Complex Variables Section 44 Homework

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March 15, 2016

p. 135 # 4. p. 149 # 2. p. 160 # 4.

1 p. 135 # 4

$$\begin{aligned}
\int_C f(z)\ dz &= \int_{-1}^{1} f(z(t))z'(t)\ dt \\
&= \int_{-1}^{1} f(t^3+ti)\cdot[3t^2+i]\ dt \\
&= \int_{-1}^{0} f(t^3+ti)\cdot[3t^2+i]\ dt + \int_{0}^{1} f(t^3+ti)\cdot[3t^2+i]\ dt
\end{aligned}$$

When $t > 0$, $y > 0$. Similarly, when $t < 0$, $y < 0$. So,

$$\begin{aligned}
&= \int_{-1}^{0} 1\cdot[3t^2+i]\ dt + \int_{0}^{1} 4t\cdot[3t^2+i]\ dt \\
&= \int_{-1}^{0} 3t^2\ dt + i\int_{-1}^{0} 1\ dt + \int_{0}^{1} 12t^3\ dt + i\int_{0}^{1} 4t\ dt \\
&= \left[t^3\right]_{-1}^{0} + i[1] + 3\left[t^4\right]_0^1 + 2i\left[t^2\right]_0^1 \\
&= -1 + i + 3 + 2i \\
&= 2 + 3i
\end{aligned}$$

2 p. 148 # 2

2.1 Part a

$e^{\pi z}$ is entire so it has an antiderivative. We define an antiderivative by the integral from 0 to z along the polygonal path C from 0 to $x + 0i$ to $x + yi = z$.

$$\begin{aligned}
F(z) &= \int_C f(z)\ dz \\
&= \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz \\
&= \int_0^x f(t)(1)\ dt + \int_0^y f(x+ti)(1)\ dt \\
&= \int_0^x e^{\pi t}\ dt + \int_0^y e^{\pi x}e^{i\pi y}\ dt
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{e^{\pi x}}{\pi} - \frac{1}{\pi}\right] + e^{\pi x}\left[\frac{e^{\pi t i}}{\pi}\right]_0^y \\
&= \left[\frac{e^{\pi x}}{\pi} - \frac{1}{\pi}\right] + \frac{e^{\pi x}}{\pi}\left[e^{i\pi y} - 1\right] \\
&= \frac{1}{\pi}\left[e^{\pi x} - 1 + e^{\pi z} - e^{\pi x}\right] \\
&= \frac{e^{\pi z}}{\pi} - \frac{1}{\pi}
\end{aligned}
$$

Then $\int_i^{i/2} e^{\pi z}\, dz = F(i/2) - F(i) = \pi^{-1}e^{\pi i/2} - \pi^{-1}e^{\pi i} = \frac{i}{\pi} - \frac{-1}{\pi} = (1+i)/\pi$.

2.2 Part b

By a similar construction,

$$
\begin{aligned}
F(z) &= \int_0^x \cos(t/2)\, dt + \int_0^y \cos\left(\frac{x+ti}{2}\right) i\, dt \\
&= 2\sin\left(\frac{x}{2}\right) + i\int_0^y \cos\left(\frac{x+ti}{2}\right)\, dt \\
&= 2\sin\left(\frac{x}{2}\right) + i\frac{1}{2}\int_0^y \exp\left(\frac{ix-t}{2}\right) + \exp\left(\frac{t-ix}{2}\right)\, dt \\
&= 2\sin\left(\frac{x}{2}\right) + i\frac{1}{2}\int_0^y \exp\left(\frac{ix-t}{2}\right)\, dt + i\frac{1}{2}\int_0^y \exp\left(\frac{t-ix}{2}\right)\, dt \\
&= 2\sin\left(\frac{x}{2}\right) + i\frac{1}{2}e^{ix/2}\int_0^y \exp\left(-\frac{t}{2}\right)\, dt + i\frac{1}{2}e^{-ix/2}\int_0^y \exp\left(\frac{t}{2}\right)\, dt \\
&= 2\sin\left(\frac{x}{2}\right) + i\frac{1}{2}e^{ix/2}\left[-2e^{-t/2}\right]_0^y + i\frac{1}{2}e^{-ix/2}\left[2e^{t/2}\right]_0^y \\
&= 2\sin\left(\frac{x}{2}\right) + ie^{-ix/2}\left[e^{t/2}\right]_0^y - ie^{ix/2}\left[e^{-t/2}\right]_0^y \\
&= 2\sin\left(\frac{x}{2}\right) + ie^{-ix/2}\left[e^{y/2} - 1\right] - ie^{ix/2}\left[e^{-y/2} - 1\right]
\end{aligned}
$$

Then

$$
\begin{aligned}
F(\pi+2i) - F(0) &= \left[2\sin(\pi/2) + ie^{-i\pi/2}(e-1) - ie^{i\pi/2}(e^{-1}-1)\right] - \left[2\sin(0/2) + ie^{-i\pi}(1-1) - ie^{i\pi}(1-1)\right] \\
&= \left[2 + i(-i)(e-1) - i(i)(e^{-1}-1)\right] - 0 \\
&= 2 + e - 1 + e^{-1} - 1 \\
&= e + e^{-1}
\end{aligned}
$$

2.3 Part c

$$
\begin{aligned}
F(z) &= \int_0^x (t-2)^3\, dt + \int_0^y (x+ti-2)^3)i\, dt \\
&= \int_0^x (t-2)^3\, dt + \int_{x-2}^{y+x-2} (ti)^3 i\, dt \\
&= \int_{-2}^{x-2} t^3\, dt + \int_{x-2}^{y+x-2} t^3\, dt \\
&= \int_{-2}^{y+x-2} t^3\, dt \\
&= \left[\frac{t^4}{4}\right]_{-2}^{y+x-2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(y+x-2)^4}{4} - \frac{32}{4} \\
&= \frac{(y+x-2)^4}{4} - 8
\end{aligned}
$$

So $F(3) - F(1) = 1^4/4 - 8 - (-1)^4/4 + 8 = 0$.

3 p. 160 # 4

3.1 4.a

Label the right edge C_1, the top C_2, left C_3, and bottom C_4.

To obtain the top edge C_2,

$$
\begin{aligned}
\int_{C_2} e^{-z^2}\, dz &= \int_a^{-a} e^{-(x+bi)^2}\, dx \\
&= -\int_{-a}^{a} e^{-x^2+b^2-2xbi}\, dx \\
&= -\int_{-a}^{a} e^{b^2-x^2-2xbi}\, dx \\
&= -\int_{-a}^{a} e^{b^2} e^{-x^2} e^{-2xbi}\, dx \\
&= -e^{b^2} \int_{-a}^{a} e^{-x^2} e^{-2xbi}\, dx \\
&= -e^{b^2} \int_{-a}^{a} e^{-x^2} \left[\cos(2xb) - i \sin(2xb)\right]\, dx \\
&= -e^{b^2} \int_{-a}^{a} e^{-x^2} \cos(2xb)\, dx - i \int_{-a}^{a} e^{-x^2} \sin(2xb)\, dx
\end{aligned}
$$

Since e^{-x^2} is even and $\sin(2xb)$ is odd, the right (imaginary) half integrates to 0. Similarly, the left side is all even, so it may be subdivided:

$$
= -2e^{b^2} \int_0^a e^{-x^2} \cos(2xb)\, dx
$$

We obtain C_4 by flipping the sign (to reverse direction) and setting $b \leftarrow 0$, obtaining $2\int_0^a e^{-x^2}\, dx$. Thus, the sum of the horizontal edges is

$$
2\int_0^a e^{-x^2}\, dx - 2e^{b^2} \int_0^a e^{-x^2} \cos(2xb)\, dx
$$

For C_1,

$$
\begin{aligned}
\int_{C_1} e^{-z^2}\, dz &= \int_0^b e^{-(a+ti)^2} i\, dz \\
&= i \int_0^b e^{-a^2+y^2-2ayi}\, dy \\
&= i \int_0^b e^{-a^2} e^{y^2} e^{-2ayi}\, dy \\
&= ie^{-a^2} \int_0^b e^{y^2} e^{-2ayi}\, dy
\end{aligned}
$$

To obtain C_3, negate the integral for C_1 and $a \leftarrow -a$, yielding

$$\int_{C_3} e^{-z^2}\ dz = -ie^{-a^2} \int_0^b e^{y^2} e^{2ayi}\ dy$$

Summing these yields

$$ie^{-a^2} \int_0^b e^{y^2} e^{-2ayi}\ dy - ie^{-a^2} \int_0^b e^{y^2} e^{2ayi}\ dy$$

By the Cauchy-Goursat theorem, these sum to 0:

$$0 = 2\int_0^a e^{-x^2}\ dx - 2e^{b^2} \int_0^a e^{-x^2} \cos(2xb)\ dx + ie^{-a^2} \int_0^b e^{y^2} e^{-2ayi}\ dy - ie^{-a^2} \int_0^b e^{y^2} e^{2ayi}\ dy$$

$$2e^{b^2} \int_0^a e^{-x^2} \cos(2xb)\ dx = 2\int_0^a e^{-x^2}\ dx + ie^{-a^2} \int_0^b e^{y^2} e^{-2ayi}\ dy - ie^{-a^2} \int_0^b e^{y^2} e^{2ayi}\ dy$$

$$\int_0^a e^{-x^2} \cos(2xb)\ dx = e^{-b^2} \int_0^a e^{-x^2}\ dx + 0.5ie^{-(a^2+b^2)} \int_0^b e^{y^2} e^{-2ayi}\ dy - 0.5ie^{(-a^2+b^2)} \int_0^b e^{y^2} e^{2ayi}\ dy$$

$$\int_0^a e^{-x^2} \cos(2xb)\ dx = e^{-b^2} \int_0^a e^{-x^2}\ dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin(2ay)\ dy$$

3.2 4.b

$$\begin{aligned}
\lim_{a\to\infty} \int_0^a e^{-x^2} \cos(2xb)\ dx &= \lim_{a\to\infty} e^{-b^2} \int_0^a e^{-x^2}\ dx + \lim_{a\to\infty} e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin(2ay)\ dy \\
&= e^{-b^2} \lim_{a\to\infty} \int_0^a e^{-x^2}\ dx + \lim_{a\to\infty} e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin(2ay)\ dy \\
&= e^{-b^2} \frac{\pi}{2} + \lim_{a\to\infty} e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin(2ay)\ dy
\end{aligned}$$

Since $\left|\int_0^b e^{y^2} \sin(2ay)\ dy\right| \le \int_0^b e^{y^2}\ dy$, this integral is bounded as a function in a. Since $\lim_{a\to\infty} e^{-(a^2+b^2)} = 0$, the limit of the product of a bounded function and a 0-limit function is 0. Thus,

$$\begin{aligned}
&= \frac{\pi e^{-b^2}}{2} + 0 \\
&= \frac{\pi e^{-b^2}}{2}
\end{aligned}$$