Singular Sturm-Liouville Eigenvalue Problems: Theory and Computation

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Introduction

- Throughout the research process, we studied the boundary value problem

$$
\begin{aligned}
&(-x^n y')' = \lambda x^m y \ \ in\ (0,1) \\
&y(0) = 0, y(1) = 0
\end{aligned}
\tag{1}
$$

- Note: this boundary value problem is indeed an eigenvalue problem
- With the following goals:
  - determine the λ values such that (1) has a non-trivial solution
  - explore how the parameters n and m influence those eigenvalues

Motivation

- Our research is motivated by the work of Paul Binding and Hans Volkmer who, in [1], discussed the following problem

$$-(p(x)y')' + q(x)y = (\lambda r(x) + \mu)y, \text{ for } a \leq x \leq b$$

with the separated boundary conditions

$$\cos(\alpha)y(a) - \sin(\alpha)p(a)y'(a) = 0 \text{ and } \cos(\beta)y(b) - \sin(\beta)p(b)y'(b) = 0$$

The Initial Value Problem

- In order to gain a better understanding of (1), we investigated solutions to the following *Initial Value Problem*

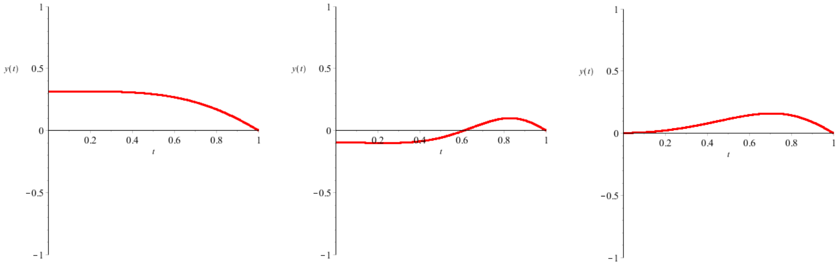
$$\begin{aligned} &(-x^n y')' = \lambda x^m y \\ &y'(1) = -1, y(1) = 0 \end{aligned} \tag{2}$$

- Defined on the interval $(0, 1)$
- Through experimentation, examined the affects of changing the value of m and n on the eigenvalues and eigenfunctions that were obtained

Basic idea:

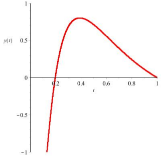
- Employed the mathematical software Maple in order to plot solutions of the Initial Value Problem (2) for varying values of m, n, and λ
- Three situations

"Overshoot" "Undershoot" "Just right"

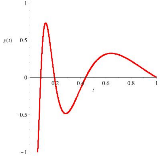


First Experimental Result: No Eigenvalues

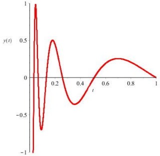
- $n = 2,\ m = 0,$ and $\lambda = 4$



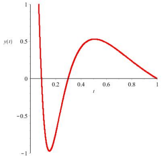
- $n = 2,\ m = 0,$ and $\lambda = 15$



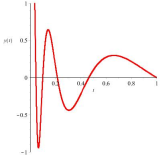
- $n = 2,\ m = 0,$ and $\lambda = 22$



- $n = 2,\ m = 0,$ and $\lambda = 7$



- $n = 2,\ m = 0,$ and $\lambda = 17$



First Theoretical Result: Euler Equations

Observation 1

For positive m and n values, asymptotic behaviour, specifically a vertical asymptote, occurs along the $x - axis$

- We first considered solutions that were in the form of *Euler equations*
  - Are of the form $x^2y'' + \alpha xy' + \beta y = 0$, where α and β are real constants
  - Have a point at which the Euler equation cannot be represented by a power series of the form $\sum_{n=0}^{\infty} a_n x^n$

First Theoretical Result: Euler Equations (continued)

Obtaining solutions:

1. We start with

$$(-x^n y')' = \lambda x^m y, \text{ subject to the boundary conditions } y(0) = 0, y(1) = 0$$

2. Then, after computing derivatives and re-arranging we get

$$-x^n y'' - nx^{n-1} y' - \lambda x^m y = 0$$

3. We next suppose that solutions are of the form $y = x^r$, calculate more derivatives, plug-in our derivatives appropriately, and after some algebra, we have

$$r(r-1) + nr + \lambda = 0, \text{ which is our } \textit{characteristic} \text{ equation}$$

First Theoretical Result: Euler Equations (continued)

4. We use the quadratic formula after plugging in our well-chosen value of $n = -1$ to find the roots of our characteristic equation

$$\frac{-1}{2} \pm \sqrt{\frac{1}{4} - \lambda}$$

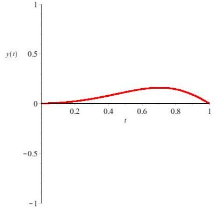
5. Finally, after some substitution, using our initial conditions, and using *Euler's formula* to extract the real (non-complex) roots, we have our final solutions

$$y_1(x) = x^{-\frac{1}{2}} \cos\left(\sqrt{\lambda - \frac{1}{4}} \ln(x)\right)$$

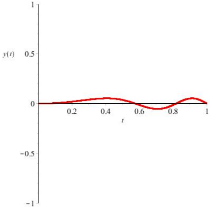
$$y_2(x) = x^{-\frac{1}{2}} sin\left(\sqrt{\lambda - \frac{1}{4}} \ln(x)\right)$$

Second Experimental Result: Existence of Eigenvalues

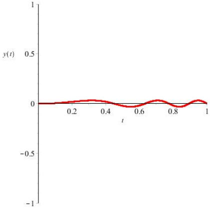
- $n = -1$, $m = 1$, and $\lambda = 39$



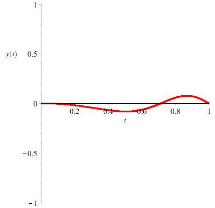
- $n = -1$, $m = 1$, and $\lambda = 355$



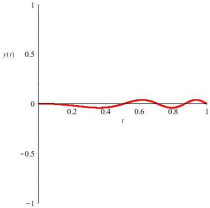
- $n = -1$, $m = 1$, and $\lambda = 990$



- $n = -1$, $m = 1$, and $\lambda = 160$



- $n = -1$, $m = 1$, and $\lambda = 640$



Second Theoretical Result: Power Series Solutions and the Method of Frobenius

Observation 2

When m is greater than n, n is negative or zero, and as lambda values increase, the roots become clustered closer together.

- Previously, we had considered finding solutions using *Euler equations*

$$x^2y'' + \alpha xy' + \beta y = 0, \text{ where } \alpha \text{ and } \beta \text{ are real constants}$$

- With the method of Frobenius, however, we replace α and β in the above equation with coefficients that have "nice" series representations

$$x^2y'' + x[\alpha(x)]y' + [\beta(x)]y = 0$$

Second Theoretical Result: Power Series Solutions and the Method of Frobenius (continued)

Deriving solutions:

1. As before, we begin with

$(-x^n y')' = \lambda x^m y$, subject to the boundary conditions $y(0) = 0, y(1) = 0$

2. Then, after computing derivatives and re-arranging we get

$$-x^n y'' - nx^{n-1} y' - \lambda x^m y = 0$$

3. If we make the substitution for our well-chosen values of $n = -1$ and $m = 1$, multiply through by x^3, and simplify, we have

$$x^2 y'' - xy' + \lambda x^4 y = 0$$

Second Theoretical Result: Power Series Solutions and the Method of Froenius (continued)

4. Then, if we compare the general equation

$$x^2y'' + x[\alpha(x)]y' + [\beta(x)]y = 0$$

With our equation

$$x^2y'' - xy' + \lambda x^4 y = 0$$

It is clear that

$$\alpha(x) = -1 \qquad \text{and} \qquad \beta(x) = \lambda x^4$$

5. Then, we confirm that the point $x = 0$ is indeed a regular singular point by computing the following limits and showing that they exist

$$\lim_{x\to 0} \alpha(x) = -1 \qquad \text{and} \qquad \lim_{x\to 0} \beta(x) = 0$$

Second Theoretical Result: Power Series Solutions and the Method of Frobenius (continued)

6. Now that we have shown that $x = 0$ is a regular singular point, we know that our solutions will be of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

7. We also know that $\alpha(x)$ and $\beta(x)$ can be defined as follows

$$\alpha(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \text{and} \qquad \beta(x) = \sum_{n=0}^{\infty} q_n x^n$$

8. Similar to the *characteristic equation* we had with Euler equations, here, we have our *indicial equation*

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

Second Theoretical Result: Power Series Solutions and the Method of Frobenius (continued)

9. If we then evaluate $\alpha(x) = \sum_{n=0}^{\infty} p_n x^n$ and $\beta(x) = \sum_{n=0}^{\infty} q_n x^n$ at $n = 0$, plug this into our *indicial equation*, and solve for r, we find

$$r_1 = 2 \qquad \text{and} \qquad r_2 = 0$$

10. At this point, we know that our solutions will be of the form

$$y_1(x) = |x|^{r_1} \left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right]$$

and

$$y_2(x) = a y_1(x) \ln|x| + |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} c_n(r_2) x^n \right]$$

Second Theoretical Result: Power Series Solutions and the Method of Frobenius (continued)

11. Finally, after substituting in for r_1, r_2, and $y_1(x)$ and simplifying

$$y_1(x) = |x|^2 \left[1 + \sum_{n=1}^{\infty} 2a_n x^n\right]$$

and

$$y_2(x) = a|x|^2 \left[1 + \sum_{n=1}^{\infty} 2a_n x^n\right] \ln|x| + 1$$

Future ideas

1. Explore a greater variety of values for m and n
2. Investigate fractional differences between m and n

Acknowledgements

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References

Binding, Paul; Volkmer, Hans *Eigencurves for two-parameter Sturm-Liouville equations. SIAM Rev. 38 (1996), no. 1, 27—48.*

Boyce, William E.; DiPrima, Richard C. *Elementary differential equations and boundary value problems. John Wiley & Sons, Inc., New York-London-Sydney 1965 xi+485 pp. 34.00.*