Notation: $\eta_{ij} = \text{diag}(1,1,1,-1)$, so $p^2 = -m^2$ and $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$.

A useful Lorentz Transformation: $L(\boldsymbol{p}) = R(\hat{\boldsymbol{p}})B(|\boldsymbol{p}|)R^{-1}(\hat{\boldsymbol{p}})$,

$$R(\hat{\boldsymbol{p}}) = \exp(-i\varphi J^3)\exp(-i\theta J^2),$$

and

$$B(|\boldsymbol{p}|) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \sqrt{1+(|\boldsymbol{p}|/m)^2} & |\boldsymbol{p}|/m \\ & & |\boldsymbol{p}|/m & \sqrt{1+(|\boldsymbol{p}|/m)^2} \end{pmatrix}.$$

Lorentz-invariance of a quantum field φ_l:

$$U_0(\Lambda,a)\varphi_l(x)U_0^{-1}(\Lambda,a) = \sum_{l'} D_{ll'}(\Lambda^{-1})\varphi_{l'}(\Lambda x + a),$$

where D furnish a representation of the homogeneous Lorentz group.

For space-inversion, charge-conjugation and time-inversion operator,

$$\begin{aligned} \mathrm{P}a(\boldsymbol{p})\mathrm{P}^{-1} &= \eta^* a(-\boldsymbol{p}), \\ \mathrm{C}a(\boldsymbol{p})\mathrm{C}^{-1} &= \xi^* a^c(\boldsymbol{p}), \\ \mathrm{T}a(\boldsymbol{p})\mathrm{T}^{-1} &= \zeta^* a(-\boldsymbol{p}). \end{aligned}$$

1 Causal Scalar Field

It is the case that $D = 1$, a general scalar field is spin-zero field and can be written as

$$\phi(x) = \phi^+(x) + \phi^{c+\dagger}(x),$$

where

$$\phi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{ip\cdot x}\mathrm{d}^3\boldsymbol{p}}{2\sqrt{\boldsymbol{p}^2+m^2}} a(\boldsymbol{p}),$$

and

$$\phi^{c+}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{ip\cdot x}\mathrm{d}^3\boldsymbol{p}}{2\sqrt{\boldsymbol{p}^2+m^2}} a^c(\boldsymbol{p}).$$

What's more,

$$[\varphi(x), \varphi^\dagger(y)] = \Delta(x-y) = \Delta_+(x-y) - \Delta_+(y-x),$$

where

$$\Delta_+(x) = \frac{1}{(2\pi)^3} \int \frac{e^{ip\cdot x}\mathrm{d}^3\boldsymbol{p}}{2\sqrt{\boldsymbol{p}^2+m^2}} = \frac{m}{(2\pi)^2\sqrt{x^2}} \int_0^\infty \frac{u\mathrm{d}u}{\sqrt{u^2+1}} \sin(m\sqrt{x^2}u).$$

It is obviously even when $x^2 > 0$, i.e. x is space-like.

Propagator:

$$\Delta(x,y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 q \frac{e^{iq\cdot(x-y)}}{q^2+m^2-i\epsilon}.$$

For internal symmetry, $\eta^c = \eta^*$, $\xi^c = \xi^*$ and $\zeta^c = \zeta^*$,

$$\begin{aligned}
\mathrm{P}\phi(x)\mathrm{P}^{-1} &= \eta^* \phi(\mathscr{P}x), \\
\mathrm{C}\phi(x)\mathrm{C}^{-1} &= \xi^* \phi^\dagger(x), \\
\mathrm{T}\phi(x)\mathrm{T}^{-1} &= \zeta^* \phi(-\mathscr{P}x).
\end{aligned}$$

2 Causal Vector Field

It is the case that $D(\Lambda) = \Lambda$. A general vector field can be spin zero and spin one, so it is a boson. If spin zero, it is just the derivative of scalar field, $\psi^\mu(x) = \partial^\mu \phi(x)$.

Spin one: Define

$$\begin{aligned}
e^\mu(0, \pm 1) &= -\frac{\sqrt{2}}{2}(1, \pm i, 0, 0), \\
e^\mu(0, 0) &= (0, 0, 1, 0)
\end{aligned}$$

and $e^\mu(\boldsymbol{p}, \sigma) = L^\mu{}_\nu(\boldsymbol{p}) e^\mu(\boldsymbol{p}, 0)$, then the field is written as

$$v^\mu(x) = \phi^{+\mu} + \phi^{+c\mu\dagger},$$

where

$$\phi^{+\mu} = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{e^{ip\cdot x} \mathrm{d}^3\boldsymbol{p}}{2\sqrt{\boldsymbol{p}^2 + m^2}} e^\mu(\boldsymbol{p}, \sigma) a(\boldsymbol{p}, \sigma),$$

and

$$\phi^{+c\mu} = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{e^{ip\cdot x} \mathrm{d}^3\boldsymbol{p}}{2\sqrt{\boldsymbol{p}^2 + m^2}} e^\mu(\boldsymbol{p}, \sigma) a^c(\boldsymbol{p}, \sigma).$$

This may be useful

$$\sum_\sigma e^\mu(\boldsymbol{p}, \sigma) e^{\nu *}(\boldsymbol{p}, \sigma) = \eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2},$$

and then

$$[v^\mu(x), v^{\nu\dagger}(y)] = \left(\eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2}\right) \Delta(x - y).$$

Propagator:

$$\Delta_{\mu\nu}(x, y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 q \frac{e^{iq\cdot(x-y)} P_{\mu\nu}(q)}{q^2 + m^2 - i\epsilon} + m^{-2} \delta^4(x - y) \delta^0_\mu \delta^0_\nu,$$

where $P_{\mu\nu}(q) = \eta_{\mu\nu} + m^{-2} q_\mu q_\nu$.

For internal symmetry, $\eta^c = \eta^*$, $\xi^c = \xi^*$ and $\zeta^c = \zeta^*$,

$$\begin{aligned}
\mathrm{P}v^\nu(x)\mathrm{P}^{-1} &= -\eta^* \mathscr{P}^\mu{}_\nu v^\nu(\mathscr{P}x), \\
\mathrm{C}v^\nu(x)\mathrm{C}^{-1} &= \xi^* v^{\nu\dagger}(x), \\
\mathrm{T}v^\nu(x)\mathrm{T}^{-1} &= \zeta^* \mathscr{P}^\mu{}_\nu v^\nu(-\mathscr{P}x).
\end{aligned}$$

3 Causal Dirac Field

It is the case that D is the spin representation. If $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ is an infinitesimal tranformation, then

$$D(\Lambda) = 1 + \frac{i}{2}\omega_{\mu\nu}\mathscr{J}^{\mu\nu},$$

and the commutation relations of $\mathscr{J}$ is

$$i\,[\mathscr{J}^{\mu\nu}, \mathscr{J}^{\rho\sigma}] = \eta^{\nu\rho}\mathscr{J}^{\mu\sigma} - \eta^{\mu\rho}\mathscr{J}^{\nu\sigma} - \eta^{\sigma\mu}\mathscr{J}^{\rho\nu} + \eta^{\sigma\nu}\mathscr{J}^{\rho\mu}.$$

To find such a set of matrices, suppose we first construct matrics γ^μ that satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$

and tentatively define

$$\mathscr{J}^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu],$$

so then using

$$[\mathscr{J}^{\mu\nu}, \gamma^\rho] = -i\gamma^\mu\eta^{\nu\rho} + i\gamma^\nu\eta^{\mu\rho},$$

it is not difficult to vertify the commutation relations of $\mathscr{J}$.

Some useful relations:
(1) γ is a vector:

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda) = (\Lambda^{-1})^\rho{}_\sigma\gamma^\sigma.$$

(2) $\mathscr{J}$ is an anticommutatic tensor:

$$D(\Lambda)\mathscr{J}^{\rho\sigma}D^{-1}(\Lambda) = (\Lambda^{-1})^\rho{}_\mu(\Lambda^{-1})^\sigma{}_\nu\mathscr{J}^{\mu\nu}.$$

(3) Define $\beta = i\gamma^0$, then β is the space-inversion operator, i.e.

$$\beta\gamma^\mu\beta^{-1} = \mathscr{P}^\mu{}_\nu\gamma^\nu,$$

$$\beta\mathscr{J}^{ij}\beta^{-1} = \mathscr{J}^{ij}, \quad \beta\mathscr{J}^{0i}\beta^{-1} = -\mathscr{J}^{0i}.$$

Define

$$v_1 \wedge \cdots \wedge v_n = \sum_{\sigma\in S^n}(-1)^{\mathrm{sign}(\sigma)}v_{\sigma_1}\cdots v_{\sigma_n},$$

where S^n is the symmetry group, so that $[v_1, v_2] = v_1 \wedge v_2$. If $v_i = v_j$ in $v_1 \wedge \cdots \wedge v_n$, then $v_1 \wedge \cdots \wedge v_n = 0$.

Suppose $\{\gamma^\mu : 0 \le \mu \le n-1\}$ is a basis of a $\mathbb{C}$-vector space V, define $\Lambda^k V$ the space which spanned by $\{\gamma^{i_1}\wedge\cdots\wedge\gamma^{i_k}\}$ and $\Lambda^0 V := \mathbb{C}$, then its dimension is $\binom{n}{k}$. Denote that $\Lambda V = \oplus_{k=0}^n \Lambda^k V$, its dimension is $\sum_{k=0}^n \binom{n}{k} = 2^n$.

When $n = 4$, then $\dim \Lambda V = 2^4 = 16$. If we want to construct matrices γ^μ to furnish this representation, since the dimension of the space of $N \times N$ matrices is N^2, it is clear that the minimal N is $\sqrt{16} = 4$.

One very convenient choice (it is not unique) of γ^μ is that

$$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = -i \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix},$$

where 1 is the unit 2×2 matrix, and $\boldsymbol{\sigma}$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which have the relations that $[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk}\sigma_k$, and $(\sigma_i)^{-1} = \sigma_i$. So,

$$\mathscr{J}^{ij} = \frac{1}{2} \sum_k \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad \mathscr{J}^{i0} = \frac{i}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}.$$

This matrix representation is not irreducible.

Define $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\beta = i\gamma^0$, so that

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Another choice of γ^μ is $\gamma^0 \mapsto \gamma_5$, $\gamma^i \mapsto i\gamma^i$, where $\eta_{\mu\nu} = \mathrm{diag}(-1,-1,-1,1)$.

Some useful relations:

$$\begin{aligned}
&(\gamma^\mu)^{-1} = \gamma^\mu, \quad (\gamma^\mu)^T = (-1)^\mu \gamma^\mu, \quad \gamma_5^2 = 1, \quad \beta = \beta^{-1} = \beta^T, \\
&\{\gamma_5, \gamma^\mu\} = 0, \quad [\gamma_5, \mathscr{J}^{\mu\nu}] = 0, \quad [\gamma_5, D(\Lambda)] = 0, \\
&\{\beta, \gamma^i\} = 0, \quad \beta\gamma^{\mu\dagger}\beta = -\gamma^\mu, \quad \beta\mathscr{J}^{\mu\nu\dagger}\beta = \mathscr{J}^{\mu\nu}, \\
&\beta D(\Lambda)^\dagger \beta = D(\Lambda)^{-1}, \quad \beta(\gamma_5\gamma^\mu)^\dagger\beta = -\gamma_5\gamma^\mu.
\end{aligned}$$

Define

$$\mathscr{C} := \gamma_2\beta = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix},$$

then

$$\begin{aligned}
&\mathscr{C}^T = -\mathscr{C} = \mathscr{C}^{-1}, \quad (\beta\mathscr{C})^{-1} = \beta\mathscr{C} = -\mathscr{C}\beta, \\
&\gamma_\mu^T = -\mathscr{C}\gamma_\mu\mathscr{C}^{-1}, \\
&\gamma_5^T = \mathscr{C}\gamma_5\mathscr{C}^{-1}, \quad (\gamma_5\gamma_\mu)^T = \mathscr{C}\gamma_5\gamma_\mu\mathscr{C}^{-1}.
\end{aligned}$$

Using the relation that $A^* = (A^T)^\dagger = (A^\dagger)^T$, then

$$\begin{aligned}
\gamma_\mu^* &= \beta\mathscr{C}\gamma_\mu\mathscr{C}^{-1}\beta, \\
\mathscr{J}_{\mu\nu}^* &= -\beta\mathscr{C}\mathscr{J}_{\mu\nu}\mathscr{C}^{-1}\beta, \\
\gamma_5^* &= -\beta\mathscr{C}\gamma_5\mathscr{C}^{-1}\beta, \\
(\gamma_5\gamma_\mu)^* &= -\beta\mathscr{C}\gamma_5\gamma_\mu\mathscr{C}^{-1}\beta.
\end{aligned}$$

The Dirac field describes spin-1/2 particles, they are fermions. A general Dirac field is written as

$$\psi_l(x) = \psi_l^+(x) + \psi_l^{-c}(x),$$

where

$$\psi_l^+(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \mathrm{d}^3\boldsymbol{p}\, u_l(\boldsymbol{p},\sigma) e^{ip\cdot x} a(\boldsymbol{p},\sigma),$$

and

$$\psi_l^{-c}(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \mathrm{d}^3\boldsymbol{p}\, v_l(\boldsymbol{p},\sigma) e^{ip\cdot x} a^{c\dagger}(\boldsymbol{p},\sigma).$$

The coefficient u and v is defined by

$$u(\boldsymbol{p},\sigma) = \sqrt{\frac{m}{p^0}} D(L(p)) u(0,\sigma),$$
$$v(\boldsymbol{p},\sigma) = \sqrt{\frac{m}{p^0}} D(L(p)) v(0,\sigma),$$

where

$$u(0,1/2) = \frac{1}{\sqrt{2}}(1,0,1,0),$$
$$u(0,-1/2) = \frac{1}{\sqrt{2}}(0,1,0,1),$$
$$v(0,1/2) = \frac{1}{\sqrt{2}}(0,1,0,-1),$$
$$v(0,-1/2) = \frac{-1}{\sqrt{2}}(1,0,-1,0).$$

Some useful relations:

$$D(L(p))\beta D(L(p))^{-1} = -i\frac{p_\mu\gamma^\mu}{m} = -i\frac{\not{p}}{m},$$
$$M(\boldsymbol{p})_{ll'} = \sum_\sigma u_l(\boldsymbol{p},\sigma) u_{l'}(\boldsymbol{p},\sigma) = \left(\frac{1}{2p^0}(-i\not{p}+m)\beta\right)_{ll'},$$
$$N(\boldsymbol{p})_{ll'} = \sum_\sigma v_l(\boldsymbol{p},\sigma) v_{l'}(\boldsymbol{p},\sigma) = \left(\frac{1}{2p^0}(-i\not{p}-m)\beta\right)_{ll'},$$

so

$$\{\psi_l(x), \psi_{l'}^\dagger(y)\} = ((-\not{\partial}+m)\beta)_{ll'}\,\Delta(x-y).$$

Propagator:

$$\Delta(x,y) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4q \frac{e^{iq\cdot(x-y)}(-i\not{q}+m)\beta}{q^2+m^2-i\epsilon}.$$

For internal symmetry, $\eta^c = -\eta^*$, $\xi^c = \xi^*$ and $\zeta^c = \zeta^*$,

$$\mathrm{P}\psi(x)\mathrm{P}^{-1} = \eta^*\beta\psi(\mathscr{P}x),$$
$$\mathrm{C}\psi(x)\mathrm{C}^{-1} = -\xi^*\beta\mathscr{C}\psi^*(x),$$
$$\mathrm{T}\psi(x)\mathrm{T}^{-1} = -\zeta^*\gamma_5\mathscr{C}\psi(-\mathscr{P}x).$$

Define $\bar{\psi} := \psi^\dagger\beta$, then $\bar{\psi}(x)M\psi(x)$ has the Lorentz transformation property, i.e.

$$U_0(\Lambda)[\bar{\psi}(x)M\psi(x)]U_0^{-1}(\Lambda) = \bar{\psi}(\Lambda x)D(\Lambda)MD^{-1}(\Lambda)\psi(\Lambda x),$$

and under a space inversion

$$\mathrm{P}[\bar{\psi}(x)M\psi(x)]\mathrm{P}^{-1} = \bar{\psi}(\mathscr{P}x)\beta M\beta\psi(\mathscr{P}x).$$

Taking $M = 1$, γ^μ, $\mathscr{J}^{\mu\nu}$, $\gamma_5\gamma^\mu$, γ_5, yield a bilinear $\bar{\psi}M\psi$ transforms as a scalar, vector, tensor, axial vector, and pseudoscalar, respectively.

Define

$$L = \frac{1+\gamma_5}{2}, \quad R = \frac{1-\gamma_5}{2},$$

then $L^2 = L$, $R^2 = L$, $LR = RL = 0$ and $R + L = 1$, so they are projective operators. Since $[L, U(\Lambda)] = 0$ and $[R, U(\Lambda)] = 0$, then

$$LU_0(\Lambda, a)\psi(x)U_0^{-1}(\Lambda, a) = LD(\Lambda^{-1})\psi(\Lambda x) = D(\Lambda^{-1})L\psi(\Lambda x) = U_0(\Lambda, a)L\psi(x)U_0^{-1}(\Lambda, a)$$

and

$$RU_0(\Lambda, a)\psi(x)U_0^{-1}(\Lambda, a) = U_0(\Lambda, a)R\psi(x)U_0^{-1}(\Lambda, a).$$

If a field ψ satisfied $L\psi = \psi$ ($R\psi = \psi$), it is called left-chiral (right-chiral) field. A left-chiral (right-chiral) field will still be left-chiral (right-chiral) after a Lorentz transformation. Any field can be decomposed into left-chiral field and right-chiral field such that

$$\psi = L\psi + R\psi = \psi_L + \psi_R.$$

Since

$$U_0(\Lambda, a)\psi_L(x)U_0^{-1}(\Lambda, a) = D(\Lambda^{-1})L\psi_L(\Lambda x) = LD(\Lambda^{-1})\psi_L(\Lambda x),$$

where $D_L(\Lambda) := LD(\Lambda)$ is another representation because

$$D_L(\Lambda_1)D_L(\Lambda_2) = LD(\Lambda_1)LD(\Lambda_2) = L^2D(\Lambda_1)D(\Lambda_2) = D_L(\Lambda_1\Lambda_2),$$

so ψ_L furnish the left-chiral spin representation. Similarly for ψ_R. Thus any spin representation can be decomposed that

$$D(\Lambda) = D_L(\Lambda) + D_R(\Lambda),$$

i.e. $D = D_L \oplus D_R$. The representation D_L is usually denoted by $(1/2, 0)$ representation, and D_R is denoted by $(0, 1/2)$, thus the Dirac spin representaion is denoted by $(1/2, 0) \oplus (0, 1/2)$.

Using the matrix representation,

$$L = \frac{1+\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \frac{1-\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so

$$D_L\left(J^{ij}\right) = \frac{1}{2}\sum_k \epsilon_{ijk}\sigma_k, \quad D_L\left(J^{i0}\right) = \frac{i}{2}\sigma_i,$$

and

$$D_R\left(J^{ij}\right) = \frac{1}{2}\sum_k \epsilon_{ijk}\sigma_k, \quad D_R\left(J^{i0}\right) = -\frac{i}{2}\sigma_i.$$

It is easy to see that D_L or D_R contains a spin-1/2 representation of $\mathfrak{su}(2)$.

The Lie algebra representation $D_L \otimes D_R := D_L \otimes \mathrm{id} + \mathrm{id} \otimes D_R$ is denoted by $(1/2, 1/2)$. In fact, $D_L \otimes D_R$ is a Lie algebra isomorphism, so it is just the vector representation. To prove this, since $D_L \otimes D_R$ has been a Lie algebra morphism, we only need to show that it is an isomorphism between vector spaces, and it is because that $\{(D_L \otimes D_R)(J^{\mu\nu})\}$ are linearly independent (this can be vertified by directly calculation).