Complex Variables Final Exam

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1 Problem 1

We decompose S into four portions.

$$\begin{aligned}
\int_S \bar{z}\ dz &= \int_0^2 x\ dx + \int_0^2 2 + yi\ dy + \int_2^0 x + 2i\ dx + \int_2^0 yi\ dy \\
&= \int_0^2 x\ dx + 4 + i\int_0^2 y\ dy - \int_0^2 x\ dx - 4i - i\int_0^2 y\ dy \\
&= 4 - 0 + 4 + i(4-0) - (4-0) - 4i - i(4-0) \\
&= 4 + 4 + 4i - 4 - 4i - 4i \\
&= 4 - 4i
\end{aligned}$$

2 Problem 2

The value of this integral over the spiral-shaped contour may be simplified. Namely, z^2 is entire; thus we may define a loop homotopy of C into $C'(t) = 1 + t$ by

$$H : [1,2] \times [0,1] \to \mathbb{C} \qquad\qquad H(t,s) \stackrel{\text{def}}{=} (1-s) \cdot te^{2\pi i t} + s \cdot t$$

Thus,

$$\begin{aligned}
\int_C z^2\ dz &= \int_{C'} z^2\ dz \\
&= \int_1^2 x^2\ dx \\
&= \left.\frac{x^3}{3}\right|_1^2 \\
&= \frac{2^3}{3} - \frac{1^3}{3} \\
&= \frac{8-1}{3} \\
&= \frac{7}{3}
\end{aligned}$$

3 Problem 3

$$\begin{aligned}
\int_C \frac{\cos 2z}{(z-\pi)^3}\,dz &= \frac{2\pi i}{2!}\left[D_z^2(z \mapsto \cos(2z))\right](\pi) \\
&= \pi i \left[D_z(z \mapsto -2\sin(2z))\right](\pi) \\
&= \pi i \left[z \mapsto -4\cos(2z)\right](\pi) \\
&= -4\pi i \cos(2\pi) \\
&= -4\pi i
\end{aligned}$$

4 Problem 4

5 Problem 5

6 Problem 6

7 Problem 7

8 Problem 8

9 Problem 9

10 Problem 10

11 Problem 11

Theorem 11.1. *Suppose that f is an entire function satisfying*

$$f(z+i) = f(z) \text{ and } f(z+1) = f(z) \text{ for all } z$$

Then f is constant.

Proof. Note that if f is entire it is continuous. Thus it takes compact subsets of $\mathbb{C}$ to compact subsets of $\mathbb{C}$. Consequently, the image of $[0,1]+[0,1]i$ under f, $f([0,1]+[0,1]i)$, is bounded; that is,

$$|f([0,1]+[0,1]i)| < B$$

for some bound $B \in \mathbb{R}$.

But then if $f([n,n+1]+[m,m+1]i)$ is bounded, we also know that by the presuppositions of the theorem, all its neighboring cells share the same bound:

$$\begin{aligned}
|f([n,n+1]+[m,m+1]i| &= |f([n+1,n+2]+[m,m+1]i| < B \\
|f([n,n+1]+[m,m+1]i| &= |f([n-1,n]+[m,m+1]i| < B \\
|f([n,n+1]+[m,m+1]i| &= |f([n,n+1]+[m+1,m+2]i| < B \\
|f([n,n+1]+[m,m+1]i| &= |f([n,n+1]+[m-1,m]i| < B
\end{aligned}$$

Since every point of $\mathbb{C}$ lies in some integer unit square in the complex plane, and all of these by induction using the equations above are bounded in modulus by B, f is bounded in modulus by B over all of $\mathbb{C}$. Since f is bounded in modulus by B, by the maximum modulus principle, f must be constant. □