Complex Variables p. 225 Assignment

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p. 225, # 5.
Evaluate

$$\int_C \frac{5}{e^z - e^{-z}}\, dz$$

around the unit circle.

1 Problem #5.

$$\begin{aligned}
\frac{1}{1 + z^2/3! + z^4/5! + \cdots} &= d_0 + d_1 z^1 + d_2 z^2 + d_3 z^3 + \cdots \\
1 = \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} d_n z^n\right) &= \sum_{n=0}^{\infty}\left(\sum_{2a+b=n} \frac{d_b}{(2a+1)!}\right) z^{2a+b} \\
= \left(\frac{d_0}{1!}\right) z^0 + \left(\frac{d_1}{1!}\right) z^1 + \left(\frac{d_0}{3!} + \frac{d_2}{1!}\right) z^2 &+ \left(\frac{d_1}{3!} + \frac{d_3}{1!}\right) z^3 + \left(\frac{d_0}{5!} + \frac{d_2}{3!} + \frac{d_4}{1!}\right) z^4 + \cdots
\end{aligned}$$

Thus,

$$\begin{aligned}
1 &= \frac{d_0}{1!} = d_0 && d_0 = 1 \\
0 &= \frac{d_1}{1!} = d_1 && d_1 = 0 \\
0 &= \frac{d_0}{3!} + \frac{d_2}{1!} = 1/3 + d_2 && d_2 = -1/3 \\
0 &= 0/3! + d_3/1! = d_3 && d_3 = 0 \\
0 &= d_0/5! + d_2/3! + d_4/1! = 1/5! - (1/3)/3! + d_4 && \\
&= 1/120 - 1/18 + d_4 && d_4 = 17/360
\end{aligned}$$

2 Evaluate

$$\int_C \frac{5}{e^z - e^{-z}}\, dz = \int_C \frac{10}{\sinh z}\, dz = 10 \int_C \frac{1}{\sum_{n=0}^{\infty} z^{2n+1}/(2n+1)!}\, dz$$

This is basically the same Laurent series as in Problem #5, so we'll just copy that and multiply by $1/z$.

$$= 10 \int_C \frac{1}{z}\left(1 - \frac{1}{3} z^2 + \frac{17}{360} z^4 + \cdots\right) dz = 10 \int_C \left(\frac{1}{z} - \frac{1}{3} z + \frac{17}{360} z^3 + \cdots\right) dz$$

Breaking the integral over the sum, only the first term has nontrivial integral (it is the only term which is not analytic on all of $\mathbb{C}$) and therefore

$$= 10 \int_C \frac{1}{z}\, dz + 0 + 0 + \cdots = 10(2\pi i) = 20\pi i$$