Binary geometries: Associahedra, Cyclohedra and Generalized Permutahedra

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Abstract

In [] the study of stringy canonical forms and binary geometries with "perfect" u equations, associated with the scattering of particles and strings was initiated. In this paper we continue the study of binary geometries and find two large classes of new examples. The first class corresponds to degenerations of A_n and B_n (associahedra and cyclohedra respectively) which have perfect u-equations. The second class corresponds to a large subset of generalized permutahedra which can be realised as degenerations of the permutahedron P_n which are binary positive geometries despite not having perfect u-equations. Both these large classes of examples have stringy integrals which factorise at any finite α' on all the massless poles.

Contents

1 Introduction

In [] the notions of stringy canonical forms and binary geometries were introduced which helped in understanding the configuration spaces of clusters as generalisations of moduli space for scattering of particles and strings .

It is a natural question to ask if binary geometries whose stringy integrals factorise at any finite α' are extremely special and are associated to only the classical A_n, B_n, C_n, D_n or Exceptional E_6, E_7, E_8, F_2, G_4 type clusters as discussed in []. In this paper we answer this question in the negative by providing infinitely many counter examples. These fall broadly into two classes, the first of which corresponds to the permutahedron P_n and more generally generalised permutahedra which do not have perfect u-equations but are still binary. The second class corresponds to various degenerations of the associahedra and cyclohedra which are binary geometries with perfect u-equations. In both these cases the configuration space can be realised as hyperplane arrangements and we discuss how this allows to understand why some degenerations give us binary geometries with perfect u-equations while others do not.

1.1 Invitation: stringy canonical forms and cluster configuration spaces

review of stringy canonical forms, mention big polytopes and u variables in general; then specify to cluster string integrals, then cluster configuration spaces with perfect u equations, do A_n, B_n examples; define "binary geometries"

2 Stringy canonical forms and u-equations for generalized permutahedra

In this section we shall argue that a large subset of generalised permutahedra which are realised as degenerations of permutahderon P_n are binary geometries. Before we proceed we shall review some details about the generalised permutahedra which we shall use throughout the paper.

2.1 Generalized permutohedra

the definition, facets and combinatorial factorizations, do P_n example in full details.

2.2 Natural stringy integrals with linear factors

introduce a natural stringy integrals for any generalized stringy canonical forms: variables $x_0 = 1, x_1, \cdots, x_n$ and define $x_I = \sum_{i \in I} x_i$, the factors are $x_I^{\alpha' S_I}$ (we can say monomials $S\{i\} = X_i$ and polynomials $S_I = -c_I$. $N = n + m$ and big polytope is a simplex, then general formula for u variables, which is equivalent to ABHY conditions; again P_n in full details and others as degenerations.

2.3 Binary geometries and u equations

argue in general to have binary geometries we must have $1 - u = \prod u' p(\{u\})$ with $p(u = 0) = 1$, conjecture that this is true fo ALL gen. perm. Show it for gen. perm. more examples?

3 Configuration spaces with perfect u-equations from degenerating A_n and B_n

In this section we shall consider some degenerations of A_n and B_n and show that these form an infinite class of examples of binary geometries with perfect u-equations. To do this we shall use the fact that both A_n and B_n are generalised permutahedra and can be realised as a Minkowski sum of coordinate simplices.

3.1 A_n and B_n as generalized permutohedra

linear factors, (standard) string integrals, ABHY and u variables, talk about hyperplane arrangement

3.2 Degenerations of A_n and B_n with perfect u equations

things I wrote in my handwritten notes...

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Permutahedron

For $x_1, x_2, \cdots, x_{n+1} \in \mathbb{R}$, the permutahedron $P_n(x_1, \cdots, x_{n+1})$ is a convex polytope in $\mathbb{R}^{n+1}$ defined as the convex hull of all vectors obtained from $(x_1, x_2, \cdots, x_{n+1})$ by permutations of the coordinates:

$$P_n(x_1, x_2, \cdots, x_{n+1}) = ConvexHull\{(x_{w(1)}, x_{w(2)}, \cdots, x_{w(n+1)}) \mid w \in S_{n+1}\},$$

where S_{n+1} is the symmetric group.

The permutahedron has $(n+1)!$ vertices and is of dimension at most n since it lies in the hyperplane

$$H_c = \{(t_1, t_2, \cdots, t_{n+1}) | t_1 + t_2 + \cdots + t_{n+1} = c\} \subset \mathbb{R}^{n+1},$$

where $c = x_1 + x_2 + \cdots + x_{n+1}$

For $n = 2$ and distinct x_1, x_2, x_3 the permutahedron $P_2(x_1, x_2, x_3)$ is a hexagon. If two of the x_i's are equal then the permutahedron degenerates into a triangle and if $x_1 = x_2 = x_3$ then its degenerates into a single point.

We shall state a few results about permutahedra:

Rado's theorem: For any $x_1 \geq x_2 \cdots \geq x_{n+1}$ a point $(t_1, t_2, \cdots, t_{n+1}) \in \mathbb{R}^{n+1}$ belongs to the permutahedron $P_n(t_1, t_2, \cdots, t_{n+1})$ if and only if

$$t_1 + \cdots + t_{n+1} = x_1 + \cdots + x_{n+1}$$

and for any nonemepty subset $\{i_1, \cdots, i_k\} \subset \{1, \cdots, n+1\}$, we have

$$t_{i_1} + \cdots + t_{i_k} \leq x_1 + \cdots + x_k$$

The combinatorial structure of the permutahedron $P_n(x_1, \cdots, x_{n+1})$ does not depend on $x_1, \cdots, x_{n+1}$ as long as all these are distinct.

Proposition 2: For any $x_1 > \cdots > x_{n+1}$. The d-dimensional faces of $P_n(x_1, \cdots, x_{n+1})$ are in one-to-one correspondence with the disjoint subdivisions of the corresponding set $\{x_1, \cdots, x_{n+1}\}$ into nonempty ordered blocks $B_1 \cup B_2 \cup \cdots \cup B_{n+1-d} = \{1, \cdots, n+1\}$. The face corresponding to $B_1, \cdots, B_{n+1-d}$ is given by the $n+1-d$ linear equations

$$\sum_{i \in B_1 \cup \cdots \cup B_k} t_i = x_1 + \cdots + x_{|B_1 \cup \cdots \cup B_k|}, \quad for\ k = 1, \cdots, n+1-d$$

Generalized permutahedra

Generalized permutahedra are polytopes which are deformations of the usual permutahedron i.e., obtained by moving the vertices of the usual permutahedron so that the directions of all the edges are preserved though some of the edges may degenerate into points.

Since each generalised permutahedron is obtained by parallel translation of facets of the usual permutahedron it is parametrized by a collection $\{z_I\}$ of $2^{n+1} - 1$ coordinates, for non-empty sets of $I \subset [n+1] := \{1, \cdots, n+1\}$

$$P_n^z(\{z_I\}) = \left\{ (t_1, t_2, \cdots, t_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} t_i = z_{[n+1]},\ \sum_{i \in I} t_i \geq z_I,\ for\ subsets\ I \right\}$$

If $z_I = z_J$ whenever $|I| = |J|$, then $P_n^z(\{z_I\})$ is the usual permutahedron.

A different construction of the generalised permutahedron is the following :

Let, $\Delta_{[n+1]} = ConvexHull(e_1, \cdots, e_n)$ be the standard coordinate simplex in $\mathbb{R}^{n+1}$. For any $I \subset [n+1]$ let $\Delta_I = ConvexHull(e_i \mid i \in I)$ denote the face of the $\Delta_{[n+1]}$. The polytope $P_n^y(\{y_I\})$ obtained as the Minkowski sum of simplices Δ_I scaled by parameters $y_I \geq 0$ for all nonempty subsets $I \subset [n+1]$

$$P_n^y(\{y_I\}) = \sum_{I \subset [n+1]} y_I . \Delta_I$$

is the generalised permutahedron $P_n^z(\{z_I\})$ provided $z_I = \sum_{J \subset I} y_J\ for\ all\ nonempty\ I \subset [n+1]$. Note that all generalised permutahedra cannot be written as Minkowski sum of coordinate simplices and we shall restrict ourselves to the large class of generalised permutahedra which admit such a realisation.

Nested Complex

Since the combinatorial structure of the generalised permutahedron depends only on the set B of nonemepty subsets $I \subset [n+1]$ such that $y_I \geq 0$ which is called the *building set*. We can describe the combinatorial structure when B additionally satisfies the following:
(1) If $I, J \in B$ and $I \cap J \neq \phi$, then $I \cup J \in B$.
(2) B contains all singletons $\{i\}$ for $i \in S$.

A subset N in the building set B is called a *nested set* if it satisfies the following conditions:
(1) For any $I, J \in N$, we either have $I \subset J$ or $J \subset I$ or I and J are disjoint.
(2) For any collection of $k \geq 2$ disjoint subsets $J_1, J_2, \cdots, J_k \in N$ their union $J_1 \cup \cdots \cup J_k$ is not in B.
(3) N contains all maximal elements of B.

The *nested complex* $\mathcal{N}(B)$ is defined as the poset of the set of all nested sets in B ordered by inclusion.

Theorem 1: Let us assume that the set B associated with a generalised permutahedron P_n^y is a building set on $[n+1]$. Then the poset of faces P_n^y ordered by reverse inclusion is isomorphic to the nested complex $\mathcal{N}(B)$.

Theorem 2: Let us assume that the set B associated with a generalised permutahedron P_n^y is a building set on $[n+1]$. The face P_N of $P_n^y(y_I)$ associated with the nested set $N \in \mathcal{N}(B)$ is given by:

$$P_N = \{(t_1, \cdots, t_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i \in I} t_i = y_I \ for\ I \in N;\ \sum_{i \in J} t_i \geq y_J, for J \in B)\} \tag{1}$$

In particular the dimension of the face P_N equals $n + 1 - |N|$.

In summary the above results imply that we can look at any collection of subsets of $[n+1]$ which form a building set B and associate coordinate simplex Δ_I for each $I \in B$ and resulting Minkowski sum with positive weights y_I generates a generalised permutahedron associated with the building set. Further, the number of facets of the generalised permutahedron just correspond to the set of all non-singlet elements in B. We shall use $\{0, 1, \cdots, n\}$ instead of $[n+1]$ with $x_0 = 1$ from now on. Since, the number of singlets correspond to the dimension of the generalised permutahedron this implies that:

Number of facets = Number of linear equation + dimension of gen permutahedron

Thus, generalised permutahedra have "Big Polytope" which correspond to a simplex and we can write down the stringy canonical forms and solve for the u-variables and examine if they satisfy some kind of u-equations.

Here are some interesting examples of generalised permutahedra:

(1) If B consists of only singlets i.e., $B = \{\{0,1,\cdots,n\},\{0\},\{1\},\cdots,\{n\}\}$ then the generalised permutahedron is a Simplex. In this case the relevant x variables are $x_i,\ i = 0,\cdots,n$ and $\sum_{i=0}^{n} x_i$. The Newton polytope of the Minkowski sum is $\prod_{i=1}^{n} x_i(1+\sum_{j=1}^{n} x_j)$ and u-variables are $u_i = \frac{x_i}{1+\sum_{i=1}^{n} x_i}$ which satisfy $\sum u_i = 1$ as their only u-equation.

(2) If $B = \{[i]|i = 1,\cdots,n+1\}$ is the complete flag of intervals, then $P_n(\mathbf{Y})$ is the Stanley-Pitman polytope or Hypercube. The Newton polytope of the Minkowski sum is $x_1\cdots x_n(1+x_1)\cdots(1+x_1+\cdots+x_n)$ and u-variables are $u_i = \frac{1}{1+\sum_{j=1}^{i} x_j}, \quad u_i^{'} = \frac{\sum_{j=1}^{i} x_j}{1+\sum_{j=1}^{i} x_j}$ for $j = 1,\cdots,n$ which satisfy $u_i + u_i^{'} = 1$ as their u-equation.

(3) If B corresponds to all the non empty subsets of $\{0,1,\cdots,n\}$ and $Y_I = y_{|I|}$ i.e., the variables Y_I are equal for all subsets of the same cardinality, then $P_n(\mathbf{Y})$ is the usual permutahedron P_n.

ABHY like realisation for the Permutahedron

The following set of equations to define the n-dimensional Permutahedron:

$$C_I = (-1)^{|I|}\left(\sum_{i\in I} X_i - \sum_{\substack{i<j\\ i,j\in I}} X_{ij} + \cdots + (-1)^{|I|}X_I\right)$$

for all non-empty subsets $I \subset \{0,1,\cdots,n\}$ with the understanding that $C_I = 0$ for all singlets I and $X_I = 0$ for $I = \{0,1,\cdots,n\}$.

It is clear that

$$\begin{aligned} m = \text{No. of C's} &= 2^{n+1} - (n+2) \\ N = \text{No. of X's} &= 2^{n+1} - 2 \end{aligned}$$

Thus we see that $N = d + m$ and hence the "Big polytope" in this case is again a simplex. We can write down the stringy integral as usual to be

$$\int_{\mathbb{R}_+^n} \prod_{i=1}^{n} dx_i x_i^{\alpha^{'} X_i - 1} \prod_I \left(\sum_{a\in I} x_a\right)^{-\alpha^{'} C_I}$$

where the product is over all non-singlets $I \subset \{0,1,\cdots,n\}$ with $x_0 = 1$.

We can then solve for the corresponding $u's$:

$$
\begin{aligned}
u_J &= \prod_{J\subset I} x_I^{(-1)^{|I|-|J|}} \\
u_J^{'} &= \prod_{\substack{0\in I \\ \{0,\cdots,n\}-I\subset J}} x_I^{(-1)^{|I|-|J|+mod(n,2)}}
\end{aligned}
$$

2d permutahedron

In this case the building set $B = \{\{0,1,2\},\{0,1\},\{0,2\},\{1,2\},\{0\},\{1\},\{2\}\}$ and the relevant x variables are $x_0 = 1,\ x_1 = x,\ x_2 = y,\ x_{01} = 1+x,\ x_{02} = 1+y,\ x_{12} = x+y,\ x_{012} = 1+x+y$.

We can write down the Newton polynomial for the Minkowski sum as $x_1\ x_2\ (1+x_1)\ (1+x_2)\ (x_1+x_2)\ (1+x_1+x_2)$.

The u-variables can be written in terms of x variables as:

$$
\begin{aligned}
u_1 &= \frac{x(1+x+y)}{(x+y)(1+x)},\ u_2 = \frac{y(1+x+y)}{(x+y)(1+y)},\ u_{12} = \frac{(x+y)}{(1+x+y)} \\
u_1^{'} &= \frac{(1+y)}{(1+x+y)}, \qquad u_2^{'} = \frac{(1+x)}{(1+x+y)}, \qquad u_{12}^{'} = \frac{(1+x+y)}{(1+x)(1+y)}
\end{aligned}
$$

The u-equations are:

$$
\begin{aligned}
1-u_i &= (u_i^{'})^2 u_j u_{12}^{'}, \quad 1-u_{12} = u_1^{'}u_2^{'}u_{12}^{'} \\
1-u_i^{'} &= u_i u_j^{'} u_{12}, \qquad 1-u_{12}^{'} = u_1 u_2 (u_{12})^2
\end{aligned}
$$

where $i=1$ implies $j=2$ and vice versa. Let us look at the $n=3$ example.

3d permutahedron

In this case the building set $B = \{\{0,1,2,3\},\{0,1,2\},\{0,2,3\},\{0,1,3\},\{1,2,3\},\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{2,3\},\{1,3\},\{0\},\{1\},\{2\},\{3\}\}$ and the relevant x variables are $x_0 = 1,\ x_1 = x,\ x_2 = y,\ x_3 = z,\ x_{01} = 1+x,\ x_{02} = 1+y,\ x_{03} = 1+z,\ x_{12} = x+y,\ x_{13} = x+z,\ x_{23} = y+z,\ x_{012} = 1+x+y,\ x_{013} = 1+x+z,\ x_{023} = 1+y+z,\ x_{123} = x+y+z,\ x_{0123} = 1+x+y+z$.

The u and u'-variables can be written interms of x variables as:

$$
\begin{aligned}
u_1 &= \frac{x(1+x+y)(1+x+z)(x+y+z)}{(x+y)(x+z)(1+x)(1+x+y+z)}, \; u_2 = \frac{y(1+x+y)(1+y+z)(x+y+z)}{(x+y)(y+z)(1+y)(1+x+y+z)}, \; u_3 = \frac{z(1+x+z)(1+y+z)(x+y+z)}{(x+z)(y+z)(1+z)(1+x+y+z)},\\
u_1^{'} &= \frac{(1+y+z)}{(1+x+y+z)}, \qquad u_2^{'} = \frac{(1+x+z)}{(1+x+y+z)}, \qquad u_3^{'} = \frac{(1+x+y)}{(1+x+y+z)},\\
u_{12} &= \frac{(x+y)(1+x+y+z)}{(1+x+y)(x+y+z)}, \qquad u_{23} = \frac{(y+z)(1+x+y+z)}{(1+y+z)(x+y+z)}, \qquad u_{13} = \frac{(x+z)(1+x+y+z)}{(1+x+z)(x+y+z)},\\
u_{12}^{'} &= \frac{(1+z)(1+x+y+z)}{(1+y+z)(1+x+z)}, \qquad u_{23}^{'} = \frac{(1+x)(1+x+y+z)}{(1+x+y)(1+x+z)}, \qquad u_{13}^{'} = \frac{(1+y)(1+x+y+z)}{(1+x+y))(1+y+z)},\\
u_{123} &= \frac{(x+y+z)}{(1+x+y+z)}, \qquad u_{123}^{'} = \frac{(1+x+y)(1+y+z)}{(1+x)(1+y)(1+z)(1+x+y+z)}
\end{aligned}
$$

The u-equations in this case are:

$$
\begin{aligned}
1-u_i &= u_j u_k (u_{jk})^2 (u_i^{'})^3 (u_{ij}^{'})^2 (u_{ik}^{'})^2 u_{123}^{'} \left(1+u_i u_{ij} u_{ik} u_j^{'} u_k^{'} u_{jk}^{'}\right), \quad 1-u_{ij} = u_k u_{ik} u_{jk} (u_i^{'})^2 (u_j^{'})^2 (u_{ij}^{'})^2 u_{ik}^{'} u_{jk}^{'} u_{123}^{'}\\
1-u_i^{'} &= u_i u_{ij} u_{ik} u_j^{'} u_k^{'} u_{jk}^{'}, \qquad 1-u_{ij}^{'} = u_i u_j (u_{ij})^2 u_{ik} u_{jk} (u_{123})^2 (u_k^{'})^2 u_{ik}^{'} u_{jk}^{'}\\
1-u_{123}^{'} &= u_1 u_2 u_3 (u_{12})^2 (u_{23})^2 (u_{13})^2 (u_{123})^3 \left(1+u_{123}^{'} u_1^{'} u_2^{'} u_3^{'} u_{12}^{'} u_{23}^{'} u_{13}^{'}\right), 1-u_{123} = u_1^{'} u_2^{'} u_3^{'} u_{12}^{'} u_{23}^{'} u_{13}^{'} u_{123}^{'}
\end{aligned}
$$

where $i, j, k \in 1, 2, 3$ and $i \neq j \neq k$.

The 8 facets of the 3d permutahedron corresponding to $u_i \to 0, u_i^{'} \to 0, u_{123} \to 0$ *and* $u_{123}^{'} \to 0$ are all B_2's. Similarly, the 6 facets corresponding to $u_{ij} \to 0, u_{ij}^{'} \to 0$ are all A_1^2.

Thus, the 3d permutahedron integral factorizes nicely on all massless poles at finite $\alpha^{'}$!!

4d permutahedron

The u variables for the 4d permutahedron are:

$$u_1 = \frac{x(w+x+1)(x+y+1)(w+x+y)(x+z+1)(w+x+z)(x+y+z)(w+x+y+z+1)}{(x+1)(w+x)(x+y)(w+x+y+1)(x+z)(w+x+z+1)(x+y+z+1)(w+x+y+z)},$$
$$u_2 = \frac{y(w+y+1)(x+y+1)(w+x+y)(y+z+1)(w+y+z)(x+y+z)(w+x+y+z+1)}{(y+1)(w+y)(x+y)(w+x+y+1)(y+z)(w+y+z+1)(x+y+z+1)(w+x+y+z)}$$
$$u_3 = \frac{z(w+z+1)(x+z+1)(w+x+z)(y+z+1)(w+y+z)(x+y+z)(w+x+y+z+1)}{(z+1)(w+z)(x+z)(w+x+z+1)(y+z)(w+y+z+1)(x+y+z+1)(w+x+y+z)},$$
$$u_4 = \frac{w(w+x+1)(w+y+1)(w+x+y)(w+z+1)(w+x+z)(w+y+z)(w+x+y+z+1)}{(w+1)(w+x)(w+y)(w+x+y+1)(w+z)(w+x+z+1)(w+y+z+1)(w+x+y+z)},$$
$$u_{12} = \frac{(x+y)(w+x+y+1)(x+y+z+1)(w+x+y+z)}{(x+y+1)(w+x+y)(x+y+z)(w+x+y+z+1)},\ u_{13} = \frac{(x+z)(w+x+z+1)(x+y+z+1)(w+x+y+z)}{(x+z+1)(w+x+z)(x+y+z)(w+x+y+z+1)},$$
$$u_{14} = \frac{(w+x)(w+x+y+1)(w+x+z+1)(w+x+y+z)}{(w+x+1)(w+x+y)(w+x+z)(w+x+y+z+1)}, u_{24} = \frac{(w+y)(w+x+y+1)(w+y+z+1)(w+x+y+z)}{(w+y+1)(w+x+y)(w+y+z)(w+x+y+z+1)},$$
$$u_{23} = \frac{(y+z)(w+y+z+1)(x+y+z+1)(w+x+y+z)}{(y+z+1)(w+y+z)(x+y+z)(w+x+y+z+1)},\ u_{34} = \frac{(w+z)(w+x+z+1)(w+y+z+1)(w+x+y+z)}{(w+z+1)(w+x+z)(w+y+z)(w+x+y+z+1)},$$
$$u_{123} = \frac{(x+y+z)(w+x+y+z+1)}{(x+y+z+1)(w+x+y+z)},\ u_{134} = \frac{(w+x+z)(w+x+y+z+1)}{(w+x+z+1)(w+x+y+z)},\ u_{124} = \frac{(w+x+y)(w+x+y+z+1)}{(w+x+y+1)(w+x+y+z)},\ u_{234} = \frac{(w+y+z)(w+x+y+z+1)}{(w+y+z+1)(w+x+y+z)},$$
$$u_{1234} = \frac{w+x+y+z}{w+x+y+z+1},$$
$$u'_1 = \frac{w+y+z+1}{w+x+y+z+1},\quad u'_2 = \frac{w+x+z+1}{w+x+y+z+1}, u'_3 = \frac{w+x+y+1}{w+x+y+z+1}, u'_4 = \frac{x+y+z+1}{w+x+y+z+1},$$
$$u'_{12} = \frac{(w+z+1)(w+x+y+z+1)}{(w+x+z+1)(w+y+z+1)}, u'_{13} = \frac{(w+y+1)(w+x+y+z+1)}{(w+x+y+1)(w+y+z+1)},$$
$$u'_{14} = \frac{(y+z+1)(w+x+y+z+1)}{(w+y+z+1)(x+y+z+1)}, u'_{23} = \frac{(w+x+1)(w+x+y+z+1)}{(w+x+y+1)(w+x+z+1)},$$
$$u'_{24} = \frac{(x+z+1)(w+x+y+z+1)}{(w+x+z+1)(x+y+z+1)}, u'_{34} = \frac{(x+y+1)(w+x+y+z+1)}{(w+x+y+1)(x+y+z+1)},$$
$$u'_{123} = \frac{(w+1)(w+x+y+1)(w+x+z+1)(w+y+z+1)}{(w+x+1)(w+y+1)(w+z+1)(w+x+y+z+1)},\ u'_{124} = \frac{(z+1)(w+x+z+1)(w+y+z+1)(x+y+z+1)}{(w+z+1)(x+z+1)(y+z+1)(w+x+y+z+1)},$$
$$u'_{134} = \frac{(y+1)(w+x+y+1)(w+y+z+1)(x+y+z+1)}{(w+y+1)(x+y+1)(y+z+1)(w+x+y+z+1)}, u'_{234} = \frac{(x+1)(w+x+y+1)(w+x+z+1)(x+y+z+1)}{(w+x+1)(x+y+1)(x+z+1)(w+x+y+z+1)},$$
$$u'_{1234} = \frac{(w+x+1)(w+y+1)(x+y+1)(w+z+1)(x+z+1)(y+z+1)(w+x+y+z+1)}{(w+1)(x+1)(y+1)(w+x+y+1)(z+1)(w+x+z+1)(w+y+z+1)(x+y+z+1)}$$

The u equations are

$$
\begin{aligned}
1-u_1 &= u_2u_3u_4u_{2,3}^2u_{24}^2u_{34}^2u_{234}^4\left(u'\right)_1^8\left(u'\right)_{12}^4\left(u'\right)_{13}^4\left(u'\right)_{14}^4\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{134}^2u'_{1234}+6\,u_1^2u_2u_3u_4u_{12}^2u_{13}^2u_{14}^2u_{23}^2u_{24}^2u_{34}^2u_{123}^2u_{124}^2u_{134}^2u_{234}^3u_{1234}u'_2u'_3u'_4\left(u'\right)_1^4\\
&\quad\left(u'\right)_{12}^3\left(u'\right)_{13}^3\left(u'\right)_{14}^3\left(u'\right)_{23}^2\left(u'\right)_{24}^2\left(u'\right)_{34}^2\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2u'_{1234}+\ 2\ u_1u_2u_3u_4u_{12}u_{13}u_{14}u_{23}^2u_{24}^2u_{34}^2u_{123}u_{124}u_{134}u_{234}^3\left(u'\right)_1^4\left(u'\right)_{12}^3\left(u'\right)_{13}^3\\
&\quad\left(u'\right)_{14}^3u'_{23}u'_{24}u'_{34}\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{134}^2u'_{234}\left(u_{234}\left(1-u_{1234}\right)u_{1234}u'_1+u'_2u'_3\left(1-u'_{23}\right)+u'_2u'_4\left(1-u'_{24}\right)+u'_3u'_4\left(1-u'_{34}\right)+\left(u'\right)_1^2\right)u'_{1234},\\
1-u_2 &= u_1u_3u_4u_{13}^2u_{14}^2u_{34}^2u_{134}^4\left(u'\right)_2^8\left(u'\right)_{12}^4\left(u'\right)_{23}^4\left(u'\right)_{24}^4\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{234}^2u'_{1234}+\ 6u_1u_2^2u_3u_4u_{1,2}^2u_{13}^2u_{14}^2u_{23}^2u_{24}^2u_{34}^2u_{123}^2u_{124}^2u_{134}^3u_{234}^2u_{1234}u'_1u'_3u'_4\left(u'\right)_2^4\\
&\quad\left(u'\right)_{12}^3\left(u'\right)_{13}^2\left(u'\right)_{14}^2\left(u'\right)_{23}^3\left(u'\right)_{24}^3\left(u'\right)_{34}^2\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2u'_{1234}+\ 2\ u_1u_2u_3u_4u_{12}u_{13}^2u_{14}^2u_{23}u_{24}u_{34}^2u_{123}u_{124}u_{134}^3u_{234}\left(u'\right)_2^4\left(u'\right)_{12}^3u'_{13}u'_{14}\\
&\quad\left(u'\right)_{23}^3\left(u'\right)_{24}^3u'_{34}\left(u'\right)_{123}^2\left(u'\right)_{124}^2u'_{134}\left(u'\right)_{234}^2\left(u_{134}\left(1-u_{1234}\right)u_{1234}u'_2+u'_1u'_3\left(1-u'_{13}\right)+u'_1u'_4\left(1-u'_{14}\right)+u'_3u'_4\left(1-u'_{34}\right)+\left(u'\right)_2^2\right)u'_{1234},\\
1-u_3 &= u_1u_2u_4u_{12}^2u_{14}^2u_{24}^2u_{124}^4\left(u'\right)_3^8\left(u'\right)_{13}^4\left(u'\right)_{23}^4\left(u'\right)_{34}^4\left(u'\right)_{123}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2u'_{1234}+\ 6u_1u_2u_3^2u_4u_{12}^2u_{13}^2u_{14}^2u_{23}^2u_{24}^2u_{34}^2u_{123}^2u_{124}^3u_{134}^2u_{234}^2u_{1234}u'_1u'_2u'_4\left(u'\right)_3^4\\
&\quad\left(u'\right)_{12}^2\left(u'\right)_{13}^3\left(u'\right)_{14}^2\left(u'\right)_{23}^3\left(u'\right)_{24}^2\left(u'\right)_{34}^3\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2u'_{1234}+\ 2\ u_1u_2u_3u_4u_{1,2}^2u_{13}u_{14}^2u_{23}u_{24}^2u_{34}u_{123}u_{124}^3u_{134}u_{234}\left(u'\right)_3^4u'_{12}\left(u'\right)_{13}^3u'_{14}\\
&\quad\left(u'\right)_{23}^3u'_{24}\left(u'\right)_{34}^3\left(u'\right)_{123}^2u'_{124}\left(u'\right)_{134}^2\left(u'\right)_{234}^2\left(u_{124}\left(1-u_{1234}\right)u_{1234}u'_3+u'_1u'_2\left(1-u'_{12}\right)+u'_1u'_4\left(1-u'_{14}\right)+u'_2u'_4\left(1-u'_{24}\right)+\left(u'\right)_3^2\right)u'_{1234},\\
1-u_4 &= u_1u_2u_3u_{12}^2u_{13}^2u_{23}^2u_{123}^4\left(u'\right)_4^8\left(u'\right)_{14}^4\left(u'\right)_{24}^4\left(u'\right)_{34}^4\left(u'\right)_{124}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2u'_{1234}+\ 6u_1u_2u_3u_4^2u_{12}^2u_{13}^2u_{14}^2u_{23}^2u_{24}^2u_{34}^2u_{123}^3u_{124}^2u_{134}^2u_{234}^2u_{1234}u'_1u'_2u'_3\left(u'\right)_4^4\\
&\quad\left(u'\right)_{12}^2\left(u'\right)_{13}^2\left(u'\right)_{14}^3\left(u'\right)_{23}^2\left(u'\right)_{24}^3\left(u'\right)_{34}^3\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2u'_{1234}+\ 2u_1u_2u_3u_4u_{12}^2u_{13}^2u_{14}u_{23}^2u_{24}u_{34}u_{123}^3u_{124}u_{134}u_{234}\left(u'\right)_4^4u'_{12}u'_{13}\left(u'\right)_{14}^3\\
&\quad u'_{23}\left(u'\right)_{24}^3\left(u'\right)_{34}^3u'_{123}\left(u'\right)_{124}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2\left(u_{123}\left(1-u_{1234}\right)u_{1234}u'_4+u'_1u'_2\left(1-u'_{12}\right)+u'_1u'_3\left(1-u'_{13}\right)+u'_2u'_3\left(1-u'_{23}\right)+\left(u'\right)_4^2\right)u'_{1234},\\
1-u_{12} &= u_3u_4u_{13}u_{14}u_{23}u_{24}u_{34}^2u_{134}^2u_{234}^2\left(u'\right)_1^3\left(u'\right)_2^3\left(u'\right)_{12}^3\left(u'\right)_{13}^2\left(u'\right)_{14}^2\left(u'\right)_{23}^2\left(u'\right)_{24}^2\left(1+u_{12}u_{123}u_{124}u_{1234}u'_3u'_4u'_{34}\right)\left(u'\right)_{123}^2\left(u'\right)_{124}^2u'_{134}u'_{234}u'_{1234},\\
1-u_{13} &= u_2u_4u_{12}u_{14}u_{23}u_{24}^2u_{34}u_{124}^2u_{234}^2\left(u'\right)_1^3\left(u'\right)_3^3\left(u'\right)_{12}^2\left(u'\right)_{13}^3\left(u'\right)_{14}^2\left(u'\right)_{23}^2\left(1+u_{13}u_{123}u_{134}u_{1234}u'_2u'_4u'_{24}\right)\left(u'\right)_{34}^2\left(u'\right)_{123}^2u'_{124}\left(u'\right)_{134}^2u'_{234}u'_{1234},\\
1-u_{14} &= u_2u_3u_{12}u_{13}u_{23}^2u_{24}u_{34}u_{123}^2u_{234}^2\left(u'\right)_1^3\left(u'\right)_4^3\left(u'\right)_{12}^2\left(u'\right)_{13}^2\left(u'\right)_{14}^3\left(1+u_{14}u_{124}u_{134}u_{1234}u'_2u'_3u'_{23}\right)\left(u'\right)_{24}^2\left(u'\right)_{34}^2u'_{123}\left(u'\right)_{124}^2\left(u'\right)_{134}^2u'_{234}u'_{1234},\\
1-u_{34} &= u_1u_2u_{12}^2u_{13}u_{14}u_{23}u_{24}u_{123}^2u_{124}^2\left(u'\right)_3^3\left(u'\right)_4^3\left(1+u_{34}u_{134}u_{234}u_{1234}u'_1u'_2u'_{12}\right)\left(u'\right)_{13}^2\left(u'\right)_{14}^2\left(u'\right)_{23}^2\left(u'\right)_{24}^2\left(u'\right)_{34}^3u'_{123}u'_{124}\left(u'\right)_{134}^2\left(u'\right)_{234}^2u'_{1234},\\
1-u_{24} &= u_1u_3u_{12}u_{13}^2u_{14}u_{23}u_{34}u_{123}^2u_{134}^2\left(u'\right)_2^3\left(u'\right)_4^3\left(u'\right)_{12}^2\left(1+u_{24}u_{124}u_{234}u_{1234}u'_1u'_3u'_{13}\right)\left(u'\right)_{14}^2\left(u'\right)_{23}^2\left(u'\right)_{24}^3\left(u'\right)_{34}^2u'_{123}\left(u'\right)_{124}^2u'_{134}\left(u'\right)_{234}^2u'_{1234},\\
1-u_{23} &= u_1u_4u_{12}u_{13}u_{14}^2u_{24}u_{34}u_{124}^2u_{134}^2\left(u'\right)_2^3\left(u'\right)_3^3\left(u'\right)_{12}^2\left(u'\right)_{13}^2\left(1+u_{23}u_{123}u_{234}u_{1234}u'_1u'_4u'_{14}\right)\left(u'\right)_{23}^3\left(u'\right)_{24}^2\left(u'\right)_{34}^2\left(u'\right)_{123}^2u'_{124}u'_{134}\left(u'\right)_{234}^2u'_{1234},\\
1-u_{124} &= u_3u_{13}u_{23}u_{34}u_{123}u_{134}u_{234}\left(u'\right)_1^2\left(u'\right)_2^2\left(u'\right)_4^2\left(u'\right)_{12}^2u'_{13}\left(u'\right)_{14}^2u'_{23}\left(u'\right)_{24}^2u'_{34}u'_{123}\left(u'\right)_{124}^2u'_{134}u'_{234}u'_{1234},\\
1-u_{123} &= u_4u_{14}u_{24}u_{34}u_{124}u_{134}u_{234}\left(u'\right)_1^2\left(u'\right)_2^2\left(u'\right)_3^2\left(u'\right)_{12}^2\left(u'\right)_{13}^2u'_{14}\left(u'\right)_{23}^2u'_{24}u'_{34}\left(u'\right)_{123}^2u'_{124}u'_{134}u'_{234}u'_{1234},\\
1-u_{134} &= u_2u_{12}u_{23}u_{24}u_{123}u_{124}u_{234}\left(u'\right)_1^2\left(u'\right)_3^2\left(u'\right)_4^2u'_{12}\left(u'\right)_{13}^2\left(u'\right)_{14}^2u'_{23}u'_{24}\left(u'\right)_{34}^2u'_{123}u'_{124}\left(u'\right)_{134}^2u'_{234}u'_{1234},\\
1-u_{234} &= u_1u_{12}u_{13}u_{14}u_{123}u_{124}u_{134}\left(u'\right)_2^2\left(u'\right)_3^2\left(u'\right)_4^2u'_{12}u'_{13}u'_{14}\left(u'\right)_{23}^2\left(u'\right)_{24}^2\left(u'\right)_{34}^2u'_{123}u'_{124}u'_{134}\left(u'\right)_{234}^2u'_{1234},\\
1-u_{1234} &= u'_1u'_2u'_3u'_4u'_{12}u'_{13}u'_{14}u'_{23}u'_{24}u'_{34}u'_{123}u'_{124}u'_{134}u'_{234}u'_{1234},\\
1-u'_1 &= u_1u_{12}u_{13}u_{14}u_{123}u_{124}u_{134}u_{1234}u'_2u'_3u'_4u'_{23}u'_{24}u'_{34}u'_{234},\\
1-u'_2 &= u_2u_{12}u_{23}u_{24}u_{123}u_{124}u_{234}u_{1234}u'_1u'_3u'_4u'_{13}u'_{14}u'_{34}u'_{134},\\
1-u'_3 &= u_3u_{13}u_{23}u_{34}u_{123}u_{134}u_{234}u_{1234}u'_1u'_2u'_4u'_{12}u'_{14}u'_{24}u'_{124},\\
1-u'_4 &= u_4u_{14}u_{24}u_{34}u_{124}u_{134}u_{234}u_{1234}u'_1u'_2u'_3u'_{12}u'_{13}u'_{23}u'_{123},\\
1-u'_{12} &= u_1u_2u_{12}^2u_{13}u_{14}u_{23}u_{24}u_{123}^2u_{124}^2u_{134}u_{234}u_{1234}^2\left(u'\right)_3^2\left(u'\right)_4^2u'_{13}u'_{14}u'_{23}u'_{24}\left(u'\right)_{34}^2u'_{134}u'_{234},\\
1-u'_{13} &= u_1u_3u_{12}u_{13}^2u_{14}u_{23}u_{34}u_{123}^2u_{124}u_{134}^2u_{234}u_{1234}^2\left(u'\right)_2^2\left(u'\right)_4^2u'_{12}u'_{14}u'_{23}\left(u'\right)_{24}^2u'_{34}u'_{124}u'_{234},\\
1-u'_{14} &= u_1u_4u_{12}u_{13}u_{14}^2u_{24}u_{34}u_{123}u_{124}^2u_{134}^2u_{234}u_{1234}^2\left(u'\right)_2^2\left(u'\right)_3^2u'_{12}u'_{13}\left(u'\right)_{23}^2u'_{24}u'_{34}u'_{123}u'_{234},\\
1-u'_{23} &= u_2u_3u_{12}u_{13}u_{23}^2u_{24}u_{34}u_{123}^2u_{124}u_{134}u_{234}^2u_{1234}^2\left(u'\right)_1^2\left(u'\right)_4^2u'_{12}u'_{13}\left(u'\right)_{14}^2u'_{24}u'_{34}u'_{124}u'_{134},\\
1-u'_{24} &= u_2u_4u_{12}u_{14}u_{23}u_{24}^2u_{34}u_{123}u_{124}^2u_{134}u_{234}^2u_{1234}^2\left(u'\right)_1^2\left(u'\right)_3^2u'_{12}\left(u'\right)_{13}^2u'_{14}u'_{23}u'_{34}u'_{123}u'_{134},\\
1-u'_{34} &= u_3u_4u_{13}u_{14}u_{23}u_{24}u_{34}^2u_{123}u_{124}u_{134}^2u_{234}^2u_{1234}^2\left(u'\right)_1^2\left(u'\right)_2^2\left(u'\right)_{12}^2u'_{13}u'_{14}u'_{23}u'_{24}u'_{123}u'_{124},\\
1-u'_{123} &= u_1u_2u_3u_{12}^2u_{13}^2u_{14}u_{23}^2u_{24}u_{34}u_{123}^3u_{124}^2u_{134}^2u_{234}^2u_{1234}^3\left(u'\right)_4^3\left(u'\right)_{14}^2\left(u'\right)_{24}^2\left(u'\right)_{34}^2\left(1+u'_1u'_2u'_3u'_{12}u'_{13}u'_{23}u'_{123}\right)u'_{124}u'_{134}u'_{234},\\
1-u'_{124} &= u_1u_2u_4u_{12}^2u_{13}u_{14}^2u_{23}u_{24}^2u_{34}u_{123}^2u_{124}^3u_{134}^2u_{234}^2u_{1234}^3\left(u'\right)_3^3\left(u'\right)_{13}^2\left(u'\right)_{23}^2\left(u'\right)_{34}^2u'_{123}\left(1+u'_1u'_2u'_4u'_{12}u'_{14}u'_{24}u'_{124}\right)u'_{134}u'_{234},\\
1-u'_{134} &= u_1u_3u_4u_{12}u_{13}^2u_{14}^2u_{23}u_{24}u_{34}^2u_{123}^2u_{124}^2u_{134}^3u_{234}^2u_{1234}^3\left(u'\right)_2^3\left(u'\right)_{12}^2\left(u'\right)_{23}^2\left(u'\right)_{24}^2u'_{123}u'_{124}\left(1+u'_1u'_3u'_4u'_{13}u'_{14}u'_{34}u'_{134}\right)u'_{234},\\
1-u'_{234} &= u_2u_3u_4u_{12}u_{13}u_{14}u_{23}^2u_{24}^2u_{34}^2u_{123}^2u_{124}^2u_{134}^2u_{234}^3u_{1234}^3\left(u'\right)_1^3\left(u'\right)_{12}^2\left(u'\right)_{13}^2\left(u'\right)_{14}^2u'_{123}u'_{124}u'_{134}\left(u'_2u'_3u'_4u'_{23}u'_{24}u'_{34}u'_{234}+1\right),\\
1-u'_{1234} &= 6u_1u_2u_3u_4u_{12}^2u_{13}^2u_{14}^2u_{23}^2u_{24}^2u_{34}^2u_{123}^3u_{124}^3u_{134}^3u_{234}^3u_{1234}^4u'_1u'_2u'_3u'_4\left(u'\right)_{12}^2\left(u'\right)_{13}^2\left(u'\right)_{14}^2\left(u'\right)_{23}^2\left(u'\right)_{24}^2\left(u'\right)_{34}^2\left(u'\right)_{123}^2\left(u'\right)_{124}^2\left(u'\right)_{134}^2\left(u'\right)_{234}^2\left(u'\right)_{1234}^2-\\
&\quad u_1u_2u_3u_4u_{12}^2u_{13}^2u_{14}^2u_{23}^2u_{24}^2u_{34}^2u_{123}^3u_{124}^3u_{134}^3u_{234}^3u_{1234}^4u'_{12}u'_{13}u'_{14}u'_{23}u'_{24}u'_{34}u'_{123}u'_{124}u'_{134}u'_{234}\left(-3u_{1234}^2+\left(1-u'_1\right)^2+\left(1-u'_2\right)^2+\left(1-u'_3\right)^2+\left(1-u'_4\right)^2\right)u'_{1234}\\
&\quad+u_1u_2u_3u_4u_{12}^2u_{13}^2u_{14}^2u_{23}^2u_{24}^2u_{34}^2u_{123}^4u_{124}^4u_{134}^4u_{234}^4u_{1234}^8
\end{aligned}
$$

The u-equations for $u_1, u_2, u_3, u_4, u_{1234}^{'}$ are quadratic in the coresponding u's. The 10 facets obtained by setting $u_i^{'},\ u_i, u_{1234}^{'}, u_{1234} \to 0$ are all P_3 and the 20 facets obtained by setting $u_{ij}^{'},\ u_{ij},\ u_{ijk}^{'},\ u_{ijk} \to 0$ are all $A_1 \times P_2$. Thus, 4d permutahderon is indeed a binary positive geometry!

More generally the following are some of the u-equations for the n-dimensional permutahderon:

For $|I| = n-1$ and $s_K = \begin{cases} 1, & if K \not\subset I \\ 2, & if K \subset I \end{cases}$

$$\begin{aligned} 1 - u_I &= \prod_{S-I \subset J} u_J \prod_{K \cap I \neq \Phi} (u_K^{'})^{s_K} \\ 1 - u_{12\cdots n} &= \prod_{\substack{I \subset S \\ I \neq \Phi}} u_I^{'} \end{aligned}$$

For $t_J = \begin{cases} 1, & if I \not\subset J \\ 2, & if I \subset J \end{cases}$

$$\begin{aligned} 1 - u_i^{'} &= \prod_{\{i\} \subset J} u_J \prod_{K \subset S - \{i\}} u_K^{'} \\ 1 - u_{ij}^{'} &= \prod_{\{i,j\} \cap J \neq \Phi} u_J^{t_J} \prod_{K \subset S - \{i,j\}} (u_K^{'})^2 u_{K\cup i}^{'} u_{K \cup j}^{'} \end{aligned}$$

and for $|I| = n-2$ with $S - I = \{i,j\}$

$$1 - u_I = u_i u_j u_{ij}^2 \prod_{J \subset I} u_{J\cup\{i\}} u_{J\cup\{j\}} u_{J\cup\{i,j\}}^2 \prod_{K \subset S - I} (u_K^{'})^3 (u_{K\cup\{i\}}^{'})^2 (u_{K\cup\{j\}}^{'})^2 u_{K\cup\{i,j\}}^{'} \left(1 + u_i^{'} u_j^{'} u_{ij}^{'} \prod_{I \subset L} u_L \right)$$

The other u-equations are not two or three term equations and seem to be some higher order polynomials in the corresponding u's and it would be nice if we could classify all of them.

Associahedra

If $B = \{\{i, i+1, \cdots, j\} | 1 \leq i \leq j \leq n\}$ is the set of consecutive intervals, then $P_n(\mathbf{Y})$ is the associahedron .

2d associahedron as generalized permutahedron

In this case the building set $B = \{\{0,1,2\},\{0,1\},\{1,2\},\{0\},\{1\},\{2\}\}$ and the relevant x variables are $x_0 = 1,\ x_1 = x,\ x_2 = y,\ x_{01} = 1+x,\ x_{12} = x+y,\ x_{012} = 1+x+y$.

We can write down the Newton polynomial for the Minkowski sum as $x_1\ x_2\ (1+x_1)\ (x_1+x_2)\ (1+x_1+x_2)$.

The u-variables can be written in terms of x variables as:

$$u_1 = \frac{x(x+y+1)}{(x+1)(x+y)}, u_2 = \frac{x+y}{x+y+1}, u_3 = \frac{y}{x+y}, u_4 = \frac{x+1}{x+y+1}, u_5 = \frac{1}{x+1}$$

The u-equations are:

$$1-u_1 = u_3u_5, 1-u_2 = u_4u_5, 1-u_3 = u_1u_4, 1-u_4 = u_2u_3, 1-u_5 = u_1u_2$$

Notice that these are precisely the u equations for the ABHY realisation of A_2.

3d associahedron as generalized permutahedron

In this case the building set $B = \{\{0,1,2,3\},\{0,1,2\},\{1,2,3\},\{0,1\},\{1,2\},\{2,3\},\{0\},\{1\},\{2\},\{3\}\}$ and the relevant x variables are $x_0 = 1,\ x_1 = x,\ x_2 = y,\ x_3 = y,\ x_{01} = 1+x,\ x_{12} = x+y,\ x_{23} = y+z,\ x_{012} = 1+x+y,\ x_{123} = x+y+z,\ x_{0123} = 1+x+y+z$.

We can write down the Newton polynomial for the Minkowski sum as $x_1\ x_2\ x_3\ (1+x_1)\ (x_1+x_2)\ (x_2+x_3)\ (1+x_1+x_2)\ (x_1+x_2+x_3)\ (1+x_1+x_2+x_3)$.

The u-variables can be written in terms of x variables as:

$$\begin{aligned} u_1 &= \frac{x(x+y+1)}{(x+1)(x+y)},\ u_2 = \frac{(x+y)(x+y+z+1)}{(x+y+1)(x+y+z)},\ u_3 = \frac{x+y+z}{x+y+z+1},\ u_4 = \frac{y}{x+z}, \\ u_5 &= \frac{y(x+y+z)}{(x+y)(y+z)},\ u_6 = \frac{y+z}{x+y+z}, u_7 = \frac{x+y+1}{x+y+z+1}, u_8 = \frac{x+1}{x+y+1}, u_9 = \frac{1}{x+1} \end{aligned}$$

and satisfy the u-equations

$$\begin{aligned} 1-u_1 &= u_5u_6u_9,\ 1-u_2 = u_4u_6u_8u_9,\ 1-u_3 = u_7u_8u_9,\ 1-u_4 = u_2u_5u_7, \\ 1-u_5 &= u_1u_4u_8, 1-u_6 = u_1u_2u_7u_8, 1-u_7 = u_3u_4u_6, 1-u_8 = u_2u_3u_5u_6, 1-u_9 = u_1u_2u_3 \end{aligned}$$

We see again that these are precisely the u equations of ABHY realisation of A_3 !

This is not a coincidence and it extends to all n since the corresponding Newton polytope is given by $\prod_{1\le i\le j\le n}(x_i + x_{i+1} + \cdots + x_j)$ is nothing but the Loday's realisation of the associahedron which is equivalent to the ABHY type A_n associahedron.

Thus remarkably both the generalised permutahedron realisation and ABHY realisation of the associahedra A_n are the same.

Cyclohedra

If B is the set of cyclic intervals, then $P_n(\mathbf{Y})$ is a cyclohedron.

3d cyclohedron

The 2d cyclohedron is the same as the 2d permutahedron as they both have the same building sets.

Let us look at the 3d case. In this case the building set $B = \{\{0,1,2,3\},\{0,1,2\},\{1,2,3\},\{2,3,0\},\{3,0,1\},\{0,1\},\{1,2\},\{2,3\},\{0,3\},\{0\},\{1\},\{2\},\{3\}\}$.

The Newton polynomial of the Minkowski sum is $xyz(1+x)(x+y)(y+z)(1+z)(1+x+y)(x+y+z)(y+z+1)(z+1+x)(1+x+y+z)$ and

we find the relevant u's as:

$$\begin{aligned}
u_1 &= \frac{x(x+y+1)}{(x+1)(x+y)},\ u_2 = \frac{(x+y)(x+y+z+1)}{(x+y+1)(x+y+z)},\ u_3 = \frac{x+y+z}{x+y+z+1}, \qquad u_4 = \frac{z(y+z+1)}{(z+1)(y+z)},\\
u_5 &= \frac{y(x+y+z)}{(x+y)(y+z)},\ u_6 = \frac{(y+z)(x+y+z+1)}{(y+z+1)(x+y+z)},\ u_7 = \frac{y+z+1}{x+y+z+1}, \qquad u_8 = \frac{x+z+1}{x+y+z+1},\\
u_9 &= \frac{x+y+1}{x+y+z+1},\ u_{10} = \frac{(z+1)(x+y+z+1)}{(x+z+1)(y+z+1)},\ u_{11} = \frac{(x+1)(x+y+z+1)}{(x+y+1)(x+z+1)},\ u_{12} = \frac{x+z+1}{(x+1)(z+1)}
\end{aligned}$$

and the u-equations are:

$$\begin{aligned}
1-u_1 &= u_5u_6u_7^2u_{10}u_{12},\ 1-u_2 = u_4u_6u_7^2u_8^2u_{10}^2u_{11}u_{12},\ 1-u_3 = u_7u_8u_9u_{10}u_{11}u_{12},\ 1-u_4 = u_2u_5u_9^2u_{11}u_{12},\\
1-u_5 &= u_1u_4u_8^2u_{10}u_{11},\ 1-u_6 = u_1u_2u_8^2u_9^2u_{10}u_{11}^2u_{12},\ 1-u_7 = u_1u_2u_3u_8u_9u_{11},\ 1-u_8 = u_2u_3u_5u_6u_7u_9,\\
1-u_9 &= u_3u_4u_6u_7u_8u_{10},\ 1-u_{10} = u_1u_2^2u_3^2u_5u_6u_9^2u_{11},\ 1-u_{11} = u_2u_3^2u_4u_5u_6^2u_7^2u_{10},\ 1-u_{12} = u_1u_2u_3^2u_4u_6
\end{aligned}$$

The u-equations above are that of ABHY realisation B_3 of the cyclohedron. This fact also generalizes to all n.

Thus, the gen.permutahedron realisation and ABHY type B_n realisation of the cyclohedron are one and the same.

We can now consider several degenerations of both A_n and B_n and show that they are binary geometries with perfect u-equations.

Minkowski sum of n A2's

We can also consider general Minkowski sums of non-simplices and see if these also give us any instance of polytopes with perfect u equations. One such case is the following:

We consider the Minkowski sum of $n-1$ A_2's with following Newton polynomial :

$$\prod_{i=1}^{n}(1+x_i)\prod_{i=1}^{n-1}(1+x_i+x_ix_{i+1}) \tag{2}$$

This gives a family of simple n-dimensional polytopes with $3n-1$ facets and Pell number P_n (recursively defined as $P_n = 2P_{n-1}+P_{n-2}$ with $P_1 = 1,\ P_2 = 2$) of vertices which we shall call X_n.

We can solve for u-variables for this family and we get :

$$\begin{aligned} u_1 &= \frac{p_n}{q_n},\ u_2 = \frac{p_{n-1}q_n}{p_{n-1,n}},\ u_3 = \frac{p_{n-2}q_{n-1}}{p_{n-2,n-1}},\cdots,u_{n-2} = \frac{p_1q_2}{p_{12}},\ u_{n-1} = \frac{p_2q_3}{p_{23}},\ u_n = \frac{p_3q_4}{p_{34}}, \\ u_{n+1} &= \frac{1}{q_1},\ u_{n+2} = \frac{p_{12}}{q_1q_2},\ u_{n+3} = \frac{q_1}{p_{23}},\cdots,u_{3n-1} = \frac{q_{n-1}}{p_{n-1n}} \end{aligned}$$

where $p_i = x_i$, $q_i = 1+x_i$ and $p_{i\ i+1} = 1+x_i+x_ix_{i+1}$.

The u equations obtained from these u variables are of three types viz. $1-u$ being the product of two, three or four u's for any n.

There are exactly 4 u's which have $1-u$ is product of 2 u's:

$$1-u_1 = u_{3n-2}u_{3n-1},\ 1-u_{n-2} = u_{n+1}u_{n+3},\ 1-u_{n+1} = u_{n-2}u_{n+2},\ 1-u_{3n-1} = \begin{cases} u_2u_3, n=3 \\ u_1u_4, n=4 \\ u_1u_2, n\geq 5 \end{cases}$$

The facets corresponding to setting any of these $u\to 0$ is X_{n-1}.

There are exactly $n-3$ u's which have $1-u$ is product of 4 u's:

$$1-u_{n+4} = u_nu_{n+2}u_{n+4}u_{n+6}\ \ and\ \ 1-u_{n+6+2i} = u_{n-3-i}u_{n+4+2i}u_{n+5+2i}u_{n+8+2i}\ for\ i=0,\cdots,(n-5)$$

The facets corresponding to setting any of these $u\to 0$ is $A_1^m\times X_{n-m-1}$.

All the other $2n-2$ u's correspond to $1-u$ is product of 3 u's and we do not yet have a complete classification of the facets.

We can also replace the first the first term in the Newton polynomial (**??**) with $(1+x_1+x_2)$ and we get an identical system of u-equations

$$
\begin{aligned}
u_1 &= \frac{p_n}{q_n},\ u_2 = \frac{p_{n-1}q_n}{p_{n-1,n}},\ u_3 = \frac{p_{n-2}q_{n-1}}{p_{n-2,n-1}}, \cdots, u_{n-3} = \frac{p_4q_5}{p_{45}},\ u_{n-2} = \frac{p_1}{q_1},\ u_{n-1} = \frac{p_2q_3}{p_{23}},\ u_n = \frac{p_3q_4}{p_{34}}, \\
u_{n+1} &= \frac{q_2}{p_{12}},\ u_{n+2} = \frac{q_1}{p_{12}},\ u_{n+3} = \frac{p_{23}}{q_2q_3}, \cdots, u_{3n-1} = \frac{p_{12}}{q_1q_2}
\end{aligned}
$$

where $p_i = x_i$, $q_i = 1 + x_i$ and $p_{i\ i+1} = 1 + x_i + x_i x_{i+1}$ with $p_{12} = 1 + x_1 + x_2$.

The u equations obtained from these u variables are of three types viz. $1-u$ being the product of two, three or four u's for any n.

There are exactly 4 u's which have $1-u$ is product of 2 u's:

$$
1-u_1 = u_{3n-2}u_{3n-3},\ 1-u_{n-2} = u_{n+1}u_{3n-1},\ 1-u_{n+1} = u_{n-2}u_{n+2},\ 1-u_{3n-1} = \begin{cases} u_2u_3, n=3 \\ u_1u_4, n=4 \\ u_1u_2, n \geq 5 \end{cases}
$$

The facets corresponding to setting any of these $u \to 0$ is X_{n-1}.

There are exactly $n-3$ u's which have $1-u$ is product of 4 u's:

$$
1-u_{n+3} = u_nu_{n+2}u_{n+5}u_{3n-1} \ \ and \ \ 1-u_{n+5+2i} = u_{n-3-i}u_{n+3+2i}u_{n+4+2i}u_{n+7+2i}\ for\ i = 0, \cdots, (n-5)
$$

It would be nice to see if these also correspond to some degeneration of the associahedron or if they are generalised permutahedra.

4 Discussions

There are several interesting open questions, like the question of whether degenerations of other cluster types (in particular C_n and D_n can also give binary geometries with (perfect?) u-equations.

Though many examples of degenerations of A_n and B_n were products of lower dimensional objects of the same type, there were examples where we got non-trivial degenerations which did not factor. We would like to completely classify these cases and this would settle the question of identifying all the "atoms" of binary geometries with perfect u-equations. One class of examples which is certainly of both mathematical and physical interest in this context are the Stokes polytopes (and more generally Accordiohedra).