Complex Variables Homework Section 15

Adam Buskirk

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Prove $\lim_{z\to 0}|z|/z$ does not exist, find $\lim_{z\to i} z^2+1+i$ and use $\epsilon-\delta$ proof to prove that answer correct.

Theorem 1. $\lim_{z\to 0}|z|/z$ *does not exist.*

Proof. Consider two sequences, $(1/n)$ and (i/n).

$$\lim_n \frac{|1/n|}{1/n} = \lim_n \frac{1/n}{1/n} = \lim_n 1 = 1$$

$$\lim_n \frac{|i/n|}{i/n} = \lim_n \frac{\sqrt{(i/n)(-i/n)}}{i/n} = \lim_n \frac{\sqrt{-(-1)/n^2}}{i/n} = \lim_n \frac{1/n}{i/n} = \lim_n \frac{1}{i}\frac{-i}{-i} = \lim_n \frac{-i}{1} = -i$$

Since these two sequences both converge to 0 but they converge to different images, the limit at 0 is not well-defined and does not exist. □

Theorem 2.

$$\lim_{z\to i} z^2+1+i = i$$

Proof. Suppose we are given some $\epsilon > 0$. Then let $\delta = \sqrt{1+\epsilon}-1$.

Then suppose $|z-i| < \delta$. So

$$\begin{aligned}
|z-i| &< \sqrt{1+\epsilon}-1 \qquad (1)\\
|z+i-2i| &< \sqrt{1+\epsilon}-1\\
|z+i|-|2i| \leq \big||z+i|-|2i|\big| &< \sqrt{1+\epsilon}-1\\
|z+i|-2 &< \sqrt{1+\epsilon}-1\\
|z+i| &< \sqrt{1+\epsilon}+1 \qquad (2)
\end{aligned}$$

Combining the inequalities (1) and (2),

$$\begin{aligned}
|z-i|\cdot|z+i| &< (\sqrt{1+\epsilon}-1)(\sqrt{1+\epsilon}+1)\\
|(z-i)(z+i)| &< (\sqrt{1+\epsilon})^2-1\\
|(z+i)(z-i)| &< 1+\epsilon-1\\
|z^2+1| &< \epsilon\\
|(z^2+1+i)-i| &< \epsilon
\end{aligned}$$

Hence if $|z-i| < \delta = \sqrt{1+\epsilon}-1$, then $|(z^2+1+i)-i| < \epsilon$. Therefore, $\lim_{z\to i} z^2+1+i = i$. □