

Week 7 Notes

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1 Connectedness

Definition 1.1 (Separation). Let X be a topological space. Then a **separation** of X is a partition of $X = A \cup B$, where A, B are disjoint, open and nonempty sets.

Note that this definition may be called a *disconnection* of X by some authors (c.f. [Lee11]). A space X is **connected** if and only if there exists no separation of X .

A set is said to be clopen if it is both open and closed.

Proposition 1.2. Let X be a topological space. Then X is connected if and only if it has no nontrivial clopen subsets, i.e. the only clopen subsets of X are \emptyset and X .

Proof. Obviously. □

Definition 1.3 (Path). Let X be a topological space and $x, y \in X$. Then, a **path** from x to y is a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$.

Note that the domain can be replaced with any closed interval $[a, b]$ since all closed intervals are homeomorphic to $[0, 1]$.

A space X is said to be **path-connected** if given any $x, y \in X$, there exists a path from x to y .

Theorem 1.4 (Path connectedness implies connectedness). Let X be a path-connected space. Then X is connected.

Proof. If not, let A, B be a separation of X . Let $a \in A, b \in B$ and p is a path from a to b . Then $p[[0, 1]] \cap A$ and $p[[0, 1]] \cap B$ is a separation of $p[[0, 1]]$ which contradicts the connectedness of $p[[0, 1]]$. □

Note that we have made use of the fact that intervals are connected, and the image of a connected space under a continuous function is connected.

Definition 1.5. A space X is totally disconnected if the only connected subspaces of X are singletons.

Clearly the discrete topology on a space with more than one point is totally disconnected. However, not every totally disconnected space has the discrete topology.

Example 1.6 (The rationals are totally disconnected). Let $X = \mathbb{Q}$ considered as a subspace of \mathbb{R} . Then X is totally disconnected since given p, q where $p \neq q$, we can partition $X = (X \cap (-\infty, p)) \cup ((p, \infty) \cap X)$. //

Recall that if $X \subseteq \mathbb{R}$, a subset $A \subseteq X$ is said to be *convex* if given $a, b \in A$, we have that $[a, b] \subseteq X$. We shall now prove that intervals in \mathbb{R} are connected. Before we begin the proof, note the properties of the real numbers that we make use of: the fact that supremums exist, and between any two reals, we can find another real.

Theorem 1.7. Let $X \subseteq \mathbb{R}$. Then X is connected if and only if it is convex.

Proof. If X is not convex, then let $a, b \in X$ and $z \in \mathbb{R}$ be such that $a < z < b$, and $z \notin X$. Then X can be separated by $X = (X \cap (-\infty, z)) \cup ((z, \infty) \cap X)$.

Suppose X is convex but that $X = A \cup B$ is a separation. Let $a \in A, b \in B$ and suppose without loss of generality that $a < b$. Since X is convex, $[a, b] \subseteq X$. We thus separate $[a, b] = (A \cap [a, b]) \cup (B \cap [a, b])$. Let $A_0 = (A \cap [a, b]), B_0 = (B \cap [a, b])$. Let $c = \sup A_0$. Then $c \in X$ and $c \in A_0$ as A_0 is closed. Since A_0 is open there is some ε such that $c + \varepsilon \in A_0$. But $c + \varepsilon > c$ which contradicts c being $\sup A_0$. Oopsies! \square

Example 1.8 (Topologist's sine curve). The topologist's sine curve is an example of a space which is connected, but not path connected. Let $f : (0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = \sin(1/x)$. Let $S \subseteq \mathbb{R}^2$ be the graph of f . The topologist's sine curve is thus defined to be \bar{S} . It is connected, because it is the closure of the image of a connected space under a continuous function. (The function is $x \mapsto (x, f(x))$). However, we run into an issue when trying to construct a path from $x \in S$ to the set of limit points of S . For concreteness, let us suppose we are trying to connect x to $(0, 0)$. Suppose we somehow have a path $p : [0, 1] \rightarrow \bar{S}$ from x to $(0, 0)$. Let $L = \{0\} \times [-1, 1]$ (which is the set of limit points of S). L is closed in \bar{S} , so $p^{-1}(L)$ is closed too.

See [Mun00, Example 7, pp. 156–157] for full argument. //

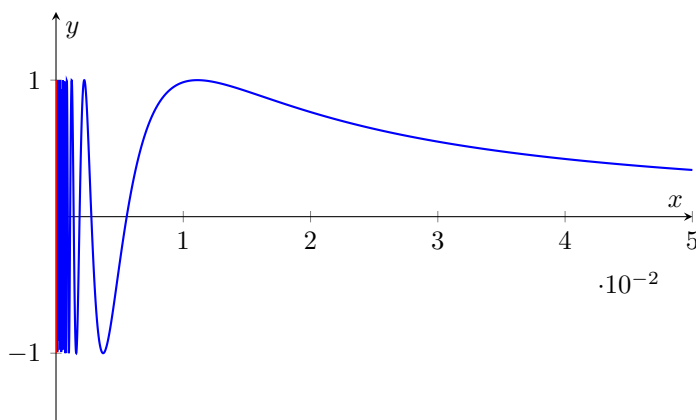


Figure 1: Topologist's sine curve

Proposition 1.9. Let X be a topological space and let Y be a (path) connected subspace of X . If $A \cup B$ is a separation of X , then either $Y \subseteq A$ or $Y \subseteq B$.

Proof. If Y is not fully contained within either A or B then we can separate Y with $(Y \cap A) \cup (Y \cap B)$. \square

Proposition 1.10. Let X be a topological space and let A_α be a collection of (path) connected spaces and suppose $z \in A_\alpha$ for all α , so the A_α 's have a common point. Then, $\bigcup A_\alpha$ is (path) connected.

Proof. We first prove it for path connectedness. Let z be a point in common. If $x, y \in \bigcup A_\alpha$, say $x \in A_\alpha$ and $y \in A_\beta$. Then, glue a path from x to z and a path from z to y together. This one is easy to visualize by drawing a picture.

Let us now prove it for connectedness. Suppose that $\bigcup A_\alpha$ is the union of disjoint open sets $A \cup B$. Then $z \in A$ or $z \in B$. Suppose without loss of generality that $z \in A$. Then for all α , A_α must intersect A . By the previous proposition, all $A_\alpha \subseteq A$. So this means B is empty. Thus there is no separation of $\bigcup A_\alpha$. \square

Proposition 1.11. Suppose A is a connected subspace of X and B is a set such that $A \subseteq B \subseteq \bar{A}$. Then, B is connected.

Proof. Use Proposition 1.9. (For full proof, see [Mun00] or [Lee11, Prop 4.9, p. 88].) \square

It is important to note that this proposition is untrue if A is path connected. See Example 1.8 for this happening.

Theorem 1.12 (Main theorem on connectedness). Let X be a connected space and let $f : X \rightarrow Y$ be a continuous function. Then $f[X]$ is connected.

Proof. If not, let A, B be a separation of $f[X]$. Then $f^{-1}(A), f^{-1}(B)$ separate X . \square

Note here that A, B are considered as open/closed sets in the subspace topology on $f[X]$. The above theorem is also true with path-connectedness in place of connectedness. The proof is obvious, as you can simply compose the path with f .

Corollary 1.13 (Intermediate value theorem). *Let $f : X \rightarrow \mathbb{R}$ and suppose X is connected. If $p, q \in X$ then f attains every value between $f(p)$ and $f(q)$.*

Proof. Suppose without loss of generality that $f(p) < f(q)$. Then $f[X]$ is connected so it must contain $[f(p), f(q)]$. \square

See [Lee11, Thm 4.12, p. 89] for further details.

Warning. The preimage of a connected or path-connected space need not be connected. Take $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the trivial topology. Then the identity is continuous from X to Y but the preimage of Y is disconnected.