1) W know that any matrix in $O(2)$ must either take the form of

$$\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$$

or

$$\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$$

And, since any matrix in $O(2)$ either reflects or rotates a vector in $\mathbb{R}^2$, the determinant of any matrix in $O(2)$ will be ± 1. Therefore, the only elements that commute are $Z(G) = \{\pm I\}$, or the identity matrix.

2)

Proof. Doing the prime-factorization of 2016, we get $2016 = 2^5 \cdot 3^2 \cdot 7$. So, by the *Fundamental Theorem of Finitely Generated Abelian Groups*, we have

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_7$$
$$\mathbb{Z}_2 \times \mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$
$$\mathbb{Z}_2 \times \mathbb{Z}_{16} \times \mathbb{Z}_9 \times \mathbb{Z}_7$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_7$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 x \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_7$$
$$\mathbb{Z}_{32} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$
$$\mathbb{Z}_{32} \times \mathbb{Z}_9 \times \mathbb{Z}_7$$

□

3)

4)

Proof. Let the order of G be $m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. It is known that G is a direct product of p-groups, say:

$$G = G_1 \times \dots \times G_k$$

where each G_i is a p_i-group. By the *fundamental theorem of finite abelian groups*, each G_i is isomorphic to a direct product of cyclic groups of the form

$$\mathbb{Z}_{p_i{}^{\beta_1}} \times \ldots \times \mathbb{Z}_{p_i{}^{\beta_l}},$$
$$\text{where } \beta_1, \ldots, \beta_l \text{ are positive integers such that } \sum_{j=1}^{l} \beta_j = \alpha_i.$$

Now if n divides m, then we must have

$$n = p_1^{\gamma_1} \ldots p_k^{\gamma_k}$$
$$\text{for some } \gamma_1, \ldots, \gamma_k \text{ with } 0 \leq \gamma_i \leq \alpha_i.$$

Which brings us to the following claim: *Each G_i has a subgroup of order $p_i^{\gamma_i}$*
Proof: As above, we have that
$G_i \cong \mathbb{Z}_{p_i{}^{\beta_1}} \times \ldots \times \mathbb{Z}_{p_i{}^{\beta_l}}$ where $\beta_1, \ldots, \beta_l$ are positive integers such that $\displaystyle\sum_{j=1}^{l} \beta_j = \alpha_i$. Now since $0 \leq \gamma_i \leq \alpha_i$, we can find$l$ numbers $\delta_1, \ldots, \delta_l$ such that $\gamma_i = \displaystyle\sum_{j=1}^{l} \delta_j$, and $0 \leq \delta_j \leq \beta_j$. (This choice of numbers is not necessarily unique). Then, $p_i^{\delta_j} | p_i^{\beta_j}$ for each $j = 1, \ldots, l$. Hence, for each factor $\mathbb{Z}_{p_i{}^{\beta_j}}$, there exists a subgroup of order $p_i^{\delta_j}$, namely $\mathbb{Z}_{p_i{}^{\delta_j}}$ (using the fact that the converse of Lagrange's theorem is true for finite cyclic groups). Taking the direct product of each of these subgroups, we get a new subgroup G_i' of G_i:

$$G_i' \cong \mathbb{Z}_{p_i{}^{\delta_1}} \times \ldots \times \mathbb{Z}_{p_i{}^{\delta_l}}$$

The order of this subgroup is $p_i^{\delta_1} \times \ldots \times p_i^{\delta_l} = p_i^{\delta_1 + \ldots + \delta_l} = p_i^{\gamma_i}$. So, we have found a subgroup of G_i of order $p_i^{\gamma_i}$, as required. So, each factor G_i in the product $G = G_1 \times \ldots \times G_k$ has a subgroup G_i' of order $p_i^{\gamma_i}$. Therefore, G has a subgroup

$$G_1' \times G_2' \times \ldots \times G_k'$$
$$\text{of order } p_1^{\gamma_i} ... p_k^{\gamma_k} = n,$$

which completes the proof.

□

5)

Proof. Similar to problem 2, we first compute the prime factorization of 16

$$16 = 2^4$$

Then, by the *Fundamental Theorem of Finitely Generated Abelian Groups*, we have

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$$
$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_8$$
$$G \cong \mathbb{Z}_4 \times \mathbb{Z}_4$$
$$G \cong \mathbb{Z}_{16}$$

Now, if we can rule out $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, and $\mathbb{Z}_{16}$, as $\mathbb{Z}_4$, $\mathbb{Z}_8$, and $\mathbb{Z}_{16}$ each have elements of order 2, but $a^2 = b^2$ in each case. However, in $\mathbb{Z}_4 \times \mathbb{Z}_4$, the order of $(1, 0)$ is 4 and $(1, 0)^2 = (2, 0)$, and for $(0, 1)$, the order is also 4. But, $(0, 1)^2 = (0, 2) \neq (2, 0)$. Therefore, the isomorphism class of G is $\mathbb{Z}_4 \times \mathbb{Z}_4$.
$\square$