1 Useful Gaussian Integrals

1.1 Simple univariate Gaussian integral

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx \\
\rightarrow I^2 &= \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2\sigma^2} dxdy \\
&= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2/2\sigma^2} rdrd\theta = -2\pi\sigma^2 \int_{0}^{\infty} e^{-r^2/2\sigma^2} \frac{-r}{\sigma^2} dr = 2\pi\sigma^2 \\
\Rightarrow I &= \sigma\sqrt{2\pi}
\end{aligned} \tag{1}
$$

This also implies

$$
\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = \sigma\sqrt{2\pi} \tag{2}
$$

This can be shown by making the substitution $x \rightarrow x-\mu$; but, intuitively this makes sense, as the area under a Gaussian curve does not depend on where the curve is located.

1.2 Product of two univariate Gaussian integrals

$$
\begin{aligned}
f(x) &= \frac{1}{\sigma_f\sqrt{2\pi}} \exp\left[-\frac{(x-\mu_f)^2}{2\sigma_f^2}\right] \\
g(x) &= \frac{1}{\sigma_g\sqrt{2\pi}} \exp\left[-\frac{(x-\mu_g)^2}{2\sigma_g^2}\right] \\
\Rightarrow f(x)g(x) &= \frac{1}{2\pi\sigma_f\sigma_g} \exp\left[-\frac{(x-\mu_f)^2}{2\sigma_f^2} - \frac{(x-\mu_g)^2}{2\sigma_g^2}\right] \\
&\equiv \frac{1}{2\pi\sigma_f\sigma_g} \exp[-\beta]
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\beta &= \frac{(x-\mu_f)^2}{2\sigma_f^2} + \frac{(x-\mu_g)^2}{2\sigma_g^2} \\
&= \frac{2\sigma_g^2[x^2 - 2\mu_f x + \mu_f^2] + 2\sigma_f^2[x^2 - 2\mu_g x + \mu_g^2]}{4\sigma_f^2\sigma_g^2} \\
&= \frac{(\sigma_f^2+\sigma_g^2)x^2 - 2(\mu_f\sigma_g^2 + \mu_g\sigma_f^2)x + \mu_f^2\sigma_g^2 + \mu_g\sigma_f^2}{2\sigma_f^2\sigma_g^2} \\
&= \frac{x^2 - 2\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}x + \frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \\
\text{Define} : \alpha &\equiv \frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} \\
\Rightarrow \beta &= \frac{(x^2 - 2\alpha x + \alpha^2) - \alpha^2 + \frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}-\alpha^2 &= \frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}-\left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2 \\
&= \frac{1}{(\sigma_f^2+\sigma_g^2)^2}\left[(\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2)(\sigma_f^2+\sigma_g^2)-(\mu_f\sigma_g^2+\mu_g\sigma_f^2)^2\right] \\
&= \frac{1}{(\sigma_f^2+\sigma_g^2)^2}\left[\mu_f^2\sigma_f^2\sigma_g^2+\mu_g^2\sigma_f^2\sigma_g^2+\mu_f^2\sigma_g^4+\mu_g^2\sigma_f^4-\ldots\right. \\
&\qquad \left.-\mu_f^2\sigma_g^4-\mu_g^2\sigma_f^4-2\mu_f\mu_g\sigma_f^2\sigma_g^2\right] \\
&= \frac{\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)^2}\left[\mu_f^2-2\mu_f\mu_g+\mu_g^2\right] \\
&= \frac{\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)^2}(\mu_f-\mu_g)^2
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\beta &= \frac{(x^2-2\alpha x+\alpha^2)-\alpha^2+\frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \\
&= \frac{(x-\alpha)^2+\frac{\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)^2}(\mu_f-\mu_g)^2}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \\
\text{Define}: \gamma^2 &= \frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2} \\
\beta &= \frac{(x-\alpha)^2}{2\gamma^2}+\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
f(x)g(x) &= \frac{1}{2\pi\sigma_f\sigma_g}e^{-\beta} = \frac{1}{2\pi\sigma_f\sigma_g}\exp\left[-\frac{(x-\alpha)^2}{2\gamma^2}\right]\exp\left[\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right] \\
&= \frac{1}{\sigma_f\sigma_g}\gamma\sqrt{(\sigma_f^2+\sigma_g^2)}\times\frac{1}{\gamma\sqrt{2\pi}}\exp\left[-\frac{(x-\alpha)^2}{2\gamma^2}\right]\times\ldots \\
&\quad \times\frac{1}{\sqrt{(\sigma_f^2+\sigma_g^2)}\sqrt{2\pi}}\exp\left[\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right] \\
\Rightarrow f(x)g(x) &= \frac{1}{\gamma\sqrt{2\pi}}\exp\left[-\frac{(x-\alpha)^2}{2\gamma^2}\right]S_{fg} \\
S_{fg} &= \frac{\gamma\sqrt{(\sigma_f^2+\sigma_g^2)}}{\sigma_f\sigma_g}\frac{1}{\sqrt{(\sigma_f^2+\sigma_g^2)}\sqrt{2\pi}}\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right] \\
&= \frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}}\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right]
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)g(x)dx = \frac{S_{fg}}{\gamma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\alpha)^2}{2\gamma^2}\right] = S_{fg} \\
= \frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}} \exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right]
\end{aligned} \tag{8}
$$

1.3 Univariate Gaussian with linear term in exponential

$$
I = \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2\sigma^2} + Jx\right] \tag{9}
$$

Just as above, the idea is to complete the square

$$
\begin{aligned}
\frac{x^2}{2\sigma^2} - Jx = \frac{1}{2\sigma^2}\left[x^2 - 2\sigma^2 Jx + (\sigma^2 J)^2 - (\sigma^2 J)^2\right] \\
= \frac{1}{2\sigma^2}\left(x-\sigma^2 J\right)^2 - \frac{1}{2\sigma^2}(\sigma^2 J)^2 = \frac{\left(x-\sigma^2 J\right)^2}{2\sigma^2} - \frac{(\sigma J)^2}{2}
\end{aligned} \tag{10}
$$

Therefore, I find:

$$
\begin{aligned}
I = \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2\sigma^2} + Jx\right] = \int_{-\infty}^{\infty} \exp\left[\frac{\left(x-\sigma^2 J\right)^2}{2\sigma^2} - \frac{(\sigma J)^2}{2}\right] \\
= \sigma\sqrt{2\pi} \exp\left[\frac{(\sigma J)^2}{2}\right]
\end{aligned} \tag{11}
$$