

# Algebra

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### **Abstract**

This is a collection of notes for abstract algebra, but it could be interpreted as a textbook, as we include exercises. We will cover the theory of groups, rings, fields as well as modules.

The only prerequisite for this book is good mathematical maturity. It would help slightly if you have had linear algebra, as some of our examples depend on linear algebra.

Corrections and improvements are always appreciated. Please email me with any corrections.

# Contents

<b>1</b>	<b>Groups</b>	<b>2</b>
1.1	Groups . . . . .	2
1.1.1	Problems . . . . .	4
1.2	Subgroups . . . . .	5
1.2.1	Problems . . . . .	5
1.3	Cyclic groups . . . . .	5
<b>2</b>	<b>Rings</b>	<b>6</b>
<b>3</b>	<b>Field Extensions and Splitting Fields</b>	<b>7</b>
3.1	Extension fields . . . . .	7
3.2	Splitting Fields . . . . .	7
	<b>References</b>	<b>8</b>

# Chapter 1

## Groups

### 1.1 Groups

There are many ways to motivate what a group is meant to do. However, we believe that the reader is already sufficiently motivated to learn the content.

**Definition 1.1.1 (Group).** A group is a set  $G$  with a binary operation  $\cdot : G \times G \rightarrow G$  such that

1. **(Associativity)** For all  $x, y, z \in G$ ,  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ .
2. **(Identity)** There exists  $e \in G$  such that for all  $g \in G$ ,  $e \cdot g = g \cdot e = g$ .
3. **(Inverses)** For all  $g \in G$ , there exists  $h \in G$  such that  $g \cdot h = h \cdot g = e$ .

Note that the order of properties 2 and 3 do matter. We cannot write property 3 before property 2.

To make notation clearer, we shall write  $gh$  for  $g \cdot h$ . We may sometimes use addition to denote the group operation as well, writing  $g + h$ . Additionally, because of associativity, we can drop any brackets. This means that there is no ambiguity about what  $xyz$  is. Recall that when adding numbers,  $(2 + 3 + 4) + 5 = (2 + 3) + (4 + 5)$ . Of course, it follows that you can drop the brackets for finitely many elements.

**Exercise 1.1.2.** Let  $G$  be a group. Prove that associativity holds for finitely many elements  $x_1, \dots, x_n \in G$ . For example,  $(xy)(zw) = x((yz)w)$ .

Additionally, if we can commute elements under the group operation, the group is called Abelian.

**Definition 1.1.3 (Abelian group).** Let  $G$  be a group. Then  $G$  is Abelian if for every  $g, h \in G$ , we have  $gh = hg$ .

We will immediately prove some properties about groups.

**Theorem 1.1.4 (Basic properties of groups).** Let  $G$  be a group. Then, the following are true.

1. The identity of  $G$  is unique.
2. If  $g \in G$  has an inverse  $h$ , then it is unique.

*Proof.* (1) Let  $e, e' \in G$  and suppose both  $e, e'$  are identities. Keeping in mind that they satisfy the property of being an identity, we have,

$$e = ee' = e'e = e'.$$

(2) Suppose  $h, h'$  are both inverses of  $g$ . Again keeping in mind that  $h, h'$  both satisfy the properties of being an inverse for  $g$ .

$$h = h(h'g) = h(gh') = (hg)h' = h'.$$

□

Henceforth we shall talk about "the" identity of a group, and "the" inverse of an element. Additionally, if  $g \in G$ , then we shall denote the inverse of  $g$  by  $g^{-1}$ .

Let us now see some examples of groups.

**Example 1.1.5.** The integers form a group under usual addition. Clearly the identity is 0. Inverses are obvious. //

**Example 1.1.6.** The set of integers under usual multiplication is not a group. There is no multiplicative inverse for 2. //

**Example 1.1.7.** Let  $V$  be a vector space over  $\mathbb{R}$ . Then  $V$  is a group under vector addition. //

**Example 1.1.8.** Let  $\text{GL}_n(\mathbb{R})$  denote the set of  $n \times n$  invertible matrices with real entries. Then this set is a group under the operation of matrix multiplication. //

**Example 1.1.9.** Let  $n$  be an integer. Let  $D_n$  be the set of symmetries of a regular  $n$ -sided polygon. **TODO: This example needs to be improved** //

**Example 1.1.10.** The real numbers form a group under usual addition. The real numbers without 0 form another group under usual multiplication. //

Note that the previous example illustrates an important point. *The same (similar) set can be a different group when the operation is replaced.* This tells us that to specify a group, we need both the set, as well as the group operation. However, if the operation does not matter, or it is clear from context, we shall simply say that  $G$  is a group.

**Exercise 1.1.11.** Verify that all of the above examples which are claimed to be groups are indeed groups.

**Exercise 1.1.12.** Groups can be finite or infinite in size. Identify which of the above groups are finite and which are not.

**Exercise 1.1.13.** Not every group is Abelian. Identify which of the groups above are abelian and which are not.

We state a few more properties of groups. While reading the proof, the reader should keep in the mind the proof strategies being deployed here.

**Theorem 1.1.14.** Let  $G$  be a group. Then, the following are true.

1. **(Generalized associativity)** For any  $x_1, \dots, x_n \in G$ , the value of  $x_1 \cdots x_n$  is independent of how it is bracketed.
2. If  $g \in G$ , then  $(g^{-1})^{-1} = g$ .
3. **(Socks-shoes property)** If  $g, h \in G$ , then  $(gh)^{-1} = h^{-1}g^{-1}$ .
4. **(Cancellation)** Let  $g, h, h' \in G$ . If  $gh = gh'$  then  $h = h'$ . This is called left cancellation. Additionally, if  $hg = h'g$ , then  $h = h'$ . This is called right cancellation.

*Proof.* (1) is [Exercise 1.1.2](#). If you could not do this, see [DF04, Prop 1, p. 19].

(2) Write

$$(g^{-1})(g^{-1})^{-1} = e = g^{-1}g.$$

Then the result follows by uniqueness of inverses.

(3)

$$(gh)^{-1}(gh) = e = (h^{-1}g^{-1})(gh) = h^{-1}(g^{-1}g)h.$$

(4) Exercise for reader. □

To ensure that the reader is adequately familiar with the techniques of the proof above, we include the following simple exercises.

**Exercise 1.1.15.** Prove part (4) of [Theorem 1.1.14](#).

The next example is an important infinite family of finite groups.

**Example 1.1.16.** Let  $\mathbb{Z}_n = \{0, \dots, n-1\}$  be equipped with the operation of addition modulo  $n$ . That is, we define  $+$  on  $\mathbb{Z}_n$  to be given by  $a + b = (a + b) \bmod n$ . This is called the *group of integers modulo  $n$* , or alternatively *the cyclic group of order  $n$* . We will soon see what this means. //

**Exercise 1.1.17.** Verify that  $\mathbb{Z}_n$  with the operation as defined above is indeed a group.

**Example 1.1.18.** Let  $U(n)$  denote the set of all nonnegative integers  $k \leq n$  such that  $\gcd(k, n) = 1$ . Then  $U(n)$  is a group under the operation of multiplication modulo  $n$ . That is, if  $a, b \in U(n)$ ,  $ab = a \cdot b \bmod n$ . //

The next example is also rather important.

**Example 1.1.19** (Symmetric groups). Let  $S = \{1, \dots, n\}$ . Then consider the set of all permutations of  $S$  (bijective functions from  $S$  to  $S$ ). We shall call this set  $S_n$ , which stands for *symmetric group on  $n$  things*. This set is a group under function composition. //

**Exercise 1.1.20.** Prove that  $S_n$  is a group under function composition.

We now introduce some notation. Let  $G$  be a group and  $g \in G$ . We shall write

$$g^n = \underbrace{gg \cdots g}_{n \text{ times}}$$

to mean  $g$  multiplied by itself  $n$  times.

### 1.1.1 Problems

**Problem 1.1.1.** Let  $G$  be a group. Prove that if for every  $g \in G$ , we have  $g^2 = e$ , then  $G$  is Abelian.

## 1.2 Subgroups

We can essentially understand a subgroup as a smaller group that's contained within a group.

**Definition 1.2.1 (Subgroup).** Let  $G$  be a group. A subset  $H \subseteq G$  is a **subgroup** of  $G$  if the following properties hold.

1. The identity of  $G$  is in  $H$ .
2. For all  $x, y \in H$ ,  $xy \in H$ .
3. For all  $x \in H$ ,  $x^{-1} \in H$ .

This tells us that if we restrict the operation of  $G$  to  $H$ , then  $H$  is still a group. If  $H$  is a *proper* subgroup of  $G$ , it means that  $H$  is a proper subset of  $G$ .

**Theorem 1.2.2 (Subgroup tests).** Let  $G$  be a group and  $H \subseteq G$ . Then, the following are equivalent.

1.  $H$  is a subgroup of  $G$ .
2.  $H$  is nonempty, and for all  $x, y \in H$ , we have  $xy^{-1} \in H$ .

*Proof.* We will not insult the reader's intelligence by providing a proof. □

**Exercise 1.2.3.** Prove [Theorem 1.2.2](#).

### 1.2.1 Problems

**Problem 1.2.1.** Prove that no group is the union of 2 proper subgroups. (No cheating and looking this up)

## 1.3 Cyclic groups

**Definition 1.3.1 (Cyclic group).** Let  $x$  be anything.

## Chapter 2

# Rings

TBD



# Chapter 3

## Field Extensions and Splitting Fields

### 3.1 Extension fields

Given a polynomial, is it possible to find a field in which that polynomial has a root? For example, consider the polynomial  $x^2 + 1$ .

**Definition 3.1.1.** Let  $F$  be a field. If  $E \supseteq F$  is a field and the operations of  $E$  restricted to  $F$  are the same as the operations of  $F$ , then  $E$  is an **extension field** of  $F$ .

If  $E$  is an extension field of  $F$ , we can say that  $E$  is an extension of  $F$ , or  $E$  extends  $F$ . Note the abuse of notation here again:  $F$  may not actually be a subset of  $E$ , but if it is isomorphic to a subfield of  $E$  it is good enough.

**Example 3.1.2.**  $\mathbb{C}$  is clearly an extension field of  $\mathbb{R}$ . Additionally,  $\mathbb{R}$  is an extension field of  $\mathbb{Q}$ . //

**Example 3.1.3.** Let  $F$  be a field and let  $p \in F[x]$  be irreducible over  $F$ . Then,  $F[x]/\langle p \rangle$  is an extension field of  $F$ . Notice that we can embed  $F$  as a subfield of  $F[x]/\langle p \rangle$  by the map

$$x \mapsto x + \langle p \rangle.$$

It is not too hard to see that this map is an isomorphism onto its image. We will use this example to motivate the following theorem. //

**Theorem 3.1.4 (Existence of Extension Fields).** Let  $F$  be a field and let  $f \in F[x]$  be a nonconstant polynomial. Then there exists an extension field  $E$  of  $F$  such that  $f$  has a root in  $E$ .

*Proof.* Let  $p(x)$  be an irreducible factor of  $f$ . This exists as  $F[x]$  is a UFD. It suffices to produce an extension field of  $F$  where  $p$  has a root in. Let  $E = F[x]/\langle p \rangle$ . Then  $F$  embeds into  $E$ . Now, we see that  $x + \langle p \rangle$  is a root of  $p$  in  $E$ . Write  $p(x) = \sum_{i=0}^n a_i x^i$ , then

$$p(x + \langle p \rangle) = \sum_{i=0}^n a_i (x + \langle p \rangle)^i = \left( \sum_{i=0}^n a_i x^i \right) + \langle p \rangle = \langle p \rangle.$$

□

Note that if  $D$  is an integral domain and  $p \in D[x]$ , then there is an extension field of  $Q(D)$  that contains a root of  $p$ . This means that there is an extension field that contains  $D$ . This need not be true if  $D$  is not an integral domain.

**Example 3.1.5.** Let  $f(x) = 2x + 1$  in  $\mathbb{Z}_4[x]$ . Then given any ring  $R \supseteq \mathbb{Z}_4$ ,  $f$  has no roots in  $R$ . //

### 3.2 Splitting Fields

**Definition 3.2.1.** Let  $F$  be a field, and let  $E$  be an extension of  $F$ . Then we *define*  $F(a_1, \dots, a_n)$  to be the *smallest* subfield of  $E$  that contains  $F$  and  $\{a_1, \dots, a_n\}$ .

It immediately follows that  $F(a_1, \dots, a_n)$  is the intersection of all subfields of  $E$  that contain  $F$  and  $\{a_1, \dots, a_n\}$ . We warn the reader that it is important that we have an extension field to talk about. For example, it is nonsensical to write something like  $\mathbb{Q}(\text{apple})$  when we don't have any field that contains apple in it.

**Definition 3.2.2 (Polynomial splitting).** Let  $F$  be a field and let  $E$  be an extension of  $F$ . Let  $f \in F[x]$ . Then  $f$  **splits** in  $E$  if it can be factorized into linear factors, i.e. we have  $a \in F$ ,  $a_i \in E$  such that

$$f(x) = a(x - a_1) \cdots (x - a_n).$$

We say that  $E$  is a **splitting field** for  $f$  if  $E = F(a_1, \dots, a_n)$ .

In other words,  $E$  is a splitting field for  $f$  it is the smallest field that contains  $F$  and all roots of  $f$ . We remark that whether a polynomial splits depends on which field the polynomial comes from.

**Example 3.2.3.** Let  $f(x) = x^2 + 1$  in  $\mathbb{Q}[x]$ . Then  $\mathbb{C}$  is *not* a splitting field of  $f$  over  $\mathbb{Q}$ , since we can find a smaller field that still contains roots of  $f$ , namely,  $\mathbb{Q}[x]/\langle f \rangle$ . //

It would be pretty stupid if splitting fields did not exist. Luckily they do.

**Theorem 3.2.4 (Splitting fields exist).** Let  $F$  be a field and  $f \in F[x]$  be nonconstant. Then there is a splitting field of  $f$  over  $F$ .

The proof of the theorem is simple: induction on  $\deg f$  and use [Theorem 3.1.4](#).

*Proof.* We go by induction<sup>1</sup> on  $\deg f$ . If  $\deg f = 1$  it is trivial:  $f(x) = (x - a)$  for some  $a \in F$ . Now suppose the theorem is true for all polynomials of degree less than  $\deg f$  and all fields. By [Theorem 3.1.4](#), there is an extension field  $E \supseteq F$  such that  $f$  has a root in  $E$ . Let this root be  $a_1$ . Then we factorize  $f$  over  $E$ , so write  $f(x) = (x - a_1)g(x)$ , where  $g(x) \in E[x]$ . Thus there is a splitting field  $K \supseteq E$  of  $g$  over  $E$ .  $K$  has all roots of  $g$ , say they are  $a_2, \dots, a_n$ . Since  $E \supseteq F$ ,  $K$  contains  $a_1$ ,  $F$  and  $a_2, \dots, a_n$ . So we can take the splitting field to be  $F(a_1, \dots, a_n)$ .  $\square$

Now we can finally give some examples of splitting fields.

**Example 3.2.5.** Let  $f(x) = x^2 + 1$ , but this time considered as an element of  $\mathbb{R}[x]$ . Then  $\mathbb{C}$  is a splitting field of  $f$  over  $\mathbb{R}$ . Notice that  $\mathbb{R}[x]/\langle f \rangle$  also is a splitting field of  $f$ . Are these the same splitting field? We will answer this soon. //

<sup>1</sup>Note that strong induction is used here, since  $\deg g$  may not necessarily be  $\deg f - 1$ . If I am wrong, please correct me.

# Bibliography

- [DF04] David Steven Dummit and Richard M. Foote. *Abstract algebra*. 3rd ed. Hoboken, NJ: Wiley, 2004. ISBN: 9780471433347.
- [Gal20] Joseph A. Gallian. *Contemporary abstract algebra*. Tenth edition. Boca Raton: Chapman & Hall/CRC, 2020. ISBN: 9781003142331.