Group Quiz 1

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1)

The displacement vector starting at P_1 and ending at Q_2:

$$Q_2 - P_1 = (L_1, L_2, L_3) - (L_1, 0, 0)$$
$$Q_2 - P_1 = (0, L_2, L_3)$$
$$\vec{v}_{1,2} = (0, L_2, L_3)$$

The displacement vector starting at P_1 and ending at Q_3:

$$Q_3 - P_1 = (0, L_2, L_3) - (L_1, 0, 0)$$
$$Q_3 - P_1 = (-L, L_2, L_3)$$
$$\vec{v}_{1,3} = (-L, L_2, L_3)$$

The displacement vector starting at P_1 and ending at Q_4:

$$Q_4 - P_1 = (0, 0, L_3) - (L_1, 0, 0)$$
$$Q_4 - P_1 = (-L, 0, L_3)$$
$$\vec{v}_{1,4} = (-L, 0, L_3)$$

The displacement vector starting at P_3 and ending at Q_2:

$$Q_2 - P_3 = (L, L_2, L_3) - (0, L_2, 0)$$
$$Q_2 - P_3 = (L_1, 0, L_3)$$
$$\vec{v}_{2,3} = (L_1, 0, L_3)$$

The displacement vector starting at P_3 and ending at Q_3:

$$Q_3 - P_3 = (0, L_2, L_3) - (0, L_2, 0)$$
$$Q_3 - P_3 = (0, 0, L_3)$$
$$\vec{v}_{3,3} = (0, 0, L_3)$$

The displacement vector starting at P_3 and ending at Q_4:

$$Q_4 - P_3 = (0, 0, L_3) - (0, L_2, 0)$$
$$Q_4 - P_3 = (0, -L_2, L_3)$$
$$\vec{v}_{4,3} = (0, -L_2, L_3)$$

2)Let the differential equation $u''(x) + \lambda u(x) = 0$ for $0 \leq x \leq L$, and subject to the *Dirichlet boundary conditions* $u(0) = 0$ and $u(L_1) = 0$ be given.

Next, suppose that the solutions are of the form $u(x) = e^{kx}$. Then, $u'(x) = ke^{kx}$ and $u''(x) = k^2e^{kx}$. Plugging back in, we have:

$$k^2e^{kx} + \lambda e^{kx} = 0$$
$$e^{kx}(k^2 + \lambda) = 0$$
$$k^2 + \lambda = 0, \text{ which is our auxiliary equation.}$$

3)Consider $\lambda < 0$. Using the auxiliary equation we obtained in question 2, we have

$$k^2 + \lambda = 0$$
$$k^2 = -\lambda$$
$$k = \pm\sqrt{\lambda}$$

So, $v_1(x) = e^{\sqrt{-\lambda}\,x}$ and $v_2 = e^{-\sqrt{-\lambda}\,x}$.

Now, check the wronskian:

$$w(v_1, v_2) = \begin{vmatrix} e^{kx} & e^{-kx} \\ ke^{kx} & -ke^{kx} \end{vmatrix}$$
$$w(v_1, v_2) = (e^{kx} \cdot -ke^{kx}) - (e^{-kx} \cdot ke^{kx})$$
$$w(v_1, v_2) = (-k(e^{kx})(e^{-kx})) - (k(e^{-kx})(e^{kx}))$$
$$w(v_1, v_2) = (-k \cdot 1) - (k \cdot 1) = (-k)(k) = -k^2 \neq 0$$

Since $w(v_1, v2) \neq 0$, the solutions $v_1(x)$ and $v_2(x)$ are linearly independent and thus form a fundamental set of solutions. Now, the astute reader can observer that plugging in any values (say, $\sqrt{-\lambda}$) will have the same result.

4)Now, consider $\lambda = 0$. Then, the differential equation becomes

$$u''(x) = 0$$

Then, if we integrate, we have:

$$\begin{gathered} u'(x) = c \\ u(x) = c_1 x + c_0, \text{ for some constants } c_1 \textit{ and } c_0 \end{gathered}$$

Using $u(0) = 0$, we have:

$$\begin{gathered} u(0) = c_1(0) + c_0 \\ 0 = 0 \end{gathered}$$

Using $u(L_1) = 0$, we have:

$$\begin{gathered} u(L_1) = c_1(L_1) + c_0 \\ 0 = c_0 \end{gathered}$$

So, our two solutions are $h_1(x) = 1$ and $h_2(x) = x$

Now, check the wronskian:

$$\begin{gathered} w(h_1(x), h_2(x)) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} \\ w(h_1(x), h_2(x)) = (1 \cdot 1) - (0 \cdot x) \\ w(h_1(x), h_2(x)) = 1 - 0 = 1 \neq 0 \end{gathered}$$

Since $w(h_1(x), h_2(x)) \neq 0$, our two solutions are linearly independent and thus form a fundamental set of solutions.
Using $u(0) = 0$, we have:

$$\begin{aligned} h_1(0) &= 1 & \qquad h_2(0) &= 1 \\ 0 &= 1 & 0 &= 0 \end{aligned}$$

$$\begin{aligned} h_1(L_1) &= 1 & \qquad h_2(L_1) &= 1 \\ 0 &= 1 & 0 &= L_1 \end{aligned}$$

Since the functions don't satisfy the boundary conditions, they are not *Dirichlet-Laplacian eigenfunctions.*

5)Now, consider $\lambda > 0$. Then, suppose the solutions are of the form $u(x) = e^{kx}$. Then, $u'(x) = ke^{kx}$ and $u''(x) = k^2e^{kx}$. Plugging in, we have:

$$k^2e^{kx} + \lambda e^{kx} = 0$$
$$e^{kx}(k^2 + \lambda) = 0$$
$$k^2 + \lambda = 0$$
$$k = \pm\sqrt{-\lambda}$$
$$k = \pm i\sqrt{\lambda}$$

So, $u_1(x) = e^{i\sqrt{\lambda}}$ and $u_2(x) = e^{-i\sqrt{\lambda}}$.

Next, let $u_\lambda(x) = \dfrac{1}{2}\Bigg[u_1(x) + u_2(x)\Bigg]$.

Then,

$u_\lambda(x) = \dfrac{1}{2}\Bigg[\cos(\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x) + \cos(-\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x)\Bigg]$

$u_\lambda(x) = \cos(\sqrt{\lambda}\,x)$

Let $v_\lambda(x) = \dfrac{1}{2i}\Bigg[\cos(\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x) - \cos(-\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x)\Bigg]$

$u_\lambda(x) = \sin(\sqrt{\lambda}\,x)$

Now, consider $u_\lambda(x) = \cos(\sqrt{\lambda}\,x)$. Using $u(0) = 0$, we have:

$$u_\lambda(0) = \cos(\sqrt{\lambda}\,(0))$$
$$0 = \cos(0)$$
$$0 = 1$$

Then, using $u(L_1) = 0$, we have:

$$u_\lambda(0) = \cos(\sqrt{\lambda}\,L_1)$$
$$0 = \cos(\sqrt{\lambda}\,L_1)$$
$$\sqrt{\lambda}\,L_1 = 0$$

Now, consider $u_\lambda(x) = \sin(\sqrt{\lambda}\, x)$. Using, Using $u(0) = 0$, we have:

$$u_\lambda(0) = \sin(\sqrt{\lambda}\,(0))$$
$$0 = 0$$

Then, using $u(L_1) = 0$, we have:

$$u_\lambda(0) = \sin(\sqrt{\lambda}\, L_1)$$
$$0 = \sin(\sqrt{\lambda}\, L_1)$$
$$\sqrt{\lambda}\, L_1,\ n\pi, \text{ where } n \in \mathbb{N}$$

So, $\lambda = \left(\dfrac{n\pi}{L_1}\right)^2$, where $n \in \mathbb{N}$. These are the allowable eigenvalues. Denote this result as $\lambda_{n,D}$. Finally, let $u_{n,D}(x) = \sin\left(\dfrac{n\pi x}{L_1}\right)$, where $n \in \mathbb{N}$.

6) Let the differential equation $u''(y) + \lambda u(y) = 0$ for $0 \leq y \leq L_2$ and subject to the *Neumann boundary conditions* $u'(0) = 0$ and $u'(L_2) = 0$ be given. Next, suppose that the solutions are of the form $u(x) = e^{kx}$. Then, $u'(x) = ke^{kx}$ and $u''(x) = k^2e^{kx}$. Plugging in, we have:

$$k^2e^{kx} + \lambda e^{kx} = 0$$
$$e^{kx}(k^2 + \lambda) = 0$$
$$k^2 + \lambda = 0, \text{ which is, again, our auxiliary equation.}$$

Note how this auxiliary equation is exactly the same as the one we obtained in number two.

7)Consider $\lambda < 0$. Using the auxiliary equation we obtained in question six, we have

$$k^2 + \lambda = 0$$
$$k^2 = -\lambda$$
$$k = \pm\sqrt{\lambda}$$

So, $v_1(y) = e^{\sqrt{-\lambda}\,x}$ and $v_2(y) = e^{-\sqrt{-\lambda}\,x}$.

Now, check the wronskian:

$$w(v_1(y), v_2(y)) = \begin{vmatrix} e^{ky} & e^{-ky} \\ ke^{ky} & -ke^{ky} \end{vmatrix}$$
$$w(v_1(y), v_2(y)) = (e^{ky} \cdot -ke^{ky}) - (e^{-ky} \cdot ke^{ky})$$
$$w(v_1(y), v_2(y)) = (-k(e^{ky})(e^{-ky})) - (k(e^{-ky})(e^{ky}))$$
$$w(v_1(y), v_2(y)) = (-k \cdot 1) - (k \cdot 1) = (-k)(k) = -k^2 \neq 0$$

Since $w(v_1(y), v2(y)) \neq 0$, the solutions $v_1(y)$ and $v_2(y)$ are linearly independent and thus form a fundamental set of solutions. Now, the astute reader can observer that plugging in any values (say, $\sqrt{-\lambda}$) will have the same result.

Next, taking derivatives, we have:

$$v_1'(y) = \sqrt{-\lambda}e^{\sqrt{-\lambda}\,y} \text{ and } v_2'(y) = -\sqrt{-\lambda}e^{-\sqrt{-\lambda}\,y}.$$

Using $u'(0) = 0$, we have:

$$\begin{aligned} v_1'(0) &= \sqrt{-\lambda}e^{\sqrt{-\lambda}\,(0)} \\ v_1'(0) &= \sqrt{-\lambda} \\ 0 &= \sqrt{-\lambda} \end{aligned} \qquad \begin{aligned} v_2'(0) &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}\,(0)} \\ v_2'(0) &= -\sqrt{-\lambda} \\ 0 &= -\sqrt{-\lambda} \end{aligned}$$

Using $u'(L_2) = 0$, we have:

$$\begin{aligned} v_1'(L_2) &= \sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L_2)} \\ 0 &= \sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L_2)} \end{aligned} \qquad \begin{aligned} v_2'(0) &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L_2)} \\ 0 &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L_2)} \end{aligned}$$

Since the functions $v_1(y)$ and $v_2(y)$ do not satisfy the boundary conditions, they are not *Neumann-Laplacian eigenfunctions.*

8)Now, consider $\lambda = 0$. Then, the differential equation becomes

$$u''(y) = 0$$

Then, if we integrate, we have:

$$u'(y) = c$$
$$u(y) = c_1 y + c_0, \text{ for some constants } c_1 \; and \; c_0$$

Using $u(0) = 0$, we have:

$$u(0) = c_1(0) + c_0$$
$$0 = 0$$

Using $u(L_1) = 0$, we have:

$$u(L_1) = c_1(L_1) + c_0$$
$$0 = c_0$$

Similar to the *Dirichlet-Laplacian eigenproblem*, our two solutions are $h_1(y) = 1$ and $h_2(y) = y$

Now, check the wronskian:

$$w(h_1(y), h_2(y)) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$
$$w(h_1(y), h_2(y)) = (1 \cdot 1) - (0 \cdot x)$$
$$w(h_1(y), h_2(y)) = 1 - 0 = 1 \neq 0$$

Since $w(h_1(y), h_2(y)) \neq 0$, our two solutions are linearly independent and thus form a fundamental set of solutions.
Calculating some derivatives, we have:

$$h_1'(y) = 0 \quad and \quad h_2'(y) = 1$$

Using $u'(0) = 0$, we have:

$$h_1'(0) = 0 \qquad\qquad h_2'(0) = 1$$
$$0 = 0 \qquad\qquad 0 = 1$$

Using $u'(L_2) = 0$, we have:

$$h_1'(L_2) = 0 \qquad\qquad h_2'(L_2) = 1$$
$$0 = 0 \qquad\qquad 0 = 1$$

Just as in the *Dirichlet-Laplacian eigenproblem*, the corresponding harmonic functions do not satisfy the boundary conditions, and in this case, are not *Neumann-Laplacian eigenfunctions.*

9)Now, consider $\lambda > 0$. Then, suppose the solutions are of the form $u(y) = e^{ky}$. Then, $u'(y) = ke^{ky}$ and $u''(y) = k^2e^{ky}$. Plugging in, we have:

$$
\begin{gathered}
k^2e^{ky} + \lambda e^{ky} = 0 \\
e^{ky}(k^2 + \lambda) = 0 \\
k^2 + \lambda = 0 \\
k = \pm\sqrt{-\lambda} \\
k = \pm i\sqrt{\lambda}
\end{gathered}
$$

So, $u_1(y) = e^{i\sqrt{\lambda}}$ and $u_2(y) = e^{-i\sqrt{\lambda}}$.

Next, let $u_\lambda(y) = \dfrac{1}{2}\Big[u_1(y) + u_2(y)\Big]$.

Then,

$u_\lambda(y) = \dfrac{1}{2}\Big[\cos(\sqrt{\lambda}\, y) + i\sin(\sqrt{\lambda}\, y) + \cos(-\sqrt{\lambda}\, y) + i\sin(\sqrt{\lambda}\, y)\Big]$

$u_\lambda(y) = \cos(\sqrt{\lambda}\, y)$

Let $v_\lambda(y) = \dfrac{1}{2i}\Big[\cos(\sqrt{\lambda}\, y) + i\sin(\sqrt{\lambda}\, y) - \cos(-\sqrt{\lambda}\, y) + i\sin(\sqrt{\lambda}\, y)\Big]$

$v_\lambda(y) = \sin(\sqrt{\lambda}\, y)$

Now, calculating some derivatives, we have:

$$
\begin{gathered}
u_\lambda'(y) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\, y) \\
v_\lambda(y) = \sqrt{\lambda}\cos(\sqrt{\lambda}\, y)
\end{gathered}
$$

Now, consider $u_\lambda(x) = \cos(\sqrt{\lambda}\, x)$. Using $u'(0) = 0$, we have:

$$
\begin{gathered}
u_\lambda'(0) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\, y) \\
u_\lambda'(0) = 0 \\
0 = 0
\end{gathered}
$$

Then, using $u'(L_2) = 0$, we have:

$$
\begin{gathered}
u_\lambda'(L_2) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\, l_2) \\
0 = \sqrt{\lambda}\sin(\sqrt{\lambda}\, L_2) \\
0 = \sin(\sqrt{\lambda}\, L_2) \\
\sqrt{\lambda}\, L_1,\ n\pi, \text{ where } n \in \mathbb{N}
\end{gathered}
$$

So, $\lambda = \left(\dfrac{n\pi}{L_2}\right)^2$, where $n \in \mathbb{N}$. These are the allowable eigenvalues.

Denote this result as $\lambda_{n,N}$. Finally, let $u_{n,N}(y) = \sin\left(\dfrac{n\pi y}{L_2}\right)$, where $n \in \mathbb{N}$.

10)The two sets of *eigendata* that were obtained from the two problems for the one-dimensional *Laplacian* are identical except for some notation changes here and there.