Point Set Topology Notes

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June 2024

Abstract

Note that this is still a work in progress. Any corrections are welcome and should be directed to robert [dot] xiu [at] mail [dot] utoronto [dot] ca. Alternatively, if you have me on your discord friend's list, let me know directly.

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Chapter 1

Topologies and bases

Definition 1.0.1 (Topology). A topology on X is a set $\tau \subseteq \mathcal{P}(X)$ such that

1. $\varnothing, X \in \tau$,
2. If $\{ U_\alpha : \alpha \in \Lambda \} \subseteq \tau$, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$.
3. If $U_1, \ldots, U_n \in \tau$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

When talking to your friends, you might introduce it as ”a topology is a subset of the power set of X that contains the empty set, X, and it is closed under arbitrary unions and finite intersections.

A topological space X is a set with a topology τ. When we are only working with one topology, we will abuse notation and simply call a set X a topological space with no explicit mention of the topology.

When we say $\mathbb{R}$ is a topological space, it is usually understood that the topology on $\mathbb{R}$ is the usual topology on $\mathbb{R}$.

Example 1.0.2. Take $\mathbb{R}$ and the topology on $\mathbb{R}$ to be arbitrary unions and finite intersections of open intervals. //

Example 1.0.3. Let X be any set and let $\tau = \{ \varnothing, X \}$. This is a topology on X, it is the smallest topology on X and it is called the **trivial/indiscrete topology** on X. //

Example 1.0.4. Let X be any set and let $\tau = \mathcal{P}(X)$. Then this is a topology on X and it is the largest topology on X. It is called the **discrete topology** on X. //

Definition 1.0.5 (Open set). Let X be a set with topology τ. An element $O \in \tau$ is called an **open subset of** X, or an **open set**. If $O \subseteq X$ and we say that O is open in X, it means that $O \in \tau$.

Example 1.0.6. Every open interval in $\mathbb{R}$ is open. //

Definition 1.0.7 (Neighborhood). Let X be a topological space with topology τ and $x \in X$. Then a set U is a **(open) neighborhood of** x if $x \in U$ and $U \in \tau$.

Note that we sometimes will simply say neighborhood of x. Neighborhoods are always understood to be open. If the neighborhood is not open, it will be stated explicitly.

1.1 Basis and subbasis

Definition 1.1.1. Let X be a set. Then $\mathcal{B} \subseteq \mathcal{P}(X)$ is a **basis** (for a topology) on X if

- For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.
- For every $B_1, B_2 \in \mathcal{B}$, for every $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Lemma 1.1.2. *Let X be a set and τ, τ' be topologies on X. Let $\mathcal{B}$ be a basis for τ and $\mathcal{B}'$ be a basis for τ'. Then, $\tau \subseteq \tau'$ if and only if for every $x \in X$ and every $B \in \mathcal{B}$ such that $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Loosely speaking, this says that τ' is finer than τ if and only if given any basis element B of τ we can find a basis element of τ' that is contained within B.

Note that it is sufficient to find $U \in \tau'$ such that $x \in U$ and $U \subseteq B$, as every open set is the union of basis elements.

Definition 1.1.3. Let X be a set. $\mathcal{S} \subseteq \mathcal{P}(X)$ is a **subbasis** if $\bigcup \mathcal{S} = X$. The topology generated by $\mathcal{S}$ is the unions of finite intersections of elements of $\mathcal{S}$.

Note that a subbasis can be used to generate a basis as well.
Example 1.1.4. The lower limit topology on $\mathbb{R}$ is finer than the usual topology. Let $\mathcal{B} = \{ [a, b) : a < b, a, b \in \mathbb{R} \}$ be the usual basis of the lower limit topology. The usual topology has a basis of open intervals, $\{ (a, b) : a < b, a, b \in \mathbb{R} \}$. Take any $x \in \mathbb{R}$ and any open interval (a, b) that contains x. Then notice that $[x, b) \subseteq (a, b)$ and $[x, b) \in \mathcal{B}$. //

1.2 Closed sets, interiors and boundaries

Definition 1.2.1 (Closed set)**.** Let X be a topological space. A set $C \subseteq X$ is **closed** if $X \setminus X$ is open.

Note that a set can be both open and closed at the same time.
Example 1.2.2. In any topological space X, $\varnothing$ and X are closed. //
Example 1.2.3. In $\mathbb{R}$, any closed interval $[a, b]$ is closed. //

Definition 1.2.4 (Interior)**.** Let X be a topological space and $A \subseteq X$. Then the **interior of** A, denoted A° is defined to be

$$A^\circ = \bigcup \{ U \subseteq A : U \text{ is open } \}.$$

The definition says that the interior of A is the union of all the open sets that are contained within A. Put differently, the interior of a set A is the largest open set contained in A
Exercise 1.2.5. Prove the remark above.

Definition 1.2.6 (Closure)**.** The **closure** of $A \subseteq X$, denoted $\overline{A}$, is defined to be

$$\overline{A} = \bigcap \{ C \supseteq A : C \text{ is closed } \}.$$

The above definition simply says that the closure of A is the intersection of all closed sets that contain A. Thus the closure of a set is the smallest closed set contained in A.
Exercise 1.2.7. Prove the remark above.

Definition 1.2.8 (Boundary)**.** Let X be a topological space. Let $A \subseteq X$. Then the **boundary of** A, denoted ∂A, is defined as $\overline{A} \cap \overline{X \setminus A}$.

Lemma 1.2.9. *$x \in \partial A$ if and only if any open neighborhood of x intersects A and $X \setminus A$.*

Proof. Left to reader. □

Definition 1.2.10 (Limit point)**.** Let X be a topological space and $A \subseteq X$. A point $x \in X$ is a **limit point of** A if any neighborhood of x intersects $A \setminus \{ x \}$.

Proposition 1.2.11. *Let X be a topological space and $A \subseteq X$. Then $\overline{A} = A \cup L_A$, where L_A is the set of limit points of A.*

Proof. Exercise. □

Definition 1.2.12 (Convergence). Let $(x_n) \subseteq X$ be a sequence. Then, (x_n) is said to **converge to** x, and we write $(x_n) \to x$ (or $\lim_{n\to\infty} x_n = x$) if given any neighborhood U of x, there is an $N \in \mathbb{N}$ such that if $n \geq N$, $x_n \in U$. We say that x is *a* **limit** of x_n.

Note how similar this is to the definition of convergence in $\mathbb{R}$ (or $\mathbb{R}^n$). In fact, they are equivalent. We leave it to the reader to prove this.

Exercise 1.2.13. Show that in $\mathbb{R}$ with the usual topology, a sequence x_n converges to x in the usual epsilon-N sense if and only if it converges in the topological definition.

We must warn the reader that *limits may not be unique.* That is why we say "a" limit, not "the" limit.

Example 1.2.14. Let $X = \mathbb{R}$ and give it the trivial topology. Then any sequence in X converges to any point in X. //

We shall quickly summarize some useful results from this section.

Proposition 1.2.15 (Criterion for openness)**.** *Let $A \subseteq X$. Then, the following are equivalent:*

- *A is open.*
- *$A = A^\circ$.*
- *A has no boundary points.*
- *Every point of A has a neighborhood contained in A.*

Proof. Relatively easy. □

Proposition 1.2.16 (Criterion for closedness)**.** *Let $A \subseteq X$. Then, the following are equivalent:*

- *A is closed.*
- *$A = \overline{A}$.*
- *Every boundary point of A is in A.*
- *Every point of A^c has a neighborhood contained in A^c.*

Proof. Not hard. □

1.3 Hausdorff spaces

Definition 1.3.1 (Hausdorff space)**.** A topological space X is called Hausdorff if given $x, y \in X$ such that $x \neq y$, there exists disjoint open neighborhoods U of x and V of y such that $U \cap V = \varnothing$.

Example 1.3.2. $\mathbb{R}$ is a Hausdorff space. Any $\mathbb{R}^n$ is a Hausdorff space. Any metric space is Hausdorff. Given $x \neq y$ in a metric space, let $r = d(x, y)/2$. Then a ball of radius r around x and around y are disjoint. //

Example 1.3.3. Let X be a set with more than one point and give it the trivial topology. Then X is not Hausdorff. //

Lemma 1.3.4. *If X is a Hausdorff space and $x \in X$, then $\{ x \}$ is closed.*

Proof. We simply need to show that $X \setminus \{ x \}$ is open. Let $y \in X \setminus \{ x \}$. Then $y \neq x$, so there are disjoint neighborhoods U, V of y and x respectively. Then $U \subseteq X \setminus \{ x \}$. □

It immediately follows that finite point sets are closed.

One useful property of a Hausdorff space is the fact that if we have a sequence, it has a unique limit.

Proposition 1.3.5. *In a Hausdorff space, limits are unique.*

Proof. Suppose not. Let $(x_n) \to x$ and $(x_n) \to y$ where $x \neq y$. Let U, V be disjoint neighborhoods of x and y respectively. Then, there is N_1 such that $x_{N_1} \in U$ and N_2 such that $x_{N_2} \in V$. Let $N = \max\{ N_1, N_2 \}$. Then $x_N \in U \cap V$. But U, V are disjoint. Oops! □

Since $\mathbb{R}$ is a Hausdorff space this proves that limits are unique in $\mathbb{R}$.

1.4 Continuous functions

Definition 1.4.1 (Continuity)**.** Let X, Y be topological spaces and $f : X \to Y$. Then f is said to be **continuous** if for every open subset O of Y, $f^{-1}(O)$ is open in X.

Proposition 1.4.2. *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is continuous in the epsilon-delta sense if and only if f is continuous in the topological sense.*

Proposition 1.4.3 (Continuity with closed sets)**.** *Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if given a closed set $C \subseteq Y$, $f^{-1}(C)$ is closed in X.*

Chapter 2

Constructing spaces

2.1 Subspace topology

Definition 2.1.1 (Subspace topology)**.** Let X be a topological space with topology τ and Y *any* subset of X. Then the set

$$\{U \cap Y : U \in \tau\}$$

is a basis for a topology on Y, and the topology it generates is called the **subspace topology on** Y (from X). We then call Y a **subspace of** X.

It is important to note that this is very much unlike what subspaces are in linear algebra. We emphasize that *any subset* of a topological space whatsoever can be given the subspace topology.

Example 2.1.2. Let $X = \mathbb{R}$ with the usual topology and $Y = (0, 1)$, giving Y the subspace topology from X. What do the open sets of Y look like? Well, if $(a, b) \subseteq Y$, then (a, b) is open in Y as $(a, b) = (a, b) \cap Y$. //

We warn that

Theorem 2.1.3 (Open and closed subspaces)**.** Let X be a topological space and Y a subspace of X.

Theorem 2.1.4 (Properties of subspaces)**.** Let X be a topological space and $Y \subseteq X$ a subspace of X. The following statements are true.

1. If X is Hausdorff, then Y is too.

Proof. (1) is trivial. **TODO: finish theorem and proofs** □

Theorem 2.1.5 (Continuous functions and subspaces)**.** Let X, Y be topological spaces, and $f : X \to Y$ a continuous function. Let $A \subseteq X$ be a subspace of X. Then, the following are true.

1. The restriction of f to A, denoted $f\,|_A$, is continuous.
2. If $T \subseteq Y$ and $T \supseteq f[X]$, then $f : X \to T$ is continuous.
3. If Y is a subspace of Z, then $f : X \to Z$ is continuous.

2.2 Product topology

Definition 2.2.1 (Product topology)**.** Let X and Y be topological spaces with topologies τ_X and τ_Y respectively.

Then the set
$$\{U \times V : U \in \tau_X, V \in \tau_Y\}$$
is a basis for a topology on $X \times Y$, and the topology that it generates is called the *product topology on* $X \times Y$.

TODO: Put stuff

Theorem 2.2.2 (Properties of product topology)**.** Let X, Y be topological spaces.

2.3 Order topology

Definition 2.3.1 (Partial order)**.** Let X be a set. A **partial order on** X is a set $R \subseteq X \times X$ such that the following are true:

1. For all $x \in X$, xRx.
2. For all $x, y \in X$, if xRy and yRx then $x = y$
3. For all $x, y, z \in X$, if xRy and yRz then xRz.

Property 1 is called reflexivity. Property 2 is called antisymmetry. Property 3 is called transitivity. We can thus summarize a partial order as being a reflexive, transitive and antisymmetric relation.

We will immediately give some examples to help the reader better understand this.
Example 2.3.2. Let $X = \mathbb{N}$ and consider the relation $\subseteq$ defined on $\mathcal{P}(X)$ by aRb if and only if $a \subseteq b$. This is a partial order. //
Example 2.3.3. The real numbers $\mathbb{R}$ is equipped with a usual partial order, $\leq$. It is not too hard to verify that this is a partial order. //

It turns out that the partial order on $\mathbb{R}$ satisfies an additional property. For instance, given any pair of real numbers, we can tell which one is the bigger one amongst them.

Definition 2.3.4 (Total order)**.** Let X be a set. A **total order on** X is a partial order R on X such that for any $x, y \in X$, either xRy or yRx.

We will also sometimes call a total order a "simple order". A set X with a total order on it is said to be *totally ordered*, or *simply ordered*, or just an *ordered set*.
Example 2.3.5. Referring back to Example 2.3.2, we notice that the subset relation is a partial order, but definitely not a total order. For example, we cannot compare $\{1\}$ and $\{2\}$ with $\subseteq$, neither is a subset of the other. //

Definition 2.3.6 (Order topology)**.** Let $(X, \leq)$ be a totally ordered set. Then the **order topology** on X is generated by the basis consisting of

- (a, b) where $a, b \in X$, $a < b$
- $[a, b)$ if a is a minimal element of X.
- $(a, b]$ if b is a maximal element of X.

Example 2.3.7. The order topology on $\mathbb{R}$ is equal to the usual topology on $\mathbb{R}$. We leave the reader to check this. //

Definition 2.3.8 (Convex (sub)set)**.** Let X be a totally ordered set. Then $S \subseteq X$ is said to be **convex** if given any $x, y \in S$, $[x, y] \subseteq X$.

Definition 2.3.9 (Linear continuum)**.** Let L be a totally ordered set. Then, L is called a **linear continuum** if L has the least upper bound property, and given $x, y \in L$, there exists z such that $x < z < y$.

We will use this definition to prove that intervals and rays are connected in $\mathbb{R}$ later on.

TODO: Put more stuff

2.4 Initial and final topologies

TODO: Put stuff

Chapter 3

Connectedness and Compactness

3.1 Connected spaces

Definition 3.1.1 (Disconnected). Let X be a topological space. Then X is said to be **disconnected** if there exists sets U, V that are open in X, disjoint and nonempty and $X = U \cup V$. The sets U, V are said to **disconnect** X.

A space X is said to be **connected** if it is not disconnected. So the definition of connectedness is literally just the negation of disconnectedness. A good tip is that if you need to prove something is connected, using contradiction or contrapositive will work rather well: Simply assume it is disconnected.

We provide some alternative classifications of connected spaces.

Proposition 3.1.2. *X is connected if and only if the only sets that are both open and closed in X are $\varnothing$ and X.*

Proof. If X is disconnected let U, V be open, disjoint and nonempty, such that $U \cup V = X$. Then U^c is closed and it is V, so V is closed. Thus V is a set that is both closed and open and it is not the empty set or all of X. □

Definition 3.1.3 (Separation). A **separation/disconnection of** X is a pair of disjoint nonempty sets $A, B \subseteq X$ such that $A \cup B = X$, and $\overline{A} \cap B = \varnothing$, $A \cap \overline{B} = \varnothing$.

Proposition 3.1.4. *X is disconnected if and only if there is a separation of X.*

Proof. Exercise. □

The next theorem is a generalization of the fact that any interval in $\mathbb{R}$ is connected. It turns out that when proving that any interval or ray is connected in $\mathbb{R}$, we only needed the fact that $\mathbb{R}$ has least upper bounds, and between any 2 distinct elements we can find another element between them.

Theorem 3.1.5. Let X be a linear continuum with the order topology. Then, any $C \subseteq X$ that is convex is connected.

Proof. Let $C \subseteq X$ be convex. Suppose that C is not connected. Let $C = A \cup B$, where A, B are open in C, nonempty and disjoint sets. □

We state a sort-of converse to the theorem above, but for $\mathbb{R}$.

Proposition 3.1.6. *If $Y \subseteq \mathbb{R}$ is connected, then Y is a singleton, interval or ray.*

We shall proceed with some examples of connected spaces.

Example 3.1.7. $\mathbb{R}$ with the usual topology is connected. In fact, any interval in $\mathbb{R}$ is connected. //

Example 3.1.8. Let X be a set with at least 2 points and give it the trivial topology. Then X is connected. //

Example 3.1.9. Let X be a singleton. Then X is connected, no matter what the topology on X is. //

Example 3.1.10. $(0,1) \cup (2,3)$, considered as a subspace of $\mathbb{R}$ is clearly disconnected. //

And now some examples of disconnected spaces.

Example 3.1.11. $\mathbb{Q}$ is disconnected by $\mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q}(\sqrt{2}, \infty)$. You can play the same trick with any subset of $\mathbb{Q}$, except singletons. //

Example 3.1.12. The lower limit topology $\mathbb{R}_l$ is not connected. To see this, note that

$$\mathbb{R} = (-\infty, 0) \cup [0, \infty).$$

We know that $(-\infty, 0)$ is open, it is the union of open intervals, which is open in $\mathbb{R}_l$. Additionally, $[0, \infty]$ is the union of half open intervals so it is also open. //

3.2 Properties of connectedness

If X is disconnected by C, D but Y is connected in X, then it makes sense that Y is either in C or in D. The next lemma makes this precise.

Lemma 3.2.1. *Let X be separated by C, D. If $Y \subseteq X$ is a connected subspce, then $Y \subseteq C$ or $Y \subseteq D$ (but not both).*

Proof. If not then $Y \cap C, Y \cap D$ disconnects Y. □

Note that in the previous proof, we are using the fact that f restricted to Y is still continuous.

The next theorem can arguably be called the fundamental theorem of connectedness. It tells us that the image of a connected space is connected.

Theorem 3.2.2 (Connectedness is a topological invariant). Let $f : X \to Y$ be a continuous function and X be a connected space. Then, $f[X]$ is connected.

Proof. Suppose that $f[X]$ is disconnected by C, D. Then $f^{-1}(C)$ and $f^{-1}(D)$ disconnect X. □

Corollary 3.2.3. *If X is homeomorphic to Y and X is connected, so is Y.*

Now we get the intermediate value theorem for free. Really, this should be a corollary.

Theorem 3.2.4 (Intermediate value theorem). Let X be a connected space and Y be a linear continuum. Let $f : X \to Y$ be a continuous function. For any $a, b \in X$, and $c \in Y$ such that $f(a) < c < f(b)$, there is $x \in X$ such that $f(x) = c$.

Proof. Follows from Theorem 3.2.2 and Theorem 3.2.2. □

For the usual intermediate value theorem of calculus, put X to be an interval and Y to be $\mathbb{R}$.

We then provide some ways to construct new connected spaces.

Theorem 3.2.5 (Properties of connectedness). Let X be a topological space.

1. If $\{ Y_\alpha \}_{\alpha \in \Lambda}$ is a family of connected subspaces of X and $y \in Y_\alpha$ for all α, then $\bigcup_{\alpha \in \Lambda} Y_\alpha$ is connected.
2. If $A \subseteq X$ is connected, then $\overline{A}$ is connected. Additionally, if B is such that $A \subseteq B \subseteq \overline{A}$, then B is connected.
3. If X, Y are connected spaces, then $X \times Y$ is connected.

Proof. For (1), suppose the union is disconnected by C, D. Let $\alpha \in \Lambda$ be whatever. Then by Lemma 3.2.1, and without loss of generality, $Y_\alpha \subseteq C$. Since D is nonempty there must be some β such that $Y_\beta \subseteq D$. But this means $a \in C$ and $a \in D$. Oops!

For (2), suppose $\overline{A}$ is disconnected by C, D. Since A is connected, without loss of generality suppose $A \subseteq C$. Then as D is nonempty, let $y \in D$. Write $D = U \cap \overline{A}$ where U is open in X. But U is an open neighborhood of y that does not intersect A, a contradiction. The additional remark is left to the reader.

For (3), let $(a, b) \in X \times Y$. Notice that $X \times \{ b \}$ is connected as it is homeomorphic to X. Now, let $x \in X$, then we see that $\{ x \} \times Y$ is connected too as it is homeomorphic to Y. Define

$$T_x = (X \times \{ b \}) \cup (\{ x \} \times Y).$$

Then T_x is connected as the point (a, b) is in $(X \times \{ b \})$ and $(\{ x \} \times Y)$ (use part 1). Then $X = \bigcup_{x \in X} T_x$. This is connected as it is the union of a collection of connected subspaces with a point in common. For if $x, x' \in X$, then the point $(a, b) \in T_x \cap T_{x'}$ by definition. ☐

Since $(X_1 \times \cdots \times X_n) \times X_{n+1}$ is homeomorphic to $X_1 \times \cdots \times X_{n+1}$, finite products of connected spaces are connected.

It is important to note that connectedness may not extend to arbitrary products.
Example 3.2.6. $\mathbb{R}^\omega$ with the box topology is not connected. **TODO: Include proof** //
Example 3.2.7. $\mathbb{R}^\omega$ with the product topology is connected. **TODO: Include proof** //

3.3 Path connectedness

Path connectedness is a more intuitive notion of connectedness. It essentially says, given any 2 points in a topological space, if we can draw a line between them, and the line stays in the topological space, then it is connected.

Definition 3.3.1 (Path). Let X be a topological space and $x, y \in X$. A **path** from x to y is a continuous function $p : [0, 1] \to X$ such that $p(0) = x$ and $p(1) = y$.

Definition 3.3.2 (Path-connectedness). Let X be a topological space. Then X is **path-connected** if given any $x, y \in X$, there is a path p from x to y such that $p[[0, 1]] \subseteq X$.

Path connectedness is a sufficient condition for connectedness.

Theorem 3.3.3 (Path-connectedness implies connectedness). Let X be a topological space. If X is path connected, then X is connected.

Proof. Suppose not. Let C, D disconnect X. Let $c \in C$ and $d \in D$, since both are nonempty. Since X is path connected let $p : [0, 1] \to X$ be a path from c to d. $[0, 1]$ is connected so $p[[0, 1]]$ is also connected. However,

$$p[[0, 1]] = (C \cap p[[0, 1]]) \cup (D \cap p[[0, 1]]).$$

This is a disconnection of $p[[0, 1]]$, contradicting the fact that $p[[0, 1]]$ is connected. ☐

Example 3.3.4. Now that we have path connectedness, it is easy to prove that $\mathbb{R}^n$ is connected. Pick any $x, y \in \mathbb{R}^n$ and define $p(t) = (1 - t)x + ty$. Observe that p is a straight line path from x to y and the straight line lies in $\mathbb{R}^n$. It is also continuous. //

The converse of this theorem is untrue. The most famous example of this is called the topologist's sine curve.
Example 3.3.5 (The topologist's sine curve). Let $T_0 = \{ (x, \sin 1/x) : x \in (0, 1] \}$ and let $T_1 = \{ (0, y) : y \in [-1, 1] \}$. The topologist's sine curve is defined to be $T = T_0 \cup T_1$. It is not hard to see that T is connected. T_0 is the image of $(0, 1]$ under a continuous function, so it is connected. Notice that T_1 is the set of limit points of T_0. By part (2) of Theorem 3.2.5, T is connected.

We shall now show that T is not path connected. **TODO: finish** //

Theorem 3.3.6 (Properties of path-connected spaces). Let X be a topological space. **TODO: finish**

3.4 Compactness

Compactness is arguably the most important concept in all of topology. Compactness captures the idea of what it means for a set to be "finite"-ish.

Definition 3.4.1 (Open cover)**.** Let X be a topological space. An **open cover of** X is a collection $\mathcal{U}$ of open subsets of X such that $\bigcup \mathcal{U} = X$.

We can get the definition for a cover of X by removing all mentions of the word "open" from the above definition.

To help with digesting this definition we give some examples of open covers.
Example 3.4.2. Let $X = \mathbb{R}$. Consider the collection $\mathcal{U} = \{ (n, n+1) : n \in \mathbb{Z} \}$ of open intervals with integer endpoints. This is an open cover of X. **TODO: Draw a picture** //

Definition 3.4.3 (Finite subcover)**.** If $\mathcal{U}$ is an open cover of X, then a **finite subcover** is a collection of sets $U_1, \ldots, U_n \in \mathcal{U}$ such that $\bigcup_{i=1}^{n} U_i = X$.

Definition 3.4.4 (Compactness)**.** Let X be a topological space. Then X is said to be **compact** if given *any* open cover of X, there is a finite subcover.

We emphasize here that for a set to be compact, you must be able to extract a finite subcover from *any* open cover whatsoever.
Example 3.4.5. Let X be any finite set and give X any topology. Then X is compact. //
Example 3.4.6. The set $\mathbb{R}$ with the usual topology is definitely not compact. The cover $\mathcal{U} = \{ (n, n+1) : n \in \mathbb{Z} \}$ has no finite subcover. //

If we have a subspace, then the following proposition provides a more convenient way to characterize whether a subspace is compact in the subspace topology.
Proposition 3.4.7 (Compactness in subspace)**.** *Let X be a topological space and Y be a subspace of X. Then Y is compact in the subspace topology if and only if every cover of Y by open sets of X has a finite subcover.*

Proof. We shall not insult the reader's intelligence by providing a proof of this. □

Example 3.4.8. Take $\mathbb{R}$ with the usual topology and consider the open interval $(0, 1)$. We will show this is not compact. Consider the open cover $\mathcal{U} = \{ (1/n, 1) : n \in \mathbb{N} \}$. Indeed, $\bigcup \mathcal{U} = (0, 1)$, but any finite subcover will be missing points of the form $1/k$. //