Complex Analysis Homework, Section 10

Adam Buskirk

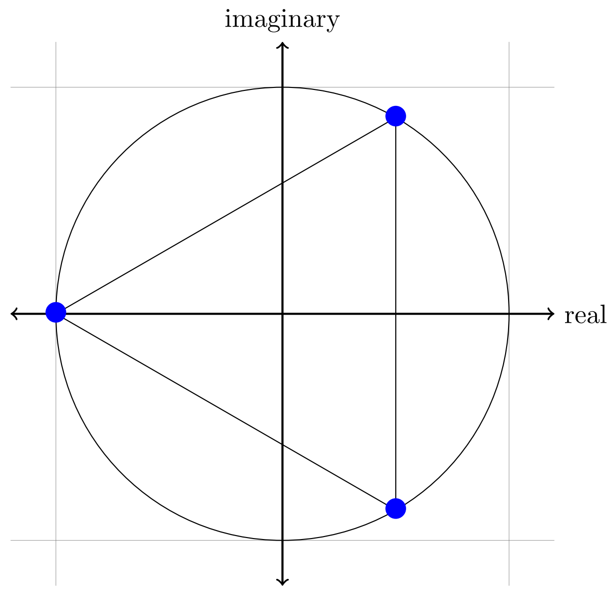
January 19, 2016

1 Problem 10.3.a

The roots will be

$$z = 1 \exp\left[i\left(\frac{\pi + 2k\pi}{3}\right)\right], \qquad k = 0, 1, 2$$

Which yields roots $(\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3})) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ (for $k = 0$), -1 (for $k = 1$), and $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ (for $k = 2$).



The principle root is $\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

2 Factoring Problem

Factor $x^4 + 2$ over $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$.

The roots of $x^4 + 2$ are the solutions to $x^4 + 2 = 0$.

$$x^4 + 2 = 0$$

$$x^4 = -2$$

This has roots

$$z = \sqrt[4]{2} \exp\left[i\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)\right], \qquad k = 0, 1, 2, 3$$

Which reduces to

$$z = \sqrt[4]{2}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right), \sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

For $k = 0$, this is $\sqrt[4]{2}(\cos(\pi/4), \sin(\pi/4)) = \sqrt[4]{2}(\sqrt{2}/2, \sqrt{2}/2) = 2^{1/4}\left(2^{-1/2}, 2^{-1/2}\right) = (2^{-1/4}, 2^{-1/4}) = \sqrt[4]{1/2} + i\sqrt[4]{1/2}$.

For $k = 1$, this is $\sqrt[4]{2}(\cos(3\pi/4), \sin(3\pi/4)) = \sqrt[4]{2}(-\sqrt{2}/2, \sqrt{2}/2) = 2^{1/4}(-2^{-1/2}, 2^{-1/2}) = (-2^{-1/4}, 2^{-1/4}) = -\sqrt[4]{1/2} + i\sqrt[4]{1/2}$.

By symmetry, for $k = 2$, the root is $(-2^{-1/4}, -2^{-1/4})$. Similarly, for $k = 3$, the root is $(2^{-1/4}, -2^{-1/4})$.

Thus, in $\mathbb{C}[x]$,

$$x^4 + 2 = (x + 2^{-1/4} + i2^{-1/4})(x - 2^{-1/4} + i2^{-1/4})(x + 2^{-1/4} - i2^{-1/4})(x - 2^{-1/4} - i2^{-1/4}) \tag{1}$$

To factor this in $\mathbb{R}[x]$, we must multiply the pairs of complex conjugate terms above, which yield

$$(x + 2^{-1/4} + i2^{-1/4})(x + 2^{-1/4} - i2^{-1/4}) = x^2 + 2^{3/4}x + 2^{1/2}$$

$$(x - 2^{-1/4} + i2^{-1/4})(x - 2^{-1/4} - i2^{-1/4}) = x^2 - 2^{3/4}x + 2^{1/2}$$

Hence,

$$x^4 + 2 = (x^2 + 2^{3/4}x + 2^{1/2})(x^2 - 2^{3/4}x + 2^{1/2}) \tag{2}$$

Neither of these factor polynomials is in $\mathbb{Q}[x]$, however. Thus we conclude that

$$x^4 + 2 = x^4 + 2 \tag{3}$$

is the only factorization of $x^4 + 2$ in $\mathbb{Q}[x]$.