Complex Variables Homework Section 34

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p. 108 #7 (Just do $|\sin z|^2$). Find $\cos^{-1}(i)$.

1 Problem 7

$$\begin{aligned}
|\sin z|^2 = \sin z \cdot \overline{\sin z} &= (\sin x \cosh y + i \cos x \sinh y) \cdot \overline{(\sin x \cosh y - i \cos x \sinh y)} \\
&= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^2 y \\
&= \sin^2 x \cosh^2 y + \sinh^2 y - \sin^2 x \sinh^2 y \\
&= \sin^2 x \left(\cosh^2 y - \sinh^2 y\right) + \sinh^2 y \\
&= \sin^2 x + \sinh^2 y
\end{aligned}$$

2 $\cos^{-1}(i)$

$$\begin{aligned}
\cos^{-1} i &= -i \log\left(i + i(1 - i^2)^{1/2}\right) \\
&= -i \log\left(i(1 + 2^{1/2})\right) \\
&= -i \log\left(i(1 \pm \sqrt{2})\right)
\end{aligned}$$

For the positive square root,

$$\begin{aligned}
(+) &= -i \log(i(1 + \sqrt{2})) \\
&= -i \left(\ln(1 + \sqrt{2}) + i\left(\frac{\pi}{4} + 2n\pi\right)\right) && (n \in \mathbb{Z}) \\
&= -i \ln(1 + \sqrt{2}) + \frac{\pi}{4} + 2n\pi && (n \in \mathbb{Z})
\end{aligned}$$

For the negative square root,

$$\begin{aligned}
(-) &= -i \log(i(1 - \sqrt{2})) = -i \log(-i(\sqrt{2} - 1)) \\
&= -i \left(\ln(\sqrt{2} - 1) + i\left(-\frac{\pi}{4} + 2n\pi\right)\right) && (n \in \mathbb{Z}) \\
&= -i \ln(\sqrt{2} - 1) - \frac{\pi}{4} + 2n\pi && (n \in \mathbb{Z}) \\
&= i \ln(1 + \sqrt{2}) - \frac{\pi}{4} + 2n\pi && (n \in \mathbb{Z})
\end{aligned}$$

Therefore,

$$\cos^{-1} i = \pm i \ln(1 + \sqrt{2}) + \frac{\pi}{4} + 2n\pi \qquad (n \in \mathbb{Z})$$