

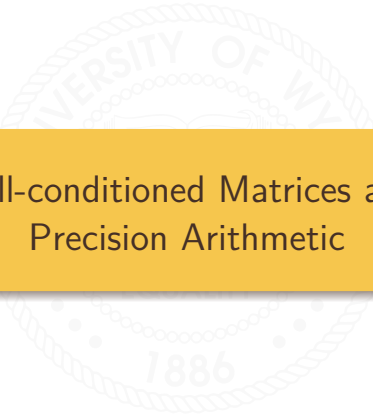
MATH 3341: Introduction to Scientific Computing Lab

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Lab 09: Ill-conditioned Matrices and Finite Precision Arithmetic



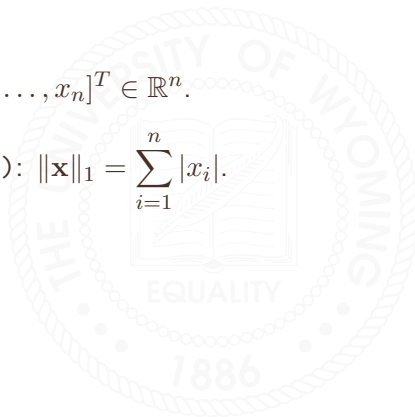
Ill-Conditioned Matrices



Vector Norm

Let $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$.

- $\text{norm}(\mathbf{x}, 1)$: $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$.



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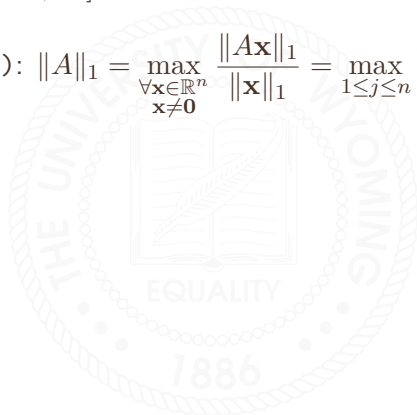
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- $\text{norm}(\mathbf{x}, \text{inf})$: $\|\mathbf{x}\|_\infty = \max_{i=1, \dots, n} \{|x_i|\}$.



Matrix Norm

Let $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$.

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- $\text{norm}(A, \text{'fro'})$: $\|A\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \right)^{1/2}$, Frobenius norm.



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The *condition number* of nonsingular matrix A relative to the norm $\|\cdot\|$ is

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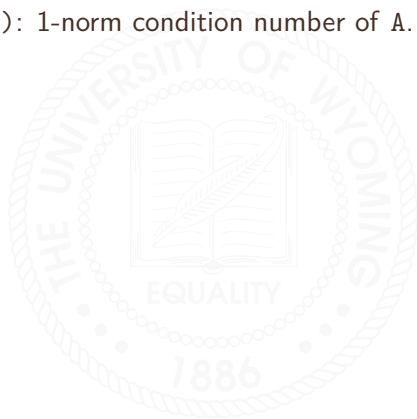
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- If $\kappa(A) = \infty$, then the matrix A is singular, i.e., the matrix is not invertible.



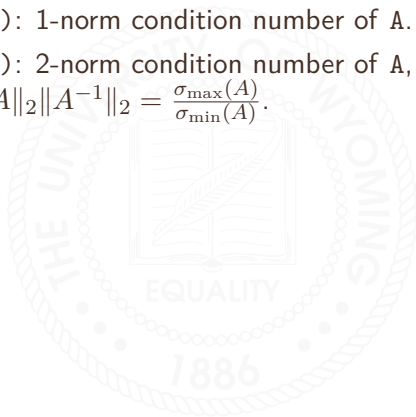
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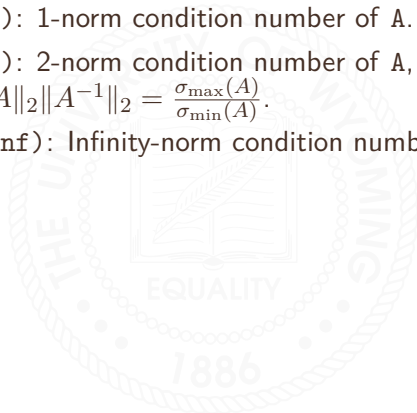
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- Example

```
A = magic(5);  
condA1 = cond(A, 2)  
condA2 = norm(A, 2) * norm(inv(A), 2)  
condA3 = max(sqrt(eig(A'*A))) * max(sqrt(eig(inv(A'*A))))  
condA4 = max(sqrt(eig(A'*A))) / min(sqrt(eig(A'*A)))
```



Ill-Conditioned Matrix: Hilbert Matrix

A Hilbert matrix is a square matrix with elements defined by

$$H_{ij} = \frac{1}{i + j - 1}.$$

For example, a 3×3 Hilbert matrix is

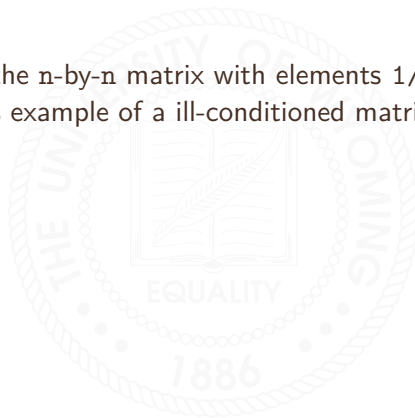
$$H_{3 \times 3} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

Note that this matrix is symmetric and positive definite.



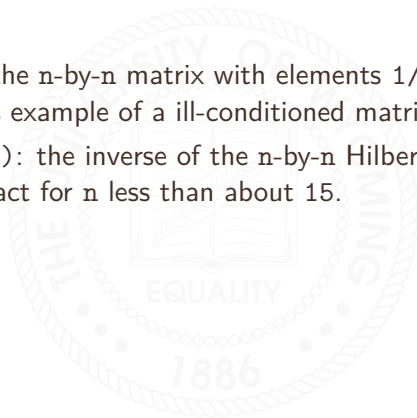
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- Example:

```
H = hilb(10);  
invH1 = inv(H);  
invH2 = invhilb(10);  
norm(invH1 - invH2)
```



Ill-Conditioned Linear System

A linear system $A\mathbf{x} = \mathbf{b}$ is said to be ill-conditioned if A is a ill-conditioned matrix. The typical numerical methods for solving linear systems such as Jacobi method, Gauss-Seidel method would become unreliable. Example: $H\mathbf{x} = \mathbf{b} \implies \mathbf{x} = H^{-1}\mathbf{b}$.

```
n = 10;  
H = hilb(n);  
invH = invhilb(n);  
b = rand(n, 1);  
x = invH * b;  
x1 = inv(H) * b;  
x2 = H \ b;  
norm(x - x1)  
norm(x - x2)
```



The background of the slide features a large, faint watermark of the University of Wyoming seal. The seal is circular with a rope-like border. Inside the border, the words "UNIVERSITY OF WYOMING" are at the top, "EQUALITY" is in the center, and "1886" is at the bottom. In the middle of the seal is an illustration of an open book.

Finite Precision Arithmetic



Finite Precision Arithmetic

Computers can only store values up to a certain level of accuracy. Past this level, the computer will round values, thus causes the round-off error. What this means is that arithmetic does not work exactly as we expect. Namely, arithmetic is no longer commutative, associative, or distributive. The lab exercises will demonstrate some of the issues that arise.



IEEE 754

<https://babbage.cs.qc.cuny.edu/IEEE-754/>

