Section 2-2

1.

① problem 14

$(B-C)D = 0$

$(B-C)D \cdot D^{-1} = 0 \cdot D^{-1} \Rightarrow 0$

$(B-C)I = 0$

$B - C = 0$

$B = C$ hence, proved

② problem 16

B invert

AB invert

Let $C = AB$

$CB^{-1} = AB \cdot B^{-1}$

$CB^{-1} = A$

Since AB is invertible, C is invertible, B^{-1} is invertible as well. Hence A is invertible because C and B^{-1} is invertible, then so is CB^{-1} (Theorem 6)

③ Problem 18

$A = PBP^{-1}$

$AP = PB \cdot (P^{-1} \cdot P)$

$AP = PB \cdot I$

$AP = PB$

$P^{-1}AP = P^{-1}P \cdot B$

$P^{-1} \cdot AP = IB$

Hence, $B = P^{-1} \cdot AP$

④ Problem 20

a). $(A-AX)^{-1} = X^{-1}B$

$X(A-AX)^{-1} = X \cdot X^{-1} \cdot B$

$X(A-AX)^{-1} = B$

Since X and A-AX is invertible, B is invertible.

b.)

$(A-AX)^{-1} = X^{-1}B$

$A-AX = B^{-1}X$

$A = B^{-1}X + AX$

$A = X(B^{-1}+A)$

$A \cdot X^{-1} = B^{-1}+A$ (A and X invertible
$B^{-1}+A$ is invertible as well)

$X^{-1} = (B^{-1}+A) \cdot A^{-1}$

$$\boxed{X = (B^{-1}+A)^{-1} \cdot A}$$

4/18

Problem 26

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Hence A^{-1} works

Problem 38

$AD = I_2$ $\quad A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $\quad$ just make matrix like $\begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & 1 & \boxed{1} \end{bmatrix}$ this

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1+b_1+c_1 & a_2+b_2+c_2 \\ b_1+c_1+d_1 & b_2+c_2+d_2 \end{bmatrix} = I_2$$

$CA = I_4$

$$\Rightarrow CA = \left[C\begin{bmatrix} 1 \\ 0 \end{bmatrix} \; C\begin{bmatrix} 1 \\ 1 \end{bmatrix} \; C\begin{bmatrix} 1 \\ 1 \end{bmatrix} \; C\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

In order to I_4, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = C\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = C\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

However, when we calculate CA again, $CA \neq I_4$. So no such C exist $\longrightarrow$ that $CA = I_4$

2.3

2.

① Problem 18

$Cx = v$ is unique

Proof: We know $Cx=v$ is consistent for every v in R^6, then C is invertible by the Invertible Matrix Theorem. Then we know $Cx=v$ have unique soluton (Theorem 5)

② Problem 22

$Hx=C \rightarrow$ incosistent for some c in R^n, the Invertible Matrix Theorem doesn't apply. $Hx=0$ doesn't have a trivial solution

③ Problem 24

Since this is a $N \times N$ matrix, so it's equiliviate to say $Ax=b$ has a solution for every b in R^n. At same time, the system doesn't have free variable, the columns of L span R^n.

④ Problem 26

A is linearly independent $\rightarrow$ A is invertible

$\rightarrow$ A^2 is invertible as well

$\rightarrow$ Therefore, by Invertible Matrix Theorem, the columns of A^2 span R^n.

(5) problem 28

Let $X = (AB)^{-1}$ (invertible)

then $AB \cdot X = I$

$\Rightarrow I = AX \cdot B$, hence B is invertible

$B^{-1} = AX$.

$ABW = I$

$(AB)W = A(BW)$

$BW = A^{-1}$

A is invertible

4. Problem 2

a) $\begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} = P$

b) $Px = x$

$Px - Ix = 0$

$(P - I)x = 0$

$$\left(\begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.50 & 0.25 \\ 0.25 & 0.25 & -0.50 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.5 & 0.25 & 0.25 & 0 \\ 0.25 & -0.50 & 0.25 & 0 \\ 0.25 & 0.25 & -0.50 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{-0.50} & 0.25 & 0.25 & 0 \\ 0 & \boxed{-0.375} & 0.375 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases} \Rightarrow$ we can get vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In order to calculat probability we divide $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by the sum of entries

$x = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

Hence the animal ~~is~~ is alway likely to chose any of three foods equally. and the probability is 0.33....

5. Problem 4

a)

Good	Indifference	Bad	To
0.60	0.40	0.4	good
0.30	0.30	0.5	Indiff
0.10	0.30	0.1	Bad

$$P = \begin{bmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix}$$

b). According to question, we get initial vector $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$

$$x_1 = P x_0 = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}$$

Therefore, the chance of bad weather tomorrow will be 20% (0.2)

c). $x_0 = \begin{bmatrix} 0 \\ 0.4 \\ 0.6 \end{bmatrix}$

$$x_2 = P \cdot P x_0 = \begin{bmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.336 \\ 0.184 \end{bmatrix}$$

The chance of good wether Wed will be 48% (0.48)

6. Problem 10

$$P = \begin{bmatrix} 1 & 0.2 \\ 0 & 0.8 \end{bmatrix}$$

$$P^2 = P \cdot P = \begin{bmatrix} 1 & 0.2 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} 1 & 0.2 \\ 0 & 0.8 \end{bmatrix} = \begin{bmatrix} 1 & 0.36 \\ 0 & 0.64 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = P^2 \cdot \begin{bmatrix} 1 & 0.2 \\ 0 & 0.8 \end{bmatrix} = \begin{bmatrix} 1 & 0.488 \\ 0 & 0.512 \end{bmatrix}$$

One of P^k entry is zero. Therefore this is not a regular matrix.

7. Problem 12 $\quad (A-I)X=0$

$$\begin{bmatrix} 0.6 & 0.4 & 0.4 & 0 \\ 0.3 & 0.3 & 0.5 & 0 \\ 0.1 & 0.3 & 0.1 & 0 \end{bmatrix} \xrightarrow{} \sim \begin{bmatrix} -0.4 & 0.4 & 0.4 & 0 \\ 0.3 & -0.7 & 0.5 & 0 \\ 0.1 & 0.3 & -0.9 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3X_3 \\ 2X_3 \\ X_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$

The chance of good weather in long-run is $\frac{1}{2}$

$\boxed{(50\%)}$

8. ① Problem 19

a) Inorder to be a probalitility vector, it must has non-negatie entiies with the sum of 1,

$S = [1\ 1 \cdots 1]$

Let $X = (X1, x_2, \cdots x_n)^T \in R^n$ be a probability vector

$$X_1 + X_2 \cdots + X_n = 1$$

$$(1) \cdot X_1 + (1) \cdot x_2 \cdots + (1) X_n = 1$$

$$[1\ 1 \cdots 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 1 \Rightarrow Sx = 1$$

Hence, we can tell entry of x must be nonnegatie and $Sx = 1$

b) P is $n \times n$ stochastic matrix

$P = [P_1, P_2 \cdots P_n]$

so, each p_n should be a probability vector

$SP = S[p_1, \cdots p_n]$

$= [Sp_1, Sp_2, \cdots Sp_n] = [1, 1, 1, 1 \ldots]$

Since, from (a), S is $Sx = 1$

There fore, $SP = S$

1(c), $S(Px) = (SP)x$

From (a) and (b),

$S(Px) = Sx = 1$

Comparing this to $Sx = 1$, Px is probability vector as well.

2) Problem 20

$P = [P_1, P_2 \cdots P_n]$ each P_n is probability vectors

$P \cdot P = P^2$

$= P[P_1, P_2 \cdots P_n]$

$= [Pp_1, Pp_2 \cdots Pp_n]$

Therefore, Pp_n are probability vectors

=> Each column of P^2 are probability vectors

then P^2 is a stochastic matrix

=> Hence P^2 is stochastic when P is.

1.

Problem 36 at Page 112:

```
[[  1.5         -4.5       ]
 [-72.16666667 219.5       ]
 [ 22.66666667 -69.        ]]
```

2.

Problem 16 at Page 263:

First we return to row reduced echelon form:

```
[[ 1.          0.         -0.91919192  0.        ]
 [ 0.          1.         -0.19191919 -0.        ]
 [ 0.          0.          0.          0.        ]]
```

Then we know, y3 is free, y2 = 0.191919y3, y1 = 0.919192y3

According to (P - I) y = 0, y= t [[0.919192],this is also the basis for solution space for (P-I)y =0
[0.191919]
[1]]

Now, we can find the probability vector

```
q = ½.111111  [[ 0.919192],   = [[0.435407],
               [0.191919]        [0.090909],
               [1]]              [0.473684]]
```

(0.090909)* 2000 = 182

Hence, the number of cars will be rented or available for rent from the downtown location is 182.

3.

Problem 21 at Page 263:

Part A:

k = 2 the matirx P:

```
[[0.2779 0.278  0.2803 0.2941]
 [0.3368 0.3355 0.3357 0.3335]
 [0.1847 0.1861 0.1833 0.1697]
 [0.2005 0.2004 0.2007 0.2027]]
```

k = 3 the matirx P:
[[0.2817 0.2817 0.2817 0.2814]
 [0.3356 0.3356 0.3355 0.3352]
 [0.1817 0.1817 0.1819 0.1825]
 [0.201 0.201 0.201 0.2009]]

k = 4 the matirx P:
[[0.2816 0.2816 0.2816 0.2816]
 [0.3355 0.3355 0.3355 0.3355]
 [0.1819 0.1819 0.1819 0.1819]
 [0.2009 0.2009 0.2009 0.2009]]

k = 5 the matirx P:
[[0.2816 0.2816 0.2816 0.2816]
 [0.3355 0.3355 0.3355 0.3355]
 [0.1819 0.1819 0.1819 0.1819]
 [0.2009 0.2009 0.2009 0.2009]]

By comparing P^2, P^3, P^4 P^5, P^4 and P^5 columns of the matrix are same, and the rest of columns are converges to [[0.2816]
 [0.3355]
 [0.1819]
 [0.2009]]
the steady-state vector for P:
[[0.2816]
 [0.3355]
 [0.1819]
 [0.2009]]

Part B:

k= 10 the matirx Q:
[[0.8222 0.4044 0.5385]
 [0.0324 0.3966 0.1666]
 [0.1453 0.199 0.2949]]

k= 20 the matirx Q:
[[0.7674 0.6   0.669]
 [0.0637 0.2036 0.1326]
 [0.1688 0.1964 0.1984]]

k= 30 the matirx Q:

[[0.7477 0.6815 0.7105]
 [0.0783 0.1329 0.1074]
 [0.174 0.1856 0.1821]]

k= 40 the matirx Q:
[[0.7401 0.714 0.7257]
 [0.0843 0.1057 0.096]
 [0.1756 0.1802 0.1783]]

k= 50 the matirx Q:
[[0.7372 0.7269 0.7315]
 [0.0867 0.0951 0.0913]
 [0.1761 0.178 0.1772]]

k= 60 the matirx Q:
[[0.736 0.732 0.7338]
 [0.0876 0.0909 0.0894]
 [0.1763 0.1771 0.1767]]

k= 70 the matirx Q:
[[0.7356 0.734 0.7347]
 [0.088 0.0893 0.0887]
 [0.1764 0.1767 0.1766]]

k= 80 the matirx Q:
[[0.7354 0.7348 0.7351]
 [0.0881 0.0887 0.0884]
 [0.1764 0.1766 0.1765]]

 when k= 116 the matirx Q:
[[0.7353 0.7353 0.7353]
 [0.0882 0.0882 0.0882]
 [0.1765 0.1765 0.1765]]
 when k= 117 the matirx Q:
[[0.7353 0.7353 0.7353]
 [0.0882 0.0882 0.0882]
 [0.1765 0.1765 0.1765]]

Comparing all of matrix above, Q^116 and Q^177 columns are same, and the rest of columns are converges to [[0.2816],   so this is the steady state vector.
[0.3355],
[0.1819],
[0.2009]]

Conjecture: Let A be regular stochastic, A^k is its power, then the columns of martix A^k is converges to its steady state vector as k get bigger and bigger.

Part C:

From part a and part c, we can observe that, P and Q regular stochastic matrix, then the columns of matrix p^k is converges to steady state vector of P as k increase. (which is Theorem 18)
x0
*Xk = P^k *x0 (k is inifinite)*

*As k increase, q = P^k *ei*
And ith column of matrix pk converges to q(which is steady state vector)

```python
def p36():
    A = np.array([[-25,-9,-27],[546,180,537],[154,50,149]])
    Ainverse = np.linalg.inv(A)
    Secondandthird = Ainverse[:,[1,2]]
    print(Secondandthird)


#4-9 Problem 16 and 21


def p16():
    P = np.array([[0.90,0.01,0.09],[0.01,0.90,0.01],[0.09,0.09,0.90]])
    I = np.identity(3)
    PsubI = np.subtract(P,I)
    zeros = np.zeros((3,1))
    afterappend = np.append(PsubI, zeros,axis=1)
    forward = forwardElimination(afterappend)
    back = backsubstitution(forward)

    print("First we return to row reduced echelon form: \n" + str(back))
    print()



def p21():
    P = np.array([[0.3355,0.3682,0.3067,0.0389],[0.2663,0.2723,0.3277,0.5451],
    Q = np.array([[0.97, 0.05, 0.10],[0,0.90,0.05],[0.03,0.05,0.85]])

    #
    print("Part A: \n")
    for i in range(2, 6):
        Pori = np.linalg.matrix_power(P,i-1)
        Pk = np.linalg.matrix_power(P,i)
        result = np.around(Pk, 4)
        print("k = " + str(i) + " the matirx P: \n" + str(result))

    print()
    # b
    print("Part B: \n")
    for j in range(10,90,10):
        Qori = np.linalg.matrix_power(Q,j-1)
        Qk = np.linalg.matrix_power(Q,j)
        result = np.around(Qk, 4)
        print("k= " + str(j) + " the matirx Q: \n" + str(result))
    Q116 = np.linalg.matrix_power(Q, 116)
    result116 = np.around(Q116, 4)
    Q117 = np.linalg.matrix_power(Q, 117)
    result117 = np.around(Q117, 4)
    print(" when k= 116 the matirx Q: \n" + str(result116))
    print(" when k= 117 the matirx Q: \n" + str(result117))
```