Complex Variables Section 26 Homework

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Find harmonic conjugates for

$$u(x, y) = \frac{x}{x^2 + y^2}$$

and

$$u(x, y) = \ln \sqrt{x^2 + y^2}$$

For the first,

$$\begin{aligned}
u_x &= D_x x(x^2 + y^2)^{-1} \\
&= (x^2 + y^2)^{-1} + x(-1)(x^2 + y^2)^{-2}(2x) \\
&= \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} \\
&= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} \\
&= \frac{y^2 - x^2}{x^2 + y^2} \\
u_y &= D_y x(x^2 + y^2)^{-1} \\
&= x(-1)(x^2 + y^2)^{-2} 2y \\
&= -\frac{2xy}{(x^2 + y^2)^2}
\end{aligned}$$

Thus, for a harmonic conjugate v,

$$\begin{aligned}
v_x &= \frac{2xy}{(x^2 + y^2)^2} \\
\int v_x \, dx &= \int \frac{2xy}{(x^2 + y^2)^2} \, dx \\
&= \int 2xy(x^2 + y^2)^{-2} \, dx \\
&= \int 2x \frac{D_x y(x^2 + y^2)^{-1}}{-2x} \, dx \\
&= -\int D_x y(x^2 + y^2) \, dx \\
&= -\frac{y}{x^2 + y^2} + C(y) \\
v_y &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \\
\int v_y \, dy &= \int \frac{y^2 - x^2}{(x^2 + y^2)^2} \, dy \\
&= -\int \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy
\end{aligned}$$

$$
\begin{aligned}
&= -\int \frac{x^2+y^2}{(x^2+y^2)^2} + \frac{-2y^2}{(x^2+y^2)^2}\ dy \\
&= -\int \frac{1}{x^2+y^2} + \frac{-2y^2}{(x^2+y^2)^2}\ dy \\
&= -\int [D_y y]\,(x^2+y^2)^{-1} + y(2y)(-1)(x^2+y^2)^{-2}\ dy \\
&= -\int [D_y y]\,(x^2+y^2)^{-1} + y\left[D_y(x^2+y^2)^{-1}\right]\ dy \\
&= -\int D_y\left[y(x^2+y^2)^{-1}\right]\ dy \\
&= -\frac{y}{x^2+y^2} + C(x)
\end{aligned}
$$

Thus, the function

$$v(x,y) = \frac{-y}{x^2+y^2}$$

is a harmonic conjugate to u defined above.

For the second problem,

$$u(x,y) = \ln\sqrt{x^2+y^2}$$

$$
\begin{aligned}
u_x &= \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \cdot 2x \\
&= \frac{x}{x^2+y^2} \\
u_y &= \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \cdot 2y \\
&= \frac{y}{x^2+y^2} \\
v_x &= -u_y \\
&= -\frac{y}{x^2+y^2} \\
v &= \int v_x\ dx \\
&= \int -\frac{y}{x^2+y^2}\ dx \\
&= -y\int \frac{1}{x^2+y^2}\ dx \\
&= -y\left[\frac{1}{y}\tan^{-1}\frac{x}{y}\right] + C(y) \\
&= -\tan^{-1}\frac{x}{y} + C(y) \\
&= \tan^{-1}\frac{y}{x} + \frac{\pi}{2} + C(y) \\
v_y &= u_x \\
&= \frac{x}{x^2+y^2} \\
v &= \int \frac{x}{x^2+y^2}\ dy \\
&= x\int \frac{1}{x^2+y^2}\ dy
\end{aligned}
$$

$$
\begin{aligned}
&= x\left[\frac{1}{x}\tan^{-1}\frac{y}{x} + C(x)\right] \\
&= \tan^{-1}\frac{y}{x} + C(x)
\end{aligned}
$$

Observe that these two conditions are satisfied if $C(x) = \pi/2$ and $C(y) = 0$. Thus (u, v) is a harmonic conjugate pair for $v(x, y) = \tan^{-1}(y/x) + \pi/2$.