2) Given Charlie's chimney design, we can now explicitly write out the boundary conditions for the electric potential. For the right walls, we have

$$\begin{cases} u(l_1, y) = \varphi(y) \\ u(-l_1, y) = \varphi(y) \end{cases}$$

And, the front and back walls have the following boundary conditions

$$\begin{cases} u(x, y) = 0 \\ u(x, l_2) = 0 \end{cases}$$

3) Now, we want to find all separated solutions $u(x, y) = f(x)g(y)$ to Laplace's equation that also satisfy the appropriate *zero-dirichlet boundary conditions*. First, suppose that $u(x, y) = f(x)g(y)$, which are separated solutions. We begin with $\Delta_2 = u_{xx} + u_{yy} = 0 \in \Omega$. Then, recall that $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ and $u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$. Moving on, we do the separation of variables by taking the second partial derivatives of $u(x, y)$ with respect to $x$ and $y$.

$$\frac{\partial u}{\partial x} = f'(x)g(y)$$
$$\frac{\partial u}{\partial y} = f(x)g'(y)$$
$$\frac{\partial^2 u}{\partial x^2} = f''(x)g(y)$$
$$\frac{\partial^2 u}{\partial y^2} = f(x)g''(y)$$

Thus, substituting the partial derivatives into the PDE, we get

$$f(x)g''(x) = f''(x)g(y)$$

Now, putting all the $x$ parts on one side and all the $y$ parts on the other,

$$\frac{g''(y)}{g(y)} + \frac{f''(x)}{f(x)} = 0$$

Assume that $g(y)$ and $f(x)$ are not equal to zero so we can divide them over. Then, define a function

$$\lambda(x,y) = -\frac{1}{g(y)}g''(y) = -\frac{1}{f(x)}f''(x)$$

we can write $\lambda$ in two ways, either in $x$ terms or in terms of $y$. First, we look at $\lambda$ in terms of $y$, and take the partial derivative with respect to $x$

$$\frac{\partial \lambda}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{1}{g(y)}g''(y)\right] = 0$$

Now, take the partial derivative of the "$x$-piece" with respect to $y$

$$\frac{\partial \lambda}{\partial y} = \frac{\partial}{\partial x}\left[-\frac{1}{f(x)}f''(x)\right] = 0$$

Thus, $\lambda(x,y) = \lambda$, so $\lambda$ is a constant. Thus, $\lambda = -\frac{1}{g(y)}g''(y)$. Hence, $g''(y) + \lambda g(y) = 0$. This looks a lot like the *Dirichlet-Laplacian* eigenvalue problem. Recall from our previous work on the *Dirichlet-Laplacian* eigenvalue problem. We denote the *Dirichlet-Laplacian* eigenvalues as

$$\lambda_{n,D} = \left(\frac{ni\pi}{L}\right)^2, n \in \mathbb{N}$$

We denote the We denote the *Dirichlet-Laplacian* eigenfunctions as

$$u_{n,D}(x) = \sin\left(\frac{n\pi}{L}x\right), n \in \mathbb{N}$$

In accordance with our problem, we re-label $u_{n,D}(x)$ as

$$g_{n,D}(y) = \sin\left(\frac{n\pi}{l_2}\right), n \in \mathbb{N}$$

Now that we have the $g-problem$, we must do the $f$-problem: consider $\lambda = -\frac{1}{f(x)}f''(x)$. Thus, $f''(x) - \lambda f(x) = 0$. Then, we suppose that $f(x) = e^{\omega x}$ for some $\omega$. Plugging this in, we get

$$\omega^2 e^{\omega x} - \lambda e^{\omega x} = 0$$

We divide out $e^{\omega x}$ to get: $e^{\omega x}[\omega^2 - \lambda] = 0$. So, $\omega^2 - \lambda = 0$. So, $\omega = \pm\sqrt{\lambda}$. Then, recall that $\lambda = \left(\frac{n\pi}{l_2}x\right)^2$. So, our solution is

$$f_{n,D}(x) = A_n e^{\left(\frac{n\pi}{l_2}x\right)} + B_n e^{-\left(\frac{ni\pi}{l_2}x\right)}$$

Finally, recall that we want $u(x,y) = f(x)g(y)$, so

$$u(x,y) = \left[\sin\left(\frac{n\pi}{l_2}y\right)\right]\left[A_n e^{\left(\frac{n\pi}{l_2}x\right)} + B_n e^{-\left(\frac{n\pi}{l_2}x\right)}\right] \ \forall\ n \in \mathbb{N}$$

are the separable solutions.

4) Now, we want to write the general solution $u(x,y)$ using the separable variables solution found in question 3:

$$u(x,y) = \sum_{n=1}^{\infty}\left[\left(A_n e^{\left(\frac{n\pi}{l_2}x\right)} + B_n e^{-\left(\frac{n\pi}{l_2}x\right)}\right)\left(\sin\left(\frac{n\pi}{l_2}y\right)\right)\right]$$

Recall our nonzero boundary conditions:

$$u(-l,y,z) = \varphi(y)$$
$$u(l_1,y,z) = \varphi(y)$$

Now, we can say:

$$\varphi(y) = \sum_{n=1}^{\infty}\left[\left(A_n e^{\left(\frac{n\pi i_1}{l_2}\right)} + B_n e^{-\left(\frac{n\pi i_1}{l_2}x\right)}\right)\left(\sin\left(\frac{n\pi}{l_2}y\right)\right)\right]$$

Because $u$ is even in $x$, $A_n = B_n$. Therefore,

$$\varphi(y) = \sum_{n=1}^{\infty}A_n\left[\left(e^{\left(\frac{n\pi i_1}{l_2}\right)} + e^{-\left(\frac{n\pi i_1}{l_2}x\right)}\right)\left(\sin\left(\frac{n\pi}{l_2}y\right)\right)\right]$$

Now, using the hint $2\cosh(\theta) = e^{\theta} + e^{-\theta}$, we can re-write:

$$\varphi(y) = \sum_{n=1}^{\infty}A_n\left(2\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\sin\left(\frac{n\pi}{l_2}y\right)$$

Since $A_n$ is a constant that we don't know, two times $A_n$ is still a constant we don't know. So, we can let $A_n$ equal $2A_n$. This gives

$$\varphi(y) = \sum_{n=1}^{\infty}A_n\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\sin\left(\frac{n\pi}{l_2}y\right) \quad (*)$$

Now, we multiply $(*)$ by $\sin\left(\frac{m\pi}{l_2}y\right)$ and integrate from 0 to $l_2$

$$\int_0^{l_2} \varphi(y) \sin\left(\frac{m\pi}{l_2}y\right) = \int_0^{l_2} \sum_{n=1}^{\infty} A_n \left( \cosh\left(\frac{n\pi l_1}{l_2}\right)\right) \sin\left(\frac{n\pi}{l_2}y\right) \cdot \sin\left(\frac{m\pi}{l_2}y\right) dy$$

Since we are integrating with respect to $y$, we can factor out everything that doesn't depend on $y$

$$= \sum_{n=1}^{\infty} A_n \left( \cosh\left(\frac{n\pi l_1}{l_2}\right)\right) \int_0^{l_2} \sin\left(\frac{n\pi}{l_2}y\right) \sin\left(\frac{m\pi}{l_2}y\right) dy \quad (1)$$

Here, we have two cases, when $n \neq m$ and when $n = m$. When $n \neq m$, we know that $\int_0^{l_2} \sin\left(\frac{n\pi}{l_2}y\right) \sin\left(\frac{m\pi}{l_2}y\right) dy$ is equal to zero because of orthogonality. When $m = n$, we have $\int_0^{l_2} \sin^2\left(\frac{n\pi}{l_2}y\right)$, which is equal to $\frac{2}{L}$ by previous group quizzes. Now, if we replace the $n's$ with $m's$ in (1), we have

$$\sum_{n=1}^{\infty} A_m \left( \cosh\left(\frac{m\pi l_1}{l_2}\right)\right) \int_0^{l_2} \sin\left(\frac{m\pi}{l_2}y\right) \sin\left(\frac{m\pi}{l_2}y\right) dy + \sum_{n=1}^{\infty} A_n \left( \cosh\left(\frac{n\pi l_1}{l_2}\right)\right) \int_0^{l_2} \sin\left(\frac{n\pi}{l_2}y\right) \sin\left(\frac{n\pi}{l_2}y\right) dy$$

Because $n \neq m$, $\int_0^{l_2} \sin\left(\frac{n\pi}{l_2}y\right) \sin\left(\frac{m\pi}{l_2}y\right) dy = 0$ because it is orthogonal.
Because $n = m$, $\int_0^{l_2} \sin\left(\frac{m\pi}{l_2}y\right) \sin\left(\frac{m\pi}{l_2}y\right) dy = \frac{2}{L}$. Thus, we can re-write:

$$\int_0^{l_2} \varphi(y) \sin\left(\frac{m\pi}{l_2}y\right) dy = A_m \left( \cosh\left(\frac{m\pi l_1}{l_2}\right)\right) \left(\frac{2}{L}\right)$$

$$A_m = \frac{\int_0^{l_2} \varphi(y) \sin\left(\frac{m\pi}{l_2}y\right) dy}{\left(\cosh\left(\frac{m\pi l_1}{l_2}\right)\right)\left(\frac{2}{L}\right)}$$

Now, we can change our $m's$ back to $n's$

$$A_n = \frac{\int_0^{l_2} \varphi(y) \sin\left(\frac{n\pi}{l_2}y\right) dy}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\left(\frac{2}{L}\right)}$$

Continuing on, recall that we are given that $u(-x, y) = u(x, y)$. And, so, we can now write

$$u(-x,y)=\sum_{n=1}^{\infty}\left[\left(A_n e^{\left(\frac{n\pi}{l_2}x\right)}+B_n e^{-\left(\frac{n\pi}{l_2}x\right)}\right)\left(\sin\left(\frac{n\pi}{l_2}y\right)\right)\right]$$

$$u(x,y)=\sum_{n=1}^{\infty}\left[\left(A_n e^{\left(\frac{n\pi}{l_2}x\right)}+B_n e^{-\left(\frac{n\pi}{l_2}x\right)}\right)\left(\sin\left(\frac{n\pi}{l_2}y\right)\right)\right]$$

Since we know that $u(-x,y)=u(x,y)$, we have $2u(x,y)$ if we add $u(x,y)$ and $u(-x,y)$. Adding together, we obtain

$$\sum_{n=1}^{\infty}\left[\left(2A_n\left(\frac{e^{\left(\frac{n\pi}{l_2}x\right)}+e^{\left(\frac{n\pi}{l_2}-x\right)}}{2}\right)+2B_n\left(\frac{e^{-\left(\frac{n\pi}{l_2}x\right)}+e^{-\left(\frac{n\pi}{l_2}-x\right)}}{2}\right)\left(\sin\left(\frac{n\pi}{l_2}y\right)\right)\right)\right]$$

$$=\sum_{n=1}^{\infty}\left[2A_n\cosh\left(\frac{n\pi x}{l_2}\right)+2B_n\cosh\left(\frac{n\pi}{l_2}x\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)$$

$$=\sum_{n=0}^{\infty}\left[\hat{A_n}\cosh\left(\frac{n\pi}{l_2}x\right)\right]\sin\left(\frac{n\pi}{l_2}\right)$$

Then, we know that

$$2u(x,y)=\sum_{n=0}^{\infty}\left[\hat{A_n}\cosh\left(\frac{n\pi}{l_2}x\right)\right]\sin\left(\frac{n\pi}{l_2}\right)$$

$$\implies u(x,y)=\frac{\sum_{n=0}^{\infty}\left[\hat{A_n}\cosh\left(\frac{n\pi}{l_2}x\right)\right]\sin\left(\frac{n\pi}{l_2}\right)}{2}$$

$$\implies u(x,y)=\sum_{n=0}^{\infty}\left[\frac{\hat{A_n}}{2}\cosh\left(\frac{n\pi}{l_2}x\right)\right]\sin\left(\frac{n\pi}{l_2}\right)$$

Remembering that $\hat{A_n}=A_n$, we have

$$u(x,y)=\sum_{n=0}^{\infty}\left[A_n\cosh\left(\frac{n\pi}{l_2}x\right)\right]\sin\left(\frac{n\pi}{l_2}\right)$$

Let $u(x,y)$ be defined as above with $A_n$ equal to

$$\frac{\int_0^{l_2}\varphi(y)\sin\left(\frac{m\pi}{l_2}y\right)\,dy}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\left(\frac{2}{L}\right)}$$

5) Recall that the unique electric potential equation, $u(x,y)$, up to this point is defined as

$$u(x,y)=\sum_{n=0}^{\infty}\left[\frac{\int_0^{l_2}\varphi(y)\sin\left(\frac{m\pi}{l_2}y\right)\,dy}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\left(\frac{2}{L}\right)}\left(\cosh\left(\frac{n\pi}{l_2}x\right)\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)$$

Now, if we make the substitution $\varphi(y)=k_0$, where $k_0>0$, we have

$$u(x,y)=\sum_{n=0}^{\infty}\left[\frac{\int_0^{l_2}k_0\sin\left(\frac{m\pi}{l_2}y\right)\,dy}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\left(\frac{2}{L}\right)}\left(\cosh\left(\frac{n\pi}{l_2}x\right)\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)$$

Then, if we factor out the constants $k_0$, $\cosh\left(\frac{n\pi l_1}{l_2}\right)\left(\frac{2}{L}\right)$ since they don't "depend" on $y$, we have

$$u(x,y)=\sum_{n=0}^{\infty}\left[\int_0^{l_2}\sin\left(\frac{m\pi}{l_2}y\right)\,dy\left(\cosh\left(\frac{n\pi}{l_2}x\right)\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)$$

Then, evaluating the integral $\displaystyle\int_0^{l_2}\sin\left(\frac{m\pi}{l_2}y\right)\,dy$, we have

$$u=\frac{n\pi}{l_2}y \qquad du=\frac{n\pi}{l_2}dy$$

$$\int_0^{l_2}\sin\left(\frac{m\pi}{l_2}y\right)\,dy=\frac{l_2}{n\pi}\int_0^{l_2}\sin(u)\,du=-\frac{l_2}{\pi n}\cos(u)_0^{l_2}=$$
$$\left[-\frac{l_2}{\pi n}\cos\left(\frac{\pi n}{l_2}y\right)\right]_0^{l_2}$$

Evaluating from 0 to $l_2$, we have

$$\left[-\frac{l_2}{\pi n}\cos\left(\frac{\pi n}{l_2}(l_2)\right)\right]-\left[-\frac{l_2}{\pi n}\cos\left(\frac{\pi n}{l_2}(0)\right)\right]$$
$$=\left[-\frac{l_2}{\pi n}\cos(n\pi)\right]-\left[-\frac{l_2}{\pi n}(1)\right]$$
$$=\left[-\frac{l_2}{\pi n}(-1^n)\right]-\left[-\frac{l_2}{\pi n}\right]=\left[\frac{l_2}{\pi n}(1^n)\right]+\frac{l_2}{\pi n}=\frac{2l_2}{\pi n}$$

Plugging this result in, we have

$$u(x,y)=\sum_{n=0}^{\infty}\left[\frac{2l_2}{\pi n}\left(\cosh\left(\frac{n\pi}{l_2}x\right)\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)$$

Putting our constants back in, we have

$$u(x,y)=\sum_{n=0}^{\infty}\left[\frac{\frac{2l_2}{\pi n}}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\left(\frac{2}{L}\right)}\left(\cosh\left(\frac{n\pi}{l_2}x\right)\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)$$

Then, multiplying by $\frac{L}{2}$ on the top and bottom of $\frac{\frac{2l_2}{\pi n}}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)\left(\frac{2}{L}\right)}$ gives

$$u(x,y)=\sum_{n=0}^{\infty}\left[\frac{\frac{l_2 l_1}{\pi n}}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)}\left(\cosh\left(\frac{n\pi}{l_2}x\right)\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)$$

6) Recall that the unique electric potential formula $u(x,y)$ for when $\varphi(y)=k-0$ is

$$u(x,y)=\sum_{n=0}^{\infty}\left[\frac{\frac{l_2 l_1}{\pi n}}{\left(\cosh\left(\frac{n\pi l_1}{l_2}\right)\right)}\left(\cosh\left(\frac{n\pi}{l_2}x\right)\right)\right]\sin\left(\frac{n\pi}{l_2}y\right)\qquad(*)$$

Before we proceed, let $-l_1=1$ and $l_1=1$. Now, we want to approximate twelve points. Since we want to choose points between $-1$ and 1, then we will start with the point $(0.9,0.9)$, then $(0.85,0.85)$, then $(0.75,0.75)$, …, till we have twelve points. We will then approximate them by plugging the points into $(*)$.

For the point $(0.9,0.9)$:

$n=1 \implies u(0.9,0.9)=$

$$\left[\frac{\frac{(1)(1)}{\pi(1)}}{\left(\cosh\left(\frac{(1)\pi(1)}{(1)}\right)\right)}\left(\cosh\left(\frac{(1)\pi}{(1)}(0.9)\right)\right)\right]\sin\left(\frac{(1)\pi}{(1)}(0.9)\right)\approx 0.0719618$$

$n = 2 \implies u(0.9, 0.9) =$

$$\left[ \frac{\frac{(1)(1)}{\pi(2)}}{\left(\cosh\left(\frac{(2)\pi(1)}{(1)}\right)\right)} \left(\cosh\left(\frac{(2)\pi}{(1)}(0.9)\right)\right)\right] \sin\left(\frac{(2)\pi}{(1)}(0.9)\right) \approx -0.492569$$

$n = 3 \implies u(0.9, 0.9) =$

$$\left[ \frac{\frac{(1)(1)}{\pi(3)}}{\left(\cosh\left(\frac{(3)\pi(1)}{(1)}\right)\right)} \left(\cosh\left(\frac{(3)\pi}{(1)}(0.9)\right)\right)\right] \sin\left(\frac{(3)\pi}{(1)}(0.9)\right) \approx 0.330121$$

$n = 4 \implies u(0.9, 0.9) =$

$$\left[ \frac{\frac{(1)(1)}{\pi(4)}}{\left(\cosh\left(\frac{(4)\pi(1)}{(1)}\right)\right)} \left(\cosh\left(\frac{(4)\pi}{(1)}(0.9)\right)\right)\right] \sin\left(\frac{43)\pi}{(1)}(0.9)\right) \approx -0.212591$$

$n = 5 \implies u(0.9, 0.9) =$

$$\left[ \frac{\frac{(1)(1)}{\pi(5)}}{\left(\cosh\left(\frac{(1)\pi(1)}{(1)}\right)\right)} \left(\cosh\left(\frac{(5)\pi}{(1)}(0.9)\right)\right)\right] \sin\left(\frac{(5)\pi}{(1)}(0.9)\right) \approx 0.130615$$

Now, from here on out, the same **exact** formula will be used. The only things that will change are the $n$ values and the $x$ and $y$ coordinates. Therefore, to save paper and prevent the student from contracting carpal tunnel syndrome, only the results from the formula will be written.

For the point $(0.85, 0.85)$:

$n = 1 \rightarrow\approx 0.0904704$ $n = 2 \rightarrow\approx -0.495191$ $n = 3 \rightarrow\approx 0.251582$
$n = 4 \rightarrow\approx 0.0904704$ $n = 5 \rightarrow\approx 0.0421097$

For the point $(0.8, 0.8)$:

$n = 1 \rightarrow\approx 0.100282$ $n = 2 \rightarrow\approx -0.425200$ $n = 3 \rightarrow\approx 0.151220$
$n = 4 \rightarrow\approx -0.0373945$ $n = 5 \rightarrow\approx 0$

For the point $(0.75, 0.75)$:

$n = 1 \rightarrow\approx 0.103351$ $n = 2 \rightarrow\approx -0.326562$ $n = 3 \rightarrow\approx 0.0701830$ $n = 4 \rightarrow\approx 0$
$n = 5 \rightarrow\approx -0.00875376$

For the point $(0.7, 0.7)$:

$n = 1 \rightarrow\approx 0.101390$ $n = 2 \rightarrow\approx -0.226863$ $n = 3 \rightarrow\approx 0.0191458$
$n = 4 \rightarrow\approx 0.0106428$ $n = 5 \rightarrow\approx -0.00564437$

For the point $(0.65, 0.65)$:

$n = 1 \rightarrow\approx 0.0958608$ $n = 2 \rightarrow\approx -0.140976$ $n = 3 \rightarrow\approx -0.00605018$
$n = 4 \rightarrow\approx 0.00918691$ $n = 5 \rightarrow\approx -0.00181973$

The sum for $(0.9, 0.9)$ is $\approx -0.1724622$
The sum for $(0.85, 0.85)$ is $\approx -0.2241739$
The sum for $(0.8, 0.8)$ is $\approx -0.2110925$
The sum for $(0.75, 0.75)$ is $\approx -0.1617876$
The sum for $(0.7, 0.7)$ is $\approx -0.1013287$
The sum for $(0.65, 0.65)$ is $\approx -0.0437982$
The total some for the first six points is $\approx -0.8133144$

We continue plugging in points into $(*)$. We now consider the point $(0.6, 0.6)$:

$$u(0.6, 0.6) = \frac{\frac{1}{\pi}\left(\cosh\left(0.6\pi\right)\right)}{\cosh(\pi)}\left(\sin(0.6\pi)\right) + \frac{\frac{1}{2\pi}\left(\cosh\left((0.6)2\pi\right)\right)}{\cosh(2\pi)}\left(\sin((2\pi)0.6)\right) + \frac{\frac{1}{3\pi}\left(\cosh\left((0.6)3\pi\right)\right)}{\cosh(3\pi)}\left(\sin((3\pi)0.6)\right) + \frac{\frac{1}{4\pi}\left(\cosh\left((0.6)4\pi\right)\right)}{\cosh(4\pi)}\left(\sin((4\pi)0.6)\right) + \frac{\frac{1}{5\pi}\left(\cosh\left((0.6)5\pi\right)\right)}{\cosh(5\pi)}\left(\sin((5\pi)0.6)\right) \approx 0.0879821 - 0.0748284 - 0.0141906 + 0.0049011 + 0 \approx 0.00386421$$

Now, we use the point $(0.55, 0.55)$:

$$u(0.6, 0.6) = \frac{\frac{1}{\pi}\left(\cosh\left(0.55\pi\right)\right)}{\cosh(\pi)}\left(\sin(0.55\pi)\right) + \frac{\frac{1}{2\pi}\left(\cosh\left((0.55)2\pi\right)\right)}{\cosh(2\pi)}\left(\sin((2\pi)0.55)\right) + \frac{\frac{1}{3\pi}\left(\cosh\left((0.55)3\pi\right)\right)}{\cosh(3\pi)}\left(\sin((3\pi)0.55)\right) + \frac{\frac{1}{4\pi}\left(\cosh\left((0.55)4\pi\right)\right)}{\cosh(4\pi)}\left(\sin((4\pi)0.55)\right) + \frac{\frac{1}{5\pi}\left(\cosh\left((0.55)5\pi\right)\right)}{\cosh(5\pi)}\left(\sin((5\pi)0.55)\right) \approx 0.777117 - 0.0287471 - 0.0134281 + 0.00161596 + 0.7369306 \approx 0.7369306$$

Using the same formulas and methodology of the proceeding two points, we can continue this processes for the following 4 points:

$$u(0.5, 0.5) \approx 0.680025 + 0 - 0.00940804 + 0 + 0.000243915 \approx 0.670860875$$

$$u(0.45, 0.45) \approx 0.582795 + 0.0153746 - 0.00523333 - 0.000459924 + 0.0000786376 \approx 0.593549836$$

$$u(0.4, 0.4) \approx 0.489493 + 0.0214252 - 0.00215576 - 0.000397019 + 0 \approx 0.508365421$$

$$u(0.35, 0.35) \approx 0.402765 + 0.0216619 - 0.000358442 - 0.000211828 - 0.00001634774 \approx 0.6187973826$$

7) Recall that the electrostatic field in $\Omega$ due to the surface electric charge is given by

$$\vec{E}(x, y, z) = -\nabla(x, y, z)$$

To do this, we use the points we approximated values for in question 6

8) I think Charlie's chimney is a very interesting mathematical problem because of the four boundary conditions. However, I think Charlie's chimney was very difficult to work with because of the fact that it had four boundary conditions when we usually only have two. I also think it is remarkable that Charlie was able to find a battery powerful enough to maintain a specific potential $\varphi = \varphi(y)$ on the left and right walls. I must say, however, I do not think it is wise that anyone is standing inside the electrostatic chimney. This seems like a safety hazard.

9) $\dfrac{\partial u}{\partial t} = -\nabla u$ for $t > 0$, $(x, y, z) \in \mathbb{R}^3$

$u(x, y, z = 1$ for $(x, y, z) \in \mathbb{R}^3$ is our intial condition. To show that Charlie's though experiment is ill-posed, we want to show that it doesn't meet one of our conditions for well-posedness. The condition we focus on is continuous dependence. So, our problem is finding all $u(x, y, z)$ satisfying $\dfrac{\partial u}{\partial t} = -\nabla t$ for $t > 0$, $(x, y, z) \in \mathbb{R}^3$. To make the problem more pleasing to the eye, let $w(x, y, z, t)$ be $u(x, y, z, t)$ for $(x, y, z) \in \mathbb{R}^3$, $t > 0$. First, we take the derivative with respect to $t$:

$$\frac{\partial w}{\partial t} = \frac{\partial u}{\partial t}$$

Then, take the $2^{nd}$ partial derivative with respect to $x$, then, $y$, then $z$:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial^2 w}{\partial z^2} = \frac{\partial^2 u}{\partial z^2}$$

Now, the $u - problem$ becomes

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \text{ with } w(x, y, z, 0) = 1$$

Now, consider $w_0(x, t) = 1 + \dfrac{1}{n} e^{\alpha n^2 t} \sin(n(x + y + z))$.

$$\frac{\partial wn}{\partial t} = \frac{-n^2 t}{ne \sin(n(x + y + z))} || \frac{\partial^2 wn}{\partial x^2} = -ne^{n^2 t} \sin(n(x + y + z))$$
$$\frac{\partial wn}{\partial y^2} = -ne \sin(n(x + y + z)) || \frac{\partial^2 wn}{\partial z^2} = -ne^{n^2 t} \sin(n(x + y + z))$$

So, $w_n$ satisfies

$$\begin{cases} \dfrac{\partial w_n}{\partial t} = \dfrac{\partial^2 w_n}{\partial x^2} + \dfrac{\partial^2 w_n}{\partial y^2} + \dfrac{\partial^2 w_n}{\partial z^2} \\ w_n(x,y,z,0) = 1 + \dfrac{1}{n}\sin(n(x+y+z)) \end{cases}$$

Now, we want to compare the $w$-problem with the $u_n$-problem. Since they are the same differential equation, we compare the initial conditions

$$\begin{aligned} \left|w_n(x,y,z,0) - w(x,y,z,0)\right| &= \left|\frac{1}{n}\sin(n(x+y+z))\right| \\ &\leq \frac{1}{n}\left|\sin(n(x+y+z))\right| \\ &\leq \frac{1}{n} \to 0 \text{ as } n \to 0 \end{aligned}$$

Then, compare the solutions of the $w_n$-function to the $w$-functions:

$$\begin{aligned} \left|w_n(x,y,z,t) - w_0(x,y,z,t)\right| &= \left|1 + \frac{1}{n}e^{n^2 t}\sin(n(x+y+z)) - 1\right| \\ &\leq \frac{1}{n}e^{n^2 t}\left|\sin(n(x+y+z))\right| \\ &\leq \frac{1}{n}e^{n^2 t} \to \infty \end{aligned}$$

Thus, in conclusion, for $n$ that is really, really, really big, the initial conditions are close, but the solutions $w$ and $w_n$ are very far apart. Therefore, the problem is ill-posed.

10) We can interpret our results for the ill-posed problem using the following schematic diagrams for the initial conditions function space and solutions space, respectively:

These diagrams show that in the initial conditions space, the $w$ initial conditions are pretty close to the $w_n$ initial conditions. We can observe this by following the arrows. However, in the solution space, we can see that the $w$ solutions get progressively farther away from the $w_n$ solutions as $n$ increases. If we follow the arrows, we have $w$, then $w_n$, where $n = 1$, then $w_n$, where $n = 2$, and so on. As $n$ increases, $w$ and $w_n$ get farther and farther apart. This is the reason that the problem is ill-posed.