MATH 3341: Introduction to Scientific Computing Lab

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Lab 09: Ill-conditioned Matrices and Finite Precision Arithmetic

Ill-Conditioned Matrices

Vector Norm

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- `norm(A, 'fro')`: $\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$, Frobenius norm.

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- If $\kappa(A) = \infty$, then the matrix A is singular, i.e., the matrix is not invertible.

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- Example

```
A = magic(5);
condA1 = cond(A, 2)
condA2 = norm(A, 2) * norm(inv(A), 2)
condA3 = max(sqrt(eig(A'*A))) * max(sqrt(eig(inv(A'*A))
condA4 = max(sqrt(eig(A'*A))) / min(sqrt(eig(A'*A)))
```

Ill-Conditioned Matrix: Hilbert Matrix

A Hilbert matrix is a square matrix with elements defined by

$$H_{ij} = \frac{1}{i+j-1}.$$

For example, a 3×3 Hilbert matrix is

$$H_{3\times 3} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

Note that this matrix is symmetric and positive definite.

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- Example:

```
H = hilb(10);
invH1 = inv(H);
invH2 = invhilb(10);
norm(invH1 - invH2)
```

Ill-Conditioned Linear System

A linear system $A\mathbf{x} = \mathbf{b}$ is said to be ill-conditioned if A is a ill-conditioned matrix. The typical numerical methods for solving linear systems such as Jacobi method, Gauss-Seidel method would become unreliable. Example: $H\mathbf{x} = \mathbf{b} \implies \mathbf{x} = H^{-1}\mathbf{b}$.

```
n = 10;
H = hilb(n);
invH = invhilb(n);
b = rand(n, 1);
x = invH * b;
x1 = inv(H) * b;
x2 = H \ b;
norm(x - x1)
norm(x - x2)
```

Finite Precision Arithmetic

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Computers can only store values up to a certain level of accuracy. Past this level, the computer will round values, thus causes the round-off error. What this means is that arithmetic does not work exactly as we expect. Namely, arithmetic is no longer commutative, associative, or distributive. The lab exercises will demonstrate some of the issues that arise.

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https://babbage.cs.qc.cuny.edu/IEEE-754/