MATH 3341: Introduction to Scientific Computing Lab

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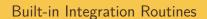
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Lab 11: MATLAB Integration Routines & Gauss Quadrature







polyint: Indefinite Integral

- I = polyint(p, c): indefinite integral of polynomial p with c being the constant.
- I = polyint(p): same as polyint(p, 0).



polyint: Definite Integral

• Fundamental Theorem of Calculus (FTOC):

$$\int_{a}^{b} p'(x) \, dx = p(x) \Big|_{a}^{b} = p(b) - p(a).$$



trapz: Trapezoidal numerical integration

- I = trapz(x, y) computes the integral of y with respect to x using the trapezoidal method, x and y must be vectors of the same length.
- Let $X=[x_1,x_2]$, $Y=[y_1,y_2]$, it is actually a trapezoid, where y_1 and y_2 are the lengths for the bases and x_2-x_1 is the height. Then

$$I = \frac{(x_2 - x_1)(y_1 + y_2)}{2}.$$

• Let $X = [x_1, x_2, \dots, x_n], Y = [y_1, y_2, \dots, y_n],$ then

$$I = \sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)(y_{i+1} + y_i)}{2} = \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)(y_{i+1} + y_i).$$



cumtrapz: Cumulative trapezoidal numerical integratio

- I = cumtrapz(x, y) computes the cumulative integral of y
 with respect to x using trapezoidal integration.
- Example:



Numerically evaluate integral - 1D

- I = integral(f, a, b) approximates the integral of function f from a to b using global adaptive quadrature and default error tolerances. f must be a function handle, a and b can be -Inf or Inf.
- Example: $\int_0^2 3x^2 + 2x + 1 \, dx = x^3 + x^2 + x + C \Big|_{x=0}^{x=2} = 14.$ f = @(x) 3 * x.^2 + 2 * x + 1;
 a = 0;
 b = 2;
 I = integral(f, a, b) % 14.0000



Numerically evaluate integral - 2D

- I = integral2(f,xmin,xmax,ymin,ymax) approximates the integral of f(x,y) over the planar region xmin $\leq x \leq$ xmax and ymin(x) \leq y \leq ymax(x). f is a function handle, ymin and ymax may each be a scalar value or a function handle.
- Example:

$$\int_{0}^{2} \int_{0}^{x} 6y + 2 \, dy \, dx = \int_{0}^{2} 3x^{2} + 2x \, dx = x^{3} + x^{2} + C \Big|_{x=0}^{x=2} = 12.$$

$$f = @(x, y) 6 * y + 2;$$

$$xmin = 0;$$

$$xmax = 2;$$

$$ymin = 0;$$

$$ymax = @(x) x;$$

$$I = integral2(f, xmin, xmax, ymin, ymax); % 12.000(f)$$

Numerically evaluate integral - 3D

- I = integral3(f,xmin,xmax,ymin,ymax,zmin,zmax)
 approximates the integral of f(x,y,z) over the region xmin
 <= x <= xmax, ymin(x) <= y <= ymax(x), and zmin(x,y)
 <= z <= zmax(x,y). f is a function handle, ymin, ymax,
 zmin, and zmax may each be a scalar value or a function
 handle.</pre>
- Example: $\int_0^2 \int_0^x \int_{-2y+2}^{4y+4} 1 \, dz \, dy \, dx = \int_0^2 \int_0^x 6y + 2 \, dy \, dx = \int_0^2 3x^2 + 2x \, dx = x^3 + x^2 + C \Big|_{x=0}^{x=2} = 12.$ f = @(x, y, z) ones(size(z)); xmin = 0; xmax = 2; ymin = 0; ymax = @(x) x; zmin = @(x,y) -2 * y + 4; zmax = @(x,y) 4 * y + 4;





Gauss-Legendre Quadrature on [-1, 1]

Integration of $f(\boldsymbol{x})$ on the interval [-1,1] using Gauss Quadrature is given by

$$\int_{-1}^{1} f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where w_i and x_i are chosen so the integration rule is exact for the largest class of polynomials. f(x) is well-approximated by polynomial on [-1,1], the associated orthogonal polynomials are Legendre polynomial, denoted by $P_n(x)$. With the n-th polynomial normalized to give $P_n(1)=1$, the i-th Gauss node, x_i , is the i-th root of P_n and the weights are given by the formula (Abramowitz & Stegun 1972, p. 887):

$$w_i = \frac{2}{(1 - x_i^2)[P_n'(x_i)]^2}.$$



Gauss-Legendre Quadrature on [a, b]

To approximate the integral on the general interval [a,b], we need to use the change of variables as follows:

$$\frac{t-a}{b-a} = \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2} \implies t = \frac{b-a}{2}x + \frac{b+a}{2}, -1 \le x \le 1$$
$$\implies dt = \frac{b-a}{2}dx.$$

So the Gauss Quadrature on a general interval $\left[a,b\right]$ is given by

$$\int_{a}^{b} f(t) dt = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \frac{b-a}{2} dx$$
$$\approx \sum_{i=1}^{n} w_{i} f\left(\frac{b-a}{2}x_{i} + \frac{b+a}{2}\right) \frac{b-a}{2}.$$



Gauss-Legendre Quadrature on [a, b]

Let

$$g(x) = f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2},$$

then

$$\int_{a}^{b} f(t) dt = \int_{-1}^{1} g(x) dx \approx \sum_{i=1}^{n} w_{i} g(x_{i}).$$

