Complex Variables Homework Section 54

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1 Problem 6

Theorem 1.1. *Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Then the component function $u(x, y)$ has a minimum value in R which occurs on the boundary of R and never in the interior.*

Proof. The function u maps a compact subset of $\mathbb{R}^2$ to $\mathbb{R}$, and thus since u is continuous, $u(R)$ is compact and thus contains its boundary. Thus, there must exist some $z' \in R$ such that $u(z') = \min u(R)$.

Consider $g(z) = \exp(f(z))$. g is the composition of two analytic functions and thus is analytic. So the minimum of $|g(z)|$ on R occurs on the boundary. But $|g(z)| = |\exp(u(x, y) + iv(x, y))| = |\exp(u(x, y))| \cdot |\exp(iv(x, y))| = |\exp u(x, y)| = \exp u(x, y)$. Since exp is a monotonically increasing function on $\mathbb{R}$, the minimum value of $|g(z)| = \exp u(x, y)$ occurs on the boundary of R and not in the interior. $\square$

2 Problem 7

Let $f(z) = e^z$ and R the region $0 \le x \le 1$, $0 \le y \le \pi$. Find points in R where $u(x, y)$ reaches its maximum and minimum values.

Decompose the region R with the cell decomposition (viewing, momentarily, the complex number line as $\mathbb{R}^2$ equipped with complex multiplication),

$$\left\{ \underbrace{\{(0,0)\}, \{(1,0)\}, \{(1,\pi)\}, \{(0,\pi)\}}_{\text{0-cells}}, \underbrace{(0,1)\times\{0\}, (0,1)\times\{\pi\}, \{0\}\times(0,\pi), \{1\}\times(0,\pi)}_{\text{1-cells}}, \underbrace{(0,1)\times(0,\pi)}_{\text{2-cell}} \right\}$$

We know the maximum and minimum values cannot occur on the 2-cell, since this entirely is in the interior of R.

The 0-cells have values $e^0 = 1$, $e^1 = e$, $e^{1+\pi i} = e(-1) = -e$, and $e^{\pi i} = -1$. Since these are all real, the max and min of u on the 0-skeleton of R are e and $-e$ respectively.

$f(z) = e^z = e^{x+yi} = e^x [\cos(y) + i \sin(y)]$. So $u_x = e^x \cos(y)$, and $u_y = -e^x \sin(y)$.

The first 1-cell listed above has extrema where $u_x(t, 0) = 0 = e^t \cos(0) = e^t$. This cannot occur.

The second 1-cell has extrema where $u_x(t, \pi) = 0 = e^t \cos(\pi) = -e^t$. Again, this is impossible.

The third 1-cell has extrema where $u_y(0, t) = 0 = -e^0 \sin(t) = -\sin(t)$. This only occurs at multiples of π, which this cell contains none of.

The fourth 1-cell has extrema where $u_y(1, t) = 0 = -e^1 \sin(t) = -e \sin(t)$. This also only occurs at multiples of π, which this cell contains none of.

Thus, the 1-skeleton of this cell decomposition of R contains no extrema.

Since R has 0-skeleton maximum at $(1, 0)$ and minimum at $(1, \pi)$, and no n-skeleton for $n > 0$ contains any extrema, $(1, 0)$ is the maximum and $(1, \pi)$ is the minimum for u on R.