MATH 3341: Introduction to Scientific Computing Lab

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Lab 11: MATLAB Integration Routines & Gauss Quadrature

Built-in Integration Routines

`polyint`: Indefinite Integral

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polyint: Indefinite Integral

- `I = polyint(p, c)`: indefinite integral of polynomial `p` with `c` being the constant.
- `I = polyint(p)`: same as `polyint(p, 0)`.
- Example: $\int 3x^2 + 2x + 1 \, dx = x^3 + x^2 + x + C$.

```
p = [3,2,1]
I1 = polyint(p, 1)  % [1,1,1,1]
I2 = polyint(p, 2)  % [1,1,1,2]
I3 = polyint(p)     % [1,1,1,0]
polyval(I1, 0)      % 1
polyval(I2, 0)      % 2
polyval(I3, 0)      % 0
```

`polyint`: Definite Integral

- Fundamental Theorem of Calculus (FTOC):

$$\int_a^b p'(x)\,dx = p(x)\Big|_a^b = p(b) - p(a).$$

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- Example: $\displaystyle\int_0^2 3x^2 + 2x + 1\,dx = x^3 + x^2 + x + C\Big|_{x=0}^{x=2} = 14.$

```
p = [3,2,1]
P = polyint(p)                          % [1,1,1,0]
I = polyval(P, 2) - polyval(P, 0)       % 14
```

`trapz`: Trapezoidal numerical integration

- `I = trapz(x, y)` computes the integral of `y` with respect to `x` using the trapezoidal method, `x` and `y` must be vectors of the same length.
- Let $X = [x_1, x_2]$, $Y = [y_1, y_2]$, it is actually a trapezoid, where y_1 and y_2 are the lengths for the bases and $x_2 - x_1$ is the height. Then
$$I = \frac{(x_2 - x_1)(y_1 + y_2)}{2}.$$
- Let $X = [x_1, x_2, \ldots, x_n]$, $Y = [y_1, y_2, \ldots, y_n]$, then

$$I = \sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)(y_{i+1} + y_i)}{2} = \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i)(y_{i+1} + y_i).$$

`cumtrapz`: Cumulative trapezoidal numerical integration

- `I = cumtrapz(x, y)` computes the cumulative integral of `y` with respect to `x` using trapezoidal integration.

cumtrapz: Cumulative trapezoidal numerical integration

- `I = cumtrapz(x, y)` computes the cumulative integral of `y` with respect to `x` using trapezoidal integration.
- Example:

```
x = [1,2,3];
y = [1,2,3];
I1 = cumtrapz(x, y)  % [0, 1.5000, 4.0000]
I2 = [trapz([1], [1]),
      trapz([1,2], [1,2]),
      trapz([1,2,3], [1,2,3])]  % [0, 1.5000, 4.0000]
```

Numerically evaluate integral - 1D

- `I = integral(f, a, b)` approximates the integral of function `f` from `a` to `b` using global adaptive quadrature and default error tolerances. `f` must be a function handle, `a` and `b` can be `-Inf` or `Inf`.

Numerically evaluate integral - 1D

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- Example: $\int_0^2 3x^2 + 2x + 1\, dx = x^3 + x^2 + x + C\Big|_{x=0}^{x=2} = 14.$

```
f = @(x) 3 * x.^2 + 2 * x + 1;
a = 0;
b = 2;
I = integral(f, a, b)              % 14.0000
```

Numerically evaluate integral - 2D

- `I = integral2(f,xmin,xmax,ymin,ymax)` approximates the integral of `f(x,y)` over the planar region `xmin <= x <= xmax` and `ymin(x) <= y <= ymax(x)`. `f` is a function handle, `ymin` and `ymax` may each be a scalar value or a function handle.

Numerically evaluate integral - 2D

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- Example:
$$\int_0^2 \int_0^x 6y + 2\, dy\, dx = \int_0^2 3x^2 + 2x\, dx = x^3 + x^2 + C\Big|_{x=0}^{x=2} = 12.$$

```
f = @(x, y) 6 * y + 2;
xmin = 0;
xmax = 2;
ymin = 0;
ymax = @(x) x;
I = integral2(f, xmin, xmax, ymin, ymax);   % 12.0000
```

Numerically evaluate integral - 3D

- `I = integral3(f,xmin,xmax,ymin,ymax,zmin,zmax)` approximates the integral of `f(x,y,z)` over the region `xmin <= x <= xmax`, `ymin(x) <= y <= ymax(x)`, and `zmin(x,y) <= z <= zmax(x,y)`. `f` is a function handle, `ymin`, `ymax`, `zmin`, and `zmax` may each be a scalar value or a function handle.

Numerically evaluate integral - 3D

- `I = integral3(f,xmin,xmax,ymin,ymax,zmin,zmax)` approximates the integral of `f(x,y,z)` over the region `xmin <= x <= xmax`, `ymin(x) <= y <= ymax(x)`, and `zmin(x,y) <= z <= zmax(x,y)`. `f` is a function handle, `ymin`, `ymax`, `zmin`, and `zmax` may each be a scalar value or a function handle.
- Example: $\int_0^2 \int_0^x \int_{-2y+2}^{4y+4} 1\,dz\,dy\,dx = \int_0^2 \int_0^x 6y + 2\,dy\,dx = \int_0^2 3x^2 + 2x\,dx = x^3 + x^2 + C\Big|_{x=0}^{x=2} = 12.$

```
f = @(x, y, z) ones(size(z));
xmin = 0; xmax = 2;
ymin = 0; ymax = @(x) x;
zmin = @(x,y) -2 * y + 4;
zmax = @(x,y) 4 * y + 4;
I = integral3(f, xmin, xmax, ymin, ymax, zmin, zmax);
```

Gauss-Legendre Quadrature

Gauss-Legendre Quadrature on $[-1, 1]$

Integration of $f(x)$ on the interval $[-1, 1]$ using Gauss Quadrature is given by

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where w_i and x_i are chosen so the integration rule is exact for the largest class of polynomials. $f(x)$ is well-approximated by polynomial on $[-1, 1]$, the associated orthogonal polynomials are *Legendre polynomial*, denoted by $P_n(x)$. With the n-th polynomial normalized to give $P_n(1) = 1$, the i-th Gauss node, x_i, is the i-th root of P_n and the weights are given by the formula (Abramowitz & Stegun 1972, p. 887):

$$w_i = \frac{2}{(1 - x_i^2)[P_n'(x_i)]^2}.$$

Gauss-Legendre Quadrature on $[a, b]$

To approximate the integral on the general interval $[a, b]$, we need to use the change of variables as follows:

$$\frac{t-a}{b-a} = \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2} \implies t = \frac{b-a}{2}x + \frac{b+a}{2}, -1 \le x \le 1$$
$$\implies dt = \frac{b-a}{2}dx.$$

So the Gauss Quadrature on a general interval $[a, b]$ is given by

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}\,dx \\ &\approx \sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right)\frac{b-a}{2}.\end{aligned}$$

Gauss-Legendre Quadrature on $[a, b]$

Let

$$g(x) = f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2},$$

then

$$\int_a^b f(t)\,dt = \int_{-1}^{1} g(x)\,dx \approx \sum_{i=1}^{n} w_i g(x_i).$$