Transversality & symmetry for pseudoholom covers

Agenda of the talk:

- Relevant background on J-holom curves
    - nonlinear CR eqn.
    - moduli space of simple curves
- The case for multiple covers
    - Equivariant transversality in finite dimensions
- The main results & stratification theory
    - [- outline of splitting of CR operators for double covers]

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Let (M, J) an almost cplx mfld

(Σ, j) a closed Riemann surface

<u>smooth</u> $u : (\Sigma, j) \to (M, J)$

s.t. $\underline{J \circ du = du \circ j}$ <u>J-holom map.</u>

<u>global perspective:</u>

$u \in \mathcal{B} = C^\infty(\Sigma, M)$

$\mathcal{E} \rightsquigarrow$ "infinite dim'l v.b"

$$\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^* TM) \Big\} \longrightarrow \begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathcal{B} \arrow[u, bend right, "\boxed{\bar\partial_J}"'] \end{tikzcd}$$

$$\bar\partial_J(u) = \frac{1}{2}(du + J \circ du \circ j)$$

$$\{\text{JH maps}\} = \bar\partial_J^{-1}(0).$$

$\underline{\bar\partial_J(u) = 0}$ $\longrightarrow$ <u>nonlinear</u> CR equation.

Simple curves:

$u: (\Sigma, j) \longrightarrow (M, J)$

$$
\begin{tikzcd}
(\Sigma, j) \arrow[d, "\varphi"'] \arrow[dr, bend left, "u"] & \\
(\Sigma', j') \arrow[r, "v"'] & (M, J)
\end{tikzcd}
$$

$\cancel{v}$ u $\leftrightarrow$ simple

$\cancel{u}$ v

holom cover.

simple curves

$\equiv$

somewhere injective curves

$\Downarrow$

$\exists\, z$, $du(z) \neq 0$.

& $u^{-1}(u(z)) = z$.

moduli space of simple curves:

(M, ω, J) compact symplectic manifold, J ω-compatible

(Σ, j) closed R.S.

$A \in H_2(M)$, $[u] = A$.

For generic J,

$$\mathcal{M}^*(A) = \{ \text{simple JH curves} \}$$

is a smooth manifold, of dim given by an index formula.

What about multiple covers?

unbranched cover

$$\begin{tikzcd} & (\tilde{\Sigma}, \tilde{j}) \arrow[dl, "d:1"', "\varphi"] \arrow[dr, dotted, "u \circ \varphi"] & \\ u : (\Sigma, j) \arrow[rr] & & (M, J) \end{tikzcd}$$

is closed JH

virtual dimensions of moduli spaces containing u
"expected"
& $u \circ \varphi$, also known as indices of these 2 curves.

$$\mathrm{ind}(u \circ \varphi) = d \cdot \underbrace{\mathrm{ind}(u)} \geq 0 .$$

There is no reason why $u \circ \varphi$ couldnot achieve transversality.

For branched covers: things are more subtle.

multiple covers

symmetry

Example (finite dimensions)

Say $f : \mathbb{R}^2 \xrightarrow[\mathcal{C}^\infty]{} \mathbb{R}^2$ $\mathbb{Z}_2$ equivariant

if $f(x, -y) = -f(x, y)$.

Can show: every such map admits no! $\mathcal{C}^\infty$-close $\mathbb{Z}_2$-equivariant perturbations for which $\vec{0}$ is a regular value.

Note that: $\mathbb{R} \times \{0\} \subset f^{-1}(\vec{0})$ $\forall$ such f

So, $\partial_x f = \vec{0}$ along x-axis, thus transversality fails.

Best case scenario: generic f intersect 0 cleanly,

i.e., all components of $f^{-1}(0)$ are submfds

with $T_x(f^{-1}(0)) = \ker df(x)$

clean intersection:

$M, N \subseteq X$

$M \pitchfork N$,

$x \in M \cap N$,

$T_x M \oplus T_x N = T_x X$.

clean intersection

$T_x(M \cap N) = T_x M \cap T_x N$

$M \cap N$ is a smooth submfd.

Equivariant transversality in finite dimensions

Fix n-dim'l orbifold M,
orbibundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ of rk m.

Every $x \in M$ has a finite grp G_x, and a nbhd $U_x \subset M$ st
$$E|_{U_x} \simeq (\mathcal{O} \times \mathbb{R}^m)/G_x$$
for some linear action of G_x on $\mathbb{R}^m$ and a nbhd $\mathcal{O} \subset \mathbb{R}^n$ of $\vec{0}$.

<u>Question</u>: Do generic $\sigma \in \Gamma(E)$ intersect the zero section transversely (or atleast cleanly)?

<u>Sample thm 1</u>: If $\dim M = \operatorname{rk} E$ and $|G_x| \leq 3$ $\forall x$, then generic sections of E intersect 0 cleanly.

<u>Sample thm 2</u>: Generic smooth fns on an orbifold are Morse.
(cf. Wasserman '69, Hepworth '09).

Ingredient A : Stratification by symmetry.

For finite grp G, representations $\rho : G \to GL(n, \mathbb{R})$;
$\tau : G \to GL(m, \mathbb{R})$.

define submfd

$$M_{\rho,\tau} = \left\{ x \in M \;\middle|\; \begin{array}{l} G_x \simeq \underline{G}, \text{ acts on } T_x M \text{ as } \underline{\rho}, \\ \text{on } E_x \text{ as } \underline{\tau} \end{array} \right\}$$

and subbundle

$$E_{\rho,\tau} = \left\{ v \in E|_{M_{\rho,\tau}} \;\middle|\; \begin{array}{c} G \text{ acts trivially} \\ \text{on } v \end{array} \right\}$$

Let $\{\theta_i : G \to \mathrm{Aut}_{\mathbb{R}}(W_i)\}_{i=1}^{N}$ denote the real irreps of G, $\quad \theta_1 =$ trivial repr.,
$m_i(\rho) =$ mult. of θ_i in ρ.

Then, $\dim M_{\rho,\tau} = m_1(\rho)$, $\operatorname{rk} E_{\rho,\tau} = m_1(\tau)$.

The orbifold M is thus a countable union of disjoint smooth submfds $M_{\rho,\tau}$ with distinguished subbundles $E_{\rho,\tau} \subset E|_{M_{\rho,\tau}}$.

Notice: $\forall\ \sigma \in \Gamma(E)$, $\sigma(M_{\rho,\tau}) \subset E_{\rho,\tau}$.

$\Rightarrow$ $\sigma \not\pitchfork 0$ at $x \in M_{\rho,\tau}$ unless τ is trivial.

Lemma (standard transversality).

For generic $\sigma \in \Gamma(E)$, $\sigma|_{M_{\rho,\tau}}$ is transverse to the zero section of $E_{\rho,\tau} \to M_{\rho,\tau}$ $\forall\ G, \rho, \tau$.

$\Longrightarrow$ Cannot conclude that $M(\sigma)$ is a smooth orbifold.

Ingredient B: Splitting the linearization.

At $x \in M_{\rho,\tau}(\sigma)$,

$$D_x := D\sigma(x) : T_x M \to E_x$$

linearized operator.

$$\left(\sigma = \bar\partial_J \quad \begin{matrix} \mathcal{E} \\ \downarrow \\ \mathcal{B} \end{matrix}\right).$$

Recall Θ_i & $d_i = \dim W_i$.

Since D_x is G_x-equivariant,

by Schur's lemma,

D_x splits wrt isotypic decompositions

$$T_x M = \bigoplus T_x M^i \text{ of } \rho \ \& \ E_x = \bigoplus E_x^i \text{ of } \tau,$$

$$\underline{\underline{D_x}} = \underline{\underline{D_x^1}} \oplus \cdots \oplus \underline{\underline{D_x^N}}$$

These operators have Fredholm indices

$$\text{ind } D_x^i = \underline{d_i} \, [\underline{m_i(\rho)} - \underline{m_i(\tau)}]$$

and D_x^1 is surj. if σ is generic.

↑ finite dim'l setup for $\bar{\partial}_J$

main results of the paper:

Thm B (transversality, unbranched)

"For generic J, $\forall$ simple JH curves $u : (\Sigma, j) \to (M, J)$ and every unbranched cover $\varphi : (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$, of closed R.S.,

the curve $u \circ \varphi$ is Fredholm regular".

Generally, harder to achieve transversality

$u \circ \varphi$

u: closed, simple

φ: deg (d) branched cover

$Z(d\varphi)$ = alg. count of branch pts.

Riemann Hurwitz: $-\chi(\tilde{\Sigma}) + d\chi(\Sigma) = Z(d\varphi)$.

standard index formula for closed curves.

$$\operatorname{ind}(u \circ \varphi) = d \cdot \operatorname{ind}(u) - (n-3)\, Z(d\varphi).$$

where $\dim_{\mathbb{R}} M = 2n$

if $\underline{\mathrm{ind}(u)=0,\ n>3,\ \mathrm{ind}(u\circ\varphi)<0}$.

$\underline{\mathrm{ind}(u\circ\varphi)\geq \mathrm{ind}(u)+2Z(d\varphi)}$.

$2Z(d\varphi)$.

$\varphi\in\mathcal{M}$

$u\circ\varphi \quad u\in\mathcal{M}$.

<u>necessary</u> to have

if $u\circ\varphi$ were Fredholm regular.

If φ has $r\geq 0$ critical values,

$\underline{\mathrm{ind}(u)\geq(n-1)r}$.

<u>Thm C</u> (transversality, branched)

" For generic J,

For branched covers, can find C^∞-close regular covers".

$\mathrm{ind}(u)=\mathrm{ind}(D_u)$.

<u>If u is immersed</u>, can consider $\boxed{\underline{D_u^N}}=\underline{D_u}|_{N_u}$.

$\longrightarrow \mathrm{ind}(D^N_{u\circ\varphi})=\underbrace{d\cdot\mathrm{ind}(D_u^N)}-(n-1)Z(d\varphi)\leq 0$

so, $\underline{D^N_{u\circ\varphi}}$ can be <u>injective</u>

$D^N_{u\circ\varphi}$ injective : $u\circ\varphi$ can never be the limit of a seq. of s.i. curves.

i.e. only other curves near $u\circ\varphi$ are other branched covers $u\circ\varphi'$ for φ' near φ.

Thm A (super-rigidity)

If $\dim M \geq 6$, for generic J,

every simple ind(0) JH curve is super-rigid.

conj. (Bryan-Pandharipande 2001)

$\bar\partial_J \pitchfork 0$ cleanly.

defn: closed, connected, simple

- $\operatorname{ind}(u) \geq 0$
- $u : \Sigma \to M$ immersion
- $\forall$ covers $\tilde u = u\circ\varphi$, $D^N_{\tilde u}$ is injective

Thm D implies all of the above

Building walls (in the sense of "crossing").

$$D_x = D_x^1 \oplus \cdots \oplus D_x^N.$$

$$\begin{tikzcd} E \arrow[d] \arrow[loop left, "\sigma"] \\ M \end{tikzcd}$$

G_x acts $\dfrac{\ker D_x^i}{\operatorname{coker} D_x^i}$ as irrep Θ_i with multiplicities.

$$\vec{k} = (k_1, \ldots, k_N), \quad \vec{c} = (c_1, \ldots, c_N).$$

$$M_{\rho,\tau}(\sigma; \vec{k}, \vec{c}) = \{ x \in M_{\rho,\tau}(\sigma) \mid \dim \ker D_x^i = d_i k_i; \ \sim \operatorname{coker} = d_i c_i \}.$$

Workhorse thm:

$\forall$ generic σ, $\forall$ choices $G, \rho, \tau, \vec{k}, \vec{c}$,

$M_{\rho,\tau}(\sigma : \vec{k}, \vec{c}) \subset M_{\rho,\tau}(\sigma)$.

is a smooth submfd, codim $= \sum t_i k_i c_i$

type of Θ_i