Discrete Mathematics for Computer Science

Department of Computer Science

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Reference Book: Discrete Mathematics and its applications BY Kenneth H. Rosen – 8th edition

2.1 Sets

- A ***set*** is a new type of structure, representing an ***unordered*** collection (group) of zero or more ***distinct*** (different) objects. The objects are called ***elements*** or ***members*** of the set.
  - Notation: $x \in S$
- Set theory deals with operations between, relations among, and statements about sets.

2.1 Sets

- The objects are called the elements or members of the set.
- Sets are denoted by capital letters A, B, C …, X, Y, Z.
- The elements of a set are represented by lower case letters a, b, c, … , x, y, z.
- If an object x is a member of a set A we write x Î A, which reads “x belongs to A” or “x is in A” or “x is an element of A”, otherwise we write x ÏA, which reads “x does not belong to A” or “x is not in A” or “x is not an element of A”.

Basic Properties of Sets

- Sets are inherently ***unordered***:
  - No matter what objects a, b, and c denote, $\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\} = \{c, a, b\} = \{c, b, a\}$.
- All elements are ***distinct*** (unequal); multiple listings make no difference!
  - If $a = b$, then $\{a, b, c\} = \{a, c\} = \{b, c\} = \{a, a, b, a, b, c, c, c, c\}$.
  - This set contains (at most) 2 elements!

Basic Notations for Sets

- **TABULAR FORM**

Listing all the elements of a set, separated by commas and enclosed within braces or curly brackets{}.

EXAMPLES

A = {1, 2, 3, 4, 5} is the set of first five **Natural Numbers**.

B = {2, 4, 6, 8, …, 50} is the set of **Even numbers** up to 50.

C = {1, 3, 5, 7, 9, …} is the set of **positive odd numbers.**

NOTE

The symbol "…" is called an ellipsis. It is a short for "and so forth."

We can denote a set *S* in writing by listing all of its elements in curly braces:

- {*a*,*b*,*c*} is the set whose elements are *a*, *b*, and *c*

Basic Notations for Sets

- **DESCRIPTIVE FORM:**

Stating in words the elements of a set.

EXAMPLES

A = {1, 2, 3, 4, 5}

A = set of first five Natural Numbers(Descriptive Form)

B = {2, 4, 6, 8, …, 50}

B = set of positive even integers less or equal to fifty.

C = {1, 3, 5, 7, 9, …}

C = set of positive odd integers. (Descriptive Form)

Basic Notations for Sets

- ***Set builder notation***:
  - For any statement $P(x)$ over any domain,
    $\{x \mid P(x)\}$ is *the set of all x such that P(x)* is true
  - Example: {1, 2, 3, 4}
    $= \{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5\}$
    $= \{x \in \mathbf{Z} \mid x > 0 \text{ and } x < 5\}$

SUBSET

- **SUBSET:** If A & B are two sets, A is called a subset of B, written $A \subseteq B$, if, and only if, any element of A is also an element of B. **Symbolically:** $A \subseteq B \Leftrightarrow$ if $x \in A$ then $x \in B$
- **REMARK:**

1. When $A \subseteq B$, then B is called a superset of A.
2. When A is not subset of B, then there exist at least one $x \in A$ such that $x \notin B$.
3. Every set is a subset of itself.

- **EXAMPLES:**

A = {1, 3, 5} B = {1, 2, 3, 4, 5} C = {1, 2, 3, 4} D = {3, 1, 5}

$A \subseteq B$ (Because every element of A is in B)

$C \subseteq B$ (Because every element of C is also an element of B)

Proper SUBSET

- **SUBSET:** Let A and B be sets. A is a proper subset of B, if, and only if, every element of A is in B but there is at least one element of B that is not in A, and is denoted as $A \subset B$.

EXAMPLE:

Let A = {1, 3, 5} B = {1, 2, 3, 5}

then $A \subset B$ (Because there is an element 2 of B which is not in A).

Try Yourself! Prove that empty set is a subset of any set.

Definition of Set Equality

- Two sets are declared to be equal *if and only if* they contain exactly the same elements.

- In particular, it does not matter *how the set is defined or denoted.*

- Example:

  The set $\{1, 2, 3, 4\}$
  $= \{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5\}$
  $= \{x \mid x \text{ is a positive integer where } x^2 < 20\}$

Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).
- Symbols for some special infinite sets:

  $\mathbf{N} = \{0, 1, 2, \ldots\}$ the set of **N**atural numbers.

  $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the set of **I**ntegers.

  $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ the set of positive Integers.

  $\mathbf{Q} = \{p/q \mid p,q \in \mathbf{Z}, \text{ and } q \neq 0\}$

  the set of **R**ational numbers.

  $\mathbf{R}$ = the set of "**R**eal" numbers.

Basic Set Relations

- $x \in S$ ("*x* is in *S*") is the proposition that object *x* is an $\in$*lement* or *member* of set *S*.
  - *e.g.* $3 \in \mathbf{N}$,

    $a \in \{x \mid x \text{ is a letter of the alphabet}\}$
  - Can define set equality in terms of $\in$ relation:

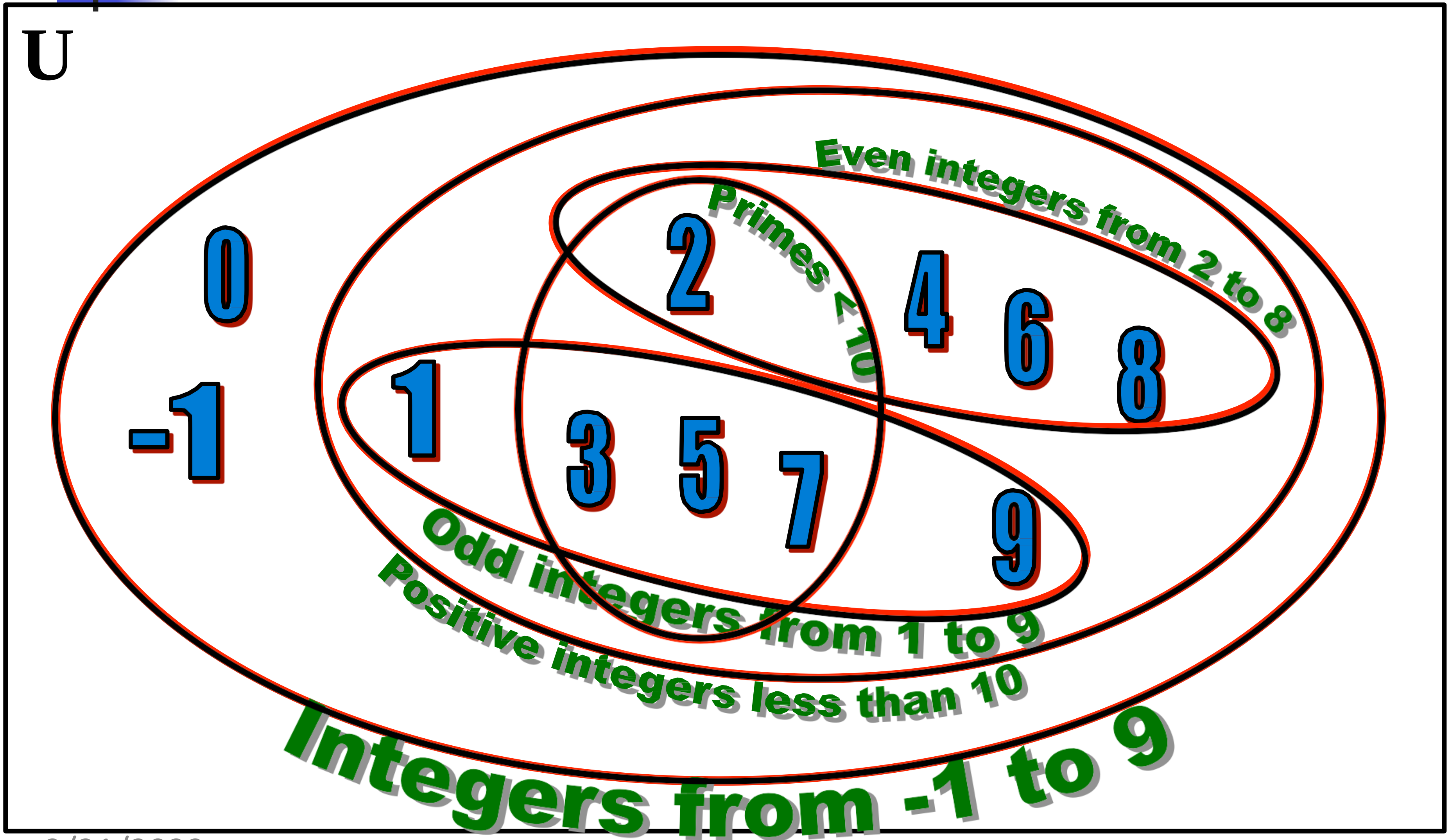
    $\forall S,T: S = T \leftrightarrow [\forall x\ (x \in S \leftrightarrow x \in T)]$

    "Two sets are equal iff they have all the same members."
- $x \notin S \equiv \neg(x \in S)$ "*x* is not in *S*"

The Empty Set

- $\varnothing$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\varnothing = \{\ \} = \{x \mid \textbf{False}\}$
- No matter the domain of discourse, we have the axiom $\neg\exists x{:}\ x \in \varnothing$.
- $\{\ \} \neq \{\varnothing\} = \{\ \{\ \}\ \}$
  - $\{\varnothing\}$ it isn't empty because it has $\varnothing$ as a member!

Venn Diagrams



Subset and Superset

- $S \subseteq T$ ("S is a subset of T") means that every element of S is also an element of T.
- $S \subseteq T \equiv \forall x\, (x \in S \rightarrow x \in T)$
- $\emptyset \subseteq S$, $S \subseteq S$
- $S \supseteq T$ ("S is a superset of T") means $T \subseteq S$
- Note $(S = T) \equiv (S \subseteq T \wedge T \subseteq S)$

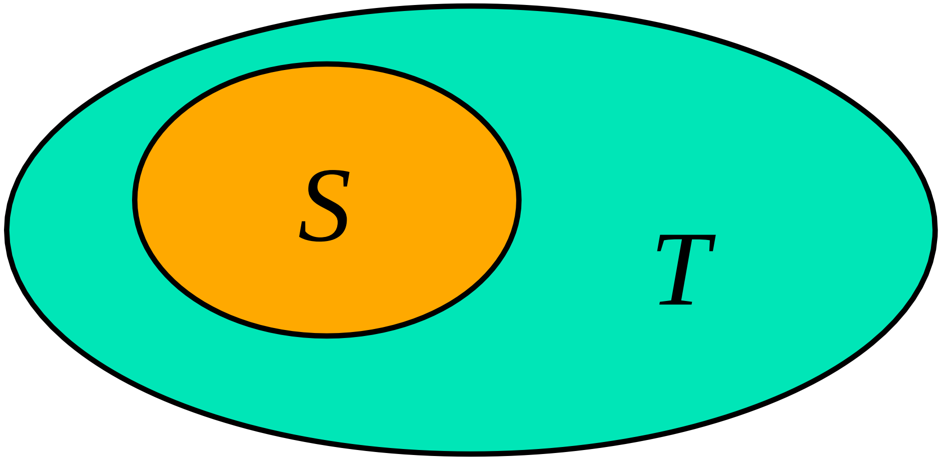
$$\equiv \forall x(x \in S \rightarrow x \in T) \wedge \forall x(x \in T \rightarrow x \in S)$$

$$\equiv \forall x(x \in S \leftrightarrow x \in T)$$

- $S \not\subseteq T$ means $\neg(S \subseteq T)$, *i.e.* $\exists x(x \in S \wedge x \notin T)$

Proper (Strict) Subsets & Supersets

- $S \subset T$ ("S is a proper subset of T") means that $S \subseteq T$ but $T \nsubseteq S$. Similar for $S \supset T$.
- Example:
  $\{1, 2\} \subset \{1, 2, 3\}$



Venn Diagram of $S \subset T$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- Example:

  Let $S = \{x \mid x \subseteq \{1, 2, 3\}\}$

  then $S = \{ \varnothing,$

  $\{1\}, \{2\}, \{3\},$

  $\{1, 2\}, \{1, 3\}, \{2, 3\},$

  $\{1, 2, 3\} \}$

- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Very Important!

Cardinality and Finiteness

- $|S|$ (read "the *cardinality of S*") is a measure of how many different elements S has.
- *E.g.*, $|\varnothing| = 0$, $|\{1, 2, 3\}| = 3$, $|\{a, b\}| = 2$,

  $|\{\{1, 2, 3\}, \{4, 5\}\}| =$ __2__

The *Power Set* Operation

- The ***power set*** $P(S)$ of a set S is the set of all subsets of S. $P(S) = \{x \mid x \subseteq S\}$.
- Examples
  - $P(\{a, b\}) = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \}$
  - $S = \{0, 1, 2\}$

    $P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\},$
    $\{0, 1, 2\}\}$
  - $P(\emptyset) = \{\emptyset\}$
  - $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
- Note that for finite S, $|P(S)| = 2^{|S|}$.
- It turns out $\forall S\ (|P(S)| > |S|)$, *e.g.* $|P(\mathbf{N})| > |\mathbf{N}|$.

Cartesian Products of Sets

- For sets A and B, their ***Cartesian product*** denoted by $\boldsymbol{A \times B}$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

  $$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}.$$

  - *E.g.* $\{a, b\} \times \{1, 2\}$

    $= \{ (a, 1), (a, 2), (b, 1), (b, 2) \}$

- Note that for finite A, B, $|A \times B| = |A||B|$.
- Note that the Cartesian product is ***not*** commutative: *i.e.*, $\neg\forall A,B\ (A \times B = B \times A)$.
- Extends to $A_1 \times A_2 \times \ldots \times A_n$

  $= \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$

The Union Operator

- For sets *A* and *B*, their ***union*** $A \cup B$ is the set containing all elements that are either in *A*, **or** ("$\vee$") in *B* (or, of course, in both).

- Formally, $\forall A,B\colon A \cup B = \{x \mid x \in A \vee x \in B\}$.

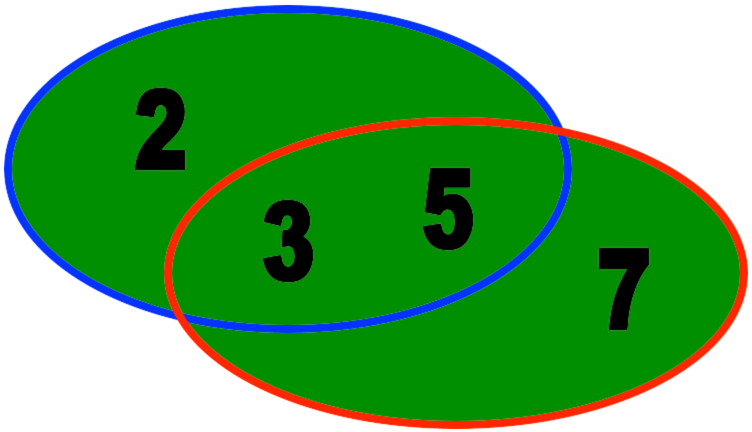
- Note that $A \cup B$ is a **superset** of both *A* and *B* (in fact, it is the smallest such superset):

$$\forall A,B\colon (A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$$

Union Examples

- $\{a, b, c\} \cup \{2, 3\} = \{a, b, c, 2, 3\}$
- $\{2, 3, 5\} \cup \{3, 5, 7\} = \{2, 3, 5, 3, 5, 7\}$
  $= \{2, 3, 5, 7\}$

Required Form



The Intersection Operator

- For sets A and B, their ***intersection*** $A \cap B$ is the set containing all elements that are simultaneously in A **and** ("$\wedge$") in B.

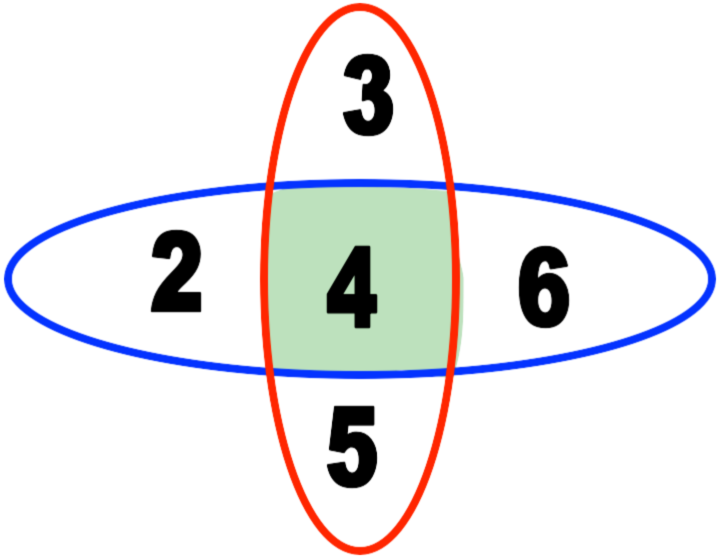
- Formally, $\forall A,B$: $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

- Note that $A \cap B$ is a **subset** of both A and B (in fact it is the largest such subset):

$$\forall A,B: (A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$$

Intersection Examples

- $\{a, b, c\} \cap \{2, 3\} = \underline{\ \varnothing\ }$
- $\{2, 4, 6\} \cap \{3, 4, 5\} = \underline{\ \{4\}\ }$



Disjointedness

- Two sets *A*, *B* are called ***disjoint*** (*i.e.*, unjoined) iff their intersection is empty. ($A \cap B = \varnothing$)

- Example: the set of even integers is disjoint with the set of odd integers.

Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?
  $|A \cup B| = |A| + |B| - |A \cap B|$

- Example: How many students in the class major in Computer Science or Mathematics?
  - Consider set $E = C \cup M$,
    $C = \{s \mid s \text{ is a Computer Science major}\}$
    $M = \{s \mid s \text{ is a Mathematics major}\}$

  - Some students are joint majors!
    $|E| = |C \cup M| = |C| + |M| - |C \cap M|$

Set Difference

- For sets *A* and *B*, the ***difference of A and B***, written *A* - *B*, is the set of all elements that are in *A* but not *B*.

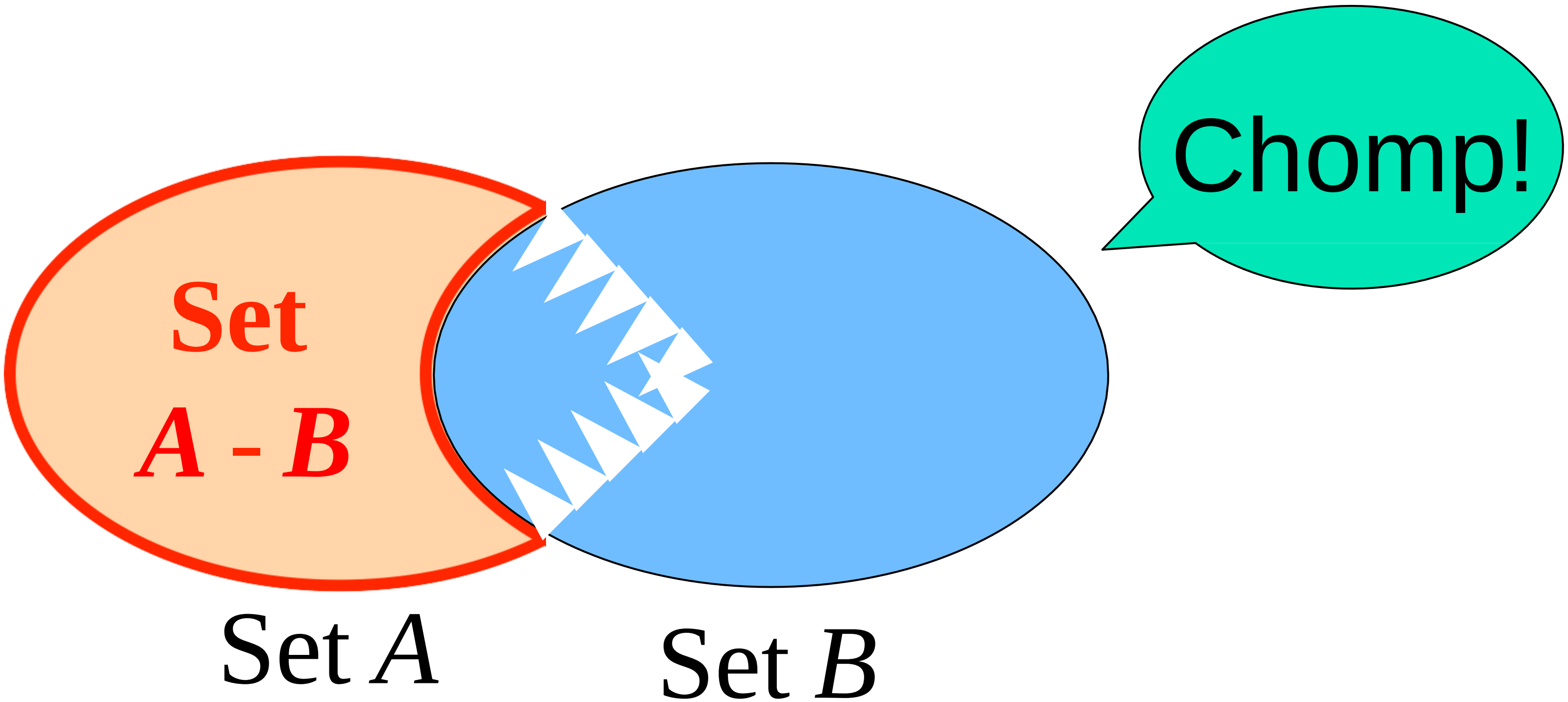
- Formally:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$
$$= \{x \mid \neg(x \in A \rightarrow x \in B)\}$$

- Also called:
  The ***complement of B with respect to A***.

Set Difference: Venn Diagram

- *A* − *B*
  is what's left after *B* "takes a bite out of *A*"



Set Difference Examples

- {1, $\cancel{2}$, $\cancel{3}$, 4, $\cancel{5}$, 6} - {2, 3, 5, 7, 9, 11} = {1, 4, 6}

- **Z** - **N** = {… , −1, 0, 1, 2, … } - {0, 1, … }
  = {*x* | *x* is an integer but not a natural #}
  = {… , −3, −2, −1}
  = {*x* | *x* is a negative integer}

Set Complements

- The *universe of discourse* (or the *domain*) can itself be considered a set, call it U.

- When the context clearly defines U, we say that for any set $A \subseteq U$, the ***complement*** of A, written as $\overline{A}$, is the complement of A with respect to U, *i.e.*, it is U - A.

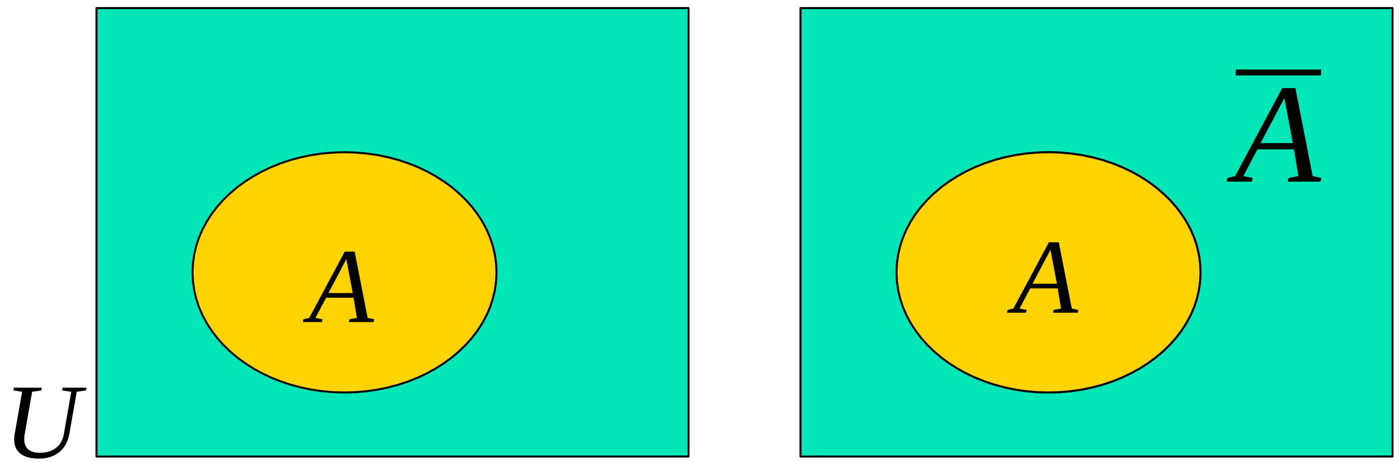
- *E.g.*, If U = **N**,

$$\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$$

More on Set Complements

- An equivalent definition, when *U* is obvious:

$$\overline{A} = \{x \mid x \notin A\}$$



Computer Representation of Sets

- **How to represent sets in the computer?**

• **One solution: Data structures like a list**

• **A better solution:**

• First, specify an arbitrary ordering of the elements of U, for instance $a1, a2, \ldots, an$. Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if ai belongs to A and is 0 if ai does not belong to A.

Example: All possible elements: U={1 2 3 4 5}

• Assume A={2,5}

– Computer representation: A = 01001

• Assume B={1,5}

– Computer representation: B = 10001

Interval Notation

- $a, b \in \mathbf{R}$, and $a < b$ then
  - $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$ Open Interval
  - $[a, b] = \{x \in \mathbf{R} \mid a \le x \le b\}$ Closed Interval
  - $(a, b] = \{x \in \mathbf{R} \mid a < x \le b\}$
  - $(-\infty, b] = \{x \in \mathbf{R} \mid x \le b\}$
  - $[a, \infty) = \{x \in \mathbf{R} \mid a \le x\}$
  - $(a, \infty) = \{x \in \mathbf{R} \mid a < x\}$

Set Identities

- Identity: $A \cup \varnothing = A = A \cap U$
- Domination: $A \cup U = U$, $A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A$, $A \cap A = A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$,
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Absorption: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
- Complement: $A \cup \overline{A} = U$, $A \cap \overline{A} = \varnothing$

DeMorgan's Law for Sets

- Exactly analogous to (and provable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proving Set Identities

- To prove statements about sets, of the form $E_1 = E_2$ (where the Es are set expressions), here are three useful techniques:

1. Use set builder notation & logical equivalences.

2. Use a *membership table*.

3. Use a Venn diagram.

Method 1: Set Builder Notation & Logical Equivalence

- Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cap B} = \{x \mid x \notin (A \cap B)\}$	def. of complement
$= \{x \mid \neg(x \in (A \cap B))\}$	def. of "does not belong"
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	def. of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	De Morgan's law (logic)
$= \{x \mid x \notin A \vee x \notin B\}$	def. of "does not belong"
$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	def. of complement
$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	def. of union
$= \overline{A} \cup \overline{B}$	by set builder notation

Method 2: Membership Tables

- Analog to truth tables in propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use "1" to indicate membership in the derived set, "0" for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

- Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
1	1	1	0	0
1	0	1	1	1
0	1	1	0	0
0	0	0	0	0

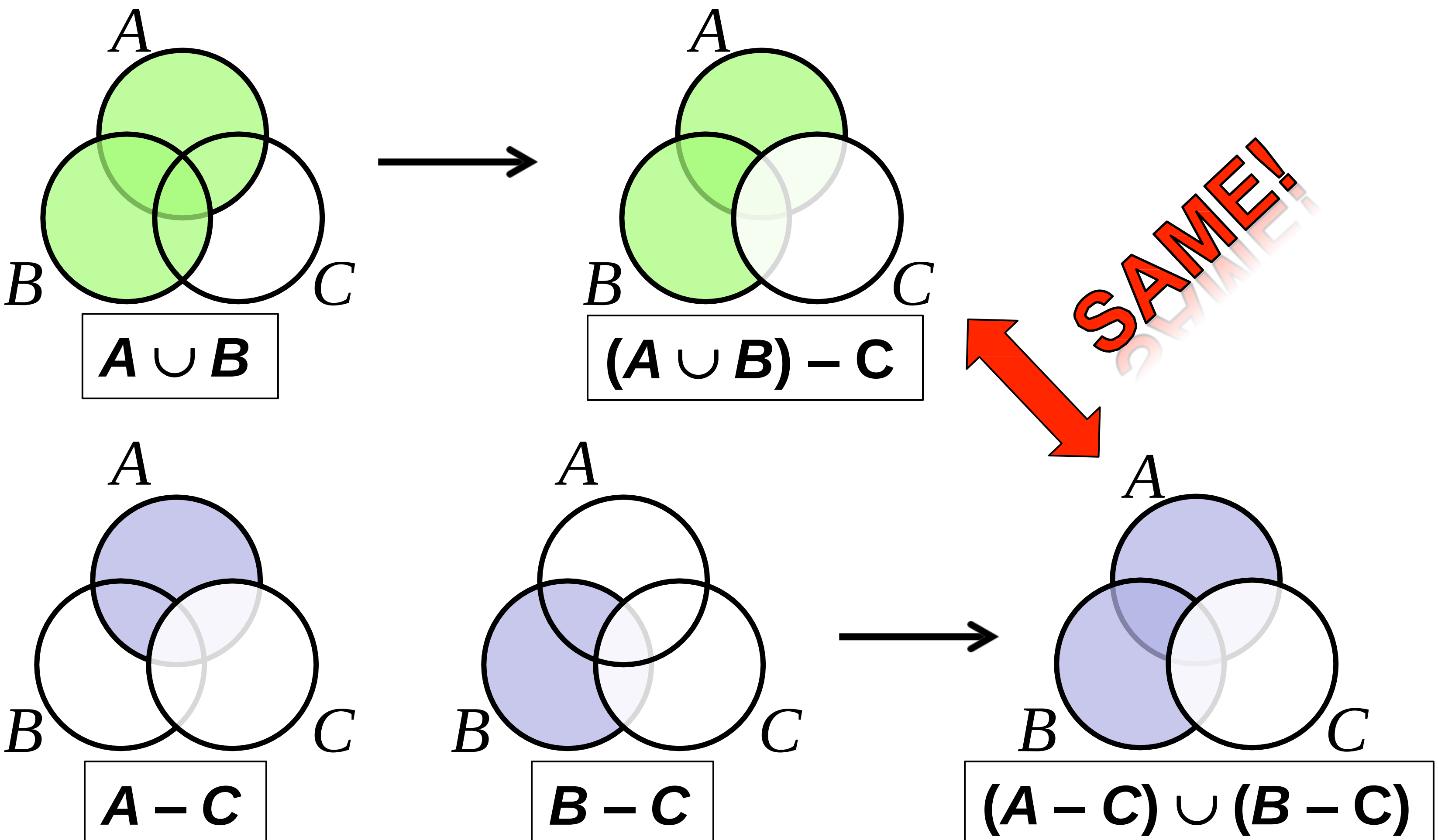
Membership Table Exercise

- Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A B C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
1 1 1	1	0	0	0	0
1 1 0	1	1	1	1	1
1 0 1	1	0	0	0	0
1 0 0	1	1	1	0	1
0 1 1	1	0	0	0	0
0 1 0	1	1	0	1	1
0 0 1	0	0	0	0	0
0 0 0	0	0	0	0	0

Method 3: Venn Diagram

- Prove $(A \cup B) - C = (A - C) \cup (B - C)$.



Venn Diagram

UNION:

Let A and B be subsets of a universal set U. The union of sets A and B is the set of all elements in U that belong to A or to B or to both, and is denoted $A \cup B$.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

EMAMPLE:

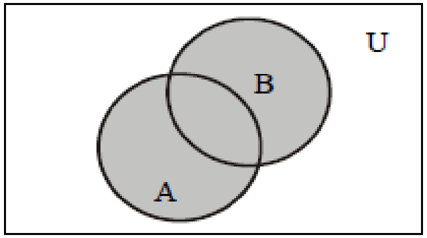
Let $U = \{a, b, c, d, e, f, g\}$

$A = \{a, c, e, g\}$, $B = \{d, e, f, g\}$

Then $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$

$= \{a, c, d, e, f, g\}$

VENN DIAGRAM FOR UNION:



$A \cup B$ is shaded

Venn Diagram

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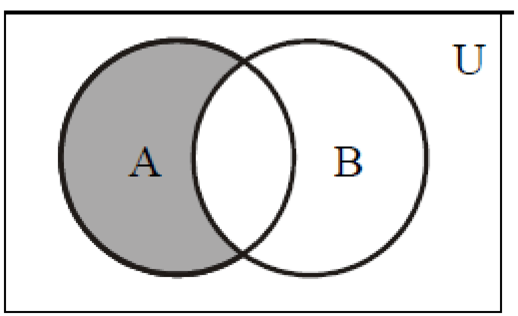
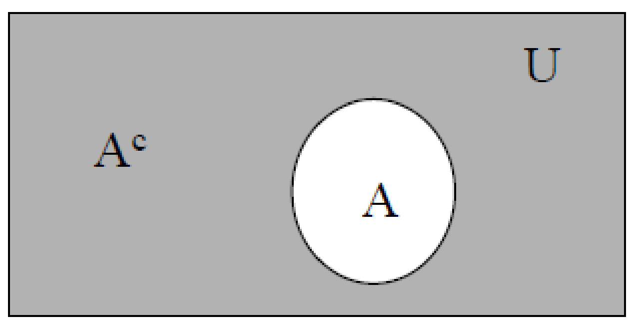
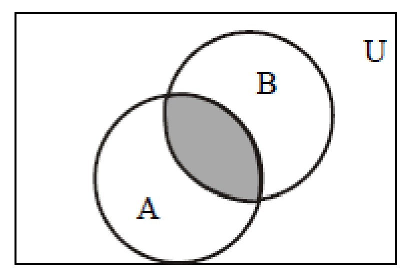


A-B is shaded



A^c is shaded



$A \cap B$ is shaded

Venn Diagram

EXERCISE:

Given the following universal set U and its two subsets P and Q, where

$U = \{x \mid x \in Z, 0 \leq x \leq 10\}$

$P = \{x \mid x \text{ is a prime number}\}$

$Q = \{x \mid x^2 < 70\}$

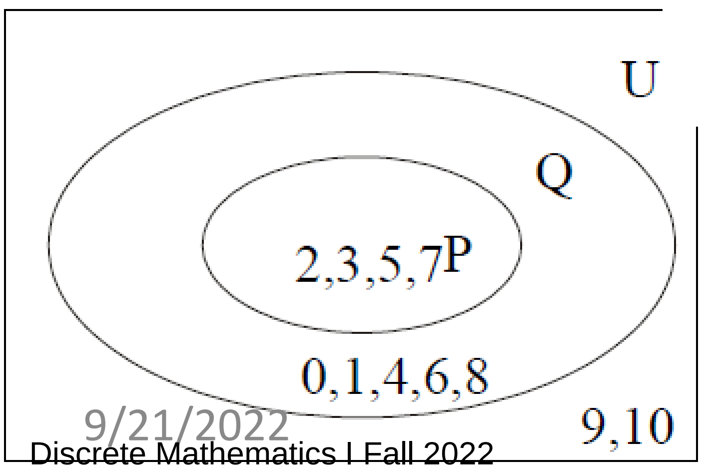
(i) Draw a Venn diagram for the above

(ii) List the elements in $Pc \cap Q$

$P^c = U - P = \{0, 1, 2, 3, \ldots, 10\} - \{2, 3, 5, 7\}$
$= \{0, 1, 4, 6, 8, 9, 10\}$

and

$P^c \cap Q = \{0, 1, 4, 6, 8, 9, 10\} \cap \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$= \{0, 1, 4, 6, 8\}$



Proving Set Identities by Venn Diagram

- Prove the following using Venn Diagrams:

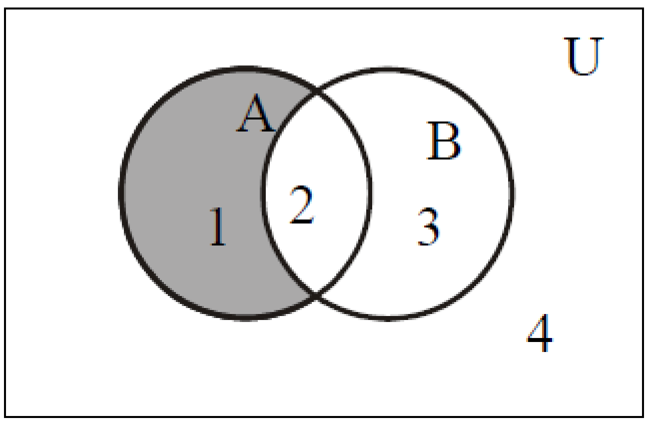
(i) $A - (A - B) = A \cap B$

(ii) $(A \cap B)^c = A^c \cup B^c$

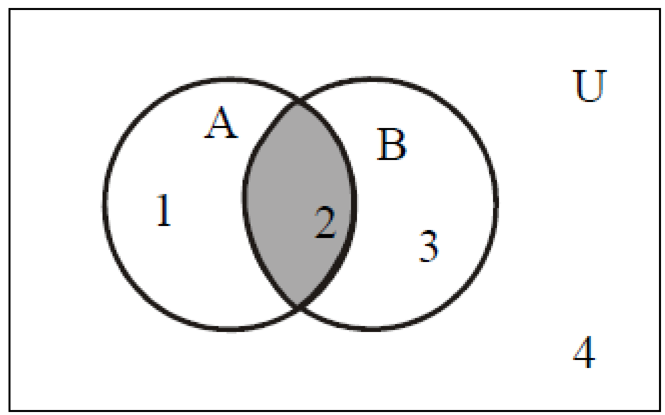
(iii) $A - B = A \cap B^c$

$A = \{ 1, 2 \}$

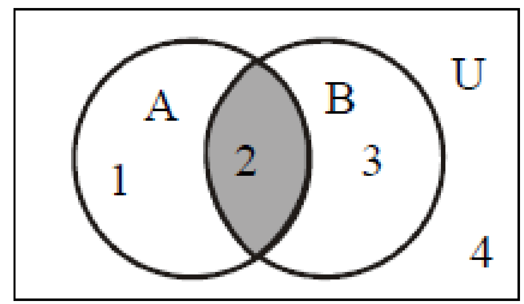
$B = \{ 2, 3 \}$



$A - B$ is shaded



$A - (A - B)$ is shaded



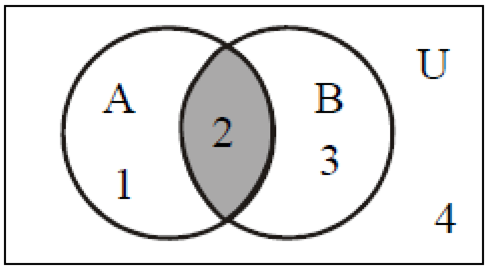
$A \cap B$ is shaded

Proving Set Identities by Venn Diagram

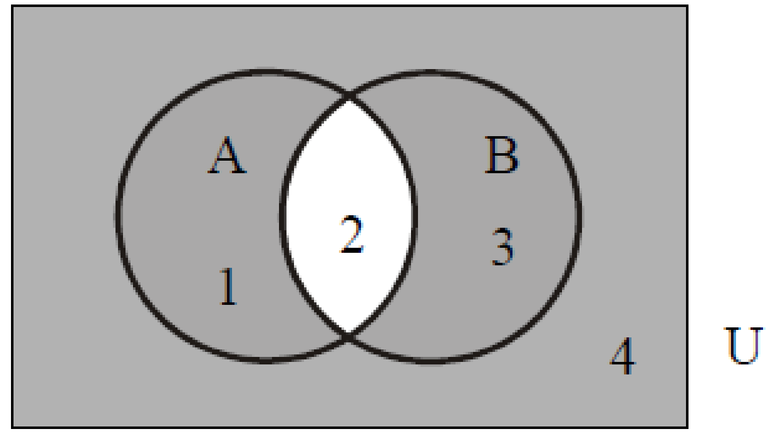
SOLUTION (ii)

$$(A \cap B)^c = A^c \cup B^c$$

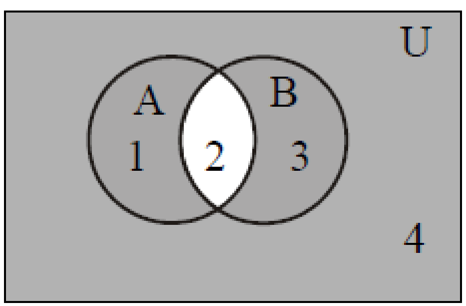
(a)



$A \cap B$



(b)



$(A \cap B)^c$

$A^c \cup B^c$ is shaded.

Proving Set Identities by Membership Table

- Prove the following using Membership Table:

(i) $A - (A - B) = A \cap B$

(ii) $(A \cap B)^c = A^c \cup B^c$

(iii) $A - B = A \cap B^c$

$A = \{ 1, 2 \}$

$B = \{ 2, 3 \}$

$A - (A - B) = A \cap B$

A	B	A-B	A-(A-B)	$A \cap B$
1	1	0	1	1
1	0	1	0	0
0	1	0	0	0
0	0	0	0	0

$(A \cap B)^c = A^c \cup B^c$

A	B	$A \cap B$	$(A \cap B)^c$	A^c	B^c	$A^c \cup B^c$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Applications of Venn Diagram

- A number of computer users are surveyed to find out if they have a printer, modem or scanner. Draw separate Venn diagrams and shade the areas,

which represent the following configurations:

(i) modem and printer but no scanner

(ii) scanner but no printer and no modem

(iii) scanner or printer but no modem.

(iv) no modem and no printer.

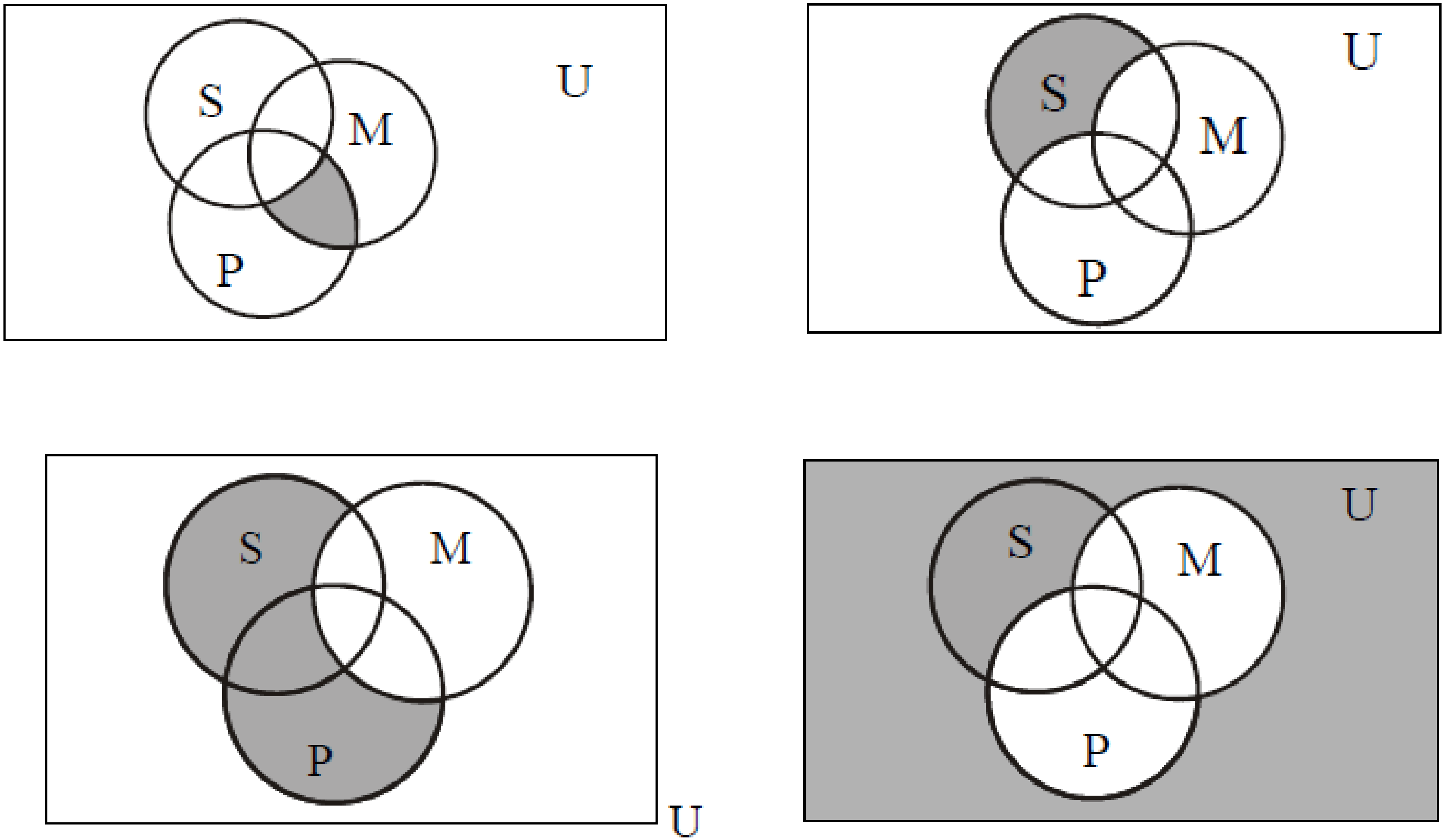
Let

P represent the set of computer users having printer.

M represent the set of computer users having modem.

S represent the set of computer users having scanner.

Applications of Venn Diagram



Applications of Venn Diagram

- Do Yourself!

Of **21 typists** in an office, **5** use all **manual typewriters (M), electronic typewriters (E) and word processors (W); 9** use **E** and **W**; **7** use **M** and **W**; **6** use **M** and **E**; but **no one** uses **M** only.

(i) Represent this information in a Venn Diagram.

(ii) If the same number of typists use electronic as use word processors, then

1. (a) How many use word processors only,
2. (b) How many use electronic typewriters?