Week 8 Notes

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1 Connected components

Definition 1.1 (Connected components)**.** Let X be a topological space. Define the equivalence relation $\sim_C$ on X by $x \sim_C y$ if there exists a connected subspace $A \subseteq X$ such that $x, y \in A$.

An equivalence class under $\sim_C$ is called a **component**.

The set of equivalence classes under $\sim_C$ is called the *components* of X. The definition for path components is similar. Note that the empty topological space has no components or path components.

Definition 1.2 (Path components)**.** Let X be a topological space. Define the equivalence relation $\sim_P$ on X by $x \sim_P y$ if there exists a path from x to y.

An equivalence class under $\sim_P$ is called a **path component**.

Proposition 1.3 (Properties of connected components)**.** *Let X be a topological space. Then, the following are true.*

1. *Components form a partition of X*
2. *Components are connected*
3. *Connected[1] subspaces of X intersect at most one component.*
4. *Components are closed.*

Proof. (1) follows immediately since $\sim_C$ is an equivalence relation.

(2) Suppose $[x]_{\sim_C} = C \cup D$ where C, D are disjoint open sets. Suppose that $x \in C$. Let $y \in [x]_{\sim_C}$ be arbitrary. Since $x \sim_C y$, there is a connected subspace A such that $x, y \in A$. Since C and D are disjoint, $A \subseteq C$, so $y \in C$. Since y is arbitrary, every point of $[x]_{\sim_C}$ is contained in C, so D is empty. So there is no separation of $[x]_{\sim_C}$.

(3) Let A be a nonempty connected subspace of X and let $x \in A$. Then, A intersects $[x]_{\sim_C}$. If $y \in A$ then by definition of $\sim_C$ we have $x \sim_C y$, so $y \in [x]_{\sim_C}$. Thus every point in A is contained in $[x]_{\sim_C}$.

(4) Given a component $E = [x]_{\sim_C}$, then the closure $\overline{E}$ is also connected, and $\overline{E}$ intersects at most one component by (3). Since it intersects E, it is contained in E and thus equal to E. □

We remark that part (3) is actually stronger than stated. If A is a connected subspace of X that intersects a component C then A is completely contained in C.

Note that (4) does not hold for path components as the closure of a path-connected space is not necessarily connected (topologist's sine curve, see last week's notes).

Proposition 1.4 (Properties of path components)**.** *Let X be a topological space. Then, the following are true.*

1. *Path form a partition of X*
2. *Path components are path conenected*
3. *Path connected subspaces of X intersect at most one component.*

[1]These have to be nonempty

4. Each path component is contained in a component.

Proof. (1) Follows from definition of $\sim_P$.

(2) Given $y, z \in [x]_{\sim_P}$, there is a path from y to z by definition of $\sim_P$ so it is path-connected.

(3) This is a similar argument to Proposition 1.3 part (3). If A is a path-connected subspace of X and $x \in A$ then A intersects $[x]_{\sim_P}$. If y is any point in A then there is a path from x to y since A is path-connnected, so it follows that $y \sim_P x$.

(4) It is clear that if $x \sim_P y$ then $x \sim_C y$, since if p is a path from x to y, then x, y are elements of the image of the path, which is connected. (Recall that the domain of a path is a closed interval which is connected.) The result follows by thinking about the definition of an equivalence class. □

A few remarks are in order. Components are maximal connected subsets of X. This means that if C is a component and A is a connected subspace of X that contains C, then $A = C$. This is actually equivalent to Definition 1.1. Note that if X is connected then X only has a single component, namely X. A similar proposition can be formulated for path components.

2 Local connectedness

Definition 2.1 (Local basis). Let X be a topological space. A **local basis** around a point $x \in X$ is a collection $\mathcal{B}$ of *open* neighborhoods of x such that given any open neighborhood U of x, there is a $B \in \mathcal{B}$ such that $B \subseteq U$.

It is important to note that the neighborhoods in the local basis must be open.

Example 2.2 (Local basis of a point in $\mathbb{R}$)**.** Let $x \in \mathbb{R}$ be any point and consider the collection of ε balls around x,

$$\mathcal{B} = \{ (x - \varepsilon, x + \varepsilon) : \varepsilon \in \mathbb{R}, \varepsilon > 0 \}.$$

Then this forms a local basis of x. //

Definition 2.3 (Local connectedness). A topological space is **locally connected** if given any point $x \in X$ there is a local basis at x of connected sets.

Note that this is not the same as saying that given a point x it has a connected neighborhood.

Proposition 2.4 (Equivalent definition of local connectedness)**.** *A space X is locally (path)-connected if and only if given any open subset $U \subseteq X$, every (path) component of U is open in X.*

Proof. Let us prove it first for connectedness. First suppose X is locally connected. Let U be an open subset of X. Let $x \in U$ and let C be the[2] component of U that contains x. Let $c \in C$ be a point, we shall prove that there is an open neighborhood[3] of c that is completely contained in C. Since X is locally connected, there is a local basis around x of connected sets, call it $\mathcal{B}$. Then there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$ by definition. As B is connected, $B \subseteq C$ by Proposition 1.3 (3). This completes the proof of the forward direction.

Now suppose that given any open set U of X every component of U is open in X. Let $x \in X$. For each open neighborhood U of x let C_U be the component of U that contains x. Keep in mind that C_U is open in X and $x \in C_U \subseteq U$. Then let $\mathcal{B}$ be the set of all such C_U's, one for every open neighborhood U of X. To see this, note that given an open neighborhood V of x, C_V is open in X and $x \in C_V \subseteq V$. But this is exactly the definition of local connectedness. □

The proof for path components is similar.

A nice property of a space X being locally path connected is that there is no difference between looking at the components of X and the path components of X. They are the same set.

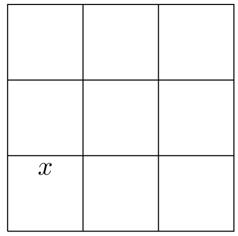
Corollary 2.5 (Components equal path components if locally connected)**.** *If X is locally path connected, then every component is a path component and conversely. Therefore, the set of components and the set of path components of X are equal.*

[2] A point can be contained in precisely one component since equivalence classes are equal or disjoint

[3] This neighborhood is open in X

Proof. Let C be a component and let $x \in C$. Then $C = [x]_{\sim_P} \cup Q$, where $Q = \bigcup_{y \in C, y \neq x} [y]_{\sim_P}$. So Q is the union of all the other path components of C. By Proposition 2.4, as each $[y]_{\sim_P}$ is open, Q is open. If Q is not empty, then $[x]_{\sim_P}$ and Q separate C, which contradicts Q being connected. $\square$

We remark that a space being connected does not mean it is locally connected.



3 Filters and Ultrafilters

Definition 3.1. A collection $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a **filter** if

1. $\mathbb{N} \subseteq \mathcal{F}$, and $\varnothing \notin \mathcal{F}$. A filter does *not* contain the empty set.
2. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ (closed under *finite* intersections).
3. If $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$ (closed upwards).

An **ultrafilter** is a filter which if $A \subseteq \mathbb{N}$, then either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A$ in $\mathcal{F}$.

So an ultrafilter is a maximal filter, that is, there is no filter which properly contains it (besides $\mathcal{P}(\mathbb{N})$). Note that a filter cannot contain both A and $\mathbb{N} \setminus A$, for if it does, then it contains their intersection, which is the empty set, violating condition (1).

Example 3.2 (Trivial filter)**.** The set $\{\mathbb{N}\}$ forms a filter. //

Example 3.3 (Filter generated by A)**.** Let $A \subseteq \mathbb{N}$. Define $\mathcal{F}_A = \{ B \subseteq \mathbb{N} : A \subseteq B \}$. //

Note that I made up the name of this example. I'm not sure if this is actually what it is called.

Example 3.4 (Frechet filter)**.** Define

$$\mathrm{Fr} = \{ A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite } \}.$$

This is called the *Frechet filter*. It contains all subsets of the naturals with finite complement. Note that this is not an ultrafilter, since the set of even numbers is not cofinite, and their complement is also not cofinite. //

Example 3.5 (The power set is not a filter)**.** The power set cannot be a filter, as it contains the empty set. //

Example 3.6 (Principal ultrafilters.)**.** Let $n \in \mathbb{N}$ be fixed. Let

$$\mathcal{U}_n = \{ A \subseteq \mathbb{N} : n \in A \}.$$

This is called a **principal ultrafilter**. So $\mathcal{U}_n$ is the set of all subsets of the naturals that contain n. This is a filter, and it is an ultrafilter, since if S does not contain n then its complement contains n. //

Remark 3.7. It is not possible to construct a ultrafilter which is not principal without the axiom of choice.

Theorem 3.8 (Ultrafilter Theorem (Tarski, 1930))**.** Every filter is contained in an ultrafilter.

Proof. Let $\mathcal{F}$ be a filter. Consider the set

$$\mathbb{P}_{\mathcal{F}} = \{ G \subseteq \mathcal{P}(\mathbb{N}) : \ G \text{ is a filter and } \mathcal{F} \subseteq G \ \}.$$

Order $\mathbb{P}_{\mathcal{F}}$ by subset inclusion. We shall use Zorn's lemma to extract a maximal element (and we shall prove that is an ultrafilter). First of all $\mathbb{P}_{\mathcal{F}}$ is nonempty, as $F \in \mathbb{P}_{\mathcal{F}}$. Now, let $\mathcal{C} \subseteq \mathbb{P}_{\mathcal{F}}$ be a chain. We shall prove that $\mathcal{C}$ is bounded. Let $\mathcal{H} = \bigcup_{G \in \mathcal{C}} G$, the union of all the filters in $\mathcal{C}$.

We claim that $\mathcal{H}$ is a filter, and thus it is a bound for $\mathcal{C}$, since every filter in $\mathcal{C}$ will be contained in $\mathcal{H}$. Clearly, $\mathbb{N} \in \mathcal{H}$ as $\mathbb{N} \in \mathcal{F} \subseteq \mathcal{H}$. If $\varnothing \in \mathcal{H}$ then $\varnothing \in G$ where $G \in \mathcal{C}$, but this is not possible as G is a filter.

Now, let $A, B \in \mathcal{H}$. Then, there are filters $G_A, G_B \in \mathcal{C}$ such that $A \in G_A$ and $B \in G_B$. Since $\mathcal{C}$ is a chain, either $G_A \subseteq G_B$ or $G_B \subseteq G_A$. Without loss of generality say $G_A \subseteq G_B$. Then, both A, B are in G_B, so $A \cap B \in G_B \subseteq \mathcal{H}$.

Let $A \in \mathcal{H}$ and suppose $B \supseteq A$. By definition of $\mathcal{H}$ there is a filter $G_A \in \mathcal{C}$ such that $A \in G_A$, then $B \in G_A$ so $B \in \mathcal{H}$.

This shows that an arbitrary chain $\mathcal{C}$ is bounded. Thus, by Zorn's lemma, there is a maximal (for $\subseteq$) $U \in \mathbb{P}_{\mathcal{F}}$. We shall show that U is an ultrafilter. Suppose not. Then, there is $A \subseteq \mathbb{N}$ such that $A \notin U$ and $\mathbb{N} \setminus A \notin U$. We claim that there is a filter containing $U \cup \{ A \}$. To see this, let **TODO: finish proof.** $\square$

We shall now use this construct a universal[4] compact Hausdorff space from X, given a topological space X.

Example 3.9 (Stone-cech compactification of $\mathbb{N}$)**.** Define

$$\beta\mathbb{N} = \{ U \subseteq \mathcal{P}(\mathbb{N}) : U \text{ is an ultrafilter } \}.$$

Give $\beta\mathbb{N}$ the topology which is generated by sets $[A]$[5]. To be more precise, we shall define $[A]$ like so. Fix $A \subseteq \mathbb{N}$, let $[A] = \{ U \in \beta\mathbb{N} : A \in U \}$. Notice $[A] \subseteq \beta\mathbb{N}$, so $[A]$ is the set of all ultrafilters on the naturals that contain A as an element. Then, define

$$\mathcal{B} = \{ [A] : A \in \mathcal{P}(\mathbb{N}) \}.$$

We claim that $\mathcal{B}$ is a basis for a topology on $\beta\mathbb{N}$. To see this, observe that $[\mathbb{N}] = \beta\mathbb{N}$ and $[\varnothing] = \varnothing$. Now let $[A], [B] \in \mathcal{B}$. Let $U \in [A] \cap [B]$. So U is an ultrafilter that contains both A and B. Thus, U contains $A \cap B$, so $U \in [A \cap B] = [A] \cap [B]$. To see why $[A \cap B] = [A] \cap [B]$, notice that if G is an ultrafilter such that $A \cap B \in G$, then $A \supseteq A \cap B$ so $A \in G$, and likewise $B \in G$. If $G \in [A] \cap [B]$, then $A, B \in G$ so $A \cap B \in G$ as G is a filter.

Notice that $\beta\mathbb{N}$ is **zero-dimensional**, meaning it has a basis which consists of clopen sets. Given $[A] \in \mathcal{B}$ (which is open), $\beta\mathbb{N} \setminus [A] = [\mathbb{N} \setminus A]$, which is open, so $[A]$ is closed too. //

Note that this technique really only works on discrete spaces.

[4] Ask me directly if you want to find out what universal means.

[5] This has nothing to do with equivalence classes.