Homework 2016-04-12

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1 Problem 5, p. 206

$$f(z) = \frac{z+1}{z-1}$$

1.1 Part a

To obtain a Maclaurin series for this, first we must compute the derivatives.

$$\begin{aligned}
f(z) &= \frac{z+1}{z-1} \\
f'(z) &= \frac{1}{z-1} + (-1)\frac{z+1}{(z-1)^2} \\
&= \frac{z-1}{(z-1)^2} - \frac{z+1}{(z-1)^2} \\
&= 2\frac{(-1)^1 \cdot 1!}{(z-1)^{1+1}}
\end{aligned}$$

Suppose then that $f^{(n)}(z)$ is of the form $2(-1)^n n!/(z-1)^{n+1}$, as is above for $n = 1$.

$$\begin{aligned}
f^{(n+1)}(z) &= (-n-1) \cdot 2\frac{(-1)^n \cdot n!}{(z-1)^{n+2}} \\
&= 2\frac{(-1)^{n+1} \cdot (n+1)!}{(z-1)^{n+2}}
\end{aligned}$$

which is also of that form. Thus, by induction, for $n \geq 1$,

$$f^{(n)}(z) = 2\frac{(-1)^n \cdot n!}{(z-1)^{n+1}}$$

Thus, the coefficients a_n of our Maclaurin series are $a_0 = -1$ and

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!} \cdot 2\frac{(-1)^n \cdot (n!)}{(0-1)^{n+1}} = -2$$

Thus,

$$f(z) = -1 - 2\sum_{n=1}^{\infty} z^n$$

and since the analytic circle around this only ends at $R_0 = 1$, this Maclaurin series holds for the disk $|z| < 1$.

1.2 Part b

2 Problem 1

Evaluate

$$\int_C \frac{z+1}{z-1}\, dz$$

where C is a positively oriented contour around 1.

Observe first that

$$\begin{aligned}
\frac{z+1}{z-1} &= \frac{z}{z-1} + \frac{1}{z-1} \\
&= \frac{1}{\frac{z}{z} - \frac{1}{z}} + \frac{1}{z-1} \\
&= \frac{1}{1-z^{-1}} + \frac{1}{z-1} \\
&= \frac{1}{1-z^{-1}} - \frac{1}{1-z} \\
&= \sum_{n=0}^{\infty} z^{-n} - \sum_{n=0}^{\infty} z^n
\end{aligned}$$

3 Problem 2

Find the Laurent series for

$$f(z) = \frac{2}{1-z^2} = \frac{1}{1-z} + \frac{1}{1+z}$$

We utilize example 4 in the text, on page 194.

$$\begin{aligned}
f(z) &= \frac{1}{1-z} + \frac{1}{1+z} \\
&= \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} (-z)^n \\
&= \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} (-1)^n z^n \\
&= \sum_{n \in 2\mathbb{N}} z^n && \text{(Note the change of index! We assume } 0 \in \mathbb{N}\text{.)} \\
&= \sum_{n=0}^{\infty} z^{2n} \\
&= \sum_{n=0}^{\infty} (z^2)^n \\
&= 2\frac{1}{1-z^2} \\
&= \frac{2}{1-z^2} \\
&= f(z)
\end{aligned}$$

Thus, the Laurent series for $f(z)$ is $\sum_{n=0}^{\infty} z^{2n}$.