Lab 09: Ill-conditioned Matrices and Finite Precision Arithmetic

Melissa Butler

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Part 1: Question (g)

Question: Next, go to `lab_09_script.m`, create a vector named `n`: `n = [9, 11, 13, 15]`. Using a for-loop to call `lab_09_function` by passing each entry of `n` as the input argument. What do you notice about the results?
Answer: As we can see from the output, the condition number increases as n increases. In other words, as the linear system gets more ill-conditioned (i.e., larger condition number), the linear system becomes harder to solve (the bigger error).

Script and Output

Output file: `lab_09_output.txt`

```
lab_09_script
========================================
 Part I Output
========================================
----------------------------------------
Hilbert Matrix with n = 9
----------------------------------------
 true x      | backslash   | inv
----------------------------------------
  9.0000e+00 |  8.9999e+00 |  8.9999e+00
 -7.2000e+02 | -7.1999e+02 | -7.2000e+02
  1.3860e+04 |  1.3860e+04 |  1.3860e+04
 -1.1088e+05 | -1.1088e+05 | -1.1088e+05
  4.5045e+05 |  4.5045e+05 |  4.5045e+05
 -1.0090e+06 | -1.0090e+06 | -1.0090e+06
  1.2613e+06 |  1.2613e+06 |  1.2613e+06
 -8.2368e+05 | -8.2368e+05 | -8.2368e+05
  2.1879e+05 |  2.1879e+05 |  2.1879e+05
----------------------------------------
Error using backslash : 5.0648e-06
Error using inv       : 4.1829e-06
Condition number of A : 1.0996e+12
----------------------------------------
----------------------------------------
Hilbert Matrix with n = 11
----------------------------------------
 true x      | backslash   | inv
----------------------------------------
  1.1000e+01 |  1.1017e+01 |  1.0955e+01
 -1.3200e+03 | -1.3220e+03 | -1.3154e+03
  3.8610e+04 |  3.8665e+04 |  3.8491e+04
 -4.8048e+05 | -4.8113e+05 | -4.7915e+05
  3.1532e+06 |  3.1572e+06 |  3.1453e+06
 -1.2108e+07 | -1.2123e+07 | -1.2080e+07
  2.8589e+07 |  2.8622e+07 |  2.8528e+07
 -4.2008e+07 | -4.2054e+07 | -4.1926e+07
  3.7413e+07 |  3.7452e+07 |  3.7345e+07
 -1.8476e+07 | -1.8494e+07 | -1.8444e+07
  3.8799e+06 |  3.8836e+06 |  3.8737e+06
----------------------------------------
Error using backslash : 1.0974e-03
Error using inv       : 1.9478e-03
Condition number of A : 1.2311e+15
----------------------------------------
```

```
----------------------------------------
Hilbert Matrix with n = 13
----------------------------------------
 true x      | backslash   | inv
----------------------------------------
  1.3000e+01 | -3.0730e-01 | -7.9476e+01
 -2.1840e+03 | -2.2206e+02 |  1.2271e+04
  9.0090e+04 |  1.8008e+04 | -4.6646e+05
 -1.6016e+06 | -4.4674e+05 |  7.6657e+06
  1.5315e+07 |  5.2671e+06 | -6.7970e+07
 -8.8216e+07 | -3.5132e+07 |  3.6409e+08
  3.2591e+08 |  1.4482e+08 | -1.2543e+09
 -7.9815e+08 | -3.8628e+08 |  2.8715e+09
  1.3095e+09 |  6.7877e+08 | -4.4140e+09
 -1.4226e+09 | -7.8028e+08 |  4.5023e+09
  9.8161e+08 |  5.6451e+08 | -2.9222e+09
 -3.8940e+08 | -2.3316e+08 |  1.0923e+09
  6.7604e+07 |  4.1911e+07 | -1.7897e+08
----------------------------------------
Error using backslash : 4.5152e-01
Error using inv       : 4.1648e+00
Condition number of A : 3.7271e+18
----------------------------------------
----------------------------------------
Hilbert Matrix with n = 15
----------------------------------------
 true x      | backslash   | inv
----------------------------------------
  1.5000e+01 |  1.0461e+01 |  9.9845e+00
 -3.3600e+03 | -1.4988e+03 | -1.3902e+03
  1.8564e+05 |  5.1797e+04 |  4.6098e+04
 -4.4554e+06 | -7.4690e+05 | -6.2201e+05
  5.8198e+07 |  5.4714e+06 |  4.0198e+06
 -4.6559e+08 | -2.1654e+07 | -1.1517e+07
  2.4443e+09 |  4.2474e+07 | -3.2337e+06
 -8.7796e+09 | -1.1580e+07 |  1.2723e+08
  2.2086e+10 | -1.1811e+08 | -4.0846e+08
 -3.9264e+10 |  2.0545e+08 |  6.2623e+08
  4.9080e+10 | -1.0285e+07 | -4.2829e+08
 -4.2185e+10 | -3.5262e+08 | -7.6960e+07
  2.3729e+10 |  4.6207e+08 |  3.4926e+08
 -7.8629e+09 | -2.5468e+08 | -2.2986e+08
  1.1634e+09 |  5.4166e+07 |  5.2158e+07
----------------------------------------
Error using backslash : 1.0002e+00
Error using inv       : 1.0087e+00
Condition number of A : 1.7360e+18
----------------------------------------
```

```
93  ==========================================
94   Part II Output
95  ==========================================
96  ------------------ (1) ------------------
97          s =  0.99999999999999988898
98      |s-1| =  0.00000000000000011102
99  ------------------ (2) ------------------
100         b =  1.00000000000000022204
101     |b-1| =  0.00000000000000022204
102 ------------------ (3) ------------------
103       lhs =  1.42752999999999996561
104       rhs =  1.42753000000000018765
105 |lhs-rhs| =  0.00000000000000022204
106 ------------------ (4) ------------------
107         x =  0.00000000000000000000
108         y =  1.00000000000000000000
109         z =  0.00000000000000000000
110 ------------------ (5) ------------------
111         u =  0.10000000000000008882
112         v =  0.10000000000000000555
113         w =  0.09999999999999997780
114 Comparing u,v,w:
115       u-v =  8.32667e-17
116       v-w =  2.77556e-17
117       w-u = -1.11022e-16
118 diary off
```

SCRIPT FILE: lab_09_script.m

```
% Math 3341, Fall 2021
% Lab 09: Ill-conditioned Matrices and Finite Precision Arithmetic
% Author: Melissa Butler
% Date: 10/18/2021

clc; clear; warning off;

%% 1 Ill-Conditioned Systems
disp('========================================')
disp(' 1 Ill-Conditioned Systems')
disp('========================================')

% 1(f)
n = [9, 11, 13, 15];          % size of desired matrix

for i = 1:length(n)
    lab_09_function(n(i));
end

%% 2 Finite Precision Arithmetic
disp('========================================')
disp(' 2 Finite Precision Arithmetic ')
disp('========================================')

% 2(a)
s = sum(ones(10, 1) * 0.1);
error_1 = abs(s - 1);

% Output for 2(a)
disp('------------------ (a) ------------------')
fprintf('%9s = % .20f \n','s', s)
fprintf('%9s = % .20f \n', '|s-1|', error_1)

% 2(b)
b = 2 - 3 * (4 / 3 - 1);
error_2 = abs(b - 1);

% Output for 2(b)
disp('------------------ (b) ------------------')
fprintf('%9s = % .20f \n','b', b)
fprintf('%9s = % .20f \n', '|b-1|', error_2)

% 2(c)
a = 0.3;
lhs = 0;
for i = 0:5
    lhs = lhs + a ^ i;
end
rhs = (1 - a^6) / (1 - a);
error_3 = abs(lhs-rhs);

% Output for 2(c)
```

```
disp('------------------ (c) ------------------')
fprintf('%9s = % .20f \n', 'lhs', lhs)
fprintf('%9s = % .20f \n', 'rhs', rhs)
fprintf('%9s = % .20f \n', '|lhs-rhs|', error_3)

% 2(d)
x = 1e16 + 1 - 1e16;
y = 1e16 - 1e16 + 1;
z = 1e16 - (1e16 - 1);

% Output for 2(d)
disp('------------------ (d) ------------------')
fprintf('%9s = % .20f \n', 'x', x)
fprintf('%9s = % .20f \n', 'y', y)
fprintf('%9s = % .20f \n', 'z', z)

% 2(e)
u = 1 + 0.1 - 1;
v = 1 - 1 + 0.1;
w = 1 - (1 - 0.1);

% Output for 2(e)
disp('------------------ (e) ------------------')
fprintf('%9s = % .20f \n', 'u', u)
fprintf('%9s = % .20f \n',  'v', v)
fprintf('%9s = % .20f \n', ' w', w)
disp('Comparing u,v,w:')
fprintf('%9s = % g \n', 'u-v', u-v)
fprintf('%9s = % g \n', 'v-w', v-w)
fprintf('%9s = % g \n', 'w-u', w-u)
```

FUNCTION FILE: lab_09_function.m

```
function lab_09_function(n)

% 1(b): Generate an n-by-n Hilbert matrix A
A = hilb(n);

% 1(c): Create an n-by-1 all-one vector b
b = ones(n,1);

% 1(d): Solving the system Ax = b
% Find the exact solution using `invhilb`
x_exact = invhilb(n) * b;
% Find the approximate solution using `\`
x_backslash = A \ b;
% Find approximate solution using `inv`
x_inv = inv(A) * b;

% 1(e): Calculate relative error of each solution
% relative error of solution obtained by `\` in infinity norm
error_backslash = norm(x_backslash - x_exact, Inf) / norm( x_exact, Inf);
% relative error of solution obtained by `inv` in infinity norm
error_inv = norm(x_inv - x_exact, Inf) / norm(x_exact, Inf);

% 1(f): Calculate the condition number of A
cond_A = cond(A,1);

%% Print results
disp('-----------------------------------------')
fprintf('Hilbert Matrix with n = %d \n', n)
disp('-----------------------------------------')
fprintf(' %-11s | %-11s | % -11s \n','exact x','backslash', 'inv')
disp('-----------------------------------------')
fprintf(' % 9.4e | % 9.4e | % 9.4e \n', [x_exact, x_backslash, x_inv]')
disp('-----------------------------------------')
fprintf('Error using backslash : %8.4e\n', error_backslash);
fprintf('Error using inv       : %8.4e\n', error_inv);
fprintf('Condition number of A : %8.4e\n', cond_A);
disp('-----------------------------------------')

end
```