1) Let the differentiable equation $f''(x) + \lambda f(x) = 0$ be given and let $g(x)$ be a twice differentiable function.

Proof. Now, multiply $f''(x) + \lambda f(x) = 0$ by $g(x)$ to obtain

$$f''(x)g(x) + \lambda f(x)g(x) = 0$$

Now, we want to integrate $f''(x)g(x) + \lambda f(x)g(x) = 0$ by parts. But, before we do, let use the *properties of integrals* to obtain

$$\int_0^L f''(x)g(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Okay, first, consider the case where $\lambda > 0$. Next, integrate $\displaystyle\int_0^L f''(x)g(x)\ dx$ by parts

$$u = g(x) \qquad du = g''(x) \qquad dv = f''(x) \qquad v = \int f''(x) = f'(x) + c$$

$$\int_0^L f''(x)g(x) = \left[g(x)f'(x)\right]\Bigg|_0^L - \int_0^L f'(x)g'(x)\ dx$$

Substituting in for $\displaystyle\int_0^L f''(x)g(x)\ dx$, we have

$$\int_0^L f''(x)g(x)\ dx + \int_0^L \lambda f(x)g(x) = 0$$

$$= \left[g(x)f'(x)\right]_0^L - \int_0^L f'(x)g'(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Now, if we use $f'_{n,N}(x) = -\sqrt{\lambda_{n,D}}f_{n,D}(x)$, and $g'_{n,M}(x) = -\sqrt{\lambda_{m,D}}f_{m,D}(x)$, we get

$$\left[g(x)(-\sqrt{\lambda_{n,D}}f_{n,D}(x))\right]_0^L - \int_0^L -\sqrt{\lambda_{n,D}}f_{n,D}(x) \cdot -\sqrt{\lambda_{m,D}}f_{m,D}(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Then, making the substitution $f(x) = f_{n,N}(x)$ and $g(x) = f_{m,N}(x)$, we have

$$\left[f_{m,N}(x)(-\sqrt{\lambda_{n,D}}f_{n,D}(x))\right]_0^L - \int_0^L -\sqrt{\lambda_{n,D}}f_{n,D}(x) \cdot -\sqrt{\lambda_{m,D}}f_{m,D}(x)\ dx + \int_0^L \lambda f_{n,N}(x)f_{m,N}(x)\ dx = 0$$

Simplifying, we have

$$\left[f_{m,N}(x)(-\sqrt{\lambda_{n,D}}f_{n,D}(x))\right]_0^L - (\sqrt{\lambda_{n,D}} \cdot \sqrt{\lambda_{m,D}}) \int_0^L f_{n,D}(x) \cdot f_{m,D}(x)\ dx + \int_0^L \lambda f_{n,N}(x) f_{m,N}(x)\ dx = 0$$

Since we know that the *Dirichlet-Laplacian* eigendata inner-product is zero, the

$$(\sqrt{\lambda_{n,D}} \cdot \sqrt{\lambda_{m,D}}) \int_0^L f_{n,D}(x) \cdot f_{m,D}(x)\ dx$$

term will be zero, leaving us with

$$\left[f_{m,N}(x)(-\sqrt{\lambda_{n,D}}f_{n,D}(x))\right]_0^L + \int_0^L \lambda f_{n,N}(x) f_{m,N}(x)\ dx = 0$$

Now, if you evaluate $-\sqrt{\lambda_{n,D}}f_{n,D}(x)$ from 0 to L, we know that by the properties of *Dirichlet-Laplacian* eigendata, this term will go to zero. This leaves us with

$$\int_0^L \lambda f_{n,N}(x) f_{m,N}(x)\ dx = 0$$

$$= \lambda \int_0^L f_{n,N}(x) f_{m,N}(x)\ dx = 0$$

$$= \int_0^L f_{n,N}(x) f_{m,N}(x)\ dx = 0$$

as desired.

Now, we turn our attention to calculating the *norm* for the *Neumann-Laplacian* eigenfunction(s). First, let the differentiable equation $f''(x) + \lambda f(x) = 0$ be given and let $g(x)$ be a twice differentiable function. Now, multiply $f''(x) + \lambda f(x) = 0$ by $g(x)$ to obtain

$$f''(x)g(x) + \lambda f(x)g(x) = 0$$

Now, we want to integrate $f''(x)g(x) + \lambda f(x)g(x) = 0$ by parts. But, before we do, let's use the properties of integrals to obtain

$$\int_0^L f''(x)g(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Next, we integrate $\displaystyle\int_0^L f''(x)g(x)\ dx$ by parts

$$u = g(x) \quad du = g'(x) \quad dv = f''(x) \quad v = \int f''(x) = f'(x) + c$$

We can now re-write $\displaystyle\int_0^L f''(x)g(x)\ dx$ as

$$\int_0^L f''(x)g(x)\ dx = \Big[g(x)f'(x)\Big]_0^L - \int_0^L f'(x)g'(x)\ dx$$

If we then substitute in for $\displaystyle\int_0^L f''(x)g(x)\ dx$, we have

$$\int_0^L f''(x)g(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$
$$= \Big[g(x)f'(x)\Big]_0^L - \int_0^L f'(x)g'(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Now, if we use $f'_{n,N}(x) = -\sqrt{\lambda_{n,D}}f_{n,D}$ and $g'_{n,N}(x) = -\sqrt{\lambda_{n,D}}f_{n,D}$, we have

$$\Big[g(x)(\sqrt{\lambda_{n,D}}f_{n,D}(x))\Big]_0^L - \int_0^L -\sqrt{\lambda_{n,D}}f_{n,D}(x)\cdot\sqrt{\lambda_{n,D}}\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

If we evaluate $\sqrt{\lambda_{n,D}}f_{n,D}(x)$ from 0 to L, we know that by the properties of *Dirichlet-Laplacian* eigendata, this term will go to zero, leaving us with

$$-\int_0^L -\sqrt{\lambda_{n,D}}f_{n,D}(x)\cdot\sqrt{\lambda_{n,D}}\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Then, if we factor our constants, we have

$$-(-\sqrt{\lambda_{n,D}})(-\sqrt{\lambda_{n,D}})\int_0^L f_{n,D}(x)f_{n,D}(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Then, if we combine our integrals, we have

$$-(-\sqrt{\lambda_{n,D}})(-\sqrt{\lambda_{n,D}})\int_0^L f_{n,D}(x)f_{n,D}(x) + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Dividing out our constants, we find

$$\int_0^L f_{n,D}(x)f_{n,D}(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Re-expanding our integrals, we have

$$-\int_0^L f_{n,D}(x)f_{n,D}(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

$$\implies \int_0^L \lambda f(x)g(x)\ dx = \int_0^L f_{n,D}(x)f_{n,D}(x)\ dx$$

$$\implies \int_0^L \lambda f(x)g(x)\ dx = \sqrt{\frac{L}{2}},\ \textit{since we know that}\ \|f_{n,D}\| = \sqrt{\frac{L}{2}}$$

Finally, making the substitution $f(x) = f)n, N(x)$ and $g(x) = f_{n,N}(x)$, we have

$$\int_0^L \lambda f(x)g(x)\ dx = \sqrt{\frac{L}{2}}, \text{ as desired.}$$

□

2)

Proof. First, we begin by supposing that the solution $f(x)$ has the specific form $f(x) = e^{kx}$, so $f''(x) = k^2e^{kx}$. We then plug-in $f(x)$ and $f''(x)$ into our ODE giving us

$$k^2e^{kx} + \lambda e^{kx} = 0$$

Next, we factor our the e^{kx} term, leaving us with

$$e^{kx}\left[k^2 + \lambda\right] = 0$$

Which simplifies to

$$k^2 + \lambda = 0, \text{ which is our } \textit{auxiliary equation}$$

Now, we want to consider the case where $\lambda < 0$, or when λ is strictly negative. Using the auxiliary equation $k^2 + \lambda = 0$, we solve for k

$$k = \pm\sqrt{-\lambda}, \text{ when } \lambda < 0$$

This gives us the following solutions

$$v_1(x) = e^{\sqrt{-\lambda}x} \qquad \text{and} \qquad v_2(x) = e^{-\sqrt{-\lambda}x}$$

We save showing linear independence for later and instead, impose our *mixed-boundary* conditions

$$v_1(x) = e^{\sqrt{-\lambda}x} \quad \text{and} \quad v_2(x) = e^{-\sqrt{-\lambda}x}$$
$$v_1(0) = e^{\sqrt{-\lambda}(0)} \quad \text{and} \quad v_2(x) = e^{-\sqrt{-\lambda}(0)}$$
$$v_1(0) = e^0 \quad \text{and} \quad v_2(x) = e^0$$
$$v_1(0) = 1 \neq 0 \quad \text{and} \quad v_2(x) = 1 \neq 0$$

Therefore, neither $v_1(x)$ nor $v_2(x)$ are eigenfunctions because they do not satisfy the *mixed-boundary* conditions

Moving on, we consider the case where $\lambda = 0$. When $\lambda = 0$, our ODE becomes $f''(x) = 0$. We can solve for $f(x)$ by integrating twice

$$f'(x) = c_1$$
$$f''(x) = c_1x + c_2$$

This gives us the following two solutions

$$h_1(x) = x \quad \text{and} \quad h_2(x) = 1$$

Now, we impose our *mixed-boundary* conditions

$$h_1(x) = x \quad \text{and} \quad h_2(x) = 1$$
$$h_1(0) = 0 \quad \text{and} \quad h_2(0) = 1 \neq 0$$
$$h_1(1) = 1 \neq 0$$

Therefore, neither $h_1(x)$ or $h_2(x)$ are eigenfunctions because they don't satisfy our *mixed-boundary* conditions

Finally, we consider the case where $\lambda > 0$, or λ is strictly positive. Using the previously calculated equation, $k^2 + \lambda = 0$, we solve for k

$$k = \pm\sqrt{-\lambda} \quad \text{or} \quad k = \pm i\sqrt{\lambda}$$

So, we obtain the solutions

$$u_1(x) = e^{i\sqrt{\lambda}x} \quad \text{and} \quad u_2(x) = e^{-i\sqrt{\lambda}x}$$

However, we only want real valued solutions. So, we use *Euler's formula*

$$\tilde{u}_1(x) = \frac{1}{2}\left[u_1(x) + u_2(x)\right]$$

$$\tilde{u}_1(x) = \frac{1}{2}\left[\cos\left(\sqrt{\lambda}x\right) + i\sin\left(\sqrt{\lambda x}\right) + \cos\left(\sqrt{-\lambda x}\right) + i\sin\left(\sqrt{-\lambda x}\right)\right]$$

$$\tilde{u}_1(x) = \cos\left(\sqrt{\lambda x}\right)$$

$$\tilde{u}_2(x) = \frac{1}{2i}\left[\cos\left(\sqrt{\lambda}x\right) + i\sin\left(\sqrt{\lambda x}\right) - \cos\left(\sqrt{-\lambda x}\right) + i\sin\left(\sqrt{-\lambda x}\right)\right]$$

$$\tilde{u}_2(x) = \sin\left(\sqrt{\lambda x}\right)$$

Now, we want to impose our *mixed-boundary* conditions

$$\tilde{u}_1(x) = \cos\left(\sqrt{\lambda x}\right) \qquad \tilde{u}_2(x) = \sin\left(\sqrt{\lambda x}\right)$$

$$\tilde{u}_1(0) = 1 \neq 0 \qquad \tilde{u}_2(x) = \sin(0) = 0$$

Because $\tilde{u}_2(x) = \sin\left(\sqrt{\lambda x}\right)$ satisfies our first boundary condition, we want to make it so that $\tilde{u}_2'(L) = 0$. So

$$\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right) = 0$$

Solving for λ, we have

$$\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right) = 0$$

$$\cos\left(\sqrt{\lambda}L\right) = 0$$

$$\cos^{-1}\left(\cos\left(\sqrt{\lambda L}\right)\right) = \cos^{-1}(0)$$

So, $(\lambda L) = (2n-1)\dfrac{\pi}{2}$, where $2n-1$ are odd numbers. So,
$\lambda = \left(\dfrac{(n\pi - \frac{\pi}{2})}{L}\right)^2, \ \forall\ n \in \mathbb{N}$
Therefore, our *DN-Laplacian* eigenvalues are denoted as:

$$\lambda_{n,DN} = \left(\frac{(n\pi - \frac{\pi}{2})}{L}\right)^2, \ \forall\ n \in \mathbb{N}$$

Thus, our *DN-Laplacian* eigenfunctions are

$$f_{n,DN}(x) = \sin\left(\frac{(n\pi - \frac{\pi}{2})}{L}x\right), \ \forall\ n \in \mathbb{N}$$

□

3)

Proof. First, we suppose that the solutions are of the form $y(x) = e^{kx}$. Now, we have already demonstrated the derivation of our *auxiliary equation* in the previous question, so we will not repeat that work again here. So, using our previously obtained *auxiliary equation*

$$k^2 + \lambda = 0$$

and the work shown in the previous question, we know that our two solutions are

$$y_1(x) = e^{\sqrt{-\lambda}x} \qquad \text{and} \qquad y_2(x) = e^{-\sqrt{-\lambda}x}$$

Now, to apply our *mixed-boundary conditions*, we take some derivatives

$$y_1'(x) = \sqrt{-\lambda}e^{\sqrt{-\lambda}x} \qquad \text{and} \qquad y_2'(x) = -\sqrt{-\lambda}e^{\sqrt{-\lambda}x}$$

Applying our *mixed-boundary conditions*, we have

$$\begin{aligned} y_1'(x) &= \sqrt{-\lambda}e^{\sqrt{-\lambda}x} &\quad \text{and} \quad y_2'(x) &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}x} \\ y_1'(0) &= \sqrt{-\lambda}e^{\sqrt{-\lambda}(0)} &\quad \text{and} \quad y_2'(0) &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}(0)} \\ 0 &= \sqrt{-\lambda} &\quad \text{and} \quad 0 &= -\sqrt{-\lambda} \end{aligned}$$

Therefore, neither $y_1(x)$ nor $y_2(x)$ are eigenfunctions since they do not satisfy our *mixed-boundary conditions*

Moving on, we consider the case where $\lambda = 0$. Plugging-in $\lambda = 0$, our ODE becomes $f''(x) = 0$. The astute reader might also notice that we have done the work to derive our solutions

$$h_1(x) = x \qquad \text{and} \qquad h_2(x) = 1$$

in the previous question, so we won't repeat said work here. Again, in order to apply our *mixed-boundary conditions*, we will have to calculate some derivatives

$$\begin{aligned} h_1(x) &= x &\quad \text{and} \quad h_2(x) &= 1 \\ h_1'(x) &= 1 &\quad \text{and} \quad h_2'(x) &= 0 \end{aligned}$$

Applying our *mixed-boundary conditions*, we find that

$$h_1'(x) = 1 \quad \text{and} \quad h_2'(x) = 0$$
$$h_1'(0) = 1 \quad \text{and} \quad h_2'(0) = 0$$

Therefore, $h_2(x)$ is the only solution that satisfies our *mixed-boundary conditions*

Finally, we want to consider the case where $\lambda > 0$. Again, using our work from the previous question, we know that our solutions are of the form

$$z_1(x) = \cos\left(\sqrt{\lambda}x\right) \quad \text{and} \quad z_2(x) = \sin\left(\sqrt{\lambda}x\right)$$

Taking some derivatives, since that allows us to apply our *mixed-boundary conditions*, we find that

$$z_1(x) = \cos\left(\sqrt{\lambda}x\right) \quad \text{and} \quad z_2(x) = \sin\left(\sqrt{\lambda}x\right)$$
$$z_1'(x) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) \quad \text{and} \quad z_2'(x) = \sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

Applying our *mixed-boundary conditions* we have

$$z_1'(x) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) \quad \text{and} \quad z_2'(x) = \sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$
$$z_1'(0) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}(0)\right) \quad \text{and} \quad z_2'(0) = \sqrt{\lambda}\cos\left(\sqrt{\lambda}(0)\right)$$
$$z_1'(0) = -\sqrt{\lambda}\sin(0) \quad \text{and} \quad z_2'(0) = \sqrt{\lambda}\cos(0)$$
$$0 = 0 \quad \text{and} \quad 0 \neq 1$$

Since $z_1(x) = \cos\left(\sqrt{\lambda}x\right)$ satisfies our first *mixed-boundary condition*, we want to make it so that $z_1'(L) = 0$

$$z_1'(x) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)$$
$$z_1'(L) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)$$
$$0 = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)$$

$$0 = \sin\left(\sqrt{\lambda}L\right)$$

$$\sin^{-1}(0) = \sin^{-1}\left(\sin\left(\sqrt{\lambda}L\right)\right)$$

So, $(\lambda L) = (2n-1)\dfrac{\pi}{2}$, where $2n-1$ are odd numbers. So,
$\lambda = \left(\dfrac{(n\pi - \frac{\pi}{2})}{L}\right)^2$, $\forall\, n \in \mathbb{N}$
Thus, our *ND-Laplacian* eigenfunctions are

$$f_{n,DN}(x) = \cos\left(\frac{(n\pi - \frac{\pi}{2})}{L}x\right),\ \forall\, n \in \mathbb{N}$$

□

4) Let the differentiable equation $f''(x) + \lambda f(x) = 0$ be given and let $g(x)$ be a twice differentiable function.

Proof. Now, multiply $f''(x) + \lambda f(x) = 0$ by $g(x)$ to obtain

$$f''(x)g(x) + \lambda f(x)g(x) = 0$$

Now, we want to integrate $f''(x)g(x) + \lambda f(x)g(x) = 0$ by parts. But, before we do, let use the *properties of integrals* to obtain

$$\int_0^L f''(x)g(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

Next, integrate $\displaystyle\int_0^L f''(x)g(x)\ dx$ by parts

$$u = g(x) \qquad du = g''(x) \qquad dv = f''(x) \qquad v = \int f''(x) = f'(x) + c$$

$$\int_0^L f''(x)g(x) = \left[g(x)f'(x)\right]\Big|_0^L - \int_0^L f'(x)g'(x)\ dx$$

Substituting in for $\displaystyle\int_0^L f''(x)g(x)\ dx$ with the above result, we have

$$\int_0^L f''(x)g(x)\ dx = \left[g(x)f'(x)\right]\Big|_0^L - \int_0^L f'(x)g'(x)\ dx$$

We can now re-write $\displaystyle\int_0^L f''(x)g(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$ as

$$\left[g(x)f'(x)\right]\Bigg|_0^L - \int_0^L f'(x)g'(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

To continue, we want to integrate by parts yet again

$$u = g'(x) \qquad du = g''(x) \qquad dv = f'(x) \qquad v = \int f'(x)\ dx = f(x)$$

We can now re-write $\int_0^L f'(x)g'(x)\ dx$ as

$$\left[g'(x)f(x)\right]_0^L - \int_0^L f(x)g''(x)\ dx$$

Substituting back-in, we can re-write
$\left[g(x)f'(x)\right]\Bigg|_0^L - \int_0^L f'(x)g'(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$ as

$$\left[g(x)f'(x)\right]_0^L \left[g'(x)f(x)\right]_0^L - \int_0^L f(x)g''(x)\ dx + \int_0^L \lambda f(x)g(x)\ dx = 0$$

which is the *Lagrange-Identity* from the beginning of the semester

Now, we turn our attention to showing the orthogonality of the mixed *DN-Laplacian* eigenfunctions. First, let the differentiable equation $f''(x) + \lambda f(x) = 0$ be given and let $g(x)$ be a twice differentiable function. Now, multiply $f''(x) + \lambda f(x)$ to obtain

$$f''(x)g(x) + \lambda f(x)g(x)$$

Now, we want to integrate $f''(x)g(x)$ by parts

$$u = g(x) \quad du = g'(x) \quad dv = f''(x) \quad v = \int f''(x) = f'(x)$$

We can now re-write $f''(x)g(x)$ as

$$f''(x)g(x) = \left[g(x)f'(x)\right]_0^L + \int_0^L f'(x)g'(x)\ dx$$

To continue, we want to integrate by parts yet again

$$u = g'(x) \quad du = g''(x) \quad dv = f'(x) \quad v = \int f'(x) = f(x)$$

We can now re-write $\left[g(x)f'(x)\right]_0^L + \int_0^L f'(x)g'(x)\ dx$ as

$$\left[g(x)f'(x)\right]_0^L - \left[g'(x)f(x)\right]_0^L - \int_0^L f(x)g''(x)\ dx$$

So, at this point, we have

$$f''(x)g(x) = \left[g(x)f'(x)\right]_0^L - \left[g'(x)f(x)\right]_0^L - \int_0^L f(x)g''(x)\ dx$$

Now, if we make the substitution $g(x) = f_{m,DN}(x)$ and $f(x) = f_{n,DN}(x)$, we have

$$\int_0^L f''_{n,DN}(x)f_{m,DN}(x)\ dx =$$
$$\left[f_{m,DN}(x)f'_{n,DN}(x)\right]_0^L - \left[f'm,DN(x)f_{n,DN}(x)\right]_0^L - \int_0^L f_{n,DN}(x)f''_{m,DN}(x)\ dx$$

Looking at the $\left[f_{m,DN}(x)f'_{n,DN}(x)\right]_0^L$ term, if we apply the boundary conditions

$$\begin{cases} f(0) = 0 \\ f'(L) = 0 \end{cases}$$

then this term will go to zero. This leaves us with

$$\int_0^L f''_{n,DN}(x)f_{m,DN}(x)\ dx =$$
$$-\left[f'm,DN(x)f_{n,DN}(x)\right]_0^L - \int_0^L f_{n,DN}(x)f''_{m,DN}(x)\ dx$$

Then, if we look at the $-\left[f'm,DN(x)f_{n,DN}(x)\right]_0^L$ term, we can see that if we again apply the boundary conditions

$$\begin{cases} f(0) = 0 \\ f'(L) = 0 \end{cases}$$

this term will also go to zero. So, are we are left with is

$$\int_0^L f''_{n,DN}(x)f_{m,DN}(x)\ dx = -\int_0^L f_{n,DN}(x)f''_{m,DN}(x)\ dx$$

Now, if we return to our twice differentiable function

$$f''(x) + \lambda f(x) = 0$$

and re-arrange terms, we have

$$f''(x) = -\lambda f(x)$$

If we substitute this into our integral, we have

$$\int_0^L -\lambda f_{n,DN}(x) f_{m,DN}(x) = -\int_0^L f_{n,DN}(x) \cdot -\lambda f_{m,DN}(x)\ dx$$

Simplifying, we have

$$-\int_0^L f_{n,DN}(x) \cdot -\lambda f_{m,DN}(x)\ dx = \int_0^L -\lambda f_{n,DN}(x) f_{m,DN}(x)\ dx$$

$$\implies -(-\lambda) \int_0^L f_{n,DN}(x) f_{m,DN}(x)\ dx = -\lambda \int_0^L f_{n,DN}(x) f_{m,DN}(x)\ dx$$

$$\implies -(-\lambda_n) \int_0^L f_{n,DN}(x) f_{m,DN}(x)\ dx + \lambda_m \lambda \int_0^L f_{n,DN}(x) f_{m,DN}(x)\ dx$$

$$\implies -(-\lambda_n) \int_0^L \lambda_m f_{n,DN}(x) f_{m,DN}(x)\ dx = 0$$

$$\implies -(\lambda_m - \lambda_n) \int_0^L \lambda_m f_{n,DN}(x) f_{m,DN}(x)\ dx = 0$$

Now, since m and n are assumed to be different, we know that $(\lambda_m - \lambda_n)$ cannot be equal to zero. Thus, $\int_0^L \lambda_m f_{n,DN}(x) f_{m,DN}(x)\ dx$ has to be equal to zero, as desired. Hence, we have shown the orthogonality of the mixed *DN-Laplacian* eigenfunctions.

We now move on to show the orthogonality of the mixed *ND-Laplacian* eigenfunctions. Like the calculation we just finished, let the differentiable equation $f''(x) + \lambda f(x) = 0$ be given and let $g(x)$ be a twice differentiable function. Now, multiply $f''(x) + \lambda f(x)$ to obtain

$$f''(x)g(x) + \lambda f(x)g(x)$$

Now, we want to integrate $f''(x)g(x)$ by parts

$$u = g(x) \quad du = g'(x) \quad dv = f''(x) \quad v = \int f''(x) = f'(x)$$

We can now re-write $f''(x)g(x)$ as

$$f''(x)g(x) = \left[g(x)f'(x)\right]_0^L + \int_0^L f'(x)g'(x)\ dx$$

To continue, we want to integrate by parts yet again

$$u = g'(x) \quad du = g''(x) \quad dv = f'(x) \quad v = \int f'(x) = f(x)$$

We can now re-write $\left[g(x)f'(x)\right]_0^L + \int_0^L f'(x)g'(x)\ dx$ as

$$\left[g(x)f'(x)\right]_0^L - \left[g'(x)f(x)\right]_0^L - \int_0^L f(x)g''(x)\ dx$$

So, at this point, we have

$$f''(x)g(x) = \left[g(x)f'(x)\right]_0^L - \left[g'(x)f(x)\right]_0^L - \int_0^L f(x)g''(x)\ dx$$

Now, if we make the substitution $g(x) = f_{m,ND}(x)$ and $f(x) = f_{n,ND}(x)$, we have

$$\int_0^L f''_{n,ND}(x)f_{m,ND}(x)\ dx =$$
$$\left[f_{m,ND}(x)f'_{n,ND}(x)\right]_0^L - \left[f'm, ND(x)f_{n,ND}(x)\right]_0^L - \int_0^L f_{n,DN}(x)f''_{m,ND}(x)\ dx$$

Looking at the $\left[f_{m,ND}(x)f'_{n,ND}(x)\right]_0^L$ term, if we apply the boundary conditions

$$\begin{cases} f(0) = 0 \\ f'(L) = 0 \end{cases}$$

then this term will go to zero. This leaves us with

$$\int_0^L f''_{n,ND}(x)f_{m,ND}(x)\ dx =$$
$$-\left[f'm, ND(x)f_{n,ND}(x)\right]_0^L - \int_0^L f_{n,ND}(x)f''_{m,ND}(x)\ dx$$

Then, if we look at the $-\Big[f'm, ND(x) f_{n,ND}(x)\Big]_0^L$ term, we can see that if we again apply the boundary conditions

$$\begin{cases} f(0) = 0 \\ f'(L) = 0 \end{cases}$$

this term will also go to zero. So, are we are left with is

$$\int_0^L f''_{n,ND}(x) f_{m,ND}(x)\ dx = -\int_0^L f_{n,ND}(x) f''_{m,ND}(x)\ dx$$

Now, if we return to our twice differentiable function

$$f''(x) + \lambda f(x) = 0$$

and re-arrange terms, we have

$$f''(x) = -\lambda f(x)$$

If we substitute this into our integral, we have

$$\int_0^L -\lambda f_{n,ND}(x) f_{m,ND}(x) = -\int_0^L f_{n,ND}(x) \cdot -\lambda f_{m,ND}(x)\ dx$$

Simplifying, we have

$$\begin{aligned}
-\int_0^L f_{n,ND}(x) \cdot -\lambda f_{m,ND}(x)\ dx &= \int_0^L -\lambda f_{n,DN}(x) f_{m,ND}(x)\ dx \\
\implies -(-\lambda) \int_0^L f_{n,ND}(x) f_{m,ND}(x)\ dx &= -\lambda \int_0^L f_{n,ND}(x) f_{m,ND}(x)\ dx \\
\implies -(-\lambda_n) \int_0^L f_{n,ND}(x) f_{m,ND}(x)\ dx &+ \lambda_m \lambda \int_0^L f_{n,ND}(x) f_{m,ND}(x)\ dx \\
\implies -(-\lambda_n) \int_0^L \lambda_m f_{n,ND}(x) f_{m,ND}(x)\ dx &= 0 \\
\implies -(\lambda_m - \lambda_n) \int_0^L \lambda_m f_{n,ND}(x) f_{m,ND}(x)\ dx &= 0
\end{aligned}$$

Now, since m and n are assumed to be different, we know that $(\lambda_m - \lambda_n)$ cannot be equal to zero. Thus, $\int_0^L \lambda_m f_{n,DN}(x) f_{m,DN}(x)\ dx$ has to be equal to zero, as desired. Thus, we have shown the orthogonality of the mixed *ND-Laplacian* eigenfunctions.

□

5) Looking at the *eigendata* from these two mixed problems, we notice that they are similar in that they both are of the form

$$\lambda = \left(\frac{(n\pi - \frac{\pi}{2})}{L}\right)^2,\ \forall\ n \in \mathbb{N}$$

yet they differ in the form of their associated *eigenfunctions*.

6)Looking at the *eigendata* from our mixed *DN-Laplacian* eigenproblem and our regular *Dirichlet* eigenproblem, we see that for the *DN-Laplacian* eigenproblem our n values in the λ expression are only odd-numbers $\in \mathbb{N}$, while for the *Dirichlet* eigenproblem, we considered all values of $n \in \mathbb{N}$.