Complex Variables Final Exam

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1 Problem 1

We decompose S into four portions.

From 0 to 2 (S_1): The integral is 2, since this reduces to a simple real integral $\int_0^2 x\ dx$.

From 2 to $2+2i$ (S_2):

$$\begin{aligned}
z(t) &= 2(1-t) + (2+2i)t \\
&= 2 - 2t + 2t + 2it \\
&= 2 + 2it \\
z'(t) &= 2i \\
\int_{S_2} f(z)\ dz &= \int_0^1 f(z(t))z'(t)\ dt \\
&= \int_0^1 \overline{2+2it} \cdot 2i\ dt \\
&= \int_0^1 (2-2it) \cdot 2i\ dt \\
&= \int_0^1 4i + 4t\ dt \\
&= \int_0^1 4t + 4i\ dt \\
&= 4\int_0^1 t\ dt + 4i \int_0^1\ dt \\
&= 4\left[1/2\right] + 4i\left[1\right] \\
&= 2 + 4i
\end{aligned}$$

From $2+2i$ to $2i$ (S_3):

$$\begin{aligned}
z(t) &= (2+2i)(1-t) + 2it \\
&= 2 + 2i - 2t - 2it + 2it \\
&= 2 + 2i - 2t \\
z'(t) &= -2 \\
\int_{S_3} f(z)\ dz &= \int_0^1 f(z(t))z'(t)\ dt \\
&= \int_0^1 \overline{2+2i-2t} \cdot (-2)\ dt \\
&= -2\int_0^1 (2-2t-2i)\ dt
\end{aligned}$$

$$
\begin{aligned}
&= -2\left[(2-2i)\int_0^1 dt - 2\int_0^1 t\ dt\right] \\
&= -2\left[(2-2i)1 - 2(1/2)\right] \\
&= -2\left[2-2i-1\right] \\
&= -2\left[1-2i\right] \\
&= -2+4i
\end{aligned}
$$

From $2i$ to 0 (S_4):

$$
\begin{aligned}
z(t) &= 2i(1-t) + 0t \\
&= 2i - 2it \\
z'(t) &= -2i \\
\int_{S_4} f(z)\ dz &= \int_0^1 f(z(t))z'(t)\ dt \\
&= \int_0^1 \overline{2i-2it}\cdot(-2i)\ dt \\
&= \int_0^1 (-2i+2it)\cdot(-2i)\ dt \\
&= \int_0^1 -4+4t\ dt \\
&= -4\int_0^1 dt + 4\int_0^1 t\ dt \\
&= -4(1)+4(1/2) \\
&= -4+2
\end{aligned}
$$

Thus,

$$\int_S f(z)\ dz = [2]+[2+4i]+[-2+4i]+[-4+2] = [2+2-2-4+2]+[4i+4i] = 8i$$

2 Problem 2

The value of this integral over the spiral-shaped contour may be simplified. Namely, z^2 is entire; thus we may define a loop homotopy of C into $C'(t) = 1+t$ by

$$H : [1,2]\times[0,1] \to \mathbb{C} \qquad\qquad H(t,s) \overset{\text{def}}{=} (1-s)\cdot te^{2\pi it} + s\cdot t$$

Thus,

$$
\begin{aligned}
\int_C z^2\ dz &= \int_{C'} z^2\ dz \\
&= \int_1^2 x^2\ dx \\
&= \left.\frac{x^3}{3}\right|_1^2 \\
&= \frac{2^3}{3} - \frac{1^3}{3} \\
&= \frac{8-1}{3} \\
&= \frac{7}{3}
\end{aligned}
$$

3 Problem 3

$$\begin{aligned}
\int_C \frac{\cos 2z}{(z-\pi)^3}\,dz &= \frac{2\pi i}{2!}\left[D_z^2(z \mapsto \cos(2z))\right](\pi) \\
&= \pi i\left[D_z(z \mapsto -2\sin(2z))\right](\pi) \\
&= \pi i\left[z \mapsto -4\cos(2z)\right](\pi) \\
&= -4\pi i\cos(2\pi) \\
&= -4\pi i
\end{aligned}$$

4 Problem 4

By the maximum modulus principle, the greatest modulus for e^z must occur on the boundary. Therefore, we must only examine the three exposed edges the square as well as the circle boundary.

$$|e^z| = |e^x(\cos y + i\sin y)| = e^x(\cos^2 y + \sin^2 y) = e^x$$

Consequently, $|e^z| = e^x$. Since this is monotonically increasing in z, exactly the point $z = 1$ is the maximum, with value e. Similarly, $|e^z|$ is minimized on the entire edge $-2 + [-1,1]i$, where it obtains the value $\left|e^{-2+yi}\right| = e^{-2}$.

5 Problem 5

Consider $g(z) = -if(z) = v(x,y) - iu(x,y)$, This is analytic by multiplication. Then suppose $|v(x,y)| \leq \exp 10$ is bounded. Then $\exp|v(x,y)| \leq 10$. But

$$\exp 10 \geq \exp|v(x,y)| \geq |\exp v(x,y)| = |\exp(v(x,y) - iu(x,y))| = |\exp g(z)|$$

For all $z \in \mathbb{C}$. Thus, $\exp g(z)$, an entire analytic function, is bounded. Thus it is constant by Liouville's Theorem. But this requires that $g(z)$ was constant. However, g is also injective; thus, g must also be constant. Consequently, if $f(z) = u(x,y) + iv(x,y)$ is an entire function and $|v(x,y)| \leq 10$, then f is constant.

6 Problem 6

Since $D_z^{(n)} e^z = e^z$, and $e^{2\pi i} = -1$,

$$a_n = -\frac{1}{n!}, \qquad n = 0, 1, 2, \cdots$$

Thus, by Taylor's theorem, since e^z is entire, we have the Taylor series

$$e^z = \sum_{n=0}^{\infty} -\frac{1}{n!}(z - 2\pi i)^n$$

which holds for all $z \in \mathbb{C}$.

7 Problem 7

We assume we wish for the annulus $1 < |z| < 5$.

The first portion, $\frac{1}{z+5}$, will yield Taylor series (for $|z| < 5$),

$$\frac{1}{z+5} = \frac{1}{5}\frac{1}{1-(-z/5)} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^n = \sum_{n=0}^{\infty}\frac{1}{5^{n+1}}z^n$$

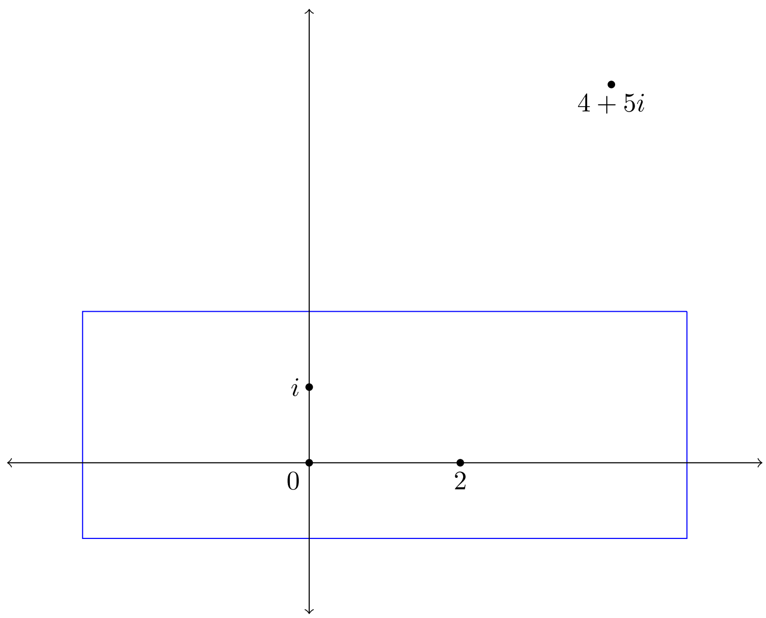
The second portion, $2/(z-i)$, has Laurent series (for $|z|>1$),

$$\frac{2}{z-i} = \frac{2}{z}\frac{1}{1-i/z} = \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n = \frac{2}{z}\sum_{n=0}^{\infty} i^n z^{-n} = \sum_{n=0}^{\infty} i^n z^{-n-1} = \sum_{n=1}^{\infty} i^{n-1} z^{-n}$$

By the uniqueness of Laurent series, and since the Taylor and Laurent series above are both correct on the annulus $1<|z|<5$,

$$f(z) = \frac{3z+10-i}{z^2+(5-i)z-5i} = \frac{1}{z+5} + \frac{2}{z-i} = \sum_{n=0}^{\infty}\frac{1}{5^{n+1}}z^n + \sum_{n=1}^{\infty} i^{n-1}z^{-n}, \qquad 1<|z|<5$$

8 Problem 8



When $z=0$ or $z=2$ or $z=i$, then $\frac{2s^2+i}{s-z}$ has singularities exactly at z, and C may be continuously deformed into a unit circle around z without changing the integral. Thus, for each of these,

$$F(z) = \int \frac{2s^2+i}{s-z}\,ds = 2\pi i(2z^2+i) = 4\pi i z^2 - 2\pi$$

Thus,

$$\begin{aligned} F(0) &= 4\pi i 0^2 - 2\pi = -2\pi \\ F(2) &= 4\pi i 2^2 - 2\pi = -2\pi + 16\pi i \\ F(i) &= 4\pi i (i)^2 - 2\pi = -2\pi - 4\pi i \end{aligned}$$

In the case of $4+5i$, the loop does not contain the only singularity of the interior function, and thus in this case C is contractible, so $F(4+5i)=0$.

9 Problem 9

$$f(z) = \frac{2}{z^3-6z^2+8z} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z-4}$$
$$2 = A(z^2-6z+8) + B(z^2-4z) + C(z^2-2z) = (A+B+C)z^2 + (-6A-4B-2C)z + 8A$$

$$A = 1/4$$
$$-6(1/4) - 4B - 2C = 0$$
$$2B + C = -3/4$$
$$C = -3/4 - 2B$$
$$0 = A + B + C = 1/4 + B - 3/4 - 2B = -1/2 - B$$
$$B = -1/2$$
$$C = -3/4 - 2(-1/2) = -3/4 + 1 = 1/4$$

Thus,

$$A = 1/4 \qquad B = -1/2 \qquad C = 1/4$$

Thus, we know that

$$f(z) = \frac{1/4}{z} + \frac{-1/2}{z-2} + \frac{1/4}{z-4}$$

Using Cauchy's integral formula, a counterclockwise loop around 0, 2, and 4 yields...

...around 0:

(This doesn't matter, so I didn't bother.)

...around 2:

$$\int_C \frac{\frac{2}{z(z-4)}}{(z-2)}\,dz = 2\pi i \frac{2}{2(-2)} = 2\pi i(-1/2) = -\pi i$$

...around 4:

$$\int_C \frac{\frac{2}{z(z-2)}}{(z-4)}\,dz = 2\pi i \frac{2}{4(4-2)} = 2\pi i \frac{1}{4} = \pi i \frac{1}{2} = \frac{1}{2}\pi i$$

9.1 a

This will be, using something pretty much like the Cauchy Residue Formula,

$$-\pi i + \frac{1}{2}\pi i = -\frac{1}{2}\pi i$$

9.2 b

Breaking this up appropriately, noting 4 is circumscribed clockwise rather than counterclockwise, we add one of the twos and subtract a 4, giving us

$$-\pi i - \frac{1}{2}\pi i = -\frac{3}{2}\pi i$$

9.3 c

This can be broken up into two loops: one around 2 once, and one containing both 2 and 4. Thus, the integral is

$$(-\pi i) + \left(-\pi i + \frac{1}{2}\pi i\right) = -2\pi i + 0.5\pi i = -\frac{3}{2}\pi i$$

10 Problem 10

$$
\begin{aligned}
x^2 + 4x + 7 &= (x + 2 - i\sqrt{3})(x + 2 + i\sqrt{3}) \\
x^2 - 8x + 18 &= (x - 4 - i\sqrt{2})(x - 4 + i\sqrt{2})
\end{aligned}
$$

We evaluate this integral using residues. Let

$$
f(z) = \frac{z^2}{(z^2 + 4z + 7)(z^2 - 8z + 18)}
$$

$$
\begin{aligned}
f(z) &= \frac{\frac{z^2}{(z-(-2-i\sqrt{3}))(z^2-8z+18)}}{(z - (-2 + i\sqrt{3})} \\
\operatorname*{Res}_{z=-2+i\sqrt{3}} f(z) &= \frac{(-2 + i\sqrt{3})^2}{(-2 + i\sqrt{3} - (-2 - i\sqrt{3}))((-2 + i\sqrt{3})^2 - 8(-2 + i\sqrt{3}) + 18)} \\
&= \frac{(-2 + i\sqrt{3})^2}{2i\sqrt{3}((-2 + i\sqrt{3})^2 - 8(-2 + i\sqrt{3}) + 18)} \\
f(z) &= \frac{\frac{z^2}{(z-(4-i\sqrt{2}))(z^2+4z+7)}}{(z - (4 + i\sqrt{2})} \\
\operatorname*{Res}_{z=4+i\sqrt{2}} f(z) &= \frac{(4 + i\sqrt{2})^2}{((4 + i\sqrt{2}) - (4 - i\sqrt{2}))((4 + i\sqrt{2})^2 + 4(4 + i\sqrt{2}) + 7)} \\
&= \frac{(4 + i\sqrt{2})^2}{2i\sqrt{2}((4 + i\sqrt{2})^2 + 4(4 + i\sqrt{2}) + 7)}
\end{aligned}
$$

Since the degree of the numerator is two less than the degree of the denominator, we may use equation (10) on page 264 of the text. So

$$
\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \left[\frac{(-2 + i\sqrt{3})^2}{2i\sqrt{3}((-2 + i\sqrt{3})^2 - 8(-2 + i\sqrt{3}) + 18)} + \frac{(4 + i\sqrt{2})^2}{2i\sqrt{2}((4 + i\sqrt{2})^2 + 4(4 + i\sqrt{2}) + 7)} \right]
$$

Since this is a genuinely horrific expression, simplification has been left to the reader. However, numerical methods evaluate this to $1.14779331879550 + i \cdot 4.35983562251079 \times 10^{-17}$, which matches to precision the approximated value.

11 Problem 11

Theorem 11.1. *Suppose that f is an entire function satisfying*

$$
f(z + i) = f(z) \text{ and } f(z + 1) = f(z) \text{ for all } z
$$

Then f is constant.

Proof. Note that if f is entire it is continuous. Thus it takes compact subsets of $\mathbb{C}$ to compact subsets of $\mathbb{C}$. Consequently, the image of $[0, 1] + [0, 1]i$ under f, $f([0, 1] + [0, 1]i)$, is bounded; that is,

$$
|f([0, 1] + [0, 1]i)| < B
$$

for some bound $B \in \mathbb{R}$.

But then if $f([n, n+1]+[m, m+1]i)$ is bounded, we also know that by the presuppositions of the theorem, all its neighboring cells share the same bound:

$$
\begin{aligned}
|f([n, n+1] + [m, m+1]i| &= |f([n+1, n+2] + [m, m+1]i| < B \\
|f([n, n+1] + [m, m+1]i| &= |f([n-1, n] + [m, m+1]i| < B \\
|f([n, n+1] + [m, m+1]i| &= |f([n, n+1] + [m+1, m+2]i| < B \\
|f([n, n+1] + [m, m+1]i| &= |f([n, n+1] + [m-1, m]i| < B
\end{aligned}
$$

Since every point of $\mathbb{C}$ lies in some integer unit square in the complex plane, and all of these by induction using the equations above are bounded in modulus by B, f is bounded in modulus by B over all of $\mathbb{C}$. Since f is bounded in modulus by B, by the maximum modulus principle, f must be constant. $\square$