Table of Formulas in Vector Analysis

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Definition:

All operators are linear. d is exterior derivative, and

$$\omega^2_{A_1\times A_2} := \omega^1_{A_1} \wedge \omega^1_{A_2}$$
$$\omega^3_{A\cdot B} := \omega^1_A \wedge \omega^2_B$$
$$\omega^1_{\nabla f} := \mathrm{d}\omega^0_f$$
$$\omega^2_{\nabla\times A} := \mathrm{d}\omega^1_A$$
$$\omega^3_{\nabla\cdot B} := \mathrm{d}\omega^2_B$$

On grad:

$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla(A\cdot B) = (A\cdot\nabla)B + (B\cdot\nabla)A + A\times(\nabla\times B) + B\times(\nabla\times A)$$
$$\nabla\left(\frac{1}{|r-r'|}\right) = -\frac{r-r'}{|r-r'|^3}$$

On div:

$$\nabla\cdot(fA) = (\nabla f)\cdot A + f\nabla\cdot A$$
$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$$
$$\nabla\cdot\left(\frac{r-r'}{|r-r'|^3}\right) = 4\pi\delta^3(r-r')$$

On rot:

$$\nabla\times(fA) = f\nabla\times A + \nabla f\times A$$
$$\nabla\times(A\times B) = (B\cdot\nabla)A - (A\cdot\nabla)B + (\nabla\cdot B)A - (\nabla\cdot A)B$$
$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla\cdot(\nabla A) = \nabla(\nabla\cdot A) - \nabla^2 A$$
$\hat{n}\cdot(\nabla\times\hat{n}) = 0$ ($\hat{n}$ is the normal vector of a surface)

On ∇^2:

$$\nabla^2(fg) = g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g$$
$$\nabla\cdot(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$$
$$\nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$$
$$\nabla\cdot(\nabla^2 f) = \nabla^2(\nabla\cdot f)$$
$$\nabla^2\left(\frac{1}{|r-r'|}\right) = -4\pi\delta^3(r-r')$$

On Integral:

$$\int_S \mathrm{d}\omega = \int_{\partial S} \omega \text{ (Stokes Theorem)}$$

$$\int_V \mathrm{d}V(\nabla * f) = \int_{\partial V} \mathrm{d}\sigma * f \text{ (When } * = \cdot\text{, it is the famous Gauss Formula.)}$$

$$\int_V \mathrm{d}V(A\cdot\nabla)B = \int_{\partial V} A\cdot \mathrm{d}\sigma B \text{ (if } \nabla\cdot A = 0\text{)}$$

$$\int_S \mathrm{d}\sigma\cdot(\nabla\times A) = \int_{\partial S} \mathrm{d}l\cdot A \text{ (the traditional Stokes Formula)}$$

$$\int_S (\mathrm{d}\sigma\times\nabla)\times A = \int_{\partial S} \mathrm{d}l\times A$$

$$\int_S \mathrm{d}\sigma\times\nabla f = \int_{\partial S} \mathrm{d}l f$$

Orthogonal coordinates:

Generally for any curvilinear coordinates in $\mathbb{R}^3$, we should have $\mathrm{d}l^2 = g_{ij}(t)\mathrm{d}t^i\mathrm{d}t^j$ and $\mathrm{d}V = \sqrt{\det g_{ij}}(t)\mathrm{d}t^1\wedge\mathrm{d}t^2\wedge\mathrm{d}t^3$. However, for orthogonal coordinates, g becomes diagonal, i.e. $g = \mathrm{diag}(H_1^2, H_2^2, H_3^2)$, and $\mathrm{d}V = H_1H_2H_3\mathrm{d}t^1\wedge\mathrm{d}t^2\wedge\mathrm{d}t^3$.

Here're the most useful orthogonal coordinates:

	H_1	H_2	H_3	$H = H_1H_2H_3$
(x,y,z)	1	1	1	1
(r,θ,z)	1	r	1	r
(ρ,θ,ϕ)	1	ρ	$\rho\sin\theta$	$\rho^2\sin\theta$

$$A\cdot\mathrm{d}l := \omega_A^1 = A_1H_1\mathrm{d}t_1 + A_2H_2\mathrm{d}t_2 + A_3H_3\mathrm{d}t_3$$

$$B\cdot\mathrm{d}\sigma := \omega_B^2 = H\left(\frac{B_1}{H_1}\mathrm{d}t_2\wedge\mathrm{d}t_3 - \frac{B_2}{H_2}\mathrm{d}t_1\wedge\mathrm{d}t_3 + \frac{B_3}{H_3}\mathrm{d}t_1\wedge\mathrm{d}t_2\right)$$

$$f\mathrm{d}V := \omega_f^3 = fH\mathrm{d}t_1\wedge\mathrm{d}t_2\wedge\mathrm{d}t_3$$

where

$$\nabla f = \frac{1}{H_1}\frac{\partial f}{\partial t_1}\hat{e}_1 + \frac{1}{H_2}\frac{\partial f}{\partial t_2}\hat{e}_2 + \frac{1}{H_3}\frac{\partial f}{\partial t_3}\hat{e}_3$$

$$\nabla\times A = \frac{1}{H}\begin{vmatrix} \partial_{t_1} & \partial_{t_2} & \partial_{t_3} \\ H_1A_1 & H_2A_2 & H_3A_3 \\ H_1\hat{e}_1 & H_2\hat{e}_2 & H_3\hat{e}_3 \end{vmatrix}$$

$$\nabla\cdot A = \frac{1}{H}\left[\frac{\partial}{\partial t_1}(A_1H_2H_3) + \frac{\partial}{\partial t_2}(A_2H_3H_1) + \frac{\partial}{\partial t_3}(A_3H_1H_2)\right]$$

$$\nabla^2 f = \frac{1}{H}\left[\frac{\partial}{\partial t_1}\left(\frac{H_2H_3}{H_1}\frac{\partial f}{\partial t_1}\right) + \frac{\partial}{\partial t_2}\left(\frac{H_3H_1}{H_2}\frac{\partial f}{\partial t_2}\right) + \frac{\partial}{\partial t_3}\left(\frac{H_1H_2}{H_3}\frac{\partial f}{\partial t_3}\right)\right]$$