Suppose all topological spaces are Hausdorff in our discussion. A path is a continuous map $p : [0, 1] \to U$, a loop is a path that $p(0) = p(1)$.

A topological space with its one point (X, x_0) form the pointed topological space catagory, where a morphism between (X, x_0) and (Y, y_0) is a continuous map f s.t. $f(x_0) = y_0$.

If U and V are topological spaces and φ, $\psi : U \to V$ are continuous maps, a homotopy $h : \varphi \simeq \psi$ is a continuous map $h : U[0, 1] \to V$ such that $h(u, 0) = \varphi(u)$ and $h(u, 1) = \psi(1)$. To simplify the notation, we will denote $h(u, t)$ as $h_t(u)$ in a homotopy. Two maps φ and ψ are called homotopic if there exists a homotopy $\varphi \simeq \psi$. Homotopy is an equivalence relation.

If p and q are paths, h is a homtopy $h : p \simeq q$ and $h_t(0) = p(0) = q(0)$, $h_t(1) = p(1) = q(1)$ are valid for any $t \in [0, 1]$, we call h a path-homotopy, and we write $p \approx q$ if a path-homotopy exists.If $f : [0, 1] \to [0, 1]$ is a continuous function s.t. $f(0) = 0$ and $f(1) = 1$, then $p \approx p \circ f$ since there exists $h_t(u) = p\big(tf(u) + (1-t)u\big)$. We call $p \circ f$ is a reparametrization of p.

A topological space U is contractible if the identity map $\mathrm{id}_U : U \to U$ is homotopic to a constant map $\varphi : U \to \{x\}$. A space U is path-connected if for all $x, y \in U$ there exists a path $p : [0, 1] \to U$ such that $p(0) = x$ and $p(1) = y$. Suppose that $p : [0, 1] \to U$ and $q : [0, 1] \to U$ are two paths in the space U such that the right endpoint of p coincides with the left endpoint of q; that is, $p(1) = q(0)$. Then we can concatenate the paths to form the path $p \cdot q$ by

$$(p \cdot q)(t) = \begin{cases} p(2t) & 0 \leq t \leq 1/2, \\ q(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

It is not difficult to check that $(p \cdot q) \cdot r \approx p \cdot (q \cdot r)$ since these paths differ by a reparametrization. We can also define $\bar{p}$ as inverse path of p by $\bar{p}(t) = p(-t)$.

The space U is simply connected if it is path-connected and given any closed path (that is, any $p : [0, 1] \to U$ such that $p(0) = p(1)$), there exists a path-homotopy $f : p \approx p_0$, where p_0 is a trivial loop mapping $[0, 1]$ onto a single point. Visually, the space is simply connected if every closed path can be shrunk to a point. It may be convenient to fix a base point $x_0 \in U$. In this case, to check whether U is simply-connected or not, it is sufficient to consider loops $p : [0, 1] \to U$ such that $p(0) = p(1) = x_0$.

Definition 1. *The fundamental group $\pi_1(M, x_0)$ consists of the set of homotopy classes of loops in M with left and right endpoints equal to x_0. The multiplication in $\pi_1(M, x_0)$ is concatenation, and the inverse operation is path-reversal.*

Clearly, $\pi_1(M, x_0) = 1$ if and only if M is simply connected. Changing the base point replaces $\pi_1(M, x_0)$ by an isomorphic group, but not canonically so. Thus, $\pi_1(M, x_0)$ is a functor from the category of pointed spaces to the category of groupsnot a functor on the category of topological spaces. If M happens to be a topological group, we will always take the base point to be the identity element.