Take Home Exam One

Brendan Busey

1)

Proof. Let H be given as defined. Then, let I be $n \times n$ identity matrix. In order to show that a matrix is orthogonal, we multiply the matrix by its transpose. If the result is the identity matrix, then the matrix is orthogonal. Looking at I, we have $II^T = II = I$. Hence, I is orthogonal and so $I \in H$.

Next, let $A, B \in H$. Then, we want to show that $\left(AB^{-1}\right)^T = \left(AB^{-1}\right)^{-1}$. So, looking at $\left(AB^{-1}\right)^T$, one has

$$\left(AB^{-1}\right)^T$$

$$\implies \left(B^{-1}\right)^T \left(A\right)^T$$

$$\implies \left(B^T\right)^{-1} \left(A\right)^T$$

$$\implies \left(B^{-1}\right)^{-1} \left(A\right)^T$$

$$\implies B(A)^{-1}$$

If one examines $\left(AB^{-1}\right)^{-1}$, one sees that

$$\left(AB^{-1}\right)^{-1} = \left(B^{-1}\right)^{-1} A^{-1} = BA^{-1}.$$

Hence, we have shown that for any $A, B \in H$, we have that $\left(AB^{-1}\right)^T = \left(AB^{-1}\right)^{-1}$. Thus, we have for any $A, B \in H$, $AB^{-1} \in H$. Therefore, by the "Fastest Gun" theorem, H is a subgroup, as desired.

$\square$

2)

Proof. Let $n \geq 2$ and H be a subgroup of S_n. If all elements of H are even, then we finished. So, we assume H contains an odd permutation, call it σ. Then, let H_{even} denote the set of even permutations $\in H$ and let H_{odd} be the set of odd permutations $\in H$. Next, define a function $f : H_{even} \to H_{odd}$ by $f(p) = \sigma p$. Since H is a subgroup and because the product of an even and odd permutation is an odd permutation, we know that f is well-defined. Also, we can claim that f is a bijection.

To verify the above claim, suppose that $f(p_1) = f(p_2)$. Then, $\sigma p_1 = \sigma p_2$. Then, by cancelation in H, we have that $p_1 = p_2$. Hence, f is one-to-one. To show f is onto, let $\gamma \in H_{odd}$. Next, consider the element $p = \sigma^{-1}\gamma$. Since H is a subgroup, we know that $p \in H$. Furthermore, since σ^{-1} and γ are odd permutations we know that p is an even permutation. Therefore, $p \in H_{even}$.

Now, $f(p) = \sigma(\sigma^{-1}\gamma) = \gamma$. So, f is onto. Thus, f is a bijection between the finite sets H_{even} and H_{odd}, and we can say $|H_{even}| = |H_{odd}|$, and we are done. $\square$

3)

Proof. Let H and K be given as defined. The, let $X \in K$ be the matrix defined as

$$X = \begin{pmatrix} \cos(\theta) & \sin(\theta) & \cdots & \sin(\theta) \\ \sin(\theta) & \cos(\theta) & \cdots & \sin(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(\theta) & \sin(\theta) & \cdots & \cos(\theta) \end{pmatrix}$$

Then, letting $\theta = 0$, we will have $\cos(\theta) = \cos(0) = 1$ and $\sin(\theta) = \sin(0) = 1$, giving the identity matrix which is in $GL_n(\mathbb{R})$

Next, let $A, B \in K$. Looking at AB^{-1}, we note a couple of things. First, sicne B has a non-zero determinant by virtue of B being in K, we know that B is invertible. However, inverting an $n \times n$ matrix does not change the dimensions of the matrix, so, B^{-1} is still an $n \times n$ matrix. Then, one sees that for AB^{-1}, since we are multiplying two $n \times n$ matrices, the condition for matrix multiplication is fulfilled, so the product will be an $n \times n$ matrix, which is in $GL_n(\mathbb{R})$. Thus, by the "Fastest Gun" theorem, K is a subgroup of $GL_n(\mathbb{R})$. $\square$

5) Need to show the associative law, the existence of the identity element, and the existence of the inverse element

<u>Associative law</u>

Proof. Let $(g_1, h_1) \in (G \times H)$, $(g_2, h_2) \in (G \times H)$, and $(g_3, h_3) \in (G \times H)$. Then, one has:

$$\Big((g_1, h_1)\Big)(g_3, h_3)$$

$$\implies \Big(g_1 g_2^{h_1}, h_1 h_2\Big)(g_3, h_3)$$

$$\implies \Big((g_1 g_2^{h_1}), (h_1 h_2)\Big)(g_3, h_3)$$

$$\implies \Big((g_1 g_2^{h_1}) g_3^{(h_1 h_2)}, (h_1 h_2) h_3\Big)$$

$$\implies \Big((g_1 g_2^{h_1}) g_3^{(h_1 h_2)}, h_1 (h_3 h_2)\Big)$$

$$\implies \Big((g_1 g_2^{h_1}) g_3^{h_1} g_3^{h_2}, h_1 (h_3 h_2)\Big)$$

$$\implies \Big(g_1 \Big(g_2^{h_1} g_3^{h_1} g_3^{h_2}\Big), h_1 (h_3 h_2)\Big)$$

$\implies \Big(g_1(g_2g_3^{h_2})^{h_1}, h_1(h_2h_3)\Big)$

$\implies (g_1, h_1)\Big((g_2, h_2)(g_3, h_3)\Big)$

Therefore, the associative law holds

□

<u>Existence of the identity element</u>

Proof. Let $(g, h) \in (G \times H)$. Then, we have:

$(g, h)(e_G, e_H) = \Big(ge_G^{e_H}, he_H\Big) = (g, h) = \Big(e_G^{e_H}g^{e_H}, e_Hh\Big) = \Big(e_Gg, e_Hh\Big) = (e_G, e_H), (g, h)$

Therefore, (e_G, e_H) is the identity for this operation.

□

<u>Existence of the inverse element</u>

Proof. Let $(g, h) \in (G \times H)$. Then, consider:

$(g, h)(g^{-1}h^{-1})$

$\implies \Big(g(g^{-1})^h, hh^{-1}\Big)$

$\implies \Big(g(g^h)^{-1}, hh^{-1}\Big)$

$\implies \Big(g(g^{e_H})^{-1}, hh^{-1}\Big)$

$\implies \Big(gg^{-1}, hh^{-1}\Big)$

$\implies \Big(e_G, e_H\Big)$

$\implies \Big(g^{-1}g, h^{-1}h\Big)$

$\implies \Big(g^{-h}g^h, h^{-1}h\Big)$

$\implies \Big(g^{-h}\Big(g^1\Big)^h, h^{-1}h\Big)$

$\implies \Big(g^{-1}\Big(g^h\Big)^1, h^{-1}h\Big)$

$\implies \Big(g^{-1}g^h, h^{-1}h\Big)$

$$\implies \left(g^{-1}g^{h^{-1}}, h^{-1}h\right) = (g^{-1}h^{-1})(g,h)$$

Hence, we have found the identity element and thus, S is a group under the operation $\star$

$\square$