Complex Variables Section 56 Homework

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Prove that if $|z_n| \to 0$, then $z_n \to 0$.
Converge or diverge: $\sum_{k=1}^{\infty}(i/2)^k$, $\sum_{k=1}^{\infty}(1-i)^k$.

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Theorem 1.1. *If* $\lim_{n\to\infty} z_n = z$*, then* $\lim_{n\to\infty} |z_n| = |z|$.

Proof. Suppose $z_n \to z$. Then for any $\epsilon > 0$, there exists N such that if $n \geq N$, then

$$|z_n - z| < \epsilon$$

$$||z_n| - |z|| \leq |z_n - z| < \epsilon$$

Thus, for any ϵ there exists N such that

$$||z_n| - |z|| < \epsilon$$

so $|z_n| \to |z|$. □

2 Modulus convergence

Theorem 2.1. *If* $|z_n| \to 0$*, then* $z_n \to 0$.

Proof. For an arbitrary ϵ, there exists some N such that $n \geq N \implies |z_n - 0| = |z_n| < \epsilon$. Then $\sqrt{x_n^2 + y_n^2} < \epsilon$, so $x_n^2 + y_n^2 < \epsilon^2$ so $x_n^2 < \epsilon^2$ and $y_n^2 < \epsilon$, and thus $|x_n - 0| < \epsilon$ and $|y_n - 0| < \epsilon$. Since ϵ was arbitrary, $x_n \to 0$ and $y_n \to 0$. Thus $z_n = x_n + iy_n \to 0 + i0 = 0$. □

3 Converge or diverge

3.1 First one

$$\sum_{k=1}^{\infty}(i/2)^k = i/2 - 1/4 - i/8 + 1/16 + \cdots = \sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^k - i\frac{1}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^{k-1}$$

The summations work out individually, utilizing the ratio test, so the complex sums converge also.

3.2 Second one

This cannot converge, since the underlying sequence does not converge to 0; by the ratio test,

$$\left|\frac{(1-i)^{k+1}}{(1-i)^k}\right| = |1-i| = \sqrt{2} > 1$$

And since the sequence being summed does not converge to 0, the summation itself cannot converge.