Complex Variables Homework Section 33

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1 Problem 1

Find $(-1)^{1/3}$ and PV $(-1)^{1/3}$.

$$
\begin{aligned}
(-1)^{1/3} &= e^{1/3 \log(-1)} \\
&= e^{1/3(\ln 1 + i(\pi + 2\pi n))} \\
&= e^{1/3(i(\pi + 2\pi n))} \\
&= e^{i\pi/3 + 2/3 i\pi n} \\
&= e^{i(\pi/3 + 2/3\pi n)} \\
&= \cos\left(\frac{\pi}{3} + \frac{2}{3}\pi n\right) + i \sin\left(\frac{\pi}{3} + \frac{2}{3}\pi n\right)
\end{aligned}
$$

In particular, the principal value of $(-1)^{1/3}$ is $0.5 + \sqrt{3}/2$.

2 Problem 2

Lemma 2.1.

$$\overline{e^z} = e^{\bar{z}}$$

Proof. Let $x + yi = z$.

$$
\begin{aligned}
\overline{e^z} &= \overline{e^x} e^{yi} \\
&= \overline{e^x \left(\cos(y) + i \sin(y)\right)} \\
&= e^x \overline{\left(\cos y + i \sin y\right)} \\
&= e^x \left(\cos y - i \sin y\right) \\
&= e^x \left(\cos(-y) + i \sin(-y)\right) \\
&= e^x e^{i(-y)} \\
&= e^{x-yi} \\
&= e^{\bar{z}}
\end{aligned}
$$

$\square$

Theorem 2.2.

$$\overline{\sin(z)} = \sin(\bar{z})$$

Proof.

$$\overline{\sin(z)} = \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)}$$

$$= \frac{\overline{\left(e^{iz} - e^{-iz}\right)}}{\overline{2i}}$$

$$= \frac{\overline{e^{iz}} - \overline{e^{-iz}}}{-2i}$$

$$= \frac{e^{(\overline{iz})} - e^{(\overline{-iz})}}{-2i}$$

$$= \frac{e^{\bar{i}\bar{z}} - e^{\overline{-i}\bar{z}}}{-2i}$$

$$= \frac{e^{-i\bar{z}} - e^{i\bar{z}}}{-2i}$$

$$= \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i}$$

$$= \sin(\bar{z})$$

□