Homework 4

Brendan Busey

1)

Proof. Let $w(x,t) = u(-x,t) - u(x,t)$. Then, $w(x,t)$ satisfies the differential equation and,

$$w(x,0) = u(-x,0) - u(x,0) = \varphi(-x) - \varphi(x) = \varphi(x) - \varphi(x) = 0$$

Thus, we have the initial condition diffusion equation

$$w_t = kW_{xx} \; w(x,0) = 0$$

By uniqueness, $w(x,0) \equiv 0$. Then,

$$\begin{gathered} u(-x,t) - u(x,t) = 0 \\ u(-x,t) = u(x,t) \end{gathered}$$

Thus, $u(x,t)$ is an even function of x. Since $u(x,t)$ is an even function, we know that $u(x,t)$ will be symmetric around the graph representing the physical situation, meaning that there will be a situation where two different points in that domain will give you the same diffusion level and that at other points, the diffusion level will be zero.

□

2) Consider the Q-problem

$$\frac{\partial Q}{\partial t} = D\frac{\partial^2 Q}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty,\, t > 0$$

$$Q(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ -1 & \text{if } x > 0 \end{cases}$$

Now, suppose $Q(x,t) = g\left(\frac{x}{\sqrt{4Dt}}\right)$, for some one-variable function g.

Next, let $\xi = \frac{x}{\sqrt{4Dt}}$. Then, $\frac{\partial Q}{\partial t} = g'(\xi)\left(-\frac{1}{2}x(4Dt)^{-\frac{3}{2}}(4D)\right)$.

Looking more closely at $g'(\xi)$, we have

$$g'(\xi) = \left(-\frac{x4D}{2(4Dt)\sqrt{4Dt}}\right)$$

Returning to $\frac{\partial Q}{\partial t}$ and simplifying, we have

$$\frac{\partial Q}{\partial t} = g'(\xi)\left(- \frac{x}{2t\sqrt{4Dt}}\right)$$
$$\frac{\partial Q}{\partial t} = g'(\xi)\left(- \frac{1}{2t}\xi\right)$$

Now, if we look at derivatives with respect with to x, we get

$$\frac{\partial Q}{\partial x} = g'(\xi)\left(\frac{1}{\sqrt{4Dt}}\right) \quad \text{and} \quad \frac{\partial^2 Q}{\partial x^2} = g''(\xi)\left(\frac{1}{\sqrt{4Dt}}\right)$$

Then, if we substitute the derivatives we just calculated into the PDE, we see it transform into

$$-\frac{1}{2t}\xi g'(\xi) = D\frac{1}{4Dt}g''(\xi)$$

If we then multiply through by $4t$, we get

$$-\frac{4}{2}\xi g'(\xi) = g''(\xi)$$
$$g''(\xi) + 2\xi g'(\xi) = 0$$

Now, let $w(\xi) = g'(\xi)$ and $w'(\xi) = g''(\xi)$. Substituting these into our differential equation, we get

$$w'(\xi) + 2\xi w(\xi) = 0$$

So, if we remember our "training", in order to solve this differential equation, the first thing we need to do is calculate the integrating factor

$$e^{\int_0^\xi 2s\ ds} = e^{\xi^2}$$

Then, if we multiply through by the integrating factor, we have

$$e^{\xi^2}w'(\xi) + 2\xi w(\xi) = 0$$
$$\frac{d}{d\xi}\left[e^{\xi^2}w(\xi)\right] = 0$$
$$\int_0^\xi \frac{d}{ds}\left[e^{s^2}w(s)\right] = 0$$

Continuing on, by the *Fundamental Theorem of Calculus*, we have

$$\left[e^{s^2}w(s)\right]\Big|_{s=0}^{s=\xi}$$

Which, simplifies to the following if we evaluate it

$$e^{\xi^2}w(\xi) - w(0) = 0$$
$$w(\xi) = w(0)e^{\xi^2}$$

Now, if we back substitute for w, we get

$$g'(\xi) = g'(0)e^{\xi^2}$$
$$\int_0^\xi g'(s)ds = g'(0)\int_0^\xi e^{-s^2}ds$$
$$g(\xi) = g(0) + g'(0)\int_0^\xi e^{-s^2}$$

Now, if we recall

$$Q(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ -1 & \text{if } x > 0 \end{cases}$$

Then, for $x > 0$, we have

$$1 = Q(x,0) = \lim_{x \to 0^+} Q(x,t) = g(0) + g'(0) \int_0^{+\infty} e^{-s^2} ds$$

$$1 = g(0) + g'(0) \int_0^{+\infty} e^{-s^2} ds = g(0) + g'(0) \frac{\sqrt{\pi}}{2}$$

$$1 = g(0) + g'(0) \frac{\sqrt{\pi}}{2}$$

If we then look at the case where $x < 0$, we have

$$0 = Q(x,0) = \lim_{x \to 0^-} Q(x,t) = g(0) + g'(0) \int_0^{+\infty} e^{-s^2} ds$$

$$0 = g(0) + g'(0) \int_0^{+\infty} e^{-s^2} ds = g(0) + g'(0) \frac{\sqrt{\pi}}{2}$$

$$1 = g(0) + g'(0) \frac{\sqrt{\pi}}{2}$$

Finally, if we add our results for the above two cases we considered, we have

$$g(0) = \frac{1}{2}$$

$$g'(0) = \frac{1}{\sqrt{\pi}}$$

$$Q(x,t) = g(\xi) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\xi} e^{-s^2} ds$$

$$Q(x,t) = g(\xi) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\frac{x}{\sqrt{4Dt}}} e^{-s^2} ds$$

$$Q(x,t) = g(\xi) = \frac{1}{2} + \frac{1}{2} erf\left(\frac{x}{\sqrt{4Dt}}\right)$$

Now, if we consider

$$\varphi(x) = \begin{cases} 0 & \text{if } x < \text{-L} \\ \sqrt{\pi} & \text{if } -L \leq x \leq L \\ 0 & \text{if } x > L \end{cases}$$

we can note a few things. The first being that whenever x is less than $-L$, then we have that φ is zero which indicates that the diffusion of kool-aid through the tube is also zero. Next, we note that when $-L \leq x \leq L$, we have that φ is equal to $\sqrt{\pi}$, so we know that the diffusion throughout the tube in this case is linear. Finally, whenever x is greater than L, we have that φ is equal to zero which indicates that the diffusion of kool-aid through the tube is, again, zero.

3) Suppose that if $u_1(x,t)$ and $u_2(x,t)$ are solutions to the given problem, then $u_1(x,t) = u_2(x,t) \ \forall \ (x,t) \in \ \Omega_{L,\infty}$, suppose

Proof. To begin, define $w(x,t) = u_1(x,t) - u_2(x,t)$.

At this point, we know that w satisfies the PDE $\dfrac{\partial w}{\partial t} - D\dfrac{\partial^2}{\partial x^2} = 0$ with the initial condition of $w(x,0) = 0$ and the boundary conditions $w(0,t) = 0$ and $w(L,t) = 0$. Okay, now, multiply $\dfrac{\partial w}{\partial t} - D\dfrac{\partial^2}{\partial x^2} = 0$ by w to obtain

$$w\frac{\partial w}{\partial t} - Dw\frac{\partial^2 w}{\partial x^2} = 0$$

Now, before continuing, let us note the following:

$$1) \ \frac{\partial}{\partial t}\left[\frac{1}{2}w^2\right] = w\left(\frac{\partial w}{\partial t}\right)$$

$$2) \ \frac{\partial}{\partial x}\left[-Dw\frac{\partial w}{\partial x}\right] = -Dw\frac{\partial^2 w}{\partial x^2} - D\left(\frac{\partial w}{\partial x}\right)^2$$

$$3) \ \frac{\partial}{\partial x}\left[-Dw\frac{\partial w}{\partial x}\right] + D\left(\frac{\partial w}{\partial x}\right)^2 = -Dw\frac{\partial^2 w}{\partial x^2}$$

Using the above points we just noted, $w\dfrac{\partial w}{\partial t} - Dw\dfrac{\partial^2 w}{\partial x^2} = 0$ becomes

$$\frac{\partial}{\partial t}\left[\frac{1}{2}w^2\right] + \frac{\partial}{\partial x}\left[-Dw\left(\frac{\partial w}{\partial x}\right)\right] + D\left(\frac{\partial w}{\partial x}\right)^2 = 0$$

Next, we want to integrate over the interval $0 \leq x \leq L$, giving us

$$\int_0^L \frac{\partial}{\partial t}\left[\frac{1}{2}w^2\right]dx + \int_0^L \frac{\partial}{\partial x}\left[-Dw\left(\frac{\partial w}{\partial x}\right)\right] + D\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx = 0$$

which simplifies to, with help from the *Fundamental Theorem of Calculus*,

$$\frac{d}{dt}\int_0^L \frac{1}{2}w^2 dx + \left[-Dw\left(\frac{\partial w}{\partial x}\right)\right]\Big|_{x=0}^{x=L} + D\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx = 0$$

Next, if we use the provided boundary conditions, the $\left[-Dw\left(\dfrac{\partial w}{\partial x}\right)\right]\Big|_{x=0}^{x=L}$ vanishes since, according to our boundary conditions

when we "plug-in" both x and L into our w function, we get zero each time. So, after doing some rearrangement, we come up with the following

$$\frac{d}{dt}\int_0^L \frac{1}{2}w^2 dx = -D\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \leq 0$$

Thus, we have that the function $F(t) = \int_0^L \frac{1}{2} w(x,t)^2 dx$, by the above function we just derived, is a decreasing function. So, since $F(0) \geq F(t)$ for any $t > 0$, we know that $\int_0^L w(x,0)^2 dx \geq \int_0^L w(x,t)^2 dx$. So, $0 \geq \int_0^L w(x,t)^2 dx \; \forall \; t > 0$. The previous statements imply that $w(x,t) = 0 \; \forall \; (x,t) \in \; \Omega_{L,\infty}$, which in turn implies that $u_1(x,t) = u_2(x,t) \in \; \Omega_{L,\infty}$.

$\square$