Complex Variables Homework Section 21

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p. 71 #1 bd [Verify the C-R equations for f(x)=1/z].

1 Problem 1

1.1 Part b

For $f(z) = z - \bar{z}$,

$$u(x, y) = 0 \qquad v(x, y) = 2y$$

So $v_y = 2 \neq 0 = u_x$, contradicting the first equation in (7) in the text.

1.2 Part d

For $f(z) = e^x e^{-iy}$,

$$u(x, y) = e^x \cos(-y) = e^x \cos(y)$$
$$v(x, y) = e^x \sin(-y) = -e^x \sin(y)$$

Thus

$$u_x = e^x \cos(y)$$
$$u_y = -e^x \sin(y)$$
$$v_x = -e^x \sin(y)$$
$$v_y = -e^x \cos(y)$$

But

$$u_y = -e^x \sin y \neq -\left[-e^x \sin y\right] = -v_x$$

Unless, per chance, $y \in \pi\mathbb{Z}$. In this case, however, $\cos(y) \neq 0$, which means the second condition,

$$u_x = e^x \cos(y) \neq -e^x \cos(y) = v_y$$

cannot be satisfied. Hence f is not differentiable at any point.

2 Verify C-R

For $f(z) = 1/z$,

$$u(x, y) = \frac{x}{x^2 + y^2}$$
$$v(x, y) = \frac{-y}{x^2 + y^2}$$

Thus

$$u_x = \frac{d}{dx}\left[x(x^2 + y^2)^{-1}\right] = \frac{1}{x^2 + y^2} + x(-1)(x^2 + y^2)^{-2}(2x) = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2}$$

$$= \frac{x^2 + y^2 - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

and

$$u_y = \frac{d}{dy}\left[x(x^2+y^2)^{-1}\right] = xD_y\left[(x^2+y^2)^{-1}\right] = x\left[(-1)(x^2+y^2)^{-2}2y\right] = \frac{-2xy}{(x^2+y^2)^2}$$

and

$$v_x = D_x\left[-y(x^2+y^2)^{-1}\right] = -yD_x(x^2+y^2)^{-1} = -y(-1)(x^2+y^2)^{-2}2x = \frac{2xy}{(x^2+y^2)^2}$$

and

$$v_y = D_y\left[-y(x^2+y^2)^{-1}\right] = -\left[1\cdot(x^2+y^2)^{-1} + y(-1)(x^2+y^2)^{-2}(2y)\right] = -\left[\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]$$

$$= -\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

From these, it is evident that

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2} = v_y$$

and

$$u_y = \frac{-2xy}{(x^2+y^2)^2} = -\left(\frac{2xy}{(x^2+y^2)^2}\right) = -v_x$$

and thus $f(x) = 1/z$ satisfies the Caucy-Riemann equations.