Week 5 notes

Robert

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1 Metric Topologies

Metric spaces are a generalization of $\mathbb{R}^n$, kind of.

Definition 1.1. A function $d : X \times X \to \mathbb{R}$ is a *metric* if and only if

1. **(Positivity)** for all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. **(Symmetric)** for all $x, y \in X$, $d(x, y) = d(y, x)$.
3. **(Triangle Inequality)** for all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

Note that we call a pair (X, d) a metric space. If the metric is clear we say that X is a metric space and drop the d.

Definition 1.2. If (X, d) is a metric space then the *metric topology on* X is the topology generated by the basis of open balls around each point $x \in X$, of every possible radius $\varepsilon > 0$.

$$
\begin{aligned}
B(x, \varepsilon) &= \{\, y \in X : d(y, x) < \varepsilon \,\} \\
\mathcal{B} &= \{\, B(x, \varepsilon) : x \in X, \varepsilon > 0 \,\}
\end{aligned}
$$

Here, $\mathcal{B}$ is the basis of every single possible ball around every point of every single radius.

Definition 1.3. A topological space (X, τ) is *metrizable* if τ is a metric topology.

Note that this does not mean that X is a metric space, but that we can *define* a metric on X such that τ is generated by the basis for the metric topology on X. Of course, if we define such a metric d this does mean that (X, d) is a metric space.

Example 1.4. Consider $\mathbb{R}$ with the usual topology. Then $\mathbb{R}$ is metrizable, with the usual metric $d(x, y) = |x - y|$. Then $B(x, \varepsilon)$ would be an open interval of radius ε around x, i.e. $(x - \varepsilon, x + \varepsilon)$. //

Example 1.5. Let X be a set with the discrete topology. Then it is metrizable with the *discrete metric* defined as

$$
d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}. \tag{1}
$$

We leave it to the reader to check that d is a metric. But to quickly see why the discrete topology is indeed generated by this, note that an open ball of radius $1/2$ around any point x is a singleton. //

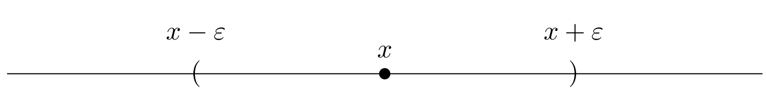
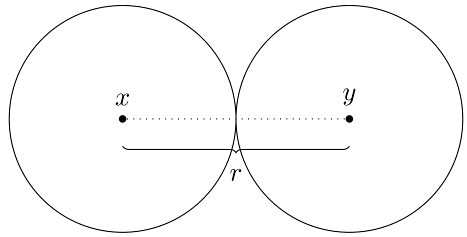


Figure 1: Open intervals in $\mathbb{R}$.



Example 1.6. Let X be a set equipped with the discrete topology. Then X is *not metrizable.*

Any metrizable space must be Hausdorff, but a space with the indiscrete topology is not Hausdorff. //

Proposition 1.7. *Let X be metrizable and $Y \subseteq X$ be a subspace. If $d : X \times X \to \mathbb{R}$ is a metric, then $d\,|_{Y\times Y}$, the restriction of d to $Y \times Y$, is also a metric.*

Proof. Just check the definition. □

Lemma 1.8. *If X is a metrizable space then X is Hausdorff.*

Proof. Let X be metrizable. Let $d : X \times X \to \mathbb{R}$ be a metric that induces the topology on X. Let $x, y \in X$ be such that $x \neq y$. Let $r = d(x, y)$, then consider the balls $B(x, r/2)$ and $B(y, r/2)$. These are disjoint, for if $z \in B(x, r/2) \cap B(y, r/2)$, then by the triangle inequality, we have

$$d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2 = r.$$

However, this contradicts $d(x, y) = r$. The balls are open because they are elements of the basis for the metric topology. □

2 Metrizing order topologies

Question: Are order topologies metrizable?

- $\mathbb{R}$ certainly is metrizable. Recall that the usual topology on $\mathbb{R}$ is the same as the order topology on $\mathbb{R}$.
- $\alpha\omega_1$ with the order topology is *not* metrizable.

Definition 2.1. A topological space X is *sequential* if for every subset $A \subseteq X$ and every $x \in \overline{A}$ there exists a sequence $(x_n : n \in \mathbb{N}) \subseteq A$ such that $x_n \to x$.

Note that if there is a sequence $(x_n : n \in \mathbb{N}) \subseteq A$ such that $x_n \to X$ then this means $x \in \overline{A}$.

Proposition 2.2. *If X is metrizable, then X is sequential.*

Proof. Let $A \subseteq X$ and let $x \in \overline{A}$. For every $n \in \mathbb{N}$, consider the ball $B(x, 1/n)$ and pick a point $y \in B(x, 1/n)$. Let $x_n = y$. Then it is easy to see that $x_n \to x$. □

3 Products and metrics

Question: Let (X, d_X) and (Y, d_y) be metric spaces. Is the product $X \times Y$ metrizable? Specifically, if we consider $X \times Y$ with the product topology, is it metrizable?

The answer is yes. Consider the function $\rho : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$\rho(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \max\{\, d_x(x_1, x_2), d_y(y_1, y_2) \,\}. \tag{2}$$

Note that ρ is called the *box distance* in Munkres.

A ball in $\mathbb{R}^2$ with the box distance would be a square, see Figure 2.

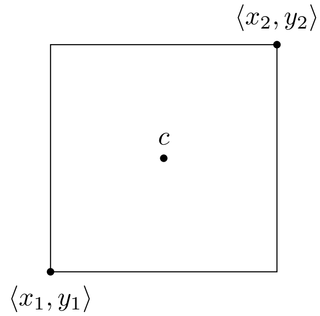


Figure 2: A ball around the point c with box distance ρ.

Proposition 3.1. *If (X, d_X) and (Y, d_Y) are metric spaces, then the product topology on $X \times Y$ is metrizable by ρ as defined in Equation (2).*

Proof. First of all, ρ is indeed a metric. Positivity and symmetry are immediate, so we only bother to check the triangle inequality. Indeed, we have

$$\begin{aligned}\rho(\mathbf{x}, \mathbf{y}) &= \max\{\, d_X(x_1, y_1), d_Y(x_2, y_2)\,\}\\ &\leq \max\{\, d_X(x_1, z_1) + d_X(z_1 + y_1), d_Y(x_2, z_2) + d_Y(z_2, y_2)\,\}\\ &\leq \rho(\mathbf{x}, \mathbf{z}) + \rho(\mathbf{z}, \mathbf{y})\end{aligned}$$

as $\max\{\, f + g\,\} \leq \max f + \max g$. Now, we shall check ρ induces the product topology. Now, $\mathbf{y} \in B_\rho(\mathbf{x}, \varepsilon)$ iff $\rho(\mathbf{x}, \mathbf{y}) < \varepsilon$, which is true iff $\max\{\, d_X(x_1, y_1), d_Y(x_2, y_2)\,\} < \varepsilon$ iff $d_X(x_1, y_1) < \varepsilon$ and $d_Y(x_2, y_2) < \varepsilon$ iff $\mathbf{y} \in B_{d_X}(x_1, \varepsilon) \times B_{d_Y}(x_2, \varepsilon)$. □

3.1 Infinite products and the uniform topology

What about infinite products? Suppose $\{\, X_\alpha\,\}_{\alpha\in\Lambda}$ is a collection of metric spaces, each with metric d_α. Take 2 points $\mathbf{x}, \mathbf{y} \in \prod_{\alpha\in\Lambda} X_\alpha$, where $\mathbf{x} = (x_\alpha : \alpha \in \Lambda)$ and $\mathbf{y} = (y_\alpha : \alpha \in \Lambda)$. We can try to define

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{\, d_\alpha(x_\alpha, y_\alpha) : \alpha \in \Lambda\,\}$$

Well, one issue is that the max may not exist. We can also try replacing it with the sup

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{\, d_\alpha(x_\alpha, y_\alpha) : \alpha \in \Lambda\,\},$$

but the sup may be infinite! This is bad. However, it turns out if we replace the metric with something that is bounded, we can indeed define a metric in this manner.

Lemma 3.2. *If (X, d) is a metric space, then $\overline{d}(x, y) = \min\{\, d(x, y), 1\,\}$ is a metric on X, and it induces the same topology that d does.*

Proof. Exercise. First check $\overline{d}$ is a metric, then check it induces the same topology on X. Use the basis lemma. □

You might worry that replacing the metric changes the topology, but fear not! This lemma assures us that replacing the metric does not actually change the topology on it, and so we can rest easy knowing that all topological properties that we care about on X are preserved.

One interesting observation that can be derived from Lemma 3.2 is that "boundedness" is not a topological notion. For example, putting this metric on $\mathbb{R}$ will mean that every point in $\mathbb{R}$ is at most a distance 1 from each other, and yet we still get the same topology that we do with the usual distance metric on $\mathbb{R}$.

Now, we can define

$$\rho(\mathbf{x}, \mathbf{y}) = \sup\{\, \overline{d_\alpha}(x_\alpha, y_\alpha) : \alpha \in \Lambda\,\}, \tag{3}$$

where $\overline{d_\alpha}$ is the bounded metric as in Lemma 3.2.

> **Definition 3.3.** Let $\{ X_\alpha \}_{\alpha\in\Lambda}$ be a collection of metrizable/metric spaces. Then the topology induced by the metric in Equation (3) on $\prod_{\alpha\in\Lambda} X_\alpha$ is called the *uniform topology.*

Now we shall see how product, uniform and box topologies are related when it comes to metric spaces.
Proposition 3.4. *Let $\{ X_\alpha \}_{\alpha\in\Lambda}$ be a collection of metrizable/metric spaces and consider $\prod_\alpha X_\alpha$ Then,*

$$\tau_{prod} \subseteq \tau_{unif} \subseteq \tau_{box}.$$

In English, this means that the box topology is finer than the uniform topology, which is finer than the product topology.

Note that if Λ is finite then they are all equal. If Λ is infinite then the inclusions are strict. See exercises in Munkres.

3.2 Metrizability of the box topology

One might wonder when the box topology is metrizable. It turns out that this is not such an easy question to answer. It may or may not be metrizable. Of course, if it is finite then the question is pretty easy: if all the spaces involved are metrizable, then the box topology certainly is.
Example 3.5. Let Λ be an infinite index set. Consider the set $\{ 0, 1 \}^\Lambda$ with the box topology. In the set $\{ 0, 1 \}$ we can give it the discrete metric (Equation (1)), which gives $\{ 0, 1 \}$ the discrete topology. Notice that $\{ 0, 1 \}^\Lambda$ now has the discrete topology, and thus is metrizable. //

It turns out that an infinite product of spaces when given the box topology is only metrizable when it is a discrete space.
Proposition 3.6. *Suppose X is metrizable, but not a discrete space. Then X^ω is not metrizable when given the box topology.*

Here, note that ω is just the natural numbers.

Proof. We shall prove that if X is not discrete, then X^ω with the box topology is not sequential (Definition 2.1). Since X is not discrete, there is some point that is not open. Let x be such a point such that $\{ x \}$ is not open. Consider any open neighborhood of x, it does not contain only x, so it must intersect $X \setminus \{ x \}$. This means that $x \in \overline{X \setminus \{ x \}}$. Now consider the constant sequence of x's given by $\mathbf{x} \in X^\omega$ such that $\mathbf{x}(n) = x$ for all $n \in \mathbb{N}$. Let $A \subseteq X^\omega$ be a set such that $\mathbf{y} \in A$ if and only if no coordinate of $\mathbf{y}$ is x (i.e. $\mathbf{y}(n) \neq x \,\forall n \in \omega$). We claim that $\mathbf{x} \in \overline{A}$. Let U be an open neighborhood of $\mathbf{x}$ in X^ω. Since $U = \prod_{n\in\omega} U_n$, and U_n is a neighborhood of x in each X, it cannot only contain x (since $\{ x \}$ is not open), and so each neighborhood must have a point that is not x. So from each of these U_n's take a point that is not x and construct a sequence using those points. This sequence is definitely an element of A as no coordinate of it is x, thus giving us a point in $U \cap A$. So we have shown that U intersects A.

We also claim that no sequence in A converges to $\mathbf{x}$. Suppose not, then there is a sequence (of sequences)[1] $(\mathbf{y}_n : n \in \mathbb{N}) \subseteq A$ in A that converges to $\mathbf{x}$. Let us keep in mind here that for each n, $\mathbf{y}_n$ is a sequence as well. For each n, we shall choose a U_n around x that does not contain the n-th coordinate of $\mathbf{y}_n$. This is possible because X is assumed to be metrizable and thus Hausdorff, so U_n can be chosen. Then $\prod_{n\in\mathbb{N}} U_n$ is an open neighborhood of x that does not contain $\mathbf{y}_n$. ☐

3.3 Metrizability of the product topology

Proposition 3.7. *Consider $\{ 0, 1 \}$ with the discrete topology. Then $\{ 0, 1 \}^\Lambda$ with the product topology is not metrizable if Λ is uncountable.*

Proof. Let

$$A = \left\{ \mathbf{x} \in \{ 0, 1 \}^\Lambda : \mathbf{x}(\alpha) = 0 \text{ for only finitely many } \alpha \right\}.$$

Let $\mathbf{0}$ be the sequence that is all zero. We claim that $\mathbf{0} \in \overline{A}$. Take any open neighborhood U of $\mathbf{0}$. Since we are working with the product topology, $U = \prod_{\alpha\in\Lambda} U_\alpha$ and $U_\alpha = \{ 0, 1 \}$ for all but finitely many α. For the U_α's that are not $\{ 0, 1 \}$, if it is just $\{ 1 \}$ we can construct a sequence that is not zero only in those finitely many α's, which would be an element of A.

[1] What do we mean here? This means $\mathbf{y}_1$ is a sequence, $\mathbf{y}_2$ is a sequence, etc. These guys are sequences in X, by the way.

We also claim that no sequence in A converges to $\mathbf{0}$. Suppose not, then we have $\mathbf{x}_n \to \mathbf{0}$. (Again here we have a sequence of sequences.) For each $n \in \mathbb{N}$, let $\Lambda_n = \{\, \alpha \in \Lambda : \mathbf{x}_n(\alpha) = 0 \,\}$. Then $\bigcup_{n\in\mathbb{N}} \Lambda_n$ is countable, so let $\alpha \in \Lambda \setminus \bigcup_{n\in\mathbb{N}} \Lambda_n$. This α is the index where for every n, we have $\mathbf{x}_n(\alpha) = 1$. Now construct the neighborhood $\{\, 1 \,\} \times \prod_{\beta\in\Lambda,\beta\neq\alpha} \{\, 0, 1 \,\}$. This is a valid open set in the product topology, and it is a neighborhood of $\mathbf{0}$, but it does not have any terms of the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ being within this neighborhood, contradicting the convergence of this sequence to $\overline{0}$. □

Corollary 3.8. *If $(X_\alpha)_{\alpha\in\Lambda}$ is an uncountable collection of metric spaces where each X_α has at least 2 points, then $prod_{\alpha\in\Lambda} X_\alpha$ with the product topology is not metrizable.*

Proof. Notice that $\{\, 0, 1 \,\}$ with the discrete is homeomorphic to a subspace of X_α for each α. If $prod_{\alpha\in\Lambda} X_\alpha$ □

Proposition 3.9. *Let (X_n, d_n) be a countable collection of metrizable (metric) spaces. Then $\prod_{n\in\mathbb{N}} X_n$ with the product topology is metrizable, with the metric*

$$D : \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n \to \mathbb{R}$$

be given by $D(\overline{x}, \overline{y}) = \sup \left\{ \frac{\overline{d_n}(x_n, y_n)}{n} : n \in \mathbb{N} \right\}$. Then D is a metric that induces the product topology on $\prod_{n\in\mathbb{N}} X_n$.

Proof. See Munkres. □