What is Floer homology?

Paramjit Singh
(HU Berlin)

Plan of talk:

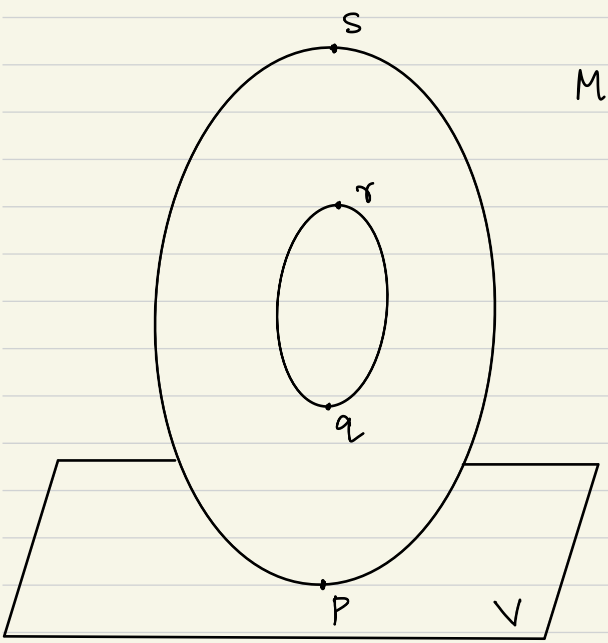
- Morse fns on a smooth manifold
- Morse homology
  - Moduli spaces of gradient flow lines
  - Morse-Smale-Witten complex
- (Hamiltonian) Floer homology
  - Hamilton's equations & Arnold conjecture
  - action functional on loop spaces
  - Floer complexes

Morse functions

Motivation: Information about a space (manifold) can be obtained by studying the structure of fns it supports.

Here, we try to obtain topological info about a mfld from the nature of critical pts of a function on it.

Example: torus

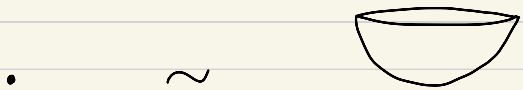


$$M^a := f^{-1}((-\infty, a])$$

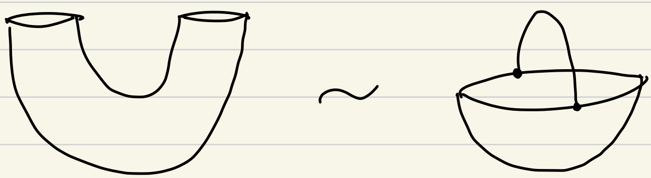
f = height above V plane

(1) $a < 0 < f(p)$ $\qquad M^a = \emptyset$

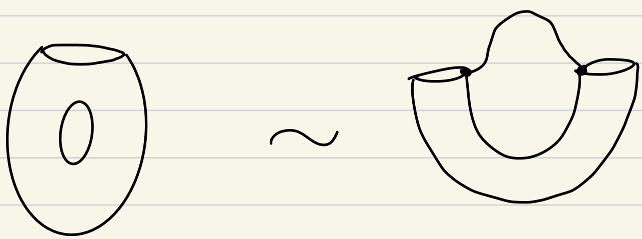
(2) $f(p) < a < f(q)$



(3) $f(q) < a < f(r)$

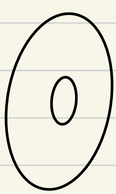


(4) $f(r) < a < f(s)$



Topology keeps changing as we go through critical pts (in fact, cells are being attached of appro. dimension)

(5) $f(s) < a$



Homotopy type of M^a changes, as a crosses p, q, r, s

Defn. $f: M \to \mathbb{R}$ is Morse if all its critical pts are nondegenerate.

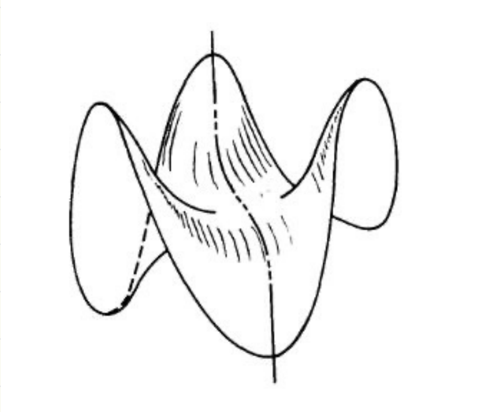
$$\mathrm{Crit}(f) = \{ x \in M : df(x) = 0 \} \to \text{all non-deg}$$

Hessian $d^2 f : T_x M \times T_x M \longrightarrow \mathbb{R}$ is non-deg $\forall \; x \in \mathrm{Crit}(f)$

Morse lemma: Nbhds of non-deg critical pts look like saddles

Let $p \in \mathrm{Crit}(f)$, f Morse

Then can find local coords $(y^1, \ldots, y^n)$ in a nbhd V of p, s.t. $y^i(p) = 0 \;\forall\; i$



$$f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$$

where λ is the index of f at p.

maximal dimension of a subspace of T_pM on which $d^2 f$ is negative definite. $\equiv$ no. of negative eigenvalues of Hessian.

Homotopy type in terms of critical values

Thm: Let $f : M \to \mathbb{R}$ Morse, $M^a = f^{-1}((-\infty, a])$

Supp. $f^{-1}[a, b]$ is compact & has no critical pts.

Then, $M^a \underset{\text{diffeo.}}{\cong} M^b$

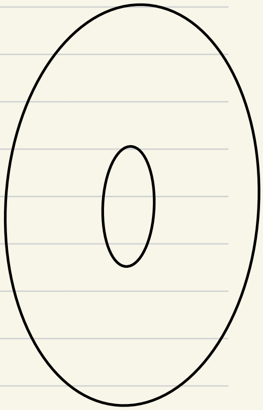
(in fact, M^a is a deformation retract of M^b)

Thm: Let $p \in \text{Crit}(f)$ with index λ. Say $f(p) = c$.

Supp $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact
& has no other critical pts,

then $M^{c+\varepsilon}$ has the homotopy type
of $M^{c-\varepsilon}$ with a k-cell attached.

(given by difference of indices)



Morse homology

Let M cpt smooth Riemannian mfld

has a notion of distance/metric, given by a symmetric 2-tensor $g : T_pM \times T_pM \to \mathbb{R}$.

$f : M \to \mathbb{R}$ Morse fn.

Consider (negative) gradient flow

$$\boxed{\dot{u} = -\nabla f(u)} \quad \text{———} \quad (*)$$

Denote by $\varphi^s : M \to M$ the flow of the above.

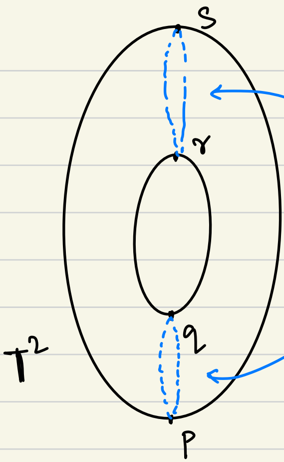
Morse condn. $\Rightarrow$ critical pts of f are (hyperbolic) fixed pts of φ^s.

Stable & unstable manifolds

$$W^s(x; f) = \left\{ z \in M \;:\; \lim_{s\to\infty} \varphi^s(z) = x \right\}$$

$$W^u(x; f) = \left\{ z \in M \;:\; \lim_{s\to -\infty} \varphi^s(z) = x \right\}$$

are smooth submflds.



$W^s(p;f) = W^u(s;f) \cong T^2 \setminus p,q,r,s$

$W^s(r;f)$

etc.

$W^u(q;f)$

Note that Morse index $= \dim W^u(x;f)$.

(*) is called a <u>Morse-Smale system</u> if for any pair of critical pts x, y of f, the stable & unstable mflds intersect transversally.

Set $\mathcal{M}(y,x;f) = W^s(x;f) \cap W^u(y;f)$

space of (connecting orbits) flow lines from y to x.

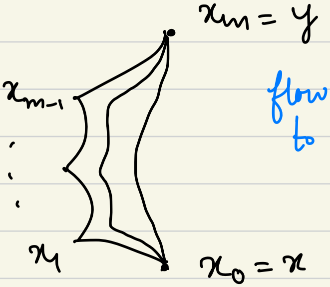
This is a smooth submfld of dim $= \mathrm{ind}_f(y) - \mathrm{ind}_f(x)$

$$\therefore \mathcal{M}(y,x;f) = \left\{ \begin{array}{l} \text{gradient flow lines } u: \mathbb{R} \to M \\ \text{running from } y = \lim_{s\to-\infty} u(s) \\ \quad\quad \text{to } x = \lim_{s\to+\infty} u(s) \end{array} \right\}$$

The group $\mathbb{R}$ acts on $\mathcal{M}(y, x; f)$ & the quotient

$\hat{\mathcal{M}}(y, x; f) = \dfrac{\mathcal{M}(y, x; f)}{\mathbb{R}}$ is a mfld of dim $\mathrm{ind}_f(y) - \mathrm{ind}_f(x) - 1$.

↳ may not be compact



flow lines converge to broken flow lines.

When the Morse indices differ by 1, $\hat{\mathcal{M}}(y, x; f)$ is finite. (so can be counted).

Fix orientations (o_x for each unstable mfld $W^u(x)$ & this gives a natural orientation for each connecting line with index diff 1)

Denote by $CM_*(f) = \bigoplus_{df(x)=0} \mathbb{Z}\langle x \rangle$ free abelian grp generated by critical pts of f.

This complex is graded by Morse index.

The boundary operator $\partial = \partial^M : CM_k(f) \to CM_{k-1}(f)$

is $$\partial^M \langle y \rangle = \sum_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(x) = k-1}} \sum_{[u] \in \hat{\mathcal{M}}(y,x)} \underbrace{\varepsilon(u)}_{\pm 1} \langle x \rangle$$

(signed count)

$(CM_\bullet(f), \partial^M) \rightsquigarrow$ <u>Morse-Smale-Witten complex</u>

<u>Thm</u>: ∂^M is indeed a boundary operator

i.e, $\partial^M \circ \partial^M = 0$

and the homology of this complex

$$HM_k(M, f; \mathbb{Z}) = \frac{\ker \partial^M}{\mathrm{im}\, \partial^M} \leftarrow \text{Morse homology.}$$

agrees with the (singular) homology of M.

Hamilton's eqns & the Arnold conjecture

Let (M,ω) be a cpt symplectic mfld.

ω is a nondeg alternating 2-tensor on TM.

Consider any smooth fn
$H : M \longrightarrow \mathbb{R}$ ("Hamiltonian fn")

Then we get a Hamiltonian v.f. $X_H : M \to TM$
defined by the eqn $\boxed{\iota(X_H)\omega = dH}$

Let $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time-dep 1-periodic family of Hamiltonian fns

& consider the differential eqn $\boxed{\dot{x}(t) = X_t(x(t))}$ $(\ddagger)$

Solns generate a family of symplectomorphisms
$\psi_t : M \to M$ via $\frac{d}{dt}\psi_t = X_t \circ \psi_t$, $\psi_0 = \mathrm{id}$.

Fixed pts of ψ_1 are in 1-1 correspondence with 1-periodic solns of $(\ddagger)$. One such soln x is called non-deg if $\det(\mathbb{1} - d\psi_1(x(0))) \neq 0$

Conjecture (Arnold): In the non-deg case, number of 1-periodic solns should be bounded below by sum of betti numbers of M.

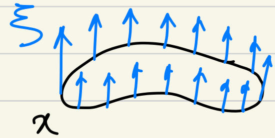
i.e., for $\mathcal{P}(H) = \{ x : \mathbb{R}/\mathbb{Z} \to M : (\ddagger) \}$,

$$\boxed{\# \mathcal{P}(H) \geq \sum_{i=0}^{2n} \dim H_i(M, \mathbb{Q})}$$

Symplectic action functional

Observation: Contractible 1-periodic solns of $(\ddagger)$ can be interpreted as critical pts of the (circle valued) symplectic action functional on the space $\mathcal{L}M$ of contractible loops in M.

Think of a loop in M
as $x:\mathbb{R}\to M$, $x(t+1)=x(t)$.



Tangent vector to $\mathcal{L}(M)$ at x is a
v.f. ξ along x, ie, $\xi:\mathbb{R}\to TM$,

$\xi(t)\in T_{x(t)}M$ & $\xi(t+1)=\xi(t)$ $\forall\, t\in\mathbb{R}$.

Denote space of such vfs by $C^\infty(\mathbb{R}/\mathbb{Z},\, x^*TM)$
$= T_x\mathcal{L}M$

$\mathcal{L}M$ carries a natural 1-form $\Psi_H : T\mathcal{L}M\to\mathbb{R}$

$$\Psi_H(x;\xi) = \int_0^1 \omega\big(\dot x(t) - X_H(x(t)),\, \xi(t)\big)\,dt$$ for $\xi\in T_x\mathcal{L}M$.

The zeroes of this 1-form are precisely the
1-per. orbits of $(\ddagger)$.

closed, but not exact

(Assume $\int_{S^2} v^*\omega\in\mathbb{Z}$ for every $v:S^2\to M$
monotonicity condn.)

However, Ψ_H is the differential of a circle valued fn $a_H : \mathcal{L}M \to \mathbb{R}/\mathbb{Z}$.

$$a_H(x) = -\int_B u^* \omega - \int_0^1 H_t(x(t))\,dt.$$

for $x \in \mathcal{L}M$, where $u : B = \{z \in \mathbb{C} : |z| \leq 1\} \to M$ with $u(e^{2\pi i t}) = x(t)$

(x contractible, so such u exist)

Symplectic action functional of the pair (x, u)

Floer's idea is to carry out Morse theory for the symplectic action functional in analogy to the Morse-Smale-Witten complex in finite dim'l Morse theory.

Consider gradient flow lines of a_H

$$\boxed{\frac{\partial u}{\partial s} + \operatorname{grad} a_H(u(s,\cdot)) = 0} \qquad (\maltese)$$

$$\Updownarrow$$

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0,$$

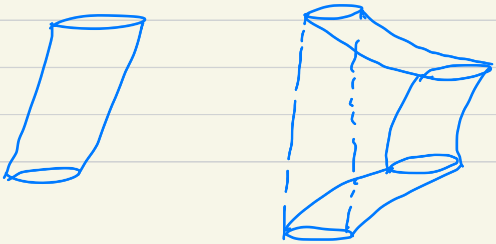
$u : \mathbb{R}^2 \to M$

$u(s,t) = u(s,t+1)$

Can define energy (area-like)

$$E(u) = \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_t(u)\right|^2 \right) ds\,dt$$

Key obs: A soln u of $(\maltese)$ has finite energy iff it converges to periodic orbits of $(\mp)$ provided that all periodic solns are non-deg.



Floer cylinders

Define moduli space $\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+; H, J)$

For generic (H, J)
modulo functional-analytic subtleties & compactness arguments
$\Bigg\}$ $\mathcal{M}(x^-, x^+)$ is a finite dim'l smooth mfld.

Floer complexes

Consider chain complex

$$CF_k(H) = \bigoplus_{\substack{x \in \mathcal{P}(H) \\ \mu_{CZ}(x; H) = k \ (\mathrm{mod}\ 2N)}} \mathbb{F}\langle x \rangle$$

$\mathbb{F} \to \mathbb{Z}_2, \mathbb{Z}$ or $\mathbb{Q}$.

Floer boundary operator

$$\partial^F \cdot \langle y \rangle = \sum_{\substack{x \in \mathcal{P}(H) \\ \mu_{CZ}(x) = k-1 \ (\mathrm{mod}\ 2N)}} \ \sum_{[u] \in \mathcal{M}^1(y, x; H, J)} \varepsilon(u) \langle x \rangle$$

↳ Floer cylinders b/w loops in 1-dim part of $\mathcal{M}(x^-, x^+)$

Fact:

$$\partial^F \circ \partial^F = 0, \quad HF_*(M, \omega, H, J; \mathbb{F}) = \ker \partial^F / \operatorname{im} \partial^F.$$

Thm (Floer): HF_* indep of (H, J) & isomorphic to $H_k(M; \mathbb{Q})$

Morse inequalities with the above prove Arnold's conjecture.