Homework 2

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1) Let the differential equation $u''(x) = 0$ for $-1 \le x \le 1$ subject to the following boundary conditions

$$-u'(-1) = \lambda u(-1) \text{ and } u'(-1) = \lambda u(1)$$

be given.

Then, we have the following:

$$u''(x) = 0$$

$$u'(x) = c, \text{ for some constant } c \in \mathbb{Z}$$

$$u(x) = c_1(x) + c_0, \text{ for some constants } c_0, c_1 \in \mathbb{Z}, \text{ which is our general solution.}$$

So, our eigenfunctions are $u(x) = x$ and $u(x) = 1$. Then, in order to use the initial conditions, we have to calculate the following:

$$u(1) = 1 \qquad u(-1) = 1 \qquad u'(x) = 1$$

Now, to solve for our λ values, we have to use the initial conditions:

For $u(x) = x$, we have:

$$\begin{aligned} u'(1) &= \lambda u(1) \\ 1 &= \lambda(1) \\ 1 &= \lambda \end{aligned} \qquad \begin{aligned} -u'(-1) &= \lambda u(-1) \\ -u'(-1) &= (1)(-1) \\ -u'(-1) &= -1 \\ u'(-1) &= 1 \end{aligned}$$

For $u(x) = 1$, we have:

$$\begin{aligned} u'(1) &= \lambda u(1) \\ u'(1) &= \lambda(1) \\ 1 &= \lambda(1) \\ \lambda &= 1 \end{aligned} \qquad \begin{aligned} -u'(-1) &= \lambda u(-1) \\ -u'(-1) &= (1)(-1) \\ -u'(-1) &= -1 \\ -u'(-1) &= -1 \end{aligned}$$

Thus, our eigenfunction(s) are $u(x) = x$ and $u(x) = 1$ and our corresponding eigendata is $\lambda = 1$.

2) For the $S - eigenvalue$ problem, the eigenfunctions that were obtained do satisfy the given boundary conditions. However, the functions obtained in the *Dirchlet-Laplacian* and the *Neumann-Laplacian* did not satisfy the respective boundary conditions. Also, the eigenfunctions obtained in this eigenvalue problem are consistent with the eigenfunctions in the other two eigenvalue problems.

3) In order to show that our two eigenfunctions are in $L^2(-1,1)$, we have to evaluate the following integral for each function:

$$\int_{-1}^{1} \left|u(x)\right|^2 dx$$

For $u(x) = x$, we have:

$$\int_{-1}^{1} \left|u(x)\right|^2 dx$$
$$\implies \int_{-1}^{1} |x|^2 \, dx$$
$$\implies \frac{1}{3}x^3\Big|_{-1}^{1}$$
$$\implies \left[\frac{1}{3}(1)^3\right] - \left[\frac{1}{3}(-1)^3\right]$$
$$\implies \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

For $u(x) = 1$, we have:

$$\int_{-1}^{1} \left|u(x)\right|^2 dx$$
$$\implies \int_{-1}^{1} |1|^2 \, dx$$
$$\implies \int_{-1}^{1} 1dx$$
$$\implies x\Big|_{-1}^{1}$$
$$\implies \left[1\right] - \left[-1\right] = 2$$

So, from the above work, we can see that only $u(x) = x \in L^2(-1,1)$.

4) To see if $u(x) = x$ and $u(x) = 1$ are orthogonal, we have to integrate their product from -1 to 1:

$$\int_{-1}^{1} (1)(x)dx$$
$$\implies \int_{-1}^{1} xdx$$
$$\implies \frac{1}{2}x^2\Big|_{-1}^{1}$$
$$\implies \left[\frac{1}{2}(1)^2\right] - \left[\frac{1}{2}(-1)2\right]$$
$$\implies \frac{1}{2} - \left(-\frac{1}{2}\right) = 0$$

So, the $S - eigenfunctions$ are orthogonal to each other since their inner (dot) product is zero. Geometrically, when we consider these functions as "vectors," this means that the "vectors" are perpendicular to each other.

5) All of the functions from classes one through four are harmonic functions since their respective $\Delta u(x, y) = 0$

6) In order to show that the functions in class-1 are in $L^2(\Omega)$, we have to evaluate the following integral for each function:

$$\int_{-1}^{1}\int_{-1}^{1} \Big|u(x, y)\Big|^2 \; dxdy$$

For $S_0(x, y) = 1$:

If we consider the outer integral, we have:

$$\int_{-1}^{1}\int_{-1}^{1} |1|^2 \; dxdy$$
$$\implies \int_{-1}^{1}\int_{-1}^{1} 1 \, dxdy$$
$$\implies x\Big|_{-1}^{1}$$
$$\implies \Big[1\Big] - \Big[-1\Big] = 2$$

Now, if we consider the integral, we have:

$$\int_{-1}^{1} 2 \; dy$$
$$\implies 2y\Big|_{-1}^{1}$$
$$\implies \Big[2(1)\Big] - \Big[2(-1)\Big] = 2(-2) = 4$$

For $S_{1,1}(x, y) = \cosh(vx)\cos(vy)$, we have:

$$\int_{-1}^{1}\int_{-1}^{1} \Big|\cosh(vx)\cos(vy)\Big|^2 \; dxdy$$
$$\implies \int_{-1}^{1}\int_{-1}^{1} \cosh^2(vx)\cos^2(vy) \, dxdy$$

Now, if we consider the inner integral, we have:

$$\int_{-1}^{1} \cosh^2(vx) \; dx$$
$$\implies v\int_{-1}^{1} \cosh^2(vx) \; dx$$
$$\implies v\int_{-1}^{1} \frac{1}{2}(1 + \cosh(2vx) \; dx$$
$$\implies v\left[\frac{1}{2}\left(x + \frac{1}{2}\sinh(2vx)\right)\Big|_{-1}^{1}\right]$$

$$\implies v\left[\frac{1}{2}\left(x+\frac{1}{2}\sinh(vx)\cosh(vx)\right)\Big|_{-1}^{1}\right]$$

$$v\left[\frac{1}{2}\left(1+\frac{1}{2}\sinh(v(1))\cosh(v(1))\right)\right]-v\left[\frac{1}{2}\left(-1+\frac{1}{2}\sinh(v(-1))\cosh(v(-1))\right)\right]$$

$$\implies v\left[\frac{1}{2}\left(1+\frac{1}{2}\sinh(v)\cosh(v)\right)\right]-v\left[\frac{1}{2}\left(-1+\frac{1}{2}\sinh(-v)\cosh(-v)\right)\right]$$

Now, if we consider the outer integral, we have:

$$\int_{-1}^{1}\cos^2(vy)\ dy$$

Now, the astute reader may recall the following (trig) identity:

$$cos^2(x)=\frac{1}{2}+\cos(2x)$$

Using this identity, our integral becomes:

$$\int_{-1}^{1}\frac{1+\cos(v\cdot vy)}{2}\ dy$$
$$\implies \frac{1}{2}\int_{-1}^{1}1+\cos(v\cdot vy)\ dy$$
$$\implies \frac{1}{2}\left(y\Big|_{-1}^{1}+\frac{\sin(2vy)}{4v}\Big|_{-1}^{1}\right)$$
$$\implies \frac{1}{2}\left(2+\left(\frac{\sin(2v)}{4v}-\frac{\sin(-2v)}{4v}\right)\right)$$
$$\implies \frac{1}{2}\left(2+2\frac{\sin(2v)}{4v}\right)$$
$$\implies \left(1+\frac{\sin(2v)}{4v}\right)$$

So, our final result is:

$$v\left[\frac{1}{2}\left(1+\frac{1}{2}\sinh(v)\cosh(v)\right)\right]-v\left[\frac{1}{2}\left(-1+\frac{1}{2}\sinh(-v)\cosh(-v)\right)\right]\cdot\left(1+\frac{\sin(2v)}{4v}\right)$$

For $S_{1,2}(x,y)=\cos(vx)\cosh(vy)$, we have:

$$\int_{-1}^{1}\int_{-1}^{1}\left|\cos(vx)\cosh(vy)\right|^2\ dxdy$$
$$\implies \int_{-1}^{1}\int_{-1}^{1}\cosh^2(vx)\cos^2(vy)\,dxdy$$

Looking at the inner integral, we have:

$$\int_{-1}^{1}\cos^2(vx)\ dx$$

and by a similar argument from the previous calculation in this question, our solution is:

$$1 + \frac{\sin(2v)}{4v}$$

Looking at the outer integral, we have:

$$\int_{-1}^{1} \cosh^2(vy)\ dx$$

and by a similar argument from the previous calculation in this question, our solution is:

$$v\left[\frac{1}{2}\left(1 + \frac{1}{2}\sinh(v)\cosh(v)\right)\right] - v\left[\frac{1}{2}\left(-1 + \frac{1}{2}\sinh(-v)\cosh(-v)\right)\right]$$

So, our final answer is

$$v\left[\frac{1}{2}\left(1 + \frac{1}{2}\sinh(v)\cosh(v)\right)\right] - v\left[\frac{1}{2}\left(-1 + \frac{1}{2}\sinh(-v)\cosh(-v)\right)\right] \cdot 1 + \frac{\sin(2v)}{4v}$$

7) Yes, I do think that the functions in classes two, three, and four are also in $L^2(\Omega)$ because they all share similar vector calculus properties such at the laplacian, gradient, etc.

8) For $S_{1,2}(x, y) = \cos(vx)\cosh(vy)$, we have:

$$\int_{-1}^{1}\int_{-1}^{1} \cos(vx)\cosh(vy)\ dxdy$$

Considering the inner integral, we have:

$$\int_{-1}^{1} \cos(vx)\ dxdy$$

$$\implies \frac{\sin(vx)}{v}\Big|_{-1}^{1}$$

$$\implies \frac{\sin(v)}{v} - \left[\frac{\sin(-v)}{v}\right] = 2\frac{\sin(v)}{v}$$

Considering the outer integral, we have:

$$\int_{-1}^{1} \cosh(vy)\ dy$$

$$\implies \frac{\sinh(vy)}{v}\Big|_{-1}^{1}$$

$$\implies \frac{\sinh(v)}{v} - \left[\frac{\sinh(-v)}{v}\right] = 2\frac{\sinh(v)}{v}$$

So, our final answer is

$$2\frac{\sin(v)}{v} \cdot 2\frac{\sinh(v)}{v} = 2\left(\frac{\sin(v)}{v} \cdot \frac{\sinh(v)}{v}\right)$$

For $S_{2,1}(x,y) = \sinh(vx)\sin(vy)$, we have:

$$\int_{-1}^{1}\int_{-1}^{1} \sinh(vx)\sin(vy)\ dxdy$$

Considering the inner integral, we have:

$$\int_{-1}^{1} \sinh(vx)\ dx$$
$$\implies \frac{\cosh(vx)}{v}\Big|_{-1}^{1}$$
$$\implies \frac{\cosh(v)}{v} - \left[\frac{\cosh(-v)}{v}\right]$$
$$\implies \frac{\cosh(v)}{v} - \left[\frac{\cosh(-v)}{v}\right] = 0$$

Considering the outer integral, we have:

$$\int_{-1}^{1} \sin(vy)\ dy$$
$$\implies -\frac{\cos(vy)}{v}\Big|_{-1}^{1}$$
$$\implies -\frac{\cos(v)}{v} - \left[-\frac{\cos(-v)}{v}\right]$$
$$\implies -\frac{\cosh(v)}{v} + \left[\frac{\cos(-v)}{v}\right] =$$
$$\implies -\frac{\cos(v)}{v} + \left[\frac{\cos(v)}{v}\right] = 0$$

So, our final answer is 0.

9)According to our calculations, we are able to find that all the functions from the four classes are orthogonal except for $S_{1,2}(x,y) = \cos(vx)\cosh(vy)$.

10) After looking over the harmonic functions for the one and two dimensional case(s), I noticed that the one-dimensional cases are either constant or linear functions, while the second-dimensional case(s) are periodic functions. However, both the one-dimensional and the two-dimensional case(s) have their laplacian equal to zero, which I was not expecting considering that the dimensions are different.