

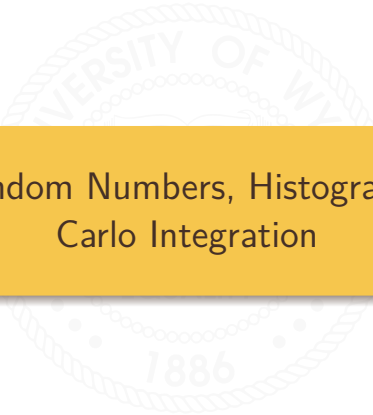
# MATH 3341: Introduction to Scientific Computing Lab

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November 22, 2021





# Lab 13: Random Numbers, Histogram & Monte Carlo Integration

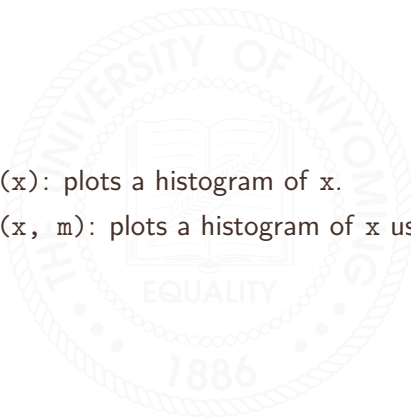


## Random Numbers and Histogram



# histogram: Plots a histogram.

- `histogram(x)`: plots a histogram of  $x$ .
- `histogram(x, m)`: plots a histogram of  $x$  using  $m$  bins.



## rand: Uniformly distributed pseudorandom numbers.

- `rand`: generate a random number that is uniformly distributed on  $(0, 1)$ .
- `rand(m, n)` or `rand([m, n])` generates a  $m$ -by- $n$  uniformly distributed random matrix.
- `rand(n)`: generates an  $n$ -by- $n$  uniform distributed random matrix.
- `rand(size(A))` generates a uniformly distributed random matrix of the same size as  $A$ .
- Example:

```
left = -2;  
right = 2;  
% Uniformly distributed on [left, right]  
numbers = rand(10, 1) * (right - left) + left;  
histogram(numbers);
```



## randn: Normally distributed pseudorandom numbers.

- `randn`: generate a random number that is normally distributed with mean 0 and standard deviation 1.
- `randn(m, n)` or `randn([m, n])` generates a  $m$ -by- $n$  normally distributed random matrix.
- `randn(n)`: generates an  $n$ -by- $n$  uniform distributed random matrix.
- `randn(size(A))` generates a normally distributed random matrix of the same size as  $A$ .
- Example:

```
mu = -2;      % mean
sigma = 2;    % standard deviation
% Normally distributed with mean -2 and standard deviation 2
numbers = randn(10, 1) * sigma + mu;
histogram(numbers);
```



The background of the slide features a large, faint watermark of the University of Wyoming seal. The seal is circular with a rope-like border. Inside the border, the words "UNIVERSITY OF WYOMING" are at the top, "EQUITY" is at the bottom, and "1886" is at the very bottom. In the center of the seal is an open book.

## Monte Carlo Integration



# 1-D Monte Carlo Integration

$$\begin{aligned}\int_a^b f(x) dx &= \int_a^b \frac{f(x)}{p(x)} p(x) dx \\ &= E[f(X)/p(X)] \\ &\approx \frac{1}{N} \sum_{i=1}^N \frac{f(x_i)}{p(x_i)} \\ &= \frac{1}{N} \sum_{i=1}^N \frac{f(x_i)}{1/(b-a)} \\ &= \frac{b-a}{N} \sum_{i=1}^N f(x_i),\end{aligned}$$

where  $x_1, x_2, \dots, x_N$  are uniformly distributed on  $[a, b]$ , hence  $p(x_i) = \frac{1}{b-a}, i = 1, 2, \dots, N$ .





## 2-D Monte Carlo Integration

$$\begin{aligned}\int_a^b \int_c^d f(x, y) dy dx &= \int_a^b \int_c^d \frac{f(x, y)}{p(x, y)} p(x, y) dy dx \\&= \int_a^b \int_c^d \frac{f(x, y)}{p_X(x)p_Y(y)} p_X(x)p_Y(y) dy dx \\&= \int_a^b \frac{1}{p_X(x)} \left[ \int_c^d \frac{f(x, y)}{p_Y(y)} p_Y(y) dy \right] p_X(x) dx \\&= E \left[ \frac{f(X, Y)}{p_Y(Y)p_X(X)} \right] \\&\approx \frac{1}{N} \sum_{i=1}^N \frac{f(x_i, y_i)}{p_X(x_i)p_Y(y_i)}.\end{aligned}$$



## 2-D Monte Carlo Integration

$$\begin{aligned}\int_a^b \int_c^d f(x, y) dy dx &\approx \frac{1}{N} \sum_{i=1}^N \frac{f(x_i, y_i)}{p_X(x_i)p_Y(y_i)} \\ &= \frac{1}{N} \sum_{i=1}^N \frac{1}{[1/(b-a)][1/(d-c)]} f(x_i, y_i) \\ &= \frac{(b-a)(d-c)}{N} \sum_{i=1}^N f(x_i, y_i),\end{aligned}$$

where  $X$  and  $Y$  are independent and identically uniformly distributed, hence  $p(x, y) = p_X(x)p_Y(y)$ , and  $p_X(x) = \frac{1}{b-a}$ ,  $p_Y(y) = \frac{1}{d-c}$ .



## 2-D Monte Carlo Integration Example

$$\int_{D:x^2+y^2 \leq 1} f(x,y) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y) dy dx$$

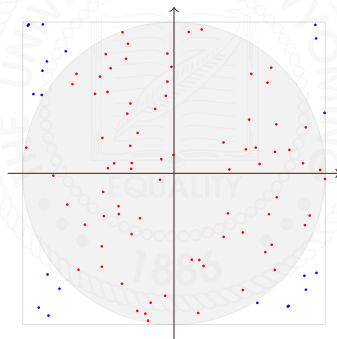


Figure 1: When  $N = 100$ .



## 2-D Monte Carlo Integration Example

$$\int_{D: x^2+y^2 \leq 1} f(x, y) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy dx$$

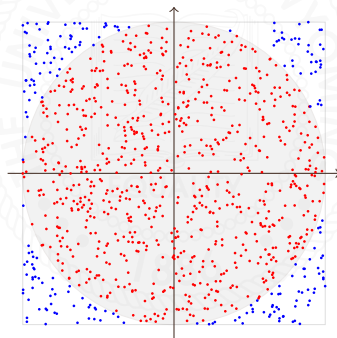


Figure 2: When  $N = 1000$ .



## 2-D Monte Carlo Integration Example

$$\int_{D:x^2+y^2 \leq 1} f(x, y) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy dx$$

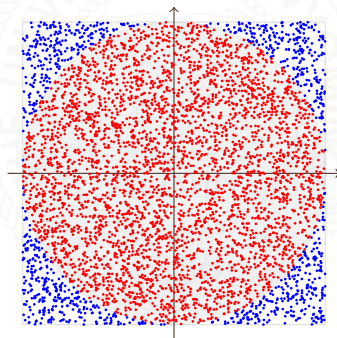


Figure 3: When  $N = 5000$ .




## 2-D Monte Carlo Integration Example

$$\int_{D: x^2+y^2 \leq 1} f(x, y) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy dx$$

- 1 Since  $-1 \leq x \leq 1$  and  $-1 \leq -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \leq 1$ , the bounding box is  $[-1, 1] \times [-1, 1]$  on which we can generate the random points  $(x_i, y_i), i = 1, 2, \dots, N$ .
- 2 We are integrating over the disk, so we need to discard the points outside the disk (blue points):

$$g(x_i, y_i) = \begin{cases} f(x_i, y_i) & \text{if } (x_i, y_i) \text{ is inside the disk,} \\ 0 & \text{otherwise.} \end{cases}$$

- 3 Therefore, we can obtain

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy dx \approx \frac{[1 - (-1)] \cdot [1 - (-1)]}{N} \sum_{i=1}^N g(x_i, y_i).$$


## 2-D Monte Carlo Integration Example

How do we check whether  $(x_i, y_i)$  is inside the disk?

$$-\sqrt{1-x_i^2} \leq y_i \leq \sqrt{1-x_i^2} \implies y_i^2 \leq 1-x_i^2 \implies x_i^2 + y_i^2 \leq 1.$$

In MATLAB: we can define  $g(x, y)$  as follows:

```
g = @(x,y) f(x,y).*(-sqrt(1-x.^2)<=y & y<=sqrt(1-x.^2));
```

or

```
g = @(x,y) f(x,y) .* (x.^2 + y.^2 <= 1);
```

