MATH 3341: Introduction to Scientific Computing Lab

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Lab 06: LU Decomposition

The LU Decomposition

Linear System and Its Matrix Form

Consider the system of equations

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -1x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

In matrix form we have the equation $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}}_{\mathbf{b}}$$

The LU decomposition allows us to factor the matrix A into two matrices, a lower triangular matrix L and an upper triangular matrix U. The LU decomposition can be viewed as the matrix form of Gaussian elimination. Computers usually solve square systems of linear equations using the LU decomposition, and it is also a key step when inverting a matrix or computing the determinant of a matrix.

$$
\begin{aligned}
A = LU &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}.
\end{aligned}
$$

Upper and lower triangular systems are easy to solve using forward or backward subsitution algorithms.

Solve the Linear System using LU Decomposition

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   where $\mathbf{z} = U\mathbf{x}$. In MATLAB, use `z = L \ b` to solve for $\mathbf{z}$ in $L\mathbf{z} = \mathbf{b}$.
3. Next, solve for $\mathbf{x}$ in $U\mathbf{x} = \mathbf{z}$, we have

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`lu` factorization.

`[L,U] = lu(A)` stores an upper triangular matrix in U and a "psychologically lower triangular matrix" (i.e. a product of lower triangular and permutation matrices) in L, so that `A = L*U`. A can be rectangular.

\: Backslash or left matrix divide

`A\B` is the matrix division of `A` into `B`, which is roughly the same as `inv(A)*B`, except it is computed in a different way. If `A` is an N-by-N matrix and `B` is a column vector with N components, or a matrix with several such columns, then `X = A\B` is the solution to the equation `A*X = B`.

Norms

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- $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} \{|x_i|\}$.

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- Example:

```
x = -ones(4, 1)              % [-1;-1;-1;-1]
xNorm1 = norm(x, 1)          % 4
xNorm2 = norm(x, 2)          % 2
xNormInf = norm(x, Inf)      % 1
xNorm2 == dot(x, x)^(1/2)    % logical 1 (true)
```