Logic I: Lecture 17

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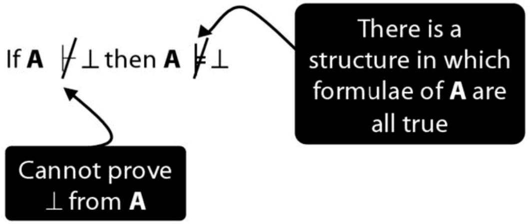
Readings refer to sections of the course textbook, *Language, Proof and Logic.*

1. The Essence of the Completeness Theorem

Reading: §8.3



Arrange the sentence letters in a series: P1, P2, P3, …

Define a structure, *h*, as follows.

Take each sentence letter, Pi, in turn.

- If **A** $\vdash$ Pi then *h*(Pi) = True

- If **A** $\vdash \neg$Pi then *h*(Pi) = False

- Otherwise *h*(Pi) = True

Every sentence of **A** is true in the structure *h*

Therefore **A** $\nvDash \bot$

2. Lemma for the Completeness Theorem

Reading: §8.3

If for every sentence letter, P, either A $\vdash$ P or A $\vdash \neg$P, then for every formula, X, either A $\vdash$ X or A $\vdash \neg$X.

Proof

Step a. Suppose (for a contradiction) that there are formulae, X, such that A $\nvdash$ X and A $\nvdash \neg$X. Take a shortest such formula, call it Y.

Step b. This formula, Y, must have one of the following forms: $\neg$P, P$\lor$Q, P$\land$Q, P$\rightarrow$Q, P$\leftrightarrow$Q, $\bot$

Step c. We can show that whichever form X has, either A $\vdash$ Y and A $\vdash \neg$Y.

Case 1: X is P$\rightarrow$Q. Then since P and Q are shorter than X, either:

(i) A $\vdash$ P and A $\vdash \neg$Q

or

(ii) A $\vdash \neg$P

or

(iii) A $\vdash$ Q

If (i), A $\vdash \neg$(P$\rightarrow$Q), that is, A $\vdash \neg$X.

If (ii), A $\vdash$ P$\rightarrow$Q, that is, A $\vdash \neg$X.

If (iii), A $\vdash$ P$\rightarrow$Q, that is, A $\vdash \neg$X.

(Here we use the last two Proofs about Proofs, see earlier)

Case 2: X is $\neg$P.

Then since P is shorter, A $\vdash$ P or A $\vdash \neg$P.

If A $\vdash$ P then A $\vdash \neg\neg$P so A $\vdash \neg$X which would contradict our assumption. This is shown in the proofs about proofs above.

If A $\vdash \neg$P then A $\vdash$ X (because X is $\neg$P), which would contradict our assumption.

Case 3: …

Step d. The demonstration in Step c contradicts our assumption, so we can conclude that it is false. That is, either A $\vdash$ X and A $\vdash \neg$X for every formula X.

3. Proof of the Completeness Theorem

Reading: §8.3, §17.1, §17.2

Suppose **A** $\nvdash \bot$. Define **A*** as every formula in **A** plus the following:

For every sentence letter, P,

if **A** $\vdash$ P, add P to **A***

if **A** $\vdash \neg$P, add $\neg$P to **A***

otherwise add P to **A***

Now **A*** $\nvdash \bot$ and **A*** contains every sentence letter or its negation

Claim: for any formula X, **A*** $\vdash$ X or **A*** $\vdash \neg$X

Define a structure, *h*, so that:

h(P)=T when P is in **A***

h(P)=F when $\neg$P is in **A***

Claim: *h*(X)=T for every X that is a logical consequence of **A*** (see Prop. 4, p. 475).

Thus **A** $\nvDash \bot$

4. Proof of Proposition 4 for the Completeness Theorem

Reading: §15.1, §15.6

5. More Records Than the KGB

Reading: §14.1, §14.3

6. The End Is Near

Reading: §14.3

'The' can be a quantifier, e.g. 'the square is broken'. How to formalise it?

The square is broken

$\dashv\vDash$ There is exactly one square and it is broken

$\dashv\vDash$ There is at most one square and there is at least one square and it is broken

$\dashv\vDash$ There is at most one square and there is at least one square and all squares are broken

$\dashv\vDash \neg\, \exists x\, \exists y\, (\, \mathrm{Square}(x) \wedge \mathrm{Square}(y) \wedge \neg x{=}y\,)$

$\quad \wedge\, \exists x\, \mathrm{Square}(x)$

$\quad \wedge\, \forall x\, (\, \mathrm{Square}(x) \longrightarrow \mathrm{Broken}(x)\,)$

Which shorter sentences are equivalent to this?

$\exists x\, (\, \mathrm{Square}(x) \wedge \forall y\, (\, \mathrm{Square}(y) \longrightarrow y{=}x\,) \wedge \mathrm{Broken}(x)\,)$

$\exists x\, (\, \forall y\, (\, \mathrm{Square}(y) \leftrightarrow y{=}x\,) \wedge \mathrm{Broken}(x)\,)$

7. Exercises

These exercises will be discussed in seminars the week after this lecture. The numbers below refer to the numbered exercises in the course textbook, e.g. '1.1' refers to exercise 1.1. on page 39 of the second edition of *Language, Proof and Logic*. Exercises marked '*' are optional.

15.33–15.40 (second edition)

14.26, 14.28