(1). Suppose $\mathcal{C}$ is a category, $\mathcal{C}^\circ$ is its dual category. Thus any contravariant functor $\varphi : \mathcal{C} \to D$ is a (covariant) functor $\varphi : \mathcal{C}^\circ \to \mathcal{D}$.

(2). Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. We can construct a category of functors form $\mathcal{C}$ to $\mathcal{D}$ by the follow steps. Firstly, the objects in *Funct*$(\mathcal{C}, \mathcal{D})$ are functors form $\mathcal{C}$ to $\mathcal{D}$. Then, a morphism f of functors from F to G (notation $f : F \to G$) is a family of morphisms in $\mathcal{D}$ that

$$f(X) : F(X) \to G(X),$$

one for each $X \in \mathcal{C}$, satisfying the following condition: for all morphism φ in $\mathcal{C}$ the following diagram

$$\begin{tikzcd}
F(X) \arrow[r, "f(X)"] \arrow[d, "F(\varphi)"'] & G(X) \arrow[d, "G(\varphi)"] \\
F(Y) \arrow[r, "f(Y)"] & G(Y)
\end{tikzcd}$$

is commutative.

If F is isomorphic to G (notation $F \cong G$), we should have morphisms f and g such that $f(X) \circ g(X) = \mathrm{id}_{G(X)}$ and $g(X) \circ f(X) = \mathrm{id}_{F(X)}$ for all $X \in \mathcal{C}$.

(3). Suppose $\hat{\mathcal{C}} =$ *Funct*$(\mathcal{C}^\circ, Set)$, $\forall X \in \mathcal{C}$, we can associate a $\hat{X} \in \hat{\mathcal{C}}$ by $\hat{X}(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and $\hat{Z}(\varphi)f = f \circ \varphi$ for $\varphi \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$. Then $\hat{Z}(\varphi) : \hat{Z}(X) \to \hat{Z}(Y)$.

Proposition 1. *Suppose X is a set in the category Set, it can be identified with the set $\hat{X}(e) = \mathrm{Hom}_{Set}(e, X)$, where e is a one-point set.*

Proof. Obviously. ◻

Indeed, in an arbitrary category $\mathcal{C}$ an analogue of one-point set does not necessarily exist. However, by considering $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ for all $Y \in \mathcal{C}$ simultaneously, we can recover complete information about an object $X \in \mathcal{C}$. This is the idea of representation theory.

Definition 1. *A functor $F \in \hat{\mathcal{C}}$ is said to be representable if $F \cong \hat{X}$ for some $X \in \mathcal{C}$. One says also that the object X represents the functor F.*

Theorem 1. $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y})$.

Proof. Let $\varphi : X \to Y$ be a morphism in $\mathcal{C}$. We associate with φ the morphism of functors $\hat{\varphi} : \hat{X} \to \hat{Y}$ by

$$\hat{\varphi}(Z) : \theta \mapsto \varphi \circ \theta \in \hat{Y}(Z),$$

where $\theta \in \hat{X}(Z)$ and X, Y, $Z \in \mathcal{C}$. It is clear that $\hat{\varphi} \circ \hat{\phi} = \widehat{\varphi \circ \phi}$.

Conversely, suppose $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y})$, define map $i : \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y}) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ by

$$i : f \mapsto f(X)(\mathrm{id}_X),$$

where $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X) = \hat{X}(X)$ and $f(\mathrm{id}_X) \in \hat{Y}(X)$. Then

$$i(\hat{\varphi}) = \hat{\varphi}(X)(\mathrm{id}_X) = \varphi \circ \mathrm{id}_X = \varphi.$$

On the other hand, we should show $\widehat{i(f)} = f$ when $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y})$, and it's equivalent to show $\widehat{i(f)}(Z) = f(Z)$ for all $Z \in \mathcal{C}$.

Now suppose a morphism $\varphi : Z \to X$,

$$i(f) \circ \varphi = \widehat{i(f)}(Z)(\varphi) = f(Z)(\varphi).$$

Using the commutativity of the diagram,

$$\begin{tikzcd}
\hat{X}(X) \arrow[r, "f(X)"] \arrow[d, "\hat{X}(\varphi)"'] & \hat{Y}(X) \arrow[d, "\hat{Y}(\varphi)"] \\
\hat{X}(Z) \arrow[r, "f(Z)"] & \hat{Y}(Z)
\end{tikzcd}$$

then

$$f(Z) \circ \hat{X}(\varphi)(\mathrm{id}_X) = \hat{Y}(\varphi) \circ f(X)(\mathrm{id}_X) = \hat{Y}(\varphi)(i(f)) = i(f) \circ \varphi$$

and

$$f(Z) \circ \hat{X}(\varphi)(\mathrm{id}_X) = f(Z)(\mathrm{id}_X \circ \varphi) = f(Z)(\varphi).$$

Thus i is an isomorphism that $i : \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y}) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$. □

If X represents the functor F, $\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Y}, F) \cong \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Y}, \hat{X}) \cong \mathrm{Hom}_{\mathcal{C}}(Y, X)$. If Y also represents F, then there exists an isomorphism $\varphi : \hat{Y} \cong F$, then $i(\varphi)$ is the according isomorphism between X and Y. Thus, the representing object of a representable functor is defined uniquely up to a isomorphism.

Definition 2. *Suppose X, $Y \in \mathcal{C}$, the direct product $X \times Y$ is (upto an isomorphism) the object Z representing the functor (if such functor is representable, or if such Z exists)*

$$W \mapsto \hat{X}(W) \times \hat{Y}(W),$$

where $\hat{X}(W) \times \hat{Y}(W)$ is the direct product of Set which has been constructed directly.

Definition 3. *Suppose X, Y, $S \in$ Set, $f : X \to S$ and $g : Y \to S$, the pullback of f and g or the fibre product of X and Y is*

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

In a general category $\mathcal{C}$, $X \times_S Y$ is defined as the object Z representing the functor

$$W \mapsto \hat{X}(W) \times_{\hat{S}(W)} \hat{Y}(W).$$

When $f = g$ are constant morphisms, $X \times_S Y \cong X \times Y$. When f and g are embedding, $X \times_S Y \cong X \cap Y$.

The universal property can be directly vertified. Since Z represents the functor $W \mapsto \hat{X}(W) \times \hat{Y}(W)$, then $\hat{Z}(W) = \hat{X}(W) \times \hat{Y}(W)$. Let $W = Z$, the image of $\mathrm{id}_Z \in \hat{Z}(Z)$ is that $\mathrm{id}_Z = (\pi_X, \pi_Y)$, where $\pi_X \in \hat{Y}(Z)$ and $\pi_Y \in \hat{Y}(Z)$. This can be writen as $X \xleftarrow{\pi_X} Z \xrightarrow{\pi_Y} Y$.

Now suppose $p_X : W \to X$ and $p_Y : W \to Y$, then there exist $\eta = (p_X, p_Y) \in \hat{X}(W) \times \hat{Y}(W) = \hat{Z}(W)$. The last thing we should to show is that $p_X = \pi_X \circ \eta$ and $p_Y = \pi_Y \circ \eta$. Indeed, for all $z \in Z$,

$$(\pi_X, \pi_Y)((p_X, p_Y)(z)) = \mathrm{id}_Z((p_X, p_Y)(z)) = (p_X, p_Y)(z).$$

The uniqueness of $X \times Y$ up to an isomorphism can be drived from the above theorem or universal property directly.

Because of some psychological reasons, we usually use a commutative diagram to visualize the universal property described above:

$$\begin{tikzcd}
X & Z \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \\
& W \arrow[ul, "p_Y"] \arrow[u, dashed, "\eta"] \arrow[ur, "p_X"'] &
\end{tikzcd}$$

Similarly, the universal property of fibre product can be written as a commutative diagram:

$$
\begin{tikzcd}
W \arrow[drr, bend left, "x"] \arrow[dr, dotted, "\varphi"] \arrow[ddr, bend right, "y"'] & & \\
& X \times_S Y \arrow[r, "p"'] \arrow[d, "q"] & X \arrow[d, "f"'] \\
& Y \arrow[r, "g"] & S
\end{tikzcd}
$$

$(A_\alpha, A_{\alpha\beta}, \rho_\alpha)$

$$
\begin{tikzcd}
A \arrow[drr, bend left, "f_\beta"] \arrow[dr, dotted, "f"] \arrow[ddr, bend right, "f_\alpha"'] & & \\
& \mathcal{F}(U) \arrow[r, "\rho_\beta"'] \arrow[d, "\rho_\alpha"] & \mathcal{F}(U_\beta) \arrow[d, "\rho_{\beta\alpha}"'] \\
& \mathcal{F}(U_\alpha) \arrow[r, "\rho_{\alpha\beta}"] & \mathcal{F}(U_\alpha \cap U_\beta)
\end{tikzcd}
$$

Given any categorical construction, we can create the dual construction by inverting all arrows in the original construction. In such a way one can obtain the coporduct $X \coprod_S Y$ by

$$
\begin{tikzcd}
W & & \\
& X \coprod_S Y \arrow[ul, dotted, "\varphi"'] & X \arrow[l, "p"] \arrow[ull, bend right, "x"'] \\
& Y \arrow[u, "q"'] \arrow[uul, bend left, "y"] & S \arrow[l, "g"'] \arrow[u, "f"]
\end{tikzcd}
$$

We fix a category $\mathcal{J}$, and define the diagonal functor $\Delta : \mathcal{C} \to \mathit{Funct}(\mathcal{J}, \mathcal{C})$ as follow:

- On objects: ΔX is the set of constant functors with the value X. In other words, $\Delta X(j) = X$ for all $j \in \mathcal{J}$, $\Delta X(\varphi) = \mathrm{id}_X$ for all morphism φ in $\mathcal{J}$.

- On morphisms: Suppose $\psi : X \to Y$ is a morphism in $\mathcal{C}$, $\Delta\psi : \Delta X \to \Delta Y$ is defined as follows: $\Delta\psi(j) : X = \Delta X(j) \to Y = \Delta Y(j)$ for all $j \in \mathcal{J}$.

It is clear that $\Delta(\varphi \circ \psi) = \Delta\varphi \circ \Delta\psi$, so Δ is indeed a functor.

Definition 4. *Suppose $F : \mathcal{J} \to \mathcal{C}$ is a functor, the (projective or inverse) limit of F in the category $\mathcal{C}$ according to $\mathcal{J}$ is an object $X \in \mathcal{C}$ representing the functor*

$$Y \mapsto \mathrm{Hom}_{\mathit{Funct}(\mathcal{J},\mathcal{C})}(\Delta Y, F) : \mathcal{C}^\circ \to \mathit{Set}.$$

The limit of F is denoted by $X = \varprojlim F$.