Complex Variables Sections 5, 8 Homework

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Section 5: # 2a, 10.
Section 8: # 1, 2.

1 Problem 5.2.a

Complex conjugation and adding imaginary numbers should not affect the real component of the values; hence $\mathrm{Re}(\bar{z} - i) = 2$ is equivalent to $\mathrm{Re}(z) = 2$, as depicted in Figure 1.

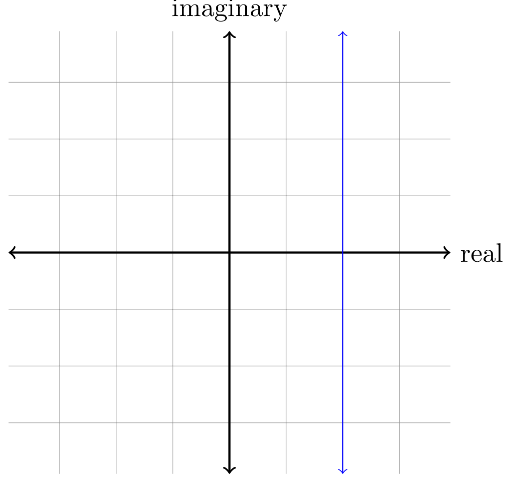

Figure 1: The solutions to 5.2.a.

2 Problem 5.10

2.1 Part 5.10.a

Theorem 2.2. *z is real if and only if $\bar{z} = z$.*

Proof. If z is real, $z = (x, 0)$, so $z = (x, 0) = (x, -0) = \bar{z}$.

If z is not real, then $z = (x, y)$ for nonzero y. Then $z = (x, y)$ and $\bar{z} = (x, -y)$, which are not equivalent since $y \neq -y$. $\square$

2.3 Part 5.10.b

Theorem 2.4. *z is either real or pure imaginary iff $\bar{z}^2 = z^2$.*

Proof. Suppose z is either real or pure imaginary. Then, regardless, z^2 is real (obviously in the real case; in the pure imaginary case, $z^2 = (0, y)^2 = (-y^2, 0)$.). Since z^2 is real, $z^2 = \overline{z^2} = \bar{z}^2$.

If $\bar{z}^2 = \overline{z^2} = z^2$, this implies z^2 must be real. So if $z = (x, y)$, then $\mathrm{Im}(z^2) = \mathrm{Im}(x^2 - y^2, 2xy) = 2xy = 0$. Thus, either $x = 0$ or $y = 0$; in the former case, z is purely imaginary, and in the latter z is real. Thus z is either real or purely imaginary. $\square$

3 Problem 8.1

3.1 Part 8.1.a

$$\begin{aligned}
\arg(z) &= \arg\left(\frac{i}{-2-2i}\right) \\
&= \arg(i) - \arg(-2-2i) \\
&= \pi/2 - (-3\pi/4) + 2\pi\mathbb{Z} \\
&= 2\pi/4 + 3\pi/4 + 2\pi\mathbb{Z} \\
&= 5\pi/4 + 2\pi\mathbb{Z} \\
&= -3\pi/4 + 2\pi\mathbb{Z} \\
\mathrm{Arg}(z) &= -3\pi/4
\end{aligned}$$

3.2 Part 8.1.b

$$\begin{aligned}
z &= \arg\left(\left(\sqrt{3}-i\right)^6\right) \\
&= 6 \cdot \arg\left(\sqrt{3}-i\right) \\
&= 6 \cdot (-\pi/6 + 2\pi\mathbb{Z}) \\
&= (-\pi/6) + (-\pi/6) + (-\pi/6) \\
&\qquad + (-\pi/6) + (-\pi/6) + (-\pi/6) + 2\pi\mathbb{Z} \\
&= -\pi + 2\pi\mathbb{Z} \\
&= \pi + 2\pi\mathbb{Z} \\
\mathrm{Arg}(z) &= \pi
\end{aligned}$$

It is essential to note that the multiplication by 6 above is not the coset $6 + 2\pi\mathbb{Z}$ in the quotient field $\mathbb{R}/2\pi\mathbb{Z}$ which the arg generally occupies, but rather the action of the integer 6 in that field on the coset-valued return of the arg above, as in a $\mathbb{Z}$-module. It does *not* stretch the coset into $-\pi + 12\pi\mathbb{Z}$.

4 Problem 8.2

4.1 Problem 8.2.a

Theorem 4.2.

$$|e^{i\theta}| = 1$$

Proof. Using the definitions of $e^{i\theta}$ and the modulus,

$$\begin{aligned}
|e^{i\theta}| &= |\cos\theta + i\sin\theta| \\
&= \sqrt{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} \\
&= \sqrt{\cos^2\theta + \sin^2\theta} \\
&= \sqrt{1} \\
&= 1
\end{aligned}$$

$\square$

4.3 Problem 8.2.b

Theorem 4.4.

$$\overline{e^{i\theta}} = e^{-i\theta}$$

Proof.

$$\begin{aligned}
\overline{e^{i\theta}} &= \bar{(}\cos\theta, \sin\theta) \\
&= (\cos\theta, -\sin\theta) \\
&= (\cos(-\theta), \sin(-\theta)) \\
&= e^{-i\theta}
\end{aligned}$$

$\square$