PH126 Logic I Lecture 14
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“Every time I go to the dentist, somebody dies.”
∀t ((Time(t) ∧ ToDentist(a,t)) → ∃x (Person(x) ∧ DiesAt(x,t)))

Truth-functional completeness (LPL §7.4)

Preliminary: How many truth-functions can be constructed using 2 sentence letters?

P	Q	P % Q
T	T	?
T	F	?
F	T	?
F	F	?

Each of these can be T or F

Truth-functional completeness: For any truth function, we can easily find a logically equivalent one using only ¬, ∨, ∧

Better: ‘A set of truth-functors is said to be *expressively adequate* (or sometimes *functionally complete*) iff, for every truth-function whatever, there is a formula containing only those truth-functors which express that truth-function, i.e. which has as its truth-table the truth-table specifying that function.’ (Bostock, *Intermediate L’gic* p. 45)

Illustration of how the proof that {¬, ∧, ∨ is truth-functionally complete}:

P	Q	P → Q	
T	T	T	[P∧Q] ∨
T	F	F	
F	T	T	[¬P∧Q] ∨
F	F	T	[¬P∧ ¬Q]

[P∧Q] ∨ [¬P∧Q] ∨ [¬P∧ ¬Q]

Exercise assuming {¬,∨,∧} is truth-functionally complete, show that {¬,∨} is?

Quantifier equivalences

P→Q ⫤ ⊨ ¬Q→¬P
∀x (Square(x) → Broken(x))
⫤ ⊨ ∀x (¬Broken(x) → ¬Square(x))

P→Q ⫤ ⊨ ¬P∨Q
∀x (Square(x) → Broken(x))
⫤ ⊨ ∀x (¬Square(x) ∨ Broken(x))

¬P∨Q ⫤ ⊨ ¬(P∧ ¬Q)
∃x ¬F(x) ⫤ ⊨ ¬∀x F(x)
∀x (¬Square(x) ∨ Broken(x))
⫤ ⊨ ∃¬x (Square(x) ∧ ¬Broken(x))

∃x ¬Created(x) ⫤ ⊨ ¬∀x Created(x)
Hybrid example

Every object is either non-square or broken
∀x (¬Square(x) ∨ Broken(x))

¬¬P ⫤ ⊨ P	⫤ ⊨ ¬¬∀x (¬Square(x) ∨ Broken(x))
¬∀x F(x) ⫤ ⊨ ∃x¬F(x)	⫤ ⊨ ¬∃x ¬(¬Square(x) ∨ Broken(x))
¬(P∨Q) ⫤ ⊨ ¬P∧¬Q	⫤ ⊨ ¬∃x (¬¬Square(x) ∧ ¬Broken(x))
¬¬P ⫤ ⊨ P	⫤ ⊨ Nothing is square and non-broken **¬∃x (Square(x) ∧ ¬Broken(x))**

Variables

Names : a, b, c, ...

Variables : x, y, z, w, ...

Variables are for saying several things about one thing even without specifying which thing

Someone is loved by everyone
∃x∀y Loves(y,x)

Someone loves everyone
∃x∀y Loves(x,y)

Optional extra proofs

Some not all are valid. Prove the valid ones, give counterexamples for the non-valid.

1. ∃x ∀y R(x,y) ⊢ ∀y ∃x R(x,y)
2. ∀x ¬F(x) ⊢ ¬∃x F(x)
3. ¬∃x F(x) ⊢ ∀x ¬F(x)
4. ∀x F(x) | ∀x G(x) ⊢ ∀x (Fx ∨ Gx)
5. ∀x (Fx ∨ Gx) ⊢ ∀x F(x) | ∀x G(x)
6. ∀x F(x) ∨ ∃y F(y) ⊢ ∃y ∀x (F(x) ∨ F(y))
7. ∀x (F(x) ↔ x=a), ∀x (G(x) ↔ x=b) ⊢ a=b ↔ G(a)
8. ∃y ∀x (F(x) ↔ x=y) ⊢ (F(a) ∧ F(b) → a=b)