Week 7 Notes

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1 Connectedness

Definition 1.1 (Separation)**.** Let X be a topological space. Then a **separation** of X is a partition of $X = A \cup B$, where A, B are disjoint, open and nonempty sets.

Note that this definition may be called a *disconnection* of X by some authors (c.f. [Lee11]). A space X is **connected** if and only if there exists no separation of X.

A set is said to be clopen if it is both open and closed.

Proposition 1.2. *Let X be a topological space. Then X is connected if and only if it has no nontrivial clopen subsets, i.e. the only clopen subsets of X are $\varnothing$ and X.*

Proof. Obviously. □

Definition 1.3 (Path)**.** Let X be a topological space and $x, y \in X$. Then, a **path** from x to y is a continuous function $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$.

Note that the domain can be replaced with any closed interval $[a, b]$ since all closed intervals are homeomorphic to $[0, 1]$.

A space X is said to be **path-connected** if given any $x, y \in X$, there exists a path from x to y.

Theorem 1.4 (Path connectedness implies connectedness)**.** Let X be a path-connected space. Then X is connected.

Proof. If not, let A, B be a separation of X. Let $a \in A, b \in B$ and p is a path from a to b. Then $p[[0,1]] \cap A$ and $p[[0,1]] \cap B$ is a separation of $p[[0,1]]$ which contradicts the connectedness of $p[[0,1]]$. □

Note that we have made used of the fact that intervals are connected, and the image of a connected space under a continuous function is connected.

Definition 1.5. A space X is totally disconnected if the only connected subspaces of X are singletons.

Clearly the discrete topology on a space with more than one point is totally disconnected. However, not every totally disconnected space has the discrete topology.

Example 1.6 (The rationals are totally disconnected)**.** Let $X = \mathbb{Q}$ considered as a subspace of $\mathbb{R}$. Then X is totally disconnected since given p, q where $p \neq q$, we can partition $X = (X \cap (-\infty, p)) \cup ((p, \infty) \cap X)$. //

Recall that if $X \subseteq \mathbb{R}$, a subset $A \subseteq X$ is said to be *convex* if given $a, b \in A$, we have that $[a, b] \subseteq X$. We shall now prove that intervals in $\mathbb{R}$ are connected. Before we begin the proof, note the properties of the real numbers that we make use of: the fact that supremums exist, and between any two reals, we can find another real.

Theorem 1.7. Let $X \subseteq \mathbb{R}$. Then X is connected if and only if it is convex.

Proof. If X is not convex, then let $a, b \in X$ and $z \in \mathbb{R}$ be such that $a < z < b$, and $z \notin X$. Then X can be separated by $X = (X \cap (-\infty, z)) \cup ((z, \infty) \cap X)$.

Suppose X is convex but that $X = A \cup B$ is a separation. Let $a \in A, b \in B$ and suppose without loss of generality that $a < b$. Since X is convex, $[a, b] \subseteq X$. We thus separate $[a, b] = (A \cap [a, b]) \cup (B \cap [a, b])$. Let $A_0 = (A \cap [a, b]), B_0 = (B \cap [a, b])$. Let $c = \sup A_0$. Then $c \in X$ and $c \in A_0$ as A_0 is closed. Since A_0 is open there is some ε such that $c + \varepsilon \in A_0$. But $c + \varepsilon > c$ which contradicts c being $\sup A_0$. Oopsies! □

Example 1.8 (Topologist's sine curve)**.** The topologist's sine curve is an example of a space which is connected, but not path connected. Let $f : (0, 1] \to \mathbb{R}$ be defined by $f(x) = \sin(1/x)$. Let $S \subseteq \mathbb{R}^2$ be the graph of f. The topologist's sine curve is thus defined to be $\overline{S}$. It is connected, because it is the closure of the image of a connected space under a continuous function. (The function is $x \mapsto (x, f(x))$). However, we run into an issue when trying to construct a path from $x \in S$ to the set of limit points of S. For concreteness, let us suppose we are trying to connect x to $(0, 0)$. Suppose we somehow have a path $p : [0, 1] \to \overline{S}$ from x to $(0, 0)$. Let $L = \{ 0 \} \times [-1, 1]$ (which is the set of limit points of S). L is closed in $\overline{S}$, so $p^{-1}(L)$ is closed too.

See [Mun00, Example 7, pp. 156–157] for full argument. (For an explicit value of u that can be chosen, you can pick $u = \frac{1}{2n\pi + \pi/2}$ so $\sin(1/u) = 1$, and $u = \frac{1}{2n\pi + (3/2)\pi}$ if you need $\sin(1/u) = -1$.) //

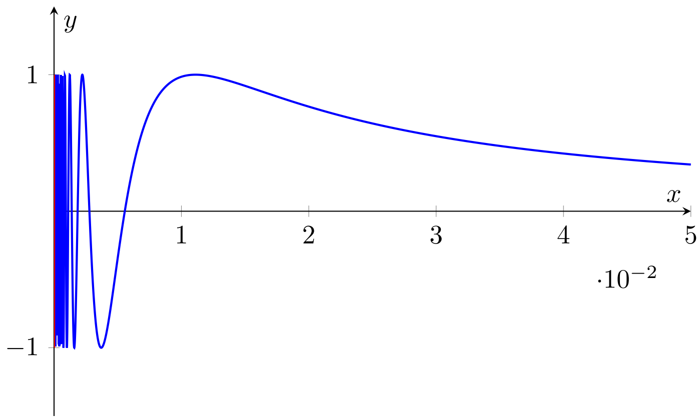

Figure 1: Topologist's sine curve

Proposition 1.9. *Let X be a topological space and let Y be a (path) connected subspace of X. If $A \cup B$ is a separation of X, then either $Y \subseteq A$ or $Y \subseteq B$.*

Proof. If Y is not fully contained within either A or B then we can separate Y with $(Y \cap A) \cup (Y \cap B)$. □

Proposition 1.10. *Let X be a topological space and let A_α be a collection of (path) connected spaces and suppose $z \in A_\alpha$ for all α, so the A_α's have a common point. Then, $\bigcup A_\alpha$ is (path) connected.*

Proof. We first prove it for path connectedness. Let z be a point in common. If $x, y \in \bigcup A_\alpha$, say $x \in A_\alpha$ and $y \in A_\beta$. Then, glue a path from x to z and a path from z to y together. This one is easy to visualize by drawing a picture.

Let us now prove it for connectedness. Suppose that $\bigcup A_\alpha$ is the union of disjoint open sets $A \cup B$. Then $z \in A$ or $z \in B$. Suppose without loss of generality that $z \in A$. Then for all α, A_α must intersect A. By the previous proposition, all $A_\alpha \subseteq A$. So this means B is empty. Thus there is no separation of $\bigcup A_\alpha$. □

Proposition 1.11. *Suppose A is a connected subspace of X and B is a set such that $A \subseteq B \subseteq \overline{A}$. Then, B is connected.*

Proof. Use Proposition 1.9. (For full proof, see [Mun00] or [Lee11, Prop 4.9, p. 88].) □

It is important to note that this proposition is untrue if A is path connected. See Example 1.8 for this happening.

> **Theorem 1.12** (Main theorem on connectedness)**.** Let X be a connected space and let $f : X \to Y$ be a continuous function. Then $f[X]$ is connected.

Proof. If not, let A, B be a separation of $f[X]$. Then $f^{-1}(A), f^{-1}(B)$ separate X. $\square$

Note here that A, B are considered as open/closed sets in the subspace topology on $f[X]$. The above theorem is also true with path-connectedness in place of connectedness. The proof is obvious, as you can simply compose the path with f.

Corollary 1.13 (Connectedness is invariant under homeomorphism)**.** *Any space homeomorphic to a connected space is connected.*

Proof. Duh. $\square$

Corollary 1.14 (Intermediate value theorem)**.** *Let $f : X \to \mathbb{R}$ and suppose X is connected. If $p, q \in X$ then f attains every value between $f(p)$ and $f(q)$.*

Proof. Suppose without loss of generality that $f(p) < f(q)$. Then $f[X]$ is connected so it must contain $[f(p), f(q)]$. $\square$

See [Lee11, Thm 4.12, p. 89] for further details.

Warning. The preimage of a connected or path-connected space need not be connected. Take $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the trivial topology. Then the identity is continuous from X to Y but the preimage of Y is disconnected.

Proposition 1.15. *If X, Y are connected spaces then $X \times Y$ is connected.*

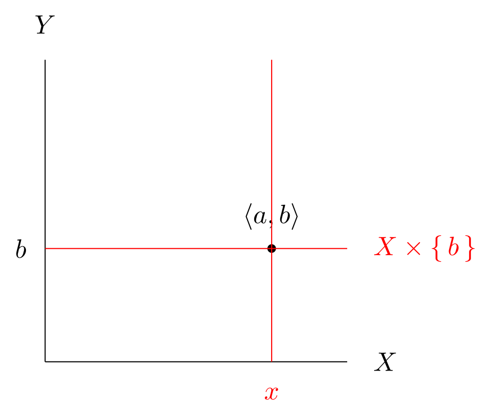

Figure 2: Proof that the finite product of connected spaces is connected

Proof. Fix a point $\langle a, b\rangle \in X \times Y$. Define $T_x = \{ x \} \times Y \cup X \times \{ b \}$. This set is connected as it is the union of 2 connected sets with the point $\langle a, b\rangle$ in common. Then $X \times Y = \bigcup_{x \in X} T_x$. This is a union of connected spaces with the point $\langle a, b\rangle$ in common. (See Section 1 for a better visualization. See [Mun00, Thm 23.6, p. 148] for complete proof.) $\square$

The product topology preserves connectedness (which is nice).

Proposition 1.16 (Product of (path) connected spaces is connected)**.** *If X_α is a collection of (path)-connected spaces, then $X = \prod X_\alpha$ is (path)-connected.*

Proof. (Path-connectedness) Let $\mathbf{x}, \mathbf{y} \in \prod X_\alpha$, writing $\mathbf{x} = \langle x_\alpha : \alpha \in \Lambda\rangle$ and $\mathbf{y} = \langle y_\alpha : \alpha \in \Lambda\rangle$. Since each X_α is path connected, for each α, let $f_\alpha : I \to X_\alpha$ be a path from x_α to y_α. We simply glue these paths together by taking $f(t) = \langle f_\alpha(t) : \alpha \in \Lambda\rangle$ which is a path from $\mathbf{x}$ to $\mathbf{y}$.

(Connectedness) Fix a point $\mathbf{a} = \langle a_\alpha : \alpha \in \Lambda\rangle$. If $F \subseteq \Lambda$ is finite, then define

$$X_F = \{ \mathbf{x} \in X : x_\alpha = a_\alpha \text{ if } \alpha \in \Lambda \setminus F \}.$$

So X_F is the set of all $\mathbf{x} \in X$ such that $x_\alpha = a_\alpha$ for all coordinates except those in F. We thus see that X_F is homeomorphic to $\prod_{\alpha \in F} X_\alpha$. Since finite products of connected spaces are connected, X_F is connected.

Now, set $Z = \bigcup_{F \subseteq \Lambda, |F| < \omega} X_F$. This is the union of X_F's across all finite subsets $F \subseteq \Lambda$. Then Z is connected, as each X_F has the point $\mathbf{a}$ in common (Proposition 1.10). Additionally, we claim that $\overline{Z} = X$. This will finish it off

(Proposition 1.11), so let us see why this is true. Pick $\mathbf{x} \in X$ and let U be a neighborhood of $\mathbf{x}$ in the product topology. We need to show that U intersects Z. Since we are in the product topology, this means that $U = \prod_{\alpha \in \Lambda} U_\alpha$ and $U_\alpha = X_\alpha$ except for finite α. Say those α's are all in the set F. Define

$$\mathbf{z}_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in F, \\ a_\alpha & \text{otherwise.} \end{cases}$$

Then $\mathbf{z} \in X_F \cap U$ so the point $\mathbf{z}$ is in the closure of Z. □

The box topology is usually not going to be connected.
Example 1.17 (Countable product of $\mathbb{R}$ with the box topology)**.** Let $X = \prod_{n \in \mathbb{N}} \mathbb{R}$ and give it the box topology. Let

$$\ell^\infty = \left\{ \mathbf{x} \in \mathbb{R}^\mathbb{N} : \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}.$$

This is the set of bounded real-valued sequences. We shall show that ℓ^∞ is clopen. Since ℓ^∞ is not all of $\mathbb{R}^\mathbb{N}$ (by obviousness) we will be done. Let $\mathbf{x} \in \ell^\infty$ be a bounded sequence. Consider the neighborhood of $\mathbf{x}$ given by $U = \prod_{n \in \mathbb{N}} B(x_n, 1)$. Notice if $\mathbf{y} \in U$, then $|y_n| < |x_n| + 1 \leq \sup_{n \in \mathbb{N}} |x_n| + 1$ so $\mathbf{y}$ must be a bounded sequence too. For being closed, notice that the complement is open. (Use the same argument). //
Remark 1.18 (Path-connectedness of finite products)**.** For the finite case the proof is very easy. Given a point $\langle x_0, y_0 \rangle \in X \times Y$ and a point $\langle x_1, y_1 \rangle \in X \times Y$, since X, Y are respectively path connected let p be a path in X from x_0 to x_1 and q be a path in Y from y_0 to y_1. Then the map $p \times q$ is the desired path. Apply induction and the fact that $(X \times Y) \times Z$ is homeomorphic to $X \times Y \times Z$.