Exam 2, Take Home Portion

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1a) Show that $\lim_{x\to\infty} erf(x) = 1$

Proof. Let the error function defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\ dx$$

be given.

Next, let $\alpha = \displaystyle\int_0^\infty e^{-x^2}\ dx$

Now, we can say that

$$\alpha^2 = \int_0^\infty e^{-x^2}\ dx\ \int_0^\infty e^{-y^2}\ dy$$
$$\alpha^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy$$
$$\alpha^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, drd\theta$$

The last line, there, because we can convert from radial coordinates, for which $dxdy = rdrd\theta$ and $r^2 = x^2 + y^2$. Now, as the inner integral does not depend on θ, we may let $r^2 = s$ (and so, $rdr = \dfrac{ds}{2}$) to get

$$\alpha^2 = \frac{\pi}{2} \int_0^\infty e^{-s}\ \frac{ds}{2}$$
$$\alpha^2 = \frac{\pi}{4}\Big[-e^{-s}\Big]\Big|_0^\infty$$
$$\alpha^2 = \frac{\pi}{4}$$

Therefore, we have that $\alpha = \sqrt{\dfrac{\pi}{4}} = \dfrac{\sqrt{\pi}}{2}$. Because of this, we have that

$$erf(\infty) = \frac{2}{\pi} \int_0^\infty e^{-x^2}\ dx$$
$$erf(\infty) = \frac{2}{\pi} \cdot \frac{\pi}{2}$$
$$erf(\infty) = 1$$

$\square$

1b) Show that $\lim_{x \to \infty} erf(x) = -1$

Proof. Let the error function defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} \, dx$$

be given.

Next, let $\alpha = \displaystyle\int_0^{-\infty} e^{-x^2} \, dx$

Now, we can say that

$$\alpha^2 = \int_0^{-\infty} e^{-x^2} \, dx \int_0^{-\infty} e^{-y^2} \, dy$$

$$\alpha^2 = \int_0^{-\infty} \int_0^{-\infty} e^{-(x^2+y^2)} \, dxdy$$

$$\alpha^2 = \int_0^{\frac{\pi}{2}} \int_0^{-\infty} e^{-r^2} \, drd\theta$$

The last line, there, because we can convert from radial coordinates, for which $dxdy = rdrd\theta$ and $r^2 = x^2 + y^2$. Now, as the inner integral does not depend on θ, we may let $r^2 = s$ (and so, $rdr = \dfrac{ds}{2}$) to get

$$\alpha^2 = \frac{\pi}{2} \int_0^{-\infty} e^{-s} \, \frac{ds}{2}$$

$$\alpha^2 = \frac{\pi}{4} \left[- e^{-s} \right] \Big|_0^{-\infty}$$

$$\alpha^2 = \frac{\pi}{4}$$

Therefore, we have that $\alpha = \sqrt{\dfrac{\pi}{4}} = -\dfrac{\sqrt{\pi}}{2}$. Because of this, we have that

$$erf(-\infty) = \frac{2}{\pi} \int_0^{-\infty} e^{-x^2} \, dx$$

$$erf(-\infty) = -\frac{2}{\pi} \cdot \frac{\pi}{2}$$

$$erf(-\infty) = -1$$

$\square$

1c) Show that $erf(0) = 0$

Proof. Let the error function be defined as given. Then, using the *Fundamental Theorem of Calculus*, we have that

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\ dx$$

Substituting in 0 for x, we get

$$erf(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-0^2}\ dx$$

$$erf(0) = \frac{2}{\sqrt{\pi}} \int_0^0 1\ dx$$

$$erf(0) = \frac{2}{\sqrt{\pi}} [x] \Big|_0^0$$

$$erf(0) = 0$$

□

2a) Show that the *error function* satisfies $erf(x_1) < erf(x_2)\ \forall\ x_1 < x_2$

Proof. Let the *error function* be given as defined. Then, by the *Fundamental Theorem of Calculus*, we have

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\ dx$$

Now, if we take a derivative, we have

$$erf'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

Now, we know that the smallest e could ever be is one, (when we have e^0), so the derivative will always be greater than zero. Next, consider two points, x_1 and x_2, such that $x_2 > x_1$. Then, if we compare $e^{-x_1^2}$ and $e^{-x_2^2}$, we have

$$\frac{e^{-x_2^2}}{e^{-x_1^2}} = \left(\frac{e^{-x_2}}{e^{-x_1}} \right)^2 > 1$$

Since we have that $x_2 > x_1$, if we start at any point in the domain (x_1) and go any distance to the right (to x_2), the function will get larger. Finally, as x tends towards ∞, the exponent in e^{-x^2} will grow very rapidly, causing the whole expression to shrink quickly. However, we know that the smallest that e will ever get is one, so our function is not increasing towards a limit of zero.

$\square$

2b) Show that the *error function* satisfies $erf(x_1) < erf(x_2)\ \forall\ x_1 < x_2$

Proof. Let the *error function* be given as defined. Then, by the *Fundamental Theorem of Calculus*, we have

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\ dx$$

Now, after substituting x_1 and x_2 in for x, we have the following two integrals

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-x_1^2}\ dx \quad \text{and} \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-x_2^2}\ dx$$

Then, using the property of integrals, we have

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \int_0^x e^{-x_2^2}\ dx &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{x_2}{2}} e^{-x_2^2}\ dx + \frac{2}{\sqrt{\pi}} \int_{\frac{x_2}{2}}^x e^{-x_2^2}\ dx \\ &\geq \frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-x_1^2}\ dx + \frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-x_1^2}\ dx \\ &= 2\left[\frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-x_1^2}\ dx\right] \end{aligned}$$

Since $\frac{2}{\sqrt{\pi}} \int_0^x e^{-x_2^2}\ dx$ is greater than $2\left[\frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-x_1^2}\ dx\right]$, it follows then that $\frac{2}{\sqrt{\pi}} \int_0^x e^{-x_2^2}\ dx$ is greater than $\frac{2}{\sqrt{\pi}} \int_0^{x_1} e^{-x_1^2}\ dx$.

$\square$

3) Let the error function be given as defined. Then, by the *Fundamental Theorem of Calculus*, we have

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\ dx$$

Then, if we take a derivative of $\int_0^x e^{-x^2}\ dx$, we get

$$erf'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

Now, in order to find inflection points, we have to calculate the second derivative. So, taking one more derivative results in

$$erf''(x) = \frac{2}{\sqrt{\pi}}(-2x)\left(e^{-x^2}\right)$$

Now, if we set the second derivative equal to zero and solve

$$\frac{2}{\sqrt{\pi}}(-2x)\left(e^{-x^2}\right) = 0$$

$$(-2x)\left(e^{-x^2}\right) = 0$$

Now, we know that the smallest e can ever be is 1 (when we have e^0), so that means $-2x$ has to be equal to zero. So,

$$-2x = 0$$

$$x = 0$$

Plugging in $x = 0$ back in to our second derivative, we get

$$erf''(0) = \frac{2}{\sqrt{\pi}}(-2(0))\left(e^{-0^2}\right)$$

$$erf''(0) = \frac{2}{\sqrt{\pi}}(0)(1)$$

$$erf''(0) = 0$$

So, our point of inflection is $(0, 0)$

Now, if we consider the graph of the *Error Function*

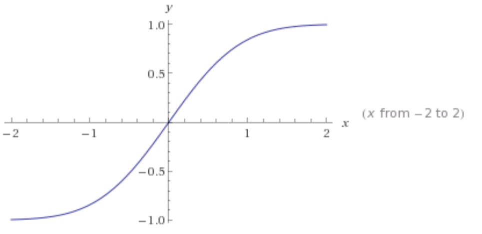

Figure 1: *The Error Function*

and the graph of $\frac{2}{\sqrt{\pi}}e^{-x^2}$

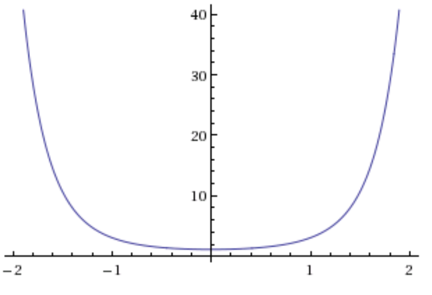

Figure 2: *The Integrand*

We note that the *error function* closely resembles the graph of the cubic function and the graph of the *integrand* is akin to the graph of x^2, so both of these functions are related in the sense that they both fall into the class of polynominal functions.

Finally, if we look at the graph of the both the *error function and integrand*, we can see that the *integrand* acts almost like an asymptote for the *error function*

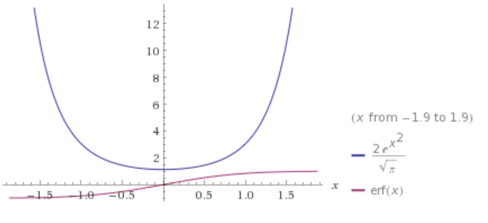

Figure 3: *The Error Function and Integrand*

4) Let the *Diffusion Equation* $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$ be given as defined.

Proof. First, suppose that solutions are of the form $u = u(x, t)$. After moving things around, we have

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0$$

Now, multiply $\dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} = 0$ through by u to get

$$u\frac{\partial u}{\partial t} - Du\frac{\partial^2 u}{\partial x^2} = 0$$

Now, before continuing, we note the following

$$\frac{\partial}{\partial t}\left[\frac{1}{2}u^2\right] = u\left(\frac{\partial u}{\partial t}\right)$$

Also,

$$\frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right] = -Du\frac{\partial^2 u}{\partial x^2} - D\left(\frac{\partial u}{\partial x}\right)^2$$
$$\frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right] + D\left(\frac{\partial u}{\partial x}\right)^2 = -Du\frac{\partial^2 u}{\partial x^2}$$

Okay, now, $u\dfrac{\partial u}{\partial t} - Du\dfrac{\partial^2 u}{\partial x^2} = 0$ becomes

$$\frac{\partial}{\partial t}\left[\frac{1}{2}u^2\right] + \frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right] + D\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

Then, we integrate over the interval $0 < x < L$ to get

$$\int_0^L \frac{\partial}{\partial t}\left[\frac{1}{2}u^2\right]\,dx + \int_0^L \frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right]\,dx + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\,dx = 0$$

After using the *Fundamental Theorem of Calculus*, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\,dx + \left[-Du\left(\frac{\partial u}{\partial x}\right)\right]\Bigg|_{x=0}^{x=L} + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\,dx = 0$$

Now, if we evaluate the middle term from $x = 0$ to $x = L$, we can see that it becomes

$$-Du(L,t)\frac{\partial u}{\partial x}(L,t) + Du(0,t)\frac{\partial u}{\partial x}(0,t)$$

So, now we have the following

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx + \Big(- Du(L,t)\frac{\partial u}{\partial x}(L,t) + Du(0,t)\frac{\partial u}{\partial x}(0,t)\Big) + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx = 0$$

Next, remembering that our boundary conditions are

$$-\frac{\partial u}{\partial x}(0,t) + b_0 u(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) + b_L u(L,t)$$

if we rearrange things a bit, we get

$$-\frac{\partial u}{\partial x}(0,t) = -b_0 u(0,t) \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = -b_L u(L,t)$$

So, upon comparing $-Du(L,t)\frac{\partial u}{\partial x}(L,t) + Du(0,t)\frac{\partial u}{\partial x}(0,t)$ with

$$-\frac{\partial u}{\partial x}(0,t) = -b_0 u(0,t) \text{ and } \frac{\partial u}{\partial x}(L,t) = -b_L u(L,t)$$

we can substitute using the boundary conditions giving us

$$Du(L,t)b_L u(L,t) + Du(0,t)b_0 u(0,t)$$

So, at this point, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx + \Big(Du(L,t)b_L u(L,t) + Du(0,t)b_0 u(0,t)\Big) + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx = 0$$

Okay, now, if we note that both the second and third term have a D term that we can factor out, after combing terms, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx + D\left[u(L,t)b_L u(L,t) + u(0,t)b_0 u(0,t) + \int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx\right] = 0$$

Then, if we move things around, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx = -D\left[u(L,t)b_L u(L,t) + u(0,t)b_0 u(0,t) + \int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx\right] \leq 0$$

Thus, the function $F(t) = \dfrac{d}{dt}\displaystyle\int_0^L \frac{1}{2}u(x,t)^2\ dx$ by the above expression is a decreasing function.

$\square$

5) We have now seen the *Energy Method* applied to *diffusion equation* problems (on a finite interval) subject to the *Dirichlet, Nuemann, and Robin* boundary conditions. When we investigated the *Dirichlet* case, when we applied the boundary conditions to the energy method, we saw a whole entire term, namely the second one, from

$$\frac{d}{dt}\int_0^L \frac{1}{2}w^2\ dx + \left[-Dw\left(\frac{\partial w}{\partial x}\right)\right]\Big|_{x=0}^{x=L} + D\int_0^L \left(\frac{\partial w}{\partial x}\right)^2\ dx = 0$$

vanish completely. Next, when we concerned ourselves with the *Robin* case, we were able to use the boundary conditions in order to transform

$$-Du(L,t)\frac{\partial u}{\partial x}(L,t) + Du(0,t)\frac{\partial u}{\partial x}(0,t)$$

into

$$Du(L,t)b_L u(L,t) + Du(0,t)b_0 u(0,t)$$

Finally, when we considered the *Nuemman* scenario, when we introduced the boundary conditions, we saw that the boundary term became zero since $\dfrac{\partial w}{\partial x}$ is zero and 0 and L.

6) Let the *Diffusion Equation* be given as defined. Now, suppose $u(x,t) = f(x)g(x)$ for some f, g. Then, define the following

$$\frac{\partial u}{\partial t} = f(x)g'(x) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = f''(x)g(t)$$

If we substitute into the given P.D.E., we have $f(x)g'(t) = vf''(x)g(t)$

Now, define the $\lambda(x,t)$ function as

$$\lambda(x,t) = -\frac{1}{vg(t)}\ g'(t) = -\frac{1}{f(x)}\ f''(x),$$
where we assume that $g(t), f(t) \neq 0$

Then, if we take some derivatives, we have

$$\frac{\partial \lambda}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{1}{v^2 g(t)}\ g''(t)\right] = 0$$
$$\frac{\partial \lambda}{\partial t} = \frac{\partial}{\partial t}\left[-\frac{1}{f(x)}\ f''(x)\right] = 0$$

Thus, $\lambda(x,t) = \lambda$. So, $\lambda = -\frac{1}{f(x)}\ f''(x)$. Then, $f''(x) + \lambda f(x) = 0$.

Now, from the boundary conditions, we have that

$$f(0)g(t) = 0 \quad \text{and} \quad f(L) = 0$$

Now, we take a slight detour so that we may derive the eigendata from scratch. So, suppose that the solutions for $f''(x) + \lambda f(x) = 0$ are of the form $u(x) = e^{kx}$. Then, $u'(x) = ke^{kx}$ and $u''(x) = k^2 e^{kx}$. Plugging back in, we have:

$$k^2 e^{kx} + \lambda e^{kx} = 0$$
$$e^{kx}(k^2 + \lambda) = 0$$
$$k^2 + \lambda = 0, \text{ which is our auxiliary equation.}$$

Next, consider $\lambda < 0$. Using the auxiliary equation we obtained in question 2, we have

$$k^2 + \lambda = 0$$
$$k^2 = -\lambda$$
$$k = \pm\sqrt{\lambda}$$

So, $v_1(x) = e^{\sqrt{-\lambda}\,x}$ and $v_2 = e^{-\sqrt{-\lambda}\,x}$.

Now, check the wronskian:

$$w(v_1, v_2) = \begin{vmatrix} e^{kx} & e^{-kx} \\ ke^{kx} & -ke^{kx} \end{vmatrix}$$
$$w(v_1, v_2) = (e^{kx} \cdot -ke^{kx}) - (e^{-kx} \cdot ke^{kx})$$
$$w(v_1, v_2) = (-k(e^{kx})(e^{-kx})) - (k(e^{-kx})(e^{kx}))$$
$$w(v_1, v_2) = (-k \cdot 1) - (k \cdot 1) = (-k)(k) = -k^2 \neq 0$$

Since $w(v_1, v2) \neq 0$, the solutions $v_1(x)$ and $v_2(x)$ are linearly independent and thus form a fundamental set of solutions. Now, the astute reader can observer that plugging in any values (say, $\sqrt{-\lambda}$) will have the same result. Now, consider $\lambda = 0$. Then, the differential equation becomes

$$f''(x) = 0$$

Then, if we integrate, we have:

$$f'(x) = c$$
$$f(x) = c_1 x + c_0, \text{ for some constants } c_1 \; and \; c_0$$

Using $f(0) = 0$, we have:

$$f(0) = c_1(0) + c_0$$
$$0 = 0$$

Using $f(L) = 0$, we have:

$$f(L) = c_1(L) + c_0$$
$$0 = c_0$$

So, our two solutions are $h_1(x) = 1$ and $h_2(x) = x$

Now, check the wronskian:

$$w(h_1(x), h_2(x)) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$
$$w(h_1(x), h_2(x)) = (1 \cdot 1) - (0 \cdot x)$$
$$w(h_1(x), h_2(x)) = 1 - 0 = 1 \neq 0$$

Since $w(h_1(x), h_2(x)) \neq 0$, our two solutions are linearly independent and thus form a fundamental set of solutions.
Using $u(0) = 0$, we have:

$$h_1(0) = 1 \qquad h_2(0) = 1$$
$$0 = 1 \qquad 0 = 0$$

$$h_1(L) = 1 \qquad h_2(L) = 1$$
$$0 = 1 \qquad 0 = L$$

Since the functions don't satisfy the boundary conditions, they are not *Dirichlet-Laplacian eigenfunctions*.

Finally, consider $\lambda > 0$. Then, suppose the solutions are of the form $u(x) = e^{kx}$. Then, $u'(x) = ke^{kx}$ and $u''(x) = k^2 e^{kx}$. Plugging in, we have:

$$k^2 e^{kx} + \lambda e^{kx} = 0$$
$$e^{kx}(k^2 + \lambda) = 0$$
$$k^2 + \lambda = 0$$
$$k = \pm\sqrt{-\lambda}$$
$$k = \pm i\sqrt{\lambda}$$

So, $u_1(x) = e^{i\sqrt{\lambda}}$ and $u_2(x) = e^{-i\sqrt{\lambda}}$.
Next, let $u_\lambda(x) = \dfrac{1}{2}\Bigg[u_1(x) + u_2(x)\Bigg]$.
Then,
$u_\lambda(x) = \dfrac{1}{2}\Bigg[\cos(\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x) + \cos(-\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x)\Bigg]$
$u_\lambda(x) = \cos(\sqrt{\lambda}\,x)$

Let $v_\lambda(x) = \dfrac{1}{2i}\Bigg[\cos(\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x) - \cos(-\sqrt{\lambda}\,x) + i\sin(\sqrt{\lambda}\,x)\Bigg]$
$u_\lambda(x) = \sin(\sqrt{\lambda}\,x)$

Now, consider $u_\lambda(x) = \cos(\sqrt{\lambda}\,x)$. Using $u(0) = 0$, we have:

$$u_\lambda(0) = \cos(\sqrt{\lambda}\,(0))$$
$$0 = \cos(0)$$
$$0 = 1$$

Then, using $u(L) = 0$, we have:

$$u_\lambda(0) = \cos(\sqrt{\lambda}\,L)$$
$$0 = \cos(\sqrt{\lambda}\,L)$$
$$\sqrt{\lambda}\,L = 0$$

Now, consider $u_\lambda(x) = \sin(\sqrt{\lambda}\,x)$. Using, Using $u(0) = 0$, we have:

$$u_\lambda(0) = \sin(\sqrt{\lambda}\,(0))$$
$$0 = 0$$

Then, using $u(L) = 0$, we have:

$$u_\lambda(0) = \sin(\sqrt{\lambda}\,L)$$
$$0 = \sin(\sqrt{\lambda}\,L)$$
$$\sqrt{\lambda}\,L,\ n\pi, \text{ where } n \in \mathbb{N}$$

So, $\lambda = \left(\dfrac{n\pi}{L}\right)^2$, where $n \in \mathbb{N}$. These are the allowable eigenvalues.

Denote this result as $\lambda_{n,D}$. Finally, let $f_{n,D}(x) = \sin\left(\dfrac{n\pi x}{L}\right)$, where $n \in \mathbb{N}$.

Okay, now that we have that out of the way, consider next,
$\lambda = \dfrac{1}{vg(t)}\ g'(t)$. Next, if we re-write things a little bit, we get

$$g'(t) + v\lambda g(t) = 0$$

Now, suppose the solutions to the above O.D.E. are of the form
$g(t) = e^{wt}$ for some w. Plugging into the O.D.E., we have

$$we^{wt} + v\lambda e^{wt} = 0$$
$$w + v\lambda = 0$$
$$w = -v\lambda$$

So, our solution is $w = -v\lambda$. But, from our derivation of the eigendata, we know that λ is defined to be $\left(\dfrac{n\pi}{L}\right)^2$. Therefore, $g_n(t) = A_n e^{-(\frac{n\pi}{L})^2 v\lambda}$.
Therefore, $u_n(x,t) = f_n(x)g_n(t)$ which gives

$$u_n(x,t) = \left(A_n e^{-(\frac{n\pi}{L})^2 v\lambda}\right)\sin\left(\frac{n\pi}{L}\ x\right),\ \forall\ n \in \mathbb{N}$$

Thus, because of the form of $f(x)$, the general solution to the given *Diffusion Equation* is

$$u_n(x,t) = \sum_{n=1}^{\infty}\left(A_n e^{-(\frac{n\pi}{L})^2 v\lambda}\right)\sin\left(\frac{n\pi}{L}\ x\right),\ \forall\ n \in \mathbb{N}$$

7)Let the *Diffusion Equation* be given as defined. Now, suppose
$u(x,t) = f(x)g(x)$ for some f, g. Then, define the following

$$\frac{\partial u}{\partial t} = f(x)g'(x) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = f''(x)g(t)$$

If we substitute into the given P.D.E., we have $f(x)g'(t) = vf''(x)g(t)$

Now, define the $\lambda(x,t)$ function as

$$\lambda(x,t) = -\frac{1}{vg(t)}\ g'(t) = -\frac{1}{f(x)}\ f''(x),$$

where we assume that $g(t), f(t) \neq 0$

Then, if we take some derivatives, we have

$$\frac{\partial\lambda}{\partial x} = \frac{\partial}{\partial x}\Big[-\frac{1}{v^2 g(t)}\ g''(t)\Big] = 0$$
$$\frac{\partial\lambda}{\partial t} = \frac{\partial}{\partial t}\Big[-\frac{1}{f(x)}\ f''(x)\Big] = 0$$

Thus, $\lambda(x,t) = \lambda$. So, $\lambda = -\frac{1}{f(x)}\ f''(x)$. Then, $f''(x) + \lambda f(x) = 0$.

Now, from the boundary conditions, we have that

$$f'(0)g(t) = 0 \quad \text{and} \quad f'(L) = 0$$

Now, we take a slight detour so that we may derive the eigendata from scratch. So, suppose that the solutions for $f''(x) + \lambda f(x) = 0$ are of the form $u(x) = e^{kx}$. Then, $u'(x) = ke^{kx}$ and $u''(x) = k^2 e^{kx}$. Plugging in, we have:

$$k^2 e^{kx} + \lambda e^{kx} = 0$$
$$e^{kx}(k^2 + \lambda) = 0$$
$$k^2 + \lambda = 0, \text{ which is our auxiliary equation.}$$

First, consider $\lambda < 0$. Using the auxiliary equation we obtained above, we have

$$k^2 + \lambda = 0$$
$$k^2 = -\lambda$$
$$k = \pm\sqrt{\lambda}$$

So, $v_1(y) = e^{\sqrt{-\lambda}\,x}$ and $v_2(y) = e^{-\sqrt{-\lambda}\,x}$.

Now, check the wronskian:

$$w(v_1(y), v_2(y)) = \begin{vmatrix} e^{ky} & e^{-ky} \\ ke^{ky} & -ke^{ky} \end{vmatrix}$$
$$w(v_1(y), v_2(y)) = (e^{ky} \cdot -ke^{ky}) - (e^{-ky} \cdot ke^{ky})$$
$$w(v_1(y), v_2(y)) = (-k(e^{ky})(e^{-ky})) - (k(e^{-ky})(e^{ky}))$$
$$w(v_1(y), v_2(y)) = (-k \cdot 1) - (k \cdot 1) = (-k)(k) = -k^2 \neq 0$$

Since $w(v_1(y), v2(y)) \neq 0$, the solutions $v_1(y)$ and $v_2(y)$ are linearly independent and thus form a fundamental set of solutions. Now, the astute reader can observer that plugging in any values (say, $\sqrt{-\lambda}$) will have the same result.

Next, taking derivatives, we have:

$$v_1'(y) = \sqrt{-\lambda}e^{\sqrt{-\lambda}\,y} \text{ and } v_2'(y) = -\sqrt{-\lambda}e^{-\sqrt{-\lambda}\,y}.$$

Using $u'(0) = 0$, we have:

$$\begin{aligned} v_1'(0) &= \sqrt{-\lambda}e^{\sqrt{-\lambda}\,(0)} & \qquad v_2'(0) &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}\,(0)} \\ v_1'(0) &= \sqrt{-\lambda} & v_2'(0) &= -\sqrt{-\lambda} \\ 0 &= \sqrt{-\lambda} & 0 &= -\sqrt{-\lambda} \end{aligned}$$

Using $u'(L) = 0$, we have:

$$\begin{aligned} v_1'(L) &= \sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L)} & \qquad v_2'(0) &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L)} \\ 0 &= \sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L)} & 0 &= -\sqrt{-\lambda}e^{\sqrt{-\lambda}\,(L)} \end{aligned}$$

Since the functions $v_1(y)$ and $v_2(y)$ do not satisfy the boundary conditions, they are not *Neumann-Laplacian eigenfunctions.* Now, consider $\lambda = 0$. Then, the differential equation becomes

$$u''(y) = 0$$

Then, if we integrate, we have:

$$\begin{gathered} u'(y) = c \\ u(y) = c_1 y + c_0, \text{ for some constants } c_1 \textit{ and } c_0 \end{gathered}$$

Using $u(0) = 0$, we have:

$$\begin{gathered} u(0) = c_1(0) + c_0 \\ 0 = 0 \end{gathered}$$

Using $u(L) = 0$, we have:

$$\begin{gathered} u(L) = c_1(L) + c_0 \\ 0 = c_0 \end{gathered}$$

Similar to the *Dirichlet-Laplacian eigenproblem*, our two solutions are $h_1(y) = 1$ and $h_2(y) = y$

Now, check the wronskian:

$$\begin{gathered} w(h_1(y), h_2(y)) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} \\ w(h_1(y), h_2(y)) = (1 \cdot 1) - (0 \cdot x) \\ w(h_1(y), h_2(y)) = 1 - 0 = 1 \neq 0 \end{gathered}$$

Since $w(h_1(y), h_2(y)) \neq 0$, our two solutions are linearly independent and thus form a fundamental set of solutions.
Calculating some derivatives, we have:

$$h_1'(y) = 0 \quad and \quad h_2'(y) = 1$$

Using $u'(0) = 0$, we have:

$$\begin{aligned} h_1'(0) &= 0 & \qquad h_2'(0) &= 1 \\ 0 &= 0 & 0 &= 1 \end{aligned}$$

Using $u'(L) = 0$, we have:

$$\begin{aligned} h_1'(L) &= 0 & \qquad h_2'(L) &= 1 \\ 0 &= 0 & 0 &= 1 \end{aligned}$$

Just as in the *Dirichlet-Laplacian eigenproblem*, the corresponding harmonic functions do not satisfy the boundary conditions, and in this case, are not *Neumann-Laplacian eigenfunctions.*
Finally, consider $\lambda > 0$. Then, suppose the solutions are of the form $u(y) = e^{ky}$. Then, $u'(y) = ke^{ky}$ and $u''(y) = k^2e^{ky}$. Plugging in, we have:

$$\begin{gathered} k^2e^{ky} + \lambda e^{ky} = 0 \\ e^{ky}(k^2 + \lambda) = 0 \\ k^2 + \lambda = 0 \\ k = \pm\sqrt{-\lambda} \\ k = \pm i\sqrt{\lambda} \end{gathered}$$

So, $u_1(y) = e^{i\sqrt{\lambda}}$ and $u_2(y) = e^{-i\sqrt{\lambda}}$.
Next, let $u_\lambda(y) = \dfrac{1}{2}\Big[u_1(y) + u_2(y)\Big]$.
Then,
$u_\lambda(y) = \dfrac{1}{2}\Big[\cos(\sqrt{\lambda}\,y) + i\sin(\sqrt{\lambda}\,y) + \cos(-\sqrt{\lambda}\,y) + i\sin(\sqrt{\lambda}\,y)\Big]$
$u_\lambda(y) = \cos(\sqrt{\lambda}\,y)$

Let $v_\lambda(y) = \dfrac{1}{2i}\Big[\cos(\sqrt{\lambda}\,y) + i\sin(\sqrt{\lambda}\,y) - \cos(-\sqrt{\lambda}\,y) + i\sin(\sqrt{\lambda}\,y)\Big]$
$v_\lambda(y) = \sin(\sqrt{\lambda}\,y)$

Now, calculating some derivatives, we have:

$$\begin{aligned} u_\lambda'(y) &= -\sqrt{\lambda}\sin(\sqrt{\lambda}\,y) \\ v_\lambda(y) &= \sqrt{\lambda}\cos(\sqrt{\lambda}\,y) \end{aligned}$$

Now, consider $u_\lambda(x) = \cos(\sqrt{\lambda}\, x)$. Using $u'(0) = 0$, we have:

$$u'_\lambda(0) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\, y)$$
$$u'_\lambda(0) = 0$$
$$0 = 0$$

Then, using $u'(L) = 0$, we have:

$$u'_\lambda(L) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\, L)$$
$$0 = \sqrt{\lambda}\sin(\sqrt{\lambda}\, L)$$
$$0 = \sin(\sqrt{\lambda}\, L)$$
$$\sqrt{\lambda}\, L,\ n\pi, \text{ where } n \in \mathbb{N}$$

So, $\lambda = \left(\dfrac{n\pi}{L}\right)^2$, where $n \in \mathbb{N}$. These are the allowable eigenvalues. Denote this result as $\lambda_{n,N}$. Finally, let $f_{n,N}(y) = \cos\left(\dfrac{n\pi y}{L}\right)$, where $n \in \mathbb{N}$.

Okay, now that we have that out of the way, consider next, $\lambda = \dfrac{1}{vg(t)}\, g'(t)$. Next, if we re-write things a little bit, we get

$$g'(t) + v\lambda g(t) = 0$$

Now, suppose the solutions to the above O.D.E. are of the form $g(t) = e^{wt}$ for some w. Plugging into the O.D.E., we have

$$we^{wt} + v\lambda e^{wt} = 0$$
$$w + v\lambda = 0$$
$$w = -v\lambda$$

So, our solution is $w = -v\lambda$. But, from our derivation of the eigendata, we know that λ is defined to be $\left(\dfrac{n\pi}{L}\right)^2$. Therefore, $g_n(t) = A_n e^{-(\frac{n\pi}{L})^2 v\lambda}$. Therefore, $u_n(x,t) = f_n(x)g_n(t)$ which gives

$$u_n(x,t) = \left(A_n e^{-(\frac{n\pi}{L})^2 v\lambda}\right)\cos\left(\frac{n\pi}{L}\, x\right),\ \forall\ n \in \mathbb{N}$$

Thus, because of the form of $f(x)$, the general solution to the given *Diffusion Equation* is

$$u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n e^{-(\frac{n\pi}{L})^2 v\lambda} \right) \cos\left(\frac{n\pi}{L}\ x \right),\ \forall\ n \in \mathbb{N}$$

So, looking back on the two different *general solutions* to the *diffusion equation*, the different boundary conditions affected the eigenfunction that was obtained in the two problems. Depending on the boundary conditions, we either had the eigenfunction defined as
$f_{n,N}(y) = \cos\left(\frac{n\pi y}{L}\right)$ (as in the *Nuemann* case) or as
$f_{n,N}(y) = \sin\left(\frac{n\pi y}{L}\right)$ (as in the *Dirichlet* case).