \text{otherwise} \end{cases}$$

$$\left(A_\tau\right)_{kj} = \begin{cases} 1 & \text{if } \tau(j) = k \\ 0 & \text{otherwise} \end{cases}$$

We are only interested in the case where $\left(A_\sigma\right)_{ik} = 1$ and $\left(A_\tau\right)_{kj} = 1$, because otherwise, the product will be zero. Now, if $\left(A_\sigma\right)_{ik} = 1$ and $\left(A_\tau\right)_{kj} = 1$, we will have

$$\sigma(k) = i \text{ and } \tau_j = k, \text{ since } \sigma \text{ and } \tau \text{ are one-to-one and onto}$$

Now, if we plug-in $\tau(j) = k$ for the k in the $\sigma(k) = i$, we obtain:

$$\begin{cases} 1 & \text{if } \sigma(\tau(j)) = i \\ 0 & \text{otherwise} \end{cases}$$

Which becomes

$$\begin{cases} 1 & \text{if } \tau(j) = k \\ 0 & \text{otherwise} \end{cases}$$

Which is $A_{\sigma\tau}$

8c)