Topology Final Exam Study Guide

Brendan Busey

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Chapter 1

1.1 Equivalence Relations

Definition 1 *A binary relation $\sim$ on a set X is an* ***equivalence relation*** *if and only if $\forall\ x, y, z \in X$ satisfies*

1. *$x \sim x$ (reflexivity)*
2. *if $x \sim y$, then $y \sim x$ (symmetry)*
3. *if $x \sim y$ and $y \sim z$, then $x \sim z$ (transitivity)*

1.2 Bijections

Definition 2 *A function $f : X \to Y$ is an injection (or a* ***one-to-one*** *function) if and only if for an $x_1, x_2 \in X$ we have*

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

The function is a ***surjection*** *(or an* ***onto*** *function) if and only if for every $y \in Y$, there is $x \in X$ with $f(x) = y$. The function is a* ***bijection*** *if and only if it is an injection and a surjection.*

Theorem 1 *Consider a function $f : X \to Y$. Then, f is a bijection if and only if f has an inverse function.*

1.3 Continuous Functions

Definition 3 *A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\forall\, x \in X$, we have the implication that $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$. A function is continuous if and only if it is continuous at each point of it s domain.*

Theorem 2 *Suppose A and B are regions of $\mathbb{R}^2$ that are bounded by polygons. Suppose $f : A \to Y$ and $g : B \to Y$ are continuous functions such that $f(x) = g(x)\ \forall x \in A \cap B$. Then, the function $h : A \cup B \to Y$ is defined by*

$$h(x) = \begin{cases} f(x) & if\ x \in A \\ g(x) & if\ x \in B \end{cases}$$

is continuous.

1.4 Topological Equivalance

Definition 4 *A **homeomorphism** (or **topological equivalence**) is a bijection $h : X \to Y$ such that both h and h^{-1} are continuous. The spaces X and Y are **homeomorphic** (or **topologically equivalent**) if and only if there is a homeomorphism from X to Y.*

Definition 5 *The standard disk is the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. A **disk** is any topological space homeomorphic to the standard disk.*

*The **standard n-dimensional ball** (or more simply, the **the standard n-ball**) is the set $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}$. An **n-ball** (or **n-cell**) is any topological space homeomorphic to the standard n-ball.*

*The **standard** n-**dimensional sphere** (or more simply, the **standard** n-**sphere**) is the set $\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}$. An n-**sphere** is any topological space homeomorphic to the standard n-sphere.*

1.5 Topological Invariants

Definition 6 *A **path** in a space X is a continuous function $\alpha : [0,1] \to X$. Consider the equivalence relation between pairs of points in a set of X defined by $x \sim y$ if and only if there is a path $\alpha : [0,1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. The equivalence classes under this relation are called **path components** of X. A set such that every two points are joined by a path is said to be **path-connected***

Theorem 3 *Suppose $\alpha : [0,1] \to A \cup B$ is a path $\alpha(0) \in A$ and $\alpha(1) \in B$. Then, there is a sequence of points of A that converges to a point of B or else there is a sequence of points of B that converges to a point in A.*

Corollary 3.1 *A homeomorphism $h : X \to Y$ induces a bijection $h_* : P(x) \to P(Y)$. In particular, the number of path components of a space is topologically invariant.*

1.6 Isotopy

Definition 7 *Suppose A and B are two subsets of a space X. An **Ambient isotopy** from A to B in X is a continuous function $h : X \times [0,1] \to X$ that satisfies the following three conditions. We denote $h(x,t)$ by $h_t(x)$.*

1. *$h_t : X \to X$ is a homeomorphism for every $t \in$ [0,1]*
2. *h_0 is the identity function on X*
3. *$h_1(A) = B$*

Chapter 2

2.1 Knots, Links, and Equivalences

Definition 8 *A **knot K** is a simple closed curve in $\mathbb{R}^3$ that can be broken into a finite number of straight line segments $e_1, e_2, \cdots, e_n$ such that the intersection of any segment with e_k with the other segments is exactly one endpoint of e_k intersecting an endpoint of e_{k-1} (or e_n if $k = 1$) and the other endpoint of e_k intersecting an endpoint of e_{k+1} (or e_1 if $k = n$).*

Definition 9 *Consider a triangle ABC with side AC matching one of the line segments of a knot K. In the plane determined by the triangle, we require that the region bounded by ABC intersects K only in the edge AC. A* ***triangular detour*** *involves replacing the edge AC of knot K with the two edges AB and BC to produce a new knot L. With the same notation, a* ***triangular shortcut*** *involves replacing the two edges AB and BC and L with the single edge of AC to produce knot K. A* ***triangular move*** *is either a triangular detour or a triangular shortcut. Two knots are* ***equivalent*** *if and only if there is a finite sequence of triangular moves that changes the first knot into the second.*

Definition 10 *A* ***Link*** *is the nonempty union of a finite number of disjoint knots.*

2.2 Knot Diagrams

Definition 11 (General Position Rule of Thumb) *Suppose two piecewise-linear objects are embedded in general position in $\mathbb{R}^n$. Suppose A is a vertex, edge, face, or analogous higher-dimensional part of one object and B is a vertex, edge, face, or analogous higher-dimensional part of the other object. If the intersection $A \cup B$ is nonempty, then.*

$$dim(A \cap B){=}dim(A){+}dim(B){-}n$$

Definition 12 *The orthogonal projection of a knot onto a plane is a* ***regular projection*** *if and only if no vertex projects to the image of another point of the knot and there are no triple points.*

Definition 13 *The* ***crossing number*** *of a knot K is the minimum number of crossing points that occur in the knot diagrams for all knots equivalent K.*

Definition 14 *The* ***unknotting number*** *is the minimum number of times the knot must be passed through itself (****crossing switch****) to untie it*

Definition 15 *A* ***trivial knot*** *is a knot that is equivalent to a triangle. A* ***trivial link*** *is a link that is equivalent to the union of disjoint triangles lying in a plane*

Definition 16 *A knot is* ***alternating*** *if and only if it is equivalent to a knot with a diagram in which underpasses alternate with overpasses as you travel around the knot*

2.3 Reidmeister Moves

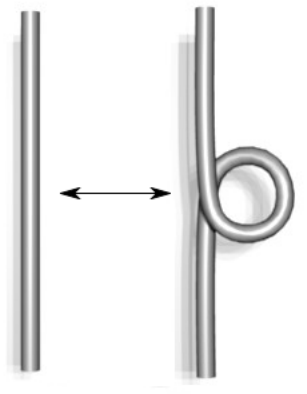

Figure 1: Type 1

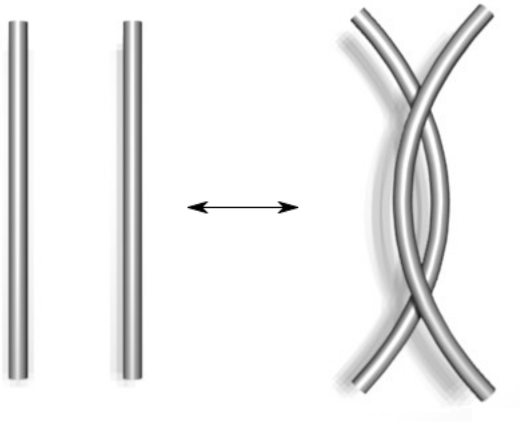

Figure 2: Type 2

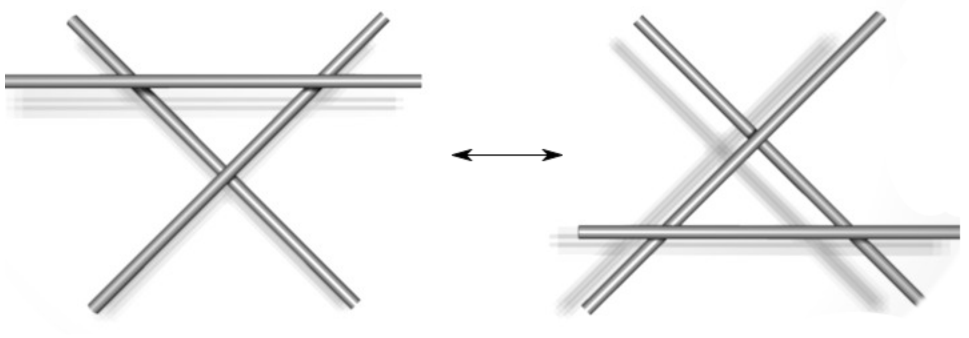

Figure 3: Type 3

Theorem 4 *If two links are equivalent, then their diagrams, subject to ambient isotopies of the plane, are related by a sequence of Reidemeister moves.*

Definition 17 *An **orientation** of a link is a choice of direction to travel around each component of the link. Consider a crossing a regular projection of an oriented link. Stand on the overpass and face in the direction of the orientation. The crossing is **right-handed** if and only if traffic on the underpass goes from right to left; the crossing is **left-handed** if and only if traffic on the underpass goes from left to right. In regular projection of an oriented link of two components, assign +1 to right-handed crossings and -1 to left-handed crossings. Add up the numbers assigned to crossings involving both components. One half this sum is the **linking number** of the two oriented components of the link.*

2.4 Colorings

Definition 18 *The diagram of a knot is **colorable** if and only if each arc can be assigned one of three colors subject to the two conditions:*

1. *At least two colors appear*
2. *At any crossing where two colors appear, all three colors appear*

Theorem 5 *The colorability of a knot diagram is an invariant property of the knot type.*

Definition 19 *Let p be an odd number greater than two. A knot is p-colorable if at every crossing:*

1. *At least two colors appear*
2. *you can solve $color1 + color2 \equiv 2x$ mod (the number of colors used), where color1 and color2 are numbers assigned to the arcs of the knot*

Theorem 6 *The colorability of a knot diagram is an invariant property of the knot type.*

Theorem 7 *The representation of a knot diagram on a wheel with p colors is an invariant property of the knot type.*

Definition 20 *The determinant of a knot is the absolute value of its Alexander Polynominal evaluated at -1 (simply, plug-in -1 for t)*

Theorem 8 *A knot is p-**colorable** for prime p greater than two if and only if p divides its determinant*

2.5 The Alexander Polynominal

Steps in computing the Alexander Polynominal:

1. Label the crossings $x_1, x_2, \cdots, x_n$
2. Label the arcs $a_1, a_2, \cdots, a_n$
3. Choose an orientation for the knot
4. As you travel around the knot in the chosen orientation, stand on the overpass of each crossing. Label the overstrand with $1 - t$, the left-end of the understrand t, and the right-end of the understrand -1
5. Create the arc/crossing matrix
6. Compute the determinant of the matrix, which is the Alexander Polynominal

Definition 21 *The projection of an oriented knot divides the plane into a number of regions. The **index** of one of these regions is the net number of times the projection winds counterclockwise around any point in the region.*

Definition 22 *The **index** of a crossing of a knot diagram is the common value of the index of two of the regions near the crossing.*

Theorem 9 *The Alexander Polynominal of an oriented knot is an invariant under Reidemeister moves.*

2.6 Skein Relations

Calculating the Δ polynominal is the same as calculating the Alexander Polynominal

2.7 The Jones Polynominal

Rules for the Bracket Polynominal *The Kauffman **Bracket Polynominal** of a regular projection of a link is a polynominal in integer powers of the variable A defined by the following three rules:*

1. $\left\langle \bigcirc \right\rangle = 1$
2. $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle L \rangle$
3. $\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

Definition 23 *The **writhe** $w(L)$ of the regular projection L of a link is the number of right-handed crossings minus the number of left-handed crossings.*

Definition 24 *The $X(L)$ polynominal is defined as:*

$X(L) = (-A)^{-3w(L)}\langle L \rangle$, *where* $\langle L \rangle$ *is the bracket polynominal for L and $w(L)$ is the writhe of L*

Steps in calculating the Jones Polynominal:

1. Calculate the Bracket Polynominal
2. Calculate the $X - polynominal$
3. Substitute $t^{-\frac{1}{4}}$ in for every "A" in the $X - polynominal$ and simplify

Chapter 3 Surfaces

3.1 Definition and Examples

Definition 25 *In a space with a way of measuring distances between points, a **neighborhood** of a point is a subset that contains all points within some positive distance of the point*

Definition 26 *A **surface** (or 2 **-manifold**) is a space that is homeomorphic to a nonempty subset of finite-dimensional Euclidean space and in which every point has a neighborhood homeomorphic to $\mathbb{R}^2$. We sometimes also wish to admit **boundary** points, which have neighborhoods homeomorphic to the half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.*

3.2 Cut-and-Paste Techniques

Definition 27 *Let S and T be path-connected surfaces Remove the interior of a disk from each surface by cutting along the boundaries of the disks. Glue the remaining surfaces together along the newly formed boundary components. The result surface is the **connected sum** of S and T. It is denoted $S\#T$.*

3.3 The Euler Characteristic and Orientability

Definition 28 *A **triangulation** of a space is a decomposition of the space into a union of disks, arcs, and points. The disks are called **faces**, the arcs are called **edges**, and the points are **vertices** of the triangulation. A face intersects other components of a triangulation only along its boundary; and the boundary of a face consists of three edges and three vertices. An edge intersects other edges and the vertices only at its endpoints; and both endpoints of an edge are vertices.*

Definition 29 *A triangulated space is **compact** if and only if it consists of a finite number of faces, edges, and vertices.*

Definition 30 *The **Euler characteristic** of a compact triangulation space S is the number of vertices minus the number of edges plus the number of faces. The Euler characteristic of S is denoted by $\chi(S)$.*

Theorem 10 *Every closed, path-connected surface is homeomorphic to exactly one of:*

1. *A 2-sphere*
2. *A connected sum of Tori*
3. *A connected sum of projective planes*

Side Note 1 *Something is orientable if it is 2-colorable or doesn't have a mobius band*

Theorem 11 *Suppose A and B are triangulated so that $A \cap B$ is also triangulated. Then, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.*

Theorem 12 *The surface formed by taking the connected sum of g tori and cutting out disks to leave b boundary components has Euler characteristic 2-2g-b. The surface formed by taking the connected sum of n projective plane and cutting out disks to leave b boundary components has Euler characteristic 2-n-b.*

Definition 31 *An **orientation** of a polygonal face of a triangulated surface is the choice of one of the two possible orientations of the boundary curve of the face. A surface is **orientable** if and only if it is possible to choose orientations of all the faces of a triangulation of the surface so that whenever two faces share a common edge, the orientation of the faces induce opposite orientations on the edge.*

Definition 32 $\chi(f_1 \# f_2) = \chi(f_1) + \chi(f_2) - 2$

Definition 33 *With the word you come up with for the surface, if every letter does not have an inverse, the surface is not orientable. If every letter does have an inverse, it is orientable*

Definition 34 *Just concatenate the two individual words of each surface to get a word for the connected sum of the two surfaces*

Side Note 2 *The table below helps you figure out what surface is being asked for based on* ***Orientability*** *and the* ***Euler characterisitic***

χ	*Oreintable*	*Non − orientable*
2	*Sphere*	
1		P
0	T	$P \# P$
-1		$P \# P \# P$
-2	$T \# T$	$P \# P \# P \# P$
-3		$P \# P \# P \# P \# P$
-4	$T \# T \# T$	$P \# P \# P \# P \# P \# P$