

# MATH 3341: Introduction to Scientific Computing Lab

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November 08, 2021



# Lab 11: MATLAB Integration Routines & Gauss Quadrature



The background of the slide features a large, faint watermark of the University of Wyoming seal. The seal is circular with a rope-like border. Inside the border, the words "UNIVERSITY OF WYOMING" are at the top, "EQUALITY" is in the center, and "1886" is at the bottom. In the middle of the seal is an open book.

## Built-in Integration Routines



## polyint: Indefinite Integral

- $I = \text{polyint}(p, c)$ : indefinite integral of polynomial  $p$  with  $c$  being the constant.
- $I = \text{polyint}(p)$ : same as  $\text{polyint}(p, 0)$ .
- Example:  $\int 3x^2 + 2x + 1 \, dx = x^3 + x^2 + x + C$ .

```
p = [3,2,1]
I1 = polyint(p, 1) % [1,1,1,1]
I2 = polyint(p, 2) % [1,1,1,2]
I3 = polyint(p)   % [1,1,1,0]
polyval(I1, 0)    % 1
polyval(I2, 0)    % 2
polyval(I3, 0)    % 0
```



# polyint: Definite Integral

- Fundamental Theorem of Calculus (FTOC):

$$\int_a^b p'(x) dx = p(x) \Big|_a^b = p(b) - p(a).$$

- Example:  $\int_0^2 3x^2 + 2x + 1 dx = x^3 + x^2 + x + C \Big|_{x=0}^{x=2} = 14.$

```
p = [3,2,1]
```

```
P = polyint(p) % [1,1,1,0]
```

```
I = polyval(P, 2) - polyval(P, 0) % 14
```



## trapz: Trapezoidal numerical integration

- $I = \text{trapz}(x, y)$  computes the integral of  $y$  with respect to  $x$  using the trapezoidal method,  $x$  and  $y$  must be vectors of the same length.
- Let  $X = [x_1, x_2]$ ,  $Y = [y_1, y_2]$ , it is actually a trapezoid, where  $y_1$  and  $y_2$  are the lengths for the bases and  $x_2 - x_1$  is the height. Then

$$I = \frac{(x_2 - x_1)(y_1 + y_2)}{2}.$$

- Let  $X = [x_1, x_2, \dots, x_n]$ ,  $Y = [y_1, y_2, \dots, y_n]$ , then

$$I = \sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)(y_{i+1} + y_i)}{2} = \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)(y_{i+1} + y_i).$$



## cumtrapz: Cumulative trapezoidal numerical integration

- $I = \text{cumtrapz}(x, y)$  computes the cumulative integral of  $y$  with respect to  $x$  using trapezoidal integration.

- Example:

```
x = [1,2,3];
```

```
y = [1,2,3];
```

```
I1 = cumtrapz(x, y) % [0, 1.5000, 4.0000]
```

```
I2 = [trapz([1], [1]),  
      trapz([1,2], [1,2]),  
      trapz([1,2,3], [1,2,3])] % [0, 1.5000, 4.0000]
```



# Numerically evaluate integral - 1D

- `I = integral(f, a, b)` approximates the integral of function `f` from `a` to `b` using global adaptive quadrature and default error tolerances. `f` must be a function handle, `a` and `b` can be `-Inf` or `Inf`.

- Example:  $\int_0^2 3x^2 + 2x + 1 dx = x^3 + x^2 + x + C \Big|_{x=0}^{x=2} = 14.$

```
f = @(x) 3 * x.^2 + 2 * x + 1;
```

```
a = 0;
```

```
b = 2;
```

```
I = integral(f, a, b) % 14.0000
```





# Numerically evaluate integral - 2D

- `I = integral2(f,xmin,xmax,ymin,ymax)` approximates the integral of  $f(x,y)$  over the planar region  $x_{\min} \leq x \leq x_{\max}$  and  $y_{\min}(x) \leq y \leq y_{\max}(x)$ .  $f$  is a function handle,  $y_{\min}$  and  $y_{\max}$  may each be a scalar value or a function handle.

- Example:

$$\int_0^2 \int_0^x 6y + 2 \, dy \, dx = \int_0^2 3x^2 + 2x \, dx = x^3 + x^2 + C \Big|_{x=0}^{x=2} = 12.$$

```
f = @(x, y) 6 * y + 2;
```

```
xmin = 0;
```

```
xmax = 2;
```

```
ymin = 0;
```

```
ymax = @(x) x;
```

```
I = integral2(f, xmin, xmax, ymin, ymax);    % 12.0000
```



# Numerically evaluate integral - 3D

- `I = integral3(f,xmin,xmax,ymin,ymax,zmin,zmax)` approximates the integral of  $f(x,y,z)$  over the region  $x_{\min} \leq x \leq x_{\max}$ ,  $y_{\min}(x) \leq y \leq y_{\max}(x)$ , and  $z_{\min}(x,y) \leq z \leq z_{\max}(x,y)$ .  $f$  is a function handle,  $y_{\min}$ ,  $y_{\max}$ ,  $z_{\min}$ , and  $z_{\max}$  may each be a scalar value or a function handle.

- Example: 
$$\int_0^2 \int_0^x \int_{-2y+2}^{4y+4} 1 \, dz \, dy \, dx = \int_0^2 \int_0^x 6y + 2 \, dy \, dx =$$

$$\int_0^2 3x^2 + 2x \, dx = x^3 + x^2 + C \Big|_{x=0}^{x=2} = 12.$$

```
f = @(x, y, z) ones(size(z));
```

```
xmin = 0; xmax = 2;
```

```
ymin = 0; ymax = @(x) x;
```

```
zmin = @(x,y) -2 * y + 4;
```

```
zmax = @(x,y) 4 * y + 4;
```

```
I = integral3(f, xmin, xmax, ymin, ymax, zmin, zmax)
```



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## Gauss-Legendre Quadrature



# Gauss-Legendre Quadrature on $[-1, 1]$

Integration of  $f(x)$  on the interval  $[-1, 1]$  using Gauss Quadrature is given by

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i),$$

where  $w_i$  and  $x_i$  are chosen so the integration rule is exact for the largest class of polynomials.  $f(x)$  is well-approximated by polynomial on  $[-1, 1]$ , the associated orthogonal polynomials are *Legendre polynomial*, denoted by  $P_n(x)$ . With the  $n$ -th polynomial normalized to give  $P_n(1) = 1$ , the  $i$ -th Gauss node,  $x_i$ , is the  $i$ -th root of  $P_n$  and the weights are given by the formula (Abramowitz & Stegun 1972, p. 887):

$$w_i = \frac{2}{(1 - x_i^2)[P'_n(x_i)]^2}.$$



# Gauss-Legendre Quadrature on $[a, b]$

To approximate the integral on the general interval  $[a, b]$ , we need to use the change of variables as follows:

$$\begin{aligned}\frac{t-a}{b-a} &= \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2} \implies t = \frac{b-a}{2}x + \frac{b+a}{2}, -1 \leq x \leq 1 \\ &\implies dt = \frac{b-a}{2}dx.\end{aligned}$$

So the Gauss Quadrature on a general interval  $[a, b]$  is given by

$$\begin{aligned}\int_a^b f(t) dt &= \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \frac{b-a}{2} dx \\ &\approx \sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right) \frac{b-a}{2}.\end{aligned}$$



# Gauss-Legendre Quadrature on $[a, b]$

Let

$$g(x) = f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \frac{b-a}{2},$$

then

$$\int_a^b f(t) dt = \int_{-1}^1 g(x) dx \approx \sum_{i=1}^n w_i g(x_i).$$

