- (1). Suppose  $\mathcal{C}$  is a category,  $\mathcal{C}^{\circ}$  is its dual category. Thus any contravariant functor  $\varphi: \mathcal{C} \to D$  is a (covariant) functor  $\varphi: \mathcal{C}^{\circ} \to \mathcal{D}$ .
- (2). Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are categories. We can construct a category of functors form  $\mathcal{C}$  to  $\mathcal{D}$  by the follow steps. Firstly, the objects in  $Funct(\mathcal{C}, \mathcal{D})$  are functors form  $\mathcal{C}$  to  $\mathcal{D}$ . Then, a morphism f of functors from F to G (notation  $f: F \to G$ ) is a family of morphisms in  $\mathcal{D}$  that

$$f(X): F(X) \to G(X),$$

one for each  $X \in \mathcal{C}$ , satisfying the following condition: for all morphism  $\varphi$  in  $\mathcal{C}$  the following diagram

$$F(X) \xrightarrow{f(X)} G(X)$$

$$F(\varphi) \downarrow \qquad \qquad \downarrow G(\varphi)$$

$$F(Y) \xrightarrow{f(Y)} G(Y)$$

is commutative.

If F is isomorphic to G (notation  $F \cong G$ ), we should have morphisms f and g such that  $f(X) \circ g(X) = \mathrm{id}_{G(X)}$  and  $g(X) \circ f(X) = \mathrm{id}_{F(X)}$  for all  $X \in \mathcal{C}$ .

(3). Suppose  $\hat{\mathcal{C}} = Funct(\mathcal{C}^{\circ}, Set)$ ,  $\forall X \in \mathcal{C}$ , we can associate a  $\hat{X} \in \hat{\mathcal{C}}$  by  $\hat{X}(Y) = \operatorname{Hom}_{\mathcal{C}}(Y, X)$  and  $\hat{Z}(\varphi)f = f \circ \varphi$  for  $\varphi \in \operatorname{Hom}_{\mathcal{C}}(Y, X)$  and  $f \in \operatorname{Hom}_{\mathcal{C}}(X, Z)$ . Then  $\hat{Z}(\varphi) : \hat{Z}(X) \to \hat{Z}(Y)$ .

**Proposition 1.** Suppose X is a set in the category Set, it can be identified with the set  $\hat{X}(e) = \text{Hom}_{Set}(e, X)$ , where e is a one-point set.

*Proof.* Obviously. 
$$\Box$$

Indeed, in an arbitrary category  $\mathcal{C}$  an analogue of one-point set does not necessarily exist. However, by considering  $\operatorname{Hom}_{\mathcal{C}}(Y,X)$  for all  $Y \in \mathcal{C}$  simultaneously, we can recover complete information about an object  $X \in \mathcal{C}$ . This is the idea of representation theory.

**Definition 1.** A functor  $F \in \hat{C}$  is said to be representable if  $F \cong \hat{X}$  for some  $X \in C$ . One says also that the object X represents the functor F.

**Theorem 1.**  $\operatorname{Hom}_{\mathcal{C}}(X,Y) \cong \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X},\hat{Y}).$ 

*Proof.* Let  $\varphi: X \to Y$  be a morphism in  $\mathcal{C}$ . We associate with  $\varphi$  the morphism of functors  $\hat{\varphi}: \hat{X} \to \hat{Y}$  by

$$\hat{\varphi}(Z): \theta \mapsto \varphi \circ \theta \in \hat{Y}(Z),$$

where  $\theta \in \hat{X}(Z)$  and  $X, Y, Z \in \mathcal{C}$ . It is clear that  $\hat{\varphi} \circ \hat{\phi} = \widehat{\varphi \circ \phi}$ .

Conversely, suppose  $f \in \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y})$ , define map  $i : \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y}) \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$  by

$$i: f \mapsto f(X)(\mathrm{id}_X),$$

where  $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X) = \hat{X}(X)$  and  $f(\mathrm{id}_X) \in \hat{Y}(X)$ . Then

$$i(\hat{\varphi}) = \hat{\varphi}(X)(\mathrm{id}_X) = \varphi \circ \mathrm{id}_X = \varphi.$$

On the other hand, we should show  $\widehat{i(f)} = f$  when  $f \in \operatorname{Hom}_{\widehat{\mathcal{C}}}(\hat{X}, \hat{Y})$ , and it's equivalent to show  $\widehat{i(f)}(Z) = f(Z)$  for all  $Z \in \mathcal{C}$ .

Now suppose a morphism  $\varphi: Z \to X$ 

$$i(f) \circ \varphi = \widehat{i(f)}(Z)(\varphi) = f(Z)(\varphi).$$

Using the commutativity of the diagram,

$$\hat{X}(X) \xrightarrow{f(X)} \hat{Y}(X)$$

$$\hat{X}(\varphi) \downarrow \qquad \qquad \downarrow \hat{Y}(\varphi)$$

$$\hat{X}(Z) \xrightarrow{f(Z)} \hat{Y}(Z)$$

then

$$f(Z) \circ \hat{X}(\varphi)(\mathrm{id}_X) = \hat{Y}(\varphi) \circ f(X)(\mathrm{id}_X) = \hat{Y}(\varphi)(i(f)) = i(f) \circ \varphi$$

and

$$f(Z) \circ \hat{X}(\varphi)(\mathrm{id}_X) = f(Z)(\mathrm{id}_X \circ \varphi) = f(Z)(\varphi).$$

Thus i is an isomorphism that  $i: \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{X}, \hat{Y}) \cong \operatorname{Hom}_{\mathcal{C}}(X, Y)$ .

If X represents the functor F,  $\operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Y},F) \cong \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Y},\hat{X}) \cong \operatorname{Hom}_{\mathcal{C}}(Y,X)$ . If Y also represents F, then there exists an isomorphism  $\varphi:\hat{Y}\cong F$ , then  $i(\varphi)$  is the according isomorphism between X and Y. Thus, the representing object of a representable functor is defined uniquely up to a isomorphism.

**Definition 2.** Suppose  $X, Y \in \mathcal{C}$ , the direct product  $X \times Y$  is (upto an isomorphism) the object Z representing the functor (if such functor is representable, or if such Z exists)

$$W \mapsto \hat{X}(W) \times \hat{Y}(W),$$

where  $\hat{X}(W) \times \hat{Y}(W)$  is the direct product of Set which has been constructed directly.

**Definition 3.** Suppose  $X, Y, S \in Set, f: X \to S$  and  $g: Y \to S$ , the pullback of f and g or the fibre product of X and Y is

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

In a general category C,  $X \times_S Y$  is defined as the object Z representing the functor

$$W \mapsto \hat{X}(W) \times_{\hat{S}(W)} \hat{Y}(W).$$

When f=g are constant morphisms,  $X\times_S Y\cong X\times Y$ . When f and g are embedding,  $X\times_S Y\cong X\cap Y$ .

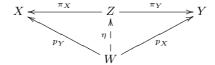
The universal property can be directly vertified. Since Z represents the functor  $W \mapsto \hat{X}(W) \times \hat{Y}(W)$ , then  $\hat{Z}(W) = \hat{X}(W) \times \hat{Y}(W)$ . Let W = Z, the image of  $\mathrm{id}_Z \in \hat{Z}(Z)$  is that  $\mathrm{id}_Z = (\pi_X, \pi_Y)$ , where  $\pi_X \in \hat{Y}(Z)$  and  $\pi_Y \in \hat{Y}(Z)$ . This can be written as  $X \xleftarrow{\pi_X} Z \xrightarrow{\pi_Y} Y$ .

Now suppose  $p_X: W \to X$  and  $p_Y: W \to Y$ , then there exist  $\eta = (p_X, p_Y) \in \hat{X}(W) \times \hat{Y}(W) = \hat{Z}(W)$ . The last thing we should to show is that  $p_X = \pi_X \circ \eta$  and  $p_Y = \pi_Y \circ \eta$ . Indeed, for all  $z \in Z$ ,

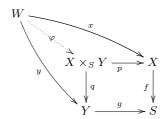
$$(\pi_X, \pi_Y)((p_X, p_Y)(z)) = \mathrm{id}_Z((p_X, p_Y)(z)) = (p_X, p_Y)(z).$$

The uniqueness of  $X \times Y$  up to an isomorphism can be drived from the above theorem or universal property directly.

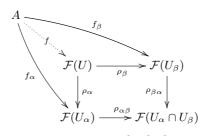
Because of some psychological reasons, we usually use a commutative diagram to visualize the universal property described above:



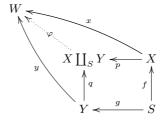
Similarly, the universal property of fibre product can be written as a commutative diagram:



 $(A_{\alpha}, A_{\alpha\beta}, \rho_{\alpha})$ 



Given any categorical construction, we can create the dual construction by inverting all arrows in the original construction. In such a way one can obtain the coporduct  $X \coprod_S Y$  by



We fix a category  $\mathcal{J}$ , and define the diagonal functor  $\Delta: \mathcal{C} \to Funct(\mathcal{J}, \mathcal{C})$  as follow:

- On objects:  $\Delta X$  is the set of constant functors with the value X. In other words,  $\Delta X(j) = X$  for all  $j \in \mathcal{J}$ ,  $\Delta X(\varphi) = \mathrm{id}_X$  for all morphism  $\varphi$  in  $\mathcal{J}$ .
- On morphisms: Suppose  $\psi: X \to Y$  is a morphism in  $\mathcal{C}$ ,  $\Delta \psi: \Delta X \to \Delta Y$  is defined as follows:  $\Delta \psi(j): X = \Delta X(j) \to Y = \Delta Y(j)$  for all  $j \in \mathcal{J}$ .

It is clear that  $\Delta(\varphi \circ \psi) = \Delta \varphi \circ \Delta \psi$ , so  $\Delta$  is indeed a functor.

**Definition 4.** Suppose  $F: \mathcal{J} \to \mathcal{C}$  is a functor, the (projective or inverse) limit of F in the category  $\mathcal{C}$  according to  $\mathcal{J}$  is an object  $X \in \mathcal{C}$  representing the functor

$$Y \mapsto \operatorname{Hom}_{Funct(\mathcal{J},\mathcal{C})}(\Delta Y, F) : \mathcal{C}^{\circ} \to Set.$$

The limit of F is denoted by  $X = \underline{\lim} F$ .