Logic I: Fast Lecture 06

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Readings refer to sections of the course textbook, *Language, Proof and Logic*.

1. Quantifier Equivalences: $\neg\forall x\ \text{Created}(x) \dashv\vDash \exists x\ \neg\text{Created}(x)$

Reading: §10.1, §10.3, §10.4

2. Proof Example: $\exists x\ \text{Dead}(x) \vdash \neg\forall x \neg\ \text{Dead}(x)$.

1.	$\exists x\ F(x)$	
2.	$\boxed{a}\ F(a)$	
3.	$\forall y \neg F(y)$	
4.	$\neg F(a)$	$\forall$Elim: 3
5.	$\bot$	$\bot$Intro: 2,4
6.	$\neg\forall y \neg F(y)$	$\neg$Intro: 3-5
7.	$\neg\forall y \neg F(y)$	$\exists$Elim: 2-6

3. Proof Example: $\neg\forall x\ \text{Dead}(x) \vdash \exists x \neg\ \text{Dead}(x)$.

1.	$\neg\forall x\ \text{Dead}(x)$	
2.	$\neg\exists x \neg\text{Dead}(x)$	
3.	$\boxed{a}$	
4.	$\neg F(a)$	
5.	$\exists x \neg\text{Dead}(x)$	$\exists$Intro: 4
6.	$\bot$	$\bot$Intro: 2,5
7.	$\neg\neg F(a)$	$\neg$Intro: 4-6
8.	$F(a)$	$\neg$Elim: 8
9.	$\forall x\ \text{Dead}(x)$	$\forall$Intro: 3-8
10.	$\bot$	$\bot$Intro: 1,9
11.	$\neg\neg\exists x \neg\text{Dead}(x)$	$\neg$Intro: 2-10
12.	$\exists x \neg\text{Dead}(x)$	$\neg$Elim: 11

4. Quantifier Equivalences: $\forall x(\text{Square}(x) \rightarrow \text{Broken}(x)) \dashv\vDash \forall x(\neg\text{Broken}(x) \rightarrow \neg\text{Square}(x))$

Reading: §10.3

P	Q	P→Q	¬Q→¬P
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

5. There Does Not Exist

Something is not dead:

$\exists x\ \neg\text{Dead}(x)$

Nothing is dead:

$\neg\exists x\ \text{Dead}(x)$

Everything is not broken:

$\forall x\ \neg\text{Broken}(x)$

Not everything is broken:

$\neg\forall x\ \text{Broken}(x)$

1.		
2.	$a=a$	=Intro
3.	$\exists x\ (x=x)$	$\exists$Intro: 2

1.	$\neg\exists x\ \text{Dead}(x)$	
2.	$\text{Dead}(a)$	
3.	$\exists x\ \text{Dead}(x)$	$\exists$Intro: 2
4.	$\bot$	$\bot$Intro: 1,3
5.	$\neg\text{Dead}(a)$	$\neg$Intro: 2-4
6.	$\exists x\ \neg\text{Dead}(x)$	$\exists$Intro: 5

1. $\exists x\, \neg \mathrm{Dead}(x)$
2. $\neg \exists x\, \mathrm{Dead}(x)$

Counterexample:
Domain: {a,b}
Dead : {b}

6. Two Things Are Broken

Reading: §14.1

To translate sentences involving number into FOL, use identity. For example,

'Two things are broken' might be translated as:

$\exists x\, \exists y\, (\, \mathrm{Broken}(x) \wedge \mathrm{Broken}(y) \wedge \neg(x=y)\,)$

7. There Is Exactly One

There is one creator (at least one, maybe more).

$\exists x\, \mathrm{Creator}(x)$

Brian is the one and only creator.

$\mathrm{Creator}(b) \wedge \forall x(\, \mathrm{Creator}(x) \longrightarrow x{=}b\,)$

All squares are broken.

$\forall x(\, \mathrm{Sqr}(x) \longrightarrow \mathrm{Brkn}(x)\,)$

There is one and only one creator.

$\exists y(\, \mathrm{Creator}(y) \wedge \forall x(\, \mathrm{Creator}(x) \longrightarrow x{=}y\,)\,)$

or:

$\exists y(\, \forall x(\, \mathrm{Creator}(x) \leftrightarrow x{=}y\,)\,)$

8. Variables

Names : a, b, c, …

Variables : x, y, z, w, …

Variables are for saying several things about one thing even without specifying which thing it is

NB: 'Some x is a horse and x is dead' ain't English.

9. More Dead Horse

Reading: §11.4, §11.5

1	$\exists y \forall x\, \mathrm{StoreFor}(y,x)$	
2	a $\forall x\, \mathrm{StoreFor}(a,x)$	
3	$\mathrm{StoreFor}(a,a)$	$\forall$Elim: 2
4	$\exists x\, \mathrm{StoreFor}(x,x)$	$\exists$Intro: 3
5	$\exists x\, \mathrm{StoreFor}(x,x)$	$\exists$Elim: 2-4

"Tesco is a store for everything"

$\forall x\, \mathrm{StoreFor}(b,x)$

Tesco is a store for everything except dead horses

$\forall x\, (\neg \mathrm{DeadHorse}(x) \longrightarrow \mathrm{StoreFor}(b,x)\,)$

Tesco is a store for everything except Tesco

$\forall x\, (\neg x{=}b \longrightarrow \mathrm{StoreFor}(b,x)\,)$

There is a store for everything except itself

$\exists y\, \forall x\, (\neg x{=}y \longrightarrow \mathrm{StoreFor}(y,x)\,)$

10. ↔ : truth tables and rules

A	B	A ↔ B
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Elimination (↔ Elim)

$P \leftrightarrow Q$ (or $Q \leftrightarrow P$)
⋮
P
⋮
▷ Q

Biconditional Introduction (↔ Intro)

P
⋮
Q

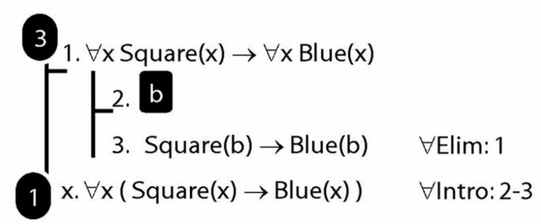
Q
⋮
P

▷ $P \leftrightarrow Q$

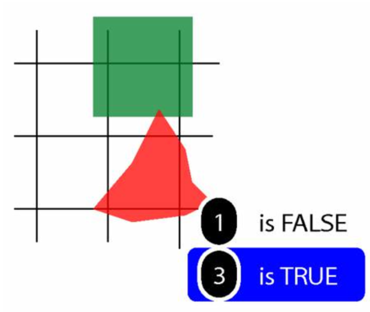
11. ∀Intro: An Incorrect Proof

Reading: §13.1, §13.2

This proof is wrong, but why?:

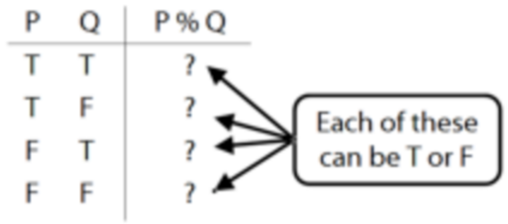


There is a counterexample to the argument:



12. How Big Is a Truth-Table?

How many truth-functions can be constructed using 2 sentence letters?



13. Truth-functional completeness

Reading: §7.4

'A set of truth-functors is said to be *expressively adequate* (or sometimes *functionally complete*) iff, for every truth-function whatever, there is a formula containing only those truth-functors which express that truth-function, i.e. which has as its truth-table the truth-table specifying that function.' (Bostock, *Intermediate Logic* p. 45)

Illustration of the proof that {¬, ∧, ∨} is truth-functionally complete:

P	Q	P → Q	
T	T	T	[P∧Q] ∨
T	F	F	
F	T	T	[¬P∧Q] ∨
F	F	T	[¬P∧ ¬Q]

[P∧Q] ∨ [¬P∧Q] ∨ [¬P∧ ¬Q]

Exercise assuming {¬,∨,∧} is truth-functionally complete, show that {¬,∨} is.

14. Soundness and Completeness: Statement of the Theorems

Reading: §8.3, §13.4

'A ⊢ B' means there is a proof of B using premises A

'⊢ B' means there is a proof of B using no premises

'A ⊨ B' means B is a logical consequence of A

'⊨ B' means B is a tautology

'A $\vDash_{TT}$ B' means B is a logical consequence of A just in virtue of the meanings of truth-functions (the textbook LPL calls this 'tautological consequence')

Soundness: If A ⊢ B then A ⊨ B

i.e. if you can prove it in Fitch, it's valid

Completeness: If A $\vDash_{TT}$ B then A ⊢ B

i.e. if it's valid just in virtue of the meanings of the truth-functional connectives, then you can prove it in Fitch.