Homework 2016-04-14

Adam Buskirk

April 19, 2016

1 Problem 5, p. 206

$$f(z) = \frac{z+1}{z-1}$$

1.1 Part a

To obtain a Maclaurin series for this, first we must compute the derivatives.

$$\begin{aligned} f(z) &= \frac{z+1}{z-1} \\ f'(z) &= \frac{1}{z-1} + (-1)\frac{z+1}{(z-1)^2} \\ &= \frac{z-1}{(z-1)^2} - \frac{z+1}{(z-1)^2} \\ &= 2\frac{(-1)^1 \cdot 1!}{(z-1)^{1+1}} \end{aligned}$$

Suppose then that $f^{(n)}(z)$ is of the form $2(-1)^n n!/(z-1)^{n+1}$, as is above for $n = 1$.

$$\begin{aligned} f^{(n+1)}(z) &= (-n-1) \cdot 2\frac{(-1)^n \cdot n!}{(z-1)^{n+2}} \\ &= 2\frac{(-1)^{n+1} \cdot (n+1)!}{(z-1)^{n+2}} \end{aligned}$$

which is also of that form. Thus, by induction, for $n \geq 1$,

$$f^{(n)}(z) = 2\frac{(-1)^n \cdot n!}{(z-1)^{n+1}}$$

Thus, the coefficients a_n of our Maclaurin series are $a_0 = -1$ and

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} \cdot 2\frac{(-1)^n \cdot (n!)}{(0-1)^{n+1}} = -2$$

Thus,

$$f(z) = -1 - 2\sum_{n=1}^{\infty} z^n$$

and since the analytic circle around this only ends at $R_0 = 1$, this Maclaurin series holds for the disk $|z| < 1$.

1.2 Part b

For the Laurent series,

$$\begin{aligned}\frac{z+1}{z-1} &= \\ &= \cdots \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}\end{aligned}$$

since

$$\begin{aligned}a_n &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-0)^{n+1}}\,dz \\ &= \frac{1}{2\pi i}\int_C \frac{\frac{z+1}{z-1}}{z^{n+1}}\,dz \\ &= \frac{1}{2\pi i}\int_C \frac{z+1}{(z-1)z^{n+1}}\,dz \\ &= \frac{1}{2\pi i}\int_C \frac{1}{(z-1)z^n} + \frac{1}{(z-1)z^{n+1}}\,dz\end{aligned}$$

I do not understand how to continue this but I also do not want to delay handing in this assignment even further. Previous attempts based on re-evaluating the interior of those integrals as

$$\frac{\frac{1}{z-1}}{z^n}$$

and applying the Cauchy integral formula lead to the conclusion that $a_n = 2$ or something of that sort, and this is evidently not the case based on the answer in the book; and no progress was made with b_n.

2 Problem 1

Evaluate

$$\int_C \frac{z+1}{z-1}\,dz$$

where C is a positively oriented contour around 1.

Expand C to be the positively oriented circle contour of radius 2 around 0, so that it lies entirely within the domain of the Laurent series representation.

$$\begin{aligned}\int_C \frac{z+1}{z-1}\,dz &= \int_C 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n}\,dz \\ &= \int_C 1\,dz + 2\sum_{n=1}^{\infty}\int_C \frac{1}{z^n}\,dz \\ &= 0 + 2\int_C \frac{1}{z}\,dz + 2\sum_{n=1}^{\infty}\int_C \frac{1}{z^{n+1}}\,dz\end{aligned}$$

Utilizing the Cauchy integral formula, and specifically example 2 on page 168,

$$\begin{aligned}&= 0 + 2(2\pi i) + 2\sum_{n=1}^{\infty} 0 \\ &= 4\pi i\end{aligned}$$

3 Problem 2

Find the Laurent series for

$$f(z) = \frac{2}{1 - z^2} = \frac{1}{1 - z} + \frac{1}{1 + z}$$

We utilize example 4 in the text, on page 194.

$$\begin{aligned}
f(z) &= \frac{1}{1-z} + \frac{1}{1+z} \\
&= \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} (-z)^n \\
&= \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} (-1)^n z^n \\
&= \sum_{n \in 2\mathbb{N}} 2z^n && \text{(Note the change of index! We assume } 0 \in \mathbb{N}.) \\
&= \sum_{n=0}^{\infty} 2z^{2n} \\
&= \sum_{n=0}^{\infty} 2(z^2)^n \\
&= 2\frac{1}{1-z^2} \\
&= \frac{2}{1-z^2} \\
&= f(z)
\end{aligned}$$

Thus, the Laurent series for $f(z)$ is $\sum_{n=0}^{\infty} z^{2n}$.