Complex Variables Midterm

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1 Problem 1

$$
\begin{aligned}
\frac{(3+2i)^2}{1-2i} &= \frac{(3+2i)^2}{1-2i} \cdot \frac{1+2i}{1+2i} \\
&= \frac{(3+2i)^2(1+2i)}{1^2+2^2} \\
&= \frac{(9+12i-4)(1+2i)}{5} \\
&= \frac{(5+12i)(1+2i)}{5} \\
&= \frac{5+12i+10i-24}{5} \\
&= \frac{-19+22i}{5} \\
&= -\frac{19}{5} + \frac{22}{5}i
\end{aligned}
$$

Thus, the real portion of the complex fraction is $-19/5$ whilst the imaginary portion is $22/5$.

2 Problem 2

The cube roots of -8 are

$$
z = \sqrt[3]{8} \exp\left[i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right] = 2\exp\left[i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0, 1, 2)
$$

Which are

$$
\begin{gathered}
2(\cos(\pi/3) + i\sin(\pi/3)) = 1 + i\sqrt{3} \\
2(\cos(\pi) + i\sin(\pi)) = -2 \\
2(\cos(5\pi/3) + i\sin(5\pi/3)) = 1 - i\sqrt{3}
\end{gathered}
$$

3 Problem 3

Let $z = 1 + i$.

3.1 Part 3.a

$\overline{z} = \overline{1+i} = 1 - i$.

3.2 Part 3.b

$|z| = \sqrt{z\overline{z}} = \sqrt{(1+i)(1-i)} = \sqrt{1+1} = \sqrt{2}$.

3.3 Part 3.c

$\operatorname{Arg}(z) = \pi/4$.

3.4 Part 3.d

$z = 1 + i = \sqrt{2}\cos(\pi/4) + i\sqrt{2}\sin(\pi/4) = \sqrt{2}\exp(i\pi/4)$.

3.5 Part 3.e

$\operatorname{Log}(z) = \ln\sqrt{2} + \pi/4$

3.6 Part 3.f

$\arg(z) = \pi/4 + 2\pi k,\ k \in \mathbb{Z}$.

3.7 Part 3.g

$\log(z) = \ln\sqrt{2} + i(\pi/4 + 2k\pi),\ k \in \mathbb{Z}$.

4 Problem 4

$$\lim_{z\to 0} \frac{z + \bar{z}}{|z^2|} = \lim_{z\to 0} \frac{x + yi + x - yi}{|x^2 - y^2 + 2xyi|} = \lim_{z\to 0} \frac{2x}{|x^2 - y^2 + 2xyi|}$$

If we approach along the line $z = \operatorname{Re} z$, then $y = 0$, so continuing the equation,

$$= \lim_{x\to 0} \frac{2x}{|x^2|} = \lim_{x\to 0} \frac{2x}{x^2} = \lim_{x\to 0} \frac{2}{2x} = \infty$$

However, if we approach along the line $z = i \operatorname{Im} z$, then

$$= \lim_{y\to 0} \frac{2 \cdot 0}{|0^2 - y^2 + 0i|} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Thus the limit is not well-defined, and consequently cannot exist.

5 Problem 5

5.1 Part 5.a

$0.5 + 0.5i$ would be an interior point.

5.2 Part 5.b

$\pi - iG$, where G is Graham's number (that is, $G = g_{64}$, where $g_1 = 3 \uparrow^4 3$ and $g_n = 3 \uparrow^{g_{n-1}} 3$, and $a \uparrow b = a^b$ and $a \uparrow^n b = [x \to a \uparrow^{n-1} x]^{b-1} a$; see Knuth's up-arrow notation) would be an exterior point, since it is a bit away from S.

5.3 Part 5.c

0 is a boundary point, since any neighborhood of it contains a point with negative imaginary and real components, outside of S.

5.4 Part 5.d

Every neighborhood of 0 contains a non-0 point of S. Thus, S is an accumulation point.

5.5 Part 5.e

No. It contains boundary point 0.

5.6 Part 5.f

No. It does not contain boundary point $1+i$.

6 Problem 6

6.1 Part 6.a

$$\begin{aligned} f(z) &= z^3 - 4z \\ &= (x+yi)^3 - 4(x+yi) \\ &= x^3 + 3x^2yi - 3xy^2 - y^3i - 4x - 4yi \\ &= (x^3 - 3xy^2 - 4x) + i(3x^2y - y^3 - 4y) \end{aligned}$$

Thus, we may define u, v to be

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 - 4x \\ v(x,y) &= 3x^2y - y^3 - 4y \end{aligned}$$

6.2 Part 6.b

$$\begin{aligned} u_x &= 3x^2 - 3y^2 - 4 \\ u_y &= -6xy \\ v_x &= 6xy = -u_y \\ v_y &= 3x^2 - 3y^2 - 4 = u_x \end{aligned}$$

And thus u and v satisfy the Cauchy-Riemann equations.

7 Problem 7

Theorem 7.1.

$$\lim_{z \to i} 2z^2 + 1 = -1$$

Proof. Suppose we are given some $\epsilon > 0$. Then if $\delta = \sqrt{1+\epsilon/2} - 1$, then if $|z-i| < \delta$, we know that

$$|z+i| \leq |z-i| + |2i| < \delta + 2$$

Thus,

$$\begin{aligned} |z-i| \cdot |z+i| &< \delta(\delta+2) \\ \left|z^2+1\right| &< \left(\sqrt{1+\epsilon/2} - 1\right)\left(\sqrt{1+\epsilon/2} + 1\right) \\ \left|z^2+1\right| &< 1 + \epsilon/2 - 1 = \epsilon/2 \\ \left|2z^2+2\right| &< \epsilon \\ \left|2z^2+1-(-1)\right| &< \epsilon \end{aligned}$$

Thus, if $|z-i| < \delta = \sqrt{1+\epsilon/2} - 1$, then $\left|(2z^2+1) - (-1)\right| < \epsilon$. Consequently, $\lim_{z \to i} 2z^2 + 1 = -1$. □

8 Problem 8

It will be differentiable where it satisfies the Cauchy-Riemann equations.

$$\begin{aligned}
u_x &= 2x \\
u_y &= 0 \\
v_x &= 0 \\
v_y &= 2y
\end{aligned}$$

This only occurs when $2x = 2y$, or $x = y$. This line is the only place where it is differentiable. However, this subset has empty interior, and thus the function is analytic nowhere.

9 Problem 9

10 Part 9.a

$$\begin{aligned}
u_x &= 2x + 2 \\
u_y &= 2y \\
v_x &= -2y && \text{(C-R)} \\
v_y &= 2x + 2 && \text{(C-R)} \\
\int v_x \, dx &= \int -2y \, dx \\
&= -2xy + C(y) \\
\int v_y \, dy &= \int 2x + 2 \, dy \\
&= 2xy + 2y + C(x)
\end{aligned}$$

These cannot meet in the same equation, since $2xy$ and $-2xy$ cannot be part of or canceled by $C(x)$ or $C(y)$. Thus, u does not have a harmonic conjugate.

11 Part 9.b

$$\begin{aligned}
u_x &= y \\
u_y &= x \\
v_x &= -x && \text{(C-R)} \\
v_y &= y && \text{(C-R)} \\
\int v_x \, dx &= \int -x \, dx \\
&= -0.5x^2 + C(y) \\
\int v_y \, dy &= \int y \, dy \\
&= 0.5y^2 + C(x) \\
v(x, y) &= -0.5x^2 + 0.5y^2 + C
\end{aligned}$$

Yes, v in the last line is a harmonic conjugate of u.

12 Problem 10

For the first one, $z = w = -i$ works, since $(-i)^2 = -1$, yet $\mathrm{Log}(-i) = -\pi/2$ and $\mathrm{Log}(-1) = i\pi \neq -i\pi/2 + -i\pi/2$.

For the second one, $z = i$ works. $\mathrm{Log}(i) = i\pi/4$, $\mathrm{Log}(-i) = -i\pi/4 \neq 3i\pi/4$.

13 Problem 11

13.1 Part 11.a

$i^4 = 1$.

13.2 Part 11.b

$4^{1/2} = \pm 2$. PV 2.

13.3 Part 11.c

$\exp(7 + \pi i) = e^7 [\cos \pi + i \sin \pi] = e^7(-1) = -e^7$

13.4 Part 11.d

$$\begin{aligned}
(1+i)^i &= \exp(i \log(1+i)) \\
&= \exp(i [\ln |1+i| + i(\pi/4 + 2\pi k)]) && (k \in \mathbb{Z}) \\
&= \exp\left(i \ln \sqrt{2} - \pi/4 - 2\pi k\right) && (k \in \mathbb{Z}) \\
&= \exp\left(i \ln \sqrt{2}\right) \exp(-\pi/4 - 2\pi k) && (k \in \mathbb{Z}) \\
&= \exp(-\pi/4 - 2\pi k) \left[\cos \ln \sqrt{2} + i \sin \ln \sqrt{2}\right] && (k \in \mathbb{Z})
\end{aligned}$$

PV is $e^{-\pi/4} \left[\cos \sqrt{2} + i \sin \sqrt{2}\right]$

13.5 Part 11.e

$$\begin{aligned}
\sin(\pi - i) &= \frac{\exp(i\pi + 1) - \exp(-i\pi - 1)}{2i} \\
&= \frac{e^1 \exp(i\pi) - e^{-1} \exp(-i\pi)}{2i} \\
&= \frac{e^1 \cos \pi + e^1 i \sin \pi - e^{-1} \cos \pi + e^{-1} i \sin \pi}{2i} \\
&= \frac{e^1(-1) - e^{-1}(-1)}{2i} \\
&= \frac{-e^1 + e^{-1}}{2i} \\
&= \frac{e^{-1} - e^1}{2i} \\
&= \frac{e^{-1} - e^1}{2i} \frac{-2i}{-2i} \\
&= \frac{-2ie^{-1} + 2ie^1}{4}
\end{aligned}$$

$$= \frac{1}{2} i e^1 - \frac{1}{2} i e^{-1}$$

13.6 Part 11.f

$$\begin{aligned}
\tan^{-1}(2i) &= \frac{i}{2} \log \frac{i+2i}{i-2i} \\
&= \frac{i}{2} \log \frac{3i}{-i} \\
&= \frac{i}{2} \log -3 \\
&= \frac{i}{2} \left[\ln 3 + i(\pi + 2\pi k)\right] \\
&= \frac{i \ln 3}{2} - \frac{\pi + 2\pi k}{2} \\
&= i \frac{\ln 3}{2} - \frac{\pi}{2} + \pi k
\end{aligned}$$

PV at $i\frac{\ln 3}{2} - \pi/2$.

14 Problem 12

Consider an arbitrary $z \in \mathbb{C}$. Then,

$$\begin{aligned}
\overline{\cos(z)} &= \overline{\left(\frac{e^{iz} + e^{-iz}}{2}\right)} \\
&= \frac{1}{2} \overline{e^{iz} + e^{-iz}} \\
&= \frac{1}{2} \left(\overline{e^{iz}} + \overline{e^{-iz}}\right) \\
&= \frac{1}{2} \left(e^{i\bar{z}} + e^{-i\bar{z}}\right) \\
&= \cos(\bar{z})
\end{aligned}$$

15 Extra credit

It doesn't. Approaching from either cardinal axis ($x = 0$, $y = 0$) yields an answer of 0. Approaching along the parabola $y = x^2$, however, yields the answer

$$\lim_{z \to 0} \frac{x^2 y}{x^4 + y^2} = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \lim_{x \to 0} \frac{x^4}{2x^4} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}$$

which is decidedly not 0. Otherwise the Riemann Hypothesis would become rather bizarre—or less bizarre, depending on how you look at it.