Hw 4 . Section 1.8

1. Problem 10

x in R^4 $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

$Ax = 0$

$$\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 0 \\ 0 & \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 + 3x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_4 = 0 \end{matrix} \Rightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

2. Problem 12

$$\begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -9 & 14 \end{bmatrix}$$

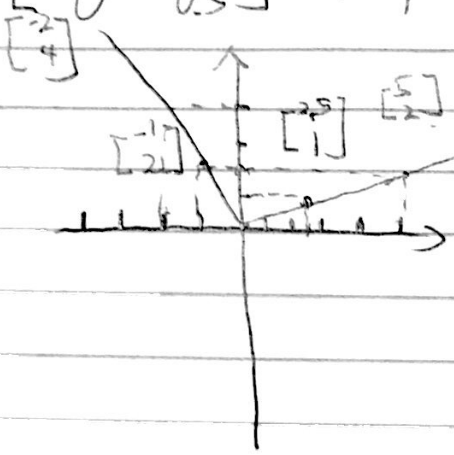
$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 17 \end{bmatrix} \Rightarrow 0 \neq 17$$

Not consistent. Therefore, b is not in range of the linear transformation $x \mapsto Ax$.

3. Problem 14

$$T(u) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

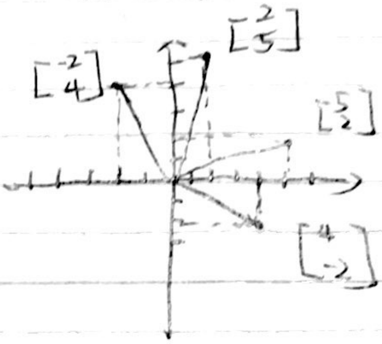
$$T(v) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



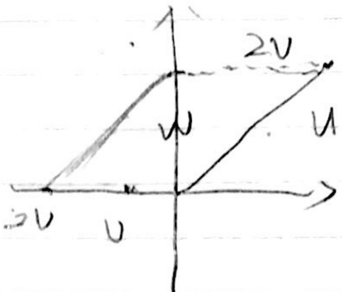
4. Problem 16

$$T(u) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

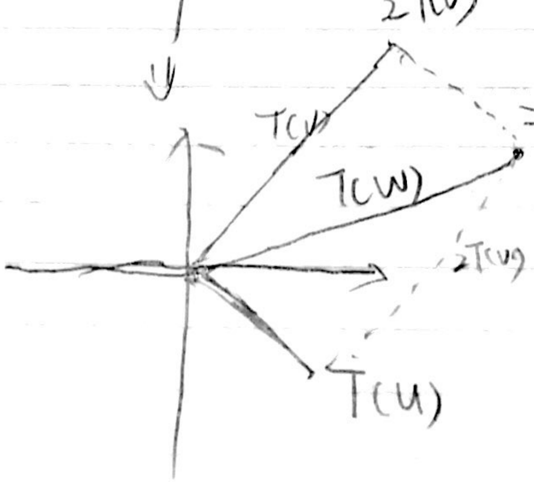


5. problem 18



$\Rightarrow$ $w = u + 2v$

$\Downarrow$



$\Rightarrow$ $T(w) = T(u + 2v)$

$= T(u) + 2T(v)$

6. Problem 20

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $v_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ $v_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$$T(x) = x_1 v_1 + x_2 v_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix} x_1 + \begin{bmatrix} 7 \\ -3 \end{bmatrix} x_2$$

$$= \begin{bmatrix} -2x_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} 7x_2 \\ -3x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2x_1 + 7x_2 \\ 5x_1 - 3x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$$

Section 1.9

7. Problem 2

$T: R^3 \Rightarrow R^2$

$[\, T(e_1) \;\; T(e_2) \;\; T(e_3) \,]$

$$\begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$$

8. Problem 4

$T: R^2 - R^2$

Rotate $-\frac{\pi}{4}$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin\frac{\pi}{4} \\ \sin(-\frac{\pi}{4}) & \cos-\frac{\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

9. Problem 6

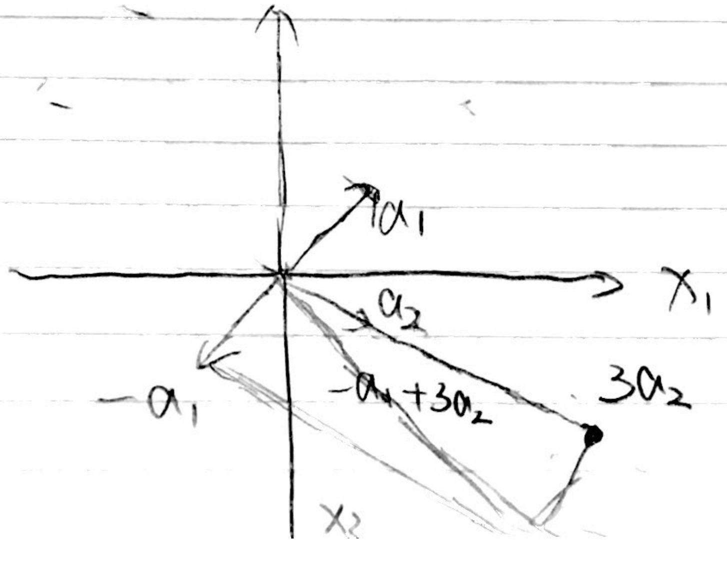
$e_1 \rightarrow$ unchange $\rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e_2 \rightarrow e_2 + 3e_1 \rightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

10. Problem 14

$$T\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = [a_1 a_2]\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -a_1 + 3a_2$$



11. Problem 18.

$T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

$$T(x) = \begin{bmatrix} 2x_2 - 3x_1 \\ x_1 - 4x_2 \\ 0 \\ 0 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

12. Problem 20

$T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4 \quad (T: R^4 \to R)$

$$A = \boxed{\begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 2x_1 + 3x_3 - 4x_4$$

13. Problem 22

$T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$

$T(x) = (-1, 4, 9)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Problem 10.

From: city suburban To:

$$\begin{bmatrix} 0.94 & 0.04 \\ 0.06 & 0.96 \end{bmatrix} \begin{array}{l} \text{city} \\ \text{suburban} \end{array}$$

$$X_0 = \begin{bmatrix} 10,000,000 \\ 800,000 \end{bmatrix}$$

2015 - 2017

$X_{k+1} = MX_k$

$$2016: Mx_1 = \begin{bmatrix} 0.94 & 0.04 \\ 0.06 & 0.96 \end{bmatrix} \begin{bmatrix} 10,000,000 \\ 800,000 \end{bmatrix} = \begin{bmatrix} 9432000 \\ 1368000 \end{bmatrix}$$

$$2017: Mx_2 = \begin{bmatrix} 0.94 & 0.04 \\ 0.06 & 0.96 \end{bmatrix} Mx_1 = \begin{bmatrix} 8920800 \\ 1879200 \end{bmatrix}$$

Therefore, in 2017, $\boxed{\text{city}}$ population will be $\boxed{8920800}$

$\boxed{\text{suburban}}$ population will be $\boxed{1879200}$

Problem 12.

From

$$\begin{array}{ccc} \text{Airport} & \text{East} & \text{West} \\ \end{array}$$
$$\begin{bmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.90 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{bmatrix} \begin{array}{l} \text{To} \\ \text{Airport} \\ \text{East} \\ \text{West} \end{array}$$

$$X_0 = \begin{bmatrix} 295 \\ 55 \\ 150 \end{bmatrix}$$

Mon thru Wed $k=2$

$$Mx_1 = \begin{bmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.90 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{bmatrix} \cdot \begin{bmatrix} 295 \\ 55 \\ 150 \end{bmatrix} = \begin{bmatrix} 303.9 \\ 57 \\ 139.1 \end{bmatrix}$$

$$Mx_2 = \begin{bmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.90 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{bmatrix} \cdot Mx_1 = \begin{bmatrix} 311.543 \\ 58.255 \\ 130.202 \end{bmatrix}$$

Therefore, there are approximately $\boxed{\text{312 car in airport}}$, $\boxed{\text{58 in East}}$

$\boxed{\text{130 in West}}$

10.

```
population = np.array([[0.94, 0.04],[0.06, 0.96]])

YearOf2015 = np.array([10000000, 800000])

population
Out[16]:
array([[0.94, 0.04],
       [0.06, 0.96]])

YearOf2015
Out[17]: array([10000000,   800000])

AfterOneYear = AxIP(population, YearOf2015)

AfterOneYear
Out[19]: array([9432000., 1368000.])

YearOf2017 = AxIP(population, AfterOneYear)

YearOf2017
Out[21]: array([8920800., 1879200.])
```

11.

```
RentCar = np.array([[0.97,0.05,0.10],[0.00,0.90,0.05],[0.03,0.05,0.85]])

InitialNum = np.array([295,55,150])

RentCar
Out[25]:
array([[0.97, 0.05, 0.1 ],
       [0.  , 0.9 , 0.05],
       [0.03, 0.05, 0.85]])

InitialNum
Out[26]: array([295,  55, 150])

newNumOfcar = AxIP(RentCar, InitialNum)

newNumOfcar
Out[28]: array([303.9,  57. , 139.1])
```

```
Finalnum = AxIP(RentCar, newNumOfcar)

Finalnum
Out[30]: array([311.543,  58.255, 130.202])
```

[ ] [ ]

Section 2.1

16.

Problem 16:

a. False

AB suppose to be a 3x3 matrix

right way to express it: $AB = [Ab_1 \quad Ab_2 \quad Ab_3]$

b. True

ith row of AB is the ith row of A times B

c. False $(AB)C \neq (AC)B$ matrix multiplication difference from algebra.

d. False $(AB)^T = B^T A^T$

e. True $(A+B)^T = A^T + B^T$

Pro 18.

$AB = [Ab_1 \quad Ab_2 \quad Ab_3 \quad \cdots \quad Ab_n]$

we already know $b_1 = b_2$

Hence $Ab_1 = Ab_2$

the first two columns of AB are equal too

Pro 20. Since the second column of AB equal A times the second column of B, the result will be $\underline{\underline{A \cdot 0 = 0}}$ since second column of B is zero.

$1/{}_{1}$ 8/20

Problem 22

When the columns of B is linearly dependent

$B = [\vec{b_1}, \cdots \vec{b_n}]$, scalar $c_1 \cdots c_n$

$c_1 b_1 + \cdots + c_n b_n = 0$ ①

$AB = A[b_1 \cdots b_n] = [Ab_1, \cdots, Ab_n]$

$A \cdot ① = c_1 A b_1 + c_2 A b_2 + c_3 A b_3 + \cdots + c_n A b_n = 0$

Hence, we get the column of AB are linearly dependent.

Problem 24

$b \to R^m$.

$AD = I_m$

$(AD)b = I_m b \Rightarrow A(Db) = b \to$ This show $x = Db$

Therefore, we can say $Ax = b$ has solution

→ Then we know there is solution in each row
In order to satisfy this, the number of column must greater or equal to number of row, since each row has to have a pivot.

→ Therefore, the matrix A cannot have more rows than columns, Problem 26

$D = [d_1 \ d_2 \ d_3]$

$AD = A[d_1 \ d_2 \ d_3]$

Assuming $AD = I = [e_1 \ e_2 \ e_3]$

$Ad_1 = e_1$

$Ad_2 = e_2$

$Ad_3 = e_3$

Since A span R^3, it must have solution It must has solution d_1, d_2, d_3 that satisfying $Ad_1 = e_1$, $Ad_2 = e_2$, $Ad_3 = e_3$

Therefore we find an $n \times 3$ matrix D such that $AD = I$.

Problem 28

$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ Now supposing $v^T = [v_1 \cdots v_n]$

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $u^T = [u_1 \cdots u_n]$

$u^T v = [u_1 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ ①

$v^T u = [v_1 \cdots v_n] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ ②

Both ① and ② equal to $u_1 v_1 + \cdots u_n v_n$

So, $\boxed{u^T v = v^T u.}$

$uv^T = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} [v_1 \cdots v_n] = \begin{bmatrix} u_1 v_1 & \cdots & u_1 v_n \\ \vdots & & \\ u_n v_1 & \cdots & u_n v_n \end{bmatrix}$

$vu^T = \begin{bmatrix} v_1 u_1 & \cdots & v_1 u_n \\ \vdots & & \\ v_n u_1 & \cdots & v_n u_n \end{bmatrix}$ Hence $\boxed{uv^T = (vu^T)^T}$

Problem35:

The answer of a:
```
[[0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]
 [0. 0. 0. 0. 0. 0.]]
```

The answer of b:
```
[[1. 1. 1. 1. 1.]
 [1. 1. 1. 1. 1.]
 [1. 1. 1. 1. 1.]]
```

The answer of c:
```
[[1. 0. 0. 0. 0. 0.]
 [0. 1. 0. 0. 0. 0.]
 [0. 0. 1. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 1.]]
```

The answer of d:
```
[[3 0 0 0 0]
 [0 5 0 0 0]
 [0 0 7 0 0]
 [0 0 0 2 0]
 [0 0 0 0 4]]
```

Problem36:

6x4 matrix with random entries in range [0,1):
```
[[0.16103371 0.63683049 0.8458108  0.45132324]
 [0.70622475 0.92945556 0.83406494 0.77650817]
 [0.43451614 0.05695421 0.20574734 0.09010021]
 [0.66178636 0.58901494 0.50433067 0.44288984]
 [0.45228542 0.97019494 0.14829375 0.80267466]
 [0.91964293 0.34483299 0.62650926 0.77417734]]
```

3x4 matrix with interger entries between -9 and 9:
```
[[ 8 -7 -6 -5]
 [-8 -9  6  1]
 [-9 -2 -9  6]]
```

Problem37:

In Example 1 random matrix A and Identity matrix I verified the formula (A+B)(A-B) - (A**2 - I) and make the zero matrix
```
[[0. 0. 0. 0.]
 [0. 0. 0. 0.]
 [0. 0. 0. 0.]
 [0. 0. 0. 0.]]
```
In Example 1 two matrics A and B fail to verified the formula (A+B)(A-B) - (A**2 - B**2) and does not make the zero matrix
```
[[ -31   40   56   97]
 [-120   20   93   40]
 [  44  -97  -10  -55]
 [ -66   60    3   21]]
```
In Example 2 random matrix A and Identity matrix I verified the formula (A+B)(A-B) - (A**2 - I) and make the zero matrix
```
[[0. 0. 0. 0.]
 [0. 0. 0. 0.]
 [0. 0. 0. 0.]
 [0. 0. 0. 0.]]
```
In Example 2 two matrics A and B fail to verified the formula (A+B)(A-B) - (A**2 - B**2) and does not make the zero matrix
```
[[  55  -11  -42   -4]
 [-115  -22  -43    8]
 [ -54  107  -39 -102]
 [ -25   -8   -9    6]]
```
In Example 3 random matrix A and Identity matrix I verified the formula (A+B)(A-B) - (A**2 - I) and make the zero matrix
```
[[0. 0. 0. 0.]
 [0. 0. 0. 0.]
 [0. 0. 0. 0.]
 [0. 0. 0. 0.]]
```
In Example 3 two matrics A and B fail to verified the formula (A+B)(A-B) - (A**2 - B**2) and does not make the zero matrix
```
[[  39  -88   41 -125]
 [ 116  168    0   14]
 [  76  -78 -111   88]
 [ -10  -36   40  -96]]
```