1 Section

1.1 Foundation

A fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition outlined below. The space B is called the base space of the bundle, E the total space, and F the fiber. The map π is called the projection map (or bundle projection). We shall assume in what follows that the base space B is connected.

We require that for every x in E, there is an open neighborhood $U \subset B$ of $\pi(p)$ (which will be called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ (where $U \times F$ is the product space) in such a way that π agrees with the projection onto the first factor. That is, the following diagram should commute: where $\mathrm{proj}_1 : U \times F \to U$ is the natural projection and $\varphi : \pi^{-1}(U) \to U \times F$ is a

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
& U &
\end{tikzcd}$$

Figure 1: Locally Trivialition

homeomorphism. The set of all (U_i, φ_i) is called a local trivialization of the bundle. Especially, if F and every $\pi^{-1}(p)$ are vector spaces, we call this bundle vector bundle.

Definition 1.1. *A section s on a bundle (E, M, π, F) is a map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_E$.*

Here's the trivial bundle $E = M \times F$ and a section on it:

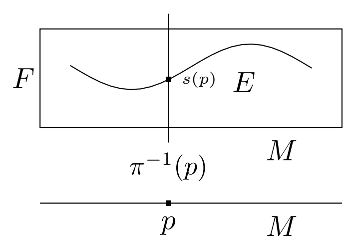

Figure 2: Trivial Bundle and its Section

For every $p \in M$, $s(p) \in \pi^{-1}(p) \cong F$, thus $s(p)$ is like an element in F. If E is a vector bundle, $s(p)$ correspond to a vector in F and s play the role of a *vector filed* on M. Naturally, we will have the definition below.

Definition 1.2. *A vector field on a manifold M is a section of vector bundle (E, M, π, F).*

Sections have the linear structure inherited from F:

$$(s_1 + s_2)(p) = s_1(p) + s_2(p),$$
$$(fs)(p) = f(p)s(p),$$

where f is a real-valued function on M.

For any vector space V, $L(V)$ is also a vector space, and so we can have another vector bundle $(E', M, \pi, L(F))$ and its section $T : M \to E'$. In vector space, it's very important for us to research the linear map acting on a vector. The analogous research here is the section fo E' acting on a section of E. We can define it by $T(s)(p) = T(p)s(p)$, and it should have

$$T(s_1 + s_2) = T(s_1) + T(s_2) \text{ and } T(fs) = fT(s).$$

From now on, we will use $\Gamma(E)$ to denote the set of all sections on bundle E and $\mathrm{End}(E)$ to replace E' used above.

Example 1.1. *Tangent bundle and Cotangent bundle:*

The tangent bundle of a differentiable manifold M is the disjoint union of the tangent spaces of M.

$$TM = \coprod_{x \in M} T_xM = \bigcup_{x \in M} \{x\} \times T_xM = \bigcup_{x \in M} \{(x, y) |\ y \in T_xM\}.$$

where T_xM denotes the tangent space to M at the point x. So, an element of TM can be thought of as a pair (x, v), where x is a point in M and v is a tangent vector to M at x.There is a natural projection

$$\pi : TM \to M$$

defined by $\pi(x, v) = x$. This projection maps each tangent space T_xM to the single point x.

Similarly, the cotangent bundle T^*M is the vector bundle of all the cotangent spaces at every point in the manifold.

1.2 Vector-valued Form on M

Let's start from the vector-valued 1-form.

A (real-valued) 1-from $\omega : TM \to \mathbb{R}$ is in T^*M. Now suppose we have a topological space E, a E-valued 1-form on $p \in M$ is locallly defined by

$$v(p) \otimes \omega(\dot{p}) : T_pM \to E \otimes \mathbb{R},$$

where $v(p) \in E$ and $\dot{p} \in T_pM$, and thus globally $v \otimes \omega : TM \to E \otimes \mathbb{R}$.

1.3 The geometry of the tangent bundle

Tangent bundle is also a manifold, so it's natural to talk about its tangent bundle. Suppose there are a manifold Q, tangent bundle TQ and its tangent bundle TTQ. Introducing the local coordinates (q^i) in Q, (q^i, v^i) in TQ and $(q^i, v^i, \dot{q}^i, \dot{v}^i)$ in TTQ, we can write their elements $v \in T_qQ$ and $V \in T_vTQ$ locally as

$$v = v^i \frac{\partial}{\partial q^i} \quad \text{and} \quad V = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{v}^i \frac{\partial}{\partial v^i},$$

and the natural projections

$$\pi_Q(q^i, v^i) = (q^i), \quad \pi_{TQ}(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, v^i) \quad \text{and} \quad (\pi_Q)_* (q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, \dot{q}^i).$$

These can be shown by the diagram:

$$\begin{tikzcd} TTQ \arrow[r, "(\pi_Q)_*"] \arrow[d, "\pi_{TQ}"] & TQ \arrow[d, "\pi_Q"'] \\ TQ \arrow[r, "\pi_Q"] & Q \end{tikzcd}$$

Figure 3: The Natural Projections

Definition 1.3. *Vertical fiber bundle* $\ker(\pi_Q)_*$ *is the disjoint union of the kernal of* $(\pi_Q)_{*v}$ *at each point* v *of* TTQ.

Definition 1.4. *Let* $v \in T_qQ$ *be vector tangent to* Q *at some point* $q \in Q$, *the vertical lift of* v *at a point* $w \in T_qQ$ *is the tangent vector* $\mathrm{Vert}_w(v) \in T_wTQ$ *given by*

$$\mathrm{Vert}_w(v)f = \frac{\mathrm{d}}{\mathrm{d}t} f(w + tv)\bigg|_{t=0}, \quad \forall f \in \mathcal{C}^\infty(T_qQ).$$

Given a smooth function $f \in \mathcal{C}^\infty(Q)$,

$$\begin{aligned} (\pi_Q)_{*w}\mathrm{Vert}_w(v)(f) &= \mathrm{Vert}_w(v)(f \circ \pi_Q) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f(\pi_Q(w + tv))\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f(q)\bigg|_{t=0} = 0. \end{aligned}$$

Thus Vert_w takes value in a fiber of vertical fiber bundle:

$$\mathrm{Vert}_w : T_qQ \to \ker(\pi_Q)_{*w} \subset T_wTQ.$$

Indeed, for each $w \in T_qQ$, the vertical lift Vert_w is a linear isomorphism between T_qQ and $\ker(\pi_Q)_{*w}$. Using local coordinates, if $v = (q^i, v^i)$ and $w = (q^i, w^i)$, then

$$\mathrm{Vert}_w(v) = (q^i, w^i, 0, v^i).$$

Definition 1.5. *The vertical endomorphism is the linear map $\mathcal{S} : TTQ \to TTQ$ that, ofr any vector $V \in TTQ$, gives the value*

$$\mathcal{S}(V) = \mathrm{Vert}_v\left((\pi_Q)_{*v} V\right),$$

where $v = \pi_{TQ}V \in TQ$.

Definition 1.6. *The Liouville or dilation vector field is the vector field Δ over TQ defined by*

$$\Delta_v = \mathrm{Vert}_v(v).$$

In adapted coordinates (q^i, v^i) of TQ , the vertical endomorphism has the local expression

$$\mathcal{S}(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, v^i, 0, \dot{q}^i) = \mathrm{d}q^i \otimes \frac{\partial}{\partial v^i},$$

the second equality is valid because

$$\left\langle \mathrm{d}q^i, \dot{q}^i \frac{\partial}{\partial q^i} + \dot{v}^i \frac{\partial}{\partial v^i} \right\rangle \frac{\partial}{\partial v^i} = \dot{q}^i \frac{\partial}{\partial v^i},$$

and the Liouville vector field is

$$\Delta = (q^i, v^i, 0, v^i) = v^i \frac{\partial}{\partial v^i}.$$

Another way to define the Liouville vector field is as the infinitesimal generator of the 1-parameter group of transformations $\phi_t : v \in TQ \mapsto e^t v \in TQ$, that is, if $v = (q^i, v^i)$, then $\phi_t : v \mapsto (q^i, e^t v^i) \in TQ$, and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \phi_t v\right|_{t=0} = (q^i, e^t v^i, 0, e^t v^i)_{t=0} = (q^i, v^i, 0, v^i).$$

This definition can easily be translated to any vector bundle.

Definition 1.7. *A second order vector field X is a section of TTQ such that $(\pi_Q)_* \circ X = \mathrm{Id}_{TQ}$.*

This mean that $(\pi_Q)_* \circ X = \pi_{TQ} \circ X$, so a second order vector field is well defined. In adapted coordinates (q^i, v^i) of TQ , X is a vector field

$$X = (q^i, v^i, v^i, a^i),$$

where (a^i) is arbitrary. Thus, neither the Liouville vector field nor the vertical lift of a vector field are second order vector fields. Even though, second order vector fields are characterized by the equation

$$\mathcal{S}(X) = \Delta,$$

locally, it is

$$\mathcal{S}(q^i, v^i, v^i, w^i) = (q^i, v^i, 0, v^i) = \Delta.$$

Definition 1.8. *Given a smooth curve $c : I \to Q$, its (first) lift to TQ is the smooth curve $c^{(1)} : I \to TQ$ such that*

$$c^{(1)}(t_0)f = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)\bigg|_{t=t_0}.$$

In local adapted coordinates

$$c^{(1)} = (c^i, \mathrm{d}c^i/\mathrm{d}t).$$

Proposition 1.1. *A vector field $X \in \Gamma(TTQ)$ is a second order vector field if and only if the integral curves of X are lifts of their own projections to Q; that is, if $\widetilde{c}$ is an integral curve of X, then*

$$\widetilde{c} = (\pi_Q \circ \widetilde{c})^{(1)}.$$

The curve $c = \pi_Q \circ \widetilde{c} : I \to Q$ is called a base integral curve of X or a solution of the second order differential equation given by X.

If $\widetilde{c} : I \to TQ$ is an integral curve of a second order vector field $X \in \Gamma(TTQ)$ locally given by $X = (q^i, v^i, v^i, a^i)$ and $c : I \to Q$ denotes its base integral curve, then

$$q^i = c^i, \quad v^i = \dot{c}^i \quad \text{and} \quad a^i = \ddot{c}^i.$$

Alternatively, the base integral curve c of $\widetilde{c}$ satisfies the system of second order differential quations

$$\ddot{c}^i = a^i(c^i, \dot{c}^i).$$