Week 6

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1 Final topologies

The final topology is the dual[1] notion of *initial topology*. With the initial topology, we have a family of maps with a common domain X, and we want to topologize X in a way that makes all the maps continuous. With the final topology, we have a family of maps with a common *codomain* Y instead, and we would like to topologize Y in a way that makes all the maps continuous.

Definition 1.1. Let Y be a set and let $\{ X_\alpha : \alpha \in \Lambda \}$ be a collection of topological spaces. Let

$$\mathcal{F} = \{ f_\alpha : X_\alpha \to Y : \alpha \in \Lambda \}$$

be a family of functions. Then the *final topology of* $\mathcal{F}$ is defined to be

$$\{ U \subseteq Y : f_\alpha^{-1}(U) \text{ is open in } X_\alpha \text{ for all } \alpha \in \Lambda \}.$$

In a sense, we are interested in providing Y with a topology that makes all the f_α's continuous. Notice here that Y is the codomain of our f_α's.

For reference, here is the definition of initial topology.

Definition 1.2. Let X be a set, and let $\{ Y_\alpha : \alpha \in \Lambda \}$ be a collection of topological spaces. Let

$$\mathcal{F} = \{ f_\alpha : X \to Y_\alpha : \alpha \in \Lambda \}$$

be a family of functions. Then the *initial topology of* $\mathcal{F}$ is defined to be

$$\bigcap \{ \tau : \tau \text{ is a topology on } X \text{ and every element of } \mathcal{F} \text{ is } \tau - \text{continuous} \}.$$

Proposition 1.3. *The final topology of $\mathcal{F}$ is the finest topology on Y where all the elements of $\mathcal{F}$ are continuous.*

Proof. Denote the final topology with $\tau_{\mathcal{F}}$. Suppose τ is a topology that makes all the f_α's continuous. Then $\tau \subseteq \tau_{\mathcal{F}}$. To see this, let $U \in \tau$. Then for every α, we have $f_\alpha^{-1}(U)$ being open in X_α, as f_α is τ continuous. This means $U \in \tau_{\mathcal{F}}$. $\square$

We can now see an application of final topologies.

[1] In this case, the duality is actually the categorical duality!

2 Quotient topology

Definition 2.1. Let X be a topological space and Y be a set. Let $q : X \to Y$ be a surjective function. Then the final topology of $\{ q \}$ is called the *quotient topology induced by* q.

If Y is a topological space, then Y is a *quotient of* X if the topology on Y is the quotient topology induced by some surjective function $q : X \to Y$.

Again, keep in mind here that Y is being topologized by the final topology induced by q. One (relatively immediate) observation is that a set $O \subseteq Y$ is open in the quotient topology on Y if and only if $q^{-1}(O)$ is open in X. In fact, this is an alternative way to define the quotient map.

We often use the quotient topology to put a topology on the set of equivalence classes. Let us recall the definition of a equivalence relation.

2.1 Equivalence relations

Definition 2.2 (Equivalence relation)**.** Let X be a set. Then an *equivalence relation* $\sim$ *on* X is a relation such that

1. (**Reflexive**) $x \sim x$,
2. (**Symmetric**) if $x \sim y$ then $y \sim x$,
3. (**Transitive**) if $x \sim y$ and $y \sim z$ then $x \sim z$.

The intuition here is that equivalence relations try to capture the notion of equality. In fact, $=$ is an equivalence relation. More examples of equivalence relations are $n \sim m$ iff $n \mod k = m \mod k$ (here, $n, m \in \mathbb{Z}$ and $k \in \mathbb{N}, k > 0$).

Given an equivalence relation on X, we can *partition*[2] the set X into *equivalence classes*. We define

$$[x]_\sim = \{ y \in X : y \sim x \} .$$

Notice that we now have the following properties:
Lemma 2.3. *Let X be a set and $\sim$ be an equivalence relation on X. Then,*

1. $X = \bigcup_{x \in X} [x]_\sim$,
2. *Equivalence classes are equal or disjoint: If $[x]_\sim \neq [y]_\sim$, then $[x]_\sim \cap [y]_\sim = \varnothing$.*

Proof. The first is obvious. For the second, we prove the contrapositive. Suppose $z \in [x]_\sim \cap [y]_\sim$. Then $z \sim x$ and $z \sim y$ by definition. By transitivity we have $x \sim y$, and by transitivity again, every element related to y is also related to x. □

Given an equivalence relation $\sim$ on X, we denote the set of equivalence classes,

$$X_\sim = X/\sim = \{ [x]_\sim : x \in X \} .$$

There is a canonical surjective function[3] from X to $X/\sim$ which sends an element $x \in X$ to its equivalence class $[x]_\sim$.

$$q(x) = [x]_\sim$$

2.2 Examples of quotient spaces

We can now see some examples of quotient spaces. The reader is encouraged to check out [Lee11, pp. 62–68] for many more examples of quotient spaces.
Example 2.4 (The sphere S^2 as a quotient space)**.** Let $D \subseteq \mathbb{R}^2$ be the unit disk, i.e. $D = \{ \langle x, y \rangle : x^2 + y^2 \leq 1 \}$.

Define $\sim$ on D by

$$\langle x, y \rangle \sim \langle z, w \rangle \text{ iff } \langle x, y \rangle = \langle z, w \rangle \text{ or } x^2 + y^2 = z^2 + w^2 = 1.$$

[2] Note that this has a rigorous definition
[3] This is actually called the natural projection

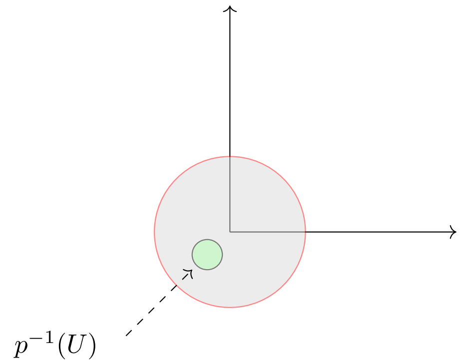

Figure 1: Unit disk $D \subseteq \mathbb{R}^2$

Intuititively, every point in the interior of D (the interior is shaded in gray) stays distinct, and every point on the boundary (colored in blue) is the "same" under $\sim$. Now, the set of equivalence classes of D, $D/\sim$ can be visualized as in Figure 2.

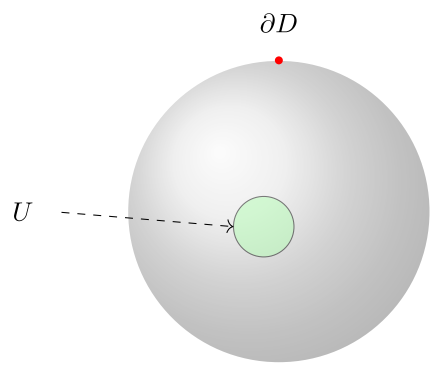

Figure 2: The sphere constructed from the unit disk

Example 2.5 (Torus as a quotient space)**.** See [Lee11, Example 3.49 on p. 66]. //

One might wonder whether quotient spaces always come from some kind of equivalence relation. The answer is yes.

Theorem 2.6. Every quotient of X is homxeomorphic to some $X/\sim$.

Proof. We would like to show that if Y is such that there exists some surjective function $q : X \to Y$ where the topology of Y is the quotient topology induced by q then there exists an equivalence relation on X such that $X/\sim$ is homeomorphic to Y. We first show the existence of such an equivalence relation. Let $\sim$ in X be defined as follows: $x \sim y$ if and only if $q(x) = q(y)$. This is easily seen to be an equivalence relation.

Now we begin constructing the homeomorphism. Let $f : X/\sim \to Y$ be defined by $f([x]_\sim) = q(x)$. Then f is a well-defined function, if we have $[x]_\sim = [x']_\sim$, then $f(x') = q(x') = q(x) = f(x)$ by definition of $\sim$. We also check that f is a bijection by finding it's inverse, $f^{-1} : Y \to X/\sim$. We'll just write it down:

$$f^{-1}(y) = \{\, x \in X : q(x) = y \,\}.$$

This is indeed an inverse. So f is a bijection. All that is left is to show that f and f^{-1} are continuous. Let $U \subseteq Y$ be open. Then

$$f^{-1}(U) = \{\, [x]_\sim : f([x]_\sim) \in U \,\} = \{\, [x]_\sim : q(x) \in U \,\}.$$

Let $p_\sim : X \to X/\sim$ be the canonical projection that sends x to $[x]_\sim$. Consider $p_\sim^{-1}(\{\, [x]_\sim : q(x) \in U \,\}) = \{\, x \in X : q(x) \in U \,\} = q^{-1}(U)$. $q^{-1}(U)$ is open in X because q is continuous, but by definition of quotient topology

$$\begin{tikzcd} X \arrow[d, two heads, "p_\sim"'] \arrow[dr, two heads, "q"] & \\ X/\sim \arrow[r, tail, two heads, "f"'] & Y \end{tikzcd}$$

Figure 3: Commutative diagram expressing the proof of Theorem 2.6

this means $\{ [x]_\sim : q(x) \in U \}$ is open. Thus we have shown that f is continuous. We leave the proof of the continuity of f^{-1} to the reader. (Just show that f is open) $\square$

2.3 Properties of quotient spaces

Unfortunately, quotient spaces are quite badly behaved. The first part where they don't play so nice is with the subspace topology. In other words, taking a quotient of a subspace is not the same as taking a subspace of a quotient space. Let $q : X \to Y$ be a surjective map. This induces the quotient topology in Y. Let $A \subseteq X$ and give A the subspace topology. Consider $q\mid_A : A \to q[A]$. There are 2 ways to think about the topology on $q[A]$: as a subspace of Y or as a quotient space of A. It turns out that these may not be equal.

Example 2.7. Let $X = [0,1] \cup [2,3] \subseteq \mathbb{R}$ with the subspace topology from $\mathbb{R}$. Let $Y = [0,2]$ with the subspace topology from $\mathbb{R}$. Let q be defined by $q(x) = x$ if $x \in [0,1]$ and $q(x) = x - 1$ if $x \in [2,3]$. Give Y the quotient topology induced by q. Then the quotient topology on Y is the same as the usual Euclidean topology on Y.

Now let $A = [0,1) \cup [2,3]$ (notice we are taking the half open interval!) and take $q\mid_A : A \to [0,2]$. Consider $q\mid_A^{-1}([1,3/2))$. The set $[1,3/2)$ is not open, but it has an open preimage. This prevents $q\mid_A$ from being a quotient map. //

Definition 2.8. Let $\sim$ be an equivalence relation on X. A subset $A \subseteq X$ is $\sim$-*saturated* if and only if

$$A = \bigcup_{x \in A} [x]_\sim .$$

Proposition 2.9. *If $A \subseteq X$ is $\sim$-saturated, then $A_\sim \subseteq X_\sim$.*

Proof. If $\sim$ is an equivalence relation on X and $A \subseteq X$, then $\sim$ induces an equivalence relation on A, call it $\sim_A$. This is simply the restriction of $\sim$ to A, i.e. $a \sim_A b \iff a \sim b$. Then $[a]_{\sim_A} = [a]_\sim$. Let $A \subseteq X$ be a subspace and let $p_\sim : A \to A_\sim$, which is really just $p_\sim : X \to X_\sim$. $\square$

Theorem 2.10. If A is open (closed) or $p_\sim$ is an open (closed) map, then the subspace topology on $A_\sim$ as a subset of $A_\sim$ is the same as the quotient topology on $A_\sim$ induced by $p_\sim$.

We encourage the reader to check out [Lee11, Proposition 3.60 on p. 69].

Proof. Let $A_\sim \cap V$ be an open subset of $A_\sim$ as a subspace of $X_\sim$. We need to prove $A_\sim \cap V$ is open in $X/\sim$, which amounts to showing that $p_\sim^{-1}(A_\sim \cap V)$ is open. Now, since $A_\sim$ is saturated, we have

$$p_\sim^{-1}(A_\sim \cap V) = p_\sim^{-1}(A) \cap p_\sim^{-1}(V) = A \cap p_\sim^{-1}(V).$$

Since $A \cap p_\sim^{-1}(V)$ is open in the subspace topology in A, this means that $A_\sim \cap V$ is open in the quotient $A_\sim$. Let $U \subseteq A_\sim$ be open in the quotient topology induced by $p_\sim\mid_A : A \to A_\sim$. We claim that if A is open and saturated, then $A \subseteq X_\sim$ is also open (proof: exercise). So U is open in the quotient if and only if $p_\sim\mid_A^{-1}(U)$ is open in A. But notice that $p_\sim\mid_A^{-1}(U) = \{ x \in X : [x]_\sim \in U \}$. This is open in A if and only if it is equal to $A \cap V$, where V is some open subset of X. Then, we leave the reader to check that

$$U = p_\sim\left(p_\sim\mid_A^{-1}(U)\right) = p_\sim(\{ x \in A : [x]_\sim \in U \}) = p_\sim(A \cap V) = A_\sim \cap p_\sim(V).$$

$\square$

References

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. en. Vol. 202. Graduate Texts in Mathematics. New York, NY: Springer New York, 2011. ISBN: 9781441979391. DOI: 10.1007/978-1-4419-7940-7. URL: https://link.springer.com/10.1007/978-1-4419-7940-7.