

# Notes on multivariable calculus

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# Contents

<b>1</b>	<b>Some prerequisite material</b>	<b>4</b>
1.1	Notation . . . . .	4
1.2	Linear algebra . . . . .	4
1.3	Topology . . . . .	5
<b>2</b>	<b>Differentiation</b>	<b>7</b>
2.1	Derivatives . . . . .	7
2.2	Consequences of differentiability . . . . .	8
2.3	The chain rule . . . . .	8
2.4	Multivariable MVT . . . . .	10
2.5	Partial derivatives . . . . .	12
2.5.1	Higher-order partial derivatives . . . . .	13
2.6	Directional derivatives . . . . .	14
2.7	Optimization . . . . .	14
2.8	Tangent vectors . . . . .	15
2.9	Smooth Manifolds . . . . .	16
2.9.1	Diffeomorphisms . . . . .	17
2.10	Implicit and inverse functions . . . . .	18
	<b>References</b>	<b>18</b>

# List of Theorems

1.2.4 Theorem (Weak Riesz Representation Theorem)	5
1.3.8 Theorem (Heine-Borel)	6
2.1.3 Theorem (Uniqueness of derivative)	7
2.3.1 Theorem (Chain rule)	8
2.3.2 Theorem (Useful properties of differentiation)	9
2.4.1 Theorem (Mean Value Theorem)	10
2.4.2 Theorem (Multivar MVT)	11
2.5.2 Theorem (Components of gradient)	12
2.5.5 Theorem (Continuous partials implies differentiable)	12
2.5.7 Theorem (Clairut's Theorem)	13
2.5.9 Theorem	13
2.7.3 Theorem (Local EVT)	15
2.7.6 Theorem (Lagrange Multipliers)	15
2.8.5 Theorem	15
2.10.1 Theorem (Implicit function theorem)	18
2.10.2 Theorem (Inverse function theorem)	18

# List of Definitions

1.2.1 Definition (Inner Product) . . . . .	4
1.3.1 Definition (Ball) . . . . .	5
1.3.2 Definition (Open set) . . . . .	5
1.3.3 Definition (Neighborhoods) . . . . .	5
1.3.4 Definition (Open covers) . . . . .	5
1.3.6 Definition (Compactness - General definition) . . . . .	6
1.3.7 Definition (Compactness - $\mathbb{R}^n$ definition) . . . . .	6
2.1.1 Definition (Derivative) . . . . .	7
2.1.2 Definition (Gradient) . . . . .	7
2.5.1 Definition (Partial derivative) . . . . .	12
2.5.6 Definition (Class $C^1$ function) . . . . .	12
2.6.1 Definition (Directional derivative) . . . . .	14
2.7.1 Definition (Local extrema) . . . . .	14
2.7.2 Definition (Critical point) . . . . .	14
2.7.5 Definition (Saddle point) . . . . .	15
2.8.1 Definition (Tangent Vector) . . . . .	15
2.8.2 Definition (Tangent space) . . . . .	15
2.8.4 Definition (Tangent plane) . . . . .	15
2.9.1 Definition (Smooth Manifold at a point $p$ ) . . . . .	16
2.9.4 Definition (Smooth Manifold) . . . . .	16
2.9.7 Definition (Diffeomorphism) . . . . .	17
2.9.11 Definition (Local Diffeomorphism) . . . . .	17

# Chapter 1

## Some prerequisite material

We will quickly remark that the prerequisite chapter is not fully complete.

### 1.1 Notation

We shall try to be as clear about the type of our variables as much as possible. In particular, vectors will be in bold:  $\mathbf{v} \in \mathbb{R}^n$ .

### 1.2 Linear algebra

We shall assume that the reader knows a little bit of linear algebra. Here, we will not bold any of our vectors, since they are all vectors.

**Definition 1.2.1 (Inner Product).** Let  $V$  be a vector space over  $\mathbb{R}$  or  $\mathbb{C}$ . Then an *inner product on  $V$*  is a map  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$  such that

1. **(Linear in the 2nd argument)**  $\langle u, \lambda v + w \rangle = \lambda \langle u, v \rangle + \langle u, w \rangle$ ,
2. **(Conjugate symmetric)**  $\langle u, v \rangle = \overline{\langle v, u \rangle}$ ,
3. **(Positive definiteness)**  $\langle v, v \rangle \geq 0$  and  $\langle v, v \rangle = 0$  if and only if  $v = 0$ .

A few quick remarks are in order. The third property of an inner product tells us that  $\langle v, v \rangle$  must be a real number, so comparison with 0 is a legal move. In addition, if  $V$  is a real vector space, conjugate symmetry is simply symmetry, i.e.  $\langle u, v \rangle = \langle v, u \rangle$  since  $\bar{r} = r$  whenever  $r$  is a real number. Thus, any inner product on a real vector space will also be linear in the first argument. In either case, linearity of the second argument allows us to define linear transformations like  $\varphi_v(w) = \langle v, w \rangle$  where  $v \in V$  is some fixed vector. We shall see that this is used later on in [Theorem 1.2.4](#), and even in the discussion after [Equation \(2.2\)](#).

We shall really only be working with one kind of inner product, the *usual inner product on  $\mathbb{R}^n$* . This is defined to be

$$\langle u, v \rangle = \sum_{i=1}^n u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \quad (1.1)$$

Given an inner product, we can define a *norm*. Again, we are mostly only interested in the *Euclidean norm on  $\mathbb{R}^n$* , which is defined by

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\sum_{i=1}^n v_i v_i}. \quad (1.2)$$

Here,  $\langle v, v \rangle$  is taking the usual inner product of  $v$  with itself.

This is a useful proposition which is used in a lot of bounding arguments. See [Spi18, Problem 1-1 on p. 4] for the problem.

**Proposition 1.2.2.** *Let  $\|\cdot\|$  be the usual Euclidean norm. Then for any  $x \in \mathbb{R}^n$  we have*

$$\|x\| \leq \sum_{i=1}^n |x_i|.$$

This proposition will come in handy quite often when proving results later on, so we state it here. See [Spi18, Problem 1-10 on p. 5] for the original exercise.

**Proposition 1.2.3.** *Suppose  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is linear. Then, there exists  $M \in \mathbb{R}$  such that  $\|T(h)\| \leq M\|h\|$  for all  $h \in \mathbb{R}^m$ .*

We may also use the Riesz Representation Theorem. For now, the weaker version will suffice. You can see [Spi18, Problem 1-12 on p. 5] for the original problem.

**Theorem 1.2.4 (Weak Riesz Representation Theorem).** Let  $V$  be a real inner product space of dimension  $n$  and let  $V^*$  denote the dual space of  $V$ . This is the set of linear functionals  $\varphi : V \rightarrow \mathbb{R}$ . Now, for each  $v \in V$ , define  $\varphi_v(w) = \langle v, w \rangle$ , and define  $F : V \rightarrow V^*$  by  $F(v) = \varphi_v$ . Then  $F$  is an isomorphism.

## 1.3 Topology

We shall only briefly touch on topology. For now, the definitions will be introduced in a less general manner. Readers who do know topology can gloss over this section.

**Definition 1.3.1 (Ball).** Let  $\mathbf{x} \in \mathbb{R}^n$  and  $\varepsilon > 0$ . Then a ball around  $\mathbf{x}$  of radius  $\varepsilon$ <sup>a</sup> is the set

$$B(\mathbf{x}, \varepsilon) := \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| < \varepsilon\}. \quad (1.3)$$

<sup>a</sup>Sometimes you will see this being called an *open ball*.

When  $n = 1$ , balls are simply open intervals. When  $n = 2$  they can be seen as circles and when  $n = 3$  they can be seen as spheres.

**Definition 1.3.2 (Open set).** Let  $U \subseteq \mathbb{R}^n$ . Then the set  $U$  is called open if for every point  $\mathbf{x} \in U$ , there exists some  $\varepsilon > 0$  such that  $B(\mathbf{x}, \varepsilon) \subseteq U$ .

It is immediate from the definition that every ball is an open set. Additionally, with a little set theory one can prove that every open set is the union of balls.

**Definition 1.3.3 (Neighborhoods).** Let  $\mathbf{x} \in \mathbb{R}^n$ . A neighborhood of  $\mathbf{x}$  is an open set  $U \subseteq \mathbb{R}^n$  such that  $\mathbf{x} \in U$ .

**Definition 1.3.4 (Open covers).** Let  $S \subseteq \mathbb{R}^n$ . An open cover of  $S$  is a collection of open sets  $\mathcal{O} = \{O_\alpha : \alpha \in \Lambda\}$  (here  $\Lambda$  is some indexing set, so we can refer to each element in  $\mathcal{O}$  by writing  $O_\alpha$ ) such that the union  $\bigcup \mathcal{O} = \bigcup_{\alpha \in \Lambda} O_\alpha$  contains  $S$ , i.e.

$$\bigcup_{\alpha \in \Lambda} O_\alpha \supseteq S.$$

If there are finitely many indices  $\alpha_1, \dots, \alpha_n$  such that  $\bigcup_{i=1}^n O_{\alpha_i} \supseteq S$ , then  $\mathcal{O}$  is said to have a finite subcover.

This definition is a little hard to get your head around so let us see a few examples.

**Example 1.3.5.** We can open cover  $\mathbb{R}$  with open intervals of the form  $(n, n+1)$ , for each  $n \in \mathbb{Z}$ . //

**Definition 1.3.6** (Compactness - General definition). Let  $S \subseteq \mathbb{R}^n$ . Then  $S$  is said to be compact if given *any* collection of open sets  $\mathcal{O} = \{O_\alpha : \alpha \in \Lambda\}$ , there is a finite subcover of  $S$ .

**Definition 1.3.7** (Compactness -  $\mathbb{R}^n$  definition). Let  $S \subseteq \mathbb{R}^n$ . Then  $S$  is said to be compact if it is closed and bounded.

Note that the equivalence of those two definitions in  $\mathbb{R}^n$  is justified by the following theorem.

**Theorem 1.3.8** (Heine-Borel). Let  $S \subseteq \mathbb{R}^n$ . Then  $S$  is compact (in the sense of Definition 1.3.6) if and only if it is closed and bounded.

*Proof.* Much shorter with topology, and thus omitted. □

You are free to use any definition of compactness that is convenient. The following property of compactness is extremely useful.

**Proposition 1.3.9.** *Let  $S \subseteq \mathbb{R}^n$  be compact. Let  $f : S \rightarrow \mathbb{R}^n$  be continuous. Then, the image of  $S$  under  $f$ ,  $f[S]$ , is also compact.*

And as a corollary, we get for free, the

**Corollary 1.3.10** (Extreme Value Theorem). *Let  $S \subseteq \mathbb{R}^n$  be compact and  $f : S \rightarrow \mathbb{R}$  be continuous. Then,  $f$  attains a maximum and minimum, i.e. there exists  $m, M \in \mathbb{R}$  such that for all  $s \in S$ ,  $m \leq f(s) \leq M$ .*

# Chapter 2

## Differentiation

### 2.1 Derivatives

We immediately begin with the most general definition of the derivative. The derivative, in essence, is trying to capture the idea of a linear approximation to a function at a point.

**Definition 2.1.1 (Derivative).** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\mathbf{a} \in \mathbb{R}^n$ . Then  $f$  is said to be differentiable at  $\mathbf{a}$  if there exists a linear transformation  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \lambda(\mathbf{h})\|}{\|\mathbf{h}\|} = 0. \quad (2.1)$$

The linear transformation is often denoted  $Df(\mathbf{a})$ .

**Warning:** This notation can get a little confusing! If we do  $Df(\mathbf{a})(\mathbf{v})$ , we are saying evaluate the linear transformation  $Df(\mathbf{a})$  at  $\mathbf{v}$ . For less confusion, the reader can insert brackets like  $(Df(\mathbf{a}))(\mathbf{v})$ . However, this notation is worth the initial confusion because we will soon see that  $Df$  itself can be a function.

When we refer to the matrix of the derivative of  $f$  at  $\mathbf{a}$ , i.e.  $\mathcal{M}(Df(\mathbf{a}))$ , it is called the *Jacobian of  $f$  at  $\mathbf{a}$* . This is denoted  $f'(\mathbf{a})$  in [Spi18].

Of course, the codomain of  $f$  need not be all of  $\mathbb{R}^n$ , just some open subset of it. When we have  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we define

**Definition 2.1.2 (Gradient).** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then the gradient of  $f$  at  $\mathbf{a}$ , denoted  $\nabla f(\mathbf{a})$  is a vector in  $\mathbb{R}^n$  such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}\|}{\|\mathbf{h}\|} = 0. \quad (2.2)$$

It is not too hard to see that the gradient is really just a special case of the derivative. In this case, the linear transformation  $\lambda$  is actually given by  $\lambda(\mathbf{h}) = \nabla f(\mathbf{a}) \cdot \mathbf{h}$ . We can appeal to [Theorem 1.2.4](#).

Of course, the derivative is unique.

**Theorem 2.1.3 (Uniqueness of derivative).** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable. Then, there is a unique linear transformation  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  that makes [Equation \(2.1\)](#) hold.

We shall refer the reader to [Spi18] for a proof for now, until I update these notes again.



## 2.2 Consequences of differentiability

Recall that in single variable calculus, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $c$ , then it is also continuous at  $c$ . The same is true in multivariable calculus

**Proposition 2.2.1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $\mathbf{a} \in \mathbb{R}^n$ . Then  $f$  is continuous at  $\mathbf{a}$ .*

*Proof.* Suppose  $f$  is differentiable at  $\mathbf{a}$ . Let  $\varepsilon > 0$ , and we would like to find a  $\delta > 0$  such that if  $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ , we have  $\|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon$ . Since  $f$  is differentiable at  $\mathbf{a}$ , there exists a linear transformation  $\lambda$  such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \lambda(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

By definition of the limit, we have some  $\delta > 0$  such that when  $0 < \|\mathbf{h}\| < \delta$ ,

$$\frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \lambda(\mathbf{h})\|}{\|\mathbf{h}\|} < \varepsilon.^1$$

Now, if we take  $\mathbf{x}$  such that  $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ , we will have

$$\frac{\|f(\mathbf{a} + (\mathbf{x} - \mathbf{a})) - f(\mathbf{a}) - \lambda(\mathbf{x} - \mathbf{a})\|}{\|\mathbf{x} - \mathbf{a}\|} < \varepsilon.$$

□

## 2.3 The chain rule

**Theorem 2.3.1 (Chain rule).** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $\mathbf{a} \in \mathbb{R}^n$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^p$  be differentiable at  $f(\mathbf{a})$ . Then  $g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is differentiable at  $\mathbf{a}$  and we have

$$D(g \circ f)(\mathbf{a}) = Dg(f(\mathbf{a})) \circ Df(\mathbf{a}). \quad (2.3)$$

*Proof. To be done*

□

In [Spi18, Theorem 2-2 in p. 19], Equation (2.3) can seem a little confusing. Let us try to explain what is going on here. First of all, we use composition because  $Dg(f(\mathbf{a}))$  and  $Df(\mathbf{a})$  are linear transformations. Next, what is the equation saying in English? Well, what we are really saying here is that the derivative of the composition  $g \circ f$  at  $\mathbf{a}$  is nothing but the derivative of  $g$  at the point  $f(\mathbf{a})$  composed with the derivative of  $f$  at  $\mathbf{a}$ . To make this less confusing, let  $\mu = Dg(f(\mathbf{a}))$  and  $\lambda = Df(\mathbf{a})$ . Then, if we evaluate  $D(g \circ f)(\mathbf{a})(\mathbf{v})$ , we get  $\mu(\lambda(\mathbf{v}))$ .

Some other useful propositions regarding differentiability are listed below. Note that if we do not quantify over  $\mathbf{a}$ , it is any vector in  $\mathbb{R}^n$ .

<sup>1</sup>We remark that no absolute value signs are needed since the norm is always positive.

**Theorem 2.3.2 (Useful properties of differentiation).** 1. Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a constant function (that is, there is some  $\mathbf{y} \in \mathbb{R}^m$  such that  $f(\mathbf{x}) = \mathbf{y}$  for all  $x \in \mathbb{R}^n$ ). Then,

$$Df(\mathbf{a}) = 0$$

(Note here that 0 refers to the linear transformation which is always 0.)

2. Suppose  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a linear transformation. Then,

$$Df(\mathbf{a}) = f.$$

This is to say, the best linear approximation of a linear transformation is itself (unsurprisingly).

3. Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ . Then  $f$  is differentiable at  $\mathbf{a} \in \mathbb{R}^n$  if and only if each component function of  $f$ ,  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  ( $i = 1, \dots, m$ ) is differentiable at  $\mathbf{a}$ . Additionally, we have

$$Df(\mathbf{a}) = (Df_1(\mathbf{a}), \dots, Df_m(\mathbf{a})).$$

4. If  $s : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by  $s(x, y) = x + y$  then  $Ds((a, b)) = s$

5. If  $p : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by  $p(x, y) = xy$  then

$$Dp((a, b))(x, y) = bx + ay.$$

This means that  $\nabla p((a, b)) = (b, a)$ .

*Proof.* (1) and (2) follow quickly from the definition of the derivative.

(3) First suppose that  $f$  is differentiable at  $\mathbf{a} \in \mathbb{R}^n$ . Recall that the projection functions  $\pi_i$  are linear and thus are differentiable everywhere. By (2) we have  $D\pi_i(f(\mathbf{a})) = \pi_i$ . By Theorem 2.3.1 we have  $f_i = \pi_i \circ f$  being differentiable at  $f(\mathbf{a})$ . Now let us suppose that each component function of  $f$  is differentiable at  $\mathbf{a}$ . Let  $\lambda = (Df_1(\mathbf{a}), \dots, Df_n(\mathbf{a}))$ . To make this easier to think about,  $\lambda$  has a matrix given by

$$\mathcal{M}(\lambda) = \begin{bmatrix} \nabla f_1(\mathbf{a}) \\ \nabla f_2(\mathbf{a}) \\ \vdots \\ \nabla f_n(\mathbf{a}) \end{bmatrix},$$

where each row of  $\mathcal{M}(\lambda)$  is given by the vector  $\nabla f_i(\mathbf{a})$ . Now, we can write

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \lambda(\mathbf{h}) = (f_1(\mathbf{a} + \mathbf{h}) - f_1(\mathbf{a}) - Df_1(\mathbf{a})(\mathbf{h}), \dots, f_n(\mathbf{a} + \mathbf{h}) - f_n(\mathbf{a}) - Df_n(\mathbf{a})(\mathbf{h})).$$

Now, let's take a look at  $\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \lambda(\mathbf{h})\|$ . By Proposition 1.2.2 we have

$$\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \lambda(\mathbf{h})\| \leq \sum_{i=1}^n |f_i(\mathbf{a} + \mathbf{h}) - f_i(\mathbf{a}) - Df_i(\mathbf{a})(\mathbf{h})|.$$

This implies that

$$\frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \lambda(\mathbf{h})\|}{\|\mathbf{h}\|} \leq \sum_{i=1}^n \frac{|f_i(\mathbf{a} + \mathbf{h}) - f_i(\mathbf{a}) - Df_i(\mathbf{a})(\mathbf{h})|}{\|\mathbf{h}\|}.$$

Taking the limit  $\mathbf{h} \rightarrow \mathbf{0}$  on both sides, the right side is 0 and thus the left side must be 0 too.

(4) follows from (2) as  $s$  is easily seen to be linear.

(5) **To be done.** See [Spi18, Theorem 2-3 on p. 21]. □

**Corollary 2.3.3.** Suppose  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  are differentiable at  $\mathbf{a}$ . Then,

$$\begin{aligned} D(f + g)(\mathbf{a}) &= Df(\mathbf{a}) + Dg(\mathbf{a}), \\ D(f \cdot g)(\mathbf{a}) &= g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}). \end{aligned}$$

If  $g(\mathbf{a}) \neq 0$  then we also have

$$D(f/g)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{(g(\mathbf{a}))^2}.$$

*Proof. To be done* □

Note here  $f + g$  is a function defined by  $(f + g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$ . The sum on the right side is actually a sum of vectors. Additionally,  $f \cdot g$  is a function defined by  $(f \cdot g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ . Keep in mind that  $f(\mathbf{a}), g(\mathbf{a})$  are scalars.

A very useful thing to note is that the determinant function is actually differentiable. The following propositions from [Spi18, Problem 2-14, 2-15 on pp. 23, 24] can be used to deduce this fact.

**Proposition 2.3.4.** *Let  $E_i, i = 1, \dots, k$  be Euclidean spaces of various dimensions. That is to say,  $E_i = \mathbb{R}^{n_i}$  for some  $n_i$ . A function  $f : E_1 \times \dots \times E_k \rightarrow \mathbb{R}^p$  is multilinear if for each  $\mathbf{x}_j \in E_j$ , where  $j \neq i$ , the function  $g : E_i \rightarrow \mathbb{R}^p$  defined by  $g(\mathbf{v}) = f(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{v}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_k)$  is linear.*

Suppose  $f$  is multilinear, and  $i \neq j$ . Then if  $\mathbf{h} = (\mathbf{h}_1, \dots, \mathbf{h}_k)$  where  $\mathbf{h}_l \in E_l$ ,

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|f(\mathbf{a}_1, \dots, \mathbf{h}_i, \dots, \mathbf{h}_j, \dots, \mathbf{a}_k)\|}{\|\mathbf{h}\|} = 0.$$

Additionally, we have

$$Df(\mathbf{a}_1, \dots, \mathbf{a}_k)(\mathbf{x}_1, \dots, \mathbf{x}_k) = \sum_{i=1}^k f(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{x}_i, \mathbf{a}_{i+1}, \dots, \mathbf{a}_k).$$

*Proof. To be done* □

**Proposition 2.3.5.** *Let  $M$  be an  $n \times n$  matrix. Treat  $M$  as an element of  $\mathbb{R}^n \times \dots \times \mathbb{R}^n$  by considering each row of  $M$  as an element of  $\mathbb{R}^n$ . Then, the determinant function  $\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable and we have*

$$D(\det)(\mathbf{r}_1, \dots, \mathbf{r}_n)(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{i=1}^n \det \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{x}_i \\ \vdots \\ \mathbf{r}_n \end{pmatrix}.$$

Here, we have  $\mathbf{x}_i$  in the  $i$ th row and everywhere else remains  $\mathbf{r}_j$  when  $j \neq i$ .

*Proof.* The determinant function is multilinear, now apply the previous proposition. □

## 2.4 Multivariable MVT

Let us first begin by recalling the mean value theorem from single variable calculus.

**Theorem 2.4.1 (Mean Value Theorem).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then, there exists  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We omit the proof as this is a single variable calculus result.

**Theorem 2.4.2 (Multivar MVT).** Let  $U \subseteq \mathbb{R}^n$  and let  $\mathbf{a}, \mathbf{b} \in U$  be such that the graph of the function  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$  defined by  $\gamma(t) = (1-t)\mathbf{a} + t\mathbf{b}$  has  $\gamma(t) \in U$  for all  $t \in [0, 1]$ . If  $f : U \rightarrow \mathbb{R}$  is differentiable on  $U$  and  $f \circ \gamma$  satisfies the hypothesis of [Theorem 2.4.1](#), then there exists  $t_0 \in (0, 1)$  such that  $\mathbf{c} = \gamma(t_0)$  and we have

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \circ (\mathbf{b} - \mathbf{a}).$$

*Proof.* Not too hard, just apply [Theorem 2.3.1](#) and [Theorem 2.4.1](#). □

Note that we do need that the codomain of  $f$  to be  $\mathbb{R}$ . The next example shows this. See [[Zam24](#), Example 4.2.3, p. 226]

**Example 2.4.3.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  be defined by  $f(t) = (\cos t, \sin t)$ . Then  $f(0) = (1, 0)$  and  $f(\pi) = (-1, 0)$  but there is no  $c \in (0, \pi)$  such that  $f(\pi) - f(0) = Df(c) \cdot (\pi - 0)$ . This is because the equation means that  $(-2, 0) = \pi(-\sin c, \cos c)$ . But if  $\cos c = 0$  then  $-\pi \sin c$  is  $\pm 1$ . //

## 2.5 Partial derivatives

**Definition 2.5.1 (Partial derivative).** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\mathbf{a} \in \mathbb{R}^n$ . The  $i$ -th **partial derivative** of  $f$  at  $\mathbf{a}$  is the limit

$$\lim_{h \rightarrow 0} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{h}, \quad (2.4)$$

if it exists. We will denote this limit with  $\partial_i f(\mathbf{a})$ , or  $D_i f(\mathbf{a})$ .

The partial derivative is actually the usual single-variable calculus derivative of a certain function. The reader has probably observed that if we define a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  by  $g(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, a_n)$  then  $D_i f(\mathbf{a}) = g'(a_i)$ .

Partial derivatives are usually quite easy to calculate. For example, if we let  $f(x, y) = x^2 y + 4y$  then we have  $D_1 f(x_0, y_0) = 2x_0 y_0$  and  $D_2 f(x_0, y_0) = x_0^2 + 4$ .

The reader will wonder how does the derivative interact with partial derivatives.

**Theorem 2.5.2 (Components of gradient).** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and suppose  $f$  is differentiable at  $\mathbf{a}$ . Then, we have

$$\nabla f(\mathbf{a}) = (\partial_1 f(\mathbf{a}), \partial_2 f(\mathbf{a}), \dots, \partial_n f(\mathbf{a})).$$

*Proof. To be done* □

If we combine **Theorem 2.5.2** and part (3) of **Theorem 2.3.2**, we can obtain the following

**Corollary 2.5.3.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and suppose  $f$  is differentiable at  $\mathbf{a}$ . Then, we have

$$\mathcal{M}(Df(\mathbf{a})) = \begin{bmatrix} \nabla f_1(\mathbf{a}) \\ \nabla f_2(\mathbf{a}) \\ \vdots \\ \nabla f_m(\mathbf{a}) \end{bmatrix} = \begin{bmatrix} \partial_1 f_1(\mathbf{a}) & \cdots & \partial_n f_1(\mathbf{a}) \\ \partial_1 f_2(\mathbf{a}) & \cdots & \partial_n f_2(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(\mathbf{a}) & \cdots & \partial_n f_m(\mathbf{a}) \end{bmatrix}$$

Additionally, we may wonder whether the existence of partial derivatives implies the existence of the derivative. This is not true.

**Example 2.5.4.** Let

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

$f$  is certainly not continuous at  $(0, 0)$  and thus is not differentiable at  $(0, 0)$ . However, it does have partial derivatives at  $(0, 0)$ . //

**Theorem 2.5.5 (Continuous partials implies differentiable).** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and we have an open set  $U$  that contains  $\mathbf{a}$ , and for all  $\mathbf{x} \in U$ , for all  $i$ ,  $D_i f(\mathbf{x})$  exists, and the function  $D_i f$  which is  $\mathbf{x} \mapsto D_i f(\mathbf{x})$  is continuous at  $\mathbf{a}$ . Then  $Df(\mathbf{a})$  exists.

If  $f$  satisfies the hypothesis of theorem above, it is called **continuously differentiable** at  $\mathbf{a}$ . Such a function is also called a  $C^1$  function.

*Proof. To be done* □

This theorem can be easily generalized for  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

**Definition 2.5.6 (Class  $C^1$  function).** Let  $A \subseteq \mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}^m$ , and let  $\mathbf{a}$  be a point in the interior of  $A$ . Then  $f$  is **continuously differentiable** at  $\mathbf{a}$  or  $f \in C^1$  at  $\mathbf{a}$  if for all  $i \in \{1, \dots, n\}$ ,  $\partial_i f$  is defined on some neighborhood of  $\mathbf{a}$  and is continuous at  $\mathbf{a}$ .

### 2.5.1 Higher-order partial derivatives

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and fix some  $i \in \{1, \dots, n\}$ . If the partial derivative of  $f$  exists everywhere (i.e.  $D_i f(\mathbf{x})$  exists for all  $\mathbf{x} \in \mathbb{R}^n$ ), then we have a function  $D_i f : \mathbb{R}^n \rightarrow \mathbb{R}$  which maps a vector to the  $i$ -th partial derivative at that vector, i.e.

$$\begin{aligned} D_i f : \mathbb{R}^n &\rightarrow \mathbb{R} \\ \mathbf{x} &\mapsto D_i f(\mathbf{x}). \end{aligned}$$

One might want to take the partial derivatives of the function  $D_i f$ . For example, the  $j$ -th partial derivative of  $D_i f$  at  $\mathbf{x}$  would be  $D_j(D_i f)(\mathbf{x})$ . It is very possible that  $D_j(D_i f)(\mathbf{x})$  exists for all  $\mathbf{x} \in \mathbb{R}^n$  too, in this case we obtain a function

$$\begin{aligned} D_j(D_i f) : \mathbb{R}^n &\rightarrow \mathbb{R} \\ \mathbf{x} &\mapsto D_j(D_i f)(\mathbf{x}). \end{aligned}$$

This function is called a **second-order (mixed) partial derivative** of  $f$ . It's not too hard to define higher order partial derivatives. In [Spi18] this function is denoted  $D_{i,j} f(\mathbf{x})$ . This notation does reverse the order of  $i$  and  $j$ , but it turns out that for most functions, this does not matter. See [Theorem 2.5.7](#).

**Theorem 2.5.7 (Clairut's Theorem).** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and let  $i, j \in \{1, \dots, n\}$ . Suppose  $Df_j(Df_i)$  and  $Df_i(Df_j)$  are continuous in some open set that contains  $\mathbf{a}$ . Then,  $Df_j(Df_i)(\mathbf{a}) = Df_i(Df_j)(\mathbf{a})$ .

*Proof.* **To be done.** □

Note that the converse of this false. The next example taken from [Spi18, Problem 2-24 on p. 29] is a counterexample to the converse of [Theorem 2.5.7](#).

**Example 2.5.8.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

We leave it to the reader show that  $D_2 f(x, 0) = x$  for all  $x$ , and  $D_1 f(0, y) = -y$  for all  $y$ . (Apply the definition of partial derivative). Then  $D_{1,2} f(0, 0) \neq D_{2,1} f(0, 0)$ . //

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . If  $f$  is differentiable everywhere and has continuous partial derivatives, we say that  $f$  is  $C^1$ . Such a function  $f$  satisfies the hypothesis of [Theorem 2.5.7](#).

**Theorem 2.5.9.** Let  $A \subseteq \mathbb{R}^n$ . If the maximum or minimum of  $f : A \rightarrow \mathbb{R}$  occurs at a point  $\mathbf{a}$  in the interior of  $A$  and  $D_i f(\mathbf{a})$  exists, then  $D_i f(\mathbf{a}) = 0$ .

*Proof.* Define  $g_i : \mathbb{R} \rightarrow \mathbb{R}$  by  $g_i(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$ . Then  $g_i$  has a maximum or minimum at  $a_i$  as  $f$  has a maximum or minimum at  $\mathbf{a}$ . Now, since  $\mathbf{a}$  is in the interior of  $A$ , there is some open ball that contains  $\mathbf{a}$  and thus there is some open interval of  $a_i$  which  $g_i$  is defined on.  $g_i$  is also differentiable because  $g'_i(a_i) = D_i f(\mathbf{a})$ <sup>2</sup>. By single variable calculus, we know that  $g'_i(a_i) = 0$  as  $g(a_i)$  is the maximum or minimum. Now,  $D_i f(\mathbf{a}) = g'_i(a_i) = 0$ . □

As an immediate corollary, we get the following:

**Corollary 2.5.10.** If  $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $\mathbf{a}$  and  $\mathbf{a}$  is in the interior of  $A$ , then  $\nabla f(\mathbf{a}) = \mathbf{0}$ .

<sup>2</sup>Technically this justification is very handwavy. However it's not too hard to repair this, so we'll do it at a later time.

## 2.6 Directional derivatives

**Definition 2.6.1** (Directional derivative). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . The directional derivative of  $f$  at  $\mathbf{a}$  in direction  $\mathbf{u}$  is defined to be

$$\lim_{t \rightarrow 0} \frac{f(\mathbf{a} + t\mathbf{x}) - f(\mathbf{a})}{t},$$

if the limit exists. We denote this limit as  $D_{\mathbf{u}}f(\mathbf{a})$ .

While this definition does not impose any conditions on the direction vector  $\mathbf{u}$ , we usually ask for it to be a unit vector to make life easier. An immediate consequence of this definition is that the  $i$ -th partial derivative of  $f$  at  $\mathbf{a}$  can be thought of as the directional derivative of  $f$  at  $\mathbf{a}$  in the direction  $\mathbf{e}_i$ , where  $\mathbf{e}_i$  is the  $i$ -th standard basis vector of  $\mathbb{R}^n$ .

**Proposition 2.6.2.** *If  $\partial_i f(\mathbf{a})$  exists then  $\partial_i f(\mathbf{a}) = D_{\mathbf{e}_i}f(\mathbf{a})$ .*

*Proof.* Apply the definition. □

Again, the reader may wonder whether the existence of directional derivatives tells us anything about the derivative. It turns out that this is not true, even if every single directional derivative exists. The following example comes from [Spi18, Prob 1-26 on p. 13, Prob 2-31 on p. 33].

**Example 2.6.3.** Let  $A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } 0 < y < x^2\}$ . To see this set better, [view the plot of this set](#).

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A. \end{cases}$$

For any  $\mathbf{h} \in \mathbb{R}^2$  we define  $g_{\mathbf{h}} : \mathbb{R} \rightarrow \mathbb{R}$  by  $g_{\mathbf{h}}(t) = f(t\mathbf{h})$ . Then  $g_{\mathbf{h}}$  is continuous at 0, but  $f$  is not continuous at  $(0, 0)$ . To see this, notice that  $g_{\mathbf{h}}$  is a function that defines a line passing through the origin. Thus, there is some open interval around 0, call it  $I$ , such that if  $t \in I$ , we have  $t\mathbf{h} \notin A$ . //

**Example 2.6.4.** Let  $n \geq 1$  and let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = \begin{cases} \frac{x^n y}{x^{n+1} + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

Then  $f$  is not continuous at 0. To see this, approach along  $y = x^n$ . Approaching along this path leads to the limit at 0 being 1.

We do have every directional derivative though. //

## 2.7 Optimization

Let us first extend the definition of maximum and minimum points to  $\mathbb{R}^n$ .

**Definition 2.7.1** (Local extrema). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then if there is a neighborhood  $U$  of  $\mathbf{a}$  such that for all  $\mathbf{x} \in U$ , we have  $f(\mathbf{x}) \leq f(\mathbf{a})$ , then  $\mathbf{a}$  is called a local maximum. If it is the case that for all  $\mathbf{x} \in U$  we have  $f(\mathbf{x}) \geq f(\mathbf{a})$ , then  $\mathbf{a}$  is called a local minimum.

Points which are local maximum or minimum points are called *local extrema*.

**Definition 2.7.2** (Critical point). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable. Then  $\mathbf{c} \in \mathbb{R}^n$  is a critical point if  $\nabla f(\mathbf{c}) = 0$  or  $\nabla f(\mathbf{c})$  does not exist. Additionally, the value  $f(\mathbf{c})$  is called a *critical value*.

Note that if the domain of  $f$  is changed to be some subset of  $\mathbb{R}^n$ , we will require that  $\mathbf{c}$  be an interior point (otherwise we cannot discuss  $\nabla f(\mathbf{c})$ ).

**Theorem 2.7.3 (Local EVT).** Let  $A \subseteq \mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}$ . If  $\mathbf{a}$  is in the interior of  $A$  and  $\mathbf{a}$  is a local extremum then  $\nabla f(\mathbf{a}) = 0$  or it does not exist.

*Proof.* This actually follows as a corollary to [Theorem 2.5.9](#). If  $\mathbf{a}$  is a local extremum then it is either a maximum or minimum, so all  $\partial_i f(\mathbf{a})$  are zero. Thus if  $\nabla f(\mathbf{a})$  exists then we must have  $\nabla f(\mathbf{a}) = 0$ .  $\square$

As a consequence we note that if  $\mathbf{a}$  is a local extremum then it is either a boundary point or a critical point.

As an example of  $\nabla f$  being undefined at a critical point, we may consider the following [[Zam24](#), Example 4.3.13, p. 234].

**Example 2.7.4.** Define  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f(\mathbf{x}) = \|\mathbf{x}\|$ . Then,  $\nabla f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ . So  $\nabla f$  is not defined at the origin, but the origin of  $f$  is a critical point of  $f$ . Observe that  $\nabla f(\mathbf{x})$  is not 0 if  $\mathbf{x}$  is not zero, so  $f$  has no local extremum on  $\mathbb{R}^n \setminus \{0\}$ . Therefore the only possible local minimum of  $f$  is at  $0$ . It is also the local minimum of  $f$ . //

**Definition 2.7.5 (Saddle point).** Let  $A \subseteq \mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}$ . Then,  $\mathbf{a}$  is a **saddle point** if it is an interior point of  $A$ ,  $\nabla f(\mathbf{a}) = 0$  and  $\mathbf{a}$  is not a local extremum.

**Theorem 2.7.6 (Lagrange Multipliers).** Let  $f, G : \mathbb{R}^n \rightarrow \mathbb{R}$  be  $C^1$  functions. Let  $S = G^{-1}(0)$ . If the restriction of  $f$  to  $S$ ,  $f : S \rightarrow \mathbb{R}$  has a maximum or minimum at  $\mathbf{c} \in S$  and  $\nabla G(\mathbf{c}) \neq 0$ , then there exists  $\lambda \in \mathbb{R}$  such that

$$\nabla f(\mathbf{c}) = \lambda \nabla G(\mathbf{c}).$$

1. Find out whether global extrema exist. Usually this involves checking if the image of the function is compact.
2. Finding the critical points on the interior. This involves calculating the gradient of the function.
3. Checking the boundary for extrema. Usually you can parameterize the boundary. Compose the parameterization with the original function to obtain a single variable function which you can optimize using single variable calculus techniques.

## 2.8 Tangent vectors

**Definition 2.8.1 (Tangent Vector).** Let  $S \subseteq \mathbb{R}^n$  and  $\mathbf{p} \in S$ . Then  $\mathbf{v}$  is a **tangent vector** of  $S$  at  $\mathbf{p}$  if there is an open interval  $I \subseteq \mathbb{R}$  containing 0 and a function  $\gamma : I \rightarrow S$  such that  $\gamma(0) = \mathbf{p}$  and  $\gamma'(0) = \mathbf{v}$ .

Note that we technically do not need that the open interval contains 0, as any open interval whatsoever is diffeomorphic to an open interval containing 0. Additionally, the condition on the codomain of  $\gamma$  can be changed to  $\gamma : I \rightarrow \mathbb{R}^n$  and  $\gamma(I) \subseteq S$ .

**Definition 2.8.2 (Tangent space).** The set of all tangent vectors of  $S$  at  $\mathbf{p}$  is called **tangent space** of  $S$  at  $\mathbf{p}$ . It is denoted  $T_{\mathbf{p}}S$ .

**Example 2.8.3.** Let  $f(x, y) = 9 - x^2 - y^2$  and let  $S = \{(x, y, f(x, y)) : x, y \in \mathbb{R}\} = \Gamma(f)$ . //

**Definition 2.8.4 (Tangent plane).** The tangent plane of  $S$  at  $\mathbf{p}$  is denoted  $\mathbf{p} + T_{\mathbf{p}}S$ .

$$\mathbf{p} + T_{\mathbf{p}}S = \{\mathbf{p} + \mathbf{v} : \mathbf{v} \in T_{\mathbf{p}}S\}.$$

If  $S$  is a graph of some  $C^1$  function the tangent space is a vector space and is easy to calculate.



**Theorem 2.8.5.** Let  $V \subseteq \mathbb{R}^k$  be an open set and  $F : V \rightarrow \mathbb{R}^{n-k}$  be a  $C^1$  function. Let  $S = \Gamma(F)$ , that is

$$S = \{ (\mathbf{v}, F(\mathbf{v})) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \mathbf{v} \in V \}.$$

Then for any  $\mathbf{a} \in V$ ,  $\mathbf{p} = (\mathbf{a}, F(\mathbf{a})) \in S$ , the tangent space of  $S$  at  $\mathbf{p}$  satisfies

$$T_{\mathbf{p}}S = \{ (\mathbf{w}, D_{\mathbf{a}}F(\mathbf{w})) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : \mathbf{w} \in \mathbb{R}^k \}.$$

Thus  $T_{\mathbf{p}}S$  is a  $k$ -dimensional vector space spanned by  $\{(\mathbf{e}_i, \partial_i F(\mathbf{a}))\}$  where  $i \in \{1, \dots, k\}$  and  $\mathbf{e}_i$  are the standard basis vectors of  $\mathbb{R}^k$ .

## 2.9 Smooth Manifolds

**Definition 2.9.1** (Smooth Manifold at a point  $\mathbf{p}$ ). Let  $S \subseteq \mathbb{R}^n$  and  $\mathbf{p} \in \mathbb{R}^n$ . Then  $S$  is a  $k$ -dimensional **smooth manifold at  $\mathbf{p}$**  if there is an open neighborhood of  $\mathbf{p}$  and a  $C^1$  function  $f : V \rightarrow \mathbb{R}^n$  where  $V$  is open such that  $S \cap U$  is the graph of  $f$ .

**Example 2.9.2** (The circle). Let  $S^1 = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$  be the unit circle. Then  $S^1$  is a smooth manifold at any point. We leave the reader to verify this fact. Notice that points  $(x, y)$  such that  $x^2 + y^2 = 1$  lie in  $S^1$ , and these are the only points. So write  $y$  in terms of  $x$ . This actually hints to [Theorem 2.10.1](#). //

**Example 2.9.3** (The real numbers).  $\mathbb{R}$  is easily seen to be a  $n$ -dimensional smooth manifold as it is the graph of the identity function from  $\mathbb{R}$  to  $\mathbb{R}$ . //

**Definition 2.9.4** (Smooth Manifold). Let  $S \subseteq \mathbb{R}^n$ . Then  $S$  is a  $k$ -dimensional smooth manifold if it is a  $k$ -dimensional smooth manifold at every point  $\mathbf{p} \in S$ .

Note that in both [Definition 2.9.4](#) and [Definition 2.9.1](#), we require that the dimension  $k$  of the manifold be lower than the dimension of the space it sits in, i.e.  $k < n$ .

Now for some non-examples of smooth manifolds.

**Example 2.9.5** (A cusp). This comes from [[Zam24](#), Example 4.6.10, p. 268]. Consider the curve determined by  $x^2 = y^3$  (see [Figure 2.1](#)).

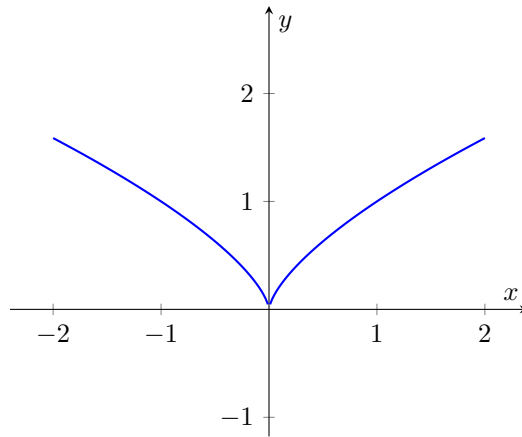
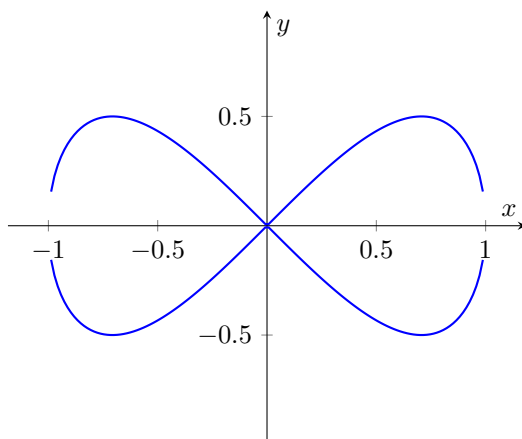


Figure 2.1: Graph of the cusp  $x^2 = y^3$

This is not a smooth manifold because it is not a smooth manifold at the origin. Notice that any ball around the origin is not the graph of a  $C^1$  function, as the function  $f(x) = x^{2/3}$  is not differentiable at the origin. A more careful proof of this fact can be found in [[Zam24](#), Example 4.6.11, p. 268]. //

**Example 2.9.6** (Figure 8). This example comes from [[Zam24](#), Example 4.6.10, p. 268] as well. Consider the figure 8 shape determined by the equation  $x^4 = x^2 - y^2$  (see [Figure 2.2](#)). Any ball at the origin intersected with this shape cannot produce a graph, since for each value of  $x$ , you need to have 2 different  $y$  values. If  $x$  is treated as a function of  $y$  instead, then 2 different  $x$  values need to be provided for a single  $y$  value. Consequently  $S$  cannot be a graph at the origin so it cannot be a smooth manifold at the origin. //

Figure 2.2: Graph of the figure 8 shape  $x^4 = x^2 - y^2$ 

Here are some non-examples of smooth manifolds in 2 dimensions. **TODO: add these**

### 2.9.1 Diffeomorphisms

Recall from linear algebra that a vector space isomorphism is a bijective linear transformation. Vector space isomorphisms preserve all the structure we care about in a vector space (algebraically speaking). A diffeomorphism is an isomorphism, but for smooth manifolds instead. It preserves the properties we care about for smooth manifolds.

**Definition 2.9.7 (Diffeomorphism).** Let  $U, V \subseteq \mathbb{R}^n$  be open sets. Then  $f : U \rightarrow V$  is a **diffeomorphism** if  $f$  is bijective,  $C^1$  and  $f^{-1} : V \rightarrow U$  is also a  $C^1$  function.

Note that the condition of  $f^{-1}$  being  $C^1$  is absolutely necessary since  $f$  being  $C^1$  does not guarantee that  $f^{-1}$  is  $C^1$ .

**Example 2.9.8.** Let  $f(x) = x^3$ . Then  $f^{-1}(y) = \sqrt[3]{y}$ .  $f^{-1}$  is differentiable, but its derivative is discontinuous at 0. For more details, see [Zam24, Example 5.5.5, p. 326]. //

**Proposition 2.9.9.** Let  $U, V \subseteq \mathbb{R}^n$  be open and  $f : U \rightarrow V$  be a diffeomorphism. Then, the following are true:

1.  $O \subseteq U$  is open (closed) iff  $f(O)$  is open (closed).
2.  $K \subseteq U$  is compact iff  $f(K)$  is compact.
3.  $S \subseteq U$  is (path)-connected iff  $f(S)$  is (path)-connected.

*Proof.* A diffeomorphism is a homeomorphism. □

**Proposition 2.9.10.** Let  $f : U \rightarrow V$  be a diffeomorphism and  $S \subseteq U$ ,  $\mathbf{p} \in S$ . Then  $\mathbf{v} \in T_{\mathbf{p}}S$  if and only if  $\mathbf{v} \in T_{f(\mathbf{p})}f(S)$ .

*Proof.* Apply the definition. You will need to use the chain rule (Theorem 2.3.1) in the proof. □

**Definition 2.9.11 (Local Diffeomorphism).** Let  $A, B \subseteq \mathbb{R}^n$  be open and  $\mathbf{a} \in A$ . Let  $f : A \rightarrow B$ . Then  $f$  is a **local diffeomorphism** at  $\mathbf{a}$  if there is an open set  $U \subseteq A$  containing  $\mathbf{a}$  such that  $f|_U : U \rightarrow f(U)$  is a diffeomorphism. The inverse  $f|_U^{-1}$  is called a *local inverse* for  $f$  at  $\mathbf{a}$ .

Clearly every global diffeomorphism is a local diffeomorphism, however the converse is untrue. Even if a function is a local diffeomorphism at every point, it may still not be a global diffeomorphism.

**Example 2.9.12.** Let  $f(x, y) = (e^x \cos y, e^x \sin y)$ . Then  $Df(x, y)$  is invertible for all  $(x, y)$  so  $f$  is a local diffeomorphism at every point in  $\mathbb{R}^2$ . However notice that  $f(x, y) = f(x, y + 2\pi)$  so  $f$  is not injective and thus is not a diffeomorphism. //

**Proposition 2.9.13.** *If  $f$  is a local diffeomorphism at  $\mathbf{a}$ , then  $Df(\mathbf{a})$  is invertible and satisfies*

$$Df^{-1}(f(x)) = (Df(x))^{-1}.$$

*Proof.* Apply the chain rule. □

The converse of this proposition is the inverse function theorem ([Theorem 2.10.2](#)). Thus we summarize that  $f$  is a local diffeomorphism at  $\mathbf{a}$  if and only if  $Df(\mathbf{a})$  is invertible.

## 2.10 Implicit and inverse functions

**Theorem 2.10.1 (Implicit function theorem).** Let  $F : \mathbb{R}^{n+k} \times \mathbb{R}^k$ ,  $F(\mathbf{x}, \mathbf{y})$  be  $C^1$  on some neighborhood  $U \subseteq \mathbb{R}^{n+k}$  of the point  $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+k}$  and suppose that  $F(\mathbf{a}, \mathbf{b}) = 0$  and  $\partial F_{\mathbf{y}}(\mathbf{a}, \mathbf{b})$  is invertible. Then there is an open ball centered at  $\mathbf{a}$  of radius  $r$  and a *unique*  $C^1$  function  $f : B(\mathbf{a}, r) \rightarrow \mathbb{R}^k$  such that  $F(\mathbf{x}, f(\mathbf{x})) = 0$  for all  $x \in B(\mathbf{a}, r)$ .

Note that

$$\partial F_{\mathbf{y}} = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial y_1} & \cdots & \frac{\partial F_k}{\partial y_k} \end{bmatrix},$$

since we have

$$Df = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n} & \frac{\partial F_k}{\partial y_1} & \cdots & \frac{\partial F_k}{\partial y_k} \end{bmatrix}$$

*Proof.* Proof from scratch To be done. □

*Proof using [Theorem 2.10.2](#).* To be done. □

**Theorem 2.10.2 (Inverse function theorem).** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be continuously differentiable in an open set containing  $\mathbf{a}$  and suppose  $Df(\mathbf{a})$  is invertible. Then, there is some open set  $V \subseteq \mathbb{R}^n$  that contains  $\mathbf{a}$  and an open set  $W \subseteq \mathbb{R}^n$  that contains  $f(\mathbf{a})$  such that  $f : V \rightarrow W$  has a continuous inverse  $f^{-1} : W \rightarrow V$ , which is differentiable and for all  $\mathbf{y} \in W$ , we have

$$D(f^{-1})(\mathbf{y}) = [Df(f^{-1}(\mathbf{y}))]^{-1}.$$

*Proof from scratch.* Very long. See [[Spi18](#), Thm 2-11]. □

*Proof using [Theorem 2.10.1](#).* Define  $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $F(\mathbf{x}, \mathbf{y}) = \mathbf{y} - f(\mathbf{x})$ . **To be completed.** □

# Bibliography

- [Spi18] Michael Spivak. *Calculus on manifolds: a modern approach to classical theorems of advanced calculus*. eng. Mathematics monograph series. Boca Raton London New York: CRC Press, Taylor & Francis Group, 2018. ISBN: 9780805390216.
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