# Week 9

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## 1 Local compactness and one-point compactifications

**Definition 1.1** (Local Compactness). A space $X$ is **locally compact at** $x \in X$ if there is a neighborhood $U$ of $x$ and a compact set $K$ containing $U$.

A space is locally compact if it is locally compact at every point.

Clearly every compact space is locally compact. We now exhibit some examples of spaces which are not compact, but locally compact. We also exhibit examples of spaces that are not locally compact.

**Example 1.2** (The real numbers)**.** The real numbers are locally compact. Given $x \in \mathbb{R}$, take an interval $(x-\varepsilon, x+\varepsilon)$ around it and the closed interval $[x-\varepsilon, x+\varepsilon]$ contains this neighborhood. //

**Example 1.3** (The rational numbers)**.** The rational numbers are *not* locally compact everywhere. If $\mathbb{Q}$ was locally compact, pick $q \in \mathbb{Q}$, then there is a neighborhood $U$ of $q$ and compact subspace $C$ such that $q \in U \subseteq C$. Now, $U$ contains some interval $(q-\varepsilon, q+\varepsilon)$. Pick an irrational number in this interval. Then choose a sequence of rational numbers converging to this irrational number[1]. The limit should be in $C$ as $C$ is closed, but the limit is irrational.

In fact, every closed interval $[a, b] \cap \mathbb{Q}$ is seen to be not compact, since $\{ (a-1, i) : i \text{ irrational }, a < i < b \}$ is an open cover of $[a, b] \cap \mathbb{Q}$[2] with no finite subcover. //

When the space $X$ is Hausdorff, we can say a bit more.

**Proposition 1.4.** *Let $X$ be Hausdorff. Then, the following are equivalent:*

1. *$X$ is locally compact at $x$;*
2. *There is a neighborhood $V$ of $x$ such that $\overline{V}$ is compact;*
3. *There is a local basis around $x$ of open sets with compact closure.*

*Proof.* It is immediate that 1 implies 2, since $\overline{U} \subseteq K$ as $K$ is closed and contains $U$. Moreover, $\overline{U}$ is closed so it is compact. For 2 implies 3, set $\mathcal{B}_x = \{ U \cap V : U \text{ is a neighborhood of } x \}$ where $V$ is given by (2). Then, $\overline{U \cap V} \subseteq \overline{V}$, so the sets $U \cap V$ have compact closure. 3 immediately implies 1. □

**Example 1.5** (Discrete topology)**.** Any space with the discrete topology is easily seen to be locally compact. //

**Theorem 1.6** (Existence of one-point compactifications). Let $X$ be a space. Then, the following are equivalent:

1. $X$ is Hausdorff and locally compact;
2. There is a compact Hausdorff space $Y$ and an embedding $i : X \to Y$ such that $Y \setminus i[X]$ is a single point;
3. $X$ is homeomorphic to an open subspace of a compact Hausdorff space.

Moreover, if there is another space $Y'$ and embedding $j$ that satisfies (2), then there is a unique homeomorphism $f : Y \to Y'$ such that $f \circ i = j$.[a]

[a]Warning: This is *not* a universal property.

[1]If $i$ is such an irrational, pick $x_n$ to be rational such that $i < x_n < i + 1/n$.
[2]We are covering it in $\mathbb{R}$, since compactness is not relative.

See [Mun00, Thm 29.1, p. 181].

*Proof.* Suppose $X$ is locally compact Hausdorff with topology $\tau$. Let[3] $\infty \notin X$. Define $Y = X \cup \{\infty\}$ and we shall define a topology $\tau_\infty$ on $Y$ with the following properties:

- Everything that is open in $X$ should be open in $Y$, i.e. $\tau \subseteq \tau_\infty$;
- We also add neighborhoods of $\infty$, they should be subsets $U$ of $Y$ such that $X \setminus U$ is compact in $X$. Thus neighborhoods of $\infty$ have compact complement in $X$.

Let us check that $\tau_\infty$ is a topology. Notice $\varnothing \in \tau_\infty$ since it is open in $X$. We also have $Y \in \tau_\infty$.

If $U, V \subseteq X$ then $U \cap V$ are open in $X$ and so open in $Y$. If $U, V$ are open neighborhoods of $\infty$ then we have $X \setminus U = K_U, X \setminus V = K_V$ being compact. Then we have $X \setminus (U \cap V) = K_U \cup K_V$ which is compact. If $U \subseteq X$ and $V$ an open neighborhood of $\infty$, then $X \setminus V$ is compact. Then $(X \cap V) \cap U$ is open. Unions are left as an exercise.

For compactness, let $\mathcal{U}$ be an open cover of $Y$. Let $U \in \mathcal{U}$ be such that $\infty \in U$. Then $X \setminus U$ is compact and so it has a finite subcover from $\mathcal{U}$.

To see $Y$ is Hausdorff, pick $x, y \in Y$. If both of $x, y$ lie in $X$ it is easy. If $y = \infty$, use the local compactness of $X$ to choose a neighborhood $U$ of $x$ with a compact subspace $C$ containing $U$. Then $Y \setminus C$ is a neighborhood of $y = \infty$ and it is disjoint from $U$.

2 implies 3 follows because the singleton $\infty$ is closed in $Y$.

For 3 implies 1, it is immediate that $X$ is Hausdorff. Let $x \in X$. Then there is some open neighborhood $U$ of $x$ such that $\overline{U} \subseteq X$[4]. Then $\overline{U}$ is compact and contains $U$.

The uniqueness condition is easy. $\square$

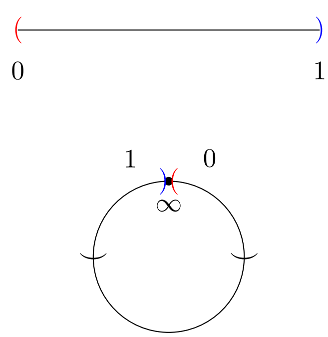


Figure 1: One point compactification of the reals. The reals are homeomorphic to $(0, 1)$.

**Remark 1.7.** Notice that we can construct a one point compactification of a space that is not necessarily locally compact or even Hausdorff.

> **Definition 1.8.** A **compactification** of a Hausdorff space $X$ is a compact Hausdorff space $Y$ such that $X$ is dense in $Y$. If $Y \setminus X$ is a single point, then $Y$ is called the **one-point compactification** of $X$, and it is denoted $\alpha X$.

Although $X$ may not actually be a subset of $Y$, we mean that there is an embedding $i : X \to Y$. Also, $Y \setminus X$ really means $Y \setminus i[X]$.

**Remark 1.9.** Constructing a one compactification of $X$ when $X$ is already compact is uninteresting since it simply adjoins an isolated point to $X$.

**Remark 1.10.** If $X$ is locally compact Hausdorff and we have a construction $Y$ that we claim to be the one point compactification of $X$, it suffices to show that $Y$ is compact Hausdorff, $Y \setminus X$ is a singleton, and $X$ embeds into $Y$ as a dense subset. This is due to the uniqueness condition of Theorem 1.6.

**Example 1.11** (One-point compactification of the naturals)**.** It is claimed that $\alpha\mathbb{N}$ is homeomorphic to $\{1/n : n \in \mathbb{N}\} \cup \{0\}$. //

[3] You can choose $\infty = X$ since $X \notin X$.

[4] See [Lee11, Lem 4.65] or Question 4a in Problem Set 7

**Example 1.12** (One-point compactification of the reals)**.** From Figure 1, you can visualize $\alpha\mathbb{R}$ as the circle. They are homeomorphic. //

**Example 1.13** (One-point compactification of $\omega_1$)**.** This has been discussed in Problem set 3, Question 8, slightly. //

# 2 Countability axioms

**Definition 2.1** (First and second countability)**.** A topological space $X$ is **first-countable** if every point has a countable local basis. It is **second-countable** if there is a countable basis of $X$.

Recall that a space is sequential if given $A \subseteq X$, and $x \in \overline{A}$, there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that $x_n \to x$.

**Proposition 2.2.** *Let $X$ be first-countable. Then, $X$ is sequential.*

*Proof.* Suppose $x \in \overline{A}$. If $\{ U_n \}$ is a local basis at $x$, then pick $x_n \in A \cap \bigcap_1^n U_i$. □

**Proposition 2.3.** *If $X$ is metrizable, then $X$ is first countable*

*Proof.* Choose balls of radius $1/n$ at each point $x$. □

**Example 2.4** (The real numbers)**.** The real numbers are easily seen to be second countable by taking the basis $\mathcal{B} = \{ (p, q) : p < q, p, q \in \mathbb{Q} \}$. //

**Proposition 2.5** (Restrictions on the cardinality of discrete subspaces)**.** *If $X$ is second countable and $A \subseteq X$ is discrete, then $A$ is countable.*

*Proof.* If $A$ is discrete, we can put a basis element $U_a$ around each point $a \in A$ that intersects $A$ only at $\{ a \}$, so we have a collection of distinct basis elements of at least the cardinality[5] of $A$. □

**Example 2.6** (Uniform topology)**.** The space $\mathbb{R}^\omega$ with the uniform topology is first countable but not second countable. It contains the subspace $\{ 0, 1 \}^\omega$, the set of binary seqeunces, which is a discrete subspace. By Proposition 2.7, the uniform topology on $\mathbb{R}^\omega$ cannot be second countable. //

**Proposition 2.7.** *Let $X$ be a second countable space. Then, the following are true:*

1. *$X$ is first countable;*
2. *$X$ satisifies the **Lindelof** property: Every open cover of $X$ has a countable subcover;*
3. *$X$ is **separable**[6]: There is a countable dense subset of $X$.*
4. *$X$ has the **countable chain condition (ccc)**: Any collection of pairwise disjoint open subsets of $X$ is countable.*

*Proof.* 1 is immediate by choosing a local basis of $x$ to be the elements of the countable basis of $X$ that contain $x$.

2 is an exercise

For 3, let $\mathcal{B}$ be a countable basis of $X$ and pick[7] a $x \in U \in \mathcal{B}$ (for each $U$ that is nonempty in $\mathcal{B}$). Then let $D$ be the collection of all these points we picked. It is clear that $D$ intersects every open set so it is dense.

4 follows by observing that you can extract a pairwise disjoint collection of basis elements from a pairwise disjoint collection of open subsets. Alternatively, you can deduce 4 from 3 by picking an element of the dense set from each open subset in the collection. □

**Remark 2.8.** Even with all 3 conditions being satisified, $X$ may still fail to be second countable. See [Mun00, Example 3, p. 192].

**Proposition 2.9.** *For metrizable spaces, second countability, possessing the Lindeloff property and separability are equivalent.*

[5] In particular, the map $a \mapsto U_a$ from $A$ into the basis of $X$ is injective.

[6] This name is quite bad, since it really has nothing to do with separations and connectedness.

[7] The axiom of countable choice will suffice