

# Complex Variables Final Exam

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## 1 Problem 1

We decompose  $S$  into four portions.

From 0 to 2 ( $S_1$ ): The integral is 2, since this reduces to a simple real integral  $\int_0^2 x \, dx$ .

From 2 to  $2 + 2i$  ( $S_2$ ):

$$\begin{aligned} z(t) &= 2(1-t) + (2+2i)t \\ &= 2 - 2t + 2t + 2it \\ &= 2 + 2it \\ z'(t) &= 2i \\ \int_{S_2} f(z) \, dz &= \int_0^1 f(z(t)) z'(t) \, dt \\ &= \int_0^1 \overline{2+2it} \cdot 2i \, dt \\ &= \int_0^1 (2-2it) \cdot 2i \, dt \\ &= \int_0^1 4i + 4t \, dt \\ &= \int_0^1 4t + 4i \, dt \\ &= 4 \int_0^1 t \, dt + 4i \int_0^1 dt \\ &= 4[1/2] + 4i[1] \\ &= 2 + 4i \end{aligned}$$

From  $2 + 2i$  to  $2i$  ( $S_3$ ):

$$\begin{aligned} z(t) &= (2+2i)(1-t) + 2it \\ &= 2 + 2i - 2t - 2it + 2it \\ &= 2 + 2i - 2t \\ z'(t) &= -2 \\ \int_{S_3} f(z) \, dz &= \int_0^1 f(z(t)) z'(t) \, dt \\ &= \int_0^1 \overline{2+2i-2t} \cdot (-2) \, dt \\ &= -2 \int_0^1 (2-2t-2i) \, dt \end{aligned}$$

$$\begin{aligned}
&= -2 \left[ (2-2i) \int_0^1 dt - 2 \int_0^1 t \, dt \right] \\
&= -2 [(2-2i)1 - 2(1/2)] \\
&= -2 [2-2i-1] \\
&= -2 [1-2i] \\
&= -2 + 4i
\end{aligned}$$

From  $2i$  to  $0$  ( $S_4$ ):

$$\begin{aligned}
z(t) &= 2i(1-t) + 0t \\
&= 2i - 2it \\
z'(t) &= -2i \\
\int_{S_4} f(z) \, dz &= \int_0^1 f(z(t)) z'(t) \, dt \\
&= \int_0^1 \overline{2i - 2it} \cdot (-2i) \, dt \\
&= \int_0^1 (-2i + 2it) \cdot (-2i) \, dt \\
&= \int_0^1 -4 + 4t \, dt \\
&= -4 \int_0^1 dt + 4 \int_0^1 t \, dt \\
&= -4(1) + 4(1/2) \\
&= -4 + 2
\end{aligned}$$

Thus,

$$\int_S f(z) \, dz = [2] + [2 + 4i] + [-2 + 4i] + [-4 + 2] = [2 + 2 - 2 - 4 + 2] + [4i + 4i] = 8i$$

## 2 Problem 2

The value of this integral over the spiral-shaped contour may be simplified. Namely,  $z^2$  is entire; thus we may define a loop homotopy of  $C$  into  $C'(t) = 1 + t$  by

$$H : [1, 2] \times [0, 1] \rightarrow \mathbb{C} \qquad H(t, s) \stackrel{\text{def}}{=} (1-s) \cdot te^{2\pi it} + s \cdot t$$

Thus,

$$\begin{aligned}
\int_C z^2 \, dz &= \int_{C'} z^2 \, dz \\
&= \int_1^2 x^2 \, dx \\
&= \left. \frac{x^3}{3} \right|_1^2 \\
&= \frac{2^3}{3} - \frac{1^3}{3} \\
&= \frac{8-1}{3} \\
&= \frac{7}{3}
\end{aligned}$$

### 3 Problem 3

$$\begin{aligned}
 \int_C \frac{\cos 2z}{(z - \pi)^3} dz &= \frac{2\pi i}{2!} [D_z^2(z \mapsto \cos(2z))] (\pi) \\
 &= \pi i [D_z(z \mapsto -2 \sin(2z))] (\pi) \\
 &= \pi i [z \mapsto -4 \cos(2z)] (\pi) \\
 &= -4\pi i \cos(2\pi) \\
 &= -4\pi i
 \end{aligned}$$

### 4 Problem 4

By the maximum modulus principle, the greatest modulus for  $e^z$  must occur on the boundary. Therefore, we must only examine the three exposed edges the square as well as the circle boundary.

$$|e^z| = |e^x(\cos y + i \sin y)| = e^x(\cos^2 y + \sin^2 y) = e^x$$

Consequently,  $|e^z| = e^x$ . Since this is monotonically increasing in  $z$ , exactly the point  $z = 1$  is the maximum, with value  $e$ . Similarly,  $|e^z|$  is minimized on the entire edge  $-2 + [-1, 1]i$ , where it obtains the value  $|e^{-2+yi}| = e^{-2}$ .

### 5 Problem 5

Consider  $g(z) = -if(z) = v(x, y) - iu(x, y)$ , This is analytic by multiplication. Then suppose  $|v(x, y)| \leq \exp 10$  is bounded. Then  $\exp |v(x, y)| \leq 10$ . But

$$\exp 10 \geq \exp |v(x, y)| \geq |\exp v(x, y)| = |\exp (v(x, y) - iu(x, y))| = |\exp g(z)|$$

For all  $z \in \mathbb{C}$ . Thus,  $\exp g(z)$ , an entire analytic function, is bounded. Thus it is constant by Liouville's Theorem. But this requires that  $g(z)$  was constant. However,  $g$  is also injective; thus,  $g$  must also be constant. Consequently, if  $f(z) = u(x, y) + iv(x, y)$  is an entire function and  $|v(x, y)| \leq 10$ , then  $f$  is constant.

### 6 Problem 6

Since  $D_z^{(n)} e^z = e^z$ , and  $e^{2\pi i} = -1$ ,

$$a_n = -\frac{1}{n!}, \quad n = 0, 1, 2, \dots$$

Thus, by Taylor's theorem, since  $e^z$  is entire, we have the Taylor series

$$e^z = \sum_{n=0}^{\infty} -\frac{1}{n!} (z - 2\pi i)^n$$

which holds for all  $z \in \mathbb{C}$ .

### 7 Problem 7

We assume we wish for the annulus  $1 < |z| < 5$ .

The first portion,  $\frac{1}{z+5}$ , will yield Taylor series (for  $|z| < 5$ ),

$$\frac{1}{z+5} = \frac{1}{5} \frac{1}{1 - (-z/5)} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n = \sum_{n=0}^{\infty} \frac{1}{5^{n+1}} z^n$$

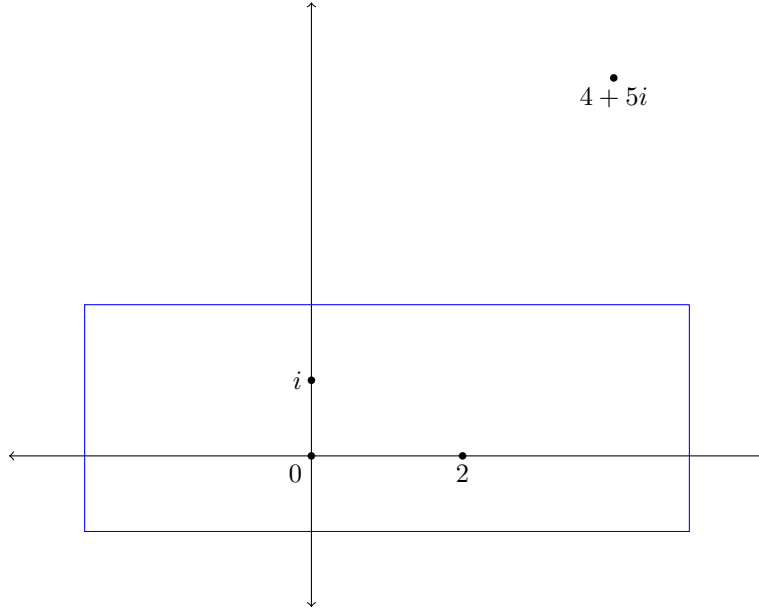
The second portion,  $2/(z-i)$ , has Laurent series (for  $|z| > 1$ ),

$$\frac{2}{z-i} = \frac{2}{z} \frac{1}{1-i/z} = \frac{2}{z} \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^n = \frac{2}{z} \sum_{n=0}^{\infty} i^n z^{-n} = \sum_{n=0}^{\infty} i^n z^{-n-1} = \sum_{n=1}^{\infty} i^{n-1} z^{-n}$$

By the uniqueness of Laurent series, and since the Taylor and Laurent series above are both correct on the annulus  $1 < |z| < 5$ ,

$$f(z) = \frac{3z+10-i}{z^2+(5-i)z-5i} = \frac{1}{z+5} + \frac{2}{z-i} = \sum_{n=0}^{\infty} \frac{1}{5^{n+1}} z^n + \sum_{n=1}^{\infty} i^{n-1} z^{-n}, \quad 1 < |z| < 5$$

## 8 Problem 8



When  $z = 0$  or  $z = 2$  or  $z = i$ , then  $\frac{2s^2+i}{s-z}$  has singularities exactly at  $z$ , and  $C$  may be continuously deformed into a unit circle around  $z$  without changing the integral. Thus, for each of these,

$$F(z) = \int \frac{2s^2+i}{s-z} ds = 2\pi i(2z^2+i) = 4\pi i z^2 - 2\pi$$

Thus,

$$F(0) = 4\pi i 0^2 - 2\pi = -2\pi$$

$$F(2) = 4\pi i 2^2 - 2\pi = 16\pi i - 2\pi$$

$$F(i) = 4\pi i (i)^2 - 2\pi = -4\pi - 2\pi = -6\pi$$

In the case of  $4+5i$ , the loop does not contain the only singularity of the interior function, and thus in this case  $C$  is contractible, so  $F(4+5i) = 0$ .

## 9 Problem 9

$$f(z) = \frac{2}{z^3-6z^2+8z} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z-4}$$

$$2 = A(z^2-6z+8) + B(z^2-4z) + C(z^2-2z) = (A+B+C)z^2 + (-6A-4B-2C)z + 8A$$

$$\begin{aligned}
A &= 1/4 \\
-6(1/4) - 4B - 2C &= 0 \\
2B + C &= -3/4 \\
C &= -3/4 - 2B \\
0 = A + B + C &= 1/4 + B - 3/4 - 2B = -1/2 - B \\
B &= -1/2 \\
C &= -3/4 - 2(-1/2) = -3/4 + 1 = 1/4
\end{aligned}$$

Thus,

$$A = 1/4 \qquad B = -1/2 \qquad C = 1/4$$

Thus, we know that

$$f(z) = \frac{1/4}{z} + \frac{-1/2}{z-2} + \frac{1/4}{z-4}$$

Using Cauchy's integral formula, a counterclockwise loop around 0, 2, and 4 yields...

...around 0:

(This doesn't matter, so I didn't bother.)

...around 2:

$$\int_C \frac{\frac{2}{z(z-4)}}{(z-2)} dz = 2\pi i \frac{2}{2(-2)} = 2\pi i(-1/2) = -\pi i$$

...around 4:

$$\int_C \frac{\frac{2}{z(z-2)}}{(z-4)} dz = 2\pi i \frac{2}{4(4-2)} = 2\pi i \frac{1}{4} = \pi i \frac{1}{2} = \frac{1}{2}\pi i$$

## 9.1 a

This will be, using something pretty much like the Cauchy Residue Formula,

$$-\pi i + \frac{1}{2}\pi i = -\frac{1}{2}\pi i$$

## 9.2 b

Breaking this up appropriately, noting 4 is circumscribed clockwise rather than counterclockwise, we add one of the twos and subtract a 4, giving us

$$-\pi i - \frac{1}{2}\pi i = -\frac{3}{2}\pi i$$

## 9.3 c

This can be broken up into two loops: one around 2 once, and one containing both 2 and 4. Thus, the integral is

$$(-\pi i) + \left(-\pi i + \frac{1}{2}\pi i\right) = -2\pi i + 0.5\pi i = -\frac{3}{2}\pi i$$

## 10 Problem 10

$$\begin{aligned}x^2 + 4x + 7 &= (x + 2 - i\sqrt{3})(x + 2 + i\sqrt{3}) \\x^2 - 8x + 18 &= (x - 4 - i\sqrt{2})(x - 4 + i\sqrt{2})\end{aligned}$$

We evaluate this integral using residues. Let

$$\begin{aligned}f(z) &= \frac{z^2}{(z^2 + 4z + 7)(z^2 - 8z + 18)} \\f(z) &= \frac{\frac{z^2}{(z - (-2 - i\sqrt{3}))(z^2 - 8z + 18)}}{(z - (-2 + i\sqrt{3}))} \\ \operatorname{Res}_{z=-2+i\sqrt{3}} f(z) &= \frac{(-2 + i\sqrt{3})^2}{(-2 + i\sqrt{3} - (-2 - i\sqrt{3}))((-2 + i\sqrt{3})^2 - 8(-2 + i\sqrt{3}) + 18)} \\ &= \frac{(-2 + i\sqrt{3})^2}{2i\sqrt{3}((-2 + i\sqrt{3})^2 - 8(-2 + i\sqrt{3}) + 18)} \\ f(z) &= \frac{\frac{z^2}{(z - (4 - i\sqrt{2}))(z^2 + 4z + 7)}}{(z - (4 + i\sqrt{2}))} \\ \operatorname{Res}_{z=4+i\sqrt{2}} f(z) &= \frac{(4 + i\sqrt{2})^2}{((4 + i\sqrt{2}) - (4 - i\sqrt{2}))((4 + i\sqrt{2})^2 + 4(4 + i\sqrt{2}) + 7)} \\ &= \frac{(4 + i\sqrt{2})^2}{2i\sqrt{2}((4 + i\sqrt{2})^2 + 4(4 + i\sqrt{2}) + 7)}\end{aligned}$$

Since the degree of the numerator is two less than the degree of the denominator, we may use equation (10) on page 264 of the text. So

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i \left[ \frac{(-2 + i\sqrt{3})^2}{2i\sqrt{3}((-2 + i\sqrt{3})^2 - 8(-2 + i\sqrt{3}) + 18)} + \frac{(4 + i\sqrt{2})^2}{2i\sqrt{2}((4 + i\sqrt{2})^2 + 4(4 + i\sqrt{2}) + 7)} \right]$$

Since this is a genuinely horrific expression, simplification has been left to the reader. However, numerical methods evaluate this to  $1.14779331879550 + i \cdot 4.35983562251079 \times 10^{-17}$ , which matches to precision the approximated value.

## 11 Problem 11

**Theorem 11.1.** *Suppose that  $f$  is an entire function satisfying*

$$f(z + i) = f(z) \text{ and } f(z + 1) = f(z) \text{ for all } z$$

*Then  $f$  is constant.*

*Proof.* Note that if  $f$  is entire it is continuous. Thus it takes compact subsets of  $\mathbb{C}$  to compact subsets of  $\mathbb{C}$ . Consequently, the image of  $[0, 1] + [0, 1]i$  under  $f$ ,  $f([0, 1] + [0, 1]i)$ , is bounded; that is,

$$|f([0, 1] + [0, 1]i)| < B$$

for some bound  $B \in \mathbb{R}$ .

But then if  $f([n, n+1] + [m, m+1]i)$  is bounded, we also know that by the presuppositions of the theorem, all its neighboring cells share the same bound:

$$\begin{aligned}
|f([n, n+1] + [m, m+1]i)| &= |f([n+1, n+2] + [m, m+1]i)| < B \\
|f([n, n+1] + [m, m+1]i)| &= |f([n-1, n] + [m, m+1]i)| < B \\
|f([n, n+1] + [m, m+1]i)| &= |f([n, n+1] + [m+1, m+2]i)| < B \\
|f([n, n+1] + [m, m+1]i)| &= |f([n, n+1] + [m-1, m]i)| < B
\end{aligned}$$

Since every point of  $\mathbb{C}$  lies in some integer unit square in the complex plane, and all of these by induction using the equations above are bounded in modulus by  $B$ ,  $f$  is bounded in modulus by  $B$  over all of  $\mathbb{C}$ . Since  $f$  is bounded in modulus by  $B$ , by the maximum modulus principle,  $f$  must be constant.  $\square$