The main reference is the *Introduction of Commutative Algebra* by Atiyah&Macdonald (for short, I will call it A&M from now on.). This article can be seem as the answers of some problems on the prime spectrum of A&M. Throughout this article the word "ring" shall mean a commutative ring with an identity element.

Proposition 1. Ex 1.15*:*

Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E, then the sets $V(E)$ satisfy the axioms for closed sets in a topological space.

Proof. We shall prove this proposition in four parts:

(0) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(1) $V(0) = X$ and $V(1) = \varnothing$.

(2) if $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

(3) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}$, $\mathfrak{b}$ of A.

Firstly, let $\mathfrak{p}$ is a prime ideal which contains E, then $\mathfrak{a} \subseteq \mathfrak{p}$ since $\forall a \in A$ we have $xa \in \mathfrak{p}$, so $V(E) \subseteq V(\mathfrak{a})$. Conversely, $V(\mathfrak{a}) \subseteq V(E)$ because $E \subseteq \mathfrak{a}$. Using that the radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals which contain $\mathfrak{a}$, $V(\mathfrak{a}) = V(r(\mathfrak{a}))$. Thus $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

Secondly, every ideal containing 0 implies $V(0) = X$, and $V(1) = V(A) = \varnothing$ is trivial by (0).

Next, an ideal contains $\cup_i E_i$ if and only if it contains each E_i.

Finally, using $r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}\mathfrak{b})$,

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(r(\mathfrak{a} \cap \mathfrak{b})) = V(r(\mathfrak{a}\mathfrak{b})) = V(\mathfrak{a}\mathfrak{b}),$$

We now should prove $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

$V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$: $\forall \mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$, $\mathfrak{p}$ contains $\mathfrak{a}$ or $\mathfrak{b}$, thus $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$ which is equivalent that $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.

$V(\mathfrak{a} \cap \mathfrak{b}) \supseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$: $\forall \mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$, thus $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$. If $\mathfrak{a} \nsubseteq \mathfrak{p}$ and $\mathfrak{b} \nsubseteq \mathfrak{p}$, there exist $x \in \mathfrak{a}$, $y \in \mathfrak{b}$ and $x, y \notin \mathfrak{p}$, and therefore $xy \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$, but $xy \notin \mathfrak{p}$ (since $\mathfrak{p}$ is prime), which means $\mathfrak{a} \cap \mathfrak{b} \nsubseteq \mathfrak{p}$. Hence if $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, $\mathfrak{p}$ contains $\mathfrak{a}$ or $\mathfrak{b}$. □

The resulting topology is called the *Zariski* topology.

Definition 1. *The topological space X is called the prime spectrum of A, and is written* Spec(A)*.*

Proposition 2. Ex 1.17*:*

For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$, i.e. $X_f = \operatorname{Spec}(A) - V(f)$. The sets X_f are open and form a basis of open sets for the Zariski topology.

Proof. X_f is open because $V(f)$ is closed. If P is a open set, it has the form $\operatorname{Spec}(A) - V(E)$ for a E, then

$$P = \operatorname{Spec}(A) - V(E) = \operatorname{Spec}(A) - \bigcap_{f \in E} V(f) = \bigcap_{f \in E} (\operatorname{Spec}(A) - V(\{f\})) = \bigcap_{f \in E} X_f.$$

□

Proposition 3. Ex 1.17*:*

(0) *$X_f \cap X_g = X_{fg}$;*
(1) *$X_f = \varnothing \Leftrightarrow f$ is nilpotent;*
(2) *$X_f = X \Leftrightarrow f$ is a unit;*
(3) *$X_f = X_g \Leftrightarrow r(f) = r(g)$;*
(4) *X is compact;*
(5) *X_f is compact;*
(6) *An open subset of X is compact if and only if it is a finite union of sets X_f.*

Proof. The sets X_f are often called basic open sets of $X = \operatorname{Spec}(A)$.
(0) $X_f \cap X_g = \operatorname{Spec}(A) - (V(f) \cup V(g)) = \operatorname{Spec}(A) - V(fg) = X_{fg}$;
(1) $X_f = \varnothing \Leftrightarrow V(f) = V(r(fA)) = X \Leftrightarrow r(f) = 0$;
(2) $X_f = X \Leftrightarrow V(f) = \varnothing \Leftrightarrow f$ is not in any maximal ideal $\Leftrightarrow f$ is a unit;
(3) $X_f = X_g \Leftrightarrow V(f) = V(g) \Leftrightarrow r(f) = r(g)$;
(4) It is enough to consider a covering of X by basic open sets X_{f_i} because $X = \bigcup_\alpha U_\alpha$ and $U_\alpha = \bigcup_\beta X_{f_{\alpha\beta}}$. Thus

$$\varnothing = V(X) = V\left(\bigcup_i f_i\right) = \bigcap_i V(f_i),$$

This means that $\{f_i\}$ generates A, so exist $\{g_i\}$ s.t.

$$\sum_i f_i g_i = 1$$

with cofinitely many of the i non-zero. Thus, X is the union of the X_{f_i} for which $g_i \neq 0$, so X is the union of finitely many U_α.

(5) Suppose $X_f \subseteq \bigcup_i X_{f_i}$, then $\bigcap_i V(f_i) \subseteq V(f)$. Let $\mathfrak{a}$ is the ideal generated by the f_i, then $f \in \mathfrak{a}$, so there is an equation

$$f^n = \sum_i f_i g_i$$

with cofinitely many of the i non-zero. Let $f_1, \ldots, f_n$ with $g_i \neq 0$, then

$$\bigcap_{i=1}^{n} V(f_i) \subseteq V(f^n) = V(f),$$

so $X_f \subseteq \bigcup_{i=1}^{n} X_{f_i}$. Now we can say X_f is compact.

(6) The union of finitely many X_f is open and compact. Conversely, suppose U is compact and open, then U is the union of some X_f as an open cover, it must have a finite subcover because of compactness. □

Definition 2. *A topological space X is said to be irreducible if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X.*

Proposition 4. Ex 1.19*:*

Spec(A) *is irreducible if and only if the nilradical of A is a prime ideal.*

Proof. If not prime, there exist f and g s.t. $fg \in \mathfrak{p}$ but $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, and

$$X_f \neq \varnothing, X_g \neq \varnothing \Rightarrow X_f \cap X_g = X_{fg} = \varnothing,$$

which means X is not irreducible.

Conversely, if not irreducible, there exist $X_f \subseteq U$, $X_g \subseteq V$ and $U \cap V = \varnothing$, thus $X_f \cap X_g = X_{fg} = \varnothing$ but neithor f or g is nilpotent. □