Seppi's Topology Notes

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Chapter 1

Manifolds

1.1 Topological Manifolds

1.2 Smooth Manifolds

1.3 Cobordism

1.4 Smooth Maps

If M and N are topological manifolds, the natural notion of a morphism is $f : M \to N$ continuous. Then two topological manifolds M, N, are equivalent if there exists a homeomorphism $f : M \to N$. Topological manifolds together with continuous maps form a category.

Definition 1.4.1. A **category** consists of objects C and arrows A. Each arrow goes from some object C (the source) to another object (the target). These arrows are often called **morphisms**. Furthermore, for each object $c \in C$ there exists a morphism $1_c \in A$ such that the source and target of 1_c are both c. In addition, morphisms can be composed and composition satisfies additivity. For $x, y \in C$ we write $\operatorname{Hom}(x, y) \subset A$ as the set of morphisms from x to y. Stated this way, composition becomes

$$\circ : \operatorname{Hom}(x, y) \times \operatorname{Hom}(y, z) \to \operatorname{Hom}(x, z)$$

$$\circ : (f, g) \mapsto g \circ f$$

Example 1.4.2. The category of topological manifolds is defined by C being the class of all topological manifolds with arrows A being continuous maps between them.

For smooth manifolds, we need the correct notion of a morphism.

Definition 1.4.3. A map $f : M \to N$, M, N smooth manifolds, is called **smooth** when for each chart (U, φ) for M and each chart (V, ψ) for N, the composition

$$\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n).$$

The set of smooth maps from M to N is denoted $C^\infty(M, N)$. A smooth map with smooth inverse is called a **diffeomorphism**.

Lemma 1.4.4. *If $g : L \to M$ and $f : M \to N$ are smooth maps, then so is $f \circ g : L \to N$.*

Chapter 2

Tangent Bundle

The motivating example for tangent bundles is the case where $U \subset V$ is open, with V a finite dimensional vector space. A tangent vector to $p \in U$ is a vector in V; we write $T_pU \simeq V$. On all of U the space $U \times V$ is called the **tangent bundle**; this is the collection of all tangent vectors on U. $TU = \coprod_{p \in U} T_pU$. $TU = U \times V$ comes with two projections:

B: *Why write it like this? What does this mean?*

$$\pi : TU \to U$$

$$S : U \to TU$$

with $\pi \circ S = \mathrm{id}$. The map S is called a **vector field** on U.

Idea 2.0.5. A tangent bundle on a manifold will locally look like the above and globally describe all tangent vectors.

B: *What above?*

Definition 2.0.6. A subspace $L \subseteq M$ of an m-manifold is called a **regular** or **embedded submanifold** of codimension k when each point $x \in L$ is contained in a chart (U, φ) of M such that

$$L \cap U = f^{-1}(0)$$

where f is the composition of φ with the projection $\mathbb{R}^m \to \mathbb{R}^k$ to the last k coordinates $(x_{m-k+1}, \cdots, x_m)$. A submanifold of codimension 1 is called a **hypersurface**.

Example 2.0.7. $S^n \subseteq \mathbb{R}^{n+1}$ is a hypersurface:

(Unintelligible diagram) $\xrightarrow{\varphi}$ $\left(\begin{array}{l}\textit{A cartesian plot with a blob marked "}\varphi(x)\textit{," "}\imath \cap\\ U\textit{," and "}\int \textit{ project onto } \mathbb{R}\textit{"}\end{array}\right)$

B: *I can't quite tell what this means...*

Now suppose $L \subseteq \mathbb{R}^m$ is a submanifold of codimension k and let φ be a diffeomorphism as in the definition. This basically sets up a "rectilinear" coordinate system on x where the first $m-k$ coordinates are in L and the last k coordinates describe directions "perpendicular" to L.

Then we say $u \in \mathbb{R}^m$ is tangent to L at p where the derivative $D\varphi(p)$ takes u to the linear subspace of $\mathbb{R}^m$ given by $x_{m-k-1} = \cdots = x_m = 0$. Then the tangent bundle TL to L is the set y pairs (p, u) where $p \in L$ and $u \in \mathbb{R}^m$ is tangent to L at p. It is a subset of $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$ and is itself a submanifold of $T\mathbb{R}^m$ of codimension $2k$.

B: *Why the ellipsis?*

2.1 The general Construction

The tangent bundle to an n-manifold M is a $2n$-manifold called TM naturally constructed in terms of M. As a set, TM is the disjoint union of the tangent spaces T_pM. We will now describe the construction in detail.

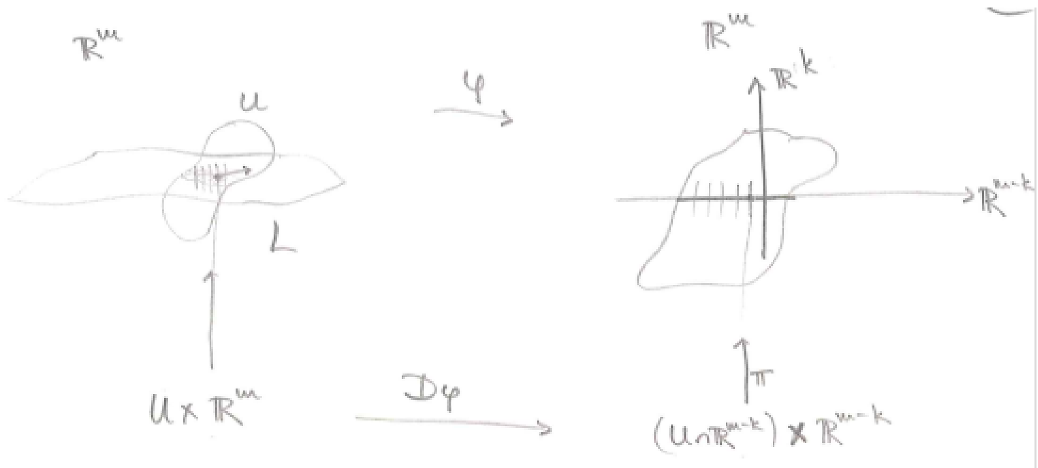

Figure 2.1: A diagram found on page 28 of the handwritten notes.

Definition 2.1.1. Let (U, φ) and (V, ψ) be charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(u)$ and $v \in T_{\psi(p)}\psi(v)$. Then (U, φ, u) and (V, ψ, v) are called equivalent when

$$D(\psi \circ \varphi^{-1})(\varphi(p))(u) = v$$

This is an equivalence relation, utilizing the chain rule.

The set of equivalence classes of such triples is called the **tangent space** to p of M, denoted T_pM.

2.2 The Derivative

Chapter 3

Transversality

Chapter 4

Vector Fields

Chapter 5

Differential Forms