Week 6

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1 Final topologies

The final topology is the dual[1] notion of *initial topology*. With the initial topology, we have a family of maps with a common domain X, and we want to topologize X in a way that makes all the maps continuous. With the final topology, we have a family of maps with a common *codomain* Y instead, and we would like to topologize Y in a way that makes all the maps continuous.

Definition 1.1. Let Y be a set and let $\{ X_\alpha : \alpha \in \Lambda \}$ be a collection of topological spaces. Let

$$\mathcal{F} = \{ f_\alpha : X_\alpha \to Y : \alpha \in \Lambda \}$$

be a family of functions. Then the *final topology of* $\mathcal{F}$ is defined to be

$$\{ U \subseteq Y : f_\alpha^{-1}(U) \text{ is open in } X_\alpha \text{ for all } \alpha \in \Lambda \}.$$

In a sense, we are interested in providing Y with a topology that makes all the f_α's continuous. Notice here that Y is the codomain of our f_α's.

For reference, here is the definition of initial topology.

Definition 1.2. Let X be a set, and let $\{ Y_\alpha : \alpha \in \Lambda \}$ be a collection of topological spaces. Let

$$\mathcal{F} = \{ f_\alpha : X \to Y_\alpha : \alpha \in \Lambda \}$$

be a family of functions. Then the *initial topology of* $\mathcal{F}$ is defined to be

$$\bigcap \{ \tau : \tau \text{ is a topology on } X \text{ and every element of } \mathcal{F} \text{ is } \tau - \text{continuous} \}.$$

Proposition 1.3. *The final topology of $\mathcal{F}$ is the finest topology on Y where all the elements of $\mathcal{F}$ are continuous.*

Proof. Denote the final topology with $\tau_{\mathcal{F}}$. Suppose τ is a topology that makes all the f_α's continuous. Then $\tau \subseteq \tau_{\mathcal{F}}$. To see this, let $U \in \tau$. Then for every α, we have $f_\alpha^{-1}(U)$ being open in X_α, as f_α is τ continuous. This means $U \in \tau_{\mathcal{F}}$. $\square$

We can now see an application of final topologies.

[1] In this case, the duality is actually the categorical duality!

2 Quotient topology

Definition 2.1. Let X be a topological space and Y be a set. Let $q : X \to Y$ be a surjective function. Then the final topology of $\{ q \}$ is called the *quotient topology induced by* q.

If Y is a topological space, then Y is a *quotient of* X if the topology on Y is the quotient topology induced by some surjective function $q : X \to Y$.

Again, keep in mind here that Y is being topologized by the final topology induced by q. One (relatively immediate) observation is that a set $O \subseteq Y$ is open in the quotient topology on Y if and only if $q^{-1}(O)$ is open in X. In fact, this is an alternative way to define the quotient map.

We often use the quotient topology to put a topology on the set of equivalence classes. Let us recall the definition of a equivalence relation.

2.1 Equivalence relations

Definition 2.2 (Equivalence relation)**.** Let X be a set. Then an *equivalence relation* $\sim$ *on* X is a relation such that

1. (**Reflexive**) $x \sim x$,
2. (**Symmetric**) if $x \sim y$ then $y \sim x$,
3. (**Transitive**) if $x \sim y$ and $y \sim z$ then $x \sim z$.

The intuition here is that equivalence relations try to capture the notion of equality. In fact, $=$ is an equivalence relation. More examples of equivalence relations are $n \sim m$ iff $n \mod k = m \mod k$ (here, $n, m \in \mathbb{Z}$ and $k \in \mathbb{N}, k > 0$).

Given an equivalence relation on X, we can *partition*[2] the set X into *equivalence classes*. We define

$$[x]_\sim = \{ y \in X : y \sim x \}.$$

Notice that we now have the following properties:

Lemma 2.3. *Let X be a set and $\sim$ be an equivalence relation on X. Then,*

1. $X = \bigcup_{x \in X} [x]_\sim$,
2. *Equivalence classes are equal or disjoint: If $[x]_\sim \neq [y]_\sim$, then $[x]_\sim \cap [y]_\sim = \varnothing$.*

Proof. The first is obvious. For the second, we prove the contrapositive. Suppose $z \in [x]_\sim \cap [y]_\sim$. Then $z \sim x$ and $z \sim y$ by definition. By transitivity we have $x \sim y$, and by transitivity again, every element related to y is also related to x. □

Given an equivalence relation $\sim$ on X, we denote the set of equivalence classes,

$$X_\sim = X/\sim = \{ [x]_\sim : x \in X \}.$$

There is a canonical surjective function[3] from X to $X_\sim$ which sends an element $x \in X$ to its equivalence class $[x]_\sim$. We shall denote it by $p_\sim$, and it is defined as

$$p_\sim(x) = [x]_\sim.$$

2.2 Examples of quotient spaces

We can now see some examples of quotient spaces. The reader is encouraged to check out [Lee11, pp. 62–68] for many more examples of quotient spaces.

Example 2.4 (The sphere S^2 as a quotient space)**.** Let $D \subseteq \mathbb{R}^2$ be the unit disk, i.e. $D = \{ \langle x, y \rangle : x^2 + y^2 \leq 1 \}$.

[2] Note that the word "partition" has a rigorous definition.

[3] Some authors call call this the natural projection.

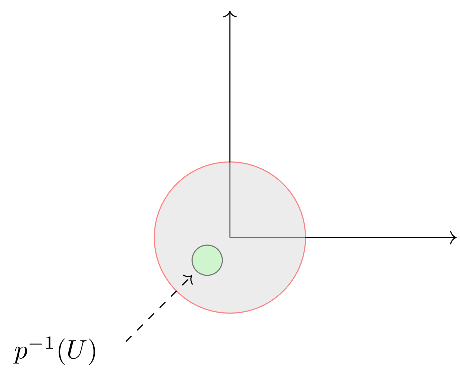

Figure 1: Unit disk $D \subseteq \mathbb{R}^2$

Define $\sim$ on D by

$$\langle x, y\rangle \sim \langle z, w\rangle \text{ iff } \langle x, y\rangle = \langle z, w\rangle \text{ or } x^2 + y^2 = z^2 + w^2 = 1.$$

Intuititively, every point in the interior of D (the interior is shaded in gray) stays distinct, and every point on the boundary (colored in blue) is the "same" under $\sim$. Now, the set of equivalence classes of D, $D/\sim$ can be visualized as in Figure 2.

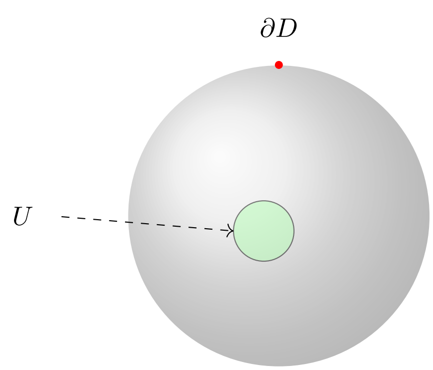

Figure 2: The sphere constructed from the unit disk

Example 2.5 (Torus as a quotient space)**.** See [Lee11, Example 3.49 on p. 66]. //

One might wonder whether quotient spaces always come from some kind of equivalence relation. The answer is yes.

Theorem 2.6. If Y is a quotient space of X, then there is an equivalence relation $\sim$ on X such that Y is homeomorphic to $X_\sim$ (endowed with the quotient topology induced by $p_\sim$).

Before we embark on the proof, readers who have had a little group theory will realize that this is basically quotienting by the kernel of a homomorphism. It turns out that this construction is valid in a lot of (concrete) categories as well

Proof. We would like to show that if Y is such that there exists some surjective function $q : X \to Y$ where the topology of Y is the quotient topology induced by q then there exists an equivalence relation on X such that $X_\sim$ is homeomorphic to Y. We first show the existence of such an equivalence relation. Let $\sim$ in X be defined as follows: $x \sim y$ if and only if $q(x) = q(y)$. This is easily seen to be an equivalence relation.

Now we begin constructing the homeomorphism. Let $f : X_\sim \to Y$ be defined by $f([x]_\sim) = q(x)$. Then f is a well-defined function, if we have $[x]_\sim = [x']_\sim$, then $f(x') = q(x') = q(x) = f(x)$ by definition of $\sim$. We also check that f is a bijection by finding it's inverse, $f^{-1} : Y \to X_\sim$. We'll just write it down:

$$f^{-1}(y) = \{\, x \in X : q(x) = y \,\}.$$

$$\begin{tikzcd} X \arrow[d, two heads, "p_\sim"'] \arrow[dr, two heads, "q"] & \\ X_\sim \arrow[r, tail, two heads, "f"'] & Y \end{tikzcd}$$

Figure 3: Commutative diagram expressing the proof of Theorem 2.6

This is indeed an inverse. So f is a bijection. All that is left is to show that f and f^{-1} are continuous. Let $U \subseteq Y$ be open. Then

$$f^{-1}(U) = \{\, [x]_\sim : f([x]_\sim) \in U \,\} = \{\, [x]_\sim : q(x) \in U \,\}.$$

Let $p_\sim : X \to X_\sim$ be the canonical projection that sends x to $[x]_\sim$. Consider $p_\sim^{-1}(\{\, [x]_\sim : q(x) \in U \,\}) = \{\, x \in X : q(x) \in U \,\} = q^{-1}(U)$. $q^{-1}(U)$ is open in X because q is continuous, but by definition of quotient topology this means $\{\, [x]_\sim : q(x) \in U \,\}$ is open. Thus we have shown that f is continuous. We leave the proof of the continuity of f^{-1} to the reader. (Just show that f is open) □

2.3 Properties of quotient spaces

Unfortunately, quotient spaces are quite badly behaved. The first part where they don't play so nice is with the subspace topology. In other words, taking a quotient of a subspace is not the same as taking a subspace of a quotient space. Let $q : X \to Y$ be a surjective map. This induces the quotient topology in Y. Let $A \subseteq X$ and give A the subspace topology. Consider $q\,|_A\colon A \to q[A]$. There are 2 ways to think about the topology on $q[A]$: as a subspace of Y or as a quotient space of A. It turns out that these may not be equal.

In the next example, we will see that the restriction of a quotient map down to a subspace may not be a quotient map. See [Lee11, Prob 3-11, p. 82] for a better statement of this result.

Example 2.7. Let $X = [0,1] \cup [2,3] \subseteq \mathbb{R}$ with the subspace topology from $\mathbb{R}$. Let $Y = [0,2]$ with the subspace topology from $\mathbb{R}$. Let q be defined by $q(x) = x$ if $x \in [0,1]$ and $q(x) = x-1$ if $x \in [2,3]$. Then q is a quotient map from X to Y.

Now let $A = [0,1) \cup [2,3]$ (notice we are taking the half open interval!) and take $q\,|_A\colon A \to [0,2]$. Consider $q\,|_A^{-1}([1,3/2)) = [2, 3/2+1)$. The set $[1,3/2)$ is not open, but it has an open preimage. This prevents $q\,|_A$ from being a quotient map as $q\,|_A$ is not continuous. //

However, it turns out if $A \subseteq X$ is open, and it is the preimage of some subset of Y, then $q\,|_A$ is a quotient map. See [Lee11, Prop 3.62, p. 70] for this result.

Definition 2.8. Let $\sim$ be an equivalence relation on X. A subset $A \subseteq X$ is *$\sim$-saturated* if and only if

$$A = \bigcup_{x \in A} [x]_\sim.$$

This definition can be alternatively thought of as follows: Let $p_\sim : X \to X_\sim$ be the map that sends an element $x \in X$ to its equivalence class $[x]_\sim$. Then $A \subseteq X$ is $\sim$-saturated iff we have $A = \bigcup_{x\in A} p_\sim^{-1}(\{\, x \,\})$. Sometimes, one might see $p_\sim^{-1}(x)$ instead of $p_\sim^{-1}(\{\, x \,\})$. In this case, they mean the same thing. We call the preimage of the singleton x the **fiber of** x. So in other words, a set A is $\sim$-saturated if and only it is the union of fibers. See [Lee11, Exercise 3.59 on p. 69] for a useful characterization of a set being saturated.

Proposition 2.9. *If $A \subseteq X$ is $\sim$-saturated, then $A_\sim \subseteq X_\sim$.*

Proof. If $\sim$ is an equivalence relation on X and $A \subseteq X$, then $\sim$ induces an equivalence relation on A, call it $\sim_A$. This is simply the restriction of $\sim$ to A, i.e. $a \sim_A b \iff a \sim b$. Then $[a]_{\sim_A} = [a]_\sim$. Let $A \subseteq X$ be a subspace and let $p_\sim : A \to A_\sim$, which is really just $p_\sim : X \to X_\sim$ but restricted. □

Theorem 2.10. If A is open (closed) or $p_\sim$ is an open (closed) map, then the subspace topology on $A_\sim$ as a subset of $A_\sim$ is the same as the quotient topology on $A_\sim$ induced by $p_\sim$.

We additionally encourage the reader to check out [Lee11, Proposition 3.60 on p. 69].

Proof. Let $A_\sim \cap V$ be an open subset of $A_\sim$ as a subspace of $X_\sim$. We need to prove $A_\sim \cap V$ is open in $X_\sim$, which amounts to showing that $p_\sim^{-1}(A_\sim \cap V)$ is open. Now, since $A_\sim$ is saturated, we have

$$p_\sim^{-1}(A_\sim \cap V) = p_\sim^{-1}(A_\sim) \cap p_\sim^{-1}(V) = A \cap p_\sim^{-1}(V).$$

Since $A \cap p_\sim^{-1}(V)$ is open in the subspace topology in A, this means that $A_\sim \cap V$ is open in the quotient $A_\sim$. Let $U \subseteq A_\sim$ be open in the quotient topology induced by $p_\sim|_A \colon A \to A_\sim$. We claim that if A is open and saturated, then $A_\sim \subseteq X_\sim$ is also open (proof: exercise). So U is open in the quotient if and only if $p_\sim|_A^{-1}(U)$ is open in A. But notice that $p_\sim|_A^{-1}(U) = \{\, x \in X : [x]_\sim \in U \,\}$. This is open in A if and only if it is equal to $A \cap V$, where V is some open subset of X. Then, we leave the reader to check that

$$U = p_\sim\left(p_\sim|_A^{-1}(U)\right) = p_\sim(\{\, x \in A : [x]_\sim \in U \,\}) = p_\sim(A \cap V) = A_\sim \cap p_\sim(V).$$

$\square$

We remark that a quotient space of a Hausdorff space may not be Hausdorff.
Example 2.11. Let $X = \mathbb{R}$ and let f be the sign function be defined by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.$$

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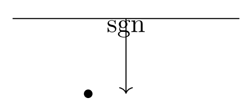

Example 2.12 (Points are closed, but not Hausdorff)**.** Let $X = \mathbb{R}_K$ (the K-topology) and define the equivalence relation on X by $a \sim b$ if and only if $a = b$ or $a, b \in K$. Then, $X_\sim$ is not Hausdorff, but points are closed. To see why this is not hausdorff, notice that we cannot find disjoint open neighborhoods of $[0]_\sim$ and $[1]_\sim$. Indeed, $[0]_\sim = \{\, 0 \,\}$ and $[1]_\sim = K$. But any neighborhood of $[1]_\sim$ must contain all the $1/n$'s (by looking at the neighborhood in X) and thus contain 0. //

Additionally, products and quotients also do not behave well. If Y is a quotient space of X, and $q : X \to Y$ is a surjective map, then it may not be true that the product topology on $Y \times Y$ is the same as the quotient topology induced by $q \times q$. That is to say, there is a difference between first putting the quotient topology on Y using q and taking the product $Y \times Y$, versus putting the quotient topology on $Y \times Y$ with $q \times q$.
Example 2.13. We make use of Example 2.12 and the following fact: the diagonal of X, which is the set $\Delta_X = \{\, \langle x, x \rangle : x \in X \,\}$ is closed in $X \times X$ if and only if X is Hausdorff. Let q be the quotient map which is given by $\sim$. It is true that Δ is closed in $X \times X$, but $\Delta_{X_\sim}$ is not closed in $X_\sim \times X_\sim$ as it is not Hausdorff. However, $(q \times q)^{-1}(\Delta_{X_\sim}) = \Delta_X$, so $q \times q$ cannot be a quotient map. //

References

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. en. Vol. 202. Graduate Texts in Mathematics. New York, NY: Springer New York, 2011. ISBN: 9781441979391. DOI: 10.1007/978-1-4419-7940-7. URL: https://link.springer.com/10.1007/978-1-4419-7940-7.