1 Internal Groups

Internal groups (or groups in a category) allows us to do group theory in categories.

Definition 1.1. An internal group in a category C with finite products is an object G together with morphisms $m : G \times G \to G$, $u : 1 \to G$, $i : G \to G$, where m is the multiplication, u is the unit and i is the inverse. Let x, y, z be generalized elements. Then $m(m(x, y), z) = m(x, m(y, z))$, $m(x, u) = x = m(u, x)$, $m(x, ix) = u = m(ix, x)$.

So now we can basically do group theory in categories. For example, if we have an internal group in **Top**, our m, i, u are continuous maps, so this basically tells us m, i are continuous. If we have an internal group in **Sets**, then this is just a usual group (A set equipped with a binary operation that makes it a group).

Remark 1.2. Now, if we have an internal group in **Grp**, the object has to be an abelian group. Also, all abelian groups in **Grp** are internal groups. This is what the statement "the groups in **Grp** are precisely the abelian groups" means.

Definition 1.3. If G, H are internal groups in C, then a homomorphism $\phi : G \to H$ is a morphism such that $h \circ m = \overline{m} \circ (h \times h)$, $\overline{u} = h \circ u$, and $h \circ \overline{i} = i \circ h$.

Remark 1.4. Normally in group theory h only has to preserve the multiplication. However, this implicitly adds in the other 2 properties when considering groups as internal groups in **Sets**

2 Kernels for categories

Kernels are very important in group theory, so we'd like to have them for categories too. In usual group theory, to quotient by a kernel is really to quotient by an equivalence relation, so we first need to define that.

Definition 2.1. Let C be a category. A congruence on C is an "equivalence relation"[1] on arrows such that if $f \sim g$ then their domains and codomains agree, and if a, b are morphisms such that the compositions bfa and bga make sense, then $bfa \sim bga$

Remark 2.2. Let us pretend for now that C is small, so we can discuss "the set of morphisms of C". In this form, congruences are equivalence relations (but the converse is not true).

If we have $\sim$ on a category C then this can be used to construct the quotient category $C/\sim$ where the objects are the same as the objects in C, and the morphisms are "equivalence classes" of morphisms in C.

$C/\sim$ is actually a coequalizer of the congruence category of C, denoted $C^{\sim}$ which has the same objects as C, but has morphisms pairs (f, g) where $f \sim g$, and composition is defined componentwise.

Let $\mathbf{C}, \mathbf{D}$ be categories and $F : \mathbf{C} \to \mathbf{D}$ be a functor. Define a congruence $\sim_F$ on $\mathbf{C}$ by declaring that $f \sim_F g$ if and only if the domains and codomains of f and g agree and they are mapped to the same morphism under F. Now the kernel category of F, denoted $\ker F$ is simply the congruence category of C with $\sim_F$.

Remark 2.3. Kernels are equalizers (I gave an example in the previous week's summary)

This theorem is basically the homomorphism theorem for groups, but for categories.

Remark 2.4. The homomorphism theorem for groups is a generalization of the first isomorphism theorem.

Theorem 2.5. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Then F has a kernel category, and if $\sim$ is a congruence on $\mathbf{C}$, then $f \sim g$ implies $f \sim_F g$ if and only if we have a factorization $\tilde{F} : \mathbf{C}/\sim \to \mathbf{D}$ such that $F = \tilde{F} \circ \pi$.

We can now use congruences to define finitely presented categories. Since I am running out of space this section will be brief.

If we have a finite graph G, let Σ be a finite set of relations $(g_1 \dots g_n) = (g'_1 \dots g'_m)$ where the domains of the rightmost morphisms agree and the codomains of g_1, g'_1 agree. Since $g_i \in G$ these guys are basically just vertices so such a relation can be intuitively viewed as an identification of 2 paths with the same endpoint and direction. If we declare $\sim_\Sigma$ to be the smallest congruence on $\mathbf{C}(G)$ such that $g \sim g'$ if and only if $g = g'$ in Σ, then $\mathbf{C}(G, \Sigma) := \mathbf{C}(G)/\sim_\Sigma$ is called a finitely presented category.

[1] I believe there is a notion of equivalence relation for classes