

"Every time I go to the dentist, somebody dies."

$\forall t ((\text{Time}(t) \wedge \text{ToDentist}(a,t)) \rightarrow \exists x (\text{Person}(x) \wedge \text{DiesAt}(x,t)))$

### Truth-functional completeness (LPL §7.4)

Preliminary: How many truth-functions can be constructed using 2 sentence letters?

P	Q	P % Q
T	T	?
T	F	?
F	T	?
F	F	?

Each of these can be T or F

*Truth-functional completeness:* For any truth function, we can easily find a logically equivalent one using only  $\neg, \vee, \wedge$

Better: 'A set of truth-functors is said to be *expressively adequate* (or sometimes *functionally complete*) iff, for every truth-function whatever, there is a formula containing only those truth-functors which express that truth-function, i.e. which has as its truth-table the truth-table specifying that function.' (Bostock, *Intermediate Logic* p. 45)

Illustration of how the proof that  $\{\neg, \wedge, \vee$  is truth-functionally complete}:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$[P \wedge Q] \vee$   
 $[\neg P \wedge Q] \vee$   
 $[\neg P \wedge \neg Q]$

$$[P \wedge Q] \vee [\neg P \wedge Q] \vee [\neg P \wedge \neg Q]$$

*Exercise* assuming  $\{\neg, \vee, \wedge\}$  is truth-functionally complete, show that  $\{\neg, \vee\}$  is?

### Quantifier equivalences

$$P \rightarrow Q \quad \equiv \quad \vdash \neg Q \rightarrow \neg P$$

$$\forall x (\text{Square}(x) \rightarrow \text{Broken}(x)) \\ \equiv \quad \vdash \forall x (\neg \text{Broken}(x) \rightarrow \neg \text{Square}(x))$$

$$P \rightarrow Q \quad \equiv \quad \vdash \neg P \vee Q$$

$$\forall x (\text{Square}(x) \rightarrow \text{Broken}(x)) \\ \equiv \quad \vdash \forall x (\neg \text{Square}(x) \vee \text{Broken}(x))$$

$$\neg P \vee Q \quad \equiv \quad \vdash \neg (P \wedge \neg Q)$$

$$\exists x \neg F(x) \quad \equiv \quad \vdash \neg \forall x F(x)$$

$$\forall x (\neg \text{Square}(x) \vee \text{Broken}(x)) \\ \equiv \quad \vdash \exists \neg x (\text{Square}(x) \wedge \neg \text{Broken}(x))$$

$$\exists x \neg \text{Created}(x) \quad \equiv \quad \vdash \neg \forall x \text{Created}(x)$$

*Hybrid example*

Every object is either non-square or broken  
 $\forall x (\neg \text{Square}(x) \vee \text{Broken}(x))$

$$\neg \neg P \quad \equiv \quad \vdash P$$

$$\neg \forall x F(x) \quad \equiv \quad \vdash \exists x \neg F(x)$$

$$\neg (P \vee Q) \quad \equiv \quad \vdash \neg P \wedge \neg Q$$

$$\neg \neg P \quad \equiv \quad \vdash P$$

$$\equiv \quad \vdash \neg \neg \forall x (\neg \text{Square}(x) \vee \text{Broken}(x))$$

$$\equiv \quad \vdash \neg \exists x \neg (\neg \text{Square}(x) \vee \text{Broken}(x))$$

$$\equiv \quad \vdash \neg \exists x (\neg \neg \text{Square}(x) \wedge \neg \text{Broken}(x))$$

$$\equiv \quad \vdash \text{Nothing is square and non-broken}$$

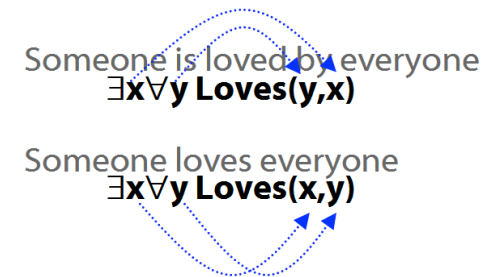
$$\equiv \quad \vdash \neg \exists x (\text{Square}(x) \wedge \neg \text{Broken}(x))$$

### Variables

Names : a, b, c, ...

Variables : x, y, z, w, ...

Variables are for saying several things about one thing even without specifying which thing



### Optional extra proofs

Some not all are valid. Prove the valid ones, give counterexamples for the non-valid.

$$1. \exists x \forall y R(x,y) \vdash \forall y \exists x R(x,y)$$

$$2. \forall x \neg F(x) \vdash \neg \exists x F(x)$$

$$3. \neg \exists x F(x) \vdash \forall x \neg F(x)$$

$$4. \forall x F(x) \mid \forall x G(x) \vdash \forall x (F(x) \vee G(x))$$

$$5. \forall x (F(x) \vee G(x)) \vdash \forall x F(x) \mid \forall x G(x)$$

$$6. \forall x F(x) \vee \exists y F(y) \vdash \exists y \forall x (F(x) \vee F(y))$$

$$7. \forall x (F(x) \leftrightarrow x=a), \forall x (G(x) \leftrightarrow x=b) \vdash \\ a=b \leftrightarrow G(a)$$

$$8. \exists y \forall x (F(x) \leftrightarrow x=y) \vdash (F(a) \wedge F(b) \rightarrow a=b)$$