

Transversality & symmetry for pseudoholom covers

Agenda of the talk:

- Relevant background on J -holom curves
 - nonlinear CR eqn.
 - moduli space of simple curves
- The case for multiple covers
 - Equivariant transversality in finite dimensions
- The main results & stratification theory
 - [- outline of splitting of CR operators for double covers]

— x —

Let (M, J) an almost cplx mfld

(Σ, j) a closed Riemann surface

smooth $u: (\Sigma, j) \rightarrow (M, J)$

s.t. $J \circ du = du \circ j$

J-holom map.

global perspective:

$u \in \mathcal{B} = \mathcal{C}^\infty(\Sigma, M)$

$\mathcal{E} \rightsquigarrow$ "infinite dim'l v.s.b."

$$\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^* TM) \} \rightarrow \begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathcal{B} \end{array} \begin{array}{c} \nearrow \\ \boxed{\bar{\partial}_J} \end{array}$$

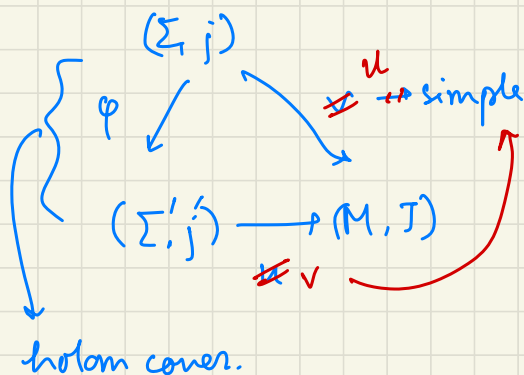
$$\bar{\partial}_J(u) = \frac{1}{2} (du + J \cdot du \circ j)$$

$$\underbrace{\{ \text{JH maps} \}} = \bar{\partial}_J^{-1}(0).$$

$$\underline{\bar{\partial}_J(u) = 0} \rightarrow \underline{\text{nonlinear}} \text{ CR equation.}$$

Simple curves :

$$u: (\Sigma, j) \longrightarrow (M, J)$$



simple curves

|||

somewhere injective
curves

$$\downarrow$$

$$\exists z, \quad du(z) \neq 0.$$

$$\& \quad u^{-1}(u(z)) = z.$$

moduli space of simple curves:

(M, ω, J) ^{compact} symplectic manifold, J ω -compatible

(Σ, j) closed R.S.

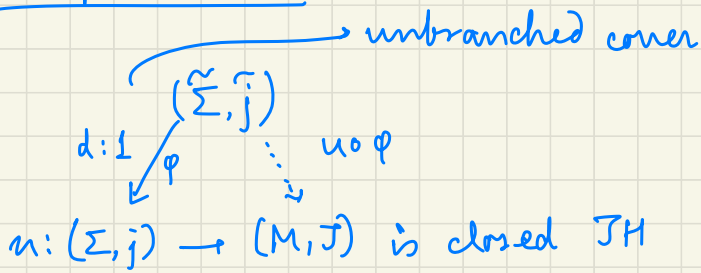
$$A \in H_2(M), \quad \underline{[u]} = A.$$

For generic J,

$$\mathcal{M}^*(A) = \{ \text{simple JH curves} \}$$

is a smooth manifold, of dim given by an
index formula.

What about
multiple covers?



virtual dimensions of moduli spaces containing u
"expected" $\& u \circ \phi$, also known as indices of these
2 curves.

$$\underbrace{\text{ind}(u \circ \phi)} = d \cdot \underbrace{\text{ind}(u)} \geq 0.$$

There is no reason why $u \circ \phi$ could not achieve transversality.

For branched covers : things are more subtle.

multiple covers

symmetry

Example (finite dimensions)

Say $f: \mathbb{R}^2 \xrightarrow{C^\infty} \mathbb{R}^2$ \mathbb{Z}_2 equivariant
if $f(x, -y) = -f(x, y)$.

Can show: every such map admits ^{no!} C^∞ -close
 \mathbb{Z}_2 -equivariant perturbations for which $\vec{0}$ is a regular
value.

Note that: $\mathbb{R} \times \{0\} \subset f^{-1}(\vec{0}) \forall$ such f
So, $\partial_x f = \vec{0}$, thus transversality fails.
along x -axis

Best case scenario: generic f intersect 0 cleanly,
i.e., all components of $f^{-1}(0)$ are submfd
with $T_x(f^{-1}(0)) = \ker df(x)$

clean intersection:

clean intersection

$$T_x(M \cap N) = T_x M \cap T_x N$$

MAN is a smooth submfd.

$$M, N \subseteq X$$

$$M \not\supset N,$$

$$x \in M \cap N,$$

$$\underline{T_x M} \oplus \underline{T_x N} = T_x X.$$

Equivariant transversality in finite dimensions

Fix n -dim'l orbifold M ,
orbibundle $\underset{M}{E} \downarrow$ of rk m .

Every $x \in M$ has a finite grp G_x , and a nbhd
 $U_x \subset M$ st $E|_{U_x} \simeq (\mathcal{O} \times \underline{\mathbb{R}^m}) / G_x$

for some linear action of G_x on \mathbb{R}^m and a
nbhd $\mathcal{O} \subset \mathbb{R}^n$ of $\vec{0}$.

Question: Do generic $\sigma \in \Gamma(E)$ intersect the zero
section transversely (or at least cleanly)?

Sample thm 1: If $\dim M = \text{rk } E$ and $|G_x| \leq 3 \ \forall x$,
then generic sections of E intersect 0 cleanly.

Sample thm 2: Generic smooth fns on an orbifold are
Morse.
(cf. Wasserman '69, Hepworth '09).

Ingredient A : Stratification by symmetry.

For finite grp G , representations $\rho: G \rightarrow GL(n, \mathbb{R})$,
 $\tau: G \rightarrow GL(m, \mathbb{R})$.

define submfld

$$M_{\rho, \tau} = \left\{ x \in M \mid \begin{array}{l} G_x \cong \underline{G}, \text{ acts on } T_x M \text{ as } \underline{\rho}, \\ \text{on } E_x \text{ as } \underline{\tau} \end{array} \right\}$$

and subbundle

$$E_{\rho, \tau} = \left\{ v \in E|_{M_{\rho, \tau}} \mid \begin{array}{l} G \text{ acts trivially} \\ \text{on } v \end{array} \right\}$$

Let $\{\theta_i: G \rightarrow \text{Aut}_{\mathbb{R}}(W_i)\}_{i=1}^N$ denote the real irreps of G , $\theta_1 = \text{trivial repr.}$,

$m_i(p) = \text{mult. of } \theta_i \text{ in } p$.

Then, $\dim M_{\rho, \tau} = m_1(p)$, $\text{rk } E_{\rho, \tau} = m_1(\tau)$.

The orbifold M is thus a countable union of disjoint smooth submflds $M_{\rho, \tau}$ with distinguished subbundles

$$E_{\rho, \tau} \subset E|_{M_{\rho, \tau}}.$$

Notice: $\forall \sigma \in \Gamma(E)$, $\sigma|_{M_{p,\tau}} \subset E_{p,\tau}$.

$\Rightarrow \sigma \not\equiv 0$ at $x \in M_{p,\tau}$ unless τ is trivial.

Lemma (standard transversality).

For generic $\sigma \in \Gamma(E)$, $\sigma|_{M_{p,\tau}}$ is transverse to
the zero section of $E_{p,\tau} \rightarrow M_{p,\tau}$ $\forall G, p, \tau$.

\Rightarrow Cannot conclude that $M(\sigma)$ is a smooth orbifold.

Ingredient B: splitting the linearization.

At $x \in M_{p,\tau}(\sigma)$,

$$D_x := D\sigma(x) : T_x M \rightarrow E_x$$

linearized operator.

$$\left(\sigma \neq \bar{\sigma}_j \quad \begin{array}{c} \mathcal{E} \\ \downarrow \\ \underline{\underline{\mathcal{B}}} \end{array} \right).$$

Recall θ_i & $d_i = \dim W_i$.

Since D_x is G_x -equivariant,
by Schur's lemma,

D_x splits into isotypic decompositions

$$T_x M = \bigoplus T_x H^i \text{ of } \rho \text{ \& } E_x = \bigoplus E_x^i \text{ of } \tau,$$

$$\underline{\underline{D_x}} = \underline{\underline{D_x^1}} \oplus \dots \oplus \underline{\underline{D_x^N}}$$

These operators have Fredholm indices

$$\text{ind } D_x^i = \underline{d_i} [\underline{m_i(\rho)} - \underline{m_i(\tau)}]$$

and D_x^i is surj. if σ is generic.

↑ finite dim'l setup for $\overline{\mathcal{D}_J}$

main results of the paper:

Thm B (transversality, unbranched)

"For generic J , \forall simple JH curves $\underline{u}: (\Sigma, j) \rightarrow (M, J)$
and every unbranched cover $\underline{\varphi}: (\tilde{\Sigma}, \tilde{j}) \rightarrow (\Sigma, j)$, of
closed R.S.,

the curve $\underline{u \circ \varphi}$ is Fredholm regular."

Generally, harder to achieve transversality

$\begin{array}{l} u \circ \varphi \\ \downarrow \quad \searrow \\ \text{closed,} \quad \text{deg}(d) \text{ branched cover} \\ \text{simple} \end{array}$

$Z(d\varphi) = \text{alg. count of branch pts.}$

$\left\{ \begin{array}{l} \text{Riemann Hurwitz: } -\chi(\underline{\tilde{\Sigma}}) + d\chi(\Sigma) = Z(d\varphi). \\ \text{standard index formula for closed curves.} \end{array} \right.$

$$\underline{\text{ind}(u \circ \varphi) = d \cdot \text{ind}(u) - (n-3) Z(d\varphi).}$$

∇

where $\dim M = 2n$
 \mathbb{R}

if $\text{ind}(u) = 0, n \geq 3, \text{ind}(u \circ \varphi) < 0$.

$$\text{ind}(u \circ \varphi) \geq \text{ind}(u) + 2Z(d\varphi).$$

$2Z(d\varphi)$.

necessary to have

if $u \circ \varphi$ were Fredholm regular.

$\varphi \in \mathcal{M}$
 $u \circ \varphi \leftarrow \begin{matrix} \varphi \in \mathcal{M} \\ u \in \mathcal{M} \end{matrix}$



if φ has $r \geq 0$ critical values,

$$\text{ind}(u) \geq (n-1)r. \quad \checkmark$$

Thm C (transversality, branched)

" For generic J ,

For branched covers, can find C^α -close regular covers".



$$\text{ind}(u) = \text{ind}(D_u).$$

if u is immersed, can consider $\left[\underline{D_u^N} \right] = \underline{D_u} \mid_{N_u}$.

$$\rightarrow \text{ind}(D_{u \circ \varphi}^N) = \underbrace{d \cdot \text{ind}(D_u^N)} - (n-1)Z(d\varphi) \leq 0$$

so, $D_{u \circ \varphi}^N$ can be injective

$D_{u \circ \varphi}^N$ injective : $u \circ \varphi$ can never be the limit of a seq. of s.i. curves.

i.e. only other curves near $u \circ \varphi$ are other branched covers $u \circ \varphi'$ for φ' near φ .

Thm A (super-rigidity)

If $\dim M \geq 6$, for generic J ,

every simple ind(0) JH curve is super-rigid.

conj. (Bryan-Pandharipande 2001)

$\overline{\mathcal{I}}_J \cap 0$

cleanly.

defn: closed, connected, simple

• $\text{ind}(u) \geq 0$

• $u: \Sigma \rightarrow M$ immersion

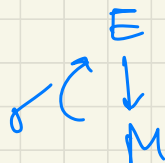
• \forall covers $\tilde{u} = u \circ \varphi$,

$D_{\tilde{u}}^N \eta$ is injective

Thm D implies all of the above

Building walls (in the sense of "crossing").

$$D_\alpha = D_\alpha^1 \oplus \dots \oplus D_\alpha^N.$$



G_α acts $\frac{\ker D_\alpha^i}{\text{coker } D_\alpha^i}$ as irrep Θ_i with multiplicities.

$$\vec{k} = (k_1, \dots, k_N), \quad \vec{c} = (c_1, \dots, c_N).$$

$$M_{p,\tau}(\sigma; \vec{k}, \vec{c}) = \left\{ x \in M_{p,\tau}(\sigma) \mid \begin{array}{l} \dim \ker D_\alpha^i = d_i k_i, \\ \sim \text{coker } = d_i c_i \end{array} \right\}.$$

Workhorse thm:

\forall generic σ , \forall choices $G, p, \tau, \vec{k}, \vec{c}$,

$$M_{p,\tau}(\sigma; \vec{k}, \vec{c}) \subset M_{p,\tau}(\sigma).$$

is a smooth submf, $\text{codim} = \sum_i t_i k_i c_i$
 \leftarrow type of Θ_i