MATH 3341: Introduction to Scientific Computing Lab

Melissa Butler

University of Wyoming

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Lab 12: Romberg Integration

Romberg Integration

Composite Trapezoidal Rule

- Composite Trapezoidal rule for approximating the integral of a function $f(x)$ on an interval $[a,b]$ using m subintervals:

$$I = \int_a^b f(x)\,dx \approx \frac{1}{2}\sum_{j=0}^{m-1}(x_{j+1}-x_j)(y_{j+1}+y_j),$$

where $a = x_0 < x_1 < \cdots < x_m = b$, $y_j = f(x_j), j = 0, \ldots, m$.

- Let $x_{j+1} - x_j = h = (b-a)/m, j = 0, 1, \ldots, m-1$.

$$I \approx \frac{1}{2}\sum_{j=0}^{m-1} h[f(x_{j+1})+f(x_j)] = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{m-1} f(x_j)\right],$$

where $x_j = x_0 + jh = a + jh$ for each $j = 0, 1, \ldots, m$.

Composite Trapezoidal Rule

- Composite Trapezoidal Rule using m subintervals:

$$I \approx \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{m-1} f(a+jh)\right].$$

- Finding approximation for $m_1 = 1, m_2 = 2, m_3 = 4, \ldots, m_n = 2^{n-1}$ for $n \in \mathbb{N}$. The corresponding step size h_k for each m_k is then given by $h_k = (b-a)/m_k = (b-a)/2^{k-1}$. The composite trapezoidal rule then becomes

$$I = \int_a^b f(x)\,dx \approx \frac{h_k}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{2^{k-1}-1} f(a+jh_k)\right].$$

Here we'll use the notation $R_{k,1}$ to denote the portion used for the trapezoidal approximation. In other words,

$$\begin{aligned}
R_{1,1} &= \frac{h_1}{2}[f(a)+f(b)] = \frac{b-a}{2}[f(a)+f(b)], \\
R_{2,1} &= \frac{h_2}{2}[f(a)+f(b)+2f(a+h_2)] \\
&= \frac{1}{2}\frac{h_1}{2}\left[f(a)+f(b)+2f\left(a+h_2\right)\right] \\
&= \frac{1}{2}\left\{\frac{h_1}{2}\left[f(a)+f(b)\right]+2\frac{h_1}{2}f\left(a+h_2\right)\right\} \\
&= \frac{1}{2}[R_{1,1}+h_1 f(a+h_2)], \\
R_{3,1} &= \frac{1}{2}\{R_{2,1}+h_2[f(a+h_3)+f(a+3h_3)]\}.
\end{aligned}$$

This leads to the Trapezoidal rule in the general form

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{j=1}^{2^{k-2}} f(a + (2j-1)h_k)\right] \quad \text{for } k = 2, 3, \ldots, n.$$

This method converges very slowly on its own. A technique called Richardson's Extrapolation is applied to speed convergence. Essentially, this performs a method of averaging previously calculated entries to obtain the next entry in the table. This is given in general form

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}.$$

This method will give us the following entries of R in a tabular format. The number of rows is determined by the value that we desire.

$$\begin{bmatrix} R_{1,1} & & & & & \\ R_{2,1} & R_{2,2} & & & & \\ R_{3,1} & R_{3,2} & R_{3,3} & & & \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ R_{n,1} & R_{n,2} & R_{n,3} & R_{n,4} & \cdots & R_{n,n} \end{bmatrix}$$

Algorithm

Algorithm 1: Romberg Integration

Function romberg(f, a, b, n)**:**

- $h \leftarrow b - a$;
- $R_{1,1} \leftarrow [f(a) + f(b)] \cdot h/2$;
- **for** $k \leftarrow 2$ **to** n **do**
  - $$R_{k,1} \leftarrow \frac{1}{2}\left[R_{k-1,1} + h\sum_{j=1}^{2^{k-2}} f(a + (2j-1)\cdot h/2)\right];$$
  - **for** $j \leftarrow 2$ **to** k **do**
    - $$R_{k,j} \leftarrow R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1};$$
  - **end**
  - $h \leftarrow h/2$;
- **end**
- **return** $[R_{1,1}, R_{2,2}, R_{3,3}, \ldots, R_{n,n}]$;

end