1. Chapter 5.3

① problem 4

$$\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3-4}\begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$A = PDP^{-1}$

$A^2 = (PDP^{-1}) \cdot (PDP^{-1})$

$= PD \cdot P^{-1} \cdot P \cdot D \cdot P^{-1}$

$= PD^2P^{-1}$ ①

$\Rightarrow$ From equation ① we know

$A^k = PD^kP^{-1}$

$$\underbrace{\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}}_{P^{-1}}$$

$$\Rightarrow \underbrace{\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}^k}_{A^k} = \underbrace{\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix}}_{D^k} \underbrace{\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}}_{P^{-1}}$$

$$A^k = \begin{bmatrix} -3 \cdot 2^k + 4 & 12 \cdot 2^k - 12 \\ -2^k + 1 & 4 \cdot 2^k - 3 \end{bmatrix}$$

$3 \cdot 2^k \quad 4^k$

2^k

② problem 6

$A = PDP^{-1}$

$$\begin{array}{cccc} A & P & D & P^{-1} \\ \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} = & \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix} \end{array}$$

By The Diagonalization Theorem, in $A = PDP^{-1}$, the diagonal entries of D are eigenvalue of A, P is eigenvector,

Hence, eigenvalue is 5, 4 ; eigenvector is

$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$; Since 5 repeated, multiplicity 2 the eigenvector for

$\lambda = 5$ is $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; for $\lambda = 4$, is $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

③ problem 8

$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$, This is a triangular matrix, eigenvalue are its entries, which is 5 ; multiplicity 2.

$(A - \lambda I)X = 0$

$\begin{bmatrix} 5-5 & 1 \\ 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = 5$, eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

dimension is 1. $1 \neq 2$

Hence, matrix A is not diagonalizable.

④ Problem 10

$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

$P = A - I\lambda$

$= \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$

$\Rightarrow (2-\lambda)(1-\lambda) - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda+2)(\lambda-5) = 0$

$\lambda = -2$ or 5

For $\lambda = -2$

$A - (-2)I = A + 2I$

$= \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & 3 & 0 \\ 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \Rightarrow$ eigenvector

For $\lambda = 5$

$A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -3 & 3 & 0 \\ 4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ eigenvector

$P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}$ $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$

According to Theorem 6, An nxn matrix with n distinct eigenvalues is diagonalizable, in the matrix A it has TWO λ, then A is $\boxed{\text{diagonalizable}}$

A ≠ PD

⑤ problem 12

$$A=\begin{bmatrix}4&2&2\\2&4&2\\2&2&4\end{bmatrix} \quad \lambda=2,8$$

For $\lambda=2$

$$A-2I=\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}$$

$$\begin{bmatrix}2&2&2&0\\2&2&2&0\\2&2&2&0\end{bmatrix}\sim\begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\Rightarrow x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_2-x_3\\x_2\\x_3\end{bmatrix}$$

$$=x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

For $\lambda=8$

$$A-8I=\begin{bmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{bmatrix}$$

$$\begin{bmatrix}-4&2&2&0\\2&-4&2&0\\2&2&-4&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$$

$$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_3\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$$

eigenvector $\Rightarrow \overset{v_1}{\begin{bmatrix}-1\\1\\0\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}, \overset{v_3}{\begin{bmatrix}1\\1\\1\end{bmatrix}}$, and $A=PDP^{-1}$

$P=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}$, $D=\begin{bmatrix}2&0&0\\0&2&0\\0&0&8\end{bmatrix}$, entries of D correspond to v_1, v_2, v_3 respectively. Hence, the matrix A is diagonalizable

(b) Problem 14

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad \lambda = 5, 4$$

for $\lambda = 5$

$$A - 5I = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

for $\lambda = 4$

$$A - 4I = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } A = PDP^{-1}$$

D correspond to v_1, v_2, v_3 respectively

Hence, diagonalizable

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⑦ Problem 22

a. False, the eigenvectors have to be linearly independent.

b. False, the matrix might have counting multiplicities.
For example, $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, only have two eigenvalues, 2 counting multiplicities, but still diagonalizable.

c. True, in $A = PDP^{-1}$, D is diagonal, the column of P is linearly independent eigenvectors of A.

d. False, A is diagonalizable when A has n linearly independent eigenvectors; invertible or not is not directly affect diagonalizable

⑧ Problem 24

matrix A is not diagonalizable, since there are only two linearly independent eigenvectors exist, each space is one-d and matrix is 3×3. The sum of the dimensions of the distinct eigenvectors need to equal 3.

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2. Chapter 6.1 $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ $w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ $x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$

① problem 6

$$\left(\frac{x \cdot w}{x \cdot x}\right) \cdot x$$

$$x \cdot w = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$$

$$= 6 \cdot 3 + 2 + (-15)$$

$$= 5$$

$$x \cdot x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$= 6 \cdot 6 + 4 + 9$$

$$= 49$$

$$\Rightarrow \frac{5}{49} \cdot \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{30}{49} \\ -\frac{10}{49} \\ \frac{15}{49} \end{bmatrix}$$

② Problem 8

$$\|x\| = \sqrt{x \cdot x}$$

$$= 7$$

$x \cdot x = 49$ (from problem 6)

③ problem 10

$$v = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

$$v \cdot v = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}^T \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

$$= 36 + 16 + 9 = 61$$

First,

$$\|v\| = \sqrt{v \cdot v}$$

$$= \sqrt{61}$$

Unit vector $u = \frac{1}{\|v\|} \cdot v = \frac{1}{\sqrt{61}} \cdot \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{\sqrt{61}} \\ \frac{4}{\sqrt{61}} \\ \frac{3}{\sqrt{61}} \end{bmatrix}$

Problem 12

$$v = \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}$$

$$v \cdot v = \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}^T \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}$$

$$= \frac{8}{3} \cdot \frac{8}{3} + 4$$

$$= \frac{100}{9}$$

$$\|v\| = \sqrt{v \cdot v}$$

$$= \sqrt{\frac{100}{9}}$$

$$= \frac{10}{3}$$

unit vector $u = \frac{1}{\|v\|} \cdot v$

$$= \frac{1}{\frac{10}{3}} \cdot \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

Problem 14

distance between $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ $z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$

$$\text{dis} = \|u - z\|$$

$$= \sqrt{(u-z) \cdot (u-z)}$$

$$= \sqrt{68} = \boxed{2\sqrt{17}}$$

$$u - z = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$$

$$(u-z) \cdot (u-z) = \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix}^T \cdot \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix}$$

$$= 16 + 16 + 36 = 68$$

Problem 16

$$u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$$ orthogonal?

$$v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$u \cdot v = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}^T \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$= 12 \cdot 2 + 3 \cdot (-3) + (-5) \cdot 3$$

$$= 24 - 9 - 15$$

$$= 0$$

Hence, u and v are orthogonal.

Problem 18

$$y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix} \quad z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

$$y \cdot z = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$$

$$= -3 - 56 + 60 + 0$$

$$= 1$$

Hence, y and z not orthogonal.

Problem 20

a. True, dot product is commutative

b. False, $\|cv\| = |c| \cdot \|v\|$, c should be faund absolute value.

c. True, this is one of definition of $W^{\perp}$

d. True, this is Pythagorean Theorem

e. True, this is Theorem 3 $(\text{Row } A)^{\perp} = \text{Nul } A$

Problem 28

y is orthogonal to every w in Span{u, v}

$\Rightarrow y \cdot u = 0 \;;\; y \cdot v = 0$

Let $w \in \text{Span}\{u, v\}$

$w = c_1 u + c_2 v$

$w \cdot y = (c_1 u + c_2 v) \cdot y$

$= c_1 u \cdot y + c_2 \cdot v \cdot y$

$= c_1 (0) + c_2 (0)$

$= 0$

Hence, y is orthogonal to w in Span{u, v}

Chapter 6.2

1. problem 2

$$u = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad x = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

$$u \cdot v = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= 0 - 2 + 2 = 0$$

u and v are orthogonal

$$u \cdot x = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

$$= -5 + 4 + 1$$

$$= 0$$

u and x are orthogonal

$$v \cdot x = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

$$= -2 + 2$$

$$= 0$$

v and x are orthogonal

Each pair are orthogonal, this is a orthogonal set

Problem 4

$u = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, z = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

$u \cdot v = \begin{bmatrix} 2 & -5 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$= 0$ (u and v are orthogonal)

$v \cdot z = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = 0$

v and z are orthogonal

$u \cdot z = \begin{bmatrix} 2 & -5 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = 8 + 10 - 18 = 0$

u and z are orthogonal

Hence, this is an orthogonal set.

Problem 6

$u = \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} \quad z = \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$

$u \cdot v = \begin{bmatrix} 5 & -4 & 0 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} = -20 - 4 + 24 = 0$

$v \cdot z = \begin{bmatrix} -4 & 1 & -3 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 5 \\ 1 \end{bmatrix} = -12 + 3 - 15 - 8 = -32$

Hence, this is not an orthogonal set.

Problem 8

$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \quad x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$u_1 \cdot u_2 = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = -6 + 6 = 0$

u_1, u_2 are an orthogonal set

$\Rightarrow u_1, u_2$ should be linearly independent, $\{u_1, u_2\}$ is a basis for $\mathbb{R}^2$.

$x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$u_1 x = -18 + 3 = -15$

$u_1 \cdot u_1 = 10$

$u_2 \cdot x = 9 + 1 = 10$

$u_2 \cdot u_2 = 36 + 4 = 40$

$x = \left(\frac{u_1 x}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{u_2 \cdot x}{u_2 \cdot u_2} u_2\right)$

$= \frac{-15}{10} \cdot u_1 + \frac{30}{40} \cdot u_2$

$= -\frac{3}{2} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3}{4} \cdot \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

Problem 10

$u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \quad x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

$u_1 \cdot u_2 = \begin{bmatrix} 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 6 - 6 + 0 = 0$

$u_1 \cdot u_3 = \begin{bmatrix} 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 3 - 3 = 0$

$u_1 x = 15 + 9 + 0 = 24$

$u_2 \cdot u_3 = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2 + 2 - 4 = 0$

$u_1 \cdot u_1 = 9 + 9 + 0 = 18$

$u_2 x = 10 - 6 - 1 = 3$

$u_2 \cdot u_2 = 9$

$u_3 x = 5 - 3 + 4 = 6$

$u_3 \cdot u_3 = 1 + 1 + 16 = 18$

$\{u_1, u_2, u_3\}$ is an orthogonal set, $\Rightarrow$ linearly independent, they are basis for $\mathbb{R}^3$.

$x = \left(\frac{u_1 x}{u_1 \cdot u_1}\right) \cdot u_1 + \left(\frac{u_2 x}{u_2 \cdot u_2}\right) \cdot u_2 + \left(\frac{u_3 x}{u_3 \cdot u_3}\right) \cdot u_3$

$= \frac{24}{18} \cdot u_1 + \frac{3}{9} \cdot u_2 + \frac{6}{18} \cdot u_3 = \frac{4}{3} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

Problem 12

$$\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u}$$

$$\vec{y} \cdot \vec{u} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= -1 - 3 = -4$$

$$\vec{u} \cdot \vec{u} = (-1) \cdot (-1) + 3 \cdot 3$$

$$= 10$$

$$= \frac{-4}{10} \cdot \vec{u}$$

$$= -\frac{2}{5} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} \frac{2}{5} \\ -\frac{6}{5} \end{bmatrix}$$

Problem 14

$$y \cdot u = \begin{bmatrix} 2 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 14 + 6 = 20$$

$$u \cdot u = 49 + 1 = 50$$

$$\hat{y} = \frac{y \cdot u}{u \cdot u} \cdot u$$

$$\hat{y} = \frac{20}{50} \cdot u = \frac{2}{5} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix}$$

$$y = \hat{y} + (y - \hat{y}) = \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix} + \begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix}$$

Problem 16

$$\hat{y} = \frac{y \cdot u}{u u} u$$

$$= \frac{15}{5} \cdot u$$

$$= 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\|y - \hat{y}\| = \sqrt{(y-\hat{y})(y-\hat{y})}$$

$$= \sqrt{45}$$

$$= \boxed{3\sqrt{5}}$$

$$y \cdot u = \begin{bmatrix} -3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= -3 + 18$$

$$= 15$$

$$u \cdot u = 1 + 4$$

$$= 5$$

$$y - \hat{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 3 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 3 \end{bmatrix} = 36 + 9 = 45$$

Problem 24

a. False, orthogonal implies linearly independent.

b. False, not every orthogonal set contain unit vectors

c. True, this is theorem 7. $\|Ux\| = \|x\|$ (orthogonal columns)

d. True, $\frac{y \cdot (cv)}{(cv) \cdot (cv)} \cdot (cv) = \frac{c^2 (y \cdot v)}{c^2 (v \cdot v)} v$

e. True, the columns are linearly independent, hence, invertible.

Problem 28

$I = UU^{-1} = UU^T$

Since U is an $n \times n$ orthogonal matrix, U^T has orthonormal columns which means these columns are linearly independent. By invertible matrix theorem, the columns of A span $\mathbb{R}^n$.

Thus, the columns of U^T form a basis for $\mathbb{R}^n$.

Hence, the rows of U form an orthonormal basis for $\mathbb{R}^n$.

Problem 30

Since the matrix U is an orthogonal matrix,

$e_i \cdot e_j = 0$ (for any i, j value)

V is formed by interchanging some columns of U.

the orthonormality won't change, since $e_i \cdot e_j = 0$ won't change.

Hence, V is an orthonormality matrix.

Problem 34

$x, y \in \mathbb{R}^3$, $c \in \mathbb{R}$

$T(x+y) = \text{refl}_L(x+y) = 2\,\text{proj}_L(x+y) - x - y = 2\dfrac{(x+y)\cdot u}{u \cdot u} \cdot u - x - y$

$= 2 \cdot \dfrac{xu + yu}{u \cdot u} \cdot u - x - y = 2\dfrac{x \cdot u}{u \cdot u} \cdot u - x + 2\dfrac{y \cdot u}{u \cdot u} \cdot u - y$

$= 2\,\text{proj}_L(x) - x + 2\,\text{proj}_L(y) - y = \text{refl}_L(x) + \text{refl}_L(y) = T(x) + T(y)$

$T(cx) = \text{refl}_L(cx) = 2\,\text{proj}_L(cx) - cx = 2c\dfrac{x \cdot u}{u \cdot u} \cdot u - cx$

$= c\left(2\dfrac{x \cdot u}{u \cdot u} \cdot u - x\right) = c(2\,\text{proj}_L(x) - x) = c\,\text{refl}_L(x) = cT(x)$

Hence, T is a linear transformation.

5.3

Problem 34

Find eigenvectors

$$A = \begin{bmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{bmatrix}$$

eigenvalue $= (-4, 24, 1, -4)$

8 28 76 80

For $\lambda = 4$ 26 16 80

$$A-(-4)\lambda = \begin{bmatrix} 4 & 13 & 8 & 4 \\ 4 & 13 & 8 & 4 \\ 8 & 6 & 16 & 8 \\ 0 & 5 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 13 & 8 & 4 & 0 \\ 4 & 13 & 8 & 4 & 0 \\ 8 & 6 & 16 & 8 & 0 \\ 0 & 5 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 13 & 8 & 4 & 0 \\ 0 & -20 & 0 & -4 & 0 \\ 0 & 5 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 13 & 2 & 1 & 0 \\ 0 & -20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 24$

$$A - 24\lambda = \begin{bmatrix} -24 & 13 & 8 & 4 \\ 4 & -15 & 8 & 4 \\ 8 & 6 & -12 & 8 \\ 0 & 5 & 0 & -28 \end{bmatrix} \Rightarrow \begin{bmatrix} -24 & 13 & 8 & 4 & 0 \\ 4 & -15 & 8 & 4 & 0 \\ 8 & 6 & -12 & 8 & 0 \\ 0 & 5 & 0 & -28 & 0 \end{bmatrix} \sim \begin{bmatrix} -24 & 13 & 8 & 4 & 0 \\ 0 & -\frac{77}{6} & \frac{28}{3} & \frac{14}{3} & 0 \\ 0 & \frac{31}{3} & -\frac{28}{3} & \frac{28}{3} & 0 \\ 0 & 5 & 0 & -28 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -24 & 13 & 8 & 4 & 0 \\ 0 & -\frac{77}{6} & \frac{28}{3} & \frac{14}{3} & 0 \\ 0 & 0 & -\frac{20}{11} & \frac{144}{11} & 0 \\ 0 & 0 & \frac{40}{11} & -\frac{286}{11} & 0 \end{bmatrix} \sim \begin{bmatrix} -24 & 13 & 8 & 4 & 0 \\ 0 & -\frac{77}{6} & \frac{28}{3} & \frac{14}{3} & 0 \\ 0 & 0 & \frac{40}{11} & -\frac{286}{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{28}{5} & 0 \\ 0 & 1 & 0 & -\frac{28}{5} & 0 \\ 0 & 0 & 1 & -\frac{36}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{28}{5}x_4 \\ \frac{28}{5}x_4 \\ \frac{36}{5}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{28}{5} \\ \frac{28}{5} \\ \frac{36}{5} \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 28 \\ 28 \\ 36 \\ 5 \end{bmatrix}$$

For $\lambda = 1$

$$A-\lambda = \begin{bmatrix} -1 & 13 & 8 & 4 \\ 4 & 8 & 8 & 4 \\ 8 & 6 & 11 & 8 \\ 0 & 5 & 0 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 13 & 8 & 4 & 0 \\ 4 & 8 & 8 & 4 & 0 \\ 8 & 6 & 11 & 8 & 0 \\ 0 & 5 & 0 & -5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & 6 & 11 & 8 & 0 \\ 0 & 5 & \frac{5}{2} & 0 & 0 \\ 0 & \frac{55}{4} & \frac{75}{8} & 5 & 0 \\ 0 & 5 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 8 & 6 & 11 & 8 & 0 \\ 0 & \frac{55}{4} & \frac{75}{8} & 5 & 0 \\ 0 & 0 & -\frac{10}{11} & -\frac{20}{11} & 0 \\ 0 & 0 & -\frac{75}{22} & -\frac{75}{11} & 0 \end{bmatrix} \sim \begin{bmatrix} 8 & 6 & 11 & 8 & 0 \\ 0 & \frac{55}{4} & \frac{75}{8} & 5 & 0 \\ 0 & 0 & -\frac{75}{22} & -\frac{75}{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_4 \\ -2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

Hence, matrix's eigenvector $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 28 \\ 28 \\ 36 \\ 5 \end{bmatrix},$

$\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ (I used my code check the correction later)

```
A = np.array([[0,13,8,4],[4,9,8,4],[8,6,12,8],[0,5,0,-4]])
Out[15]:
array([[ 0, 13,  8,  4],
       [ 4,  9,  8,  4],
       [ 8,  6, 12,  8],
       [ 0,  5,  0, -4]])
```

After we run,

```
eigen = np.linalg.eigvalues(A)

array([24., -4.,  1., -4.]),
```

For eigenvalue = -4

```
B = np.subtract(A, (-4.0*np.eye(4,4)))

B
Out[55]:
array([[ 4., 13.,  8.,  4.],
       [ 4., 13.,  8.,  4.],
       [ 8.,  6., 16.,  8.],
       [ 0.,  5.,  0.,  0.]])
```

Solve Ax= 0

```
zeros = np.zeros((4,1))
afterappend = np.append(B, zeros,axis=1)

C = forwardElimination(afterappend)
D = backsubstitution(C)
array([[ 1.,  0.,  2.,  1.,  0.],
       [ 0.,  1., -0., -0., -0.],
       [ 0.,  0.,  0.,  0.,  0.],
       [ 0.,  0.,  0.,  0.,  0.]])
```

X3 and x4 is free variable. Hence, after calculation, we can get

```
[[-2,    [[-1,
  0,       0
  1,       0
  0]]      1]]
```

```
Basis for the eigenspace of eigenvalue -4 is [[-2,   [[-1,
                                               0,      0
                                               1,      0
                                               0]]     1]]
```

For eigenvalue = 24

```
B = np.subtract(A, (24*np.eye(4,4)))

B
array([[-24.,  13.,   8.,   4.],
       [  4., -15.,   8.,   4.],
       [  8.,   6., -12.,   8.],
       [  0.,   5.,   0., -28.]])
```

Solve Ax=0

```
zeros = np.zeros((4,1))
afterappend = np.append(B, zeros,axis=1)

C = forwardElimination(afterappend)
D = backsubstitution(C)
array([[ 1. ,  0. ,  0. , -5.6,  0. ],
       [ 0. ,  1. ,  0. , -5.6,  0. ],
       [ 0. ,  0. ,  1. , -7.2, -0. ],
       [ 0. ,  0. ,  0. ,  0. ,  0. ]])
```

X4 is free, Hence, after calculation, we can get

```
[[28
  28
  36
  5]]
```

For eigenvalue = 1

```
B = np.subtract(A, (1*np.eye(4,4)))

B
array([[-1., 13.,  8.,  4.],
       [ 4.,  8.,  8.,  4.],
       [ 8.,  6., 11.,  8.],
```

```
       [ 0.,  5.,  0., -5.]])
```

Solve Ax=0

```
zeros = np.zeros((4,1))
afterappend = np.append(B, zeros,axis=1)

C = forwardElimination(afterappend)
D = backsubstitution(C)

array([[ 1.,  0.,  0., -1.,  0.],
       [ 0.,  1.,  0., -1.,  0.],
       [ 0.,  0.,  1.,  2.,  0.],
       [ 0.,  0.,  0.,  0.,  0.]])
```

X4 is free, Hence, after calculation, we can get

[[1
 1
 -2
 1]]

Overall, Basis for the eigenspace of eigenvalue -4 is

[[-2,	[[-1,	[[28 ,	[[1
0,	0	28	1
1,	0	36	-2
0]]	1]]	5]]	1]]

Thus, we construct P = [-2 -1 28 1
0 0 28 1
1 0 36 -2
0 1 5 1]

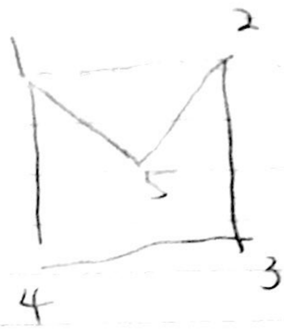
D = [-4 0 0 0
0 -4 0 0
0 0 24 0
0 0 0 1]

5.

a.

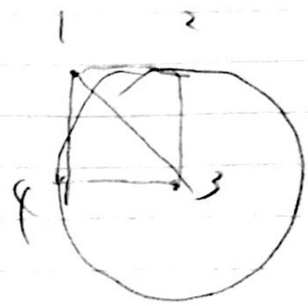
$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \\ 4 & & & & & \\ 5 & & & & & \end{array}$$

$$\begin{array}{cc} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left[\begin{array}{ccccc} 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \end{array}\right] \end{array}$$



b.

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{cccc} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right] \end{array}$$



6. a).

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \quad P' = \begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & \frac{1}{5} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$P'' = \begin{bmatrix} \frac{1}{500} & \frac{1}{500} & \frac{124}{125} & \frac{1}{500} & \frac{1}{5} \\ \frac{83}{250} & \frac{1}{500} & \frac{1}{500} & \frac{497}{1000} & \frac{1}{5} \\ \frac{83}{250} & \frac{1}{500} & \frac{1}{500} & \frac{497}{1000} & \frac{1}{5} \\ \frac{83}{250} & \frac{497}{1000} & \frac{1}{500} & \frac{1}{500} & \frac{1}{5} \\ \frac{1}{500} & \frac{497}{1000} & \frac{1}{500} & \frac{1}{500} & \frac{1}{5} \end{bmatrix}$$

b).

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad P' = \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

$$P'' = \begin{bmatrix} \frac{1}{6} & \frac{149}{300} & \frac{299}{1200} & \frac{1}{600} & \frac{1}{600} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{600} & \frac{299}{1200} & \frac{1}{600} & \frac{1}{600} & \frac{1}{6} \\ \frac{1}{6} & \frac{149}{300} & \frac{1}{600} & \frac{149}{300} & \frac{1}{600} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{600} & \frac{299}{1200} & \frac{1}{600} & \frac{149}{300} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{600} & \frac{299}{1200} & \frac{149}{300} & \frac{1}{600} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{600} & \frac{1}{600} & \frac{1}{600} & \frac{149}{300} & \frac{1}{6} \end{bmatrix}$$

6

a. P” =

```
Pa = np.array([[1./500, 1./500, 124./125, 1./500, 1./5],
            [83./250, 1./500, 1./500, 497./1000, 1./5],
            [83./250, 1./500, 1./500, 497./1000, 1./5],
            [83./250, 497./1000, 1./500, 1./500, 1./5],
            [1./500, 497./1000, 1./500, 1./500, 1./5]])
==[[0.002 0.002 0.992 0.002 0.2  ]
 [0.332 0.002 0.002 0.497 0.2  ]
 [0.332 0.002 0.002 0.497 0.2  ]
 [0.332 0.497 0.002 0.002 0.2  ]
 [0.002 0.497 0.002 0.002 0.2  ]]
```

b. P” =

```
Pb = np.array([[1./6, 149./300, 299./1200, 1./600, 1./600, 1./6],
            [1./6, 1./600, 299./1200, 1./600, 1./600, 1./6],
            [1./6, 149./300, 1./600, 149./300, 1./600, 1./6],
            [1./6, 1./600, 299./1200, 1./600, 149./300, 1./6],
            [1./6, 1./600, 299./1200, 149./300, 1./600, 1./6],
            [1./6, 1./600, 1./600, 1./600, 149./300, 1./6]])
==[[0.16666667 0.49666667 0.24916667 0.00166667 0.00166667 0.16666667]
 [0.16666667 0.00166667 0.24916667 0.00166667 0.00166667 0.16666667]
 [0.16666667 0.49666667 0.00166667 0.49666667 0.00166667 0.16666667]
 [0.16666667 0.00166667 0.24916667 0.00166667 0.49666667 0.16666667]
 [0.16666667 0.00166667 0.24916667 0.49666667 0.00166667 0.16666667]
 [0.16666667 0.00166667 0.00166667 0.00166667 0.49666667 0.16666667]]
```

Code:

```
arrays = [Pa, Pb]
```

```
for array in arrays:

    eigenvalues, eigenvectors = np.linalg.eig(array)
    indices = np.argsort(eigenvalues)
    principal = indices[-1]
    steadyState = eigenvectors[:,principal]
    steadyState = np.divide(steadyState, np.sum(steadyState))
    reverseOrder = np.argsort(steadyState)
    order = 1+reverseOrder[::-1]

    print('final order = {}'.format(order))
    print('importance = {}'.format(steadyState[order-1]))
```

After aboving calculation, we get:

For a:
final order = [1 4 3 2 5]
importance = [0.23648889 0.21038811 0.21038811 0.21038811 0.13234678]
For b:
final order = [3 4 5 1 6 2]
importance = [0.19983853+0.j 0.19944284+0.j 0.19944284+0.j 0.15057436+0.j
 0.1499828 +0.j 0.10071863+0.j]