4) Let the *Diffusion Equation* $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$ be given as defined.

*Proof.* First, suppose that solutions are of the form $u = u(x,t)$. After moving things around, we have

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0$$

Now, multiply $\dfrac{\partial u}{\partial t} - D\dfrac{\partial^2 u}{\partial x^2} = 0$ through by $u$ to get

$$u\frac{\partial u}{\partial t} - Du\frac{\partial^2 u}{\partial x^2} = 0$$

Now, before continuing, we note the following

$$\frac{\partial}{\partial t}\left[\frac{1}{2}u^2\right] = u\left(\frac{\partial u}{\partial t}\right)$$

Also,

$$\frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right] = -Du\frac{\partial^2 u}{\partial x^2} - D\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right] + D\left(\frac{\partial u}{\partial x}\right)^2 = -Du\frac{\partial^2 u}{\partial x^2}$$

Okay, now, $u\dfrac{\partial u}{\partial t} - Du\dfrac{\partial^2 u}{\partial x^2} = 0$ becomes

$$\frac{\partial}{\partial t}\left[\frac{1}{2}u^2\right] + \frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right] + D\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

Then, we integrate over the interval $0 < x < L$ to get

$$\int_0^L \frac{\partial}{\partial t}\left[\frac{1}{2}u^2\right]\,dx + \int_0^L \frac{\partial}{\partial x}\left[-Du\left(\frac{\partial u}{\partial x}\right)\right]\,dx + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\,dx = 0$$

After using the *Fundamental Theorem of Calculus*, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\,dx + \left[-Du\left(\frac{\partial u}{\partial x}\right)\right]\Bigg|_{x=0}^{x=L} + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\,dx = 0$$

Now, if we evaluate the middle term from $x = 0$ to $x = L$, we can see that it becomes

$$-Du(L,t)\frac{\partial u}{\partial x}(L,t) + Du(0,t)\frac{\partial u}{\partial x}(0,t)$$

So, now we have the following

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx + \Bigg( - Du(L,t)\frac{\partial u}{\partial x}(L,t) + Du(0,t)\frac{\partial u}{\partial x}(0,t)\Bigg) + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx = 0$$

Next, remembering that our boundary conditions are

$$-\frac{\partial u}{\partial x}(0,t) + b_0 u(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) + b_L u(L,t)$$

if we rearrange things a bit, we get

$$-\frac{\partial u}{\partial x}(0,t) = -b_0 u(0,t) \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = -b_L u(L,t)$$

So, upon comparing $-Du(L,t)\frac{\partial u}{\partial x}(L,t) + Du(0,t)\frac{\partial u}{\partial x}(0,t)$ with

$$-\frac{\partial u}{\partial x}(0,t) = -b_0 u(0,t) \text{ and } \frac{\partial u}{\partial x}(L,t) = -b_L u(L,t)$$

we can substitute using the boundary conditions giving us

$$Du(L,t)b_L u(L,t) + Du(0,t)b_0 u(0,t)$$

So, at this point, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx + \Bigg(Du(L,t)b_L u(L,t) + Du(0,t)b_0 u(0,t)\Bigg) + D\int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx = 0$$

Okay, now, if we note that both the second and third term have a $D$ term that we can factor out, after combing terms, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx + D\left[u(L,t)b_L u(L,t) + u(0,t)b_0 u(0,t) + \int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx\right] = 0$$

Then, if we move things around, we have

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2\ dx = -D\left[u(L,t)b_L u(L,t) + u(0,t)b_0 u(0,t) + \int_0^L \left(\frac{\partial u}{\partial x}\right)^2\ dx\right] \le 0$$

Thus, the function $F(t) = \dfrac{d}{dt}\displaystyle\int_0^L \frac{1}{2}u(x,t)^2\, dx$ by the above expression is a decreasing function.

$\square$