PH126 Logic I Lecture 14

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"Every time I go to the dentist, somebody dies."

$\forall t\,((\mathrm{Time}(t) \wedge \mathrm{ToDentist}(a,t)) \rightarrow \exists x\,(\mathrm{Person}(x) \wedge \mathrm{DiesAt}(x,t)))$

Truth-functional completeness (LPL §7.4)

Preliminary: How many truth-functions can be constructed using 2 sentence letters?

P	Q	P % Q
T	T	?
T	F	?
F	T	?
F	F	?

Each of these can be T or F

Truth-functional completeness: For any truth function, we can easily find a logically equivalent one using only $\neg$, $\vee$, $\wedge$

Better: 'A set of truth-functors is said to be *expressively adequate* (or sometimes *functionally complete*) iff, for every truth-function whatever, there is a formula containing only those truth-functors which express that truth-function, i.e. which has as its truth-table the truth-table specifying that function.' (Bostock, *Intermediate L'gic* p. 45)

Illustration of how the proof that {$\neg$, $\wedge$, $\vee$ is truth-functionally complete}:

P	Q	P $\rightarrow$ Q	
T	T	T	$[P \wedge Q] \vee$
T	F	F	
F	T	T	$[\neg P \wedge Q] \vee$
F	F	T	$[\neg P \wedge \neg Q]$

$[P \wedge Q] \vee [\neg P \wedge Q] \vee [\neg P \wedge \neg Q]$

Exercise assuming {$\neg$,$\vee$,$\wedge$} is truth-functionally complete, show that {$\neg$,$\vee$} is?

Quantifier equivalences (continued)

$\exists x\, \neg\mathrm{Created}(x) \dashv\vDash \neg\forall x\, \mathrm{Created}(x)$

Hybrid example

	Every object is either non-square or broken $\forall x\,(\neg \mathrm{Square}(x) \vee \mathrm{Broken}(x))$
$\neg\neg P \dashv\vDash P$	$\dashv\vDash \neg\neg\forall x\,(\neg\mathrm{Square}(x) \vee \mathrm{Broken}(x))$
$\neg\forall x\, F(x) \dashv\vDash \exists x \neg F(x)$	$\dashv\vDash \neg\exists x\, \neg(\neg\mathrm{Square}(x) \vee \mathrm{Broken}(x))$
$\neg(P \vee Q) \dashv\vDash \neg P \wedge \neg Q$	$\dashv\vDash \neg\exists x\,(\neg\neg\mathrm{Square}(x) \wedge \neg\mathrm{Broken}(x))$
$\neg\neg P \dashv\vDash P$	$\dashv\vDash$ Nothing is square and non-broken $\neg\exists x\,(\mathrm{Square}(x) \wedge \neg\mathrm{Broken}(x))$

Variables

Names : a, b, c, …

Variables : x, y, z, w, …

Variables are for saying several things about one thing even without specifying which thing

Optional extra proofs

Some not all are valid. Prove the valid ones, give counterexamples for the non-valid.

1. $\exists x\, \forall y\, R(x,y) \vdash \forall y\, \exists x\, R(x,y)$
2. $\forall x\, \neg F(x) \vdash \neg\exists x\, F(x)$
3. $\neg\exists x\, F(x) \vdash \forall x\, \neg F(x)$
4. $\forall x\, F(x) \mid \forall x\, G(x) \vdash \forall x\,(Fx \vee Gx)$
5. $\forall x\,(Fx \vee Gx) \vdash \forall x\, F(x) \mid \forall x\, G(x)$
6. $\forall x\, F(x) \vee \exists y\, F(y) \vdash \exists y\, \forall x\,(F(x) \vee F(y))$
7. $\forall x\,(F(x) \leftrightarrow x=a), \forall x\,(G(x) \leftrightarrow x=b) \vdash$ $a=b \leftrightarrow G(a)$
8. $\exists y\, \forall x\,(F(x) \leftrightarrow x=y) \vdash (F(a) \wedge F(b) \rightarrow a=b)$

Exercises 07

For your eighth seminar

Not for fast groups

9.20 (trans.)

10.24–7, *10.28–9

*11.3

11.4, 11.8, 11.9

13.1–4 (quantifier proofs)

13.10–15 (*13.16)

Despite what book says, DON'T even THINK about using TAUT CON.

Proofs too easy? Try exercise 13.51

Translations too easy? Try 14.4–5

Short of time? Skip the exercises marked *

Proofs about proofs

1. If $A \vdash B$ then $\vdash A \to B$

Proof Given a proof for $A \vdash B$ ...

1	A
	…
n	B

... we can turn it into a proof for $\vdash A \to B$:

1	A	
	…	
n	B	
	$A \to B$	$\to$Intro: 1-n

2. If $\vdash A \to B$ then $A \vdash B$

3. If $A \vdash B$ then $A \vdash \neg\neg B$

Proof:

This is the original proof of $A \vdash B$

Adding this turns it into a proof of $A \vdash \neg\neg B$

1	A	
2	…	
n	B	
p	$\neg B$	
q	$\bot$	$\bot$Intro:n,p
	$\neg\neg B$	$\neg$Intro:p-q

4. If $A \vdash C$ then $A \vdash B \to C$

5. If $A \vdash B$ and $A \vdash \neg C$ then $A \vdash \neg(B \to C)$

Lemma for Completeness Theorem

If for every sentence letter, P, either $\mathbf{A} \vdash P$ or $\mathbf{A} \vdash \neg P$, then for every formula, X, either $\mathbf{A} \vdash X$ or $\mathbf{A} \vdash \neg X$.

Proof

Step a. Suppose (for a contradiction) that there are formulae, X, such that $A \vdash X$ and $A \vdash \neg X$. Take a shortest such formula, call it Y.

Step b. This formula, Y, must have one of the following forms: $\neg P$, $P \vee Q$, $P \wedge Q$, $P \to Q$, P□Q, $\bot$

Step c. We can show that whichever form X has, either $A \vdash Y$ and $A \vdash \neg Y$.

Case 1: X is $P \to Q$. Then since P and Q are shorter than X, either:
(i) $\mathbf{A} \vdash P$ and $\mathbf{A} \vdash \neg Q$
or
(ii) $\mathbf{A} \vdash \neg P$
or
(iii) $\mathbf{A} \vdash Q$
If (i), $\mathbf{A} \vdash \neg(P \to Q)$, that is, $\mathbf{A} \vdash \neg X$.
If (ii), $\mathbf{A} \vdash P \to Q$, that is, $\mathbf{A} \vdash \neg X$.
If (iii), $\mathbf{A} \vdash P \to Q$, that is, $\mathbf{A} \vdash \neg X$.
(Here we use (4) and (5) from *Proofs about Proofs* above)

Case 2: X is $\neg P$. Then since P is shorter, $\mathbf{A} \vdash P$ or $\mathbf{A} \vdash \neg P$.
If $\mathbf{A} \vdash P$ then $\mathbf{A} \vdash \neg\neg P$ so $\mathbf{A} \vdash \neg X$ which would contradict our assumption. This is show in (3) above.
If $\mathbf{A} \vdash \neg P$ then $\mathbf{A} \vdash X$ (because X *is* $\neg P$), which would contradict our assumption.

Case 3: ...

Step d. The demonstration in Step c contradicts our assumption, so we can conclude that it is false. That is, either $A \vdash X$ and $A \vdash \neg X$ for every formula X.

Completeness (if $\mathbf{A} \vDash_T B$ then $\mathbf{A} \vdash B$)

We prove only: If $\mathbf{A} \nvdash \bot$ then $\mathbf{A} \nvDash \bot$
(See Lemma 2 p. 471 for how to generalise this result.)

Suppose $\mathbf{A} \nvdash \bot$. Define $\mathbf{A}^*$ as every formula in $\mathbf{A}$ plus the following:

For every sentence letter, P,

if $\mathbf{A} \vdash P$, add P to $\mathbf{A}^*$

if $\mathbf{A} \vdash \neg P$, add $\neg P$ to $\mathbf{A}^*$

otherwise add P to $\mathbf{A}^*$

Now $\mathbf{A}^* \nvdash \bot$ and $\mathbf{A}^*$ contains every sentence letter or its negation

Claim: for any formula X, $\mathbf{A}^* \vdash X$ or $\mathbf{A}^* \vdash \neg X$

Define a structure, *h*, so that:

h(P)=T when P is in $\mathbf{A}^*$

h(P)=F when $\neg P$ is in $\mathbf{A}^*$

Claim: *h*(X)=T for every X that is a logical consequence of $\mathbf{A}^*$ (see Prop. 4, p. 475).
Thus $\mathbf{A} \nvDash \bot$