



Discrete Mathematics for Computer Science

Department of Computer Science

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Reference Book: Discrete Mathematics and its applications BY
Kenneth H. Rosen – 8th edition



Lecture 3

Chapter 1. The Foundations

- 2. Propositional Equivalences
- 3. Predicates and Quantifiers



Truth Tables


- Truth table for $\sim p \wedge (q \vee \sim r)$

p	q	r	$\sim r$	$q \vee \sim r$	$\sim p$	$\sim p \wedge (q \vee \sim r)$
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

Double Negation

- Double Negative Property $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F



- Example:** “It is not true that I am not happy”
- Solution:
- Let p = “I am happy”
- then $\sim p$ = “I am not happy”
- and $\sim(\sim p)$ = “It is not true that I am not happy”
- Since $\sim(\sim p) \equiv p$
- Hence the given statement is equivalent to: **“I am happy”**



De Morgan's Laws

- The negation of an **and** statement is logically equivalent to the **or** statement in which each component is negated.
Symbolically $\sim(p \wedge q) \equiv \sim p \vee \sim q$.
- The negation of an **or** statement is logically equivalent to the **and** statement in which each component is negated.
Symbolically: $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

De Morgan's Laws

- How we can prove this?

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T



Same truth values



De Morgan's Laws

- Give negations for each of the following statements:
 - a. The fan is slow **or** it is very hot.
 - b. Akram is unfit **and** Saleem is injured.
 - **Solution:**
 - a. The fan is **not** slow **and** it is **not** very hot.
 - b. Akram is **not** unfit **or** Saleem is **not** injured.
 - **INEQUALITIES AND DEMORGAN'S LAWS:**
 - Use DeMorgan's Laws to write the negation of
 - 1 < x ≤ 4
 - -1 < x ≤ 4 means x > -1 **and** x ≤ 4
- By DeMorgan's Law, the negation is:
x > -1 or x ≤ 4 Which is equivalent to: x ≤ -1 **or** x > 4



1.2 Propositional Equivalence

- A **tautology** is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are! And represented by a symbol “t”
 - e.g. $p \vee \neg p$ (“Today the sun will shine or today the sun will not shine.”) [What is its truth table?]
- A **contradiction** is a compound proposition that is **false** no matter what! And represented by a symbol “c”
 - e.g. $p \wedge \neg p$ (“Today is Wednesday and today is not Wednesday.”) [Truth table?]
- A **contingency** is a compound proposition that is neither a tautology nor a contradiction.
 - e.g. $(p \vee q) \rightarrow \neg r$



Logical Equivalence

- Compound proposition p is **logically equivalent** to compound proposition q , written $p \equiv q$ or $p \leftrightarrow q$, **iff** the compound proposition $p \leftrightarrow q$ is a tautology.
- Compound propositions p and q are logically equivalent to each other **iff** p and q contain the same truth values as each other in all corresponding rows of their truth tables.

Proving Equivalence via Truth Tables

- Prove that $\neg(p \wedge q) \equiv \neg p \vee \neg q$. (De Morgan's law)

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q)$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

- Show that Check out the solution in the textbook!

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's law)
- $p \rightarrow q \equiv \neg p \vee q$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (distributive law)



Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match part of a much more complicated proposition and to find an equivalence for it and possibly simplify it.



Equivalence Laws

- *Identity:* $p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
- *Domination:* $p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
- *Idempotent:* $p \vee p \equiv p$ $p \wedge p \equiv p$
- *Double negation:* $\neg\neg p \equiv p$
- *Commutative:* $p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
- *Associative:* $(p \vee q) \vee r \equiv p \vee (q \vee r)$
 $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$



More Equivalence Laws

- *Distributive:*
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
- *De Morgan's:*
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
- *Absorption*
$$p \vee (p \wedge q) \equiv p \qquad p \wedge (p \vee q) \equiv p$$
- *Trivial tautology/contradiction:*
$$p \vee \neg p \equiv \mathbf{T} \qquad p \wedge \neg p \equiv \mathbf{F}$$

See Table 6, 7, and 8 of Section 1.2



Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$
 $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$
- Implies: $p \rightarrow q \equiv \neg p \vee q$
- Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
 $p \leftrightarrow q \equiv \neg(p \oplus q)$

This way we can “normalize” propositions



An Example Problem

- Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$$\neg(p \rightarrow q)$$

$$\equiv \neg(\neg p \vee q) \quad [\text{Expand definition of } \rightarrow]$$

$$\equiv \neg(\neg p) \wedge \neg q \quad [\text{DeMorgan's Law}]$$

$$\equiv p \wedge \neg q \quad [\text{Double negation Law}]$$



EXERCISE

- Use Logical Equivalence to rewrite each of the following sentences more simply.
- **It is not true that I am tired and you are smart.**
{I am not tired or you are not smart.}
- **It is not true that I am tired or you are smart.**
{I am not tired and you are not smart.}
- **I forgot my pen or my bag and I forgot my pen or my glasses.**
{I forgot my pen or I forgot my bag and glasses.}
- **It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.**
{It is raining and I have forgotten my umbrella or my hat.}



Negation of Implication

- Since $p \rightarrow q \equiv \sim p \vee q$ therefore
$$\begin{aligned}\sim (p \rightarrow q) &\equiv \sim (\sim p \vee q) \\ &\equiv \sim (\sim p) \wedge (\sim q) \text{ by De Morgan's law} \\ &\equiv p \wedge \sim q \text{ by the Double Negative law}\end{aligned}$$

Thus the negation of “**if p then q**” is logically equivalent to “**p and not q**”.

Example:

1. If Ali lives in Pakistan then he lives in Lahore.
Ali lives in Pakistan and he does not live in Lahore.



Example

- Show that $\sim(p \rightarrow q) \rightarrow p$ is a tautology without using truth tables.
- $\sim(p \rightarrow q) \rightarrow p$ Given statement form
 - $\equiv \sim[\sim(p \wedge \sim q)] \rightarrow p$ Implication law $p \rightarrow q \equiv \sim(p \wedge \sim q)$
 - $\equiv (p \wedge \sim q) \rightarrow p$ Double negation law
 - $\equiv \sim(p \wedge \sim q) \vee p$ Implication law $p \rightarrow q \equiv \sim p \vee q$
 - $\equiv (\sim p \vee q) \vee p$ De Morgan's law
 - $\equiv (q \vee \sim p) \vee p$ Commutative law
 - $\equiv q \vee (\sim p \vee p)$ Associative law
 - $\equiv q \vee t$ Negation law
 - $\equiv t$



Another Example Problem

- Check using a symbolic derivation whether

$$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r$$

$$(p \wedge \neg q) \rightarrow (p \oplus r) \quad [\text{Expand definition of } \rightarrow]$$

$$\equiv \neg(p \wedge \neg q) \vee (p \oplus r) \quad [\text{Expand definition of } \oplus]$$

$$\equiv \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$$

[DeMorgan's Law]

$$\equiv (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$$

cont.

Example Continued...

$$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r$$

$$\begin{aligned} & (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r)) \quad [\vee \text{ Commutative}] \\ & \equiv (q \vee \neg p) \vee ((p \vee r) \wedge \neg(p \wedge r)) \quad [\vee \text{ Associative}] \\ & \equiv q \vee (\neg p \vee ((p \vee r) \wedge \neg(p \wedge r))) \quad [\text{Distribute } \vee \text{ over } \wedge] \\ & \equiv q \vee ((\neg p \vee (p \vee r)) \wedge (\neg p \vee \neg(p \wedge r))) \quad [\vee \text{ Assoc.}] \\ & \equiv q \vee ((\neg p \vee p) \vee r) \wedge (\neg p \vee \neg(p \wedge r)) \quad [\text{Trivial taut.}] \\ & \equiv q \vee (\mathbf{T} \vee r) \wedge (\neg p \vee \neg(p \wedge r)) \quad [\text{Domination}] \\ & \equiv q \vee (\mathbf{T} \wedge (\neg p \vee \neg(p \wedge r))) \quad [\text{Identity}] \\ & \equiv q \vee (\neg p \vee \neg(p \wedge r)) \end{aligned}$$

cont.



End of Long Example

$$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r$$

$$q \vee (\neg p \vee \neg(p \wedge r)) \quad [\text{DeMorgan's Law}]$$

$$\equiv q \vee (\neg p \vee (\neg p \vee \neg r)) \quad [\vee \text{ Associative}]$$

$$\equiv q \vee ((\neg p \vee \neg p) \vee \neg r) \quad [\text{Idempotent}]$$

$$\equiv q \vee (\neg p \vee \neg r) \quad [\text{Associative}]$$

$$\equiv (q \vee \neg p) \vee \neg r \quad [\vee \text{ Commutative}]$$

$$\equiv \neg p \vee q \vee \neg r \quad \blacksquare$$



Review: Propositional Logic

- Atomic propositions: p, q, r, \dots
- Boolean operators: $\neg \wedge \vee \oplus \rightarrow \leftrightarrow$
- Compound propositions: $(p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \leftrightarrow \equiv \neg(p \rightarrow q)$
- Proving equivalences using:
 - Truth tables
 - Symbolic derivations (series of logical equivalences) $p \equiv q \equiv r \equiv \dots$



1.3 Predicate Logic

- Consider the sentence

“For every x , $x > 0$ ”

If this were a true statement about the positive integers, it could not be adequately symbolized using only statement letters, parentheses and logical connectives.

*The sentence contains two new features: a **predicate** and a **quantifier***



Subjects and Predicates

- In the sentence “The dog is sleeping”:
 - The phrase “the dog” denotes the **subject** – the *object* or *entity* that the sentence is about.
 - The phrase “is sleeping” denotes the **predicate** – a property that the subject of the statement can have.
- In predicate logic, a **predicate** is modeled as a **propositional function $P(\cdot)$** from subjects to propositions.
 - $P(x)$ = “ x is sleeping” (where x is any subject).
 - $P(\text{The cat})$ = “*The cat* is sleeping” (proposition!)



More About Predicates

- Convention: Lowercase variables $x, y, z...$ denote subjects; uppercase variables $P, Q, R...$ denote propositional functions (or predicates).
- Keep in mind that *the result of applying a predicate P to a value of subject x is the proposition*. But the predicate P , or the statement $P(x)$ **itself** (e.g. $P = \text{"is sleeping"}$ or $P(x) = \text{"x is sleeping"}$) is **not** a proposition.
 - e.g. if $P(x) = \text{"x is a prime number"}$,
 $P(3)$ is the *proposition* "3 is a prime number."



Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.
 - *e.g.:*
let $P(x,y,z)$ = “x gave y the grade z”
then if
 $x = \text{“Mike”}$, $y = \text{“Mary”}$, $z = \text{“A”}$,
then
 $P(x,y,z) = \text{“Mike gave Mary the grade A.”}$



Examples

- Let $P(x): x > 3$. Then
 - $P(4)$ is ~~TRUE~~/FALSE $4 > 3$
 - $P(2)$ is TRUE/~~FALSE~~ $2 > 3$
- Let $Q(x, y): x$ is the capital of y . Then
 - $Q(\text{Washington D.C., U.S.A.})$ is TRUE
 - $Q(\text{Hilo, Hawaii})$ is FALSE
 - $Q(\text{Massachusetts, Boston})$ is FALSE
 - $Q(\text{Denver, Colorado})$ is TRUE
 - $Q(\text{New York, New York})$ is FALSE
- Read EXAMPLE 6 (pp.33)
 - If $x > 0$ then $x := x + 1$ (in a computer program)



Universe of Discourse (U.D.)

- The power of distinguishing subjects from predicates is that it lets you state things about *many* objects at once.
- e.g., let $P(x) = "x + 1 > x"$. We can then say, "For **any** number x , $P(x)$ is true" instead of $(0 + 1 > 0) \wedge (1 + 1 > 1) \wedge (2 + 1 > 2) \wedge \dots$
- The collection of values that a variable x can take is called x 's ***universe of discourse*** or the ***domain of discourse*** (often just referred to as the ***domain***).



Quantifier Expressions

- **Quantifiers** provide a notation that allows us to *quantify (count) how many* objects in the universe of discourse satisfy the given predicate.
- “ \forall ” is the FOR \forall LL or **universal** quantifier.
 $\forall x P(x)$ means for all x in the domain, $P(x)$.
- “ \exists ” is the \exists XISTS or **existential** quantifier.
 $\exists x P(x)$ means there exists an x in the domain (that is, 1 or more) such that $P(x)$.



The Universal Quantifier \forall

- $\forall x P(x)$: *For all x in the domain, $P(x)$.*
- $\forall x P(x)$ is
 - *true* if $P(x)$ is true for every x in D (D : domain of discourse)
 - *false* if $P(x)$ is false for at least one x in D
 - For every real number x , $x^2 \geq 0$ **TRUE**
 - For every real number x , $x^2 - 1 > 0$ **FALSE**
- A **counterexample** to the statement $\forall x P(x)$ is a value x in the domain D that makes $P(x)$ false
- What is the truth value of $\forall x P(x)$ when the domain is empty? **TRUE**



The Universal Quantifier \forall

- If all the elements in the domain can be listed as x_1, x_2, \dots, x_n then, $\forall x P(x)$ is the same as the conjunction:

$$P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$$

- Example: Let the domain of x be parking spaces at UH. Let $P(x)$ be the statement “ x is full.” Then the ***universal quantification*** of $P(x)$, $\forall x P(x)$, is the *proposition*:
 - “All parking spaces at UH are full.”
 - or “Every parking space at UH is full.”
 - or “For each parking space at UH, that space is full.”



The Existential Quantifier \exists

- $\exists x P(x)$: *There exists an x in the domain (that is, 1 or more) such that $P(x)$.*
- $\exists x P(x)$ is
 - *true* if $P(x)$ is true for at least one x in the domain
 - *false* if $P(x)$ is false for every x in the domain
- What is the truth value of $\exists x P(x)$ when the domain is empty? FALSE
- If all the elements in the domain can be listed as x_1, x_2, \dots, x_n then, $\exists x P(x)$ is the same as the disjunction:

$$P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$$



The Existential Quantifier \exists

- Example:

Let the domain of x be parking spaces at UH.

Let $P(x)$ be the statement “ x is full.”

Then the ***existential quantification*** of $P(x)$, $\exists x P(x)$, is the *proposition*:

- “Some parking spaces at UH are full.”
- or “There is a parking space at UH that is full.”
- or “At least one parking space at UH is full.”





Free and Bound Variables

- An expression like $P(x)$ is said to have a **free variable** x (meaning, x is undefined).
- A quantifier (either \forall or \exists) *operates* on an expression having one or more free variables, and **binds** one or more of those variables, to produce an expression having one or more **bound variables**.



Example of Binding

- $P(x,y)$ has 2 free variables, x and y .
- $\forall x P(x,y)$ has 1 free variable , and one bound variable . [Which is which?]
- “ $P(x)$, where $x = 3$ ” is another way to bind x .
- An expression with zero free variables is a bona-fide (actual) proposition.
- An expression with one or more free variables is not a proposition:

e.g. $\square x P(x,y) = Q(y)$



Quantifiers with Restricted Domain

- Sometimes the universe of discourse is restricted within the quantification, e.g.,
 - $\forall x > 0 P(x)$ is shorthand for
“For all x that are greater than zero, $P(x)$.”
 $= \forall x (x > 0 \rightarrow P(x))$
 - $\exists x > 0 P(x)$ is shorthand for
“There is an x greater than zero such that $P(x)$.”
 $= \exists x (x > 0 \wedge P(x))$



Translating from English

- Express the statement “*Every student in this class has studied calculus*” using predicates and quantifiers.
 - Let $C(x)$ be the statement: “*x has studied calculus.*”
 - If domain for x consists of the students in this class, then
 - it can be translated as $\forall x C(x)$

or

- If domain for x consists of all people
- Let $S(x)$ be the predicate: “*x is in this class*”
- Translation: $\forall x (S(x) \rightarrow C(x))$



Translating from English

- Express the statement “*Some students in this class has visited Mexico*” using predicates and quantifiers.
 - Let $M(x)$ be the statement: “ x has visited Mexico”
 - If domain for x consists of the students in this class, then
 - it can be translated as $\exists x M(x)$
 - or
 - If domain for x consists of all people
 - Let $S(x)$ be the statement: “ x is in this class”
 - Then, the translation is $\exists x (S(x) \wedge M(x))$



Translating from English

- Express the statement “*Every student in this class has visited either Canada or Mexico*” using predicates and quantifiers.
- Let $C(x)$ be the statement: “ x has visited Canada” and $M(x)$ be the statement: “ x has visited Mexico”
- If domain for x consists of the students in this class, then
- it can be translated as $\forall x (C(x) \vee M(x))$



Negations of Quantifiers

- $\forall x P(x)$: “Every student in the class has taken a course in calculus” ($P(x)$: “ x has taken a course in calculus”)
 - “Not every student in the class ... calculus”
 $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- Consider $\exists x P(x)$: “There is a student in the class who has taken a course in calculus”
 - “There is no student in the class who has taken a course in calculus”
 $\neg \exists x P(x) \equiv \forall x \neg P(x)$



Negations of Quantifiers

- Definitions of quantifiers: If the domain = $\{a, b, c, \dots\}$
 - $\forall x P(x) \equiv P(a) \wedge P(b) \wedge P(c) \wedge \dots$
 - $\exists x P(x) \equiv P(a) \vee P(b) \vee P(c) \vee \dots$
- From those, we can prove the laws:
 - $\neg \forall x P(x) \equiv \neg(P(a) \wedge P(b) \wedge P(c) \wedge \dots)$
 $\equiv \neg P(a) \vee \neg P(b) \vee \neg P(c) \vee \dots$
 $\equiv \exists x \neg P(x)$
 - $\neg \exists x P(x) \equiv \neg(P(a) \vee P(b) \vee P(c) \vee \dots)$
 $\equiv \neg P(a) \wedge \neg P(b) \wedge \neg P(c) \wedge \dots$
 $\equiv \forall x \neg P(x)$
- Which *propositional* equivalence law was used to prove this?



Negations of Quantifiers

Theorem:

- **Generalized De Morgan's laws for logic**

1. $\neg \forall x P(x) \equiv \exists x \neg P(x)$

2. $\neg \exists x P(x) \equiv \forall x \neg P(x)$



Negations: Examples

- What are the negations of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$?
 - $\neg \forall x (x^2 > x) \equiv \exists x \neg (x^2 > x) \equiv \exists x (x^2 \leq x)$
 - $\neg \exists x (x^2 = 2) \equiv \forall x \neg (x^2 = 2) \equiv \forall x (x^2 \neq 2)$
- Show that $\neg \forall x (P(x) \rightarrow Q(x))$ and $\exists x (P(x) \wedge \neg Q(x))$ are logically equivalent.
 - $$\begin{aligned} \neg \forall x (P(x) \rightarrow Q(x)) &\equiv \exists x \neg (P(x) \rightarrow Q(x)) \\ &\equiv \exists x \neg (\neg P(x) \vee Q(x)) \\ &\equiv \exists x (P(x) \wedge \neg Q(x)) \end{aligned}$$

Summary

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TABLE 1 Quantifiers.

<i>Statement</i>	<i>When True?</i>	<i>When False?</i>
$\forall x P(x)$	$P(x)$ is true for every x .	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x .

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TABLE 2 De Morgan's Laws for Quantifiers.

<i>Negation</i>	<i>Equivalent Statement</i>	<i>When Is Negation True?</i>	<i>When False?</i>
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x , $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x .



Nesting of Quantifiers

- Example:

Let the domain of x and y be people.

Let $L(x,y)$ = “ x likes y ” (A statement with 2 free variables – not a proposition)

- Then $\exists y L(x,y)$ = “There is someone whom x likes.” (A statement with 1 free variable x – not a proposition)

- Then $\forall x (\exists y L(x,y))$ =
“Everyone has someone whom they like.”
(A Proposition with 0 free variables.)

Nested Quantifiers

- Nested quantifiers are quantifiers that occur within the scope of other quantifiers.
- The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

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TABLE 1 Quantifications of Two Variables.		
<i>Statement</i>	<i>When True?</i>	<i>When False?</i>
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y .	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y .
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y .	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y .



Nested Quantifiers

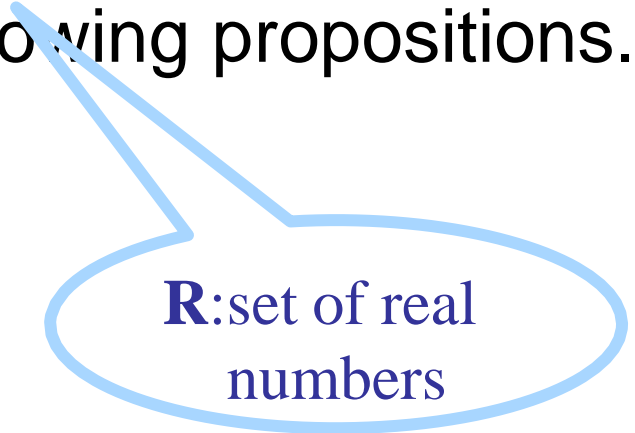
- Let the domain of x and y is \mathbf{R} , and $P(x,y): xy = 0$. Find the truth value of the following propositions.

- $\forall x \forall y P(x, y)$ (F)

- $\forall x \exists y P(x, y)$ (T)

- $\exists x \forall y P(x, y)$ (T)

- $\exists x \exists y P(x, y)$ (T)



R: set of real numbers

- $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

- For every x , there exists y such that $x + y = 0$. (T)

- There exists y such that, for every x , $x + y = 0$. (F)



Nested Quantifiers: Example

- Let the domain = $\{1, 2, 3\}$. Find an expression equivalent to $\forall x \exists y P(x,y)$ where the variables are bound by substitution instead:
 - Expand from inside out or outside in.
 - Outside in:

$$\forall x \exists y P(x,y)$$

$$\equiv \exists y P(1,y) \wedge \exists y P(2,y) \wedge \exists y P(3,y)$$

$$\equiv [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge \\ [P(2,1) \vee P(2,2) \vee P(2,3)] \wedge \\ [P(3,1) \vee P(3,2) \vee P(3,3)]$$



Quantifier Exercise

- If $R(x,y)$ = “ x relies upon y ,” express the following in unambiguous English when the domain is all people

$\forall x(\exists y R(x,y)) =$ Everyone has *someone* to rely on.

$\exists y(\forall x R(x,y)) =$

There’s a poor overburdened soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y R(x,y)) =$

There’s some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x R(x,y)) =$

Everyone has *someone* who relies upon them.

$\forall x(\forall y R(x,y)) =$

Everyone relies upon *everybody*, (including themselves)!

Negating Nested Quantifiers

- Successively apply the rules for negating statements involving a single quantifier
- Example: Express the negation of the statement $\forall x \exists y (P(x,y) \wedge \exists z R(x,y,z))$ so that all negation symbols immediately precede predicates.

$$\begin{aligned} & \neg \forall x \exists y (P(x,y) \wedge \exists z R(x,y,z)) \\ & \equiv \exists x \neg \exists y (P(x,y) \wedge \exists z R(x,y,z)) \\ & \equiv \exists x \forall y \neg (P(x,y) \wedge \exists z R(x,y,z)) \\ & \equiv \exists x \forall y (\neg P(x,y) \vee \neg \exists z R(x,y,z)) \\ & \equiv \exists x \forall y (\neg P(x,y) \vee \forall z \neg R(x,y,z)) \end{aligned}$$



Equivalence Laws

- $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

$$\exists x \exists y P(x,y) \equiv \exists y \exists x P(x,y)$$

- $\forall x (P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$

$$\exists x (P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$$

- Exercise:

See if you can prove these yourself.



Notational Conventions

- Quantifiers have higher precedence than all logical operators from propositional logic:

$$(\forall x P(x)) \wedge Q(x)$$

- Consecutive quantifiers of the same type can be combined:

$$\forall x \forall y \forall z P(x,y,z) \equiv \forall x,y,z P(x,y,z)$$

$$\text{or even } \forall xyz P(x,y,z)$$