Trees

Chapter 11

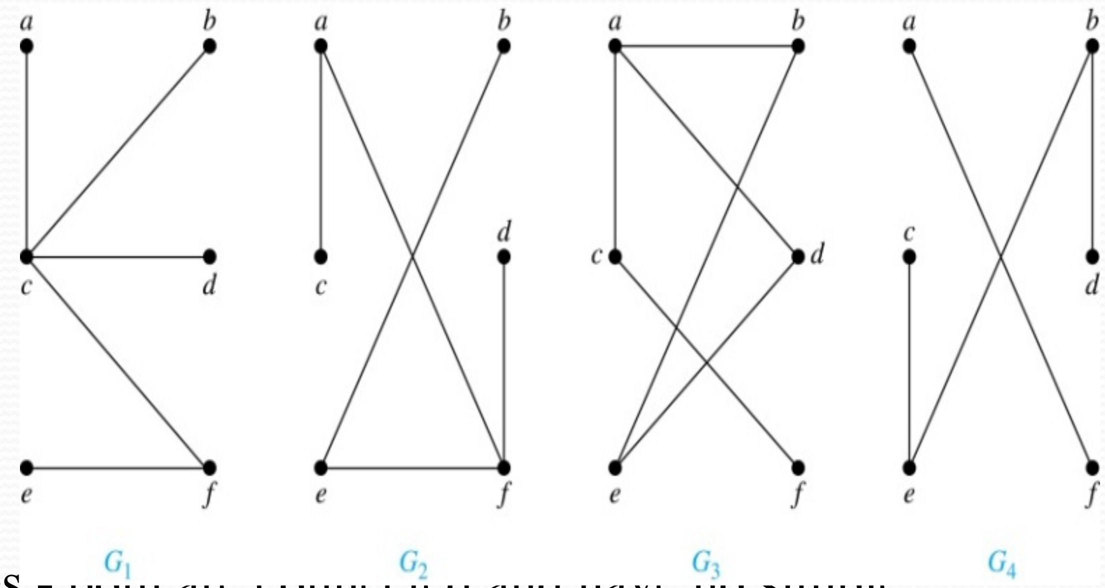
Chapter Summary

- Introduction to Trees
- Applications of Trees (*not currently included in overheads*)
- Tree Traversal
- Spanning Trees
- Minimum Spanning Trees (*not currently included in overheads*)

Trees

Definition: A *tree* is a connected undirected graph with no simple circuits.

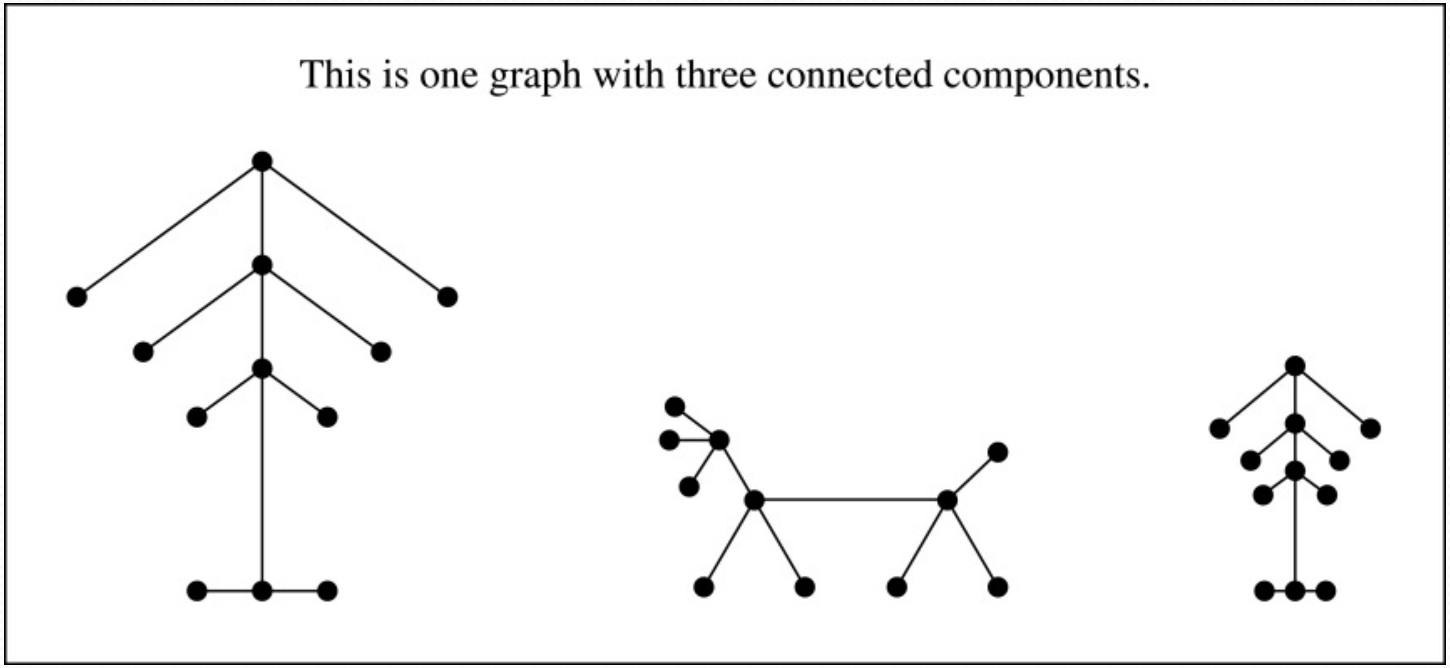
Example: Which of these graphs are trees?



Solution: G_1 and G_2 are trees - both are connected and have no simple circuits. Because e, b, a, d, e is a simple circuit, G_3 is not a tree. G_4 is not a tree because it is not connected.

Trees

Definition: A *forest* is a graph that has no simple circuit, but is not connected. Each of the connected components in a forest is a tree.



Trees (*continued*)

Theorem: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Proof: Assume that T is a tree. Then T is connected with no simple circuits. Hence, if x and y are distinct vertices of T, there is a simple path between them. This path must be unique - for if there were a second path, there would be a simple circuit in T. Hence, there is a unique simple path between any two vertices of a tree.

Now assume that there is a unique simple path between any two vertices of a graph T. Then T is connected because there is a path between any two of its vertices. Furthermore, T can have no simple circuits since if there were a simple circuit, there would be two paths between some two vertices.

Hence, a graph with a unique simple path between any two vertices is a tree.

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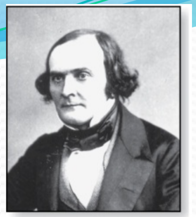
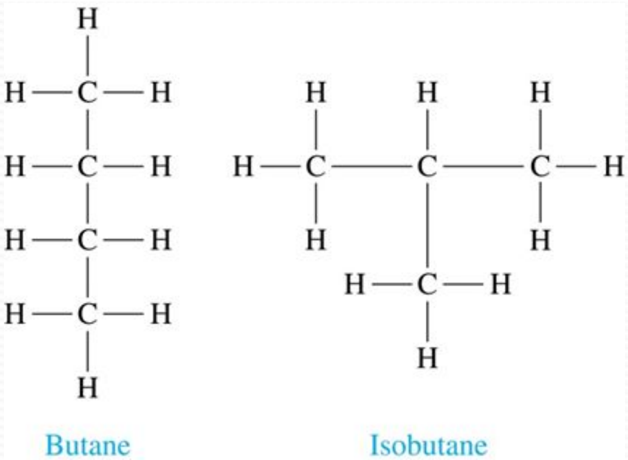
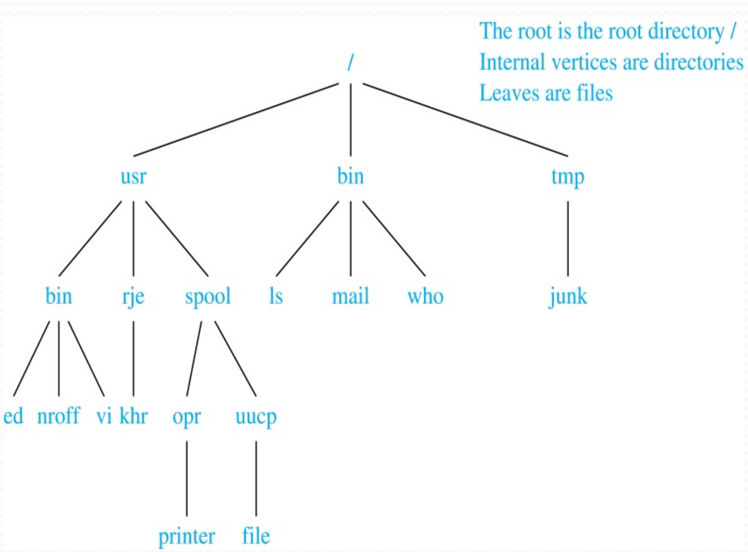
Arthur Cayley
(1821-1895)

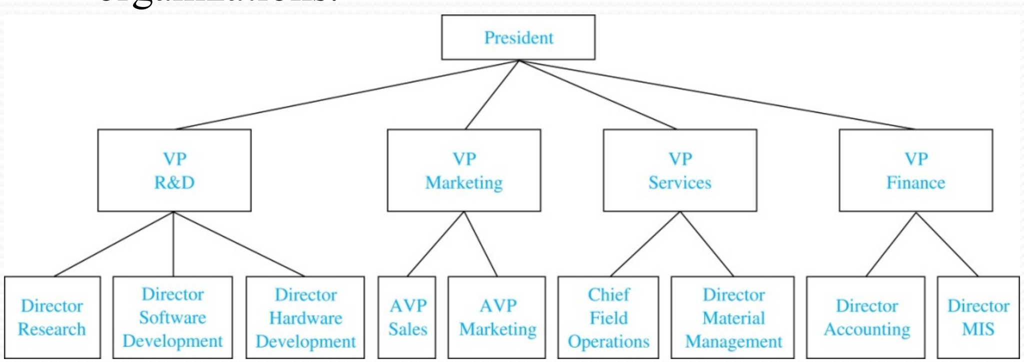


Trees as Models

- Trees are used as models in computer science, chemistry, geology, botany, psychology, and many other areas.
- Trees were introduced by the mathematician Cayley in 1857 in his work counting the number of isomers of saturated hydrocarbons. The two isomers of butane are shown at the right.
- The organization of a computer file system into directories, subdirectories, and files is naturally represented as a tree.
- Trees are used to represent the structure of organizations.



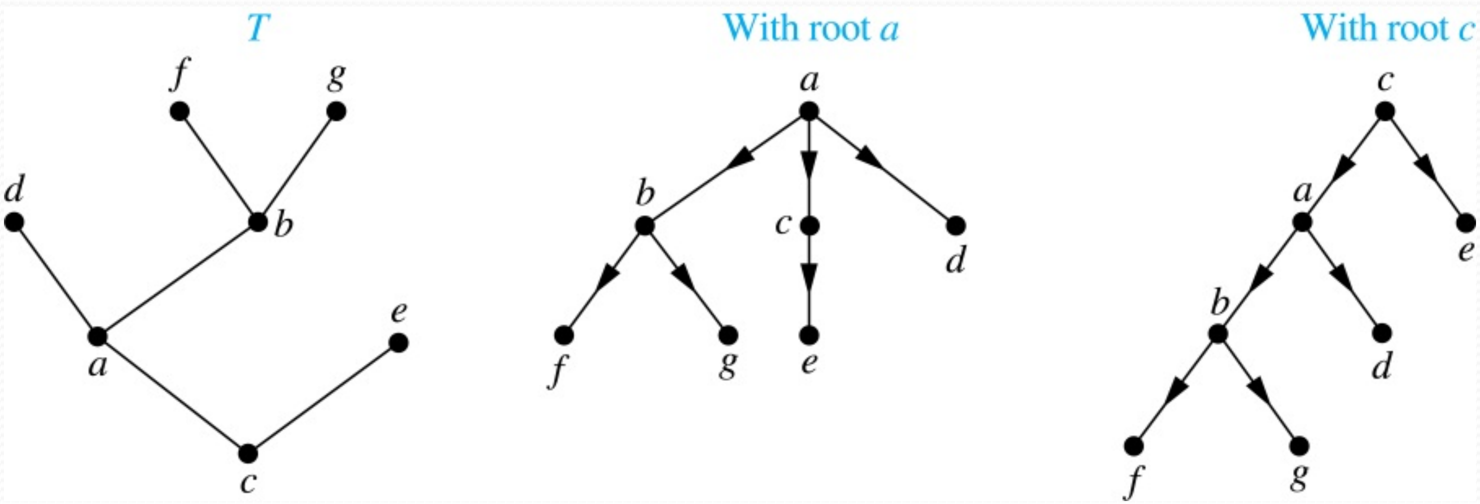




Rooted Trees

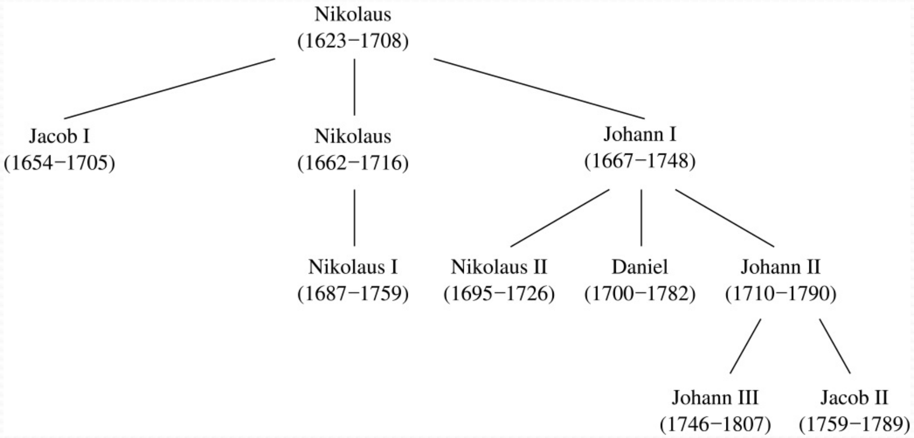
Definition: A *rooted tree* is a tree in which one vertex has been designated as the *root* and every edge is directed away from the root.

An unrooted tree is converted into different rooted trees when different vertices are chosen as the root.



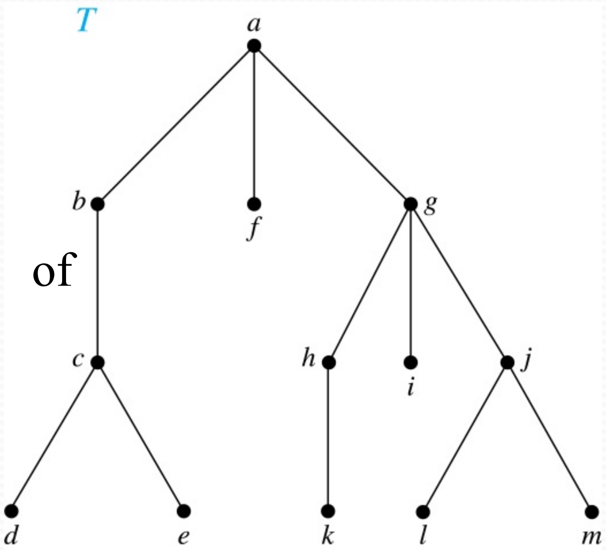
Rooted Tree Terminology

- Terminology for rooted trees is a mix from botany and genealogy (such as this family tree of the Bernoulli family of mathematicians).



- If v is a vertex of a rooted tree other than the root, the *parent* of v is the unique vertex u such that there is a directed edge from u to v. When u is a parent of v, v is called a *child* of u. Vertices with the same parent are called *siblings*.
- The *ancestors* of a vertex are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root. The *descendants* of a vertex v are those vertices that have v as an ancestor.
- A vertex of a rooted tree with no children is called a *leaf*. Vertices that have children are called *internal vertices*.
- If a is a vertex in a tree, the *subtree* with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.

Terminology for Rooted Trees



Example: In the rooted tree *T* (with root *a*):

(i) Find the parent of *c*, the children of *g*, the siblings of *h*, the ancestors of *e*, and the descendants of *b*.

(ii) Find all internal vertices and all leaves.

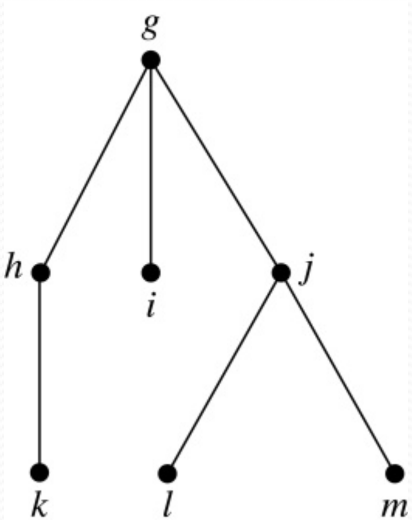
(iii) What is the subtree rooted at *G*?

Solution:

(i) The parent of *c* is *b*. The children of *g* are *h*, *i*, and *j*. The siblings of *h* are *i* and *j*. The ancestors of *e* are c, *b*, and *a*. The descendants of *b* are *c*, *d*, and *e*.

(ii) The internal vertices are *a*, *b*, *c*, *g*, *h*, and *j*. The leaves are *d*, *e*, *f*, *i*, *k*, *l*, and *m*.

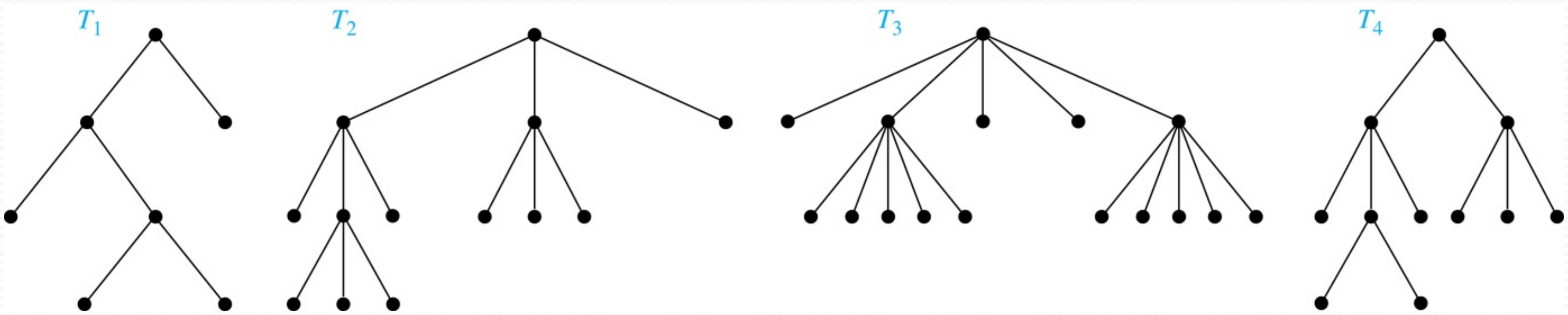
(iii) We display the subtree rooted at *g*.



m-ary Rooted Trees

Definition: A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. An m-ary tree with $m = 2$ is called a *binary* tree.

Example: Are the following rooted trees full m-ary trees for some positive integer m?



Solution: T_1 is a full binary tree because each of its internal vertices has two children. T_2 is a full 3-ary tree because each of its internal vertices has three children. In T_3 each internal vertex has five children, so T_3 is a full 5-ary tree. T_4 is not a full m-ary tree for any m because some of its internal vertices have two children and others have three children.

Ordered Rooted Trees

Definition: An *ordered rooted tree* is a rooted tree where the children of each internal vertex are ordered.

- We draw ordered rooted trees so that the children of each internal vertex are shown in order from left to right.

Definition: A *binary tree* is an ordered rooted where where each internal vertex has at most two children. If an internal vertex of a binary tree has two children, the first is called the *left child* and the second the *right child*. The tree rooted at the left child of a vertex is called the *left subtree* of this vertex, and the tree rooted at the right child of a vertex is called the *right subtree* of this vertex.

Ordered Rooted Trees

Example: Consider the binary tree T.

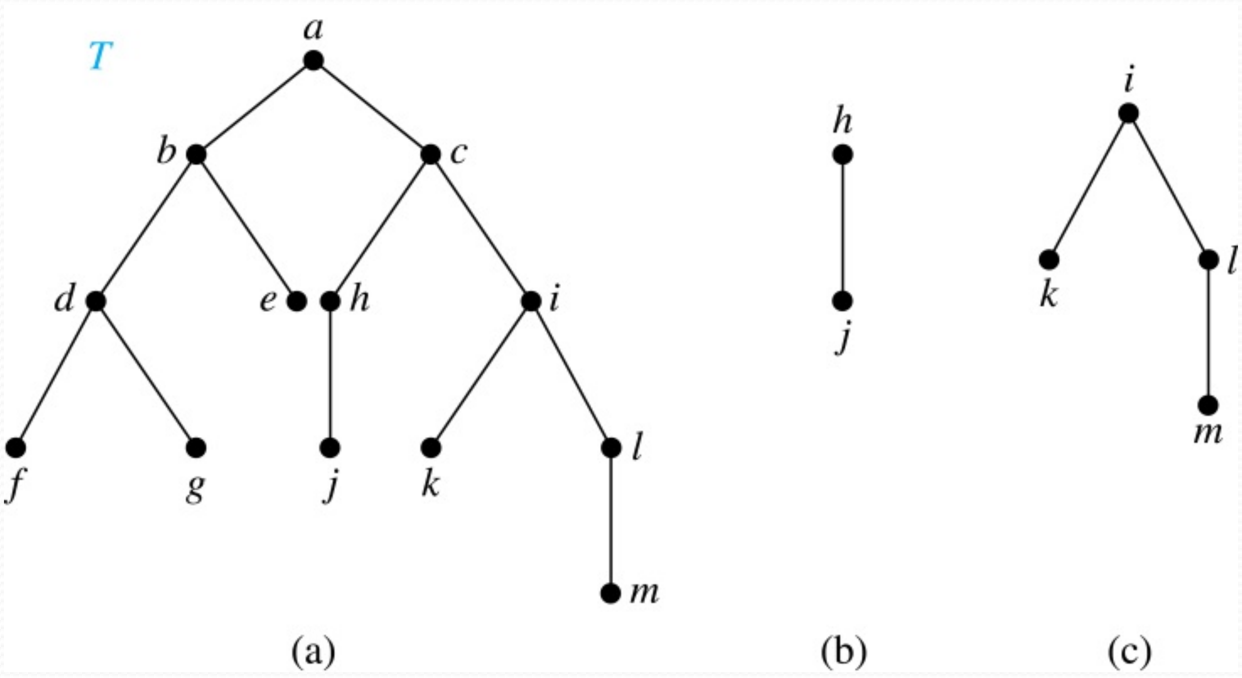
(i) What are the left and right children of d?

(ii) What are the left and right subtrees of c?

Solution:

(i) The left child of d is f and the right child is g.

(ii) The left and right subtrees of c are displayed in (b) and (c).



Properties of Trees

Theorem 2: A tree with n vertices has $n - 1$ edges.

***Proof* (*by mathematical induction*):**

BASIS STEP: When $n = 1$, a tree with one vertex has no edges. Hence, the theorem holds when $n = 1$.

INDUCTIVE STEP: Assume that every tree with k vertices has $k - 1$ edges.

Suppose that a tree T has $k + 1$ vertices and that v is a leaf of T. Let w be the parent of v. Removing the vertex v and the edge connecting w to v produces a tree T' with k vertices. By the inductive hypothesis, T' has $k - 1$ edges. Because T has one more edge than T', we see that T has k edges. This completes the inductive step. ◀

Counting Vertices in Full *m*-Ary Trees

Theorem 3: A full m-ary tree with i internal vertices has $n = mi + 1$ vertices.

Proof : Every vertex, except the root, is the child of an internal vertex. Because each of the i internal vertices has m children, there are mi vertices in the tree other than the root. Hence, the tree contains $n = mi + 1$ vertices.

◀

Counting Vertices in Full *m*-Ary Trees (*continued*)

Theorem 4: A full m-ary tree with

(*i*) n vertices has $i = (n - 1)/m$ internal vertices and $l = [(m - 1)n + 1]/m$ leaves,

(*ii*) i internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves,

(*iii*) l leaves has $n = (ml - 1)/(m - 1)$ vertices and $i = (l - 1)/(m - 1)$ internal vertices.

proofs of parts (ii) and (iii) are left as exercises

Proof **(*of part i*):** Solving for i in $n = mi + 1$ (from Theorem 3) gives $i = (n - 1)/m$. Since each vertex is either a leaf or an internal vertex, $n = l + i$. By solving for l and using the formula for i, we see that

$$l = n - i = n - (n - 1)/m = [(m - 1)n + 1]/m .$$

◀

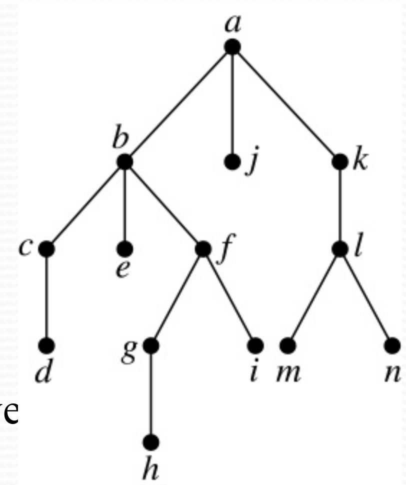
Level of vertices and height of trees

- When working with trees, we often want to have rooted trees where the subtrees at each vertex contain paths of approximately the same length.
- To make this idea precise we need some definitions:
  - The *level* of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
  - The *height* of a rooted tree is the maximum of the levels of the vertices.

Example:

(*i*) Find the level of each vertex in the tree to the right.

(*ii*) What is the height of the tree?



Solution:

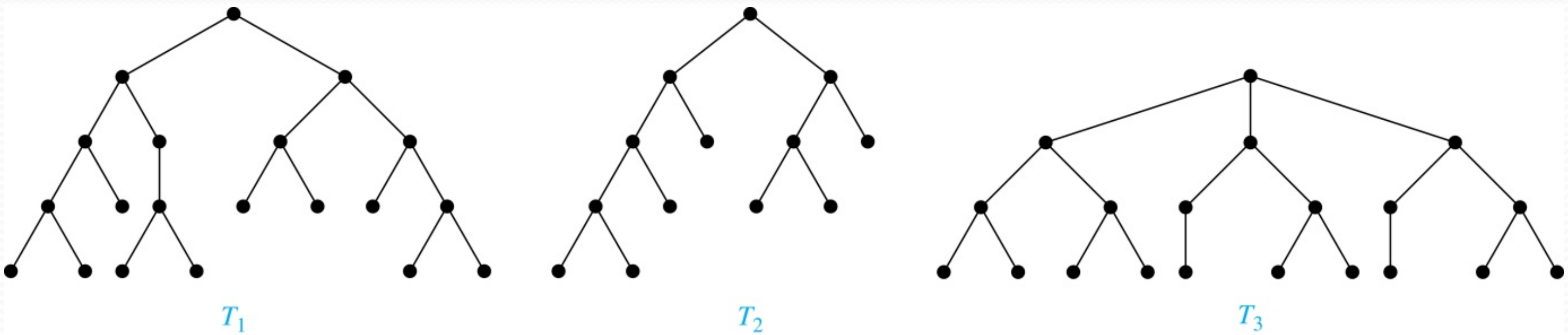
(*i*) The root a is at level 0. Vertices b, j, and k are at level 1. Vertices c, e, f, and l are at level 2. Vertices d, g, i, m, and n are at leve Vertex h is at level 4.

(*ii*) The height is 4, since 4 is the largest level of any vertex.

Balanced *m*-Ary Trees

Definition: A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h - 1$.

Example: Which of the rooted trees shown below is balanced?



Solution: T_1 and T_3 are balanced, but T_2 is not because it has leaves at levels 2, 3, and 4.

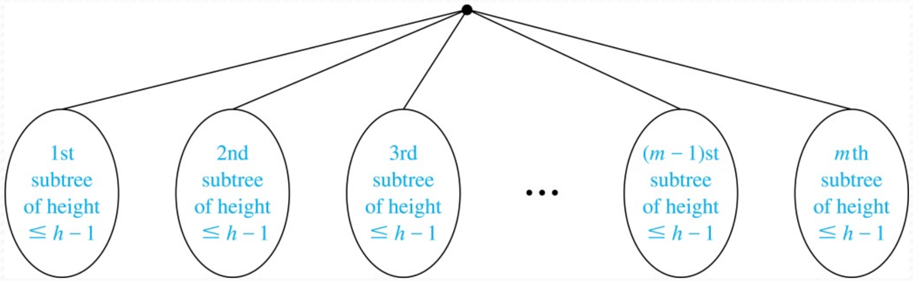
The Bound for the Number of Leaves in an m-Ary Tree

Theorem 5: There are at most m^h leaves in an m-ary tree of height h.

***Proof* (*by mathematical induction on height*):**

BASIS STEP: Consider an m-ary trees of height 1. The tree consists of a root and no more than m children, all leaves. Hence, there are no more than $m^1 = m$ leaves in an m-ary tree of height 1.

INDUCTIVE STEP: Assume the result is true for all m-ary trees of height $< h$. Let T be an m-ary tree of height h. The leaves of T are the leaves of the subtrees of T we get when we delete the edges from the root to each of the vertices of level 1.



Each of these subtrees has height $\leq h-1$. By the inductive hypothesis, each of these subtrees has at most m^{h-1} leaves. Since there are at most m such subtees, there are at most $m \cdot m^{h-1} = m^h$ leaves in the tree. ◀

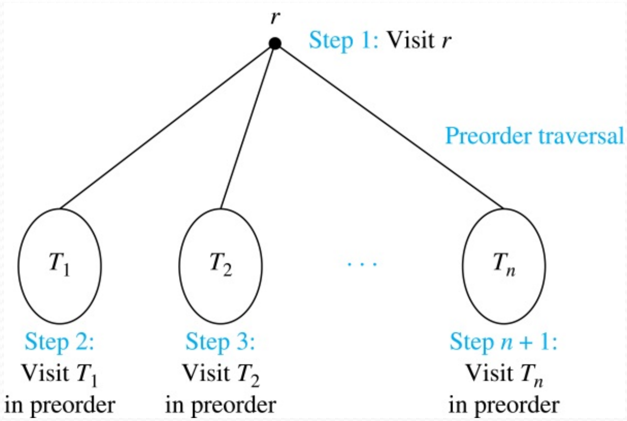
Corollary 1: If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$. If the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$. (*see text for the proof*)

Tree Traversal

- Procedures for systematically visiting every vertex of an ordered tree are called *traversals*.
- The three most commonly used *traversals* are *preorder traversal*, *inorder traversal*, and *postorder traversal*.

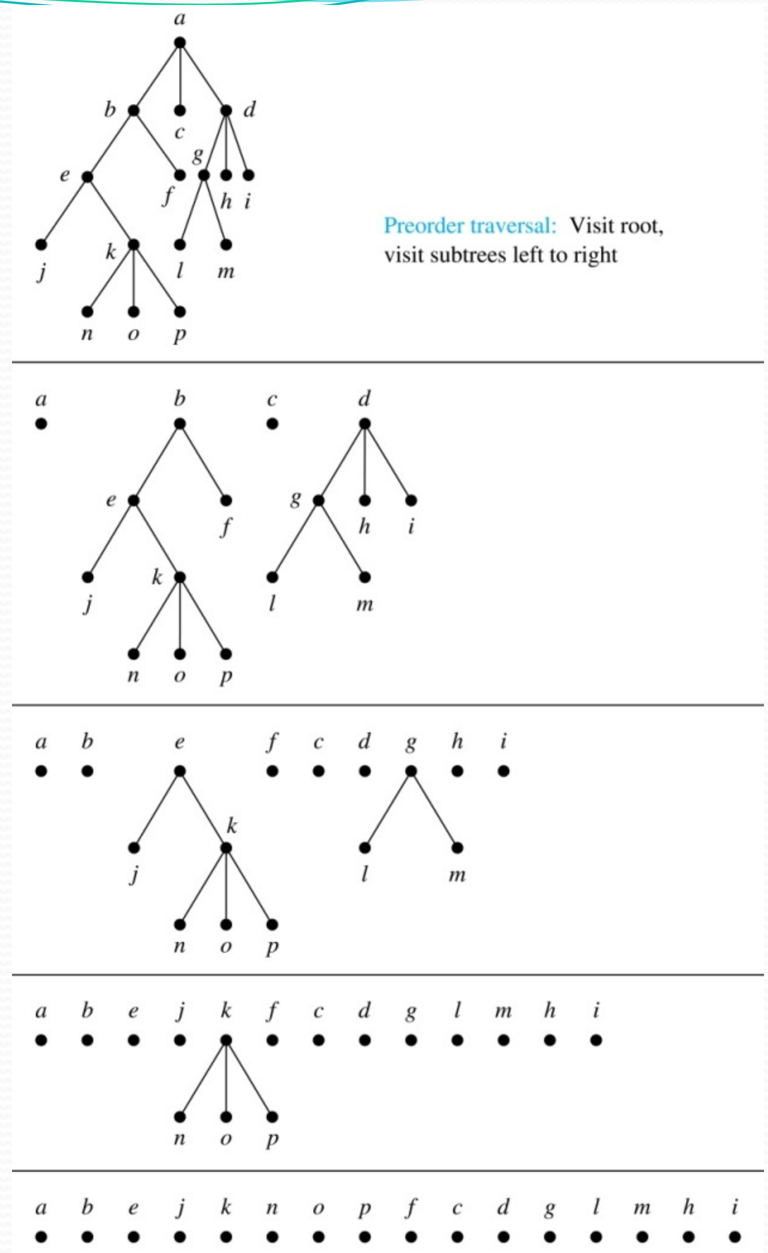
Preorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The preorder traversal begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.



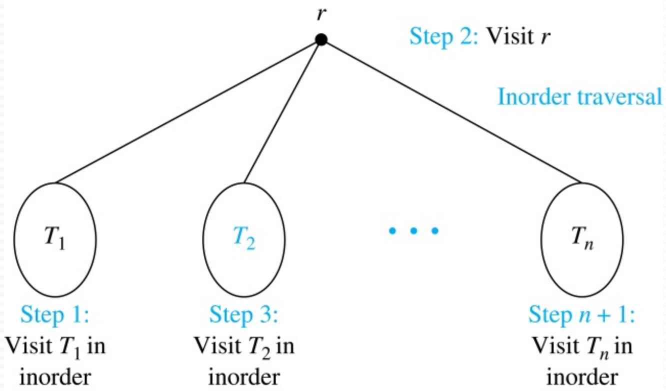
Preorder Traversal

procedure *preorder* (*T*: ordered rooted tree)
r := root of *T*
list *r*
for each child *c* of *r* from left to right
    T(*c*) := subtree with *c* as root
    preorder(*T*(*c*))



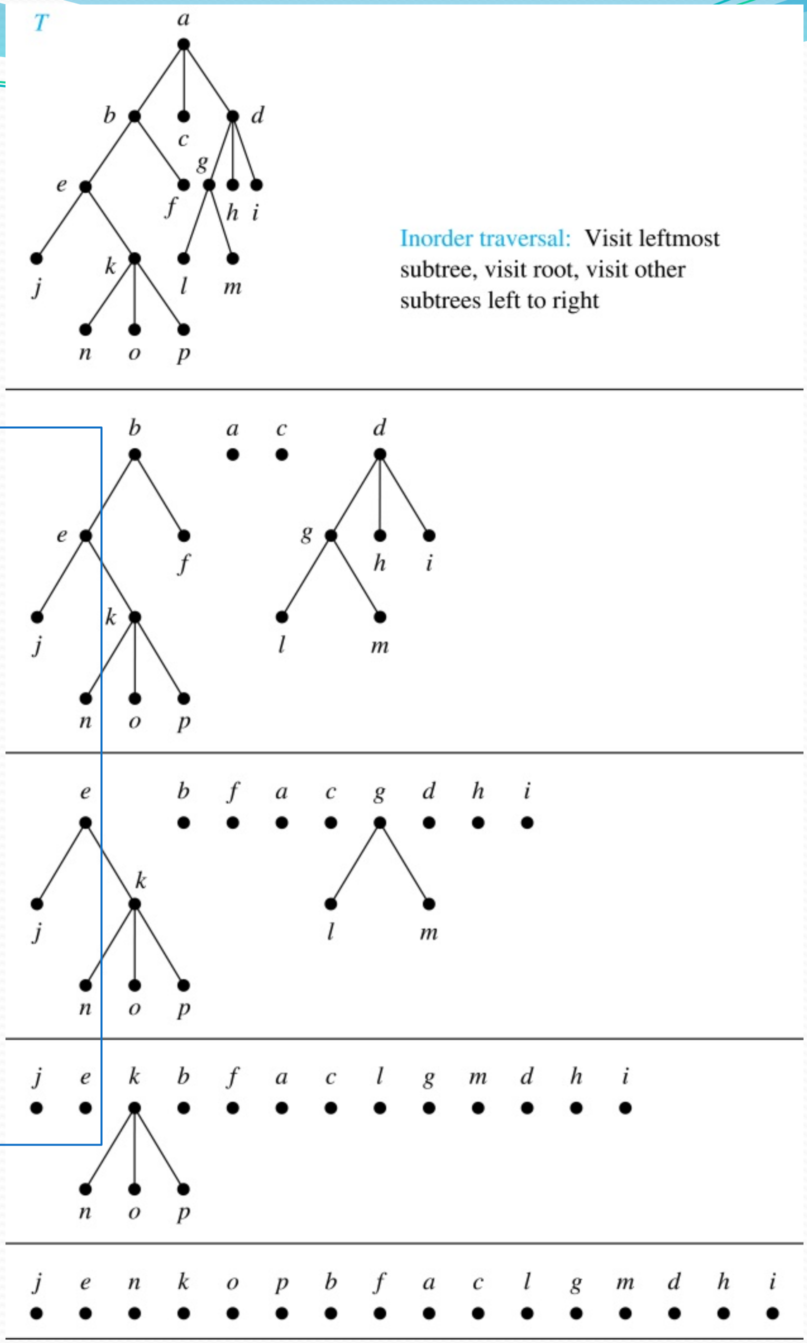
Inorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The inorder traversal begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.



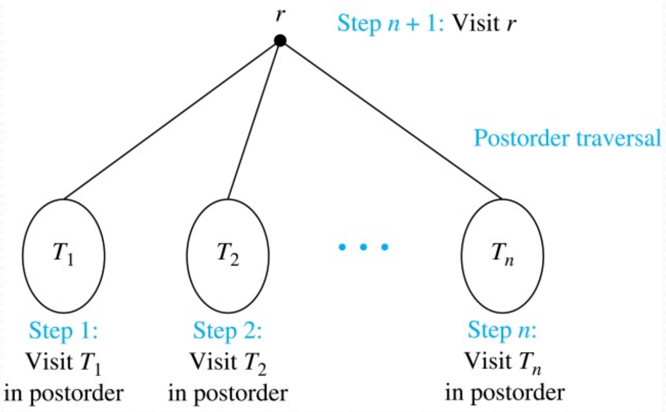
Inorder Traversal

procedure *inorder* (*T*: ordered rooted tree)
r := root of *T*
if *r* is a leaf **then** list *r*
else
    l := first child of *r* from left to right
    T(*l*) := subtree with *l* as its root
    inorder(*T*(*l*))
    list(*r*)
    for each child *c* of *r* from left to right
        T(*c*) := subtree with *c* as root
        inorder(*T*(*c*))



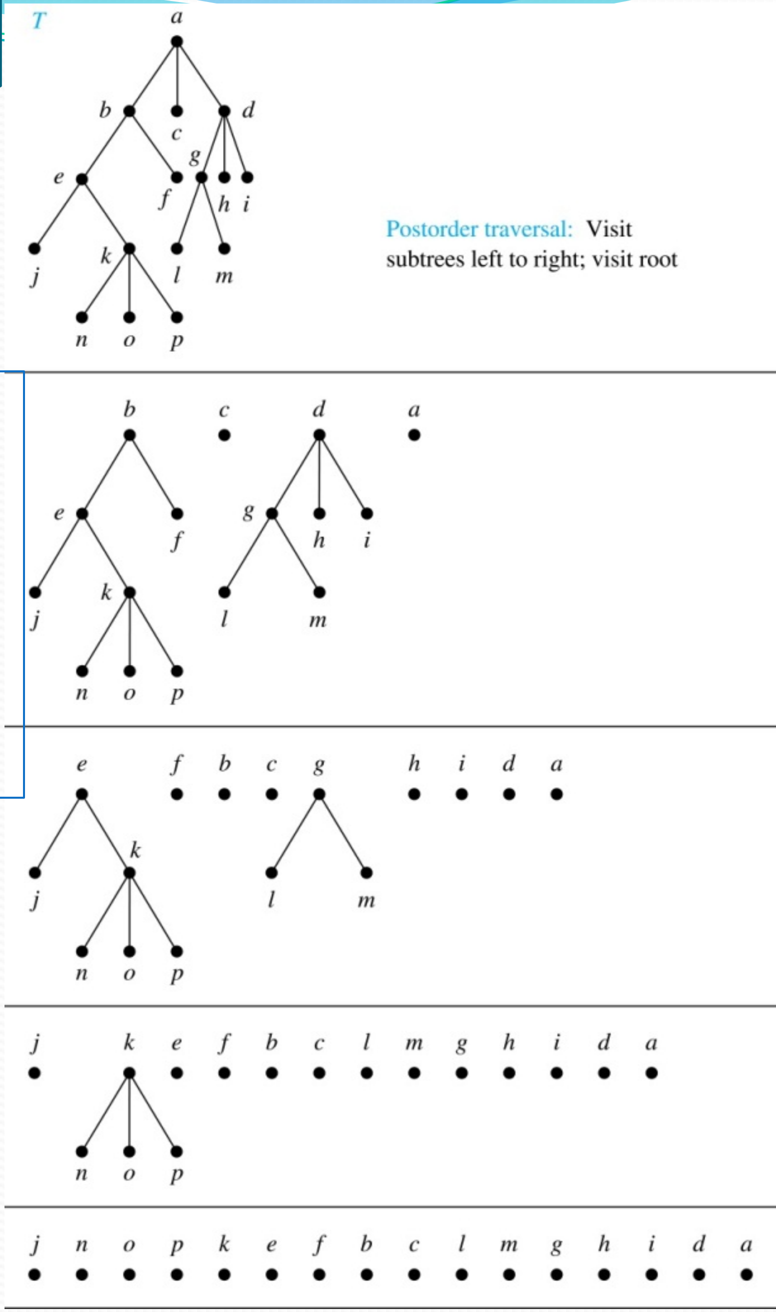
Postorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The postorder traversal begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.



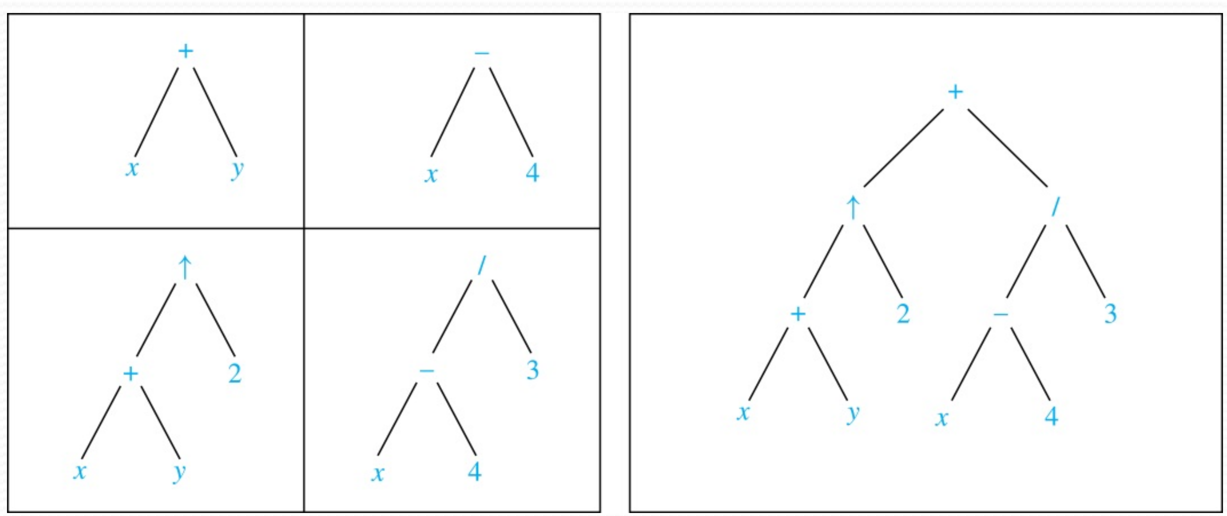
Postorder Traversal (*continued*)

procedure *postordered* (*T*: ordered rooted tree)
r := root of *T*
for each child *c* of *r* from left to right
    T(*c*) := subtree with *c* as root
    postorder(*T*(*c*))
list *r*



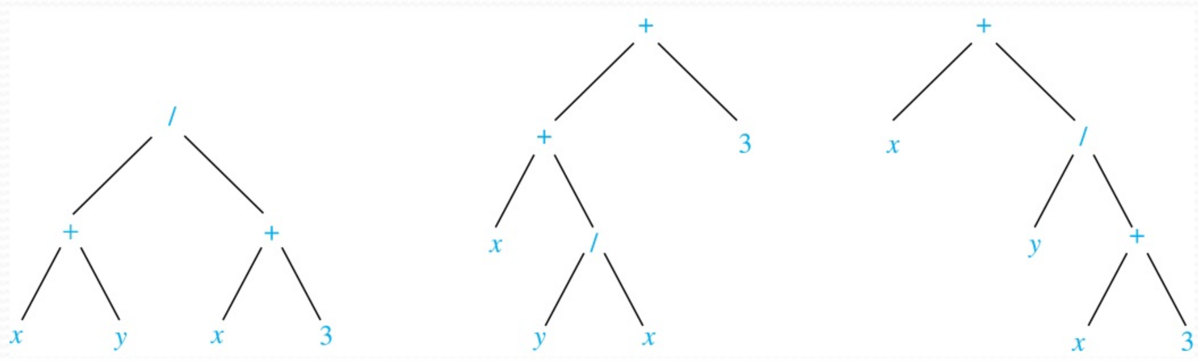
Expression Trees

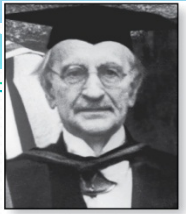
- Complex expressions can be represented using ordered rooted trees.
- Consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$.
- A binary tree for the expression can be built from the bottom up, as is illustrated here.



Infix Notation

- An inorder traversal of the tree representing an expression produces the original expression when parentheses are included except for unary operations, which now immediately follow their operands.
- We illustrate why parentheses are needed with an example that displays three trees all yield the same infix representation.



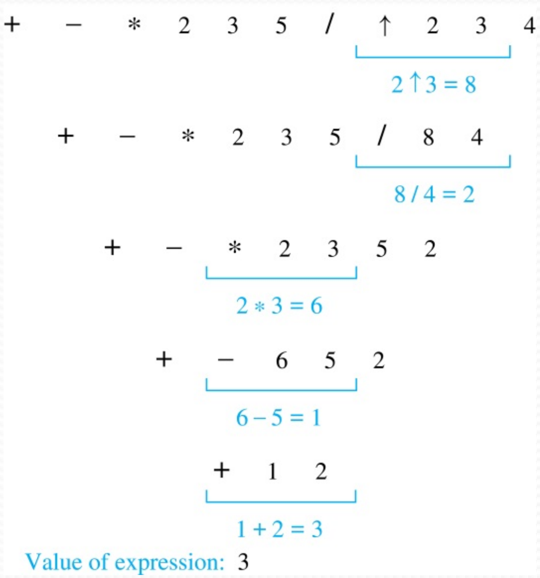


Jan Łukasiewicz
(1878-1956)

Prefix Notation

- When we traverse the rooted tree representation of an expression in preorder, we obtain the *prefix* form of the expression. Expressions in prefix form are said to be in *Polish notation*, named after the Polish logician Jan Łukasiewicz.
- Operators precede their operands in the prefix form of an expression. Parentheses are not needed as the representation is unambiguous.
- The prefix form of $((x + y) \uparrow 2) + ((x - 4)/3)$ is $+ \uparrow + x\ y\ 2\ / - x\ 4\ 3$.
- Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the corresponding operation with the two operations to the right.

Example: We show the steps used to evaluate a particular prefix expression:



Postfix Notation

- We obtain the *postfix form* of an expression by traversing its binary trees in postorder. Expressions written in postfix form are said to be in *reverse Polish notation*.
- Parentheses are not needed as the postfix form is unambiguous.
- $x\ y + 2 \uparrow x\ 4 - 3\ / +$ is the postfix form of $((x + y) \uparrow 2\ ) + ((x - 4)/3)$.
- A binary operator follows its two operands. So, to evaluate an expression one works from left to right, carrying out an operation represented by an operator on its preceding operands.

Example: We show the steps used to evaluate a particular postfix expression.

$$7\ \underbrace{2\ \ 3\ \ *}_{2 * 3 = 6}\ -\ \ 4\ \uparrow\ 9\ \ 3\ \ /\ +$$

$$\underbrace{7\ \ 6\ -}_{7 - 6 = 1}\ 4\ \uparrow\ 9\ \ 3\ \ /\ +$$

$$\underbrace{1\ \ 4\ \uparrow}_{1^4 = 1}\ 9\ \ 3\ \ /\ +$$

$$1\ \underbrace{9\ \ 3\ \ /}_{9/3 = 3}\ +$$

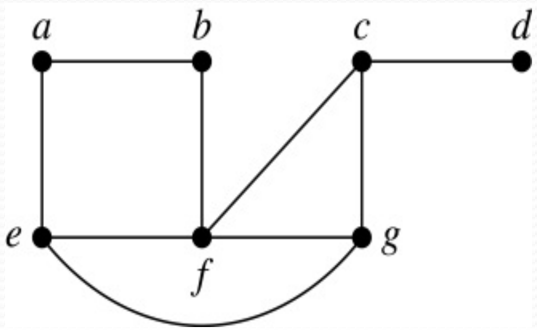
$$\underbrace{1\ \ 3\ +}_{1 + 3 = 4}$$

Value of expression: 4

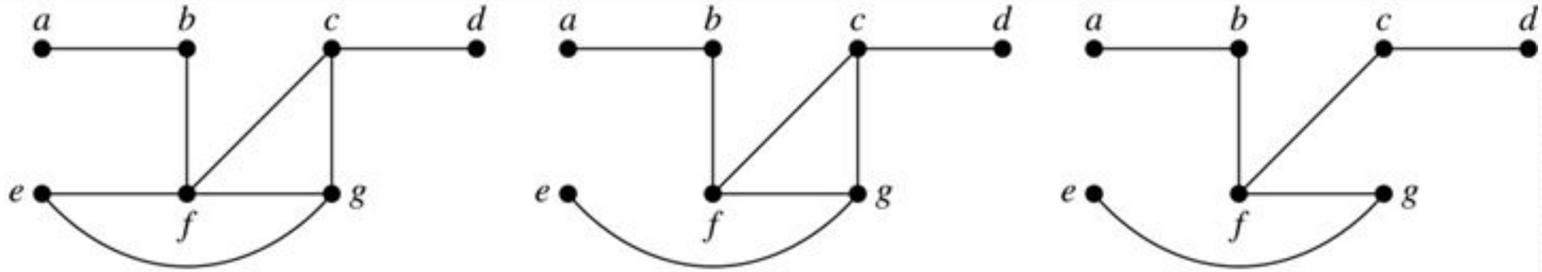
Spanning Trees

Definition: Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

Example: Find the spanning tree of this simple graph:



Solution: The graph is connected, but is not a tree because it contains simple circuits. Remove the edge $\{a, e\}$. Now one simple circuit is gone, but the remaining subgraph still has a simple circuit. Remove the edge $\{e, f\}$ and then the edge $\{c, g\}$ to produce a simple graph with no simple circuits. It is a spanning tree, because it contains every vertex of the original graph.



Edge removed: $\{a, e\}$ (a)

$\{e, f\}$ (b)

$\{c, g\}$ (c)

Spanning Trees (*continued*)

Theorem: A simple graph is connected if and only if it has a spanning tree.

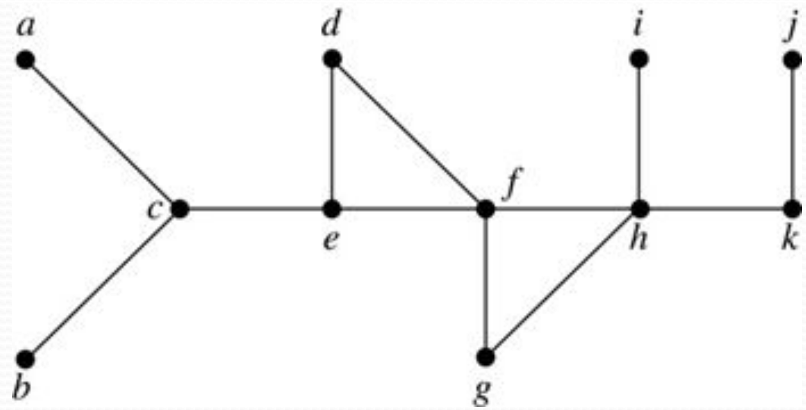
Proof: Suppose that a simple graph G has a spanning tree T. T contains every vertex of G and there is a path in T between any two of its vertices. Because T is a subgraph of G, there is a path in G between any two of its vertices. Hence, G is connected.

Now suppose that G is connected. If G is not a tree, it contains a simple circuit. Remove an edge from one of the simple circuits. The resulting subgraph is still connected because any vertices connected via a path containing the removed edge are still connected via a path with the remaining part of the simple circuit. Continue in this fashion until there are no more simple circuits. A tree is produced because the graph remains connected as edges are removed. The resulting tree is a spanning tree because it contains every vertex of G.

Depth-First Search

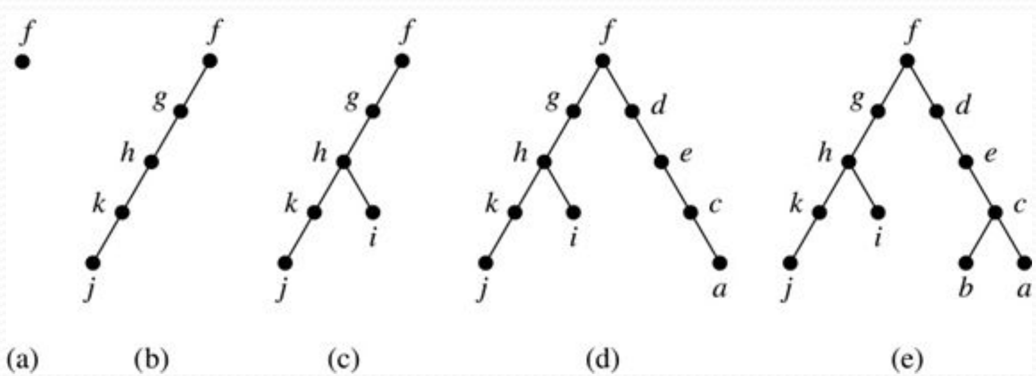
- To use *depth-first search* to build a spanning tree for a connected simple graph first arbitrarily choose a vertex of the graph as the root.
  - Form a path starting at this vertex by successively adding vertices and edges, where each new edge is incident with the last vertex in the path and a vertex not already in the path. Continue adding vertices and edges to this path as long as possible.
  - If the path goes through all vertices of the graph, the tree consisting of this path is a spanning tree.
  - Otherwise, move back to the next to the last vertex in the path, and if possible, form a new path starting at this vertex and passing through vertices not already visited. If this cannot be done, move back another vertex in the path.
  - Repeat this procedure until all vertices are included in the spanning tree.

Depth-First Search



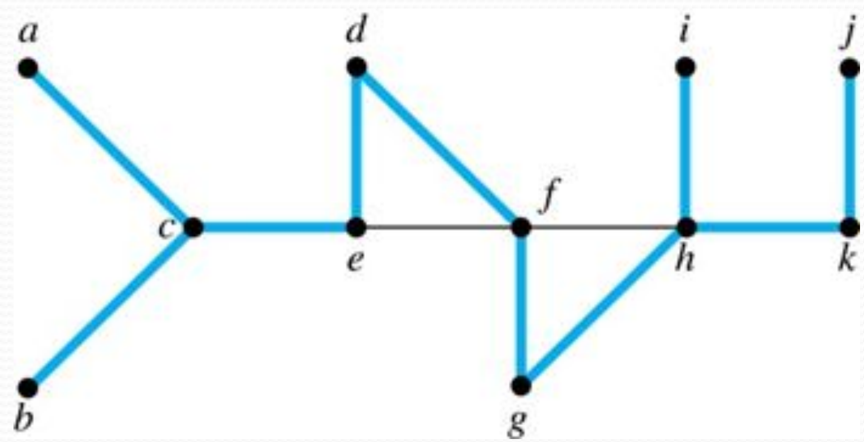
Example: Use depth-first search to find a spanning tree of this graph.

Solution: We start arbitrarily with vertex *f*. We build a path by successively adding an edge that connects the last vertex added to the path and a vertex not already in the path, as long as this is possible. The result is a path that connects *f*, *g*, *h*, *k*, and *j*. Next, we return to *k*, but find no new vertices to add. So, we return to *h* and add the path with one edge that connects *h* and *i*. We next return to *f*, and add the path connecting *f*, *d*, *e*, *c*, and *a*. Finally, we return to *c* and add the path connecting *c* and *b*. We now stop because all vertices have been added.



Depth-First Search (*continued*)

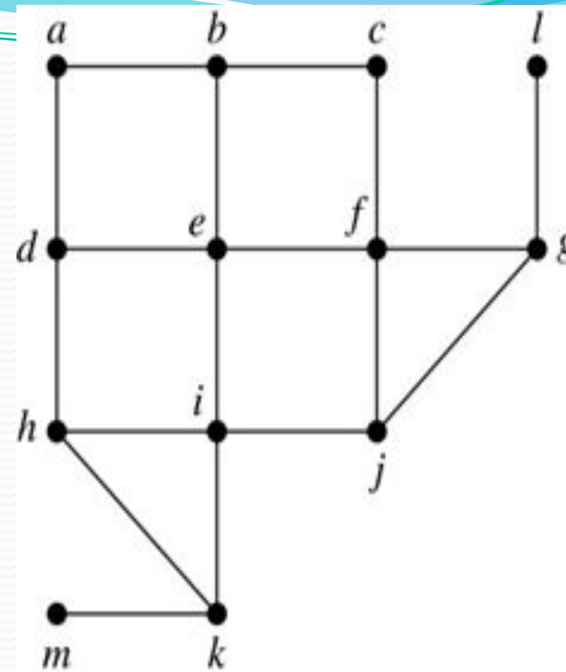
- The edges selected by depth-first search of a graph are called *tree edges*. All other edges of the graph must connect a vertex to an ancestor or descendant of the vertex in the graph. These are called *back edges*.
- In this figure, the tree edges are shown with heavy blue lines. The two thin black edges are back edges.



Breadth-First Search

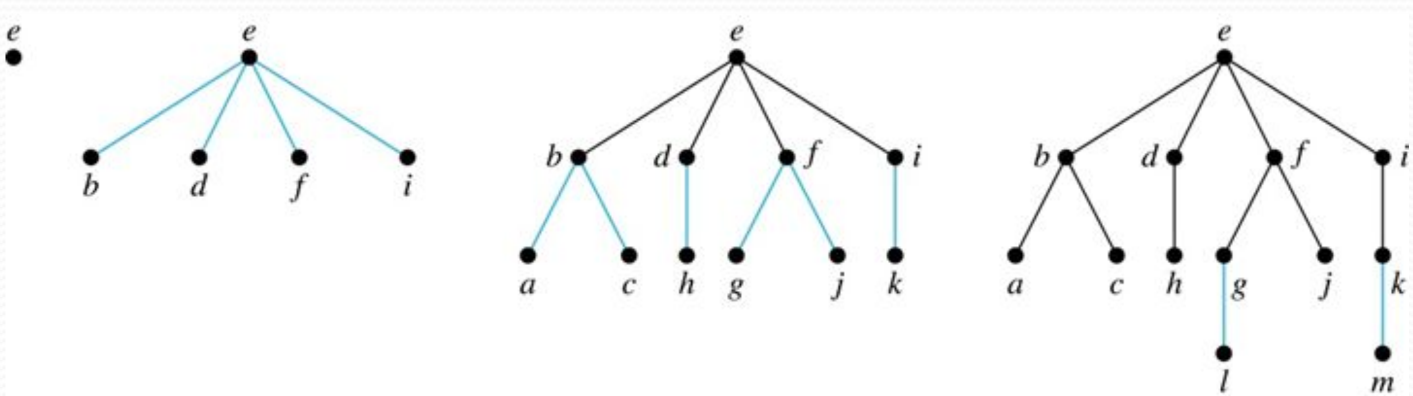
- We can construct a spanning tree using *breadth-first search*. We first arbitrarily choose a root from the vertices of the graph.
  - Then we add all of the edges incident to this vertex and the other endpoint of each of these edges. We say that these are the vertices at level 1.
  - For each vertex added at the previous level, we add each edge incident to this vertex, as long as it does not produce a simple circuit. The new vertices we find are the vertices at the next level.
  - We continue in this manner until all the vertices have been added and we have a spanning tree.

Breadth-First Search



Example: Use breadth-first search to find a spanning tree for this graph.

Solution: We arbitrarily choose vertex *e* as the root. We then add the edges from *e* to *b*, *d*, *f*, and *i*. These four vertices make up level 1 in the tree. Next, we add the edges from *b* to *a* and *c*, the edges from *d* to *h*, the edges from *f* to *j* and *g*, and the edge from *i* to *k*. The endpoints of these edges not at level 1 are at level 2. Next, add edges from these vertices to adjacent vertices not already in the graph. So, we add edges from *g* to *l* and from *k* to *m*. We see that level 3 is made up of the vertices *l* and *m*. This is the last level because there are no new vertices to find.



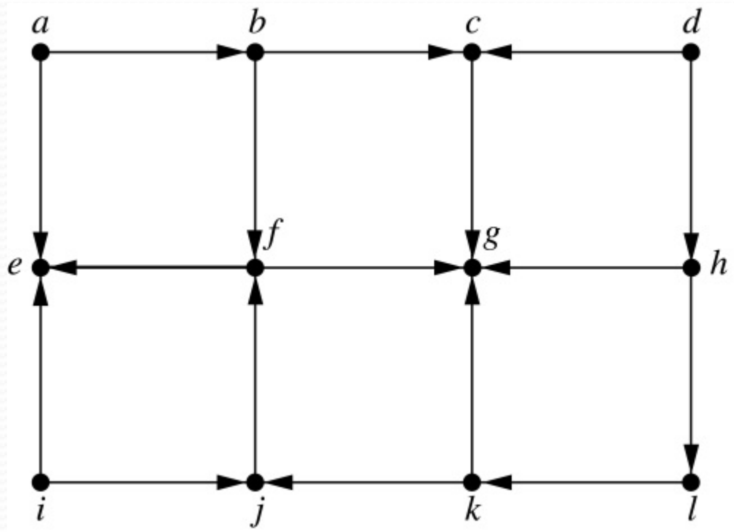
Depth-First Search in Directed Graphs

- Both depth-first search and breadth-first search can be easily modified to run on a directed graph. But the result is not necessarily a spanning tree, but rather a spanning forest.

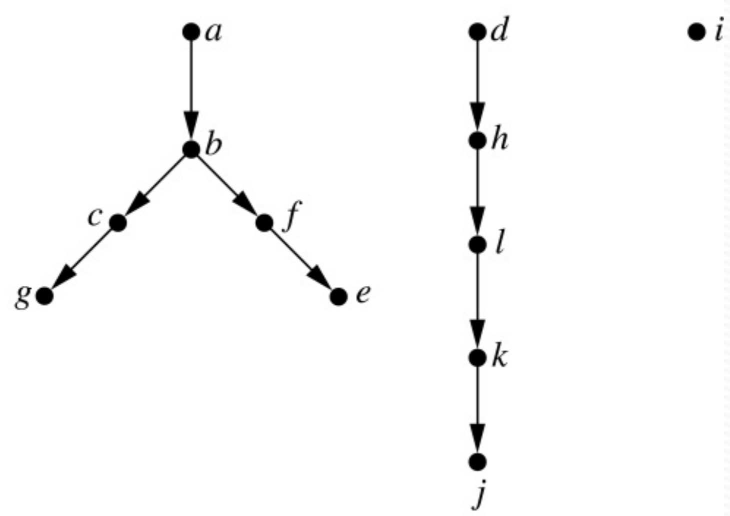
- To index websites, search engines such as Google systematically explore the web starting at known sites. The programs that do this exploration are known as *Web spiders*. They may use both breath-first search or depth-first search to explore the Web graph.

Depth-First Search in Directed Graphs

Example: For the graph in (a), if we begin at vertex *a*, depth-first search adds the path connecting *a*, *b*, *c*, and *g*. At *g*, we are blocked, so we return to *c*. Next, we add the path connecting *f* to *e*. Next, we return to *a* and find that we cannot add a new path. So, we begin another tree with *d* as its root. We find that this new tree consists of the path connecting the vertices *d*, *h*, *l*, *k*, and *j*. Finally, we add a new tree, which only contains *i*, its root.



(a)



(b)

Applications of Trees – Binary Search Tree

Example: Form a binary search tree for the words *mathematics, physics, geography, zoology, meteorology, geology, psychology*, and *chemistry* (using alphabetical order).

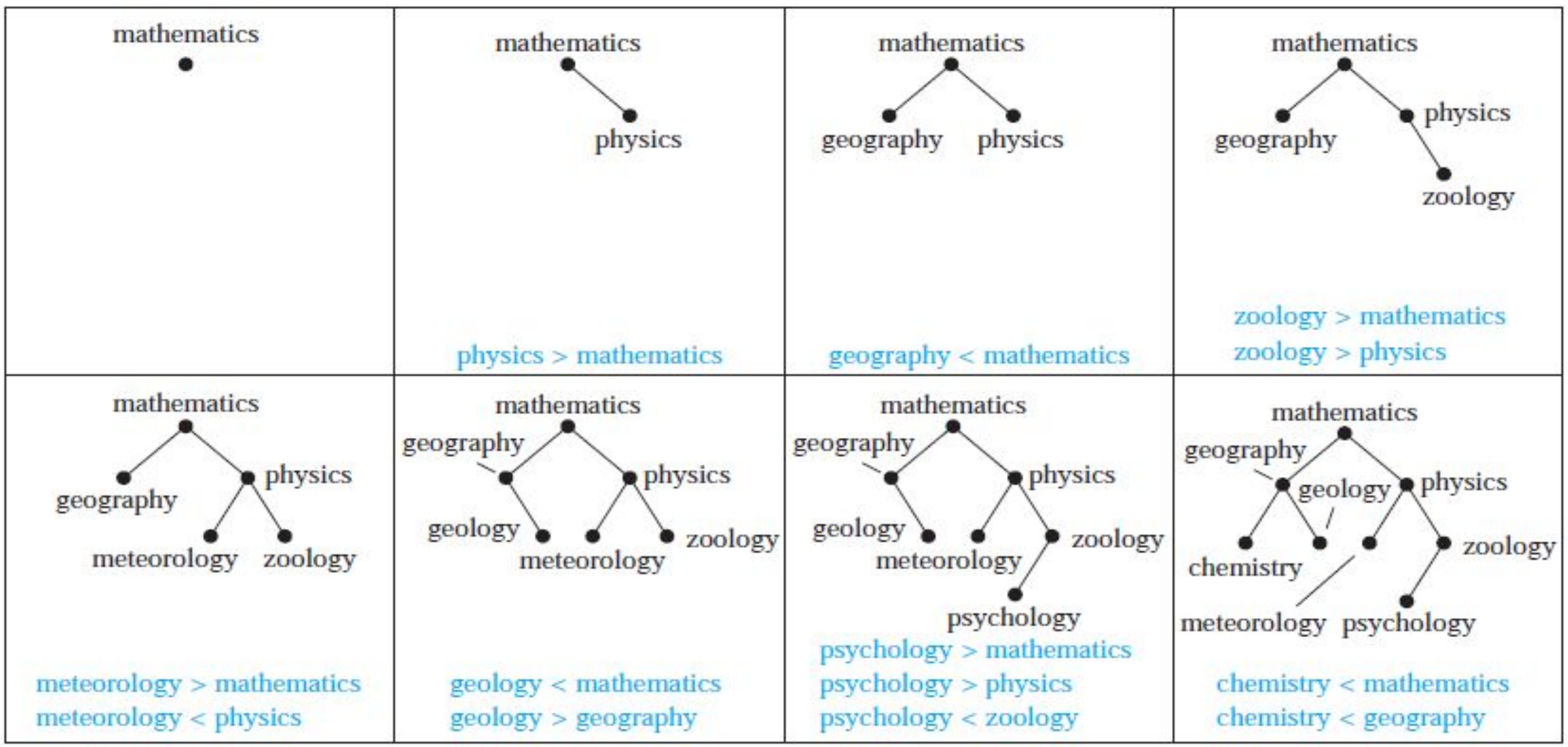


FIGURE 1 Constructing a Binary Search Tree.

Applications of Trees – Decision Tree

A rooted tree in which each internal vertex corresponds to a decision, with a subtree at these vertices for each possible outcome of the decision, is called a **decision tree**.

Example: Suppose there are seven coins, all with the same weight, and a counterfeit coin that weighs less than the others. How many weighings are necessary using a balance scale to determine which
of the eight coins is the counterfeit one?

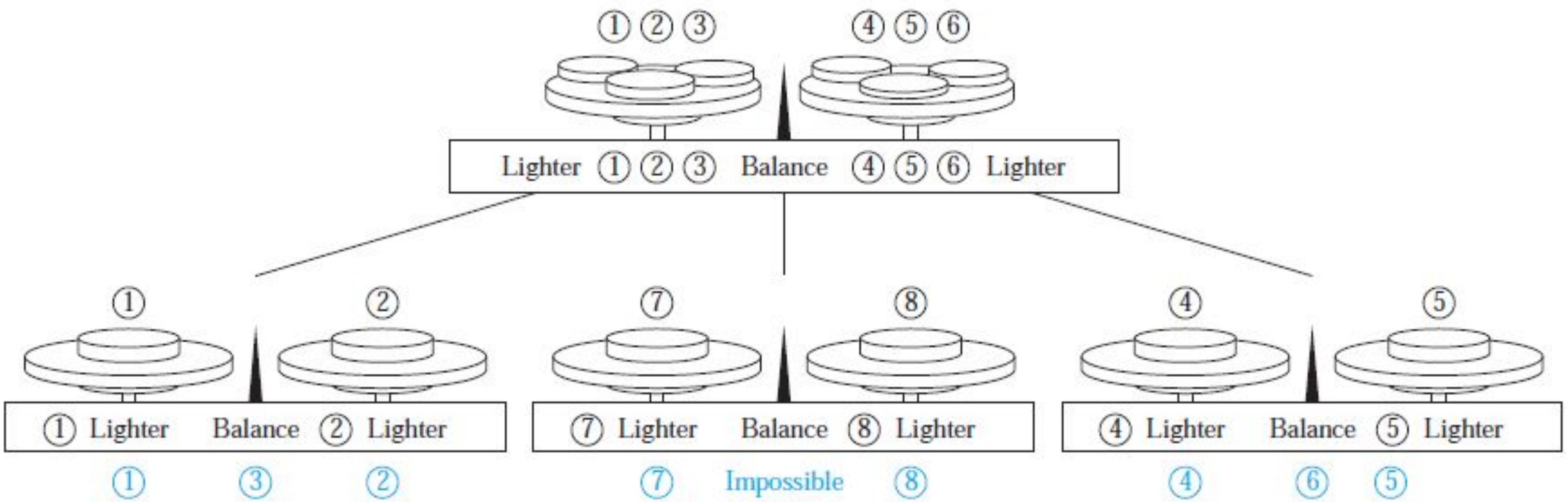


FIGURE 3 A Decision Tree for Locating a Counterfeit Coin. The counterfeit coin is shown in color below each final weighing.

Applications of Trees – Decision Tree

EXAMPLE 4 We display in Figure 4 a decision tree that orders the elements of the list a, b, c.

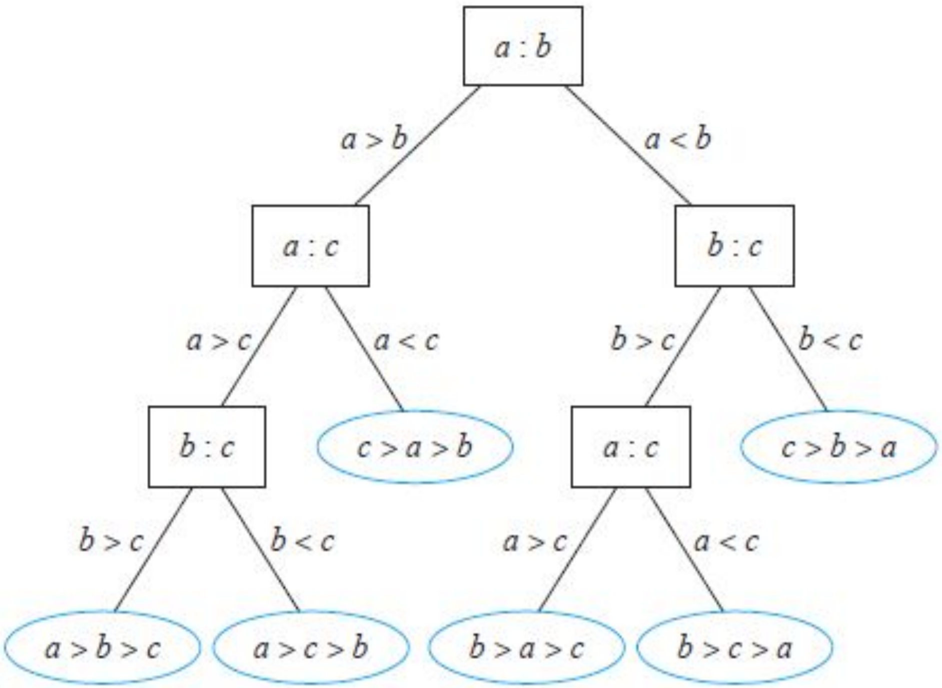


FIGURE 4 A Decision Tree for Sorting Three Distinct Elements.

Applications of Trees – Huffman Coding

Huffman coding, was developed by David Huffman in a term paper he wrote in 1951 while a graduate student at MIT. (Note that this algorithm assumes that we already know how many times each symbol occurs in the string, so we can compute the frequency of each symbol by dividing the number of times this symbol occurs by the length of the string.)

Huffman coding is a fundamental algorithm in *data compression*, the subject devoted to reducing the number of bits required to represent information.

Huffman coding is extensively used to compress bit strings representing text and it also plays an important role in compressing audio and image files.

Applications of Trees – Huffman Coding

ALGORITHM 2 Huffman Coding.

procedure *Huffman*(C: symbols a_i with frequencies w_i, $i = 1, \ldots, n$)
$F :=$ forest of n rooted trees, each consisting of the single vertex a_i and assigned weight w_i
while F is not a tree
  Replace the rooted trees T and T' of least weights from F with $w(T) \geq w(T')$ with a tree having a new root that has T as its left subtree and T' as its right subtree. Label the new edge to T with 0 and the new edge to T' with 1.
  Assign $w(T) + w(T')$ as the weight of the new tree.
{the Huffman coding for the symbol a_i is the concatenation of the labels of the edges in the unique path from the root to the vertex a_i}

Applications of Trees – Huffman Coding

Example: Use Huffman coding to encode the following symbols with the frequencies listed: A: 0.08, B: 0.10, C: 0.12, D: 0.15, E: 0.20, F: 0.35. What is the average number of bits used to encode a character?

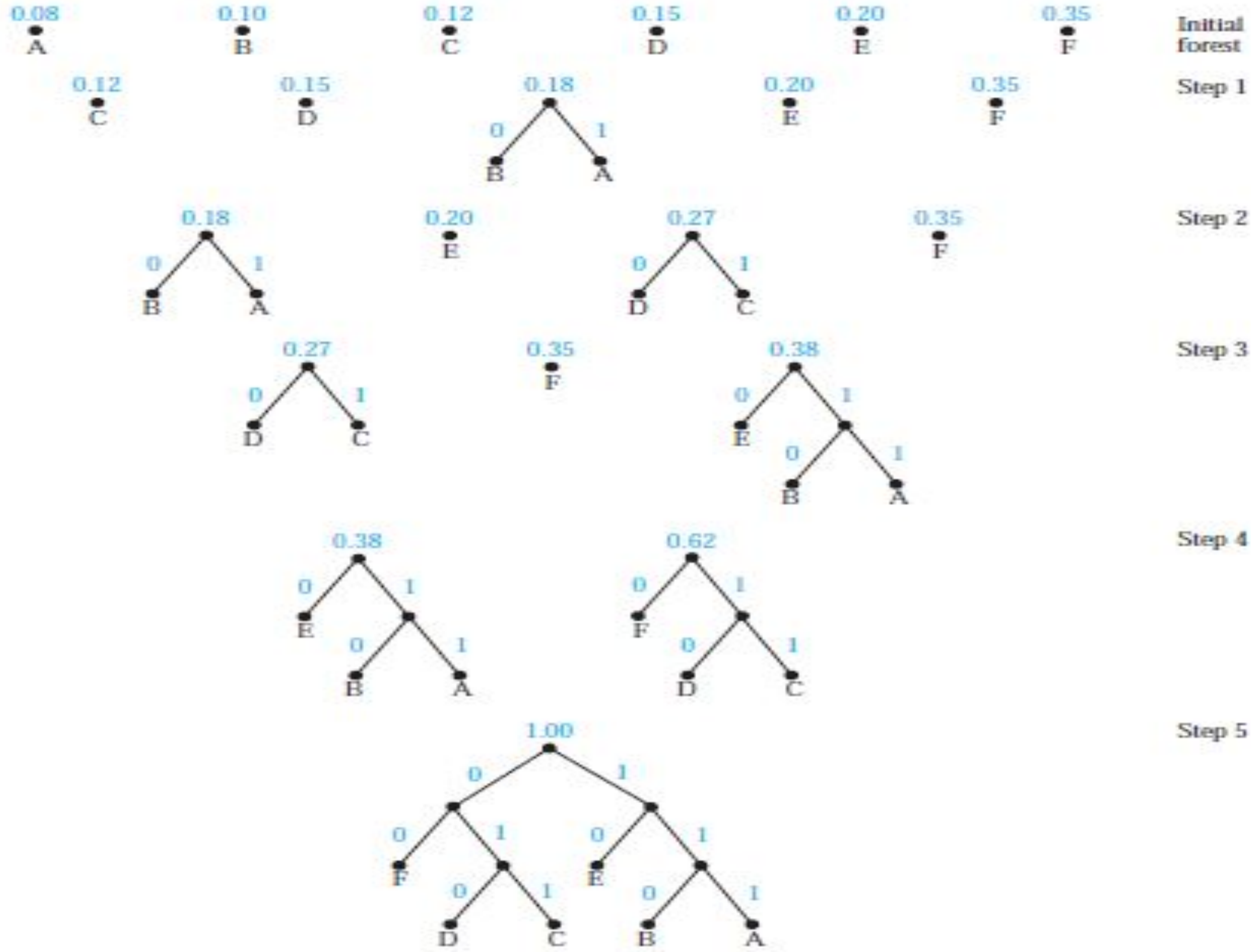


FIGURE 6 **Huffman Coding of Symbols in Example 4.**

Applications of Trees – Game Tree

Trees can be used to analyze certain types of games such as **tic-tac-toe, nim, checkers, and chess**.

In each of these games, two players take turns making moves. Each player knows the moves made by the other player and no element of chance enters into the game.

We model such games using **game trees**; the vertices of these trees represent the positions that a game can be in as it progresses; the edges represent legal moves between these positions.

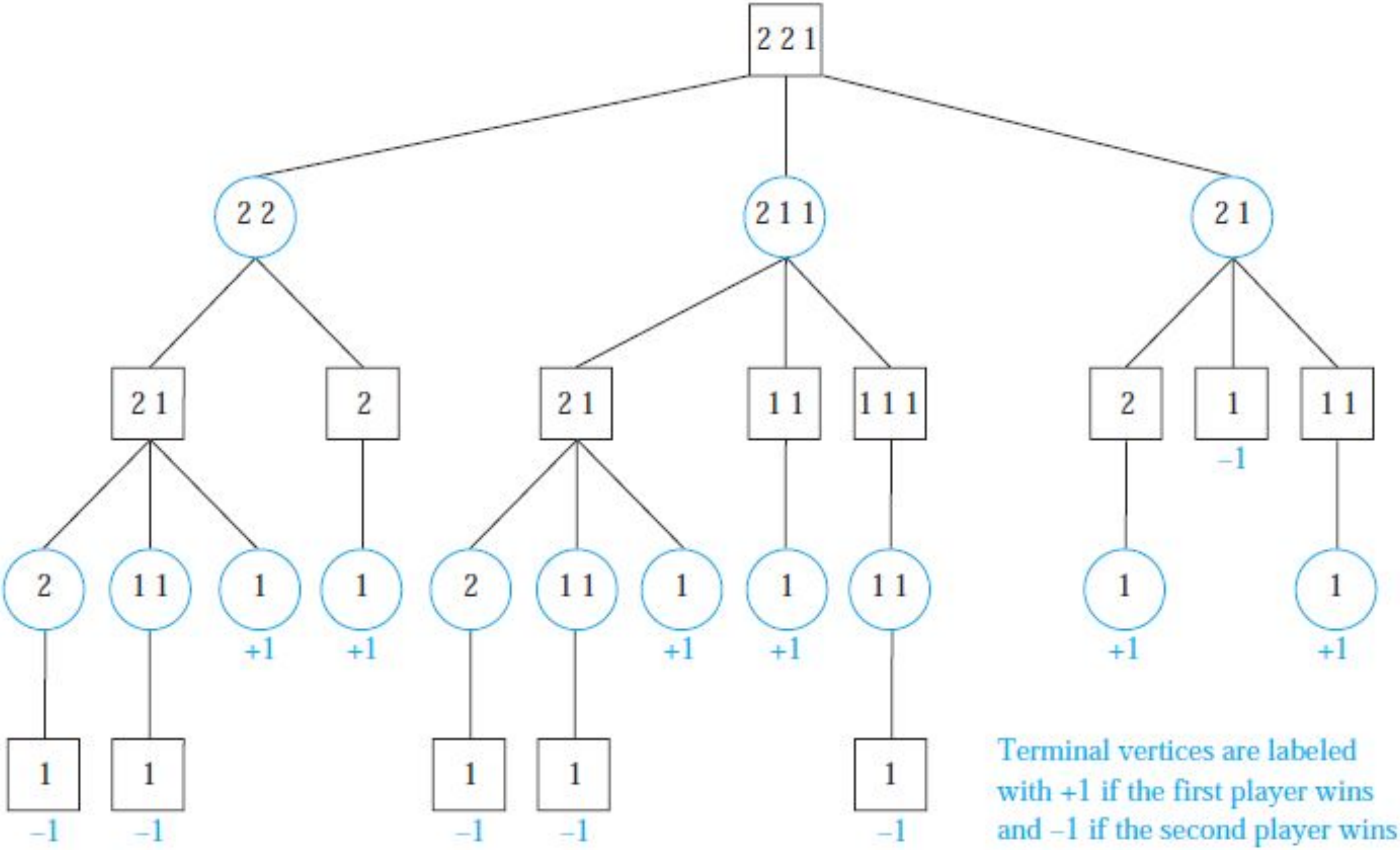


FIGURE 7 **The Game Tree for a Game of Nim.**