Floer theory : Homework 6

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Problem 1. Let $f : X \to \mathbb{R}$ be a Morse function on a compact manifold X. Let g_ν be a sequence of metrics on X that converge to g_∞ in $\mathscr{C}^\infty$. Suppose a sequence of g_ν-trajectories $\gamma_\nu \in \mathcal{M}(x, y)$ converge to a g_∞-trajectory $\gamma_\infty \in \mathcal{M}(x, y)$. If the operator D_{γ_∞} is surjective, prove that D_{γ_ν} is surjective for large ν.

Solution. We use the argument outlined in class.

Since the convergence is in $\mathscr{C}^\infty(X)$ (global $\mathscr{C}^\infty$), the limit trajectory γ_∞ is betweeen the same pair of critical points as γ_ν. Fix a metric $\bar{g}$ on X which is flat in a neighborhood of those critical points, and let $\nabla^{\bar{g}}$ denote the Levi-Civita connection of this metric.

Recall that D_γ for a trajectory γ was defined as

$$D_\gamma = D\mathscr{F}_\gamma(0) : W^{1,2}(\gamma^* TX) \to L^2(\gamma^* TX), \qquad \xi \mapsto \nabla_{\partial/\partial t}\xi + \nabla_\xi \operatorname{grad} f(\gamma(t))$$

We shall consider the above operator for γ_ν. Trivializing $\gamma_\nu^* TX \simeq \mathbb{R} \times \mathbb{R}^n$ using (parallel transport by) $\nabla^{\bar{g}}$, we obtain parametric formulations of D_{γ_ν}, that is, for $\xi : \mathbb{R} \to \mathbb{R}^n$

$$D_{\gamma_\nu}(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\xi + \nabla_\xi \operatorname{grad}^{g_\nu} f(\gamma_\nu), \quad \text{or more succinctly,} \quad D_{\gamma_\nu} = \frac{\mathrm{d}}{\mathrm{d}t} + \nabla_- \operatorname{grad}^{g_\nu} f(\gamma_\nu)$$

The above expression is of the form $\mathrm{d}/\,\mathrm{d}t + A_\nu$, where $A_\nu : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$, given by $A_\nu(\xi) = \nabla_\xi^{g_\nu} \operatorname{grad}^{g_\nu} f(\gamma_\nu)$. We want to say that since $\gamma_\nu \to \gamma_\infty$ and $g_\nu \to g_\infty$ (both in $\mathscr{C}^\infty(X)$), we have that $A_\nu \to A_\infty$ (with the appropriate A_∞ defined as above). To see this, note that

$$(A_\nu - A_\infty)(\xi) = (\nabla_\xi^{g_\nu} - \nabla_\xi^{g_\infty}) \operatorname{grad}^{g_\nu} f(\gamma_\nu) + \nabla_\xi^{g_\infty}(\operatorname{grad}^{g_\nu} - \operatorname{grad}^{g_\infty}) f(\gamma_\nu) + \nabla^{g_\infty} \operatorname{grad}^{g_\infty}(f(\gamma_\nu) - f(\gamma_\infty))$$

Now as $\gamma_\nu \to \gamma_\infty$, we have $f(\gamma_\nu) \to f(\gamma_\infty)$ as functions on $\mathbb{R}$ and hence each of the above terms can be made arbitrarily small by taking sufficiently large ν. The first term can be made small by considering the differences between the Christoffel symbols of the Levi-Civita connections for g_ν and g_∞, and from the definition of the Levi-Civita connections in terms of the metrics. For the second term, note that $g_\nu(\operatorname{grad}^{g_\nu} f, X) = \mathrm{d}f(X) = g_\infty(\operatorname{grad}^{g_\infty} f, X)$ for any functional f. Thus, $(g_\infty - g_\nu)(\operatorname{grad}^{g_\nu} f, X) = g_\infty((\operatorname{grad}^{g_\nu} - \operatorname{grad}^{g_\infty}) f, X)$, which shows that the second term can be made small. The final term can clearly be made small, as ∇^{g_∞} and $\operatorname{grad}^{g_\infty}$ are $\mathbb{R}$-linear operators.

Thus, with the above trivializations, we have expressed D_{γ_ν} and D_{γ_∞} as maps $W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$. Now since D_{γ_∞} is surjective and surjective operators between Banach spaces are open in the operator norm topology[1], it follows that D_{γ_ν} is surjective for large ν.

[1] See https://math.stackexchange.com/questions/17087/why-is-the-space-of-surjective-operators-open.