747: Topology II

Math 747: Differential Topology

Topics: 1. Manifolds: Def, properties, maps.
2. Tangent Bundle: Def, Vector Fields, Maps
3. Transversality: Sard's Theorem (Existence Thm.)
Whitney Theorems.
4. Vector Fields and Bundles.
5. Differential Forms: Def, Integration, Stokes' Thm.

1. Manifolds

locally: look like $\mathbb{R}^n$
globally: patched out of open sets $U \subset \mathbb{R}^n$.

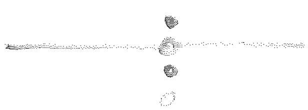
1.1 Topological Manifolds

$\boxed{\text{Def 1.1}}$ A real n-dimensional topological manifold M is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

Remark: 1) "loc. homeo to $\mathbb{R}^n$": Each point $p \in M$ has an open nbd $U \ni p$ s.t. there is a homeomorphism $\varphi: U \to V \subset \mathbb{R}^n$, V open.
The pair (U, φ) is called a coordinate chart around p: A collection $\{(U,\varphi)\}$ which covers M is called an atlas.

2) "Hausdorff": Removing this allows for spaces like $N = \mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$ where

$\mathbb{R}_1 \setminus \{0\} \ni x \sim \varphi(x) \in \mathbb{R}_2 / \{0\}$, $\varphi : \mathbb{R}_1 \to \mathbb{R}_2$ a homeomorphism with $\varphi(0) = 0$: Then



locally N is homeom. to $\mathbb{R}$, but $[0 \in \mathbb{R}_1]$, $[0 \in \mathbb{R}_2]$ can't be separated by open nbds.

3) Second countability is not as crucial, but will be needed for the Whitney Theorems.
($\exists$ ctble atlas) ($\rightsquigarrow$ paracompact $\to$ part. of unity)

Ex: 0) $\emptyset$, $\{p_1, \dots, p_n\} \in \mathbb{R}^n$, $\mathbb{R}^n$

1) $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Fix $z \in S^1$, then $z = e^{2\pi i c}$ for some $0 \le c < 1$. Define a map

$$v_z : \mathbb{R} \longrightarrow S^1$$
$$t \longmapsto e^{2\pi i t}$$

Let $I_c = (c - 1/2, c + 1/2)$; then $v_z : I_c \longrightarrow S^1 \setminus \{-z\}$

So $\varphi_z = \left(v_z|_{I_c} \right)^{-1}$ is a coordinate chart.

2) products of coord charts $\Rightarrow$ charts for prod. of mflds. (3

$\Rightarrow$ <u>n-torus</u>: $\underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$ is a topol. mfd

3) (open subsets): $U \subset M$ open subset is a topol. mfd by restriction.

$\Rightarrow$ $Mat(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$ topol. mfd.

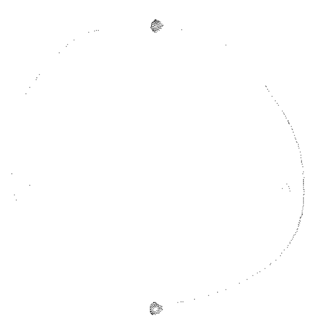
$GL(n,\mathbb{R}) = \{A \in Mat(n,\mathbb{R}) \mid \det A \neq 0\} \subset Mat(n,\mathbb{R})$

open (why?), so is a topol. mfd.

4) (Spheres)

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}.$$

$N = (1,0,\dots,0)$



$S = (-1,0,\dots,0)$

$$\varphi_N : U_N = S^n \setminus \{S\} \longrightarrow \mathbb{R}^n$$
$$(x_0, x) \longmapsto \frac{1}{1+x_0} x$$

(stereograph. projection)

$$\varphi_S : U_S = S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$$
$$(x_0, x) \longmapsto \frac{1}{1-x_0} x$$

This is only one of many possible atlases; this endows S^n with a certain topology.

<u>Q</u>: Does there exist $\tilde{S}^n$ homotopy equiv to S^n but not homeomorphic?

So $\exists$ $f: \tilde{S}^n \to S^n$, $g: S^n \to \tilde{S}^n$ s.t.

$$f \circ g \sim Id, \quad g \circ f \sim Id$$

but not homeomorphic?

No: Any homotopy n-sphere is homeom. to the n-sphere.

$n>4$: Smale (1960's) (F)

$n=4$: Freedman (1982) (F)

$n=3$: Hamilton - Perelman (2003) (- T)

$n=1,2$: class. of 1- and 2-mfds.

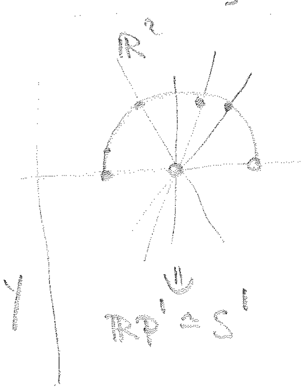
We will return to this in the smooth setting.

5) (Projective Spaces) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$$\mathbb{K}P^n = \{\text{space of lines through } \{0\} \text{ in } \mathbb{K}^{n+1}\}$$

$$= \underbrace{\mathbb{K}^{n+1} \setminus \{0\}}_{X} \Big/ x \sim y$$

where $x \sim y$ iff $\exists \lambda \in \mathbb{K} \setminus \{0\} : \lambda x = y$



give this the quotient topology.

The proj map $\pi : X \to \mathbb{K}P^n$ is an open map, so $\mathbb{K}P^n$ is second countable. To show it is Hausdorff, show

$$\Gamma_\sim = \{(x,y) \in X \times X \mid x \sim y\}$$

is closed. But we can write $\Gamma_\sim$ as the common (why?) zeros of the functions

$$f_{ij}(x,y) = x_i y_j - y_j x_i \qquad i \neq j$$

An atlas is given by the open sets

$$U_i = \pi(\hat{U}_i) \text{ where } \hat{U}_i = \{ (x_0, \dots, x_n) \in X \ , \ x_i \neq 0 \}$$

with charts

$$\varphi_i([x_0, \dots, x_n]) = \frac{1}{x_i} (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

(inverse: $(y_1, \dots, y_n) \mapsto [y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n]$)

6) (Connected Sum)

topol. mfds. $\quad M \ni p \qquad N \ni q$

charts $\quad (U, \varphi) \qquad (V, \psi)$

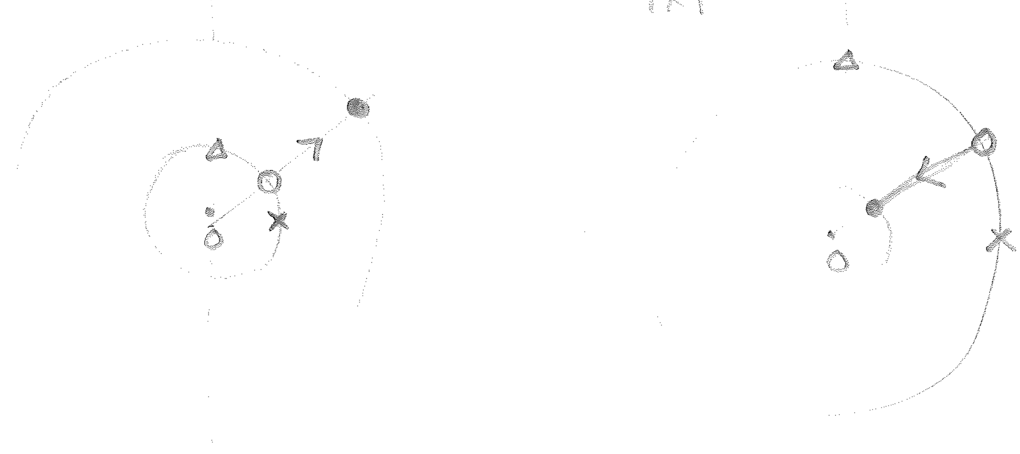
$\qquad \varphi(p) = 0 \qquad \psi(q) = 0$

Choose $\varepsilon \ll 1$ s.t. $B(0, 2\varepsilon) \subset \varphi(U)$, $B(0, 2\varepsilon) \subset \varphi(V)$.

Define a map

$$\phi : B(0, 2\varepsilon) \setminus \overline{B(0, \varepsilon)} \longrightarrow B(0, 2\varepsilon) \setminus \overline{B(0, \varepsilon)}$$

$$x \longmapsto \frac{2\varepsilon^2}{|x|^2} x$$



Define the <u>connected sum</u> $M \# N$ as the quotient

$$X/\sim$$

where

$$X = \left(M \setminus \overline{\varphi^{-1}(B(0,\varepsilon))}\right) \sqcup \left(N \setminus \overline{\psi^{-1}(B(0,\varepsilon))}\right)$$

with

$$x \sim \psi^{-1}\phi\varphi(x) \quad \text{for } x \in \varphi^{-1}(B(0,2\varepsilon)).$$

If A_M, A_N are atlases of M, N resp then

$$A_M\Big|_{M \setminus \overline{\varphi^{-1}(B(0,\varepsilon))}} \cup A_N\Big|_{N \setminus \overline{\psi^{-1}(B(0,\varepsilon))}}$$

is an atlas for $M \# N$.

<u>Remark</u>: The homeomorphism type of $M \# N$ is independent of the choices of p, q and φ, ψ except for the orientation of

$$\psi^{-1}\phi\varphi.$$

<u>Recall</u>: Any compact 2-mfd can be built from S^2, T^2, $\mathbb{RP}^2$ using the conn. sum.

7) Let F be a top. space. A <u>fiber bundle</u> with fiber F is a triple (E, p, B) where E, B are topol. spaces and $p: E \to B$ a cont. surjection such that for each $b \in B$ there is an open nbd $U \ni b$ and a homeomorphism

$$\phi: p^{-1}(U) \to U \times F$$

such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

$F \quad \{u\} \qquad \{u\} \times F \quad \{u\}$

Ex:

$$\begin{tikzcd} S^1 \arrow[r] & T^2 \arrow[d] \\ & S^1 \end{tikzcd}$$

This is often written as

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

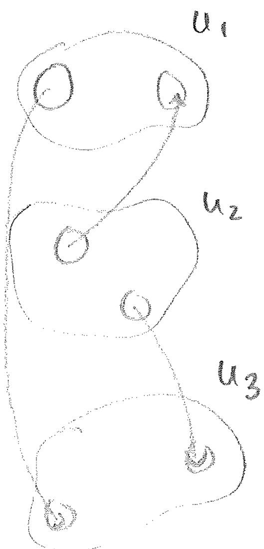
fiber — E total space, p projection, B base

when B, F are topol. mfds, then so is E.

8) (general gluing construction)

→ glue open subsets of $\mathbb{R}^n$ using homeo's ("from scratch")

Let $\{U_i\}$ countable collection of open subsets.



For each i, choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

s.t.

i) $\varphi_{ij} \varphi_{ji} = Id_{U_{ji}}$

ii) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$

and iii) φ_{ij} homeomorphisms.

Additionally, we want

$$\varphi_{ki}\,\varphi_{jk}\,\varphi_{ij} = \mathrm{id}_{U_{ij}\cap U_{jk}} \quad \forall i,j,k$$

Second countability is guaranteed as $\{U_i\}$ is countable, for Hausdorff we need

$$\{(x, \varphi_{ij}(x)) : x \in U_{ij}\} \subset U_i \times U_j$$

is closed.

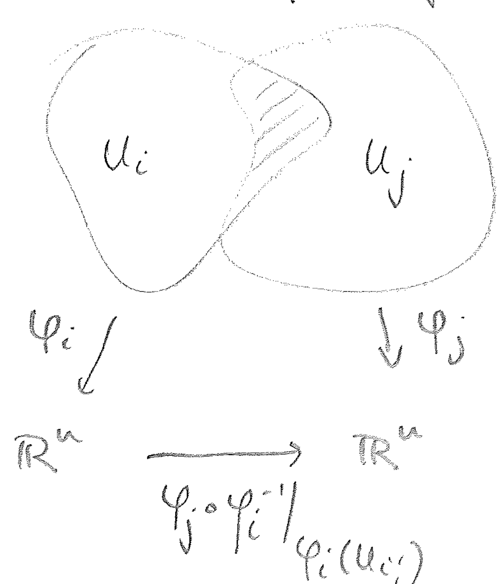
Then

$$M = \bigsqcup U_i \big/ \sim$$

with $x \sim \varphi_{ij}(x) \;\; \forall x \in U_{ij}$ is a topol. mfd.

1.2 Smooth Manifolds

On a topol. mfd: (U_i, φ_i), (U_j, φ_j)



On $U_{ij} = U_i \cap U_j$ we get a map

$$\varphi_j \circ \varphi_i^{-1}\big|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \to \varphi_j(U_{ij})$$

This map is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$\Rightarrow$ coordinate change on a topol mfd may not be differentiable, i.e. a function that is diffble in one coord chart may not be in another.

<u>Remark</u> (Some Calc 3 repackaged; an aside on smooth maps of vector spaces)

Let $U \subset V$ open subset of a finite dim vector space V and let $f : U \to W$ be a function with values in a vector space W. We say f is <u>differentiable at $p \in U$</u>, if there is a linear map

$$Df(p) : V \to W$$

which approximates f near p, meaning that

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{\| f(p+x) - f(p) - Df(p)x \|}{\|x\|} = 0 \quad \left(\begin{array}{c}\text{Fréchet} \\ \text{Deriv.}\end{array}\right)$$

This uniquely defines $Df(p)$.

- finite dim $\Rightarrow$ all norms equivalent
- infinite dim: norms need to be specified and $Df(p)$ is required to be a cont. linear operator.

Let $(e_1, .., e_n)$ be a basis for V. Then V has a corresponding dual basis $(x_1, ..., x_n)$ of linear functions, we call these coordinates:

$$x_i(e_j) = \delta_{ij}$$

Let $(y_1, \ldots, y_m)$ be coordinates on W.
Then f has m scalar components and

$$f_j = y_j \circ f$$

Relative to these chosen basis on V, W the linear map $Df(p)$ can be written as a $m \times n$ matrix called the <u>Jacobian matrix of f at p</u>:

$$Df(p) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

We say f is <u>differentiable in U</u> when it is diff'ble at each $p \in U$. It is <u>continuously diff'ble</u> when

$$Df : U \longrightarrow \mathrm{Hom}(V, W)$$

is continuous. The space of such functions is denoted by $C^1(U, W)$.

Notice that Df is a map from U to a vector space $\mathrm{Hom}(V, W)$, so if it is diff'ble we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}\big(V, \mathrm{Hom}(V, W)\big).$$

and so on.

$\rightarrow$ $C^k(U, W)$ = k-times cont. diff'ble functions $f : U \to W$.

<u>smooth maps</u>: $C^\infty(U,W)$; "all derivatives exist"

$$\Rightarrow \quad C^\infty(U,W) = \bigcap_k C^k(U,W)$$

<u>Remark</u>: In fact $D^2 f$ has values in a subspace of $V^* \otimes V^* \otimes W$, namely $\mathrm{Sym}^2(V^*) \otimes W$ due to "mixed partials are equal".

Def 1.2 A <u>smooth manifold</u> is a topological manifold equipped with an equivalence class of smooth atlases: An atlas $A = \{(U_i, \varphi_i)\}$ is called <u>smooth</u> when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}\big|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \to \varphi_j(U_{ij})$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$.
Two atlases A, A' are <u>equivalent</u> if $A \cup A'$ is itself a smooth atlas.

<u>Remark</u>: If we require the gluing maps to be C^k or real-analytic or holomorphic or ... then we define C^k, real analytic, complex, ... mfds respectively.

Ex. 1) Recall the circle and pick $z=1$, $z=-1$. Then $\{(U_1, \varphi_1), (U_{-1}, \varphi_{-1})\}$ defines a smooth structure on S^1. Notice:

$$\varphi_{-1} \circ \varphi_1^{-1} = \begin{cases} t+1 & -\frac{1}{2} < t < 0 \\ t & 0 < t < \frac{1}{2} \end{cases}$$

$\to (0,1)$ $\qquad t \mapsto e^{2\pi i t}$, $\left(-\frac{1}{2}, \frac{1}{2}\right) \to$

2) products, open subsets as before.

3) (Spheres)

$$\varphi_N \circ \varphi_S^{-1} : \overset{(x_0,x)}{\overset{\shortparallel}{z}} \longmapsto \frac{1-x_0}{1+x_0} z = \frac{(1-x_0)^2}{|x|^2} z = \frac{z}{|z|^2}$$

$$\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$$

is smooth, So S^n are smooth mfds. with respect to the atlas given.

4) (projective spaces)

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \dots, z_n) = \frac{1}{z_1}(1, z_2, \dots, z_n)$$

$$\mathbb{R}^n \setminus \{z_1 = 0\} \to \mathbb{R}^n$$

$\to$ smooth as required, the same for φ_i, φ_j

5) Connected sums smooth as $\phi(x) = \dfrac{2\varepsilon^2}{|x|^2} x$ is smooth.

$\boxed{\text{Def 1.3}}$ Let $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

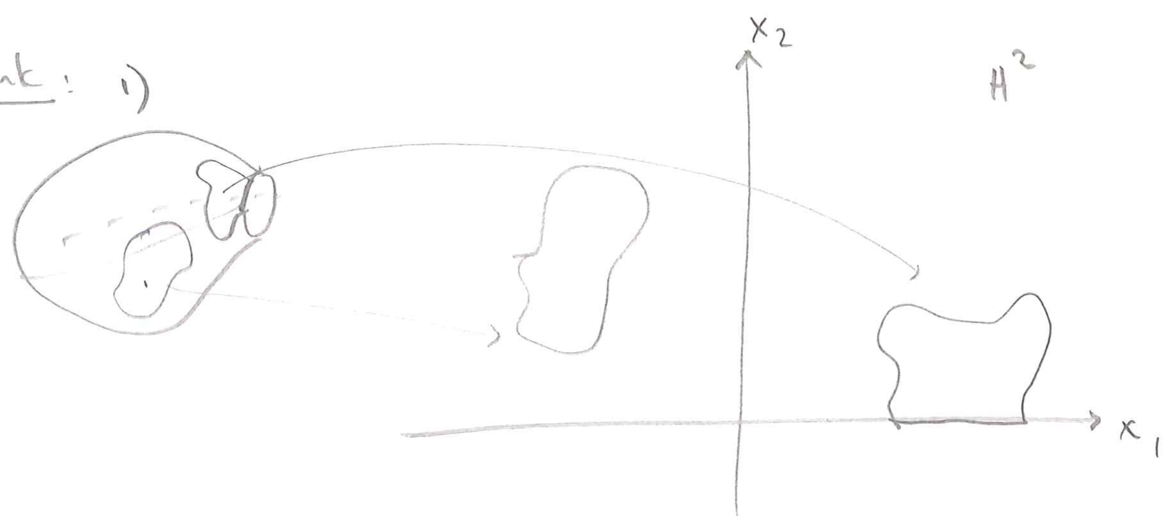
1) A <u>topological manifold M with boundary</u> is a second countable, Hausdorff topol. space loc. homeo. to $\mathbb{H}^n$. Its <u>boundary ∂M</u> is the $(n-1)$-manifold consisting of all points mapped to $x_n = 0$ by a chart and its <u>interior $\operatorname{Int} M$</u> is the set of points mapped to $x_n > 0$ by a chart. It follows that

$$M = \partial M \sqcup \operatorname{Int} M$$

2) Let V, W be finite dim vector spaces. A function $f : A \to W$, $A \subset V$ is <u>smooth</u> when it admits a smooth extension to an open nbhd $U_p \subset V$ of every point $p \in A$.

3) A <u>smooth manifold with boundary</u> is an equivalence class of smooth atlases, smooth as above.

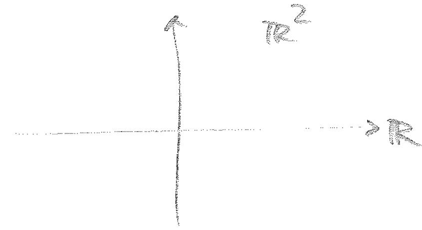
<u>Remark</u>: 1)



In particular, every mfd is a mfd with boundary (maybe with $\partial M = \emptyset$).

2) "boundary"

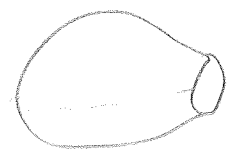
topol. bdy: Let $\mathbb{R} \subset \mathbb{R}^2$:



Then $\partial \mathbb{R} = \mathbb{R}$.

Now consider $\mathbb{R}$ as a topol. space without $\mathbb{R}^2$. Then $\partial \mathbb{R} = \emptyset$.

manifold bdy:

punctured sphere 

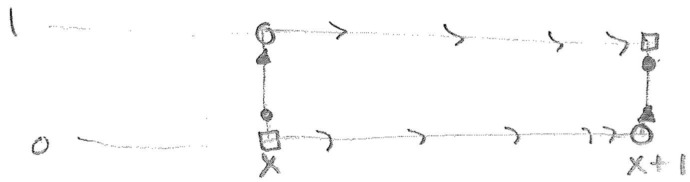
has a bdy when viewed as a subset $\subset \mathbb{R}^2$ or not! Here $\partial M \cong S^1$. $(\partial_{Top} M = M)$.

3) $f(x,y) = y$ is smooth in $\mathbb{H}^2$ but $f(x,y) = \sqrt{y}$ is not.

4) Notice: *) $\mathrm{Int}\, M$ is a mfd without boundary of the same dim as M.

*) ∂M is a $n-1$-dim mfd without bdy. (i.e. $\partial^2 = 0$)

Ex: (Möbius Strip) Consider $\mathbb{R} \times [0,1] = X$. Define a quotient $E = X/\sim$ with $(x,y) \sim (x+1, 1-y)$:



E is a mfd with boundary.

In fact, E is also a fiber bundle over S^1 via the map $\pi: [(x,y)] \mapsto e^{2\pi i x}$. The boundary ∂E is isomorphic to S^1. This is a non-trivial fiber bundle because $S^1 \times [0,1]$ has disconnected boundary.

1.3 Cobordism

Use compact $(n+1)$-mfds with bdy to define an equivalence rel. on compact n-mfds, called cobordism:

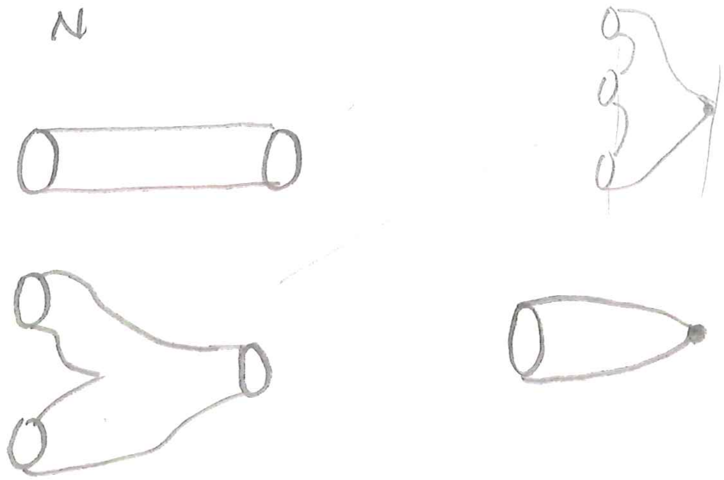
Def. 1.4 Compact n-mfds M_1, M_2 are cobordant when there exists a compact $(n+1)$-mfd w. bdy N such that $\partial N \cong M_1 \sqcup M_2$. All mfds cobordant to M form the cobordism class of M. We say M is null-cobordant if $M \cong \partial N$.

Remark: Compactness is crucial. If not cpt, then all mfds are null-cobordant via

$N = M \times [0,1)$.

Ex: points:

1-dim:



$\Rightarrow$ all cpt 1-mfds are null-cobordant.

Let $\Omega^n = \{$ cobordism classes of cpt n-mfds $\}$ including $\emptyset$

Using the disjoint union $[M_1] + [M_2] = [M_1 \sqcup M_2]$ we see that Ω^n is an abelian group with identity $[\emptyset]$.

The direct sum $\Omega = \bigoplus_{n \geq 0} \Omega^n$ is endowed with

$$[M_1] \cdot [M_2] = [M_1 \times M_2]$$

making Ω into a graded commutative ring with mult. unit $[\{*\}]$. Ω is called the <u>cobordism ring</u>.

$\boxed{\text{Lemma 1.5}}$ The cobordism ring is 2-torsion, i.e.

$$x + x = 0 \quad \forall x \in \Omega.$$

<u>Pf</u>: Let M be a mfd, the mfd w. bdy $M \times [0,1]$ has bdy $M \sqcup M$. Hence $[M]+[M] = [M \sqcup M] = [\emptyset] = 0$ as required. $\boxtimes$

<u>Ex</u>: 1) S^n is null-cobordant: $S^n \cong \overline{\partial B_{n+1}(0)}$.

2) Any oriented 2-mfd is null-cobordant:

S^2 $\quad$ T^2 $\quad$ ...

each bds a region in $\mathbb{R}^3$.
→ "inside" is mfd. w. bdy.

(so in part: all oriented 2-mfds are cobordant)

<u>Theorem 1.6</u> (Thom 1956) The cobordism ring is a (countably generated) polynomial ring over $\mathbb{F}_2$ with generators in every dimension $n \neq 2^k - 1$, i.e.

$$\Omega = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots]$$

<u>Ex</u>: <u>dim 0</u>: two classes: $\underset{[\emptyset]}{\overset{0}{\shortparallel}}$ $\quad$ $\underset{[*]}{\overset{1}{\shortparallel}}$ $\qquad \Omega^0 = \mathbb{F}_2$

<u>dim 1</u>: no classes, so all null-cobordant.
besides $[\emptyset]$. $\qquad \Omega^1 = 0$

dim 2: $\{0, x_2\}$ 2 classes (what mfds rep. x_2?)
$\Omega^2 = \mathbb{F}_2$

dim 3: $[\emptyset]$: every cpt 3-mfd is the bdy of a 4-mfd.
$\Omega^3 = 0$

dim 4: $\{0, x_2^2, x_4, x_2^2 + x_4\} \cong \mathbb{F}_2 \times \mathbb{F}_2$

Q: what are reps of each class?

1.4 Smooth Maps

→ M, N topol. mfds the natural notion of a morphism is $f: M \to N$ continuous. Two top. mfds M, N are equivalent if there exists $f: M \to N$ homeomorphism.

→ topological mfds together with continuous maps form what is called a "category":

$\boxed{\text{Def 1.7:}}$ A category consists of objects C and arrows A. Each arrow goes from an object (the source) to an object (the target). These arrows are often called morphisms. Each object has an identity morphism $id_M: M \to M$ for $M \in C$. Morphisms can be composed, and satisfy associativity. For $X, Y \in C$ we write $\text{Hom}(X, Y) \subset A$ as the set of morphisms from X to Y.

Then composition becomes

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$$
$$(f,g) \longmapsto g \circ f.$$

Ex: $C = \{$topol. mfds$\}$, $A = \{$continuous maps$\}$ defines the category of topological mfds.

For smooth mfds we need the correct notion of a morphism.

Def 1.8 A map $f: M \to N$, M, N smooth mfds, is called <u>smooth</u> when for each chart (U,φ) for M and each chart (V,ψ) for N, the composition $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$.

The set of smooth maps from M to N is denoted $C^\infty(M,N)$. A smooth map with smooth inverse is called a <u>diffeomorphism</u>.

Lemma 1.9 If $g: L \to M$ and $f: M \to N$ are smooth maps, then so is $f \circ g: L \to N$.

Pf: Choose charts

L	M	N
p	$g(p)$	$f(g(p))$
(U,φ)	(V,χ)	(X,ψ)

Then

$$\varphi \circ (f \circ g) \circ \varphi^{-1} = \underbrace{\varphi \circ f \circ \chi}_{\substack{\mathbb{R}^n \to \mathbb{R}^n \\ \text{smooth}}} \circ \underbrace{\chi^{-1} \circ g \circ \varphi^{-1}}_{\substack{\mathbb{R}^n \to \mathbb{R}^n \\ \text{smooth}}}$$

By the chain rule the RHS is smooth. $\boxtimes$

$\Rightarrow$ $C = \{\text{smooth mfds}\}$, $A = \{\text{smooth maps}\}$ is a cat.

The same holds for

$C = \{\text{sm. mfds w. bdy}\}$ $A = \{\text{smooth maps}\}$.

Here we could restrict A to only those maps which preserve the boundary.

Recall the operator ∂ which sends a mfd. w. bdy to a mfd w.o. bdy. If $\varphi : M \to N$ is a boundary preserving smooth map, then $\partial\varphi = \varphi|_{\partial M}$ is well-defined.

So $\partial : \{\text{smooth mfds w. bdy}\} \longrightarrow \{\text{smooth mfds}\}$.

$\{\text{bdy pres. smooth maps}\} \longrightarrow \{\text{smooth maps}\}$

∂ : Category of smth. mfds w. bdy $\to$ Cat of sm. mfds.

Such a map is called a <u>functor</u>.

<u>Ex</u> of smooth maps:

1) φ_z a chart for S^1, $j : S^1 \to \mathbb{C}$ the inclusion map.

Then

$$j\circ\varphi_2^{-1} : t \longmapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))$$

is a smooth map from I_c to $\mathbb{R}^2$. So j is smooth.

2) $\mathbb{C}P^1 \simeq S^2$:

Atlas for $\mathbb{C}P^1$:

$$\varphi_1 : \underbrace{\mathbb{C}P^1 \setminus [0:1]}_{U_1} \to \mathbb{C} \qquad [1:z_2] \longmapsto z_2$$

$$\varphi_2 : \underbrace{\mathbb{C}P^1 \setminus [1:0]}_{U_2} \to \mathbb{C} \qquad [z_1:1] \longmapsto z_1$$

$$\varphi_1(U_1\cap U_2) = \mathbb{C}\setminus\{0\} \overset{``}{=} \mathbb{C}^* = \varphi_2(U_1\cap U_2)$$

$$\varphi_2\circ\varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*$$
$$z \longmapsto z^{-1}$$

Atlas for S^2:

$$\varphi_1 : \underbrace{S^2\setminus(0,0,1)}_{V_1} \to \mathbb{C} \qquad (a,b,c) \longmapsto \frac{a}{1-c} + i\,\frac{b}{1-c}$$ inverse

$$\varphi_2 : \underbrace{S^2\setminus(0,0,-1)}_{V_2} \to \mathbb{C} \qquad (a,b,c) \longmapsto \frac{a}{1+c} - i\,\frac{b}{1+c}$$

$$\varphi_1(V_1\cap V_2) = \mathbb{C}^* = \varphi_2(V_1\cap V_2)$$

$$\varphi_2\circ\varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* \qquad z \longmapsto z^{-1}$$

Define $f : \mathbb{C}P^1 \to S^2$ by $f(\varphi_i^{-1}(z)) = \varphi_i^{-1}(z)$ $\forall z\in\mathbb{C}$

$$\begin{tikzcd} \mathbb{C}P^1 \arrow[rr, "f"] \arrow[dr, "\varphi_i"'] & & S^2 \arrow[dl, "\varphi_i"] \\ & \mathbb{C} & \end{tikzcd}$$

This map is well defined. It is smooth because

$$\varphi_j \circ f \circ \varphi_i^{-1}(z) = \begin{cases} z & i = j \\ \frac{1}{z} & i \neq j \end{cases}$$

The obvious inverse is also smooth. So f defines a diffeomorphism.

3) $\underline{B^n \cong \mathbb{R}^n}$:

$$F(x) = \frac{x}{\sqrt{1 - |x|^2}} \qquad B^n \to \mathbb{R}^n$$

$$G(y) = \frac{y}{\sqrt{1 + |y|^2}} \qquad \mathbb{R}^n \to B^n$$

are smooth and inverses of each other.

4) $j : S^1 \to \mathbb{C}$ induces a smooth inclusion

$$j \times j : T^2 \to \mathbb{C}^2$$

The image of this map does not include 0, so we may compose this map with the quotient map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ and the diffeomorph. $\mathbb{CP}^1 \to S^2$ to get a smooth map

$$\pi \circ (j \times j) : T^2 \longrightarrow S^2$$

Remark (Exotic smooth structures)

Q: Fix M a manifold with a smooth structure. Do there exist smooth structures on M not diffeo. to the given one? If so, how many? Does every topol. mfd. admit a smooth structure?

Assume M is connected.

0-dim: $M = \{pt\}$: only smooth structure is $\{(M, \varphi)\}$ where $\varphi : M \to \mathbb{R}^0$ is unique.

1-dim: M homeom (= diffeom.) to either $\mathbb{R}$ or S^1 w. stand. topology.

2-dim: every 2-mfd admits a unique smooth str; compact 2-mfds are classified.

3-dim: every 3-mfd admits a unique smooth str; compact 3-mfds are classified.

4-dim:
- * $\mathbb{R}^4$ (Donaldson, Gompf, Freedman): uncountably many non-diffeomorphic smooth structures (these are called exotic $\mathbb{R}^4$'s).
- * Many 4-mfds admit no smooth structure. (simply-connected: Freedman gave a classif.)
- * S^4: no one knows!? (Smooth Poincaré Conj.)
- * M compact: some have countably infinite, no classification. → exotic $\mathbb{CP}^2$?

<u>Spheres</u>: (Kervaire - Milnor) <u>$n \neq 4$</u>

$\{\text{smooth } n\text{-spheres}\}/_{\text{diffeo}}$ is a finite abelian gp.

First non-trivial order : $n = 7$ → order 28.
(exotic S^7: homeo to S^7 but not diffeo to S^7)

<u>dim ≥ 5</u>: M compact → finitely many smooth structures up to diffeo. (Milnor, Kervaire, Hirsch)

why do we need "up to diffeo"?

Let (M, A) be a smooth mfd. w. atlas, and $h: M \to M$ a homeomorphism. Then

$$h^*A = \{ (h^{-1}(U), \varphi \circ h) \}$$

is also a smooth structure on M since the overlap maps are the same as for A:

$$(\psi \circ h) \circ (\varphi \circ h)^{-1} = \psi \circ \varphi^{-1}$$

But A and h^*A are the same iff h is a diffeomorphism. However, $h^{-1}: (M, A) \to (M, h^*A)$ is a diffeomorphism.

⇒ if a topol. mfd. M admits a smooth str, it admits uncountably many diffeom. smooth structures.

Ex (Lie group)

G group, $m: G \times G \to G$ (multiplication)

$i: G \to G$ (inversion)

e: identity element.

If one endows G with a topology making G a topol. mfd. and such that m, i are continuous, then G is called a topological group.

If G is given a smooth structure and m, i are smooth maps, then G is called a Lie group.

Basic examples ⊛ $(\mathbb{R}, +)$; $(S^1$, cpx. mult.$)$ $\Rightarrow$ cart. products

⊛ $GL(n, \mathbb{R})$

Notice that the map m induces two further interesting smooth maps: Fix $g \in G$ and define

$$R_g : G \to G \qquad R_g(h) = hg$$

$$L_g : G \to G \qquad L_g(h) = gh$$

Notice further that

$$R_g L_h = L_h R_g \qquad \text{(associativity)}$$

$$((hk)g = h(kg))$$

2. Tangent Bundle

Motivation: $U \subset V$ open, V f.dim vectorspace.

- a tangent vector to $p \in U$ is simply a vector in V. we write $T_pU \cong V$.
- on all of U the space $U \times V$ is called the <u>tangent bundle</u>; this is the collection of all tangent vectors on U; $TU = \bigsqcup_{p \in U} T_pU$
- $TU = U \times V$ comes with (two) projections,

  $\pi : TU \to U$ (most important)

- a <u>vector field</u> on U is a map

  $\underbrace{s : U \to TU \quad \text{with} \quad \pi \circ s = id}_{\text{such a map is called a section}}$

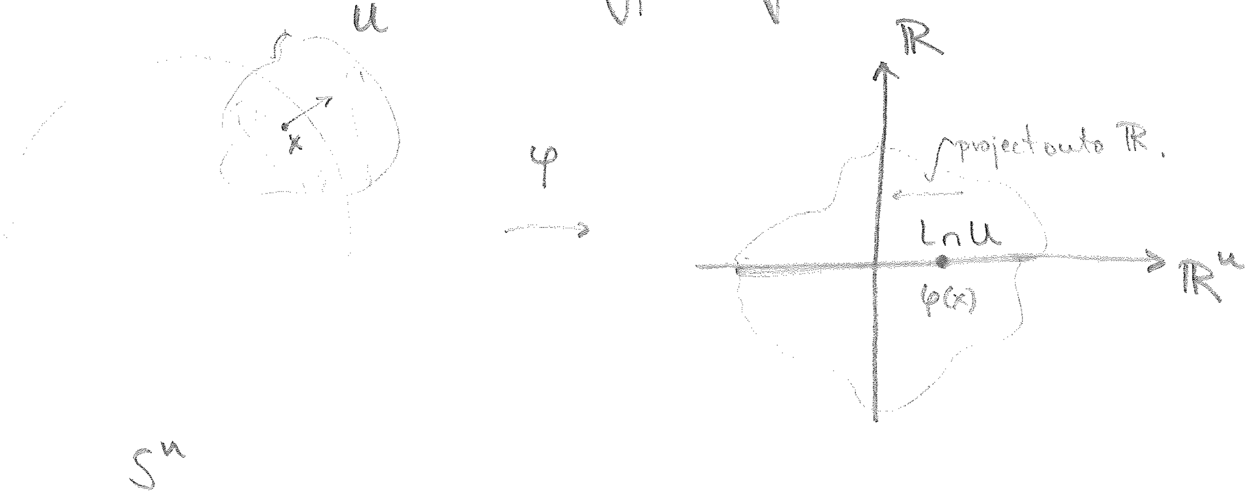
<u>Idea</u>: Tangent bundle on a manifold will locally look like the above and globally describe all tangent vectors.

$\boxed{\text{Def 2.1}}$ A subspace $L \subseteq M$ of an m-mfd is called a <u>regular</u> or <u>embedded submanifold</u> of codimension k when each point $x \in L$ is contained in a chart (U, φ) of M such that

$$L \cap U = f^{-1}(0)$$

where f is the composition of φ with the projection $\mathbb{R}^m \to \mathbb{R}^k$ to the last k coordinates $(x_{m-k+1}, \dots, x_m)$. A submanifold of codimension 1 is called a hypersurface.

Ex: $S^n \subseteq \mathbb{R}^{n+1}$ is a hypersurface:

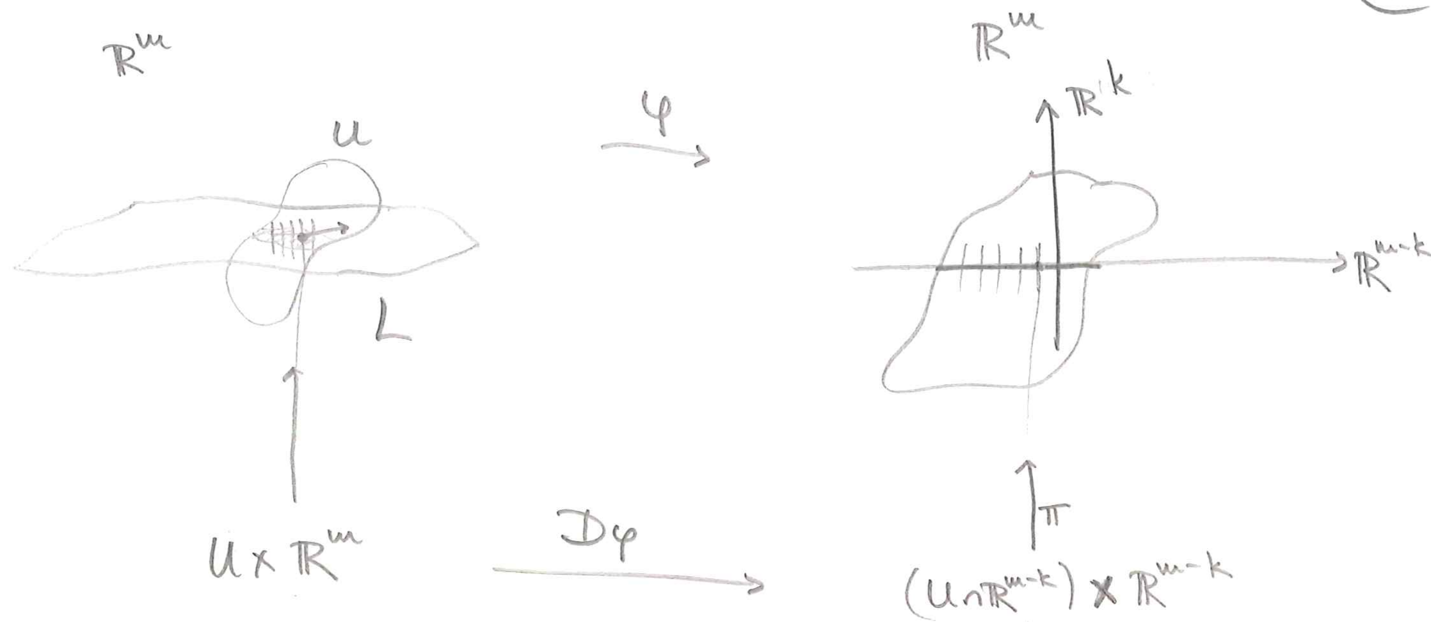


Now suppose $L \subseteq \mathbb{R}^m$ is a submfd of codim k and let φ be a diffeomorphism as in the def:

This basically sets up a "rectilinear" coord system on x where first $m-k$ coords are in L and last k coords describe directions "perpendicular" to L.

Then we say $u \in \mathbb{R}^m$ is tangent to L at p when the derivative $D\varphi(p)$ takes u to the linear subspace of $\mathbb{R}^m$ given by $x_{m-k-1} = \cdots = x_m = 0$.

Then the tangent bundle TL to L is the set of pairs (p,u) where $p \in L$ and $u \in \mathbb{R}^m$ is tangent to L at p. It is a subset of $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$ and is itself a submfd of $T\mathbb{R}^m$ of codim $2k$.



2.1 The General Construction

The tangent bundle to an n-mfd M is a $2n$-mfd, called TM, naturally constructed in terms of M. As a set, TM is the disjoint union of the tangent spaces T_pM. We will now describe this construction in detail:

Def 2.2: Let $(U, \varphi), (V, \psi)$ be charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then

(U, φ, u) ; (V, ψ, v) are called equivalent when

$$D(\psi \circ \varphi^{-1})(\varphi(p))(u) = v.$$

(This is an equiv. rel. by the chain rule → check it!)

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM.

$T_p(M)$ is a real vectorspace of dimension $\dim M$ and $D(\psi\circ\varphi^{-1})$ is a linear isomorphism.

As a set, the tangent bundle TM is

$$TM = \bigsqcup_{p\in M} T_pM$$

equipped with a natural projection $\pi: TM \to M$.

$\boxed{\text{Lemma 2.3}}$: For an n-mfd M, the set TM has a natural topology and smooth structure such that TM is a $2n$-mfd and $\pi: TM\to M$ is a smooth map.

Pf Any chart (U,φ) for M defines a bijection

$$T\varphi(U) = \varphi(U)\times\mathbb{R}^n \longrightarrow \pi^{-1}(U)$$

via

$$(p,v) \longmapsto (U,\varphi,v)\in T_pM$$

Using this, we induce a smooth mfd. str. on $\pi^{-1}(U)$ and view $(\pi^{-1}(U),\phi)$, ϕ the inverse of this map, as a chart to $\varphi(U)\times\mathbb{R}^n$.

Let (V,ψ) be another chart; let $(\pi^{-1}(V),\Psi)$ be the corresp. chart on $\pi^{-1}(V)$. Then

$$\Psi\circ\phi^{-1}: \varphi(U\cap V)\times\mathbb{R}^n \longrightarrow \psi(U\cap V)\times\mathbb{R}^n$$

$$(p,u)\longmapsto \left((\psi\circ\varphi^{-1})(p),\ D(\psi\circ\varphi^{-1})(p)\,u\right)$$

is smooth. Thus we obtain a topology and a smooth structure on TM by defining

$W \subset TM$ to be open when $\pi(W \cap \pi^{-1}(U))$ is open for every U in an atlas for M. ∎

Remark: Another (equivalent) approach: let $\{(U_i, \varphi_i)\}$ be a countable locally finite atlas of M. Glue together $U_i \times \mathbb{R}^n$ and $U_j \times \mathbb{R}^n$ by an equivalence

$$(x,u) \sim (y,v) \iff y = \varphi_j \circ \varphi_i^{-1}(x) \text{ and}$$
$$v = D(\varphi_j \circ \varphi_i^{-1})(\varphi(x))u$$

2.2 The Derivative

Let $f: M \to N$ be a smooth map of manifolds. Locally in charts (U_i, φ_i) on M and (V_α, ψ_α) on N f can be viewed as a collection of vector-valued functions

$$f_{i\alpha} = \psi_\alpha \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_\alpha(V_\alpha)$$

where

$$(\psi_\beta \circ \psi_\alpha^{-1}) \circ f_{i\alpha} = f_{j\beta} \circ (\varphi_j \circ \varphi_i^{-1})$$

Differentiating we obtain

Df locally:
$\psi_\alpha^{-1} \circ f_{i\alpha} \circ \varphi_i = f$
$\Downarrow$
$Df = D\psi_\alpha^{-1} \circ Df_{i\alpha} \circ D\varphi_i$

$$D(\psi_\beta \circ \psi_\alpha^{-1}) \circ Df_{i\alpha} = Df_{j\beta} \circ D(\varphi_j \circ \varphi_i^{-1})$$

This shows that $Df_{i\alpha}$ and $Df_{j\beta}$ glue together to define a map $Df: TM \to TN$. This map is called the <u>derivative of f</u> or the <u>push-forward of f</u>

and is denoted

$$Df = f_*$$

We have

$$\begin{array}{ccc} TM & \xrightarrow{Df} & TN \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & N \end{array}$$

On each fiber $\pi^{-1}(x) = T_xM \subset TM$ the map $Df : T_xM \to T_{f(x)}N$ is a linear map of vector spaces. (In fact, (f, Df) is a homomorphism of the fiber bundle TM to TN.)

Observe that if $f \circ g = h$, then $Df \circ Dg = Dh$ by the usual chain rule.

2.3 Vector Fields

Let $U \subset V$ open, V a vector space. A vector field is given by a vector valued function $X : U \to V$ ($V \cong T_pU \ \forall p$).

If $(x_1, \dots, x_n)$ is a basis for V^* (recall the coordinates), then the constant vector fields dual to this basis are usually denoted by

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$
$$\quad X_1 \qquad\quad X_n$$

The reason for this notation is the following: given a vector $v \in V$ we can define the directional derivative in v direction. We will thus be able to let a vector field act on a function and it will do so by derivation (product rule).

global vector field:

$$(U_i, \varphi_i) : X_i : \varphi_i(U_i) \to \mathbb{R}^n$$

$$(U_j, \varphi_j) : X_j : \varphi_j(U_j) \to \mathbb{R}^n$$

on overlap

$U_i \cap U_j$ they agree if

$$D(\varphi_j \circ \varphi_i^{-1}) : X_i \longmapsto X_j$$

$\Rightarrow$ $\{X_i \in C^\infty(U_i, \mathbb{R}^n)\}$ which glue on overlaps we get a global vector field.

$\boxed{\text{Def 2.4:}}$ A <u>smooth vector field on M</u> is a smooth map $X : M \to TM$ such that $\pi \circ X = id_M$. Denote the set of such sections by

$$\Gamma^\infty(M, TM) = \mathfrak{X}(M) \; (= C^\infty(M, TM)).$$

Ex: Let $\{(\hat{U}_i, \varphi_i)\}$ be an atlas for M,

$U_i = \varphi_i(\hat{U}_i) \subseteq \mathbb{R}^n$ and $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. Then a global vector field $X \in \Gamma^\infty(M, TM)$ is specified by a collection

$$X_i : U_i \to \mathbb{R}^n$$

with

$$D\varphi_{ij}(x)\big(X_i(x)\big) = X_j\big(\varphi_{ij}(x)\big)$$

for all $x \in \varphi_i(\hat{U}_i \cap \hat{U}_j)$.

Let $M = S^1 = U_0 \sqcup U_1 / \sim$ with $U_0 = U_1 = \mathbb{R}$ and

$$x \in U_0 \setminus \{0\} \sim y \in U_1 \setminus \{0\} \quad \text{whenever } y = \frac{1}{x}.$$

Then $\varphi_{01}(x) = \frac{1}{x}$ and $\underset{\mathbb{R} \to \mathbb{R}}{D\varphi_{01}(x)}\left(\frac{\partial}{\partial x}\right) = -\frac{1}{x^2}\left(\frac{\partial}{\partial y}\right) = -y^2 \frac{\partial}{\partial y}$

Define

$$X_0 = \frac{\partial}{\partial x}$$

$$X_1 = -y^2 \frac{\partial}{\partial y}$$

Then X_0 does not vanish on U_0, X_1 vanishes to order 2 at $y = 0 \in U_1$.

Remark: Let $f \in C^\infty(M,N)$. Then

$$Df(p) : T_pM \to T_{f(p)}N$$

However, in general Df does not map a vector field $X \in \Gamma^\infty(M,TM)$ to one on N:

$$\begin{tikzcd} TM \arrow[r, "Df"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"] & N \arrow[u, "Y"'] \end{tikzcd}$$

To define Y we need to have f bijective.

$\boxed{\text{Def. 2.5}}$ $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are <u>f-related</u> if the above diagram commutes.

2.4 Local Structure of Smooth Maps.

$\boxed{\text{Def 2.6}}$ $f: M \to N$ is called a <u>local diffeomorphism at $p \in M$</u> if there exists a neighborhood $U \subset M$ of p s.t.

$$f|_U : U \to f(U)$$

is a diffeomorphism.

<u>Ex</u>: $f: \mathbb{R}^1 \to S^1$ given by $t \mapsto (\cos t, \sin t)$ is a local diffeomorphism at every point.

<u>Remark</u>: A necessary condition for f to be a local diffeomorphism at $p \in M$ is for

$$Df(p) : T_pM \to T_{f(p)}N$$

to be an isomorphism. (HW) Remarkably enough, this condition is also sufficient.

Theorem 2.7 (The inverse function Theorem)
Let $f: M \to N$ be a smooth map. Assume at $p \in M$ the map $Df(p)$ is an isomorphism. Then f is a local diffeomorphism at p. Moreover,
$$D(f^{-1})(f(p)) = (Df(p))^{-1}$$

Pf: This is a local statement. So we may reduce the problem to working in one chart around p. Furthermore, we may reduce to considering
$$f: U \to \mathbb{R}^m, \quad U \subset \mathbb{R}^m \text{ open}.$$

For convenience, assume $0 \in U$, $f(0)=0$ and $Df(0) = Id$ (replace f by $(Df(0))^{-1} \circ f$).

Goal: Invert f. $\Leftrightarrow$ solve $y = f(x)$ uniquely for x
$\Rightarrow$ Define $g(x)$ s.t. $f(x) = x + g(x)$ i.e. $g(x)$ is the non-linear part of $f(x)$.

Claim: Let $y \in B_\varepsilon(0)$ for some $\varepsilon > 0$. Then the map $h_y : x \mapsto y - g(x)$ is a contraction on a closed ball. This then has a unique fixed point $\phi(y)$ and so $y - g(\phi(y)) = \phi(y)$.

Then $f(\phi(y)) = \phi(y) + g(\phi(y)) = y$ and

$\|x'-x\| \leq \|f(x')-f(x)\| + \|g(x')-g(x)\| \Rightarrow \frac{1}{2}\|x'-x\| \leq \|f(x')-f(x)\|$

So f injective and thus $f(\phi(f(x))) = f(x)$ implies $\phi(f(x)) = x$

Hence ϕ is an inverse for f.

0) Observe $h_y(x) = x \iff y = g(x) + x \iff f(x) = y$

i) h_y is a contraction: Note that $Dh_y(0) = 0$

and hence there is a ball $B_r(0)$ where

$$\|Dh_y\| \leq 1/2 \qquad \text{(indep of } y!)$$

The MVT then implies for $x, x' \in B_r(0)$:

$$(\|g(x)-g(x')\| =)\ \|h_y(x) - h_y(x')\| \leq \frac{1}{2}\|x-x'\|$$

This looks like a contraction, but we need to make sure h_y maps a complete metric space to itself.

Note that

$$\|h_y(x)\| \leq \|h_y(x) - h_y(0)\| + \|h_y(0)\| \leq$$
$$\leq \frac{1}{2}\|x\| + \|y\|$$

Thus taking $y \in B_{r/2}(0)$ we get

$$h_y : \overline{B_r(0)} \to \overline{B_r(0)}$$

is a contraction mapping. Let $\phi(y)$ be the unique fixed point.

i½) $\|x-x'\| \leq \|f(x)-f(x')\| + \|g(x)-g(x')\| \Rightarrow \frac{1}{2}\|x'-x\| \leq \|f(x')-f(x)\|$

Hence f is injective on $B_r(0)$. Thus $f(\phi(f(x))) = f(x)$ implies $\phi(f(x)) = x$. Hence ϕ is an inverse for f.

ii) ϕ is continuous:

$$\|\phi(y) - \phi(y')\| = \| h_y(\phi(y)) - h_y(\phi(y'))\|$$

$$\leq \| g(\phi(y)) - g(\phi(y'))\| + \|y - y'\|$$

$$\underset{Dh_y = Dg}{\leq} \frac{1}{2} \|\phi(y) - \phi(y')\| + \|y - y'\|$$

$$\Rightarrow \quad \|\phi(y) - \phi(y')\| \leq 2\,\|y - y'\|$$

iii) ϕ is differentiable: Let $x = \phi(y)$ and $x' = \phi(y')$.

Now we choose r small enough s.t. Df is non-singular on $\overline{B(0,r)}$:

$$\|\phi(y) - \phi(y') - \underbrace{(Df(x))^{-1}}_{\text{guess for derivative.}}(y - y')\| = \| x - x' - (Df(x))^{-1}(f(x) - f(x'))\|$$

$$\leq \underbrace{\|Df(x)^{-1}\|}_{\text{bounded}} \cdot \| Df(x) \underbrace{(x - x')}_{\leq 2\|y-y'\|} - (f(x) - f(x'))\|$$

Now divide by $\|y - y'\|$ and take $y' \to y$. Then note that $x' \to x$ by continuity of ϕ and hence the RHS becomes a bounded term times a null-term. Hence ϕ is diff'ble with

$$D\phi = (Df)^{-1}$$

Thus ϕ is smooth and f is a local diffeom. ∎

$\boxed{\text{Theorem 2.8}}$ (Constant Rank Theorem)

Let $f: M^m \to N^n$ be smooth such that Df has constant rank k in a nbhd of $p \in M$. Then there are charts (U, φ) and (V, ψ) around p resp. $f(p)$ such that

$$\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$$

<u>Pf</u>: WLOG: $M \subseteq \mathbb{R}^m$, $N \subseteq \mathbb{R}^n$ open. As $\operatorname{rank} Df = k$ at p there is a $k \times k$ minor of $Df(p)$ with non-zero determinant. Reorder $\mathbb{R}^m, \mathbb{R}^n$ s.t. this minor is top left and translate so that $f(0) = 0$. Label the coordinates

$$(x_1, \dots, x_k, y_1, \dots, y_{m-k}) \text{ and } (u_1, \dots, u_k, v_1, \dots, v_{n-k}).$$

Thus we may write

$$f(x, y) = (Q(x,y), R(x,y))$$

with $\frac{\partial Q}{\partial x}$ non-singular.

Consider the map

$$\phi(x,y) = (Q(x,y), y) \quad \mathbb{R}^m \to \mathbb{R}^m$$

with derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix}$$

This is nonsingular at 0, so by the IFT there is a local inverse

$$\phi^{-1}(x,y) = \left(A(x,y), B(x,y) \right).$$

This means

$$(x,y) = \phi\left(\phi^{-1}(x,y)\right) = \left(Q(A,B), B \right)$$

and so $B(x,y) = y$.

Thus

$$f \circ \phi^{-1} : (x,y) \longmapsto f\left(A(x,y), y \right) =$$

$$= \left(Q(A,y), R(A,y) \right)$$

$$= \left(x, R(A,y) \right)$$

which still has rank k. Observe that

$$D(f \circ \phi^{-1}) = \begin{bmatrix} Id_{k \times k} & 0 \\ \dfrac{\partial R(A,y)}{\partial x} & \dfrac{\partial (R(A,y))}{\partial y} \end{bmatrix}$$

and hence the lower right must vanish. This implies that

$$f \circ \phi^{-1} : (x,y) \longmapsto (x, S(x))$$

Note that $S(x)$ is smooth. Define the diffeomorphism $\sigma : (u,v) \longmapsto (u, v - S(u))$. Then

$$\sigma \circ f \circ \phi^{-1} : (x,y) \longmapsto (x, 0).$$

$\blacksquare$

$\boxed{\text{Lemma 2.9}}$ Let $f: M \to N$ be smooth with constant rank on M. For any $q \in f(M)$ the inverse image $f^{-1}(q) \subseteq M$ is a regular submanifold.

Pf: Let $x \in f^{-1}(q)$. Then there exist charts s.t.

$$\psi \circ f \circ \varphi^{-1}: (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$$

Hence $f^{-1}(q) \cap U = \{x_1 = \dots = x_k = 0\}$. ∎

Ex: 1) Define

$$f: \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, \dots, x_n) \mapsto \sum x_i^2$$

Then $Df(x) \cdot v = 2 \sum x_i v_i = 2\langle x, v\rangle$ which

has rank 1 at all pts of $\mathbb{R}^n \setminus \{0\}$.

Note that $f^{-1}(q) \ni 0 \Leftrightarrow q = 0$.

Thus for $q \neq 0$, $f^{-1}(q)$ is a regular submfd:

$q > 0 \Rightarrow$ spheres

$q = 0 \Rightarrow$ pt.

$q < 0 \Rightarrow$ empty.

In fact, this smooth structure is compatible to the one given for spheres earlier.

2) <u>Orthogonal group $O(n)$</u>

$$O(n) = \{ A \in GL(n,\mathbb{R}) \mid A^T \cdot A = Id \}$$

<u>Claim</u> $O(n)$ is a regular submfd of $GL(n,\mathbb{R})$.

<u>Proof</u>: Define

$$f : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$$
$$A \mapsto A^T A.$$

Then $O(n) = f^{-1}(Id)$. The result follows if we show that f has constant rank on $GL(n,\mathbb{R})$.

Given $A, B \in GL(n,\mathbb{R})$ there is a unique $C \in GL(n,\mathbb{R})$ s.t.

$$B = AC.$$

Note that

$$f(AC) = (AC)^T AC = C^T A^T A C = C^T f(A) C$$

so we have

$$(f \circ R_C)(A) = (L_{C^T} \circ R_C \circ f)(A) \quad \forall A \in GL(n,\mathbb{R})$$

By the chain rule

$$Df(AC) \circ (R_C)_*(A) = (L_{C^T})_*(A^T A C) \circ (R_C)_*(A^T A) \circ Df(A)$$

Now R_C, L_{C^T} are diffeos, so $(R_C)_*, (L_{C^T})_*$ are isomorphisms and hence do not change the rank.

Thus

$\operatorname{rank} Df(B) = \operatorname{rank} Df(AC) = \operatorname{rank} Df(A)$.

As A, B were arbitrary, f has constant rank. ☒

$\boxed{\text{Lemma 2.10}}$ If $f: M \to N$ is a smooth map of mfds and $Df(p)$ has rank equal to $\dim N$ along $f^{-1}(q)$, then $f^{-1}(q)$ is an embedded submfd. of M.

Pf: Since f has maximal rank along $f^{-1}(q)$ there must be an open nhd $U \subset M$ containing $f^{-1}(q)$ s.t. $f: U \to N$ has maximal rank. ☒

$\boxed{\text{Def. 2.11}}$ Let $f: M \to N$ be a smooth map.

1) If $Df(p)$ is surjective, then p is called a regular point. Otherwise p is called a critical point.

2) If all points in $f^{-1}(q)$ are regular points, then q is called a regular value. Otherwise q is a critical value.

3) If $f^{-1}(q) = \emptyset$, then q is regular.

Remark: Lemma 2.10 says $f^{-1}(\text{regular value}) = \text{regular submfd}$.

<u>claim:</u>

$$D_{max}(f) = \{ p \in M \mid Df(p) \text{ has maximal rank} \}$$

is open.

Let k = max. rank of f.. Then

$$\operatorname{rank} Df(p) = k \iff \operatorname{rank} \left[\frac{\partial f_i}{\partial x_j} \right] \Big|_p = k$$

$$\iff \operatorname{rank} \left[\frac{\partial f_i}{\partial x_j} \right] \Big|_p \geq k$$

Therefore $M \setminus D_{max}(f)$ is defined by

$$\operatorname{rank} \left[\frac{\partial f_i}{\partial x_j} \right] \Big|_p < k.$$

which means all $k \times k$ minors vanish at p.
(det of $k \times k$ matrix)
The zero set of finitely many continuous functions is closed and hence $D_{max}(f)$ is open.

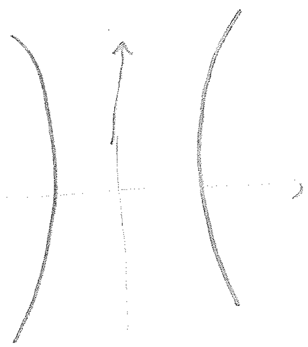
Ex: 1) $f: \mathbb{R}^2 \to \mathbb{R}$. given by $(x,y) \mapsto x^2 - y^2$

Then $Df(x,y) = \nabla f(x,y) = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$

This has rank 1 (and hence is surjective) when $(x,y) \neq (0,0)$. So $p \in \mathbb{R}^2$ is regular $\Leftrightarrow p \neq 0$.

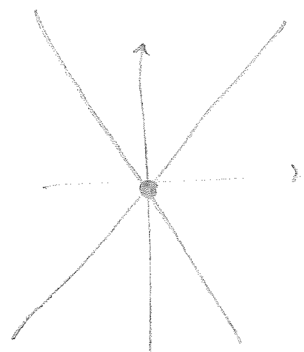
However, $q=0$ is not a regular value:

$q=1$



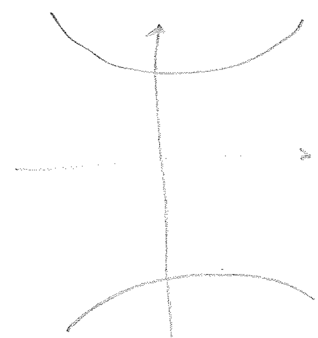
all pts are regular, $q=1$ is a regular value, $f^{-1}(q)$ is an embedded submfd.

$q=0$



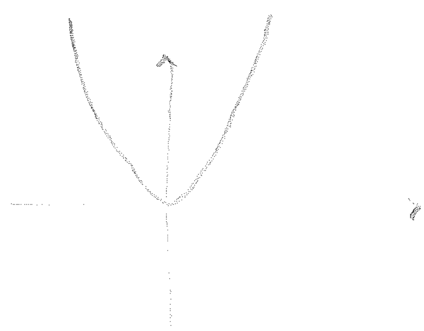
all pts $p \neq 0$ regular, $q=0$ is a critical value, $f^{-1}(0)$ is not a (sub)manifold

$q=-1$



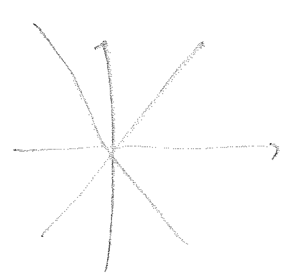
all pts regular, $q=-1$ is a regular value, $f^{-1}(q)$ is an embedded submfd.

2) $x^2 - y = 0$



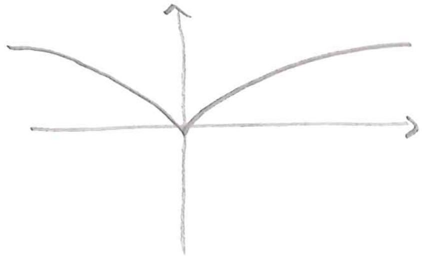
embedded submfd.

$x^2 - y^2 = 0$



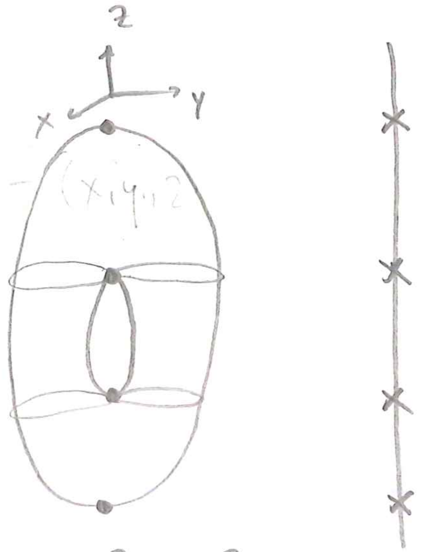
not a manifold.

$x^2 - y^3 = 0$



not an embedded submfd, surely a topol. mfd, not smooth. ($T_0 M$ undef.)

2)



x : critical value

• : critical point

$$f : \mathbb{T}^2 \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longmapsto z$$

→ Morse Theory

diff. functions on M

⇓

topology of M

(→ Grothendieck)

Maximal rank maps have special behavior. There are two ways to have maximal rank.

$\boxed{\text{Def 2.12}}$ Let $f : M^m \to N^n$ be a smooth map.

1) f is called a <u>submersion</u> if $Df(p)$ is surjective at all points p of M ($\Leftrightarrow \operatorname{rank} Df(p) = n$)

2) f is called an <u>immersion</u> if $Df(p)$ is injective at all points p of M ($\Leftrightarrow \operatorname{rank} Df(p) = m$)

3) If f is an injective immersion which is a homeomorphism onto its image (in the

subspace topology) then we call f an <u>embedding</u>.

<u>Remark</u>:

i) f submersion $\Rightarrow$ $n \leq m$ (in both cases, f has maximal rank!)

f immersion $\Rightarrow$ $m \leq n$

ii) immersion, submersion $\rightarrow$ only behavior of Df matters.

embedding $\rightarrow$ behavior of f <u>and</u> Df.

Lemma 2.13 If $f: M \to N$ is an embedding, then $f(M)$ is an embedded submanifold.

<u>Pf</u>: Let $f: M \to N$ be an embedding. Then f has maximal rank on M, hence by Thm. 2.8 there exist charts (U, φ), (V, ψ) s.t.

$$\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, 0, \dots, 0)$$

If $f(U) = f(M) \cap V$ then we've done. To make sure no other part of M gets mapped into V, note that $f(U)$ is open in the subspace topology. This means there is a smaller open subset $V' \subset V$ s.t. $V' \cap f(M) = f(U)$. Restricting to V' shows that $f(M)$ is cut out by $(x_{m+1}, \dots, x_n)$ ∎

Ex: 1) If $\iota: M \to N$ is an embedding, then $D\iota: TM \to TN$ is also an embedding. Hence TM can be viewed as a submfd of TN.

2) $M_1 \times \cdots \times M_k \xrightarrow{\pi_i} M_i$ is a smooth submersion

$TM \xrightarrow{\pi} M$ is a smooth submersion.

3) $g: \mathbb{R}^2 \to \mathbb{R}^3$

$$(u,v) \mapsto \big((2+2\cos 2\pi u)\cos 2\pi v,\ (2+2\cos 2\pi u)\sin 2\pi v,\ \sin 2\pi u \big)$$

is a smooth immersion. Its image is a torus ($(y-2)^2 + z^2 = 1$ rotated around z-axis)
Restricting $\mathbb{R}^2$ makes this into an embedding.

4) $M_i \hookrightarrow M_1 \times \cdots \times M_k$ is a smooth embedding. for each fixed choice of $p_j \in M_j$ $(i \neq j)$ via

$$q \mapsto (p_1, \ldots, p_{i-1}, q, p_{i+1}, \ldots, p_k)$$

5) If $\gamma: I \subset \mathbb{R} \to M$ is a smooth curve, then γ is a smooth immersion iff $\gamma'(t) \neq 0\ \forall t \in I$.

Thus $\gamma(t) = (t^3, 0)$ is smooth and a topol. embedding but not an immersion.

An advanced example:

Define $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ by
$$\mathbb{C}P^1 \cong S^2$$

$$[x:y] \longmapsto [x^2 : y^2 : \varepsilon xy] \quad \text{where} \quad \begin{matrix} \varepsilon \in (0,\infty) \\ x,y \in \mathbb{C} \end{matrix}$$

This defines a family of embedded spheres. To show this, we prove f is an embedding.

1) f injective: The only ambiguity is in the signs of x and y due to the x^2, y^2. If x, y have the same sign, then note $[x:y] = [-x:-y]$. If not, then note $[x:-y] = [-x:y]$, these two points are distinguished by the sign of $x \cdot y$.

2) f immersion $\Leftrightarrow$ Df has rank 2 on $\mathbb{C}P^1$.

Recall, to define f globally we patch Df on coordinate charts. Recall also, that for $\mathbb{C}P^1$ we need U_1, U_2 and on $\mathbb{C}P^2$ we have V_1, V_2, V_3 where in each U_i resp V_i the i-th coord is non-zero. The map

$$\varphi : (x,y) \longmapsto (x^2, y^2, \varepsilon xy)$$
$$\mathbb{C}^2 \longrightarrow \mathbb{C}^3$$

has

$$D\varphi = \begin{pmatrix} 2x & 0 \\ 0 & 2y \\ \varepsilon y & \varepsilon x \end{pmatrix} \quad \text{which has rank 2 whenever } x \neq 0 \text{ or } y \neq 0.$$

Note, the map $(x:y) \mapsto (x^n : y^n : \varepsilon x^k y^j)$ $k+j=n \geq 3$ does not have this property.

Thus the image of f is, for fixed ε, an embedded sphere.

We consider the limiting behavior as $\varepsilon \to 0$ and $\varepsilon \to \infty$:

$\underline{\varepsilon \to 0}$:

$$f_\varepsilon \longrightarrow f_0 : [x:y] \mapsto [x^2 : y^2 : 0]$$

We lose injectivity and in fact this is a double cover of a sphere. It is no longer an immersion

as $\quad Df_0 = \begin{pmatrix} 2x & 0 \\ 0 & 2y \\ 0 & 0 \end{pmatrix} \quad$ does not have

full rank when $x=0$ or $y=0$. I.e. it has critical points $[0:1]$ and $[1:0]$. (The so-called branch pts.)

$\underline{\varepsilon \to +\infty}$: This exhibits some wonderful behavior first described by Uhlenbeck-Sacks. Consider the two coordinate changes:

i) $\tilde{x} = \varepsilon x \Rightarrow [\tilde{x} : y] \mapsto [\tilde{x}^2/\varepsilon^2 : y^2 : \tilde{x} y]$

$$\Big\downarrow \, \varepsilon \to +\infty$$

$$[0 : y^2 : \tilde{x} y]$$

$$\|$$

$$[0 : y : \tilde{x}]$$

This is just a copy of $\mathbb{CP}^1$. $([1:0] \mapsto [0:0:1])$

ii) $\tilde{y} = \varepsilon x \; : \; [x:\tilde{y}] \longmapsto [x^2 : \tilde{y}^2/\varepsilon^2 : x\tilde{y}]$

$$\begin{aligned} &\big\downarrow \varepsilon \to +\infty \\ &[x^2 : 0 : x\tilde{y}] \\ &\quad = \\ &[x : 0 : \tilde{y}] \end{aligned}$$

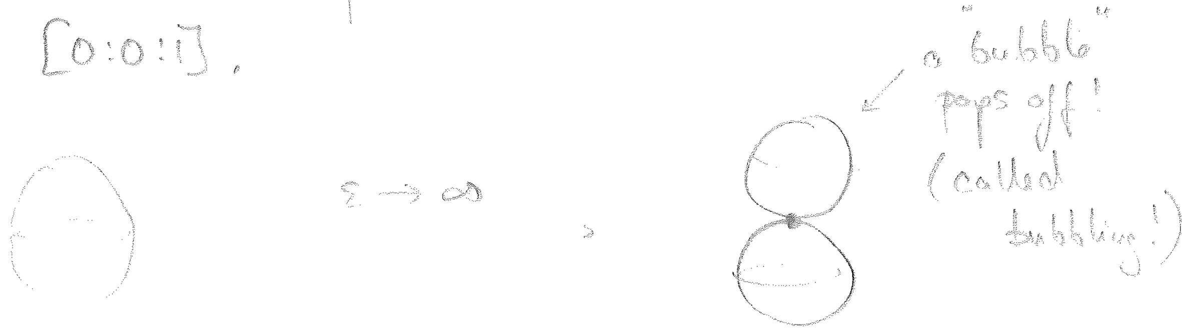
This is another copy of $\mathbb{C}P^1$ ($[0:1] \mapsto [0:0:1]$)

These two embedded spheres intersect at

$$[0:a:b] \cap [c:0:d] \neq \phi \iff a = c = 0,\ b = \lambda d$$

on the point $[0:0:1]$

Hence in the limit $\varepsilon \to +\infty$ a sphere breaks into two spheres connected at the point $[0:0:1]$.



<u>homologically:</u> $H_2(\mathbb{C}P^2) \simeq \mathbb{Z} \simeq \langle H \rangle$

Then

$$2H \xrightarrow{\varepsilon \to +\infty} H + H$$