

A topological space X can be seen as a category if we take open sets as its objects and inclusion maps as its morphisms, i.e.

$$\text{Hom}_X(U, V) = \begin{cases} \{i_{UV} : U \hookrightarrow V\}, & \text{if } U \underset{\text{open}}{\subset} V \underset{\text{open}}{\subset} X; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 1. A presheaf is a functor $\mathcal{F} : X^\circ \rightarrow AG$, where X° is the dual category of a topological space X and AG is the category of abelian groups. In other words, $\mathcal{F} : X \rightarrow AG$ is a contravariant functor. The elements $s \in \mathcal{F}(U)$ are called the sections on U . Denote that $\rho_{UV} := \mathcal{F}(i_{UV}) : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ and $\rho_{UV}(s) = s|_U$ for $s \in \mathcal{F}(V)$.

The morphisms φ between two presheaf \mathcal{F} and \mathcal{G} is just the morphisms between two functors, that is, there exist a family of morphisms $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ such that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho'_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Definition 2. If \mathcal{F} is a presheaf on X , and if p is a point on X , we define the stalk $\mathcal{F}_p \in AG$ of \mathcal{F} at p to be the direct limit¹ of the group $\mathcal{F}(U)$ for all $U \underset{\text{open}}{\subset} X$ containing p , via the restriction maps ρ .

We can directly construct $\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U)$ that

$$\mathcal{F}_p = \{ \langle U, s \rangle : p \in U \underset{\text{open}}{\subset} X, s \in \mathcal{F}(U) \} / \sim,$$

where \sim is defined as follows: Suppose $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$, if there exists a $W \underset{\text{open}}{\subset} U \cap V \neq \emptyset$ such that $s|_W = t|_W$, then $\langle U, s \rangle = \langle V, t \rangle$ or $s \sim t$. \mathcal{F}_p is indeed a group since the addition can be defined by $\langle U, s \rangle + \langle V, t \rangle = \langle W, s|_W + t|_W \rangle$.

Here $\mathcal{F}(U) \ni s \mapsto \langle U, s \rangle = s_p$ defines a family of morphisms $\rho_{Up} : \mathcal{F}(U) \rightarrow \mathcal{F}_p$ for any $U \underset{\text{open}}{\subset} X$.

We can then verify the universal property of direct product that

$$\begin{array}{ccc} & \mathcal{F}(U) & \\ \varphi_{Vp} \swarrow & \searrow \rho_{Up} & \downarrow \rho_{UV} \\ W \xleftarrow{\theta} \mathcal{F}_p & & \mathcal{F}(V) \\ \varphi_{Vp} \swarrow & \nwarrow \rho_{Up} & \end{array}$$

Now, suppose $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a functor. The universal property of direct product

$$\begin{array}{ccc} & \mathcal{F}(U) & \\ \rho'_{Up} \circ \varphi(U) \swarrow & \searrow \rho_{Up} & \downarrow \rho_{UV} \\ \mathcal{G}_p \xleftarrow{\varphi_p} \mathcal{F}_p & & \mathcal{F}(V) \\ \rho'_{Vp} \circ \varphi(V) \swarrow & \nwarrow \rho_{Up} & \end{array}$$

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\
\rho_{U,p} \downarrow & & \downarrow \rho'_{U,p} \\
\mathcal{F}_p & \xrightarrow{\varphi_p} & \mathcal{G}_p
\end{array}$$

Figure 1: The existence of φ_p

gives the existence of the morphism $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$, and the diagram is commutative.

Definition 3. A sheaf is a presheaf that for any open set $U \subset X$, the complex

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{d_0} \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{d_1} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is exact for any open cover $\{U_i\}$ of U , where

$$\begin{aligned}
d_0 : s &\mapsto \prod_{i \in I} s|_{U_i}, \\
d_1 : \prod_{i \in I} s_i &\mapsto \prod_{i,j \in I} (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}).
\end{aligned}$$

The definition can be rewritten as: For any open cover $\{U_i\}$ of any open set $U \subset X$,

- If $\forall i \in I, s|_{U_i} = 0$, then $s = 0$.
- If $\forall i, j \in I, s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there's a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

It is not so difficult to verify that this definition is equivalent to the old one.

The following proposition (which would be false for presheaves) illustrates the local nature of a sheaf.

Proposition 1. Suppose $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on a topological space X . Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$ is an isomorphism for every $p \in X$.

Proof. p.63 on Hartshorne. □

Definition 4. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves. We define the presheaf kernel of φ , presheaf of cokernel of φ , and presheaf image of φ to be the presheaves given by $U \mapsto \ker(\varphi(U))$, $U \mapsto \text{coker}(\varphi(U))$, and $U \mapsto \text{Im}(\varphi(U))$ respectively.

Proposition 2. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then the presheaf $U \mapsto \ker(\varphi(U))$ is a sheaf.

Proof. Let $\{U_i\}$ be an open cover of U , and s_i is local section on U_i .

- Suppose $s \in \ker(\varphi(U))$ and $s|_{U_i} = 0$, since \mathcal{F} is a sheaf, $s = 0$.
- Suppose $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, we need to show that there exists a global section $s \in \ker(\varphi(U))$ such that $s|_{U_i} = s_i$. Since \mathcal{F} is a sheaf, it's nature that there's $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. The last thing to verify is $\varphi(U)(s) = 0$. Restrict $\varphi(U)(s)$ on U_i , then

$$\rho'_{U, U_i} \circ \varphi(U)(s) = \varphi(U_i)(\rho_{U, U_i} s) = \varphi(U_i)(s_i) = 0,$$

so $\varphi(U)(s) \in \mathcal{G}(U)$ vanishes locally. Since \mathcal{G} is a sheaf, it also vanishes globally, i.e. $\varphi(U)(s) = 0$.

¹ X is equipped with an order that $U > V$ if $U \subset V$. Since $\mathcal{F}(U)$ are abelian groups, as \mathbb{Z} -modules, the direct limit always exists.

Thus $U \mapsto \ker(\varphi(U))$ is a sheaf. \square

However, the presheaves $\text{coker}(\varphi)$ and $\text{Im}(\varphi)$ need not to be sheaves. Actually, the key point in the proof above is that \ker is compatible with the sheaf property of \mathcal{G} . Then we come to an important notion of a sheaf associated to a presheaf, i.e. sheafification.

Roughly speaking, the sheafification of a presheaf \mathcal{F} is the "smallest" sheaf with the same stalks as \mathcal{F} . Because of the "smallest", sheafification should have the universal property.

Proposition 3. *Given a presheaf \mathcal{F} , there is a sheaf \mathcal{F}^+ and a morphism θ make the diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\ \varphi \downarrow & \swarrow \psi & \\ \mathcal{G} & & \end{array}$$

commutative for any sheaf \mathcal{G} . \mathcal{F}^+ is called the sheaf associated to the presheaf \mathcal{F} or the sheafification of \mathcal{F} .

Proof. We construct the sheaf \mathcal{F}^+ as follows. For any open set U , let $\mathcal{F}^+(U)$ be the set of functions $s : U \rightarrow \cup_{p \in U} \mathcal{F}_p$, such that

- for each $p \in U$, $s(p) \in \mathcal{F}_p$, and
- for each $p \in U$, there is a neighborhood $V \subset_{\text{open}} U$ of p , and an element $t \in \mathcal{F}(V)$, such that $\forall q \in V$, $s(q) = t_q := t^+(q)$.

The addition on $\mathcal{F}^+(U)$ is that $(s + t)(p) = s(p) + t(p)$, so $\mathcal{F}^+(U)$ is indeed a group. If $V \subset_{\text{open}} U$, there's a nature map (function restriction) $i_{UV} : \mathcal{F}^+(U) \rightarrow \mathcal{F}^+(V)$ such that $i_{UV}(s) = s|_V$, so \mathcal{F}^+ is a presheaf. Let $\{U_i\}$ be an open cover of U and $s_i \in \mathcal{F}^+(U_i)$ be local sections. If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, we can define a function $s : U \rightarrow \cup_{p \in U} \mathcal{F}_p$ by setting $s|_{U_i} = s_i$. If $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$ since it is a function. Thus \mathcal{F}^+ is a sheaf.

For each $s \in \mathcal{F}(U)$, we can associate it a section $s^+ \in \mathcal{F}^+(U)$ by $s^+(p) = s_p$, then there's a morphism $\theta(U) : \mathcal{F}(U) \rightarrow \mathcal{F}^+(U)$, since

$$i_{UV}(\theta(U)(s)) = i_{UV}(s^+) = s^+|_V$$

and

$$\theta(V)(\rho_{UV}(s)) = \theta(V)(s|_V) = s^+|_V.$$

θ is a morphism.

Let $\bar{s} \in \mathcal{F}^+(U)$, because of the construction of \mathcal{F}^+ , we can find an open cover $\{U_i\}$ of U such that $\bar{s}|_{U_i} = \bar{s}_i^+$, where $\bar{s}_i \in \mathcal{F}(U_i)$. Firstly define $\psi(U_i) : \bar{s}|_{U_i} \mapsto \varphi(U_i)(\bar{s}_i)$, and we can use the sheaf condition of \mathcal{G} to get a global section s' on U such that $s'|_{U_i} = \varphi(U_i)(\bar{s}_i)$. Then, define $\psi(U) : \bar{s} \mapsto s'$, and ψ will become the morphism $\psi : \mathcal{F}^+ \rightarrow \mathcal{G}$. Finally, because of the construction of ψ , for any $s^+ \in \mathcal{F}^+(U)$, $\psi(U)(s^+) = \varphi(U)(s)$, so that

$$\psi(U)(\theta(s)) = \psi(U)(s^+) = \varphi(U)(s)$$

makes the diagram commutative. \square

Proposition 4. $\mathcal{F}_p \cong \mathcal{F}_p^+$, so if \mathcal{F} is a sheaf, then $\mathcal{F} \cong \mathcal{F}^+$.

Proof. According to the Figure 1, there's a morphism $\theta_p : \mathcal{F}_p \rightarrow \mathcal{F}_p^+$, such that $\theta_p(\langle U, s \rangle) = \langle U, s^+ \rangle$.

- It is injective. If $\theta_p(\langle U, s \rangle) = \langle V, 0 \rangle$, then $s^+|_V = 0$, and $s_p = s^+(p) = 0$.
- It is surjective. $\forall \langle U, \bar{s} \rangle \in \mathcal{F}_p^+$, there exists an open subset V and $t \in \mathcal{F}(V)$ such that $\langle U, \bar{s} \rangle = \langle V, t^+ \rangle$, then $\theta_p(\langle V, t \rangle) = \langle U, \bar{s} \rangle$.

□

Definition 5. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, we define the kernel (respectively cokernel, image) of φ , denoted $\ker \varphi$ (respectively $\operatorname{coker} \varphi$, $\operatorname{Im} \varphi$), to be the sheaf associated to the presheaf of kernel (respectively, coker, image) of φ .

Definition 6. We say that a morphism of sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is injective (respectively, surjective) if $\ker \varphi = 0$ (respectively, $\operatorname{Im} \varphi \cong \mathcal{G}$).

Proposition 5. For any morphism of sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, $(\ker \varphi)_p = \ker(\varphi_p)$ and $(\operatorname{Im} \varphi)_p = \operatorname{Im}(\varphi_p)$ for each p .

Proof. We will show the following statements.

- $\ker(\varphi_p) \subset (\ker \varphi)_p$: Suppose $s_p = \langle U, s \rangle \in \ker(\varphi_p)$, then $\varphi_p \langle U, s \rangle = \langle U, \varphi(U)s \rangle = 0$, so there exists $W \subset_{\text{open}} U$ such that
$$(\varphi(U)s)|_W = \varphi(W)(s|_W) = 0,$$
thus $s|_W \in \ker(\varphi(W))$ and $s_p = \langle W, s|_W \rangle \in (\ker \varphi)_p$.
- $(\ker \varphi)_p \subset \ker(\varphi_p)$: Suppose $\langle U, s \rangle \in (\ker \varphi)_p$, then $\varphi_p \langle U, s \rangle = \langle U, \varphi(U)s \rangle = \langle U, 0 \rangle = 0 \in \mathcal{G}_p$, thus $\langle U, s \rangle \in \ker(\varphi_p)$.
- $\operatorname{Im}(\varphi_p) \subset (\operatorname{Im} \varphi)_p$: Suppose $t = \varphi_p s_p \in \operatorname{Im}(\varphi_p)$ and $s_p = \langle U, s \rangle$, then $t = \varphi_p s_p = \varphi_p \langle U, s \rangle = \langle U, \varphi(U)s \rangle \in (\operatorname{Im} \varphi)_p$.
- $(\operatorname{Im} \varphi)_p \subset \operatorname{Im}(\varphi_p)$: Suppose $t = \langle U, \varphi(U)s \rangle \in (\operatorname{Im} \varphi)_p$, then $t = \rho'_{U,p} \circ \varphi(U)s = \varphi_p s_p \in \operatorname{Im}(\varphi_p)$.

□

Corollary 1. For any morphism of sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, it is injective (respectively, surjective) if and only if φ_p is injective (respectively, surjective) for all p .

Proof. According to Proposition 1 and Proposition 5, $\ker \varphi = 0$ if and only if $(\ker \varphi)_p = \ker(\varphi_p) = 0$. Similarly, $\operatorname{Im} \varphi \cong \mathcal{G}$ if and only if $(\operatorname{Im} \varphi)_p = \operatorname{Im}(\varphi_p) \cong \mathcal{G}_p$. □

Corollary 2. A morphism of sheaves is an isomorphism if and only if it is injective and surjective.

Proof. A morphism of sheaves φ is an isomorphism if and only if φ_p is an isomorphism for all p . As a morphism of groups, φ_p is an isomorphism for all p if and only if it is injective and surjective for all p , and according to Corollary 1, if and only if φ is injective and surjective. □

Definition 7. Let $f : X \rightarrow Y$ be a continuous map of topological spaces. For any sheaf \mathcal{F} on X , we define the direct image sheaf $f_* \mathcal{F}$ on Y by $(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ for any open set $U \subset Y$. For any sheaf \mathcal{G} on Y , we define the inverse image sheaf $f^{-1} \mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$, where U is any open set in X , and the limit is taken over all open sets $V \subset Y$ containing $f(U)$.

Especially, if $f(U)$ is an open set, then $\varinjlim_{V \supset f(U)} \mathcal{G}(V) = \mathcal{G}(f(U))$.