- (1). Suppose \mathcal{C} is a category, \mathcal{C}° is its dual category. Thus any contravariant functor $\varphi: \mathcal{C} \to D$ is a (covariant) functor $\varphi: \mathcal{C}^{\circ} \to \mathcal{D}$.
- (2). Suppose \mathcal{C} and \mathcal{D} are categories. We can construct a category of functors form \mathcal{C} to \mathcal{D} by the follow steps. Firstly, the objects in $Funct(\mathcal{C}, \mathcal{D})$ are functors form \mathcal{C} to \mathcal{D} . Then, a morphism f of functors from F to G (notation $f: F \to G$) is a family of morphisms in \mathcal{D} that

$$f(X): F(X) \to G(X),$$

one for each $X \in \mathcal{C}$, satisfying the following condition: for all morphism φ in \mathcal{C} the following diagram

$$F(X) \xrightarrow{f(X)} G(X)$$

$$F(\varphi) \downarrow \qquad \qquad \downarrow G(\varphi)$$

$$F(Y) \xrightarrow{f(Y)} G(Y)$$

is commutative.

If F is isomorphic to G (notation $F \cong G$), we should have morphisms f and g such that $f(X) \circ g(X) = \mathrm{id}_{G(X)}$ and $g(X) \circ f(X) = \mathrm{id}_{F(X)}$ for all $X \in \mathcal{C}$.

(3). Suppose $\hat{\mathcal{C}} = Funct(\mathcal{C}^{\circ}, Set)$, $\forall X \in \mathcal{C}$, we can associate a $\hat{X} \in \hat{\mathcal{C}}$ by $\hat{X}(Y) = \operatorname{Hom}_{\mathcal{C}}(Y, X)$ and $\hat{Z}(\varphi)f = f \circ \varphi$ for $\varphi \in \operatorname{Hom}_{\mathcal{C}}(Y, X)$ and $f \in \operatorname{Hom}_{\mathcal{C}}(X, Z)$. Then $\hat{Z}(\varphi) : \hat{Z}(X) \to \hat{Z}(Y)$.

Proposition 1. Suppose X is a set in the category Set, it can be identified with the set $\hat{X}(e) = \operatorname{Hom}_{Set}(e, X)$, where e is a one-point set.

Proof. Obviously.
$$\Box$$

Indeed, in an arbitrary category \mathcal{C} an analogue of one-point set does not necessarily exist. However, by considering $\operatorname{Hom}_{\mathcal{C}}(Y,X)$ for all $Y \in \mathcal{C}$ simultaneously, we can recover complete information about an object $X \in \mathcal{C}$. This is the idea of representation theory.

Definition 1. A functor $F \in \hat{C}$ is said to be representable if $F \cong \hat{X}$ for some $X \in C$. One says also that the object X represents the functor F.

Theorem 1. $\operatorname{Hom}_{\mathcal{C}}(X,Y) \cong \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X},\hat{Y}).$

Proof. Let $\varphi: X \to Y$ be a morphism in \mathcal{C} . We associate with φ the morphism of functors $\hat{\varphi}: \hat{X} \to \hat{Y}$ by

$$\hat{\varphi}(Z): \theta \mapsto \varphi \circ \theta \in \hat{Y}(Z),$$

where $\theta \in \hat{X}(Z)$ and $X, Y, Z \in \mathcal{C}$. It is clear that $\hat{\varphi} \circ \hat{\phi} = \widehat{\varphi \circ \phi}$.

Conversely, suppose $f \in \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y})$, define map $i : \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y}) \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$ by

$$i: f \mapsto f(X)(\mathrm{id}_X),$$

where $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X) = \hat{X}(X)$ and $f(\mathrm{id}_X) \in \hat{Y}(X)$. Then

$$i(\hat{\varphi}) = \hat{\varphi}(X)(\mathrm{id}_X) = \varphi \circ \mathrm{id}_X = \varphi.$$

On the other hand, we should show $\widehat{i(f)} = f$ when $f \in \operatorname{Hom}_{\widehat{\mathcal{C}}}(\widehat{X}, \widehat{Y})$, and it's equivalent to show $\widehat{i(f)}(Z) = f(Z)$ for all $Z \in \mathcal{C}$.

Now suppose a morphism $\varphi: Z \to X$,

$$i(f) \circ \varphi = \widehat{i(f)}(Z)(\varphi) = f(Z)(\varphi).$$

Using the commutativity of the diagram,

$$\begin{array}{c|c} \hat{X}(X) & \xrightarrow{f(X)} & \hat{Y}(X) \\ \hat{X}(\varphi) \middle| & & & & & \\ \hat{X}(Z) & \xrightarrow{f(Z)} & \hat{Y}(Z) \end{array}$$

then

$$f(Z) \circ \hat{X}(\varphi)(\mathrm{id}_X) = \hat{Y}(\varphi) \circ f(X)(\mathrm{id}_X) = \hat{Y}(\varphi)(i(f)) = i(f) \circ \varphi$$

and

$$f(Z) \circ \hat{X}(\varphi)(\mathrm{id}_X) = f(Z)(\mathrm{id}_X \circ \varphi) = f(Z)(\varphi).$$

Thus i is an isomorphism that $i: \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y}) \cong \operatorname{Hom}_{\mathcal{C}}(X, Y)$.

If X represents the functor F, $\operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Y},F) \cong \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Y},\hat{X}) \cong \operatorname{Hom}_{\mathcal{C}}(Y,X)$. If Y also represents F, then there exists an isomorphism $\varphi:\hat{Y}\cong F$, then $i(\varphi)$ is the according isomorphism between X and Y. Thus, the representing object of a representable functor is defined uniquely up to a isomorphism.

Definition 2. Suppose $X, Y \in \mathcal{C}$, the direct product $X \times Y$ is (upto an isomorphism) the object Z representing the functor (if such functor is representable, or if such Z exists)

$$W \mapsto \hat{X}(W) \times \hat{Y}(W),$$

where $\hat{X}(W) \times \hat{Y}(W)$ is the direct product of Set which has been constructed directly.

Definition 3. Suppose $X, Y, S \in Set$, $f: X \to S$ and $g: Y \to S$, the pullback of f and g or the fibre product of X and Y is

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

In a general category C, $X \times_S Y$ is defined as the object Z representing the functor

$$W \mapsto \hat{X}(W) \times_{\hat{S}(W)} \hat{Y}(W).$$

When f=g are constant morphisms, $X\times_S Y\cong X\times Y$. When f and g are embedding, $X\times_S Y\cong X\cap Y$.

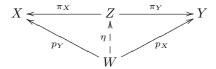
The universal property can be directly vertified. Since Z represents the functor $W \mapsto \hat{X}(W) \times \hat{Y}(W)$, then $\hat{Z}(W) = \hat{X}(W) \times \hat{Y}(W)$. Let W = Z, the image of $\mathrm{id}_Z \in \hat{Z}(Z)$ is that $\mathrm{id}_Z = (\pi_X, \pi_Y)$, where $\pi_X \in \hat{Y}(Z)$ and $\pi_Y \in \hat{Y}(Z)$. This can be writen as $X \xleftarrow{\pi_X} Z \xrightarrow{\pi_Y} Y$.

Now suppose $p_X: W \to X$ and $p_Y: W \to Y$, then there exist $\eta = (p_X, p_Y) \in \hat{X}(W) \times \hat{Y}(W) = \hat{Z}(W)$. The last thing we should to show is that $p_X = \pi_X \circ \eta$ and $p_Y = \pi_Y \circ \eta$. Indeed, for all $z \in Z$,

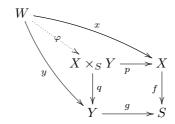
$$(\pi_X, \pi_Y)((p_X, p_Y)(z)) = \mathrm{id}_Z((p_X, p_Y)(z)) = (p_X, p_Y)(z).$$

The uniqueness of $X \times Y$ up to an isomorphism can be drived from the above theorem or universal property directly.

Because of some psychological reasons, we usually use a commutative diagram to visualize the universal property described above:

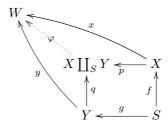


Similarly, the universal property of fibre product can be written as a commutative diagram:



 $(A_{\alpha}, A_{\alpha\beta}, \rho_{\alpha})$ $A \xrightarrow{f_{\beta}} F(U) \xrightarrow{\rho_{\beta}} F(U_{\beta})$ $\downarrow^{\rho_{\alpha}} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow^{\rho_{\beta\alpha}} \downarrow^{\rho_{\alpha}} \qquad \qquad \qquad \downarrow^{\rho_{\alpha\beta}} F(U_{\alpha} \cap U_{\beta})$

Given any categorical construction, we can create the dual construction by inverting all arrows in the original construction. In such a way one can obtain the coporduct $X \coprod_S Y$ by



We fix a category \mathcal{J} , and define the diagonal functor $\Delta: \mathcal{C} \to \mathit{Funct}(\mathcal{J}, \mathcal{C})$ as follow:

- On objects: ΔX is the set of constant functors with the value X. In other words, $\Delta X(j) = X$ for all $j \in \mathcal{J}$, $\Delta X(\varphi) = \mathrm{id}_X$ for all morphism φ in \mathcal{J} .
- On morphisms: Suppose $\psi: X \to Y$ is a morphism in \mathcal{C} , $\Delta \psi: \Delta X \to \Delta Y$ is defined as follows: $\Delta \psi(j): X = \Delta X(j) \to Y = \Delta Y(j)$ for all $j \in \mathcal{J}$.

It is clear that $\Delta(\varphi \circ \psi) = \Delta \varphi \circ \Delta \psi$, so Δ is indeed a functor.

Definition 4. Suppose $F: \mathcal{J} \to \mathcal{C}$ is a functor, the (projective or inverse) limit of F in the category \mathcal{C} according to \mathcal{J} is an object $X \in \mathcal{C}$ representing the functor

$$Y \mapsto \operatorname{Hom}_{Funct(\mathcal{J},\mathcal{C})}(\Delta Y, F) : \mathcal{C}^{\circ} \to Set.$$

The limit of F is denoted by $X = \varprojlim F$.