Summarize on $\int \epsilon \mathrm{d}\epsilon \mathrm{d}^3\chi_{n+1}$ over All N^{2}MHV Yangian invariants

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Notation

Here we require $1 \leq i_1 < i_2 < \cdots < i_{10} \leq n{+}1$. We define $X := n \wedge B$ where $\mathcal{Z}_B = \mathcal{Z}_{n-1} - C\tau Z_1$ with $C = \frac{\langle n{-}1\, n\, 2\, 3\rangle}{\langle n\, 1\, 2\, 3\rangle}$. The effect of the operation $\int \epsilon \mathrm{d}\epsilon \mathrm{d}^3\chi_{n+1}$ on NMHV Yangian invariants are known as

$$[i_1, i_2, i_3, n, n{+}1] \to \begin{cases} \mathrm{d}\log \frac{\langle X i_1 i_2\rangle}{\langle X i_2 i_3\rangle} \bar{Q} \log \frac{\langle \bar{n} i_2\rangle}{\langle \bar{n} i_1\rangle} + \mathrm{d}\log \frac{\langle X i_2 i_3\rangle}{\langle X i_1 i_3\rangle} \bar{Q} \log \frac{\langle \bar{n} i_3\rangle}{\langle \bar{n} i_1\rangle} & 1 < i_1 \text{ and } i_3 < n{-}1 \\ \mathrm{d}\log \frac{\langle X i_1 i_2\rangle}{\langle X\, n{-}2\, n{-}1\rangle} \bar{Q} \log \frac{\langle \bar{n} i_2\rangle}{\langle \bar{n} i_1\rangle} & 1 < i_1 \text{ and } i_3 = n{-}1 \\ \mathrm{d}\log \frac{\langle X i_2 i_3\rangle}{\langle X 12\rangle} \bar{Q} \log \frac{\langle \bar{n} i_3\rangle}{\langle \bar{n} i_2\rangle} & 1 = i_1 \text{ and } i_3 < n{-}1 \\ \mathrm{d}\log \frac{\langle X\, n{-}2\, n{-}1\rangle}{\langle X\, 12\rangle} \bar{Q} \log \frac{\langle \bar{n} 2\rangle}{\langle \bar{n} i_2\rangle} & 1 = i_1 \text{ and } i_3 = n{-}1 \end{cases} \tag{0.1}$$

1. $R[i_1, i_2, (i_2, i_3) \cap (i_4, i_5, i_6), (i_2, i_3, i_4) \cap (i_5, i_6), i_6] R[i_2, i_3, i_4, i_5, i_6]$

$$R[i_1, i_2, (i_2, i_3) \cap (i_4, n, n{+}1), (i_2, i_3, i_4) \cap (n, n{+}1), n{+}1] R[i_2, i_3, i_4, n, n{+}1] \to$$
$$\begin{cases} R[i_1, i_2, i_3, i_4, n] \left(\bar{Q} \log \frac{\langle n(n{-}1\, 1)(i_1\, i_2)(i_3\, i_4)\rangle}{\langle \bar{n} i_1\rangle \langle i_2 i_3 i_4 n\rangle} \mathrm{d}\log \frac{\langle X i_1 i_2\rangle}{\langle X i_3 i_4\rangle} + \bar{Q} \log \frac{\langle \bar{n} i_1\rangle}{\langle \bar{n} i_4\rangle} \mathrm{d}\log \frac{\langle X i_1 i_4\rangle}{\langle X i_3 i_4\rangle} \right) & 1 < i_1 \text{ and } i_4 < n{-}1 \\ R[i_1, i_2, i_3, n{-}1, n] \bar{Q} \log \frac{\langle \bar{n} i_1\rangle}{\langle \bar{n} i_3\rangle} \mathrm{d}\log \frac{\tau}{\langle X i_1 i_2\rangle} & 1 < i_1 \text{ and } i_4 = n{-}1 \\ R[1, i_2, i_3, i_4, n] \bar{Q} \log \frac{\langle \bar{n} i_4\rangle}{\langle \bar{n} i_2\rangle} \mathrm{d}\log \langle X i_3 i_4\rangle & i_1 = 1 \text{ and } i_4 < n{-}1 \\ R[1, i_2, i_3, n{-}1, n] \bar{Q} \log \frac{\langle \bar{n} 2\rangle}{\langle \bar{n} i_2\rangle} \mathrm{d}\log \tau & i_1 = 1 \text{ and } i_4 = n{-}1 \end{cases}$$

$$R[i_2, i_3, (i_3, i_4) \cap (n, n{+}1, i_1), (i_3, i_4, n) \cap (n{+}1, i_1), i_1] R[i_3, i_4, n, n{+}1, i_1] \to$$
$$R[i_1, i_2, i_3, i_4, n] R[i_1, (i_3, i_4) \cap (i_1, i_2, n), i_4, n, n{+}1]$$

$$R[i_3, i_4, (i_4, n) \cap (n{+}1, i_1, i_2), (i_4, n, n{+}1) \cap (i_1, i_2), i_2] R[i_3, i_4, n, n{+}1, i_1] \to$$
$$R[i_1, i_2, i_3, i_4, n] R[i_1, (i_1, i_2) \cap (i_3, i_4, n), i_4, n, n{+}1]$$

$$R[i_4, n, (n, n{+}1) \cap (i_1, i_2, i_3), (n, n{+}1, i_1) \cap (i_2, i_3), i_3] R[i_1, i_2, i_3, n, n{+}1] \to$$
$$R[i_1, i_2, (i_2, i_3) \cap (i_4, n, n{+}1), (i_2, i_3, i_4) \cap (n, n{+}1), n{+}1] R[i_2, i_3, i_4, n, n{+}1]$$

$$R[n, n{+}1, (n{+}1, i_1) \cap (i_2, i_3, i_4), (n{+}1, i_1, i_2) \cap (i_3, i_4), i_4] R[i_1, i_2, i_3, i_4, n{+}1] \to$$
$$R[(n, i_1) \cap (i_2, i_3, i_4), (n, i_1, i_2) \cap (i_3, i_4), i_4, n, n{+}1] R[i_1, i_2, i_3, i_4, n]$$

$$R[n{+}1, i_1, (i_1, i_2) \cap (i_3, i_4, n), (i_1, i_2, i_3) \cap (i_4, n), n] R[i_1, i_2, i_3, i_4, n]$$

2. $R[i_1,i_2,(i_3,i_4)\cap(i_5,i_6,i_7),(i_3,i_4,i_5)\cap(i_6,i_7),i_7]R[i_3,i_4,i_5,i_6,i_7]$

2.1 $i_1>1$ **and** $i_5<n{-}1$

$$\begin{aligned}
&R[i_1,i_2,(i_3,i_4)\cap(i_5,n,n{+}1),(i_3,i_4,i_5)\cap(n,n{+}1),n{+}1]R[i_3,i_4,i_5,n,n{+}1]\to\\
&\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_5)\rangle}{\langle\bar{n}i_5\rangle\langle i_1i_2i_3i_4\rangle}R[i_1,i_2,i_3,i_4,i_5]\mathrm{d}\log\frac{\langle Xi_5(i_1i_2)\cap(i_3i_4i_5)\rangle}{\langle Xi_1i_2\rangle}+\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_5\rangle}R[i_1,i_3,i_4,i_5,n]\mathrm{d}\log\frac{\langle Xi_1i_5\rangle}{\langle Xi_1i_2\rangle}\\
&-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_5\rangle}R[i_2,i_3,i_4,i_5,n]\mathrm{d}\log\frac{\langle Xi_2i_5\rangle}{\langle Xi_1i_2\rangle}+\bar{Q}\log\frac{\langle n(n{-}1\,1)(i_1i_2)(i_3i_5)\rangle}{\langle\bar{n}i_5\rangle\langle i_1i_2i_3n\rangle}R[i_1,i_2,i_3,i_5,n]\mathrm{d}\log\frac{\langle Xi_3i_5\rangle}{\langle Xi_1i_2\rangle}\\
&-\bar{Q}\log\frac{\langle n(n{-}1\,1)(i_1i_2)(i_4i_5)\rangle}{\langle\bar{n}i_5\rangle\langle i_1i_2i_4n\rangle}R[i_1,i_2,i_4,i_5,n]\mathrm{d}\log\frac{\langle Xi_4i_5\rangle}{\langle Xi_1i_2\rangle}\\
&-\bar{Q}\log\frac{\langle n(n{-}1\,1)(i_1i_2)(i_3i_4)\rangle}{\langle\bar{n}i_5\rangle\langle i_1i_2i_3i_4\rangle}R[i_1,i_2,i_3,i_4,n]\mathrm{d}\log\frac{\langle Xi_5(i_1i_2)\cap(i_3i_4n)\rangle}{\langle Xi_1i_2\rangle}
\end{aligned}$$

2.1 $i_1>1$ **and** $i_5=n{-}1$

$$\begin{aligned}
&R[i_1,i_2,(i_3,i_4)\cap(n{-}1,n,n{+}1),(i_3,i_4,n{-}1)\cap(n,n{+}1),n{+}1]R[i_3,i_4,n{-}1,n,n{+}1]\to\\
&\mathrm{d}\log\frac{\tau}{\langle Xi_1i_2\rangle}\bar{Q}\log\frac{\langle\bar{n}i_3\rangle}{\langle\bar{n}i_4\rangle}R[i_1,i_2,(i_3i_4)\cap(\bar{n}),n{-}1,n]
\end{aligned}$$

2.1 $i_1=1$ **and** $i_5<n{-}1$

$$\begin{aligned}
&R[1,i_2,(i_3,i_4)\cap(i_5,n,n{+}1),(i_3,i_4,i_5)\cap(n,n{+}1),n{+}1]R[i_3,i_4,i_5,n,n{+}1]\to\\
&\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_5\rangle}\Big(R[1,i_2,i_3,i_4,i_5]\mathrm{d}\log\langle Xi_5(1i_2)\cap(i_3i_4i_5)\rangle-R[1,i_2,i_3,i_4,n]\mathrm{d}\log\langle Xi_5(1i_2)\cap(i_3i_4n)\rangle\\
&-R[i_2,i_3,i_4,i_5,n]\mathrm{d}\log\langle Xi_2i_5\rangle+R[1,i_2,i_3,i_5,n]\mathrm{d}\log\langle Xi_3i_5\rangle-R[1,i_2,i_4,i_5,n]\mathrm{d}\log\langle Xi_4i_5\rangle\Big)
\end{aligned}$$

2.1 $i_1=1$ **and** $i_5=n{-}1$

$$\begin{aligned}
&R[1,i_2,(i_3,i_4)\cap(n{-}1,n,n{+}1),(i_3,i_4,n{-}1)\cap(n,n{+}1),n{+}1]R[i_3,i_4,n{-}1,n,n{+}1]\to\\
&\bar{Q}\log\frac{\langle\bar{n}2\rangle}{\langle\bar{n}i_2\rangle}R[1,i_3,i_4,n{-}1,n]\mathrm{d}\log\tau
\end{aligned}$$

2.2

$$\begin{aligned}
&R[i_2,i_3,(i_4,i_5)\cap(n,n{+}1,i_1),(i_4,i_5,n)\cap(n{+}1,i_1),i_1]R[i_1,i_4,i_5,n,n{+}1]\to\\
&R[i_1,i_2,i_3,i_4,n]R[i_1,i_4,i_5,n,n{+}1]+R[i_1,(i_4,i_5)\cap(i_2,i_3,n),i_5,n,n{+}1]R[i_2,i_3,i_4,i_5,n]\\
&-R[i_1,(i_4,i_5)\cap(i_1,i_2,i_3),i_5,n,n{+}1]R[i_1,i_2,i_3,i_4,i_5]+R[i_1,(i_4,i_5)\cap(i_1,i_2,n),i_5,n,n{+}1]R[i_1,i_2,i_4,i_5,n]\\
&-R[i_1,(i_4,i_5)\cap(i_1,i_3,n),i_5,n,n{+}1]R[i_1,i_3,i_4,i_5,n]
\end{aligned}$$

2.3

$$\begin{aligned}
&R[i_3,i_4,(i_5,n)\cap(n{+}1,i_1,i_2),(i_5,n,n{+}1)\cap(i_1,i_2),i_2]R[i_1,i_2,i_5,n,n{+}1]\to\\
&R[i_1,(i_1,i_2)\cap(i_3,i_4,n),i_5,n,n{+}1]R[i_1,i_2,i_3,i_4,n]
\end{aligned}$$

2.4 $i_1 > 1$ **and** $i_5 < n{-}1$

$$R[i_4, i_5, (n, n{+}1) \cap (i_1, i_2, i_3), (n, n{+}1, i_1) \cap (i_2, i_3), i_3]R[i_1, i_2, i_3, n, n{+}1] \to$$
$$\left(\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}(i_4i_5)\cap(i_1i_2i_3)\rangle}\mathrm{d}\log\frac{\langle X(i_4i_5)\cap(i_1i_2i_3)i_1\rangle}{\langle Xi_1i_2\rangle} + \bar{Q}\log\frac{\langle\bar{n}(i_4i_5)\cap(i_1i_2i_3)\rangle}{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_5)\rangle}\mathrm{d}\log\frac{\langle X(i_4i_5)\cap(i_1i_2i_3)i_3\rangle}{\langle Xi_1i_2\rangle}\right)$$
$$\times R[i_1, i_2, i_3, i_4, i_5]$$

2.4 $i_1 = 1$ **and** $i_5 < n{-}1$

$$R[i_4, i_5, (n, n{+}1) \cap (1, i_2, i_3), (n, n{+}1, 1) \cap (i_2, i_3), i_3]R[1, i_2, i_3, n, n{+}1] \to$$
$$\bar{Q}\log\frac{\langle 1(i_2i_3)(i_4i_5)(n{-}1\,n)\rangle}{\langle\bar{n}i_2\rangle}\mathrm{d}\log\langle X(i_4i_5)\cap(1i_2i_3)i_3\rangle R[1, i_2, i_3, i_4, i_5]$$

2.4 $i_1 > 1$ **and** $i_5 = n{-}1$

$$R[i_4, n{-}1, (n, n{+}1) \cap (i_1, i_2, i_3), (n, n{+}1, i_1) \cap (i_2, i_3), i_3]R[i_1, i_2, i_3, n, n{+}1] \to$$
$$\left(\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_4\rangle}\mathrm{d}\log\frac{\langle X(i_4\,n{-}1)\cap(i_1i_2i_3)i_1\rangle}{\langle Xi_1i_2\rangle} + \bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle n{-}1(i_1i_2)(i_3i_4)(n\,1)\rangle}\mathrm{d}\log\frac{\langle X(i_4\,n{-}1)\cap(i_1i_2i_3)i_3\rangle}{\langle Xi_1i_2\rangle}\right)$$
$$\times R[i_1, i_2, i_3, i_4, n{-}1]$$

2.4 $i_1 = 1$ **and** $i_5 = n{-}1$

$$R[i_4, n{-}1, (n, n{+}1) \cap (1, i_2, i_3), (n, n{+}1, 1) \cap (i_2, i_3), i_3]R[1, i_2, i_3, n, n{+}1] \to$$
$$\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle\bar{n}i_2\rangle}\mathrm{d}\log\langle X(i_4\,n{-}1)\cap(1i_2i_3)i_3\rangle R[1, i_2, i_3, i_4, n{-}1]$$

2.5

$$R[i_5, n, (n{+}1, i_1) \cap (i_2i_3i_4), (n{+}1\,i_1i_2) \cap (i_3i_4), i_4]R[i_1, i_2, i_3, i_4, n{+}1] \to 0$$

2.6

$$R[n, n{+}1, (i_1i_2) \cap (i_3i_4i_5), (i_1i_2i_3) \cap (i_4i_5), i_5]R[i_1, i_2, i_3, i_4, i_5]$$

2.7 $i_1 \geq 1$ **and** $i_5 < n{-}1$

$$R[n{+}1, i_1, (i_2i_3) \cap (i_4i_5n), (i_2i_3i_4) \cap (i_5n), n]R[i_2, i_3, i_4, i_5, n]$$

2.7 $i_1 > 1$ **and** $i_5 = n{-}1$

$$R[n{+}1, i_1, (i_2i_3) \cap (i_4\,n{-}1\,n), (i_2i_3i_4) \cap (n{-}1n), n]R[i_2, i_3, i_4, n{-}1, n] \to$$
$$R[i_2, i_3, i_4, n{-}1, n]\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_4\rangle}\mathrm{d}\log\frac{\tau}{\langle Xi_1(i_2i_3)\cap(i_4\,n{-}1\,n)\rangle}$$

2.7 $i_1 = 1$ **and** $i_5 = n{-}1$

$$R[n{+}1, 1, (i_2i_3) \cap (i_4\,n{-}1\,n), (i_2i_3i_4) \cap (n{-}1n), n]R[i_2, i_3, i_4, n{-}1, n] \to$$
$$R[i_2, i_3, i_4, n{-}1, n]\bar{Q}\log\frac{\langle\bar{n}2\rangle}{\langle\bar{n}i_4\rangle}\mathrm{d}\log\tau$$

3. $R[i_1,i_2,i_3,(i_3i_4i_5)\cap(i_6i_7),i_7]R[i_3,i_4,i_5,i_6,i_7]$

3.1 $i_1>1$ **and** $i_5<n{-}1$

$$R[i_1,i_2,i_3,(i_3i_4i_5)\cap(n\,n{+}1),n{+}1]R[i_3,i_4,i_5,n,n{+}1]\to$$
$$-\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_2\rangle}R[(i_1i_2)\cap(\bar{n}),i_3,i_4,i_5,n]\mathrm{d}\log\langle Xi_1i_2\rangle+\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle n(i_1i_3)(i_4i_5)(n{-}1\,1)\rangle}R[i_1,i_3,i_4,i_5,n]\mathrm{d}\log\langle Xi_1i_3\rangle$$
$$-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle n(i_2i_3)(i_4i_5)(n{-}1\,1)\rangle}R[i_2,i_3,i_4,i_5,n]\mathrm{d}\log\langle Xi_2i_3\rangle$$
$$+\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_5)\rangle}{\langle\bar{n}(i_4i_5\cap(i_1i_2i_3))\rangle}R[i_1,i_2,i_3,i_4,i_5]\mathrm{d}\log\langle i_3(i_1i_2)(i_4i_5)(X)\rangle$$
$$+\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle n(i_1i_2)(i_3i_4)(n{-}1\,1)\rangle}R[i_1,i_2,i_3,i_4,n]\mathrm{d}\log\langle Xi_3i_4\rangle+\bar{Q}\log\frac{\langle n(i_1i_2)(i_3i_5)(n{-}1\,1)\rangle}{\langle\bar{n}i_5\rangle}R[i_1,i_2,i_3,i_5,n]\mathrm{d}\log\langle Xi_3i_5\rangle$$
$$-\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle\bar{n}i_5\rangle}R[i_1,i_2,i_3,(i_4,i_5)\cap(\bar{n}),n]\mathrm{d}\log\langle Xi_4i_5\rangle$$

3.1 $i_1=1$ **and** $i_5<n{-}1$

$$R[1,i_2,i_3,(i_3i_4i_5)\cap(n\,n{+}1),n{+}1]R[i_3,i_4,i_5,n,n{+}1]\to$$
$$-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle n(i_2i_3)(i_4i_5)(n{-}1,1)\rangle}R[i_2,i_3,i_4,i_5,n]\mathrm{d}\log\langle Xi_2i_3\rangle$$
$$+\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle 1(i_2i_3)(i_4i_5)(n{-}1\,n)\rangle}R[1,i_2,i_3,i_4,i_5]\mathrm{d}\log\langle i_3(1i_2)(i_4i_5)(X)\rangle$$
$$-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_4\rangle}R[1,i_2,i_3,i_4,n]\mathrm{d}\log\langle Xi_3i_4\rangle+\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_5\rangle}R[1,i_2,i_3,i_5,n]\mathrm{d}\log\langle Xi_3i_5\rangle$$
$$-\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle\bar{n}i_5\rangle}R[1,i_2,i_3,(i_4i_5)\cap(\bar{n}),n]\mathrm{d}\log\langle Xi_4i_5\rangle$$

3.1 $i_1>1$ **and** $i_5=n{-}1$

$$R[i_1,i_2,i_3,(i_3\,i_4\,n{-}1)\cap(n\,n{+}1),n{+}1]R[i_3,i_4,n{-}1,n,n{+}1]\to$$
$$\bar{Q}\log\frac{\langle\bar{n}i_3\rangle}{\langle\bar{n}i_4\rangle}R[i_1,i_2,i_3,n{-}1\,n]\mathrm{d}\log\tau-\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_2\rangle}R[(i_1i_2)\cap(\bar{n}),i_3,i_4,n{-}1,n]\mathrm{d}\log\langle Xi_1i_2\rangle$$
$$+\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_4\rangle}R[i_1,i_3,i_4,n{-}1,n]\mathrm{d}\log\langle Xi_1i_3\rangle-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_4\rangle}R[i_2,i_3,i_4,n{-}1,n]\mathrm{d}\log\langle Xi_2i_3\rangle$$
$$-\bar{Q}\log\frac{\langle n(i_1i_2)(i_3i_4)(n{-}1\,1)\rangle}{\langle\bar{n}i_4\rangle}R[i_1,i_2,i_3,i_4,n]\mathrm{d}\log\langle Xi_3i_4\rangle$$
$$+\bar{Q}\log\frac{\langle n{-}1(i_1i_2)(i_3i_4)(n1)\rangle}{\langle\bar{n}i_4\rangle}R[i_1,i_2,i_3,i_4,n{-}1]\mathrm{d}\log\langle i_3(i_1i_2)(i_4n{-}1)(X)\rangle$$

3.1 $i_1=1$ **and** $i_5=n{-}1$

$$R[1,i_2,i_3,(i_3\,i_4\,n{-}1)\cap(n\,n{+}1),n{+}1]R[i_3,i_4,n{-}1,n,n{+}1]\to$$
$$\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_4\rangle}\Big(-R[i_2,i_3,i_4,n{-}1,n]\mathrm{d}\log\langle Xi_2i_3\rangle-R[1,i_2,i_3,i_4,n]\mathrm{d}\log\langle Xi_3i_4\rangle+R[1,i_2,i_3,n{-}1,n]\mathrm{d}\log\tau$$
$$+R[1,i_2,i_3,i_4,n{-}1]\mathrm{d}\log\langle i_3(1i_2)(i_4\,n{-}1)(X)\rangle$$

3.2

$$R[i_2,i_3,i_4,(i_4,i_5,n)\cap(n{+}1\,i_1),i_1]R[i_4,i_5,n,n{+}1,i_1]\to R[i_2,i_3,i_4,n,i_1]R[i_1,i_4,i_5,n,n{+}1]$$

3.3 $i_1>1$ **and** $i_5<n{-}1$

$$R[i_3,i_4,i_5,(i_5\,n\,n{+}1)\cap(i_1i_2),i_2]R[i_5,n,n{+}1,i_1,i_2]\to$$
$$R[i_1,i_2,i_3,i_4,i_5]\left(\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_5)\rangle}{\langle\bar{n}i_1\rangle}\mathrm{d}\log\frac{\langle Xi_1i_2\rangle}{\langle i_5(i_1i_2)(i_3i_4)(X)\rangle}+\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_5\rangle}\mathrm{d}\log\frac{\langle Xi_1i_5\rangle}{\langle i_5(i_1i_2)(i_3i_4)(X)\rangle}\right)$$

3.3 $i_1>1$ **and** $i_5=n{-}1$

$$R[i_3,i_4,n{-}1,(n{-}1\,n\,n{+}1)\cap(i_1i_2)i_2]R[n{-}1,n,n{+}1,i_1,i_2]\to$$
$$R[i_1,i_2,i_3,i_4,n{-}1]\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle n{-}1(i_1i_2)(i_3i_4)(n1)\rangle}\mathrm{d}\log\frac{\tau}{\langle Xi_1i_2\rangle}$$

3.3 $i_1=1$ **and** $i_5<n{-}1$

$$R[i_3,i_4,i_5,(i_5\,n\,n{+}1)\cap(1i_2),i_2]R[i_5,n,n{+}1,1,i_2]\to\bar{Q}\log\frac{\langle\bar{n}i_5\rangle}{\langle\bar{n}i_2\rangle}R[1,i_2,i_3,i_4,i_5]\mathrm{d}\log\langle i_5(1i_2)(i_3i_4)(X)\rangle$$

3.3 $i_1=1$ **and** $i_5=n{-}1$

$$R[i_3,i_4,n{-}1,(n{-}1\,n\,n{+}1)\cap(1i_2),i_2]R[n{-}1,n,n{+}1,1,i_2]\to\bar{Q}\log\frac{\langle\bar{n}2\rangle}{\langle\bar{n}i_2\rangle}R[1,i_2,i_3,i_4,n{-}1]\mathrm{d}\log\tau$$

3.4 $i_1>1$ **and** $i_5<n{-}1$

$$R[i_4,i_5,n,(n\,n{+}1\,i_1)\cap(i_2i_3),i_3]R[n,n{+}1,i_1,i_2,i_3]\to$$
$$R[i_2,i_3,i_4,i_5,n]\left(\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_1\rangle}\mathrm{d}\log\frac{\langle Xi_1i_2\rangle}{\langle Xi_1(i_2i_3)\cap(i_4i_5n)\rangle}+\bar{Q}\log\frac{\langle n(i_2i_3)(i_4i_5)(n{-}1\,1)\rangle}{\langle\bar{n}i_2\rangle}\mathrm{d}\log\frac{\langle Xi_2i_3\rangle}{\langle Xi_1(i_2i_3)\cap(i_4i_5n)\rangle}\right)$$

3.4 $i_1>1$ **and** $i_5=n{-}1$

$$R[i_4,n{-}1,n,(n\,n{+}1\,i_1)\cap(i_2i_3),i_3]R[n,n{+}1,i_1,i_2,i_3]\to$$
$$R[i_2,i_3,i_4,n{-}1,n]\left(\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_1\rangle}\mathrm{d}\log\frac{\langle Xi_1i_2\rangle}{\langle Xi_1(i_2i_3)\cap(i_4\,n{-}1\,n)\rangle}+\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle\bar{n}i_2\rangle}\mathrm{d}\log\frac{\langle Xi_2i_3\rangle}{\langle Xi_1(i_2i_3)\cap(i_4\,n{-}1\,n)\rangle}\right)$$

3.4 $i_1=1$ **and** $i_5<n{-}1$

$$R[i_4,i_5,n,(n\,n{+}1\,1)\cap(i_2i_3),i_3]R[n,n{+}1,1,i_2,i_3]\to R[i_2,i_3,i_4,i_5,n]\bar{Q}\log\frac{\langle n(i_2i_3)(i_4i_5)(n{-}1\,1)\rangle}{\langle\bar{n}i_2\rangle}\mathrm{d}\log\langle Xi_2i_3\rangle$$

3.4 $i_1=1$ **and** $i_5=n{-}1$

$$R[i_4,n{-}1,n,(n\,n{+}1\,1)\cap(i_2i_3),i_3]R[n,n{+}1,1,i_2,i_3]\to R[i_2,i_3,i_4,n{-}1,n]\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle\bar{n}i_2\rangle}\mathrm{d}\log\langle Xi_2i_3\rangle$$

3.5

$$R[i_5,n,n{+}1,(n{+}1\,i_1\,i_2)\cap(i_3i_4),i_4]R[n{+}1,i_1,i_2,i_3,i_4]\to R[i_1,i_2,i_3,i_4,n]R[(ni_1i_2)\cap(i_3i_4),i_4,i_5,n,n{+}1]$$

3.6

$$R[n,n{+}1,i_1,(i_1i_2i_3)\cap(i_4i_5),i_5]R[i_1,i_2,i_3,i_4,i_5]$$

3.7

$$R[n{+}1, i_1, i_2, (i_2 i_3 i_4) \cap (i_5 n) n] R[i_2, i_3, i_4, i_5, n]$$

4. $R[i_1, i_2, i_3, (i_4 i_5 i_6) \cap (i_7 i_8), i_8] R[i_4, i_5, i_6, i_7, i_8]$

4.1 $i_1 > 1$ **and** $i_5 < n{-}1$

$$\begin{aligned}
&R[i_1, i_2, i_3, (i_4 i_5 i_6) \cap (n\, n{+}1), n{+}1] R[i_4, i_5, i_6, n, n{+}1] \to \\
&\bar{Q} \log \frac{\langle \bar{n} i_1 \rangle}{\langle \bar{n} i_2 \rangle} R[(i_1 i_2) \cap (\bar{n}), i_4, i_5, i_6, n] \mathrm{d} \log \frac{\langle X (i_1 i_2 i_3) \cap (i_4 i_5 i_6) \rangle}{\langle X i_1 i_2 \rangle} \\
&- \bar{Q} \log \frac{\langle \bar{n} i_1 \rangle}{\langle \bar{n} i_3 \rangle} R[(i_1 i_3) \cap (\bar{n}), i_4, i_5, i_6, n] \mathrm{d} \log \frac{\langle X (i_1 i_2 i_3) \cap (i_4 i_5 i_6) \rangle}{\langle X i_1 i_3 \rangle} \\
&+ \bar{Q} \log \frac{\langle \bar{n} i_2 \rangle}{\langle \bar{n} i_3 \rangle} R[(i_2 i_3) \cap (\bar{n}), i_4, i_5, i_6, n] \mathrm{d} \log \frac{\langle X (i_1 i_2 i_3) \cap (i_4 i_5 i_6) \rangle}{\langle X i_2 i_3 \rangle} \\
&+ \bar{Q} \log \frac{\langle \bar{n} i_4 \rangle}{\langle \bar{n} i_5 \rangle} R[i_1, i_2, i_3, (i_4 i_5) \cap (\bar{n}), n] \mathrm{d} \log \frac{\langle X (i_1 i_2 i_3) \cap (i_4 i_5 i_6) \rangle}{\langle X i_4 i_5 \rangle} \\
&- \bar{Q} \log \frac{\langle \bar{n} i_4 \rangle}{\langle \bar{n} i_6 \rangle} R[i_1, i_2, i_3, (i_4 i_6) \cap (\bar{n}), n] \mathrm{d} \log \frac{\langle X (i_1 i_2 i_3) \cap (i_4 i_5 i_6) \rangle}{\langle X i_4 i_6 \rangle} \\
&+ \bar{Q} \log \frac{\langle \bar{n} i_5 \rangle}{\langle \bar{n} i_6 \rangle} R[i_1, i_2, i_3, (i_5 i_6) \cap (\bar{n}), n] \mathrm{d} \log \frac{\langle X (i_1 i_2 i_3) \cap (i_4 i_5 i_6) \rangle}{\langle X i_5 i_6 \rangle}
\end{aligned}$$

4.1 $i_1 = 1$ **and** $i_5 < n{-}1$

$$\begin{aligned}
&R[1, i_2, i_3, (i_4 i_5 i_6) \cap (n\, n{+}1), n{+}1] R[i_4, i_5, i_6, n, n{+}1] \to \\
&- \bar{Q} \log \frac{\langle \bar{n} i_2 \rangle}{\langle \bar{n} i_3 \rangle} R[(i_2 i_3) \cap (\bar{n}), i_4, i_5, i_6, n] \mathrm{d} \log \langle X i_2 i_3 \rangle \\
&+ \bar{Q} \log \frac{\langle \bar{n} i_2 \rangle}{\langle \bar{n} i_3 \rangle} R[1, (i_2 i_3) \cap (\bar{n}), i_4, i_5, i_6] \mathrm{d} \log \langle X (1 i_2 i_3) \cap (i_4 i_5 i_6) \rangle \\
&- \bar{Q} \log \frac{\langle \bar{n} i_4 \rangle}{\langle \bar{n} i_5 \rangle} R[1, i_2, i_3, (i_4 i_5) \cap (\bar{n}), n] \mathrm{d} \log \langle X i_4 i_5 \rangle + \bar{Q} \log \frac{\langle \bar{n} i_4 \rangle}{\langle \bar{n} i_6 \rangle} R[1, i_2, i_3, (i_4 i_6) \cap (\bar{n}), n] \mathrm{d} \log \langle X i_4 i_6 \rangle \\
&- \bar{Q} \log \frac{\langle \bar{n} i_5 \rangle}{\langle \bar{n} i_6 \rangle} R[1, i_2, i_3, (i_5 i_6) \cap (\bar{n}), n] \mathrm{d} \log \langle X i_5 i_6 \rangle
\end{aligned}$$

4.1 $i_1 > 1$ **and** $i_5 = n{-}1$

$$\begin{aligned}
&R[i_1, i_2, i_3, (i_4\, i_5\, n{-}1) \cap (n\, n{+}1), n{+}1] R[i_4, i_5, n{-}1, n, n{+}1] \to \\
&\bar{Q} \log \frac{\langle \bar{n} i_4 \rangle}{\langle \bar{n} i_5 \rangle} \Bigg(R[i_1, i_2, i_3, n{-}1\, n] \mathrm{d} \log \frac{\tau}{\langle X (i_1 i_2 i_3) \cap (i_4\, i_5\, n{-}1) \rangle} \\
&\qquad - R[i_1, i_2, (i_4 i_5) \cap (\bar{n}), n{-}1, n] \mathrm{d} \log \frac{\langle X i_1 i_2 \rangle}{\langle X (i_1 i_2 i_3) \cap (i_4\, i_5\, n{-}1) \rangle} \\
&\qquad + R[i_1, i_3, (i_4 i_5) \cap (\bar{n}), n{-}1, n] \mathrm{d} \log \frac{\langle X i_1 i_3 \rangle}{\langle X (i_1 i_2 i_3) \cap (i_4\, i_5\, n{-}1) \rangle} \\
&\qquad - R[i_2, i_3, (i_4 i_5) \cap (\bar{n}), n{-}1, n] \mathrm{d} \log \frac{\langle X i_2 i_3 \rangle}{\langle X (i_1 i_2 i_3) \cap (i_4\, i_5\, n{-}1) \rangle} \\
&\qquad - R[i_1, i_2, i_3, (i_4 i_5) \cap (\bar{n}), n] \mathrm{d} \log \frac{\langle X i_4 i_5 \rangle}{\langle X (i_1 i_2 i_3) \cap (i_4\, i_5\, n{-}1) \rangle} \Bigg)
\end{aligned}$$

4.1 $i_1 = 1$ **and** $i_5 = n{-}1$

$$R[1,i_2,i_3,(i_4\,i_5\,n{-}1)\cap(n\,n{+}1),n{+}1]R[i_4,i_5,n{-}1,n,n{+}1] \to$$
$$\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle\bar{n}i_5\rangle}\Big(R[1,i_2,i_3,n{-}1,n]\mathrm{d}\log\tau - R[i_2,i_3,(i_4i_5)\cap(\bar{n}),n{-}1,n]\mathrm{d}\log\langle Xi_2i_3\rangle$$
$$- R[1,i_2,i_3,(i_4i_5)\cap(\bar{n}),n]\mathrm{d}\log\langle Xi_4i_5\rangle + R[1,i_2,i_3,(i_4i_5)\cap(\bar{n}),n{-}1]\mathrm{d}\log\langle X(1i_2i_3)\cap(i_4\,i_5\,n{-}1)\rangle\Big)$$

4.2

$$R[i_2,i_3,i_4,(i_5i_6n)\cap(n{+}1\,i_1),i_1]R[i_5,i_6,n,n{+}1,i_1] \to R[i_1,i_2,i_3,i_4,n]R[i_1,i_5,i_6,n,n{+}1]$$

4.3

$$R[i_3,i_4,i_5,(i_6\,n\,n{+}1)\cap(i_1i_2),i_2]R[i_1,i_2,i_6,n,n{+}1] \to R[i_1,i_2,i_3,i_4,i_5]R[i_1,(i_1i_2)\cap(i_3i_4i_5),i_6,n,n{+}1]$$

4.4

$$R[i_4,i_5,i_6,(n\,n{+}1\,i_1)\cap(i_2i_3),i_3]R[n,n{+}1,i_1,i_2,i_3] \to R[i_2,i_3,i_4,i_5,i_6]R[i_1,i_2,(i_2i_3)\cap(i_4i_5i_6),n,n{+}1]$$

4.5

$$R[i_5,i_6,n,(n{+}1\,i_1\,i_2)\cap(i_3i_4),i_4]R[n{+}1,i_1,i_2,i_3,i_4] \to 0$$

4.6

$$R[i_6,n,n{+}1,(i_1i_2i_3)\cap(i_4i_5),i_5]R[i_1,i_2,i_3,i_4,i_5]$$
$$R[n,n{+}1,i_1,(i_2i_3i_4)\cap(i_5i_6),i_6]R[i_2,i_3,i_4,i_5,i_6]$$

4.7

$$R[n{+}1,i_1,i_2,(i_3i_4i_5)\cap(i_6n),n]R[i_3,i_4,i_5,i_6,n]$$

5. $R[i_1,i_2,i_3,i_4,i_8]R[i_4,i_5,i_6,i_7,i_8]$

6. $R[i_1,i_2,i_3,(i_4i_5)\cap(i_6i_7i_8),i_8]R[i_4,i_5,i_6,i_7,i_8]$

6.1 $i_1 \geq 1$ **and** $i_6 < n-1$

$$R[i_1,i_2,i_3,(i_4i_5)\cap(i_6\,n\,n{+}1),n{+}1]R[i_4,i_5,i_6,n,n{+}1] \to$$
$$\bar{Q}\log\frac{\langle\bar{n}i_4\rangle}{\langle\bar{n}i_6\rangle}R[i_1,i_2,i_3,i_4,n]\mathrm{d}\log\frac{\langle Xi_4i_6\rangle}{\langle Xi_4i_5\rangle} - \bar{Q}\log\frac{\langle\bar{n}i_5\rangle}{\langle\bar{n}i_6\rangle}R[i_1,i_2,i_3,i_5,n]\mathrm{d}\log\frac{\langle Xi_5i_6\rangle}{\langle Xi_4i_5\rangle}$$
$$-\bar{Q}\log\frac{\langle\bar{n}(i_4i_5)\cap(i_1i_2i_3)\rangle}{\langle\bar{n}i_6\rangle}R[i_1,i_2,i_3,i_4,i_5]\mathrm{d}\log\frac{\langle X(i_4i_5)\cap(i_1i_2i_3)i_6\rangle}{\langle Xi_4i_5\rangle}$$
$$+\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_4i_5n)\rangle}{\langle\bar{n}i_6\rangle}R[i_1,i_2,i_4,i_5,n]\mathrm{d}\log\frac{\langle X(i_1i_2)\cap(i_4i_5n)i_6\rangle}{\langle Xi_4i_5\rangle}$$
$$-\bar{Q}\log\frac{\langle\bar{n}(i_1i_3)\cap(i_4i_5n)\rangle}{\langle\bar{n}i_6\rangle}R[i_1,i_3,i_4,i_5,n]\mathrm{d}\log\frac{\langle X(i_1i_3)\cap(i_4i_5n)i_6\rangle}{\langle Xi_4i_5\rangle}$$
$$+\bar{Q}\log\frac{\langle\bar{n}(i_2i_3)\cap(i_4i_5n)\rangle}{\langle\bar{n}i_6\rangle}R[i_2,i_3,i_4,i_5,n]\mathrm{d}\log\frac{\langle X(i_2i_3)\cap(i_4i_5n)i_6\rangle}{\langle Xi_4i_5\rangle}$$

6.1 $i_1 \geq 1$ **and** $i_6 = n-1$

$$R[i_1, i_2, i_3, (i_4 i_5) \cap (n{-}1\, n\, n{+}1), n{+}1] R[i_4, i_5, n{-}1, n, n{+}1] \to$$
$$\bar{Q} \log \frac{\langle \bar{n} i_4 \rangle}{\langle \bar{n} i_5 \rangle} R[i_1, i_2, i_3, (i_4 i_5) \cap (\bar{n}), n] \mathrm{d} \log \frac{\tau}{\langle X i_4 i_5 \rangle} \qquad (i_1 \text{ can generally be } 1)$$

6.2 $i_1 > 1$ **and** $i_6 < n-1$

$$\begin{aligned}
&R[i_2, i_3, i_4, (i_5 i_6) \cap (n\, n{+}1\, i_1), i_1] R[i_5, i_6, n, n{+}1, i_1] \to \\
&- \bar{Q} \log \frac{\langle \bar{n} i_1 \rangle}{\langle \bar{n} i_5 \rangle} R[i_1, i_2, i_3, i_4, i_5] \mathrm{d} \log \frac{\langle X i_1 i_5 \rangle}{\langle X i_5 i_6 \rangle} + \bar{Q} \log \frac{\langle \bar{n} i_1 \rangle}{\langle \bar{n} i_6 \rangle} R[i_1, i_2, i_3, i_4, i_6] \mathrm{d} \log \frac{\langle \bar{n} i_1 i_6 \rangle}{\langle \bar{n} i_5 i_6 \rangle} \\
&+ \bar{Q} \log \frac{\langle \bar{n} (i_5 i_6) \cap (i_1 i_2 i_3) \rangle}{\langle \bar{n} i_1 \rangle} R[i_1, i_2, i_3, i_5, i_6] \mathrm{d} \log \frac{\langle i_1 (i_2 i_3)(i_5 i_6)(X) \rangle}{\langle X i_5 i_6 \rangle} \\
&- \bar{Q} \log \frac{\langle \bar{n} (i_5 i_6) \cap (i_1 i_2 i_4) \rangle}{\langle \bar{n} i_1 \rangle} R[i_1, i_2, i_4, i_5, i_6] \mathrm{d} \log \frac{\langle i_1 (i_2 i_4)(i_5 i_6)(X) \rangle}{\langle X i_5 i_6 \rangle} \\
&+ \bar{Q} \log \frac{\langle \bar{n} (i_5 i_6) \cap (i_1 i_3 i_4) \rangle}{\langle \bar{n} i_1 \rangle} R[i_1, i_3, i_4, i_5, i_6] \mathrm{d} \log \frac{\langle i_1 (i_3 i_4)(i_5 i_6)(X) \rangle}{\langle X i_5 i_6 \rangle} \\
&- \bar{Q} \log \frac{\langle \bar{n} (i_5 i_6) \cap (i_2 i_3 i_4) \rangle}{\langle \bar{n} i_1 \rangle} R[i_2, i_3, i_4, i_5, i_6] \mathrm{d} \log \frac{\langle X (i_5 i_6) \cap (i_2 i_3 i_4) i_1 \rangle}{\langle X i_5 i_6 \rangle}
\end{aligned}$$

6.2 $i_1 > 1$ **and** $i_6 = n-1$

$$\begin{aligned}
&R[i_2, i_3, i_4, (i_5\, n{-}1) \cap (n\, n{+}1\, i_1), i_1] R[i_5, n{-}1, n, n{+}1, i_1] \to \\
&\bar{Q} \log \frac{\langle \bar{n} i_1 \rangle}{\langle \bar{n} i_5 \rangle} \Bigg(R[i_1, i_2, i_3, i_4, n{-}1] \mathrm{d} \log \frac{\tau}{\langle X (i_5\, n{-}1) \cap (i_2 i_3 i_4) i_1 \rangle} \\
&- R[i_1, i_2, i_3, i_4, i_5] \mathrm{d} \log \frac{\langle X i_1 i_5 \rangle}{\langle X (i_5\, n{-}1) \cap (i_2 i_3 i_4) i_1 \rangle} \\
&- R[i_1, i_2, i_3, i_5, n{-}1] \mathrm{d} \log \frac{\langle X (i_5\, n{-}1) \cap (i_1 i_2 i_3) i_1 \rangle}{\langle X (i_5\, n{-}1) \cap (i_2 i_3 i_4) i_1 \rangle} \\
&+ R[i_1, i_2, i_4, i_5, n{-}1] \mathrm{d} \log \frac{\langle X (i_5\, n{-}1) \cap (i_1 i_2 i_4) i_1 \rangle}{\langle X (i_5\, n{-}1) \cap (i_2 i_3 i_4) i_1 \rangle} \\
&- R[i_1, i_3, i_4, i_5, n{-}1] \mathrm{d} \log \frac{\langle X (i_5\, n{-}1) \cap (i_1 i_3 i_4) i_1 \rangle}{\langle X (i_5\, n{-}1) \cap (i_2 i_3 i_4) i_1 \rangle} \Bigg)
\end{aligned}$$

6.2 $i_1 = 1$ **and** $i_6 < n-1$

$$R[i_2, i_3, i_4, (i_5 i_6) \cap (n\, n{+}1\, 1), 1] R[i_5, i_6, n, n{+}1, 1] \to - R[1, i_2, i_3, i_4, (i_5 i_6) \cap (\bar{n})] \bar{Q} \log \frac{\langle \bar{n} i_5 \rangle}{\langle \bar{n} i_6 \rangle} \mathrm{d} \log \langle X i_5 i_6 \rangle$$

6.2 $i_1 = 1$ **and** $i_6 = n-1$

$$R[i_2, i_3, i_4, (i_5\, n{-}1) \cap (n\, n{+}1\, 1), 1] R[i_5, n{-}1, n, n{+}1, 1] \to R[1, i_2, i_3, i_4, n{-}1] \bar{Q} \log \frac{\langle \bar{n} 2 \rangle}{\langle \bar{n} i_5 \rangle} \mathrm{d} \log \tau$$

6.3

$$R[i_3, i_4, i_5, (i_6 n) \cap (n{+}1\, i_1\, i_2), i_2] R[i_6, n, n{+}1, i_1, i_2] \to R[i_2, i_3, i_4, i_5, n] R[i_1, i_2, i_6, n, n{+}1]$$

6.4 $i_1 > 1$ **and** $i_6 \leq n-1$

$$
\begin{aligned}
&R[i_4,i_5,i_6,(n\,n{+}1)\cap(i_1i_2i_3),i_3]R[n,n{+}1,i_1,i_2,i_3] \to \\
&-\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}i_2\rangle}R[(i_1i_2)\cap(\bar{n}),i_3,i_4,i_5,i_6]\mathrm{d}\log\frac{\langle Xi_1i_2\rangle}{\langle X(i_1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&+\bar{Q}\log\frac{\langle\bar{n}i_1\rangle}{\langle\bar{n}(i_1i_3)\cap(i_4i_5i_6)\rangle}R[i_1,i_3,i_4,i_5,i_6]\mathrm{d}\log\frac{\langle Xi_1i_3\rangle}{\langle X(i_1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}(i_2i_3)\cap(i_4i_5i_6)\rangle}R[i_2,i_3,i_4,i_5,i_6]\mathrm{d}\log\frac{\langle Xi_2i_3\rangle}{\langle X(i_1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&+\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_5)\rangle}{\langle\bar{n}(i_4i_5)\cap(i_1i_2i_3)\rangle}R[i_1,i_2,i_3,i_4,i_5]\mathrm{d}\log\frac{\langle X(i_1i_2)\cap(i_3i_4i_5)i_3\rangle}{\langle X(i_1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&-\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_6)\rangle}{\langle\bar{n}(i_4i_6)\cap(i_1i_2i_3)\rangle}R[i_1,i_2,i_3,i_4,i_6]\mathrm{d}\log\frac{\langle X(i_1i_2)\cap(i_3i_4i_6)i_3\rangle}{\langle X(i_1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&+\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_5i_6)\rangle}{\langle\bar{n}(i_5i_6)\cap(i_1i_2i_3)\rangle}R[i_1,i_2,i_3,i_5,i_6]\mathrm{d}\log\frac{\langle X(i_1i_2)\cap(i_3i_5i_6)i_3\rangle}{\langle X(i_1i_2i_3)\cap(i_4i_5i_6)\rangle}
\end{aligned}
$$

6.4 $i_1 = 1$ **and** $i_6 \leq n-1$

$$
\begin{aligned}
&R[i_4,i_5,i_6,(n\,n{+}1)\cap(1i_2i_3),i_3]R[1,i_2,i_3,n,n{+}1] \to \\
&-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}(i_2i_3)\cap(i_4i_5i_6)\rangle}R[i_2,i_3,i_4,i_5,i_6]\mathrm{d}\log\frac{\langle Xi_2i_3\rangle}{\langle X(1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&+\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}(i_2i_3)\cap(i_4i_51)\rangle}R[1,i_2,i_3,i_4,i_5]\mathrm{d}\log\frac{\langle X(1i_2)\cap(i_3i_4i_5)i_3\rangle}{\langle X(1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}(i_2i_3)\cap(i_4i_61)\rangle}R[1,i_2,i_3,i_4,i_6]\mathrm{d}\log\frac{\langle X(1i_2)\cap(i_3i_4i_6)i_3\rangle}{\langle X(1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&+\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}(i_2i_3)\cap(i_5i_61)\rangle}R[1,i_2,i_3,i_5,i_6]\mathrm{d}\log\frac{\langle X(1i_2)\cap(i_3i_5i_6)i_3\rangle}{\langle X(1i_2i_3)\cap(i_4i_5i_6)\rangle} \\
&-\bar{Q}\log\frac{\langle\bar{n}i_2\rangle}{\langle\bar{n}i_3\rangle}R[1,i_3,i_4,i_5,i_6]\mathrm{d}\log\langle X(1i_2i_3)\cap(i_4i_5i_6)\rangle
\end{aligned}
$$

6.5

$$R[i_5,i_6,n,(n{+}1\,i_1)\cap(i_2i_3i_4),i_4]R[n{+}1,i_1,i_2,i_3,i_4] \to 0$$

6.6

$$
\begin{aligned}
R[i_6,n,n{+}1,(i_1i_2)\cap(i_3i_4i_5),i_5]R[i_1,i_2,i_3,i_4,i_5] \\
R[n,n{+}1,i_1,(i_2i_3)\cap(i_4i_5i_6),i_6]R[i_2,i_3,i_4,i_5,i_6]
\end{aligned}
$$

6.6

$$R[n{+}1,i_1,i_2,(i_3i_4)\cap(i_5i_6n),n]R[i_3,i_4,i_5,i_6,n]$$

7. $R[i_1,i_2,i_3,(i_4i_5)\cap(i_6i_7i_8),(i_4i_5i_6)\cap(i_7i_8)]R[i_4,i_5,i_6,i_7,i_8]$

Using six-term identity,

$$
\begin{aligned}
&(-R[i_1,i_2,i_3,(i_4i_5i_6)\cap(i_7i_8),i_8]+R[i_1,i_2,i_3,(i_4i_5)\cap(i_6i_7i_8),i_8] \\
&\quad+R[i_1,i_2,(i_4i_5)\cap(i_6i_7i_8),(i_4i_5i_6)\cap(i_7i_8),i_8]-R[i_1,i_3,(i_4i_5)\cap(i_6i_7i_8),(i_4i_5i_6)\cap(i_7i_8),i_8] \\
&\quad+R[i_2,i_3,(i_4i_5)\cap(i_6i_7i_8),(i_4i_5i_6)\cap(i_7i_8),i_8])R[i_4,i_5,i_6,i_7,i_8]
\end{aligned}
$$

8. $R[i_1, i_2, i_3, i_4, (i_4 i_5 i_6) \cap (i_7 i_8)] R[i_4, i_5, i_6, i_7, i_8]$

Using six-term identity,

$$\begin{aligned}
&(R[i_1, i_2, i_3, i_4, i_8] - R[i_1, i_2, i_3, (i_4 i_5 i_6) \cap (i_7 i_8), i_8] \\
&\ + R[i_1, i_2, i_4, (i_4 i_5 i_6) \cap (i_7 i_8), i_8] - R[i_1, i_3, i_4, (i_4 i_5 i_6) \cap (i_7 i_8), i_8] \\
&\ + R[i_2, i_3, i_4, (i_4 i_5 i_6) \cap (i_7 i_8), i_8]) R[i_4, i_5, i_6, i_7, i_8]
\end{aligned}$$

9. Four-mass box

This case need a special treatment.

10. $R[i_1, i_2, i_3, i_4, i_9] R[i_5, i_6, i_7, i_8, i_9]$

11. $R[i_1, i_2, i_3, i_4, (i_5 i_6 i_7) \cap (i_8 i_9)] R[i_5, i_6, i_7, i_8, i_9]$

Using six-term identity,

$$\begin{aligned}
&(R[i_1, i_2, i_3, i_4, i_9] - R[i_1, i_2, i_3, (i_5 i_6 i_7) \cap (i_8 i_9), i_9] \\
&\ + R[i_1, i_2, i_4, (i_5 i_6 i_7) \cap (i_8 i_9), i_9] - R[i_1, i_3, i_4, (i_5 i_6 i_7) \cap (i_8 i_9), i_9] \\
&\ + R[i_2, i_3, i_4, (i_5 i_6 i_7) \cap (i_8 i_9), i_9]) R[i_5, i_6, i_7, i_8, i_9]
\end{aligned}$$

12. $R[i_1, i_2, i_3, i_4, (i_5 i_6) \cap (i_7 i_8 i_9)] R[i_5, i_6, i_7, i_8, i_9]$

Using six-term identity,

$$\begin{aligned}
&(R[i_1, i_2, i_3, i_4, i_9] - R[i_1, i_2, i_3, (i_5 i_6) \cap (i_7 i_8 i_9), i_9] \\
&\ + R[i_1, i_2, i_4, (i_5 i_6) \cap (i_7 i_8 i_9), i_9] - R[i_1, i_3, i_4, (i_5 i_6) \cap (i_7 i_8 i_9), i_9] \\
&\ + R[i_2, i_3, i_4, (i_5 i_6) \cap (i_7 i_8 i_9), i_9]) R[i_5, i_6, i_7, i_8, i_9]
\end{aligned}$$

13. $\varphi R[i_1, i_2, i_3, (i_4 i_5) \cap (i_7 i_8 i_9), (i_4 i_6) \cap (i_7 i_8 i_9)] R[(i_4 i_5) \cap (i_1 i_2 i_3), (i_4 i_6) \cap (i_1 i_2 i_3), i_7, i_8, i_9]$

where

$$\varphi = \frac{\langle i_4 i_5 (i_1 i_2 i_3) \cap (i_7 i_8 i_9) \rangle \langle i_4 i_6 (i_1 i_2 i_3) \cap (i_7 i_8 i_9) \rangle}{\langle i_1 i_2 i_3 i_4 \rangle \langle i_4 i_7 i_8 i_9 \rangle \langle i_5 i_6 (i_1 i_2 i_3) \cap (i_7 i_8 i_9) \rangle}$$

This R invariant is equal to

$$\begin{aligned}
&R[i_4, i_5, i_6, (i_7 i_8) \cap (i_9 i_1 i_3), (i_7 i_8 i_9) \cap (i_1 i_3)] R[i_7, i_8, i_9, i_1, i_3] \\
&- R[i_4, i_5, i_6, (i_7 i_8) \cap (i_9 i_1 i_2), (i_7 i_8 i_9) \cap (i_1 i_2)] R[i_7, i_8, i_9, i_1, i_2] \\
&+ R[i_1, i_2, i_3, (i_4 i_5 i_6) \cap (i_8 i_9), i_9] R[i_4, i_5, i_6, i_8, i_9] + R[i_1, i_2, i_3, (i_4 i_6) \cap (i_7 i_8 i_9), i_9] R[i_4, i_6, i_7, i_8, i_9] \\
&- R[i_1, i_2, i_3, (i_4 i_5) \cap (i_7 i_8 i_9), i_9] R[i_4, i_5, i_7, i_8, i_9] - R[i_1, i_2, i_3, (i_4 i_5 i_6) \cap (i_7 i_9), i_9] R[i_4, i_5, i_6, i_7, i_9] \\
&+ R[i_9, i_1, i_2, i_3, (i_4 i_5 i_6) \cap (i_7 i_8)] R[i_4, i_5, i_6, i_7, i_8] - R[i_1, i_2, i_3, (i_5 i_6) \cap (i_7 i_8 i_9), i_9] R[i_5, i_6, i_7, i_8, i_9] \\
&- R[i_4, i_5, i_6, (i_7 i_8) \cap (i_9 i_2 i_3), (i_7 i_8 i_9) \cap (i_2 i_3)] R[i_7, i_8, i_9, i_2, i_3]
\end{aligned}$$

14. $R[i_1, i_2, i_3, i_4, i_5] R[i_6, i_7, i_8, i_9, i_{10}]$

1 The prescription for four-mass boxes

Of all one-loop NNMHV leading singularities, a special attention is needed for which corresponding to four-mass boxes, since they are R-invariants with irrational element. In general, there are two classes of four-mass boxes that have contribution to $\bar{Q}$ equations, one with mass corner $\{n, n{+}1\}$, ane the other with mass corners $\{c, n\}$ and $\{n{+}1, A\}$. (Here we follow the notation of 1303.4734). However the first classes will become rational under the collinear limit $n{+}1 \to n$. The real complexity arises from the second class. Before going into the detail, we first introduce the notation (which is almost the same as 1303.4734)

For four-mass box with mass corner $\{\{a, b{-}1\}, \{b, c{-}1\}, \{c, d{-}1\}, \{d, a{-}1\}\}$, we introduce

$$u \equiv \frac{\langle a{-}1\, a\, b{-}1\, b\rangle\langle c{-}1\, c\, d{-}1\, d\rangle}{\langle a{-}1\, a\, c{-}1\, c\rangle\langle b{-}1\, b\, d{-}1\, d\rangle} \qquad v \equiv \frac{\langle b{-}1\, b\, c{-}1\, c\rangle\langle a{-}1\, a\, d{-}1\, d\rangle}{\langle a{-}1\, a\, c{-}1\, c\rangle\langle b{-}1\, b\, d{-}1\, d\rangle}$$

and

$$\Delta \equiv \sqrt{(1-u-v)^2 - 4uv}$$

The corresponding leading singularities are

$$f^{\pm}_{a,b,c,d} = \frac{1-u-v\pm\Delta}{2\Delta} R[\alpha_{\pm}, b{-}1, b, c{-}1, c] R[\gamma_{\pm}, d{-}1, d, a{-}1, a]$$

where $\alpha_{\pm}$ and $\gamma_{\pm}$ are two solutions of

$$\alpha = (a{-}1\, a) \cap (d\, d{-}1\, \gamma) \qquad \gamma = (c{-}1\, c) \cap (b\, b{-}1\, \alpha)$$

More precisely,

$$\gamma_{\pm} = \mathcal{Z}_{c-1} + \mathcal{Z}_c \left(-\frac{\langle c{-}1\, d{-}1\, d\, (a{-}1\, a) \cap (b{-}1\, b\, c)\rangle}{\langle c\, d{-}1\, d\, (a{-}1\, a) \cap (b{-}1\, b\, c)\rangle} + \frac{\langle a{-}1\, a\, c{-}1\, c\rangle\langle b{-}1\, b\, d{-}1\, d\rangle(1-u-v\pm\Delta)}{2\langle c\, d{-}1\, d\, (a{-}1\, a) \cap (b{-}1\, b\, c)\rangle} \right).$$

1.1 Rationalize the Square root

After taking the collinear limit $\epsilon \to 0$, unlike other cases, the second kind of box will not give a rational function of τ since Δ^2 will be a quadratic polynomial of τ which is not perfect squared. To perform the τ-integration, we need rationalizing the integrand first. In other words, we need to find a variable substitution $t(\tau)$ such that Δ^2 in terms of t will be a perfect squared polynomial. This is just a classical problem to find a rational parameterization of a quadratic curve. Let us quickly review the solution of this problem. For the rational curve defined by

$$y^2 = x^2 + ax + b\,, \tag{1.1}$$

If there is a rational point (x_*, y_*) on this curve (by a rational points (x_*, y_*) we mean $x, y \in \mathbb{Q}(a, b)$), then we can insert $y = y_* + t(x - x_*)$ in eq.(1.1) to obtain the parameterization $x(t)$ and hence $y(t)$.

As for our cases, there are two kinds of obvious rational points, one have $u(\tau_*) = 0$ and the other have $v(\tau_*) = 0$. In what follows, we will denote this two points as τ_u and τ_v and take $d = n{+}1$. Depending on the values of τ_u and τ_v, the second kind of 4-mass boxes can be decomposed into 4 classes further:

i) $a = 2$ and $c = n{-}1$

In this case, we have

$$\tau_u = 0\,, \qquad \tau_v = \infty\,.$$

ii) $a = 2$ and $c < n{-}1$

In this case, we have

$$\tau_u = \frac{\langle c{-}1\, c\, n\, n{-}1\rangle\langle n\, 1\, 2\, 3\rangle}{\langle c{-}1\, c\, n\, 1\rangle\langle n{-}1\, n\, 2\, 3\rangle}\,, \qquad \tau_v = \infty\,.$$

This case first appear in one-loop 9-pt NNMHV ratio function.

iii) $a > 2$ and $c = n{-}1$

In this case, we have

$$\tau_u = 0\,, \qquad \tau_v = \frac{\langle a{-}1\, a\, n\, n{-}1\rangle\langle n\, 1\, 2\, 3\rangle}{\langle a{-}1\, a\, n\, 1\rangle\langle n{-}1\, n\, 2\, 3\rangle}\,.$$

This case first appear in one-loop 9-pt NNMHV ratio function.
iv) $a > 2$ and $c < n-1$

$$\tau_u = \frac{\langle c{-}1\, c\, n\, n{-}1\rangle\langle n\, 1\, 2\, 3\rangle}{\langle c{-}1\, c\, n\, 1\rangle\langle n{-}1\, n\, 2\, 3\rangle}\,, \qquad \tau_v = \frac{\langle a{-}1\, a\, n\, n{-}1\rangle\langle n\, 1\, 2\, 3\rangle}{\langle a{-}1\, a\, n\, 1\rangle\langle n{-}1\, n\, 2\, 3\rangle}\,.$$

This case first appear in one-loop 10-pt NNMHV ratio function.

1.2 The general four-mass box $f_{a,b,c,n+1}$: Case (iv)

Let us consider the most general case, that is, case (iv). First, since this kind of four mass box will produce two squar roots

$$\Delta_1 := \Delta_{1,a,b,c}\,, \qquad \Delta_n := \Delta_{a,b,c,n}\,,$$

we introduce

$$u_1 = \frac{x_{1a}^2 x_{bc}^2}{x_{1b}^2 x_{ac}^2}\,, \qquad v_1 = \frac{x_{ab}^2 x_{c1}^2}{x_{1b}^2 x_{ac}^2}\,, \qquad z_1\bar{z}_1 = u_1\,, \qquad (1-z_1)(1-\bar{z}_1) = v_1 \tag{1.2}$$

$$u_n = \frac{x_{ab}^2 x_{cn}^2}{x_{ac}^2 x_{bn}^2}\,, \qquad v_n = \frac{x_{bc}^2 x_{na}^2}{x_{ac}^2 x_{bn}^2}\,, \qquad z_n\bar{z}_n = u_n\,, \qquad (1-z_n)(1-\bar{z}_n) = v_n \tag{1.3}$$

In terms of these variables, the rationalize map is

$$\tau = \tau_u \frac{\big(t - u_1(1-u_n-\Delta_n)\big)\big(t - u_1(1-u_n+\Delta_n)\big)}{\big(t - v_n(1-v_1-\Delta_1)\big)\big(t - v_n(1-v_1+\Delta_1)\big)} \tag{1.4}$$

if we choose τ_u as the raitonal point, or

$$\tau = \tau_v \frac{\big(t - v_1(1-v_n-\Delta_n)\big)\big(t - v_1(1-v_n+\Delta_n)\big)}{\big(t - u_n(1-u_1-\Delta_1)\big)\big(t - u_n(1-u_1+\Delta_1)\big)} \tag{1.5}$$

if we choose τ_v as the raitonal point. We will choose the first rationalize map, then the integral interval for t is $[u_1(1-u_n+\Delta_n), v_n(1-v_1+\Delta_1)]$. Then we find

$$\begin{aligned}
&\oint_{\epsilon=0} \epsilon \mathrm{d}\epsilon \mathrm{d}\tau \int \mathrm{d}\chi_{n+1}^3 \left(f^+_{a,b,c,n+1} + f^-_{a,b,c,n+1}\right) = \\
&- \mathrm{d}\log \frac{t - u_1 v_n(1+2x_{c-1})}{t + u_1 v_n(1-2y_{c-1})} R[a{-}1,a,b{-}1,b,c{-}1]\bar{Q}\log x_{c-1} + \mathrm{d}\log \frac{t - u_1 v_n(1+2x_c)}{t + u_1 v_n(1-2y_c)} R[a{-}1,a,b{-}1,b,c]\bar{Q}\log x_c \\
&- \mathrm{d}\log \frac{t - u_1 v_n(1+2x_{b-1})}{t - u_1 v_n(1-2y_{b-1})} R[a{-}1,a,b{-}1,c{-}1,c]\bar{Q}\log x_{b-1} + \mathrm{d}\log \frac{t - u_1 v_n(1+2x_b)}{t - u_1 v_n(1-2y_b)} R[a{-}1,a,b,c{-}1,c]\bar{Q}\log x_b \\
&- \mathrm{d}\log \frac{t - u_1 v_n(1+2x_{a-1})}{t + u_1 v_n(1+2y_{a-1})} R[a{-}1,b{-}1,b,c{-}1,c]\bar{Q}\log x_{a-1} + \mathrm{d}\log \frac{t - u_1 v_n(1+2x_a)}{t + u_1 v_n(1+2y_a)} R[a,b{-}1,b,c{-}1,c]\bar{Q}\log x_a \\
&+ \mathrm{d}\log \frac{t - u_1 v_n}{t + u_1 v_n(1-2\mu)} \bar{Q}\log \frac{\langle \bar{n}\, c{-}1\rangle}{\langle \bar{n}\, c\rangle} R[a{-}1,a,b{-}1,b,(c{-}1\, c)\cap(\bar{n})] \\
&+ \mathrm{d}\log\left(t - u_1 v_n \frac{\mu+\nu}{\mu-\nu}\right) \bar{Q}\log \frac{\langle \bar{n}\, a{-}1\rangle}{\langle \bar{n}\, a\rangle} R[(a{-}1,a)\cap(\bar{n}),b{-}1,b,c{-}1,c]
\end{aligned}$$

where

$$\mu = \frac{\langle n(b{-}1\,b)(c{-}1\,c)(n{-}1\,1)\rangle\langle a{-}1\,a\,c{-}1\,c\rangle}{\langle n(a{-}1\,a)(c{-}1\,c)(n{-}1\,1)\rangle\langle b{-}1\,b\,c{-}1\,c\rangle} \qquad \nu = \frac{\langle n(b{-}1\,b)(c{-}1\,c)(n{-}1\,1)\rangle\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle n(a{-}1\,a)(b{-}1\,b)(n{-}1\,1)\rangle\langle b{-}1\,b\,c{-}1\,c\rangle}$$

$$x_{a-1} = \frac{\langle \bar{n}(c{-}1\,c)\cap(a{-}1\,b{-}1\,b)\rangle}{\langle \bar{n}\,a{-}1\rangle\langle b{-}1\,b\,c{-}1\,c\rangle} \qquad y_{a-1} = \frac{\langle a{-}1\,a\,c{-}1\,c\rangle\langle a{-}1\,b{-}1\,b\,n\rangle}{\langle a{-}1(b{-}1\,b)(c-1\,c)(a\,n)\rangle}\,, \qquad (x_a, y_a) = (x_{a-1}, y_{a-1})|_{a-1\leftrightarrow a}$$

$$x_{b-1} = \frac{\langle \bar{n}(c{-}1\,c)\cap(a{-}1\,a\,b{-}1)\rangle}{\langle \bar{n}(a{-}1\,a)\cap(b{-}1\,c{-}1\,c)\rangle} \qquad y_{b-1} = \frac{\langle a{-}1\,a\,b{-}1\,b\rangle\langle b{-}1\,c{-}1\,c\,n\rangle}{\langle a{-}1\,a\,b{-}1\,n\rangle\langle b{-}1\,b\,c{-}1\,c\rangle}\,, \qquad (x_b, y_b) = (x_{b-1}, y_b - 1)|_{b-1\leftrightarrow b}$$

$$x_{c-1} = \frac{\langle \bar{n}\,c{-}1\rangle\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle \bar{n}(a{-}1\,a)\cap(b{-}1\,b\,c{-}1)\rangle} \qquad y_{c-1} = \frac{\langle a{-}1\,a\,c{-}1\,c\rangle\langle b{-}1\,b\,c{-}1\,n\rangle}{\langle a{-}1\,a\,c{-}1\,n\rangle\langle b{-}1\,b\,c{-}1\,c\rangle}\,, \qquad (x_c, y_c) = (x_{c-1}, y_{c-1})|_{c-1\leftrightarrow c} \tag{1.6}$$

In terms of t, the box function is

$$\mathcal{I}_{a,b,c} = \mathrm{Li}_2(\zeta) - \mathrm{Li}_2(\bar{\zeta}) + \frac{1}{2}\log(\zeta\bar{\zeta})\log\frac{1-\zeta}{1-\bar{\zeta}}$$

where

$$\zeta = \frac{t - u_1 v_n}{t + u_1 v_n}\,, \qquad \bar{\zeta} = \left(\frac{b}{a}\right)\frac{t + u_1 v_n(1-2a)}{t - u_1 v_n(1+2b)}$$

Now we can easily integrate out t, before we do so, due to the following identity

$$\begin{aligned}
&\bar{Q}\log\frac{\langle \bar{n}\,a{-}1\rangle}{\langle \bar{n}\,a\rangle}R[(a{-}1,a)\cap(\bar{n}), b{-}1, b, c{-}1, c] - \bar{Q}\log\frac{\langle \bar{n}\,c{-}1\rangle}{\langle \bar{n}\,c\rangle}R[a{-}1, a, b{-}1, b, (c{-}1\,c)\cap(\bar{n})]\\
&= -R[a{-}1,a,b{-}1,b,c{-}1]\bar{Q}\log x_{c-1} + R[a{-}1,a,b{-}1,b,c]\bar{Q}\log x_c - R[a{-}1,a,b{-}1,c{-}1,c]\bar{Q}\log x_{b-1}\\
&\quad + R[a{-}1,a,b,c{-}1,c]\bar{Q}\log x_b - R[a{-}1,b{-}1,b,c{-}1,c]\bar{Q}\log x_{a-1} + R[a,b{-}1,b,c{-}1,c]\bar{Q}\log x_a,
\end{aligned}$$

we can rearrange the 8 terms to obtain 7 terms whose weight 3 symbols are relatively simple.

The result for weight-3 symbol

1. The algebraic part of the weight 3 symbol for the coefficent of $\bar{Q}\log\frac{\langle \bar{n}\,a-1\rangle}{\langle \bar{n}\,a\rangle}R[(a{-}1,a)\cap(\bar{n}), b{-}1, b, c{-}1, c]+$ $\bar{Q}\log\frac{\langle \bar{n}\,c-1\rangle}{\langle \bar{n}\,c\rangle}R[a{-}1, a, b{-}1, b, (c{-}1\,c)\cap(\bar{n})]$ is

$$\frac{1}{4}\left(u_1 \otimes \frac{1-\bar{z}_1}{1-z_1} + v_1 \otimes \frac{z_1}{\bar{z}_1}\right) \otimes \frac{u_1 - \bar{z}_1}{u_1 - z_1} - (1 \leftrightarrow n)$$

2. The algebraic part of the weight 3 symbol for the coefficent of $R[a{-}1, a, b{-}1, b, c]\bar{Q}\log x_c$

$$\begin{aligned}
&\frac{1}{4}\left(u_1 \otimes \frac{1-\bar{z}_1}{1-z_1} + v_1 \otimes \frac{z_1}{\bar{z}_1}\right) \otimes \frac{\left((x_c+1)^{-1} - z_1\right)^2 \bar{z}_1(1-\bar{z}_1)}{\left((x_c+1)^{-1} - \bar{z}_1\right)^2 z_1(1-z_1)}\\
&+ \frac{1}{4}\left(u_n \otimes \frac{1-\bar{z}_n}{1-z_n} + v_n \otimes \frac{z_n}{\bar{z}_n}\right) \otimes \frac{\left((x_c^{-1}+1)^{-1} - z_n\right)^2 \bar{z}_n(1-\bar{z}_n)}{\left((x_c^{-1}+1)^{-1} - \bar{z}_n\right)^2 z_n(1-z_n)}
\end{aligned} \tag{1.7}$$

The results for other 5 terms, that is $R[a-1,a,b-1,b,c-1]\bar{Q}\log x_{c-1}$, $R[a-1,a,b-1,c-1,c]\bar{Q}\log x_b$, and *etc.* are similar. Just note that

$$
\begin{aligned}
\left|(x_c+1)^{-1}-z_1\right|^2 &= -\frac{\langle c(A)(B)(L_1)\rangle\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle\langle AB\rangle}{\langle BL_1\rangle\langle AC\rangle\langle \bar{n}(B)\cap(Ac)\rangle^2}\\
\left|(x_c^{-1}+1)^{-1}-z_n\right|^2 &= -\frac{\langle c(A)(B)(L_n)\rangle\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle\langle AB\rangle}{\langle BL_n\rangle\langle AC\rangle\langle \bar{n}(B)\cap(Ac)\rangle^2}\\
\left|(x_b+1)^{-1}-z_1\right|^2 &= \frac{\langle b(A)(C)(L_1)\rangle\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle}{\langle BL_1\rangle\langle AC\rangle^2\langle \bar{n}b\rangle^2}\\
\left|(x_b^{-1}+1)^{-1}-z_n\right|^2 &= \frac{\langle b(A)(C)(L_n)\rangle\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle}{\langle BL_n\rangle\langle AC\rangle^2\langle \bar{n}b\rangle^2}\\
\left|(x_a+1)^{-1}-z_1\right|^2 &= -\frac{\langle a(B)(C)(L_1)\rangle\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle\langle BC\rangle}{\langle BL_1\rangle\langle AC\rangle\langle \bar{n}(B)\cap(aC)\rangle^2}\\
\left|(x_a^{-1}+1)^{-1}-z_n\right|^2 &= -\frac{\langle a(B)(C)(L_n)\rangle\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle\langle BC\rangle}{\langle BL_n\rangle\langle AC\rangle\langle \bar{n}(B)\cap(aC)\rangle^2}
\end{aligned}
$$

Here we introduce 5 lines $A=(a-1,a)$, $B=(b-1,b)$, $C=(c-1,c)$, $L_1=(n\,1)$ and $L_n=(n-1,n)$. The results for x_{a-1},x_{b-1},x_{c-1} can be easily obtained by a replacement.

note: 1. The above equations mean $\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle$ belongs to the alphabet for 2-loop n-point NMHV amplitudes, however, such a letter doesn't appear in the prediction of Landau analysis. What is more, the letters $\langle \bar{n}(B)\cap(aC)\rangle$ and $\langle \bar{n}(B)\cap(Ac)\rangle$ also appear in the rational part of the wight-3 symbol for this general box, but they are also not contained in the prediction of Landau analysis.

2. $\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle$ deserves a better notation since

$$\langle (A)\cap(\bar{n})B(C)\cap(\bar{n})\rangle=-\langle (B)\cap(\bar{n})A(C)\cap(\bar{n})\rangle=\langle (B)\cap(\bar{n})C(A)\cap(\bar{n})\rangle$$

2 Summarize on new algebraic letters in 2-loop NMHV amplitudes

In last section, we have seen how to rationalize the four-mass box $f_{a,b,c,n+1}$ with $a>2$ and $c<n-1$ and found the new algebraic letters produced by this four-mass box. However, for other two kinds of degenerate box $f_{2,b,c,n+1}$ with $c<n-1$ and $f_{a,b,n-1,n+1}$ with $a>2$, the rationalize parameterization cannot simply obtained as a degeneration of (1.4) and (1.5). But the new algebraic letters produced by these two boxex indeed can be viewed as a degeneration of (1.7)

Therefore, there are new 5 algebraic letters produced by $f_{2,b,c,n+1}$, they are the form of

$$\frac{(x_*^{-1}+1)^{-1}-z_n}{(x_*^{-1}+1)^{-1}-\bar{z}_n} \tag{2.1}$$

where z_n and $\bar{z}_n$ are defined as in eq.(1.3) with $a=2$, while the 5 x_*'s are x_a, x_{b-1}, x_b, x_{c-1} and x_c in eq.(1.6) with $a=2$. (note that $x_{a-1}^{-1}=0$ for $a=2$.)

There are new 5 algebraic letters produced by $f_{a,b,n-1,n+1}$, they are the form of

$$\frac{(x_*+1)^{-1}-z_1}{(x_*+1)^{-1}-\bar{z}_1} \tag{2.2}$$

where z_1 and $\bar{z}_1$ are defined as in eq.(1.2) with $c=n-1$, while the 5 x_*'s are x_{a-1} x_a, x_{b-1}, x_b and x_{c-1} in eq.(1.6) with $c=n-1$. (note that $x_c=0$ for $c=n-1$.)

2.1 Naive Counting(may be wrong)

Before cyclic permutation, the total numbers for $f_{2,b,c,n+1}$ and $f_{b,c,n-1,n+1}$ both are $(n-6)(n-7)/2$, each of such boxes will produce 5 new algebraic letters, the total number for $f_{a>2,b,c<n-1,n+1}$ is $(n-6)(n-7)(n-8)/6$, each of such boxes will produce 12 new algebraic letters. However, the cyclic permutation may give the same new letters, for example, the set of new algebraic letters generated by the cyclic permutation of the new algebraic letters produced by $f_{2,b,c,n+1}$ is the same with the one produced by $f_{b-1,c-1,n-1,n+1}$. Actually, they are pairing with each other. Thus,

$$\# \text{ of new algebraic letter} = \left(5(n-6)(n-7) + \frac{12(n-6)(n-7)(n-8)}{6}\right)\frac{n}{2} = \frac{n(n-6)(n-7)(2n-11)}{2}$$

Together with the algebraic letters appearing in the one-loop N^2MHV amplitudes,

$$\# \text{ of algebraic letter} = \frac{n(n-6)(n-7)(2n-11)}{2} + \frac{n(n-5)(n-6)(n-7)}{12} = \frac{n(n-6)(n-7)(13n-71)}{12}$$

Note that here we have not taken the multiplicative dependence of these letters into account.