# Week 8 Notes

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July 2024

## 1 Connected components

**Definition 1.1** (Connected components)**.** Let $X$ be a topological space. Define the equivalence relation $\sim_C$ on $X$ by $x \sim_C y$ if there exists a connected subspace $A \subseteq X$ such that $x, y \in A$.

An equivalence class under $\sim_C$ is called a **component**.

The set of equivalence classes under $\sim_C$ is called the *components* of $X$. The definition for path components is similar. Note that the empty topological space has no components or path components.

**Definition 1.2** (Path components)**.** Let $X$ be a topological space. Define the equivalence relation $\sim_P$ on $X$ by $x \sim_P y$ if there exists a path from $x$ to $y$.

An equivalence class under $\sim_P$ is called a **path component**.

**Proposition 1.3** (Properties of connected components)**.** *Let $X$ be a topological space. Then, the following are true.*

1. *Components form a partition of $X$*
2. *Components are connected*
3. *Connected[1] subspaces of $X$ intersect at most one component.*
4. *Components are closed.*

*Proof.* (1) follows immediately since $\sim_C$ is an equivalence relation.

(2) Suppose $[x]_{\sim_C} = C \cup D$ where $C, D$ are disjoint open sets. Suppose that $x \in C$. Let $y \in [x]_{\sim_C}$ be arbitrary. Since $x \sim_C y$, there is a connected subspace $A$ such that $x, y \in A$. Since $C$ and $D$ are disjoint, $A \subseteq C$, so $y \in C$. Since $y$ is arbitrary, every point of $[x]_{\sim_C}$ is contained in $C$, so $D$ is empty. So there is no separation of $[x]_{\sim_C}$.

(3) Let $A$ be a nonempty connected subspace of $X$ and let $x \in A$. Then, $A$ intersects $[x]_{\sim_C}$. If $y \in A$ then by definition of $\sim_C$ we have $x \sim_C y$, so $y \in [x]_{\sim_C}$. Thus every point in $A$ is contained in $[x]_{\sim_C}$.

(4) Given a component $E = [x]_{\sim_C}$, then the closure $\overline{E}$ is also connected, and $\overline{E}$ intersects at most one component by (3). Since it intersects $E$, it is contained in $E$ and thus equal to $E$. ☐

We remark that part (3) is actually stronger than stated. If $A$ is a connected subspace of $X$ that intersects a component $C$ then $A$ is completely contained in $C$.

Note that (4) does not hold for path components as the closure of a path-connected space is not necessarily connected (topologist's sine curve, see last week's notes).

**Proposition 1.4** (Properties of path components)**.** *Let $X$ be a topological space. Then, the following are true.*

1. *Path form a partition of $X$*
2. *Path components are path conencted*
3. *Path connected subspaces of $X$ intersect at most one component.*

[1]These have to be nonempty

4. *Each path component is contained in a component.*

*Proof.* (1) Follows from definition of $\sim_P$.

(2) Given $y, z \in [x]_{\sim_P}$, there is a path from $y$ to $z$ by definition of $\sim_P$ so it is path-connected.

(3) This is a similar argument to Proposition 1.3 part (3). If $A$ is a path-connected subspace of $X$ and $x \in A$ then $A$ intersects $[x]_{\sim_P}$. If $y$ is any point in $A$ then there is a path from $x$ to $y$ since $A$ is path-connnected, so it follows that $y \sim_P x$.

(4) It is clear that if $x \sim_P y$ then $x \sim_C y$, since if $p$ is a path from $x$ to $y$, then $x, y$ are elements of the image of the path, which is connected. (Recall that the domain of a path is a closed interval which is connected.) The result follows by thinking about the definition of an equivalence class. $\square$

A few remarks are in order. Components are maximal connected subsets of $X$. This means that if $C$ is a component and $A$ is a connected subspace of $X$ that contains $C$, then $A = C$. This is actually equivalent to Definition 1.1. Note that if $X$ is connected then $X$ only has a single component, namely $X$. A similar proposition can be formulated for path components.

## 2 Local connectedness

**Definition 2.1** (Local basis). Let $X$ be a topological space. A **local basis** around a point $x \in X$ is a collection $\mathcal{B}$ of *open* neighborhoods of $x$ such that given any open neighborhood $U$ of $x$, there is a $B \in \mathcal{B}$ such that $B \subseteq U$.

It is important to note that the neighborhoods in the local basis must be open.
**Example 2.2** (Local basis of a point in $\mathbb{R}$). Let $x \in \mathbb{R}$ be any point and consider the collection of $\varepsilon$ balls around $x$,

$$\mathcal{B} = \{ (x - \varepsilon, x + \varepsilon) : \varepsilon \in \mathbb{R}, \varepsilon > 0 \}.$$

Then this forms a local basis of $x$. //

**Definition 2.3** (Local connectedness). A topological space is **locally connected** if given any point $x \in X$ there is a local basis at $x$ of connected sets.

Note that this is not the same as saying that given a point $x$ it has a connected neighborhood.
**Proposition 2.4** (Equivalent definition of local connectedness). *A space $X$ is locally (path)-connected if and only if given any open subset $U \subseteq X$, every (path) component of $U$ is open in $X$.*

*Proof.* Let us prove it first for connectedness. First suppose $X$ is locally connected. Let $U$ be an open subset of $X$. Let $x \in U$ and let $C$ be the[2] component of $U$ that contains $x$. Let $c \in C$ be a point, we shall prove that there is an open neighborhood[3] of $c$ that is completely contained in $C$. Since $X$ is locally connected, there is a local basis around $x$ of connected sets, call it $\mathcal{B}$. Then there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$ by definition. As $B$ is connected, $B \subseteq C$ by Proposition 1.3 (3). This completes the proof of the forward direction.

Now suppose that given any open set $U$ of $X$ every component of $U$ is open in $X$. Let $x \in X$. For each open neighborhood $U$ of $x$ let $C_U$ be the component of $U$ that contains $x$. Keep in mind that $C_U$ is open in $X$ and $x \in C_U \subseteq U$. Then let $\mathcal{B}$ be the set of all such $C_U$'s, one for every open neighborhood $U$ of $X$. To see this, note that given an open neighborhood $V$ of $x$, $C_V$ is open in $X$ and $x \in C_V \subseteq V$. But this is exactly the definition of local connectedness. $\square$

The proof for path components is similar.

A nice property of a space $X$ being locally path connected is that there is no difference between looking at the components of $X$ and the path components of $X$. They are the same set.
**Corollary 2.5** (Components equal path components if locally connected). *If $X$ is locally path connected, then every component is a path component and conversely. Therefore, the set of components and the set of path components of $X$ are equal.*

[2] A point can be contained in precisely one component since equivalence classes are equal or disjoint
[3] This neighborhood is open in $X$

*Proof.* Let $C$ be a component and let $x \in C$. Then $C = [x]_{\sim_P} \cup Q$, where $Q = \bigcup_{y \in C, y \neq x} [y]_{\sim_P}$. So $Q$ is the union of all the other path components of $C$. By Proposition 2.4, as each $[y]_{\sim_P}$ is open, $Q$ is open. If $Q$ is not empty, then $[x]_{\sim_P}$ and $Q$ separate $C$, which contradicts $Q$ being connected. $\square$

We remark that a space being connected does not mean it is locally connected. See [Mak23].

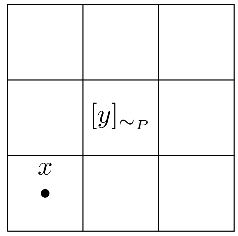


Figure 1: Proof of Corollary 2.5

## 3 Filters and Ultrafilters

**Definition 3.1.** A collection $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a **filter** (on $\mathbb{N}$)[a] if

1. $\mathbb{N} \subseteq \mathcal{F}$, and $\varnothing \notin \mathcal{F}$. A filter does *not* contain the empty set.
2. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ (closed under *finite* intersections).
3. If $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$ (closed upwards).

An **ultrafilter** is a filter which if $A \subseteq \mathbb{N}$, then either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A$ in $\mathcal{F}$.

[a] Note that filters can be defined on any nonempty collection of sets whatsoever. See [Jec03, Def 7.1, p. 73].

So an ultrafilter is a maximal filter, that is, there is no filter which properly contains it (besides $\mathcal{P}(\mathbb{N})$). Note that a filter cannot contain both $A$ and $\mathbb{N} \setminus A$, for if it does, then it contains their intersection, which is the empty set, violating condition (1).

**Example 3.2** (Trivial filter)**.** The set $\{\mathbb{N}\}$ forms a filter. //

**Example 3.3.** Let $A \subseteq \mathbb{N}$. Define $\mathcal{F}_A = \{B \subseteq \mathbb{N} : A \subseteq B\}$. //

**Example 3.4** (Frechet filter)**.** Define

$$\mathrm{Fr} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite }\}.$$

This is called the *Frechet filter*. It contains all subsets of the naturals with finite complement. Note that this is not an ultrafilter, since the set of even numbers is not cofinite, and their complement is also not cofinite. We can use the Frechet filter to reword convergence of a sequence in the language of filters instead (Proposition 3.12). //

**Example 3.5** (The power set is not a filter)**.** The power set cannot be a filter, as it contains the empty set. //

**Example 3.6** (Principal ultrafilters.)**.** Let $n \in \mathbb{N}$ be fixed. Let

$$\mathcal{U}_n = \{A \subseteq \mathbb{N} : n \in A\}.$$

This is called a **principal ultrafilter**. So $\mathcal{U}_n$ is the set of all subsets of the naturals that contain $n$. This is a filter, and it is an ultrafilter, since if $S$ does not contain $n$ then its complement contains $n$. //

**Remark 3.7.** It is not possible to prove the existence of an ultrafilter which is not principal without the axiom of choice. With the axiom of choice, we can construct an ultrafilter that contains a filter.

**Theorem 3.8** (Ultrafilter Theorem (Tarski, 1930))**.** Every filter is contained in an ultrafilter.

*Proof.* Let $\mathcal{F}$ be a filter. Consider the set

$$\mathbb{P}_{\mathcal{F}} = \{G \subseteq \mathcal{P}(\mathbb{N}) : G \text{ is a filter and } \mathcal{F} \subseteq G\}.$$

Order $\mathbb{P}_{\mathcal{F}}$ by subset inclusion. We shall use Zorn's lemma to extract a maximal element (and we shall prove that is an ultrafilter). First of all $\mathbb{P}_{\mathcal{F}}$ is nonempty, as $F \in \mathbb{P}_{\mathcal{F}}$. Now, let $\mathcal{C} \subseteq \mathbb{P}_{\mathcal{F}}$ be a chain. We shall prove that $\mathcal{C}$ is bounded. Let $\mathcal{H} = \bigcup_{G \in \mathcal{C}} G$, the union of all the filters in $\mathcal{C}$.

We claim that $\mathcal{H}$ is a filter, and thus it is a bound for $\mathcal{C}$, since every filter in $\mathcal{C}$ will be contained in $\mathcal{H}$. Clearly, $\mathbb{N} \in \mathcal{H}$ as $\mathbb{N} \in \mathcal{F} \subseteq \mathcal{H}$. If $\varnothing \in \mathcal{H}$ then $\varnothing \in G$ where $G \in \mathcal{C}$, but this is not possible as $G$ is a filter.

Now, let $A, B \in \mathcal{H}$. Then, there are filters $G_A, G_B \in \mathcal{C}$ such that $A \in G_A$ and $B \in G_B$. Since $\mathcal{C}$ is a chain, either $G_A \subseteq G_B$ or $G_B \subseteq G_A$. Without loss of generality say $G_A \subseteq G_B$. Then, both $A, B$ are in $G_B$, so $A \cap B \in G_B \subseteq \mathcal{H}$.

Let $A \in \mathcal{H}$ and suppose $B \supseteq A$. By definition of $\mathcal{H}$ there is a filter $G_A \in \mathcal{C}$ such that $A \in G_A$, then $B \in G_A$ so $B \in \mathcal{H}$.

This shows that an arbitrary chain $\mathcal{C}$ is bounded. Thus, by Zorn's lemma, there is a maximal (for $\subseteq$) $U \in \mathbb{P}_{\mathcal{F}}$. We shall show that $U$ is an ultrafilter. Suppose not. Then, there is $A \subseteq \mathbb{N}$ such that $A \notin U$ and $\mathbb{N} \setminus A \notin U$. We claim that there is a filter containing $U \cup \{ A \}$. First, observe that this set has the finite intersection property: given $X \in U$, $X \cap A$ cannot be empty, for else $X \subseteq \mathbb{N} \setminus A$ which would imply that $\mathbb{N} \setminus A \in U$. Now for the existence of the filter, let $\mathcal{G}$ be the filter gnerated by $\mathcal{F} \cup \{ A \}$. Then $\mathcal{G}$ is a filter (Lemma 3.9) that contains $U$ properly, which contradicts the maximality of $U$. $\square$

A collection of subsets of $\mathbb{N}$, $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is said to have the *finite intersection property* if given any finite collection of elements of $\mathcal{A}$, $\{ A_i \}_1^n \subseteq \mathcal{A}$, the intersection $\bigcap_{i=1}^n A_i$ is nonempty. If the intersection is infinite, $\mathcal{A}$ is said to have the *strong finite intersection property.* Every filter has the finite intersection property.

**Lemma 3.9** (Existence of filter containing a FIP collection)**.** *If $\mathcal{A}$ has the finite intersection property, then the set*

$$\mathcal{F}_{\mathcal{A}} = \left\{ A \subseteq \mathbb{N} : \exists A_1, \ldots, A_n \in \mathcal{A}, \ \bigcap_1^n A_i \subseteq A \right\}$$

*is a filter. This filter is called the* filter generated by *$\mathcal{A}$.*

*Proof.* This simply requires you to check the definitions, and thus can be safely skipped. It is immediate that $\mathbb{N} \in \mathcal{F}_{\mathcal{A}}$. If $\varnothing \in \mathcal{F}_{\mathcal{A}}$ then there is a collection $A_1, \ldots, A_n$ of elements of $\mathcal{A}$ such that $\bigcap_1^n A_i \subseteq \varnothing$ which implies they have empty intersection, which is impossible, so $\varnothing \notin \mathcal{F}_{\mathcal{A}}$. Suppose $F, G \in \mathcal{F}_{\mathcal{A}}$, then we have collections $\{ A_i \}_1^n, \{ B_i \}_1^m \subseteq \mathcal{A}$, such that $\bigcap_1^n A_i \subseteq F$ and $\bigcap_1^m B_i \subseteq G$. Then, we have $(\bigcap_1^n A_i) \cap (\bigcap_1^m B_i) \subseteq F \cap G$, and $(\bigcap_1^n A_i) \cap (\bigcap_1^m B_i) \neq \varnothing$ since $\mathcal{A}$ has the finite intersection property. Now suppose $A \in \mathcal{F}_{\mathcal{A}}$ and $B \supseteq A$. Since $A \in \mathcal{F}_{\mathcal{A}}$ by definition we have $\{ A_i \}_1^n \subseteq \mathcal{A}$ such that $\bigcap_1^n A_i \subseteq A \subseteq B$ so it follows that $B \in \mathcal{F}_{\mathcal{A}}$. $\square$

With existence of ultrafilters out of the way, we can now construct a universal[4] compact Hausdorff space from $X$, given a topological space $X$.

**Example 3.10** (Stone-cech compactification of $\mathbb{N}$)**.** Define

$$\beta\mathbb{N} = \{ U \subseteq \mathcal{P}(\mathbb{N}) : U \text{ is an ultrafilter } \}.$$

Give $\beta\mathbb{N}$ the topology which is generated by sets $[A]$[5]. To be more precise, we shall define $[A]$ like so. Fix $A \subseteq \mathbb{N}$, let $[A] = \{ U \in \beta\mathbb{N} : A \in U \}$. Notice $[A] \subseteq \beta\mathbb{N}$, so $[A]$ is the set of all ultrafilters on the naturals that contain $A$ as an element. Then, define

$$\mathcal{B} = \{ [A] : A \in \mathcal{P}(\mathbb{N}) \}.$$

We claim that $\mathcal{B}$ is a basis for a topology on $\beta\mathbb{N}$. To see this, observe that $[\mathbb{N}] = \beta\mathbb{N}$ and $[\varnothing] = \varnothing$. Now let $[A], [B] \in \mathcal{B}$. Let $U \in [A] \cap [B]$. So $U$ is an ultrafilter that contains both $A$ and $B$. Thus, $U$ contains $A \cap B$, so $U \in [A \cap B] = [A] \cap [B]$. To see why $[A \cap B] = [A] \cap [B]$, notice that if $G$ is an ultrafilter such that $A \cap B \in G$, then $A \supseteq A \cap B$ so $A \in G$, and likewise $B \in G$. If $G \in [A] \cap [B]$, then $A, B \in G$ so $A \cap B \in G$ as $G$ is a filter.

Notice that $\beta\mathbb{N}$ is **zero-dimensional**, meaning it has a basis which consists of clopen sets. Given $[A] \in \mathcal{B}$ (which is open), $\beta\mathbb{N} \setminus [A] = [\mathbb{N} \setminus A]$, which is open, so $[A]$ is closed too. //

Note that this technique really only works on discrete spaces.

[4]Ask me directly if you want to find out what universal means.

[5]This has nothing to do with equivalence classes.

### 3.1 Convergence under filters

**Definition 3.11** (Convergence under a filter)**.** Let $X$ be a topological space and let $\mathcal{F}$ be a filter on $\mathbb{N}$. A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ **$\mathcal{F}$-converges** to $x$ if given any open neighborhood $U$ of $x$, the set of indices where $x_n \in U$ is an element of $\mathcal{F}$, i.e. $\{ n \in \mathbb{N} : x_n \in U \} \in \mathcal{F}$.

We denote this by $x_n \to_{\mathcal{F}} x$ or alternatively, $x = \lim_{n\to\mathcal{F}} x_n$.

The set of limit points of a sequence $x_n$ under a filter $\mathcal{F}$ will be denoted by $\mathrm{Lim}_{\mathcal{F}}(\mathbf{x}) = \{ x \in X : x_n \to_{\mathcal{F}} x \}$.

We revisit the example of the Frechet filter (Example 3.4).
**Proposition 3.12.** *Let $x_n$ be a sequence in a topological space. A sequence $x_n$ converges to $x$ (in the usual topological sense) if and only if $x_n \to_{\mathrm{Fr}} x$ where $\mathrm{Fr}$ is the Frechet filter.*

*Proof.* If $x_n$ converges to $x$, take an open neighborhood $U$ of $x$, then there is $N \in \mathbb{N}$ such that if $n \geq N, x_n \in U$. This means that the set $A = \{ n \in \mathbb{N} : n \geq N \}$ has finite complement, so $A$ is in the Frechet filter. Conversely, if $x_n$ converges to $x$ under the Frechet filter, pick an open neighborhood $U$ of $x$. Then, let $S$ be in the Frechet filter such that if $n \in S$, $x_n \in U$. Since $S$ is in the Frechet filter, $S$ has finite complement, so let $N = \max(\mathbb{N} \setminus S)$. When $n \geq N$, it necessarily implies that $n \in S$ and so $x_n \in U$. □

The Frechet filter has the nice property that any non principal ultrafilter will contain it. Thus anything that converges in the usual sense must also converge under any non principal ultrafilter on $\mathbb{N}$. Recall that a principal ultrafilter contains singletons.
**Lemma 3.13** (Any non-principal ultrafilter has no finite sets)**.** *Let $\mathfrak{U}$ be a non principal ultrafilter on $\mathbb{N}$. Then, $\mathfrak{U}$ has no finite sets.*

*Proof.* Let $\mathfrak{U}$ be a non principal ultrafilter on $\mathbb{N}$ that contains a finite set $F$. Since $\mathfrak{U}$ is non principal, it contains no singletons. Enumerate $F = \{ a_1, \ldots, a_n \}$. For each $a_i \in F$, we notice that the set $S_i = \mathbb{N} \setminus \{ a_i \}$ is in $\mathcal{F}$, for else $\{ a_i \} \in \mathfrak{U}$, and $\mathfrak{U}$ has no singletons. Notice that $\bigcap_{i=1}^n S_i = \mathbb{N} \setminus F$, and this is in $\mathfrak{U}$ since each $S_i \in \mathfrak{U}$, and the intersection is finite. This is a contradiction. □

**Corollary 3.14.** *A filter cannot be both principal and non-principal.*
**Corollary 3.15** (Any non principal ultrafilter contains the Frechet filter)**.** *If $\mathfrak{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then $\mathrm{Fr} \subseteq \mathfrak{U}$.*

*Proof.* The Frechet filter contains all $S \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. All such $S$ is contained in $\mathfrak{U}$ (since if a cofinite set is missing then its complement, which is finite, would be in $\mathfrak{U}$). □

**Corollary 3.16.** *If $x_n$ converges to $x$ in the usual sense then it converges to $x$ under any non-principal ultrafilter.*

*Proof.* Proposition 3.12 and Corollary 3.15. □

# 4 Compactness

Compactness is apparently the most important topic in topology.

**Definition 4.1** (Open cover)**.** Let $X$ be a topological space. Then a collection $\mathcal{U} = \{ U_\alpha \}_{\alpha\in\Lambda}$ is an **open cover** of $X$ if $\bigcup_{\alpha\in\Lambda} U_\alpha = X$.

A subset of $\mathcal{U}$ that still covers $X$ is called a **subcover**.

**Example 4.2** (Open cover of a finite space)**.** Let $X = \{ 1, \ldots, n \}$ be a finite topological space endowed with the discrete topology. Then it is easy to see that the set of singletons $\{ i \}$ for each $i \in X$ is an open cover of $X$. We remark that such an open cover has a finite subcover (namely, itself). //
**Example 4.3** (Open cover of $\mathbb{R}$)**.** We can easily see that the set of open intervals of the form $(n, n + 2)$ for each $n \in \mathbb{Z}$ covers $\mathbb{R}$. //

**Definition 4.4** (Compactness)**.** Let $X$ be a topological space. Then $X$ is said to be **compact** if *every* open cover of $X$ contains a finite subcover.

We emphasize the importance that *you must be able to extract a finite subcover out of every open cover.* It is not enough to be able to extract a finite subcover out of a certain open cover, as every topological space $X$ can be trivially open covered by $\{X\}$.

**Example 4.5** (Compactness of a finite topological space)**.** Let $X$ be a finite topological space. Since any collection of subsets of a finite set must be finite, any open cover of $X$ must necessarily be finite and thus $X$ is compact. //

**Example 4.6** (Non-compactness of $\mathbb{R}$)**.** We notice that $\mathbb{R}$ is not compact, since Example 4.3 is an open cover of $\mathbb{R}$ that has no finite subcover. //

The next example illustrates that any sequence together with a limit point of the sequence is compact.

**Example 4.7** (Compactness of a convergent sequence)**.** Let $(x_n)$ be a sequence in a space $X$ that converges to $x$. We claim that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact. To see this, let $\mathcal{U}$ be an open cover of this set. Then, there is some $U_\alpha \in \mathcal{U}$ such that $U_\alpha$ contains $x$. Since $x_n \to x$, there exists some $N$ such that all $x_n$ where $n \geq N$ is contained in $U_\alpha$, so there are only finitely many $x_n$'s that are not in $U_\alpha$. We thus choose an open set $U_i \in \mathcal{U}$ for each $x_i$ that is not in $U_\alpha$. We have thus found a finite subcover for this convergent sequence. //

Just like connectedness, compactness is an intrinsic property (unlike being open). That is to say, if $A \subseteq X$ is compact in the subspace topology, then it is not going to become non-compact when considered as subset of $X$. This is unlike the notion of being open, since $U \subseteq A$ can be open in $A$, but not in $X$.

**Proposition 4.8** (Compactness is not relative.)**.** *Let $A$ be a subspace of $X$. Then, $A$ is compact (in the subspace topology) if and only if every open cover of $A$ by open sets of $X$ has a finite subcover of $A$.*

See [Lee11, Lemma 4.27, p. 94].

*Proof.* Suppose $A$ is compact in the subspace topology. Let $\mathcal{U} = \{V_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of $X$ that cover $A$. We now translate this to an open cover of $A$ by open subsets of $A$. Define $A \cap \mathcal{U} = \{A \cap V_\alpha : \alpha \in \Lambda\}$. By compactness of $A$, we have $A \cap V_1, \ldots, A \cap V_n$ that covers $A$, so $V_1, \ldots, V_n$ is a finite subcover of $A$ by subsets of $\mathcal{U}$.

The converse is trivial. □

Henceforth, when we speak of an open cover of a subspace, it does not matter whether it is an open cover of a subspace by open subsets of the subspace, or by open covers of the big space.

We now introduce an alternative characterization of compactness.

**Definition 4.9** (Finite intersection property (FIP))**.** Let $\mathcal{C} = \{C_\alpha : \alpha \in \Lambda\}$ be a collection of *closed* subsets of a topological space $X$. Then, $X$ has the **finite intersection property** if given any finite subset $F \subseteq \Lambda$, $\bigcap_{\alpha \in F} C_\alpha$ is nonempty.

Note that this definition is pretty much the same as the one given before Lemma 3.9.

**Proposition 4.10** (FIP iff compact)**.** *Let $X$ be a space. Then $X$ is compact if and only if every collection $\mathcal{C}$ of closed subsets of $X$ with the finite intersection property has a nonempty intersection, that is $\bigcap \mathcal{C} \neq \varnothing$.*

We shall sketch a proof of this. The idea is to take the contrapositive of the theorem and translate compactness into closed sets. Notice that if $\mathcal{U}$ is an open cover of $X$, then the set $X - \mathcal{U} = \{X \setminus U : U \in \mathcal{U}\}$ has the property that $\bigcap X - \mathcal{U} = \varnothing$. If additionally $\mathcal{U}$ has a finite subcover, then there is a finite number of sets in $X - \mathcal{U}$ whose intersection is empty. A space $X$ is compact if and only if given any collection $\mathcal{C}$ of closed subsets of $X$ such that $\bigcap \mathcal{C}$ is empty, any finite intersection of elements in $\mathcal{C}$ is empty.

*Proof.* See [Mun00, Thm 26.9, p. 167]. □

# 5 Properties of compactness

In this section we see how compactness interacts with other topological properties and constructions.

The most important thing about compactness is that it is preserved under continuous functions.

**Theorem 5.1** (Main theorem of compactness). Let $X$ be a compact space and $f : X \to Y$ be a continuous function. Then, $f[X]$ is compact.

*Proof.* Let $\mathcal{U} = \{ U_\alpha : \alpha \in \Lambda \}$ be an open cover of $f[X]$ by open subsets of $Y$. Then, the set $f^{-1}[\mathcal{U}] = \{ f^{-1}[U_\alpha] : \alpha \in \Lambda \}$ is an open cover of $X$, so there is a finite subcover, say $f^{-1}[U_1], \ldots, f^{-1}[U_n]$. It is now clear that $U_1, \ldots, U_n$ is a finite subcover of $f[X]$. $\square$

**Corollary 5.2** (Compactness is a topological invariance)**.** *Any space homeomorphic to a compact space is compact.*

Now we shall see how compactness interacts with other topological properties.

**Proposition 5.3** (Closed subsets of compact spaces are compact)**.** *If $X$ is a compact space and $Y \subseteq X$ is closed, then $Y$ is compact.*

*Proof.* Let $\mathcal{U} = \{ U_\alpha : \alpha \in \Lambda \}$ be an open cover of $Y$ by open sets of $X$. Then, the set $\mathcal{U} \cup \{ X \setminus Y \}$ is an open cover of $X$, so there is a finite subcover. Now, remove $X \setminus Y$ from the finite subcover of $X$ (if it is in there) to obtain a finite subcover of $Y$. $\square$

Given a Hausdorff space, we can separate a point from a compact subspace. This can be seen in [Mun00, Lem 26.4, p. 166]. A generalization of this fact can be proven ([Lee11, Lem 4.34, p. 95]) below, and it is [Mun00, Chp. 26, Ex. 5].

**Proposition 5.4** (Separating compact subspaces from compact subspaces)**.** *Let $X$ be a Hausdorff space. Let $Y$ be compact and $Z$ be disjoint compact subspaces. Then, there exists disjoint open sets $U$ and $V$ such that $Z \subseteq U$, and $Y \subseteq V$.*

*Proof.* First suppose that $Z = \{ x \}$. For each point $y \in Y$, there exists disjoint open neighborhoods $U_x$ of $x$ and $V_y$ of $y$. The collection $\{ V_y : y \in Y \}$ is an open cover of $Y$, so there is a finite subcover $V_{y_1}, \ldots, V_{y_n}$. Now set $\mathbb{U}_x = \bigcap_{i=1}^n U_{y_i}$ and $\mathbb{V}_x = \bigcup_{i=1}^n V_{y_i}$. Then, $Y \subseteq \mathbb{V}_x$, and $x \in \mathbb{U}_x$. It is also clear that $\mathbb{U}_x$ and $\mathbb{V}_x$ are disjoint.

Now suppose $Z$ is a compact subspace. For each $x \in Z$, we have disjoint open sets $\mathbb{U}_x, \mathbb{V}_x$ where $\mathbb{U}_x$ contains $x$, and $\mathbb{V}_x$ contains $Y$. The collection $\{ \mathbb{U}_x : x \in Z \}$ is a open cover of $Z$, so there is a finite subcover, say $\mathbb{U}_{x_1}, \ldots, \mathbb{U}_{x_n}$. Then, set $\mathbb{U} = \bigcup_{i=1}^n \mathbb{U}_{x_i}$ and $\mathbb{V} = \bigcap_{i=1}^n \mathbb{V}_{x_i}$. So $\mathbb{U}$ contains $Z$ and $\mathbb{V}$ contains $Y$. $\square$

**Remark 5.5.** In the lecture, it was only proved that a point can be separated from a compact subspace. The proposition above has been upgraded to separating an entire compact subset, which illustrates the fact that compactness behaves like finiteness. A property possessed by a finite set (being able to be separated from a compact subset) is also possessed by a compact subset.

If the space is Hausdorff, a compact subspace of this space has the additional property of being closed. This will be useful when proving the Heine-Borel theorem.

**Corollary 5.6** (Compactness in Hausdorff spaces)**.** *Let $X$ be a Hausdorff space and $Y$ be a compact subspace of $X$. Then, $Y$ is closed in $X$.*

*Proof.* We simply show that $X \setminus Y$ is open. Given a point $x \in X$, $x \notin Y$, apply Proposition 5.4 to find an open neighborhood of $x$ that is disjoint from $Y$. $\square$

Given a metric space, we can say more about a compact subspace of it, namely that they are bounded. If $X$ is a metric space with metric $d$, then a set $S \subseteq X$ is said to be *bounded* if there exists $M > 0$ and a point $x \in X$ such that $S \subseteq B_d(x, M)$.

**Proposition 5.7** (Compact subsets of metric spaces are bounded)**.** *Let $X$ be a metric space and $Y$ a compact subspace of $X$. Then, $Y$ is bounded.*

*Proof.* Let $x \in X$. Then consider the open cover of $Y$ by $\{ B(x, n) : n \in \mathbb{N} \}$. Clearly this must open cover $Y$, so there is a finite subcover. The largest ball contains $Y$. $\square$

You might be wondering about the product of compact spaces. Is that compact? Yes. We can prove it for finite products. However, the fact that the arbitrary product of compact spaces is compact is nontrivial. This is known as Tychonoff's Theorem.

**Proposition 5.8** (Finite product of compact spaces is compact)**.** *Suppose $X, Y$ are compact spaces. Then $X \times Y$ is compact.*

*Proof.* For this, we will need a small lemma, the tube lemma (Lemma 5.9) and we can re-use the ideas from the product of connected spaces being connected. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each point $x \in X$, the set $\{x\} \times Y$ is compact (c.f. Corollary 5.2) and so there exists a finite subcover $U_1, \ldots, U_n \in \mathcal{U}$. By Lemma 5.9, there is a neighborhood $V_x$ of $x$, which forms a tube $V_x \times Y$ such that $V_x \times Y \subseteq \bigcup_{i=1}^n U_i$. Since $X$ is compact, finitely many such $V_x$'s cover $X$, so finitely many tubes cover $Y$. Since each of the tubes is contained in finitely many $U$'s from $\mathcal{U}$, we are done. □

**Lemma 5.9** (Tube lemma)**.** *Let $X$ be a topological space and $Y$ be a compact space. Given $x \in X$, and a open set $U \subseteq X \times Y$ that contains $\{x\} \times Y$, there is an open neighborhood $V$ of $x$ such that $V \times Y \subseteq U$.*

For a picture, see [Lee11, Fig 4.5, p. 96] or [Mun00, Lemma 26.8, pp. 168–169].

*Proof.* For each $y \in Y$, there is an open set $Q \times W \subseteq U$ such that $x \in Q$ and $y \in W$. There are finitely many such $W$'s that cover $Y$ as $Y$ is compact, so set $V$ to be the union of the $Q$'s. Then $Q$ is open and $Q \times Y \subseteq U$. □

A closed interval should be compact. The next proposition proves this. For a proof that has more pictures, see [Lee11, Thm 4.39, p. 97] or [Mun00, Thm 27.1, pp. 172–173], where a generalization is proved.

**Proposition 5.10** (Closed intervals are compact)**.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ (where $a \leq b$). Then, $[a, b]$ is compact.*

*Proof.* Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $[a, b]$. Let

$$S = \{x \in [a, b] : [a, x] \text{ has a finite subcover by } \mathcal{U}\}.$$

Then $S$ is nonempty as $[a, a]$ clearly has a finite subcover. Additionally, $S$ is bounded above by $b$. So $S$ has a supremum. Let $c = \sup S$. We claim that $c = b$. Suppose not, then $c < b$. Since each $U_\alpha$ is open, there exists $\varepsilon > 0$ such that the interval $(c - \varepsilon, c + \varepsilon)$ is contained in some $U_{\alpha_0}$. Notice that $[a, c]$ can be finitely subcovered by $\mathcal{U}$. If we choose $y$ such that $c < y < c + \varepsilon$, then $U_{\alpha_0}$ contains $[c, y]$. So $[a, y]$ can be finitely subcovered, contradicting the fact that $c$ is the sup. □

We can now deduce the Heine Borel theorem for $\mathbb{R}$.

**Corollary 5.11** (Heine-Borel for $\mathbb{R}$)**.** *Any closed bounded subset of $\mathbb{R}$ is compact.*

*Proof.* If $S$ is a bounded subset of $\mathbb{R}$ then it is contained in some open interval, which is contained in some closed interval. Since the closed interval is compact and closed and $S$ is a closed subset of this closed interval, it is compact. □

We have already shown that compact subsets of metric spaces are closed and bounded. In $\mathbb{R}^n$, we can reverse the implication. This is known as the Heine-Borel theorem.

**Theorem 5.12** (Heine-Borel)**.** Let $Y$ be a closed and bounded subset of $\mathbb{R}^n$. Then, $Y$ is compact.

*Proof.* If $S$ is a closed and bounded subset of $\mathbb{R}^n$ then it is contained in some cube $[-I, I]^n$. By Proposition 5.8, and Corollary 5.11 this cube is compact. By Proposition 5.3 $S$ is compact. □

We now summarize the results about compactness.

**Theorem 5.13** (Properties of compactness)**.**

1. A closed subset of a compact space is compact.
2. A compact subspace of a Hausdorff space is closed.
3. A compact subspace of a metric space is bounded.
4. A finite product of compact spaces is compact.

*Proof.*

1. Proposition 5.3
2. Corollary 5.6
3. Proposition 5.7
4. Proposition 5.8

□

# References

[Mun00] James R. Munkres. *Topology*. 2nd ed. Upper Saddle River, NJ: Prentice Hall, Inc, 2000. ISBN: 9780131816299.

[Jec03] Tomáš Jech. *Set theory*. eng. Ed. by Springer-Verlag (Berlin). The third millenium edition, revised and expanded. Springer Monographs in Mathematics, Berlin: Springer, 2003. ISBN: 9783642078996.

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. en. Vol. 202. Graduate Texts in Mathematics. New York, NY: Springer New York, 2011. ISBN: 9781441979391. DOI: 10.1007/978-1-4419-7940-7. URL: https://link.springer.com/10.1007/978-1-4419-7940-7.

[Mak23] Sergiy Maksymenko. *Connected space being not locally connected at each point*. Forum post. May 2023. URL: https://mathoverflow.net/q/445962.