A topological space X can be seen as a category if we take open sets as its objects and inclusion maps as its morphisms, i.e.

$$\mathrm{Hom}_X(U,V) = \begin{cases} \{i_{UV} : U \hookrightarrow V\}, & \text{if } U \underset{\text{open}}{\subset} V \underset{\text{open}}{\subset} X; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Definition 1. *A presheaf is a functor $\mathcal{F} : X^\circ \to AG$, where X° is the dual category of a topological space X and AG is the category of abelian groups. In other words, $\mathcal{F} : X \to AG$ is a contravariant functor. The elements $s \in \mathcal{F}(U)$ are called the sections on U. Denote that $\rho_{UV} := \mathcal{F}(i_{UV}) : \mathcal{F}(V) \to \mathcal{F}(U)$ and $\rho_{UV}(s) = s|_V$ for $s \in \mathcal{F}(U)$.*

The morphisms φ between two presheaf $\mathcal{F}$ and $\mathcal{G}$ is just the morphisms between two functors, that is, there exist a family of morphisms $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

Definition 2. *If $\mathcal{F}$ is a presheaf on X, and if p is a point on X, we define the stalk $\mathcal{F}_p \in AG$ of $\mathcal{F}$ at p to be the direct limit*[1] *of the group $\mathcal{F}(U)$ for all $U \underset{\text{open}}{\subset} X$ containing p, via the restriction maps ρ.*

We can directly construct $\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U)$ that

$$\mathcal{F}_p = \left\{ \langle U, s\rangle : p \in U \underset{\text{open}}{\subset} X, s \in \mathcal{F}(U) \right\} / \sim,$$

where $\sim$ is defined as follows: Suppose $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$, if there exists a $W \underset{\text{open}}{\subset} U \cap V \neq \varnothing$ such that $s|_W = t|_W$, then $\langle U, s\rangle = \langle V, t\rangle$ or $s \sim t$. $\mathcal{F}_p$ is indeed a group since the addition can be defined by $\langle U, s\rangle + \langle V, t\rangle = \langle W, s|_W + t|_W\rangle$.

Here $\mathcal{F}(U) \ni s \mapsto \langle U, s\rangle = s_p$ defines a family of morphisms $\rho_{Up} : \mathcal{F}(U) \to \mathcal{F}_p$ for any $U \underset{\text{open}}{\subset} X$. We can then vertify the universal property of direct product that

$$\begin{tikzcd}
 & & \mathcal{F}(U) \arrow[dll, bend right, "\varphi_{Vp}"'] \arrow[dl, "\rho_{Up}"] \arrow[dd, "\rho_{UV}"] \\
W & \mathcal{F}_p \arrow[l, dashed, "\theta"'] & \\
 & & \mathcal{F}(V) \arrow[ull, bend left, "\varphi_{Vp}"] \arrow[ul, "\rho_{Up}"']
\end{tikzcd}$$

[1] X is equiped with an order that $U > V$ if $U \subset V$. Since $\mathcal{F}(U)$ are abelian groups, as $\mathbb{Z}$-modules, the direct limit always exists.

Now, suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a functor. The universal property of direct product

$$\begin{tikzcd}
 & & \mathcal{F}(U) \arrow[dl, "\rho_{Up}"'] \arrow[dd, "\rho_{UV}"] \arrow[dll, bend right, "\rho'_{Up}\circ\varphi(U)"'] \\
\mathcal{G}_p & \mathcal{F}_p \arrow[l, dashed, "\varphi_p"'] & \\
 & & \mathcal{F}(V) \arrow[ul, "\rho_{Vp}"] \arrow[ull, bend left, "\rho'_{Vp}\circ\varphi(V)"]
\end{tikzcd}$$

gives the existence of the morphism $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$, and the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{Up}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{Up}"] \\
\mathcal{F}_p \arrow[r, dashed, "\varphi_p"] & \mathcal{G}_p
\end{tikzcd}$$

Figure 1: The existence of φ_p

is commutative.

Definition 3. *A sheaf is a presheaf that for any open set $U \subset X$, the complex*

$$0 \to \mathcal{F}(U) \xrightarrow{d_0} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{d_1} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

is exact for any open cover $\{U_i\}$ of U, where

$$\begin{aligned}
d_0 : \quad & s & \mapsto & \prod_{i\in I} s|_{U_i}, \\
d_1 : \quad & \prod_{i\in I} s_i & \mapsto & \prod_{i,j\in I} \left(s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j}\right).
\end{aligned}$$

The definition can be rewritten as: For any open cover $\{U_i\}$ of any open set $U \subset X$,

- If $\forall i \in I$, $s|_{U_i} = 0$, then $s = 0$.
- If $\forall i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then there's a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

It is not so difficult to vertify that this definition is equivalent to the old one.

The following proposition (which would be false for presheves) illustrates the local nature of a sheaf.

Proposition 1. *Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on a topological space X. Then φ is a isomorphism if and only if the induced map on the stalk $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every $p \in X$.*

Proof. p.63 on Hartshorne. □

Definition 4. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. We define the presheaf kernel of φ, presheaf of cokernel of φ, and presheaf image of φ to be the presheaves given by $U \mapsto \ker(\varphi(U))$, $U \to \operatorname{coker}(\varphi(U))$, and $U \mapsto \operatorname{Im}(\varphi(U))$ respectively.*

Proposition 2. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the presheaf $U \mapsto \ker(\varphi(U))$ is a sheaf.*

Proof. Let $\{U_i\}$ be an open cover of U, and s_i is local section on U_i.

- Suppose $s \in \ker(\varphi(U))$ and $s|_{U_i} = 0$, since $\mathcal{F}$ is a sheaf, $s = 0$.
- Suppose $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, we need to show that there exists a global section $s \in \ker(\varphi(U))$ such that $s|_{U_i} = s_i$. Since $\mathcal{F}$ is a sheaf, it's nature that there's $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. The last thing to vetify is $\varphi(U)(s) = 0$. Restrict $\varphi(U)(s)$ on U_i, then

  $$\rho'_{UU_i} \circ \varphi(U)(s) = \varphi(U_i)(\rho_{UU_i} s) = \varphi(U_i)(s_i) = 0,$$

  so $\varphi(U)(s) \in \mathcal{G}(U)$ vanishes locally. Since $\mathcal{G}$ is a sheaf, it also vanishes globally, i.e. $\varphi(U)(s) = 0$.

Thus $U \mapsto \ker(\varphi(U))$ is a sheaf. □

However, the presheaves $\operatorname{coker}(\varphi)$ and $\operatorname{Im}(\varphi)$ need not to be sheaves. Actually, the key point in the proof above is that ker is compatible with the sheaf property of $\mathcal{G}$. Then we come to an important notion of a sheaf associated to a presheaf, i.e. sheafification.

Roughly speaking, the sheafification of a presheaf $\mathcal{F}$ is the "smallest" sheaf with the same stalks as $\mathcal{F}$. Because of the "smallest", sheafification should have the universal property.

Proposition 3. *Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism θ make the diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\theta"] \arrow[d, "\varphi"'] & \mathcal{F}^+ \arrow[dl, dashed, "\psi"] \\
\mathcal{G} &
\end{tikzcd}$$

commutative for any sheaf $\mathcal{G}$. $\mathcal{F}^+$ is called the sheaf associated to the preshead $\mathcal{F}$ or the sheafification of $\mathcal{F}$.

Proof. We construct the sheaf $\mathcal{F}^+$ as follows. For any open set U, let $\mathcal{F}^+(U)$ be the set of functions $s : U \to \cup_{p \in U} \mathcal{F}_p$, such that

- for each $p \in U$, $s(p) \in \mathcal{F}_p$, and
- for each $p \in U$, there is a neighborhood $V \underset{\text{open}}{\subset} U$ of P, and an element $t \in \mathcal{F}(V)$, such that $\forall q \in V$, $s(q) = t_q := t^+(q)$.

The addition on $\mathcal{F}^+(U)$ is that $(s+t)(p) = s(p)+t(p)$, so $\mathcal{F}^+(U)$ is indeed a group. If $V \underset{\text{open}}{\subset} U$, there's a nature map (function restriction) $i_{UV} : \mathcal{F}^+(U) \to \mathcal{F}^+(V)$ such that $i_{UV}(s) = s|_V$, so $\mathcal{F}^+$ is a presheaf. Let $\{U_i\}$ be a open cover of U and $s_i \in \mathcal{F}^+(U_i)$ be local sections. If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, we can define a function $s : U \to \cup_{p \in U} \mathcal{F}_p$ by setting $s|_{U_i} = s_i$. If $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$ since it is a function. Thus $\mathcal{F}^+$ is a sheaf.

For each $s \in \mathcal{F}(U)$, we can associate it a section $s^+ \in \mathcal{F}^+(U)$ by $s^+(p) = s_p$, then there's a morphism $\theta(U) : s \mapsto s^+$. $\forall s \in \mathcal{F}(U)$, since

$$i_{UV}\big(\theta(U)(s)\big) = i_{UV}(s^+) = s^+|_V$$

and

$$\theta(V)\big(\rho_{UV}(s)\big) = \theta(V)(s|_V) = s^+|_V.$$

θ is a morphism.

Let $\bar{s} \in \mathcal{F}^+(U)$, because of the construction of $\mathcal{F}^+$, we can find an open cover $\{U_i\}$ of U such that $\bar{s}|_{U_i} = \bar{s}_i^+$, where $\bar{s}_i \in \mathcal{F}(U_i)$. Firstly define $\psi(U_i) : \bar{s}|_{U_i} \mapsto \varphi(U_i)(\bar{s}_i)$, and we can use the sheaf condition of $\mathcal{G}$ to get a global section s' on U such that $s'|_{U_i} = \varphi(U_i)(\bar{s}_i)$. Then, define $\psi(U) : \bar{s} \mapsto s'$, and ψ will become the morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$. Finally, because of the construction of ψ, for any $s^+ \in \mathcal{F}^+(U)$, $\psi(U)(s^+) = \varphi(U)(s)$, so that

$$\psi(U)(\theta(s)) = \psi(U)(s^+) = \varphi(U)(s)$$

makes the diagram commutative. □

Proposition 4. *$\mathcal{F}_p \cong \mathcal{F}_p^+$, so if $\mathcal{F}$ is a sheaf, then $\mathcal{F} \cong \mathcal{F}^+$.*

Proof. According to the Figure 1, there's a morphism $\theta_p : \mathcal{F}_p \to \mathcal{F}_p^+$, such that $\theta_p(\langle U, s\rangle) = \langle U, s^+\rangle$.

- It is injective. If $\theta_p(\langle U, s\rangle) = \langle V, 0\rangle$, then $s^+|_V = 0$, and $s_p = s^+(p) = 0$.
- It is surjective. $\forall \langle U, \bar{s}\rangle \in \mathcal{F}_p^+$, there exists an open subset V and $t \in \mathcal{F}(V)$ such that $\langle U, \bar{s}\rangle = \langle V, t^+\rangle$, then $\theta_p(\langle V, t\rangle) = \langle U, \bar{s}\rangle$.

□

Definition 5. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define the kernel(repsectively cokernel, image) of φ, denoted $\ker\varphi$(repsectively $\operatorname{coker}\varphi$, $\operatorname{Im}\varphi$), to be the sheaf associated to the presheaf of kernel(respectively, coker, image) of φ.*

Definition 6. *We say that a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective(respectively, surjective) if $\ker\varphi = 0$(respectively, $\operatorname{Im}\varphi \cong \mathcal{G}$).*

Proposition 5. *For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, $(\ker\varphi)_p = \ker(\varphi_p)$ and $(\operatorname{Im}\varphi)_p = \operatorname{Im}(\varphi_p)$ for each p.*

Proof. We will show the following statements.

- $\ker(\varphi_p) \subset (\ker\varphi)_p$: Suppose $s_p = \langle U, s\rangle \in \ker(\varphi_p)$, then $\varphi_p\langle U, s\rangle = \langle U, \varphi(U)s\rangle = 0$, so there exists $W \underset{\text{open}}{\subset} U$ such that
$$(\varphi(U)s)|_W = \varphi(W)(s|_W) = 0,$$
thus $s|_W \in \ker(\varphi(W))$ and $s_p = \langle W, s|_W\rangle \in (\ker\varphi)_p$.
- $(\ker\varphi)_p \subset \ker(\varphi_p)$: Suppose $\langle U, s\rangle \in (\ker\varphi)_p$, then $\varphi_p\langle U, s\rangle = \langle U, \varphi(U)s\rangle = \langle U, 0\rangle = 0 \in \mathcal{G}_p$, thus $\langle U, s\rangle \in \ker(\varphi_p)$.
- $\operatorname{Im}(\varphi_p) \subset (\operatorname{Im}\varphi)_p$: Suppose $t = \varphi_p s_p \in \operatorname{Im}(\varphi_p)$ and $s_p = \langle U, s\rangle$, then $t = \varphi_p s_p = \varphi_p\langle U, s\rangle = \langle U, \varphi(U)s\rangle \in (\operatorname{Im}\varphi)_p$.

- $(\operatorname{Im}\varphi)_p \subset \operatorname{Im}(\varphi_p)$: Suppose $t = \langle U, \varphi(U)s\rangle \in (\operatorname{Im}\varphi)_p$, then $t = \rho'_{Up} \circ \varphi(U)s = \varphi_p s_p \in \operatorname{Im}(\varphi_p)$.

$\square$

Corollary 1. *For any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, it is injective(respectively, surjective) if and only if φ_p is injective(respectively, surjective) for all p.*

Proof. According to Proposition 1 and Proposition 5, $\ker\varphi = 0$ if and only if $(\ker\varphi)_p = \ker(\varphi_p) = 0$. Similarly, $\operatorname{Im}\varphi \cong \mathcal{G}$ if and only if $(\operatorname{Im}\varphi)_p = \operatorname{Im}(\varphi_p) \cong \mathcal{G}_g$. $\square$

Corollary 2. *A morphism of sheaves is an isomorphism if and only if it is injective and surjective.*

Proof. A morphism of sheaves φ is an isomorphism if and only if φ_p is an isomorphism for all p. As a morphism of groups, φ_p is an isomorphism for all p if and only if it is injective and surjective for all p, and according to Corollay 1, if and only if φ is injective and surjective. $\square$

Definition 7. *Let $f : X \to Y$ be a continuous map of topological spoaces. For any sheaf $\mathcal{F}$ on X, we define the direct image sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ for any open set $U \subset Y$. For any sheaf $\mathcal{G}$ on Y, we define the inverse image sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V\supset f(U)} \mathcal{G}(V)$, where U is any open set in X, and the limit is taken over all open sets $V \subset Y$ containing $f(U)$.*

Especially, if $f(U)$ is an open set, then $\varinjlim_{V\supset f(U)} \mathcal{G}(V) = \mathcal{G}\big(f(U)\big)$.