Week 11

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List of Definitions

List of Propositions

1 Separation axioms

Here, the word separation shall mean that we can put disjoint open neighborhoods around things.

Definition 1.1 (T_1 space). A space X is T_1 if points are closed.

Definition 1.2 (T_2 space). A space is T_2 if it is Hausdorff. Recall that a space is Hausdorff if you can separate points with disjoint closed sets.

Definition 1.3 (T_3 space). A space is **regular** or T_3 if it is T_1 and given a point x and a closed set C such that $x \notin C$, there is a neighborhood U of x and $V \supseteq C$ such that U, V are disjoint.

In other words, a space is T_3 if we can separate points from closed sets.

Definition 1.4 ($T_3.5$ space). A space is **completely regular** if it is T_1 and given a point x and a closed set C not containing x, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f[C] = \{ 1 \}$.

Definition 1.5 (T_4 space). A space is **normal** if it is T_1 and given disjoint closed sets C, D, there exists disjoint open sets U, V such that $C \subseteq U$ and $D \subseteq V$.

Clearly normal implies regular implies Hausdorff. A completely regular space is also regular. Normal also implies completely regular, but this is nontrivial (Theorem 1.15).

Proposition 1.6 (Completely regular implies regular). *A completely regular space is regular.*

Proof. Suppose X is completely regular. Let x be a point and A a closed set not containing x. Then, there is a function $f : X \to [0,1]$ such that $f(x) = 0$ and f is 1 on A. Now preimage some disjoint open neighborhoods of 0 and 1. ☐

The implications are not reversible.

Example 1.7 (T_1 but not Hausdorff). Let $X = \mathbb{R}$ with the cofinite topology. Then X is clearly T_1 since singletons are finite and are thus closed. It is easily seen to be not Hausdorff. //

Example 1.8 (Hausdorff but not regular). Let $X = \mathbb{R}_K$ be the K-topology on the reals. However, it is not regular since we cannot separate the set K from the point 0. //

Lemma 1.9 (Equivalent condition to regularity). *Let X be a topological space. Then X is regular if and only if for every $x \in X$ and neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.*

Proof. For the forward direction, apply the definition of regularity to the closed set $X \setminus U$. For the converse direction, if C is a closed set not containing x, then $X \setminus C$ is a neighborhood of x. ☐

Lemma 1.10 (Equivalent condition to normality). *Let X be a topological space. Then X is normal if and only if given a closed set $C \subseteq X$ and an open set $U \supseteq C$, there exists an open set V such that $C \subseteq V$ and $\overline{V} \subseteq U$.*

Proof. Same idea as Lemma 1.9. ☐

Proposition 1.11 (Regularity of subspace and products).

1. *A subspace of a (completely) regular space is (completely) regular.*
2. *Products of (completely) regular space is (completely) regular.*

Proof.

1. Let X be regular, and $Y \subseteq X$. Let $y \in Y$ and $A \subseteq Y$ be closed with $y \notin A$. Then there is some $C \subseteq X$ closed such that $A = C \cap Y$. Clearly $x \notin C$ so we may find disjoint U, V containing x and C respectively. Then $U \cap Y$, $V \cap Y$ are the neighborhoods as desired. If X is completely regular instead, let $f : X \to [0,1]$ be a function that separates y and C. The restriction of f to Y is still continuous and has the desired property.

2. Let X_α, $\alpha \in \Lambda$ be a collection of regular spaces. Set $X = \prod_{\alpha\in\Lambda} X_\alpha$. Given a point $\mathbf{x} = \langle x_\alpha : \alpha \in \Lambda \rangle \in X$, let $U = \prod_{\alpha\in\Lambda} U_\alpha$ be an open neighborhood of $\mathbf{x}$, so that $U_\alpha = X_\alpha$ for all but finitely many α. For those U_α which are not equal to X_α, we may find V_α such that $x_\alpha \in V_\alpha$ and $\overline{V_\alpha} \subseteq U_\alpha$ (by Lemma 1.9). For those U_α which are equal to X_α, set V_α to be X_α. Then $V := \prod_{\alpha\in\Lambda} V_\alpha$ has closure contained in U, since $\overline{\prod_{\alpha\in\Lambda} V_\alpha} = \prod_{\alpha\in\Lambda} \overline{V_\alpha}$.

   Now suppose the X_α's are now completely regularly. Let A be closed in X and not containing x. Let $U \subseteq X$ be a basic open neighborhood of $\mathbf{x}$ that is disjoint from A. (This uses regularity of X_α, but finitely many of them). Now, U is the product of open sets U_α in X_α's. Then there are U_α's which are not X_α's, finitely many of them, say $\alpha_1, \dots, \alpha_n$. For each α_i, let $f_i : X_{\alpha_i} \to [0,1]$ be a continuous function such that $f_i(x_{\alpha_i}) = 0$, and f is 1 on the set $X_{\alpha_i} \setminus U_{\alpha_i}$. Define g_i to be $f_i \circ \pi_{\alpha_i}$ and let $g(x) = \prod_{i\le n} g_i(x)$. Then g is the desired function.

□

A similar proposition is not true for normal spaces.

Example 1.12 (Product of normal may not be normal)**.** Let $X = \mathbb{R}_l$ be the Sorgenfrey line. Then X is normal. Let A, B be disjoint closed sets. For $a \in A$, let $[a, x_a) \subseteq X$ be disjoint from B. This is possible because $X \setminus A$ is open. We repeat the same trick for each $b \in B$, letting $[b, x_b) \subseteq X$ be disjoint from A. Now, $A \subseteq \bigcup_{a\in A}[a, x_a)$ and $B \subseteq \bigcup_{b\in B}[b, x_b)$. We claim that those unions are disjoint. Otherwise, let $a \in A, b \in B$ such that $[a, x_a) \cap [b, x_b) \neq \varnothing$. WLOG let us assume $a < b$, so this means $x_a > b$. But then $b \in [a, x_a)$, oops.

However, $X \times X$ is not normal. Let L be the line consisting of the points $\langle x, -x \rangle$. This set is closed[1], and it is discrete, so every subset of L is closed in $X \times X$. For all $A \subseteq L$, we have disjoint open sets U_A, V_A of $X \times X$, such that $A \subseteq U_A$, $L \setminus A \subseteq V_A$. Let us now define a function $f : \mathcal{P}(L) \to \mathcal{P}(\mathbb{Q}^2)$ as follows:

$$\begin{aligned} f(\varnothing) &= \varnothing, \\ f(L) &= \mathbb{Q}^2, \\ f(A) &= U_A \cap \mathbb{Q}^2 && \text{if } \varnothing \subset A \subset L. \end{aligned}$$

We claim that f is injective. To see this, let A be a subset of L. If A is nonempty, then U_A is nonempty. As $\mathbb{Q}^2$ is dense in $X \times X$, $f(A)$ is nonempty. If A is a proper subset of L, then $L \setminus A$ is nonempty, and so V_A is nonempty. Thus $f(A)$ is not all of $\mathbb{Q}^2$. Now suppose A, B are subsets of L and $A \neq B$. WLOG let $x \in A \setminus B$. Thus $x \in U_A$ and $x \in V_B$, so $U_A \cap V_B$ is a nonempty open set, thus it contains an element of $\mathbb{Q}^2$, say q. Then $q \in f(A) \setminus f(B)$, so $f(A) \neq f(B)$.

However, f cannot be injective, since the cardinality of $\mathcal{P}(L)$ is strictly bigger than the cardinality of $\mathcal{P}(\mathbb{Q}^2)$.

See [Mun00, Example 3, p. 198] for a full exposition. //

Example 1.13 (Subspaces of normal need not be normal)**.** The following example is called Tychonoff's plank (see [SS78, Example 87, p. 106]). Let us take $X = \alpha\omega_1 \times \alpha\omega$. Recall that αX of a locally compact Hausdorff space X is the one-point compactification of it. Clearly X is compact and Hausdorff. Now, let us set Y to be $X \setminus (\infty_{\alpha\omega_1}, \infty_{\alpha\omega})$, so we remove the point added by the one point compactification. Let $A = \omega_1 \times \{ \infty_{\alpha\omega} \}$, let $B = \{ \infty_{\alpha\omega_1} \} \times \omega$. //

The following proposition gives sufficient conditions for normality. Similar theorems can be seen in [Mun00, Chp. 32, pp. 198–203].

Proposition 1.14 (Sufficient conditions for normality)**.** *Let X be space. Then X is normal if at least one of the conditions are satisified:*

1. *X is regular and Lindelof;*
2. *X is compact and Hausdorff;*
3. *X is metrizable;*

[1] Previously proven

4. *X is a linearly ordered space (with the order topology).*

Proof. (2) See Proposition 5.4 in Week 8 Notes.

(3) To do this, we shall use the converse of Theorem 1.15. Let A, B be disjoint closed sets in X. Define

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)},$$

where $d(x, A) = \inf \{ d(x, a) : a \in A \}$. It is easy to see that f is continuous, and that f is 0 on A and 1 on B. $\square$

Theorem 1.15 (Urysohn's Lemma). Let X be a normal topological space and A, B be disjoint closed sets in X. Then, there exists a continuous function $f : X \to [0, 1]$ such that f is 0 on A and f is 1 on B.

Proof. We shall construct for each rational number r, an open set U_r. These sets will have the following properties:

1. $U_r = \varnothing$ if $r < 0$, $U_r = X$ if $r > 1$;
2. $U_0 \supseteq A$ and $U_1 = X \setminus B$;
3. If $p < q$, then $\overline{U}_p \subseteq U_q$.

Let us first define $U_1 = X \setminus B$, and U_r for the cases in (1). To find U_0, we apply Lemma 1.10 to find U_0 such that $A \subseteq U_0$ and $\overline{U}_0 \subseteq U_1$, since U_1 contains A. Thus we have satisified condition (2).

At this point, it is unclear how we can get condition (3) out. The rationals are countable, but they're not well ordered. However, we notice that it does not actually matter whether we choose them in order. So instead we shall perform induction on a sequence containing all the rational numbers. Let $(r_i)_{i \in \mathbb{N}}$ be a sequence that enumerates all the rational numbers in $(0, 1)$ *exactly once*. By normality, there exists U_{r_1} such that $\overline{U_{r_1}} \subseteq U_1$ and $\overline{U_0} \subseteq U_{r_1}$. Now, let $n \in \mathbb{N}$ and suppose we have chosen sets U_{r_i} where $i < n$ with the property that whenever $r_i < r_j$, then $\overline{U_{r_i}} \subseteq U_{r_j}$. Let p be the smallest rational number in the set $\{ 0, r_1, \ldots, r_{n-1}, 1 \}$ that is bigger than r_n and q be the largest rational number from that set that is smaller than r_n. We quickly remark that this means $q < r_n < p$. By the inductive hypothesis, $\overline{U_q} \subseteq U_p$. Normality implies that there is a open set U_{r_n} such that $\overline{U_q} \subseteq U_{r_n}$ and $\overline{U_{r_n}} \subseteq U_p$.

Now, define

$$f(x) = \inf \{ q \in \mathbb{Q} : x \in U_q \}.$$

To show f is continuous, we shall show that preimages of subbasic elements are open, i.e. $f^{-1}[(-\infty, a)]$ and $f^{-1}[(a, \infty)]$ are open. To begin, we make the following observations:

$$f(x) < a \iff x \in U_p \text{ for some } p \in \mathbb{Q}, p < a. \tag{1}$$

$$f(x) \leq a \iff x \in \overline{U_p} \text{ for all } p \in \mathbb{Q}, p > a. \tag{2}$$

The first one (Equation (1)) follows immediately by definition of inf. For Equation (2), if $f(x) \leq a$, and $r > a$ is a rational, then $\square$

The converse of Urysohn's lemma is true: if disjoint closed sets can be separated with continuous functions, then the space is normal.