

Week 6

Robert

June 2024

1 Final topologies

The final topology is the dual¹ notion of *initial topology*. With the initial topology, we have a family of maps with a common domain X , and we want to topologize X in a way that makes all the maps continuous. With the final topology, we have a family of maps with a common *codomain* Y instead, and we would like to topologize Y in a way that makes all the maps continuous.

Definition 1.1. Let Y be a set and let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Let

$$\mathcal{F} = \{f_\alpha : X_\alpha \rightarrow Y : \alpha \in \Lambda\}$$

be a family of functions. Then the *final topology of \mathcal{F}* is defined to be

$$\{U \subseteq Y : f_\alpha^{-1}(U) \text{ is open in } X_\alpha \text{ for all } \alpha \in \Lambda\}.$$

In a sense, we are interested in providing Y with a topology that makes all the f_α 's continuous. Notice here that Y is the codomain of our f_α 's.

For reference, here is the definition of initial topology.

Definition 1.2. Let X be a set, and let $\{Y_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Let

$$\mathcal{F} = \{f_\alpha : X \rightarrow Y_\alpha : \alpha \in \Lambda\}$$

be a family of functions. Then the *initial topology of \mathcal{F}* is defined to be

$$\bigcap \{\tau : \tau \text{ is a topology on } X \text{ and every element of } \mathcal{F} \text{ is } \tau\text{-continuous}\}.$$

Proposition 1.3. *The final topology of \mathcal{F} is the finest topology on Y where all the elements of \mathcal{F} are continuous.*

Proof. Denote the final topology with $\tau_{\mathcal{F}}$. Suppose τ is a topology that makes all the f_α 's continuous. Then $\tau \subseteq \tau_{\mathcal{F}}$. To see this, let $U \in \tau$. Then for every α , we have $f_\alpha^{-1}(U)$ being open in X_α , as f_α is τ continuous. This means $U \in \tau_{\mathcal{F}}$. \square

We can now see an application of final topologies.

¹In this case, the duality is actually the categorical duality!

2 Quotient topology

Definition 2.1. Let X be a topological space and Y be a set. Let $q : X \rightarrow Y$ be a surjective function. Then the final topology of $\{q\}$ is called the *quotient topology induced by q* .

If Y is a topological space, then Y is a *quotient of X* if the topology on Y is the quotient topology induced by some surjective function $q : X \rightarrow Y$.

Again, keep in mind here that Y is being topologized by the final topology induced by q . One (relatively immediate) observation is that a set $O \subseteq Y$ is open in the quotient topology on Y if and only if $q^{-1}(O)$ is open in X . In fact, this is an alternative way to define the quotient map.

We often use the quotient topology to put a topology on the set of equivalence classes. Let us recall the definition of an equivalence relation.

2.1 Equivalence relations

Definition 2.2 (Equivalence relation). Let X be a set. Then an *equivalence relation \sim on X* is a relation such that

1. **(Reflexive)** $x \sim x$,
2. **(Symmetric)** if $x \sim y$ then $y \sim x$,
3. **(Transitive)** if $x \sim y$ and $y \sim z$ then $x \sim z$.

The intuition here is that equivalence relations try to capture the notion of equality. In fact, $=$ is an equivalence relation. More examples of equivalence relations are $n \sim m$ iff $n \bmod k = m \bmod k$ (here, $n, m \in \mathbb{Z}$ and $k \in \mathbb{N}, k > 0$).

Given an equivalence relation on X , we can *partition*² the set X into *equivalence classes*. We define

$$[x]_{\sim} = \{y \in X : y \sim x\}.$$

Notice that we now have the following properties:

Lemma 2.3. Let X be a set and \sim be an equivalence relation on X . Then,

1. $X = \bigcup_{x \in X} [x]_{\sim}$,
2. *Equivalence classes are equal or disjoint: If $[x]_{\sim} \neq [y]_{\sim}$, then $[x]_{\sim} \cap [y]_{\sim} = \emptyset$.*

Proof. The first is obvious. For the second, we prove the contrapositive. Suppose $z \in [x]_{\sim} \cap [y]_{\sim}$. Then $z \sim x$ and $z \sim y$ by definition. By transitivity we have $x \sim y$, and by transitivity again, every element related to y is also related to x . \square

Given an equivalence relation \sim on X , we denote the set of equivalence classes,

$$X_{\sim} = X / \sim = \{[x]_{\sim} : x \in X\}.$$

There is a canonical surjective function³ from X to X_{\sim} which sends an element $x \in X$ to its equivalence class $[x]_{\sim}$. We shall denote it by p_{\sim} , and it is defined as

$$p_{\sim}(x) = [x]_{\sim}.$$

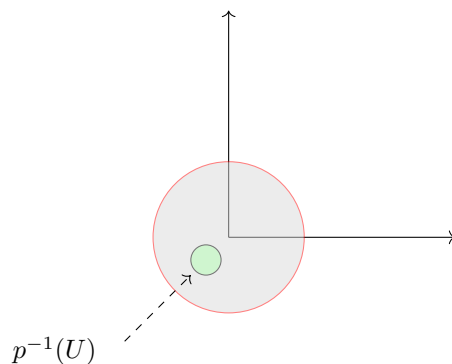
2.2 Examples of quotient spaces

We can now see some examples of quotient spaces. The reader is encouraged to check out [Lee11, pp. 62–68] for many more examples of quotient spaces.

Example 2.4 (The sphere S^2 as a quotient space). Let $D \subseteq \mathbb{R}^2$ be the unit disk, i.e. $D = \{\langle x, y \rangle : x^2 + y^2 \leq 1\}$.

²Note that the word "partition" has a rigorous definition.

³Some authors call this the natural projection.

Figure 1: Unit disk $D \subseteq \mathbb{R}^2$

Define \sim on D by

$$\langle x, y \rangle \sim \langle z, w \rangle \text{ iff } \langle x, y \rangle = \langle z, w \rangle \text{ or } x^2 + y^2 = z^2 + w^2 = 1.$$

Intuitively, every point in the interior of D (the interior is shaded in gray) stays distinct, and every point on the boundary (colored in blue) is the "same" under \sim . Now, the set of equivalence classes of D , D/\sim can be visualized as in Figure 2.

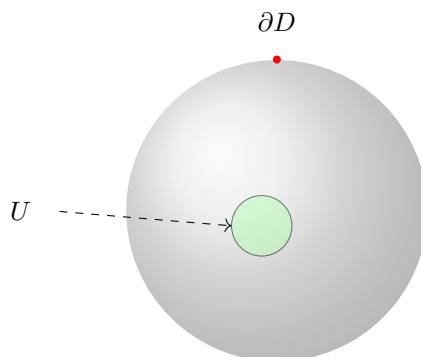


Figure 2: The sphere constructed from the unit disk

Example 2.5 (Torus as a quotient space). See [Lee11, Example 3.49 on p. 66].

One might wonder whether quotient spaces always come from some kind of equivalence relation. The answer is yes.

Theorem 2.6. If Y is a quotient space of X , then there is an equivalence relation \sim on X such that Y is homeomorphic to X_\sim (endowed with the quotient topology induced by p_\sim).

Before we embark on the proof, readers who have had a little group theory will realize that this is basically quotienting by the kernel of a homomorphism. It turns out that this construction is valid in a lot of (concrete) categories as well

Proof. We would like to show that if Y is such that there exists some surjective function $q : X \rightarrow Y$ where the topology of Y is the quotient topology induced by q then there exists an equivalence relation on X such that X_\sim is homeomorphic to Y . We first show the existence of such an equivalence relation. Let \sim in X be defined as follows: $x \sim y$ if and only if $q(x) = q(y)$. This is easily seen to be an equivalence relation.

Now we begin constructing the homeomorphism. Let $f : X_\sim \rightarrow Y$ be defined by $f([x]_\sim) = q(x)$. Then f is a well-defined function, if we have $[x]_\sim = [x']_\sim$, then $f(x') = q(x') = q(x) = f(x)$ by definition of \sim . We also check that f is a bijection by finding its inverse, $f^{-1} : Y \rightarrow X_\sim$. We'll just write it down:

$$f^{-1}(y) = \{x \in X : q(x) = y\}.$$

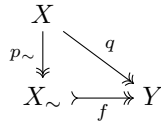


Figure 3: Commutative diagram expressing the proof of Theorem 2.6

This is indeed an inverse. So f is a bijection. All that is left is to show that f and f^{-1} are continuous. Let $U \subseteq Y$ be open. Then

$$f^{-1}(U) = \{ [x]_{\sim} : f([x]_{\sim}) \in U \} = \{ [x]_{\sim} : q(x) \in U \}.$$

Let $p_{\sim} : X \rightarrow X_{\sim}$ be the canonical projection that sends x to $[x]_{\sim}$. Consider $p_{\sim}^{-1}(\{ [x]_{\sim} : q(x) \in U \}) = \{ x \in X : q(x) \in U \} = q^{-1}(U)$. $q^{-1}(U)$ is open in X because q is continuous, but by definition of quotient topology this means $\{ [x]_{\sim} : q(x) \in U \}$ is open. Thus we have shown that f is continuous. We leave the proof of the continuity of f^{-1} to the reader. (Just show that f is open) \square

2.3 Properties of quotient spaces

Unfortunately, quotient spaces are quite badly behaved. The first part where they don't play so nice is with the subspace topology. In other words, taking a quotient of a subspace is not the same as taking a subspace of a quotient space. Let $q : X \rightarrow Y$ be a surjective map. This induces the quotient topology in Y . Let $A \subseteq X$ and give A the subspace topology. Consider $q|_A : A \rightarrow q[A]$. There are 2 ways to think about the topology on $q[A]$: as a subspace of Y or as a quotient space of A . It turns out that these may not be equal.

In the next example, we will see that the restriction of a quotient map down to a subspace may not be a quotient map. See [Lee11, Prob 3-11, p. 82] for a better statement of this result.

Example 2.7. Let $X = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$ with the subspace topology from \mathbb{R} . Let $Y = [0, 2]$ with the subspace topology from \mathbb{R} . Let q be defined by $q(x) = x$ if $x \in [0, 1]$ and $q(x) = x - 1$ if $x \in [2, 3]$. Then q is a quotient map from X to Y .

Now let $A = [0, 1) \cup [2, 3]$ (notice we are taking the half open interval!) and take $q|_A : A \rightarrow [0, 2]$. Consider $q|_A^{-1}([1, 3/2)) = [2, 3/2 + 1)$. The set $[1, 3/2)$ is not open, but it has an open preimage. This prevents $q|_A$ from being a quotient map as $q|_A$ is not continuous. //

However, it turns out if $A \subseteq X$ is open, and it is the preimage of some subset of Y , then $q|_A$ is a quotient map. See [Lee11, Prop 3.62, p. 70] for this result.

Definition 2.8. Let \sim be an equivalence relation on X . A subset $A \subseteq X$ is \sim -saturated if and only if

$$A = \bigcup_{x \in A} [x]_{\sim}.$$

This definition can be alternatively thought of as follows: Let $p_{\sim} : X \rightarrow X_{\sim}$ be the map that sends an element $x \in X$ to its equivalence class $[x]_{\sim}$. Then $A \subseteq X$ is \sim -saturated iff we have $A = \bigcup_{x \in A} p_{\sim}^{-1}(\{x\})$. Sometimes, one might see $p_{\sim}^{-1}(x)$ instead of $p_{\sim}^{-1}(\{x\})$. In this case, they mean the same thing. We call the preimage of the singleton x the **fiber** of x . So in other words, a set A is \sim -saturated if and only if it is the union of fibers. See [Lee11, Exercise 3.59 on p. 69] for a useful characterization of a set being saturated.

Proposition 2.9. If $A \subseteq X$ is \sim -saturated, then $A_{\sim} \subseteq X_{\sim}$.

Proof. If \sim is an equivalence relation on X and $A \subseteq X$, then \sim induces an equivalence relation on A , call it \sim_A . This is simply the restriction of \sim to A , i.e. $a \sim_A b \iff a \sim b$. Then $[a]_{\sim_A} = [a]_{\sim}$. Let $A \subseteq X$ be a subspace and let $p_{\sim} : A \rightarrow A_{\sim}$, which is really just $p_{\sim} : X \rightarrow X_{\sim}$ but restricted. \square

Theorem 2.10. If A is open (closed) or p_{\sim} is an open (closed) map, then the subspace topology on A_{\sim} as a subset of X_{\sim} is the same as the quotient topology on A_{\sim} induced by p_{\sim} .

We additionally encourage the reader to check out [Lee11, Proposition 3.60 on p. 69].

Proof. Let $A_\sim \cap V$ be an open subset of A_\sim as a subspace of X_\sim . We need to prove $A_\sim \cap V$ is open in X_\sim , which amounts to showing that $p_\sim^{-1}(A_\sim \cap V)$ is open. Now, since A_\sim is saturated, we have

$$p_\sim^{-1}(A_\sim \cap V) = p_\sim^{-1}(A_\sim) \cap p_\sim^{-1}(V) = A \cap p_\sim^{-1}(V).$$

Since $A \cap p_\sim^{-1}(V)$ is open in the subspace topology in A , this means that $A_\sim \cap V$ is open in the quotient A_\sim . Let $U \subseteq A_\sim$ be open in the quotient topology induced by $p_\sim|_A: A \rightarrow A_\sim$. We claim that if A is open and saturated, then $A_\sim \subseteq X_\sim$ is also open (proof: exercise). So U is open in the quotient if and only if $p_\sim|_A^{-1}(U)$ is open in A . But notice that $p_\sim|_A^{-1}(U) = \{x \in X : [x]_\sim \in U\}$. This is open in A if and only if it is equal to $A \cap V$, where V is some open subset of X . Then, we leave the reader to check that

$$U = p_\sim(p_\sim|_A^{-1}(U)) = p_\sim(\{x \in A : [x]_\sim \in U\}) = p_\sim(A \cap V) = A_\sim \cap p_\sim(V).$$

□

References

- [Lee11] John M. Lee. *Introduction to Topological Manifolds*. en. Vol. 202. Graduate Texts in Mathematics. New York, NY: Springer New York, 2011. ISBN: 9781441979391. DOI: [10.1007/978-1-4419-7940-7](https://doi.org/10.1007/978-1-4419-7940-7). URL: <https://link.springer.com/10.1007/978-1-4419-7940-7>.