Section 2.8

1. ① problem 6.

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = u$$

$$\begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & -7 & -14 & 9 \\ 0 & -5 & -10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 23 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

The system is not consistent, hence $u$ is not in subspace of $\mathbb{R}^4$ generated by $\{v_1, v_2, v_3\}$

② problem 8.

$$[A\ p] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

consistent

Therefore, $p$ is in Col $A$

Problem 10

$u = (-2, 3, 1)$

$$u^T = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

$$Au^T = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix} \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence, the vector $u = (-2\ 3\ 1)$ is in Nul A

Problem 12

the value, $p = 3$ and $q = 4$, because Nul A is a subspace of $\mathbb{R}^3$, col A is a subspace of $\mathbb{R}^4$.

Problem 14

$Ax = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = \frac{1}{3}x_3 \\ x_2 = -\frac{5}{3}x_3 \\ x_3 \text{ is free} \end{cases} \rightarrow \text{CONT}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_3 \\ -\frac{5}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \end{bmatrix}$$

$\Rightarrow$ a nonzero vector in Nul A is $\begin{bmatrix} \frac{1}{3} & -\frac{5}{3} & 1 \end{bmatrix}^T$

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-3} & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system have pivot in columns 1 and 2.

Hence $(1\ 4\ -5\ 2)^T$ or $(2\ 5\ -1\ 7)^T$ is non-zero vector in Col A

Problem 16 determine in $R^2$

$$V_1 = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \quad V_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\Rightarrow V_1 = -2V_2$$

Hence, $v_1$ and $v_2$ are linearly dependent since one is multiple of another. Therefore, $V_1$ and $V_2$ are not bases for $R^2$

Problem 20 determin in $R^3$

The vectors are linearly dependent because it has 4 vectors, in each vector the number of entries is 3. When the system have more vectors than the number of entries, the set is linearly dependent. Hence, the vectors are not base for $R^3$

Problem 24

$$A = \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A\ 0] = \begin{array}{c} \begin{bmatrix} \boxed{1} & 3 & 6 & 9 & 0 \\ 0 & 0 & \boxed{1} & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \end{array} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - \frac{3}{2}x_4 \\ x_2 \\ -\frac{5}{4}x_4 \\ x_4 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 4x_4 \begin{bmatrix} 6 \\ 0 \\ 5 \\ -4 \end{bmatrix}$$

$$\text{Nul } A = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 5 \\ -4 \end{bmatrix}$$

The pivot columns of A are column 1 and 3.

$b_2 = -3b_1 + 0b_3$

$b_4 = \frac{3}{2}b_1 + \frac{5}{4}b_3$

Therefore, the basis for Col A is $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$

Problem 26

$$A=\begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A\ 0] = \begin{bmatrix} 3 & -1 & 7 & 0 & 6 & 0 \\ 0 & 1 & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 3 & 0 & \frac{5}{2} & 0 \\ 0 & \boxed{1} & 2 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 - 2.5x_5 \\ -2x_3 - 1.5x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2.5 \\ -1.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Basis Nul A $\begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2.5 \\ -1.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Matrix A have pivot in column 1, 2, 4

Basis for col A = $\begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}$

2. Section 4.5

Problem 20

a) False, they are not subset of another.

b) False. It equals to the number of free Variable

c). False, it's impossible to span an infinite dimensional space by a finite set.

d). False, S need to have n element or S need to be linearly independent.

e). True. It spanned by three linearly independent vectors, it must be all of $R^3$

3. Section 2.9

Problem 10

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad [A\ 0] = \begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 0 & 0 \\ 0 & \boxed{1} & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

basis for Nul A is $\begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ the dimension is 2.

It has pivot at column 1, 2, 4. Then basis for

Col A is $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, the dimension is 3.

Problem 12

It has pivot at column 1, 3, 5

Col A $\begin{bmatrix} 1 \\ 5 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ -6 \end{bmatrix}$, dim = 3

$$[A\ 0] = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & -4 & 3 & 3 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} -2x_2 + 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Nul A $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, dim = 2

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Problem 14

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 3 & -6 & -9 & 15 \\ 0 & -4 & 8 & 12 & -20 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 2 & 0 & -1 & 3 \\ 0 & \boxed{-1} & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix} \quad \dim = 2$$

Problem 20

$$\text{rank } A + \dim \text{Nul} A = n$$

$$\text{rank} A = 5 - \dim \text{Nul} A$$
$$= 5 - 3$$
$$= 2$$

Problem 24

4x3 matrix rank = 1, which means have one-demenstra

column space

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}$$

4. Section 4.6

Problem 14

A 4x3

the demension of the row space of matrix A is equal to rank → equals to number of pivot

Both matrix 3X4 or 3X4 can have 3 pivot position (max), so the largest possible value for dim Row A is 3.

Problem 16

rank $A \leq 4$

When rank is the largest dim Nul A is smaller
Hence rank $A = 4$, Nul A $= 0$ (dim)

Problem 18.

a. False. The basis for the column space of A is those columns of A which have pivot position

b. False. row operation don't presere the linear dependence relation.

c. True. that are not pivot columns represent free variable, which represent dimension of null space

d. True. $A^T$ is formed by interchange the rows and columns.

e. True. when they are equivalent, you can find same echelon form.

5. Section 5.1

Problem 6

$$Av = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -2v$$

$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is eigenvector of A, and eigenvalue is -2.

Problem 8

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \lambda = 3$$

$$A-3I = \begin{bmatrix} 1-3 & 2 & 2 \\ 3 & -2-3 & 1 \\ 0 & 0 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$$

Hence, $\lambda = 3$ is an eigenvalue of A

$$Nul(A-3I_3) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

an eigenvector is $[3 \; 2 \; 1]^T$

Problem 10

$$A-4I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} A-4I & 0 \end{bmatrix} = \begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

A basis vector could be $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Problem 12

For $\lambda = 1$:

$$A-I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} A-I & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

For $\lambda = 5$:

$$\begin{bmatrix} A-5I & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Problem 14

$$A-(-2)I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix}$$

$$[A+2I \quad 0] = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & -1 & \frac{1}{3} & 0 \\ 0 & -13 & \frac{13}{3} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & -1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_3 \\ \frac{1}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Problem 22:

a. False, the vector has to be "non-zero

b. False, the linearly independent Eigenvectors may be correspond to repeated Eigenvalue

c. True, because A is a stochastic matrix

d. False, it's only Opply on triangular matrix

e. True, the eigenspace corresponding to $\lambda$ is the null space of the matrix $A-\lambda I$

Problem 24

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)^2 - 0 = 0$$

$$\lambda = 1$$

Problem 30

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let $a+b = s$
$c+d = s$

$$\Rightarrow 1 \cdot \begin{bmatrix} a \\ c \end{bmatrix} + 1 \cdot \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix}$$

$$\Rightarrow A \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ \vdots \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Hence, $s$ should be eigenvalue of $A$

1. Section 5.2

Problem 2

$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (5-\lambda)(5-\lambda) - 9$

$= \lambda^2 - 10\lambda + 16$

$= \boxed{(\lambda - 8)(\lambda - 2) = 0}$ (characteristic equation)

$\lambda = 8$ or $2$

The eigenvalues of A are 8 and 2

Problem 4.

$A = \begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (5-\lambda)(3-\lambda) - 12$

$= 15 - 5\lambda - 3\lambda + \lambda^2 - 12$

$= \lambda - 8\lambda + 3$

$\lambda - 8\lambda + 3 = 0$ (characteristic equation)

$\lambda = 4 \pm \sqrt{13}$

The eigenvalues of A are $4 \pm \sqrt{13}$.

Problem 22

a. False, $a_1, a_2, a_3$ need to be linearly independent, and matrix A is invertible.

b. False, $\det A^T = \det A$

c. True, exactly the theorem

d. False, Row operation usually change its eigenvalues

Problem 25

a). $Av_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} = v_1$ ①

$\det \begin{bmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{bmatrix} = (0.6-\lambda)(0.7-\lambda) - (0.3)(0.4)$

$= \lambda^2 - 1.3\lambda + 0.3$

$= (\lambda - 1)(\lambda - 0.3)$ ②

According to ① and ②, we know 1 and 0.3 is A's eigenvalues.

For $\lambda = 0.3$

$(A - 0.3I)x = 0$

$\begin{bmatrix} 0.3 & 0.3 & 0 \\ 0.4 & 0.4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for $\lambda = 0.3$.

b.)

$x_0 = v_1 + cv_2$

$\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow c = -\frac{1}{14}$

$\Rightarrow$ c will charge when $v_2$ charge

$\Rightarrow x_0 = v_1 - \frac{1}{14} v_2$

c) when $k=1$

$x_1 = A(v_1 + cv_2)$

$= v_1 + c \cdot 0.3 \cdot v_2$ $\quad \lambda = 1$ and $0.3$

when $k=2$

$x_2 = Ax_1$

$= A(v_1 + c \cdot 0.3 \cdot v_2)$

$= Av_1 + c(0.3) Av_2$

$= Av_1 + c(0.3)^2 v_2$

$\Rightarrow$ As we can see, as k increase

$x_k = v_1 + c(0.3)^k v_2$

9. $x_k = \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$ $x_{k+1} = Ax_k$ $f_k = f_{k-1} + f_{k-2}$

a).
$$\begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix} = A \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$$

$$\begin{bmatrix} f_k + f_{k-1} \\ f_k \end{bmatrix} = A \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix} = A \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

b)
$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\lambda - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$

c)ᵢ $\begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$

$A = PDP^{-1}$ $\quad D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$

We knew

① $x_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x_{k-1}$, $x_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x_1$ when $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

when the givening starting condition $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

which is $x_2$ in the given equation ①

d)ᵢ $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-2} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{k-2}$ $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{(-1+\sqrt{5})}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-2} = \begin{bmatrix} \frac{1+\sqrt{5}}{2}^{k-1} & \frac{1-\sqrt{5}}{2}^{k-1} \\ \frac{(1+\sqrt{5})}{2}^{k-2} & \frac{(1-\sqrt{5})}{2}^{k-2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{\sqrt{5}+1}{2\sqrt{5}} \end{bmatrix}$

$\Downarrow$

$$x_k = \frac{\begin{bmatrix} (\frac{1+\sqrt{5}}{2})^k - \frac{(1-\sqrt{5})}{2}^k \\ (\frac{1+\sqrt{5}}{2})^{k-1} - (\frac{1-\sqrt{5}}{2})^{k-1} \end{bmatrix}}{\sqrt{5}}$$

$$\Rightarrow f_k = \frac{(\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k}{\sqrt{5}} \quad \text{(closed form)}$$

$k=40$

e). $f_{40} = 1258626902.5$

$f_{50} = 154800875592.0$

$f_{60} = 19039249070913.5$

f) $f_0 = 1$ $\quad f_1 = 3$

$f_{40} = 228826127$

$f_k = x_1\left(\frac{1+\sqrt{5}}{2}\right)^k + x_2\left(\frac{1-\sqrt{5}}{2}\right)^k$

$\Rightarrow x_1 = 4-2\sqrt{5}$

$x_2 = \frac{1-3\sqrt{5}}{2}$

1. Problem 38

```
A = np.array([[9,-4,-2,-4],\
              [-56,32,-28,44],\
              [-14,-14,6,-14],\
              [42,-33,21,-45]])
  eigVals = np.linalg.eigvals(A)
  print("Eigenvalue: " + str(eigVals))
  for eigVal in eigVals:
      print("Eigenvalue: " + str(eigVal))
      B = forwardElimination(np.subtract(A, (eigVal*np.eye(4,4))))
      Bresult = backsubstitution(B)
      print(Bresult)
```

Then, we get the matrix have Eigenvalue : [ 13. -12. -12. 13.]

For, eigenvalue = 13

```
[[ 1.    0.    0.5    -0.33333333]
 [ 0.    1.    0.     1.33333333]
 [ 0.    0.    0.     0.        ]
 [ 0.    0.    0.     0.        ]]
```

```
The eigenspace [[ -1    [1
                   0    -4
                   2     0
                   0 ] ,  3]]
```

For, eigenvalue = -12

```
[[ 1.    0.    -0.28571429  0.  ]
 [ 0.    1.    -1.          1.  ]
 [ 0.    0.     0.          0.  ]
 [ 0.    0.     0.          0.  ]]
```

```
The eigenspace [[ 2    [0
                  7    -1
                  7     0
                  0 ] ,  1]]
```

2. Problem 28

```
A = np.array([[4,-2,2,-4],[6,-4,2,-4],[4,-4,2,0],[3,-1,1,-3]])

B = A.tanspose()


np.poly(A)
Out[38]: array([ 1.,  1., -2., -4., -8.])

np.poly(B)
Out[39]: array([ 1.,  1., -2., -4., -8.])


np.linalg.eigvals(A)
Out[41]:
array([ 2. +0.        j, -2. +0.        j, -0.5+1.32287566j,
       -0.5-1.32287566j])

np.linalg.eigvals(B)
Out[42]:
array([ 2. +0.        j, -0.5+1.32287566j, -0.5-1.32287566j,
       -2. +0.        j])

np.linalg.eig(A)
Out[43]:
(array([ 2. +0.        j, -2. +0.        j, -0.5+1.32287566j,
        -0.5-1.32287566j]),
 array([[ 4.86664263e-01+0.00000000e+00j, -3.30868422e-16+0.00000000e+00j,
         -6.32455532e-01-1.06003958e-15j, -6.32455532e-01+1.06003958e-15j],
        [ 4.86664263e-01+0.00000000e+00j,  7.07106781e-01+0.00000000e+00j,
         -6.32455532e-01+0.00000000e+00j, -6.32455532e-01-0.00000000e+00j],
        [ 6.48885685e-01+0.00000000e+00j,  7.07106781e-01+0.00000000e+00j,
         -1.20599666e-15+8.16911248e-16j, -1.20599666e-15-8.16911248e-16j],
        [ 3.24442842e-01+0.00000000e+00j, -1.32347369e-16+0.00000000e+00j,
         -3.95284708e-01+2.09165007e-01j, -3.95284708e-01-2.09165007e-01j]]))

np.linalg.eig(B)
Out[44]:
(array([ 2. +0.        j, -0.5+1.32287566j, -0.5-1.32287566j,
        -2. +0.        j]),
 array([[ 5.77350269e-01+0.        j, -5.25000000e-01-0.06614378j,
         -5.25000000e-01+0.06614378j,  7.07106781e-01+0.        j],
```

```
       [-5.77350269e-01+0.        j,  2.50000000e-02-0.19843135j,
         2.50000000e-02+0.19843135j, -7.07106781e-01+0.        j],
       [ 5.77350269e-01+0.        j, -2.50000000e-02+0.19843135j,
        -2.50000000e-02-0.19843135j,  4.99600361e-16+0.        j],
       [-2.69268222e-16+0.        j,  8.00000000e-01+0.        j,
         8.00000000e-01-0.        j, -3.08086889e-15+0.        j]]]))
```

Hence, A and A^T have same characteristic polynomial in 4x4 matrix,and they have same eigenvectors.

In 5x5 matrix:

```
A =
[[ 2 -4  0  4  6]
 [ 7  5  6 -2 -5]
 [-6  1  1  5 -9]
 [-6  3 -1 -3  6]
 [-3  4  6 -8 -8]]
B = A.transpose() =

[[ 2  7 -6 -6 -3]
 [-4  5  1  3  4]
 [ 0  6  1 -1  6]
 [ 4 -2  5 -3 -8]
 [ 6 -5 -9  6 -8]]

np.poly(A)
Out[52]:
array([ 1.0000e+00,  3.0000e+00,  1.5400e+02, -7.0400e+02,  5.2500e+03,
       -3.3516e+04])

np.poly(B)
Out[53]:
array([ 1.0000e+00,  3.0000e+00,  1.5400e+02, -7.0400e+02,  5.2500e+03,
       -3.3516e+04])

np.linalg.eigvals(A)
Out[54]:
array([-4.29830383+11.76853336j, -4.29830383-11.76853336j,
        0.27710052 +6.50128875j,  0.27710052 -6.50128875j,
        5.04240662 +0.        j])
```

```
np.linalg.eigvals(B)
Out[55]:
array([-4.29830383+11.76853336j, -4.29830383-11.76853336j,
        0.27710052 +6.50128875j,  0.27710052 -6.50128875j,
        5.04240662 +0.        j])
np.linalg.eig(A)
Out[56]:
(array([-4.29830383+11.76853336j, -4.29830383-11.76853336j,
         0.27710052 +6.50128875j,  0.27710052 -6.50128875j,
         5.04240662 +0.        j]),
 array([[-0.25750083-0.02847662j, -0.25750083+0.02847662j,
         -0.08753688-0.42103583j, -0.08753688+0.42103583j,
          0.07843912+0.        j],
        [ 0.3816878 +0.17769657j,  0.3816878 -0.17769657j,
         -0.43800227-0.11488919j, -0.43800227+0.11488919j,
         -0.76927982+0.        j],
        [-0.04998059+0.32813869j, -0.04998059-0.32813869j,
          0.72767844+0.        j,  0.72767844-0.        j,
         -0.42635316+0.        j],
        [-0.02709499-0.5350666 j, -0.02709499+0.5350666 j,
          0.09916983+0.16951241j,  0.09916983-0.16951241j,
         -0.43102652+0.        j],
        [ 0.59867962+0.        j,  0.59867962-0.        j,
          0.12323406-0.16355108j,  0.12323406+0.16355108j,
         -0.18572825+0.        j]]))

np.linalg.eig(B)
Out[57]:
(array([-4.29830383+11.76853336j, -4.29830383-11.76853336j,
         0.27710052 +6.50128875j,  0.27710052 -6.50128875j,
         5.04240662 +0.        j]),
 array([[-0.17206792+0.28092814j, -0.17206792-0.28092814j,
          0.7939273 +0.        j,  0.7939273 -0.        j,
          0.13133394+0.        j],
        [-0.12157828-0.19753994j, -0.12157828+0.19753994j,
         -0.01975884+0.28530709j, -0.01975884-0.28530709j,
          0.6872921 +0.        j],
        [-0.22154135-0.16195589j, -0.22154135+0.16195589j,
          0.0466213 -0.31958223j,  0.0466213 +0.31958223j,
          0.41933603+0.        j],
        [ 0.12245432+0.5638856 j,  0.12245432-0.5638856 j,
         -0.02022402-0.10123825j, -0.02022402+0.10123825j,
```

```
  0.47842619+0.        j],
 [ 0.65528345+0.        j,  0.65528345-0.        j,
  0.35705381-0.2131594 j,  0.35705381+0.2131594 j,
 -0.32503328+0.        j]]))
```

Hence, comparing above data, in 5x5 matrix, have same characteristic polynomial in 4x4 matrix, and they have same eigenvectors.

A.

```
[-0.08685593  0.16575188  0.98685593  1.03424812  0.98685593 -0.08685593]
```

```
(array([-0.08685593,  0.16575188,  0.98685593,  1.03424812,  0.98685593,
       -0.08685593]), array([[-0.80953351,  0.56906708,  0.07480866, -0.13459823, -0.05660621,
         0.1233445 ],
       [ 0.10640568, -0.09744738, -0.81306232, -0.56144158,  0.61522796,
        -0.01621249],
       [ 0.40476675,  0.56906708, -0.03740433, -0.13459823,  0.07065938,
        -0.75456328],
       [-0.05320284, -0.09744738,  0.40653116, -0.56144158, -0.76796566,
         0.09918035],
       [ 0.40476675,  0.56906708, -0.03740433, -0.13459823, -0.01405317,
         0.63121878],
       [-0.05320284, -0.09744738,  0.40653116, -0.56144158,  0.1527377 ,
        -0.08296786]]))
```

B.