PH126 Logic I Lecture 13
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This lecture is hard, skip it if you can't prove
A∨B ├ B∨A

Notation:
A ├ B means there is a proof of B using premises **A**
├ B means there is a proof of B using no premises
A ├ means the same as A ├ ⊥

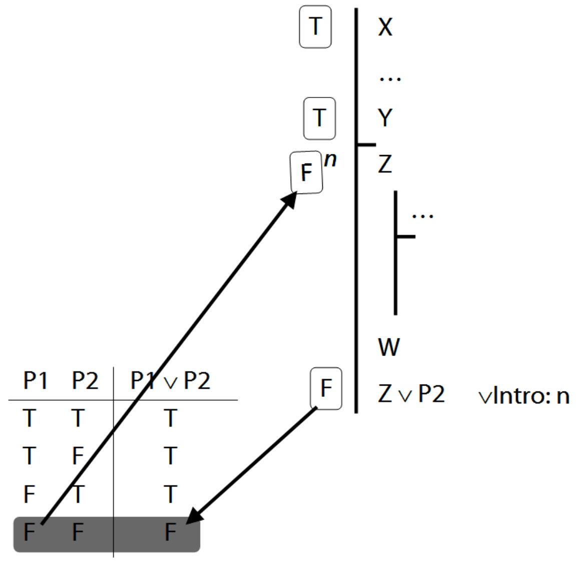
A ⊨ B means B is a logical consequence of A
⊨ B means B is a tautology
A ⊨ means the same as **A** ⊨ ⊥
A $\models_{TT}$ B means B is a logical consequence of A *just in virtue of the meanings of truth-functions* (the textbook LPL calls this 'tautological consequence')

Soundness: If **A** ├ B then **A** ⊨ B
I.e. if you can prove it in Fitch, it's valid

Completeness: If **A** $\models_{TT}$ B then **A** ├ B
I.e. if it's valid just in virtue of the meanings of truth functions, you can prove it in Fitch

Proof of Soundness (If **A** ├ B then **A** ⊨ B**)**
Useful Observation about any argument that ends with ∨Intro. Suppose this argument is not valid, i.e. the premises are true and the conclusion false. Then Z must be false. So the argument from the premises to Z (line n) is not a valid argument. So **there is a shorter proof which is not valid**.

Illustration of soundness proof: ∨Intro



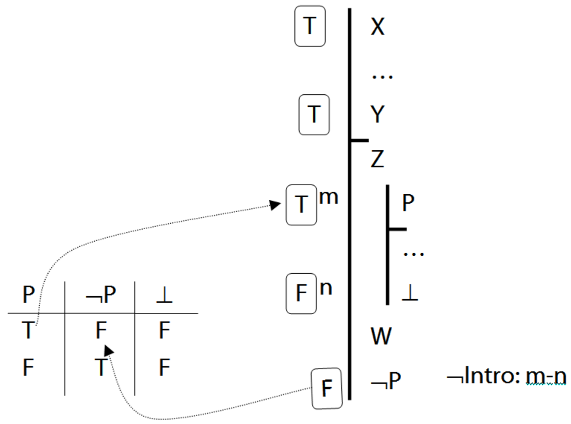
Stipulation: when I say that 'a proof is not valid', I mean that the last step of the proof is not a logical consequence of the premises (including premises of any open subproofs).

How to prove soundness? Outline

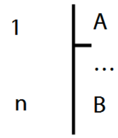
Step 1: show that each rule has this property:
- Where the last step in a proof involves that rule, if proof is not valid then there is a shorter proof which is not valid.

Step 2: Suppose (for a contradiction) that some Fitch proofs are not valid. Select one of the shortest invalid proofs. The last step must involve one of the Fitch rules. Whichever rule it involves, we know that there must be a shorter proof which is not valid. This contradicts the fact that the selected proof is a shortest invalid proof.

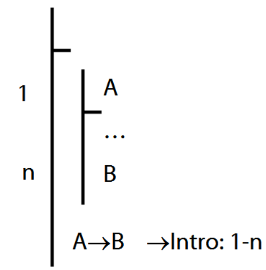
Illustration of soundness proof: ¬Intro



Proofs about proofs
1. If A ├ B then ├ A→B
Proof Given a proof for A ├ B …

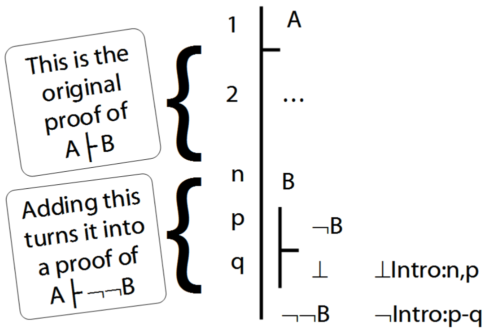


… we can turn it into a proof for ├ A→B:



2. If $\vdash A \to B$ then $A \vdash B$

3. If $A \vdash B$ then $A \vdash \neg\neg B$
Proof:



4. If $A \vdash C$ then $A \vdash B \to C$

5. If $A \vdash B$ and $A \vdash \neg C$ then $A \vdash \neg(B \to C)$

Exercises 07

For your eighth seminar

Not for fast groups

9.20 (trans.)
10.24–7, *10.28–9
*11.3
11.4, 11.8, 11.9
13.1–4 (quantifier proofs)
13.10–15 (*13.16)

Despite what book says, DON'T even THINK about using TAUT CON.

Proofs too easy? Try exercise 13.51

Translations too easy? Try 14.4–5

Short of time? Skip the exercises marked *

Lemma for Completeness Theorem

If for every sentence letter, P, either $\mathbf{A} \vdash P$ or $\mathbf{A} \vdash \neg P$, then for every formula, X, either $\mathbf{A} \vdash X$ or $\mathbf{A} \vdash \neg X$.

Proof
Step a. Suppose (for a contradiction) that there are formulae, X, such that $A \vdash X$ and $A \vdash \neg X$. Take a shortest such formula, call it Y.

Step b. This formula, Y, must have one of the following forms: $\neg P$, $P \lor Q$, $P \land Q$, $P \to Q$, P□Q, $\bot$

Step c. We can show that whichever form X has, either $A \vdash Y$ and $A \vdash \neg Y$.

Case 1: X is $P \to Q$. Then since P and Q are shorter than X, either:
(i) $\mathbf{A} \vdash P$ and $\mathbf{A} \vdash \neg Q$
or
(ii) $\mathbf{A} \vdash \neg P$
or
(iii) $\mathbf{A} \vdash Q$
If (i), $\mathbf{A} \vdash \neg(P \to Q)$, that is, $\mathbf{A} \vdash \neg X$.
If (ii), $\mathbf{A} \vdash P \to Q$, that is, $\mathbf{A} \vdash \neg X$.
If (iii), $\mathbf{A} \vdash P \to Q$, that is, $\mathbf{A} \vdash \neg X$.
(Here we use (4) and (5) from *Proofs about Proofs* above)

Case 2: X is $\neg P$. Then since P is shorter, $\mathbf{A} \vdash P$ or $\mathbf{A} \vdash \neg P$.
If $\mathbf{A} \vdash P$ then $\mathbf{A} \vdash \neg\neg P$ so $\mathbf{A} \vdash \neg X$ which would contradict our assumption. This is show in (3) above.
If $\mathbf{A} \vdash \neg P$ then $\mathbf{A} \vdash X$ (because X *is* $\neg P$), which would contradict our assumption.

Case 3: …

Step d. The demonstration in Step c contradicts our assumption, so we can conclude that it is false. That is, either $A \vdash X$ and $A \vdash \neg X$ for every formula X.

Completeness (if $\mathbf{A} \models_T B$ then $\mathbf{A} \vdash B$)

We prove only: If $\mathbf{A} \nvdash \bot$ then $\mathbf{A} \nvDash \bot$
(See Lemma 2 p. 471 for how to generalise this result.)

Suppose $\mathbf{A} \nvdash \bot$. Define $\mathbf{A}^*$ as every formula in **A** plus the following:

For every sentence letter, P,

if $\mathbf{A} \vdash P$, add P to $\mathbf{A}^*$

if $\mathbf{A} \vdash \neg P$, add $\neg P$ to $\mathbf{A}^*$

otherwise add P to $\mathbf{A}^*$

Now $\mathbf{A}^* \nvdash \bot$ and $\mathbf{A}^*$ contains every sentence letter or its negation

Claim: for any formula X, $\mathbf{A}^* \vdash X$ or $\mathbf{A}^* \vdash \neg X$

Define a structure, *h*, so that:

$h(P)=T$ when P is in $\mathbf{A}^*$

$h(P)=F$ when $\neg P$ is in $\mathbf{A}^*$

Claim: $h(X)=T$ for every X that is a logical consequence of $\mathbf{A}^*$ (see Prop. 4, p. 475).
Thus $\mathbf{A} \nvDash \bot$