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Hierarchical Structure in Social Networks

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Abstract

The measurement and characterization of power relations in social systems is a major challenge across the social and physical sciences. A number of studies have sought to introduce analytical measures of "hierarchy" in networks, yet these studies have been largely disconnected from social theories of power and hierarchy. In this study, we integrate these two traditions by linking theory and measurement. To do this, we conduct an empirical analysis of 136 social, organization, political, and economic networks. We find that the twelve analytical measures of network hierarchy we consider map onto a theoretical typology of hierarchical structures. Our results indicate that there is no one-size-fits-all measure of hierarchy, and that researchers can benefit from considering the appropriateness of applying a particular measure to their particular network.

Keywords: Hierarchy, Network, Power

1. Introduction

Hierarchy is an important feature of many organizations, such as firms, social clubs, and military units. Formally, we can define a hierarchy as a system where people or groups are ranked according to status or authority. Yet it is difficult to operationalize this definition for measurement and comparison. There has been a great deal of research on power and status in groups and organizations, but most of this research relies on measurements defined over domain specific rankings, such as job titles. At the same time, networks scholars have defined a number of broadly applicable hierarchy metrics based on network structure, but these metrics are not necessarily grounded in meaningful sociological concepts of status and authority. While some social theorists have noted that a network-oriented perspective of the "sociospatial and organizational model [of a network]" can explicate the "sources of social power," [1] they have generally not delved into the methodologies through which to fully explore such power dynamics. In this paper, we seek to bring together these two areas of research, and to develop a framework for measuring hierarchy in social networks that is both generally applicable and exhibits a high degree of construct validity.

In this study: we describe theories of hierarchy in social science, formally state and explain commonly used measures for network hierarchy, and illustrate the ways in which these measures correspond and diverge when applied to a wide variety of observed social networks. We find that the twelve measures of hierarchy we consider can be mapped onto several dimensions of hierarchy derived from the sociological literature. We also find wide variation in the theoretical and empirical suitability of the measures we consider to the measurement of hierarchy across different types of social networks (e.g. communication, friendship, exchange, etc.). We conclude that there is not one-size fits

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all measure of hierarchy, and that carefule theoretical guidance is necessary in selecting an appropriate measure or set of measures for a particular domain.

2. Theorizing Hierarchy in Networks

Social network analysis has provided researchers with invaluable tools to model dependencies in relational data and generate new scientific insights[2] Order, authority, and relative power are ubiquitous and highly salient features of nearly all social structures. In his highly influential work on the structure of society, Michael Mann argues that "Societies are constituted of multiple overlapping and intersecting sociospatial networks of power" [**?**]. When explicitly applied to networks, the sociological concept of power maps onto analytical and statistical measures of hierarchy. One common definition of hierarchy is a relational system where one actor "possesses the right to make residual decisions while the other party—the subordinate member—lacks this right" [**? ?**].

It is no surprise that this definition covers a wide variety of social structures, including firms, states, social clubs, militaries, and criminal gangs. As a basic feature of organizations, social theorists suggest that hierarchal structures exhibit significant variation. In their landmark works on organizational theory, [**? ? ?**] distinguish between U-form or unitary organizations, which are organized along distinct administrative functions, and M-form or multidivisional organizations, which are organized along autonomous and heterogeneous sub-groups.

The reason for this variation in hierarchical structures continues to puzzle social and organizational theorists. /citetlawrencelosch focused early work on the necessary trade-offs between differentiation (associated with the U-form) and integration (associated with the M-form). /citetprovankenis highlight further tensions between inclusive decision making versus efficient operation, internal versus external legitimacy, and flexibility versus stability, as well as different hierarchcal types to address these tensions. To put it simply, hierarchical structures differ according to environmental and contextual features according to which particularly hierarchical configuration ensures optimal group performance [**? ?**].

In accordance with its theoretical importance, hierarchy serves an important explanatory and outcome variable in empirical studies of social networks. Additionally, the measures used to represent hierarchy follow the heterogeneity found in the theory of social hierarchies. The technical details surrounding these different measures are provided in the following section, but it is important to also understand how these different statistical and analytical measures relate to the concepts the authors seek to operationalize [**? ?**].

Particularly within sociology, much of the initial work on hierarchy focused on the theoretically similar concept of centrality. [**?**] suggest that the most common centrality measures (degree, closeness, betweenness, and eigenvector) are misused throughout the literature, and employ a simulation study to illustrate their point. Degree centrality does not account for indirect influence or the intensity of the connection. Closeness centrality fails to handle a large number of shortest paths. Betweenness centrality does not increase monotonically with the distance of actors or with more shortest paths. Finally, eigenvector centrality is more difficult to interpret than other measures.

It is not that any hierarchy measure is inherently wrong, but there is a clear need to either specify a measurement that captures the specific heirarchical structure of interest, or account for the uncertainty in choosing from multiple, equally valid measures. For example, [**?**] mapping an adjusted eigenvector centrality to measure hierarchy as defined as "having power over someone who in turn has power over others makes one more powerful." [**?**] use betweenness centrality to represent hierarchy in an email communication network. Others are interested in modular hierarchies, similar in theoretical construction to the M-form hierarchies described above, and use the presence of triadic closures to measures this [**?**]. This linear conception of hierarchy is frequently employed in studies where edges represent dominance [**?**]. In contrast with the relational and linear hierarchical concepts discussed above, [**?**] propose their own measure for controllability, which measures the nodes necessary to control the output of an entire system. This concept of controllability is also employed in a study by [**?**]. Studies are not confined to the use of a singular hierarchical measure. [**?**] distinguish between the linearity, or extent to which the hierarchy applies to all subordinates, and steepness, or the overall differences in authority between nodes, when measuring dominance hierarchies, an approach also adopted by [**?**]. Similarly, [**?**] employ both degree and eigenvector centrality to establish the relative authority of actors in a terrorist network.

[2]For a review of many of these network structures, see [2].

To conclude, hierarchy is an important but ambiguous concept in social science theories. Analytical and statistical measurements of hierarchy exhibit considerable heterogeneity. In order to address concerns over "researcher degrees of freedom" inherent in the choice of one of these heterogeneous measures, scholars need to carefully link their theories with measurement, and understand the trade-offs inherent in the selection of a particular measurement. The following sections hope to inform that choice by first specifying the variety of different hierarchy measures used in the literature, and then describing the statistical relationships between those measures.

3. Measuring Hierarchy

A number of analytical measures of hierarchy have been proposed for directed networks. Most proposed measures return a scalar value that is meant to capture the "hierarchical-ness" of a given network [8, 9, 10, 11, 12, 13, 14]. This theoretically facilitates comparison between measures calculated on the same network, as well as comparison on the same measure across multiple networks. However some approaches to measuring hierarchy only provide a local measure of importance or position in the hierarchy for each node in the network [15]. For these measures, one can calculate a Gini coefficient [16] from the individual level scores and use these coefficients as a proxy for a global measure.

In this section, we introduce and describe twelve candidate measures of network hierarchy that have been previously used in the networks literature. It is important to note that most of the measures we consider are only defined for directed networks, and thus for the remainder of this paper we assume all networks under consideration are directed. We begin by introducing some terminology that will be common across all measures. For a given network $G = (V, E)$, let $V = \{v_i\}_{i=1}^N$ be the set of N vertices (nodes) associated with G, and $E = \{e_j\}_{j=1}^M$ be the set of M edges associated with G. Furthermore, for a given edge e_j, let $e_j^{(s)} \in 1 : N$ be the index of the sender of the edge and $e_j^{(r)} \in 1 : N$ be the index of the recipient.

One other important point is that most measures of network hierarchy are meant to be applied to networks where the edge sets capture relations other than "has power over". In this special case, it is theoretically easier to construct a measure of network hierarchy since the network must be directed and acyclic (preventing circular chains of command). However, obtaining such information is usually impossible in most cases (with military personell networks being an obvious exception). Furthermore, if the researcher has collected such an edge-set, then the need for summary measures of the "hierarchical-ness" of the network is likely obviated, as deeper insights could be gained from applying inferential network analysis tools to the raw network. Therefore, we focus or attention on the measurement of hierarchy on networks where edges are not explicitly power relations.

3.1. Analytical Measures of Hierarchy

The most basic measures of hierarchy or differential importance of a nodes in a network can all be derived from basic node-level measures of network centrality [17]. To aggregate from the node-level measures up to a single measure on a network, one can calculate C, the centralization of the network. Centralization captures the degree of inequality in the distribution of a given centrality measure over the network. In general, we should then expect that networks that are more centralized are also likely to be more hierarchical. However, this measure attains its maximal value for any centrality measure when one node has a maximal value of the given centrality measure and all the rest of the nodes have the minimal possible value. Therefore, star networks have maximal centralization scores. This implicit assumption in measurement is important to consider when evaluating the validity of centralization based measures of hierarchy. The four centrality measures we consider are: degree centrality, closeness centrality, betweenness centrality, and eigenvector centrality.

The **degree centrality** of node v_i is simply the number of outgoing or incoming edges incident to it. Formally, this can be calculated as:

$$\text{in-degree centrality}_i = \text{In}_i = \sum_{j=1}^{M} \mathbb{1}(e_j^{(r)} = i) \tag{1}$$

$$\text{out-degree centrality}_i = \text{Out}_i = \sum_{j=1}^{M} \mathbb{1}(e_j^{(s)} = i) \tag{2}$$

Degree centrality captures the number of friends or connections a node has, and intuitively, we should expect more powerful nodes will have more incoming and outgoing connections. However, degree centrality does not account for the identity of a node's alters. This can lead to difficulties when it is used to asses a node's position in a social hierarchy. For example, in a large company, we might expect that an administrative assistant may have a higher degree centrality than the CEO of a company if the edges being measured are work-related interaction. Furthermore, if there are many administrative assistants at the bottom of the power structure in an organization we might qualitatively consider to be extremely hierarchically structured, the degree centralization of the interaction network might be lower than in a comparably sized "organizationally flat" organization where a handfull of people serve as coordinators. Thus we must take great care in interpreting this statistic, depending on the type of edges it is defined over. For given node-level in-degree or out-degree centrality measures $\mathbf{c} = \{c_i\}_{i=1}^N$, the corresponding in-degree or out-degree centralization measure is defined as:

$$\text{Degree Centralization} = \frac{\sum_{i=1}^N (\max\{c_i\} - c_i)}{(N-1)(N-2)} \tag{3}$$

where $(N-1)(N-2)$ normalizes the measure for the size of the network.

The **betweenness centrality** of node v_i is a measure of the amount of influence a node has on the information transversed through it [11]. Define $D_{i,j}$ as the number of shortest paths in G between v_i and v_j, and $D_{i,j}(k)$ as the number of these shortest paths that pass through v_k, then the betweenness centrality of node k is:

$$\text{betweenness centrality}_i = \sum_{i \neq k \neq j} \left(\frac{D_{i,j}(k)}{D_{i,j}}\right) \tag{4}$$

Betweenness centrality captures how in-the-middle-of-things a node is and, when edges involve sharing information, how important the node is to information flowing quickly across the network, Intuitively, we should expect the nodes with higher betweenness centrality will be more powerful, and that greater inequality on this measure should signal a greater degree of hierarchical structure in a network. For given node-level betweenness centrality measures $\mathbf{c} = \{c_i\}_{i=1}^N$, the betweenness centralization measure is defined as:

$$\text{Betweenness Centralization} = \frac{\sum_{i=1}^N (\max\{c_i\} - c_i)}{(N-1)^2(N-2)} \tag{5}$$

where $(N-1)^2(N-2)$ normalizes the measure for the size of the network. Similarly, the **closeness centrality** of node v_i is a measure of how few intermediate edges a given node must traverse to reach all other nodes in the network. Define $d(i, j)$ as the length of the shortest path between v_i and v_j, then closeness centrality of node i is:

$$\text{closeness centrality}_i = \sum_{i \neq j} \left(\frac{1}{d(i, j)}\right) \tag{6}$$

Again, intuitively, powerful members of a hierarchy will tend to be able to reach others in the network more easily, and inequality in this measure (as captured by the closeness centralization of the network) should theoretically signal the degree of hierarchy in the network. For given node-level closeness centrality measures $\mathbf{c} = \{c_i\}_{i=1}^N$, the closeness centralization measure is defined as:

$$\text{Closeness Centralization} = \frac{\sum_{i=1}^N (\max\{c_i\} - c_i)}{\frac{(N-1)(N-2)}{(2N-3)}} \tag{7}$$

where $\frac{(N-1)(N-2)}{(2N-3)}$ normalizes the measure for the size of the network.

The last of the classical centrality based measures is **eigenvector centrality**, which is meant to capture the degree to which a node is connected to other well connected nodes [10]. Let the adjacency matrix of the network be defined as $\mathbf{A}$ and the vector of local eigenvector centrality scores for each node be defined as $\mathbf{w} = \{w(v_1), ..., w(v_V)\}$. Then to calculate the eigenvalue centralities $\mathbf{w}$, one must solve the following eigenvector equation:

$$\mathbf{A}\mathbf{w} = \lambda\mathbf{w} \tag{8}$$

where λ is a vector of positive eigenvalues. The challenge is to find the *dominant eigenvector*, as only the largest eigenvalue results in the desired centrality measure for each node. Intuitively, higher eigenvector centrality should be associated with greater social or organizational importance. For given node-level eigenvector centrality measures $\mathbf{c} = \{c_i\}_{i=1}^N$, the eigenvector centralization measure is defined as:

$$\text{Eigenvector Centralization} = \frac{\sum_{i=1}^N (\max\{c_i\} - c_i)}{(N-1)} \tag{9}$$

where $(N - 1)$ normalizes the measure for the size of the network. Interestingly, the eigenvector centralization of a network can still be large even if there are a relatively large proportion of higher degree nodes, as long as the edges are organized such that only a few nodes are connected to all of these higher degree nodes.

Landau's h and **Kendall's K** are two closely related measures of hierarchy that operate on a *dominance* network – a transformation of the weighted sociomatrix of a given network [9]. Intuitively, this *dominance* network is meant to capture dominance-subordination relationships (between animals). The $[i, j]$ entry of the corresponding sociomatrix is coded as 1 if i is dominant over j, 0.5 if i and j are equals, and 0 if j is dominant over i. If the underlying network does not capture dominance relationships, then an weighted network can be transformed into a *dominance* network by assigning a value of 1 in the $[i, j]$ entry of of the corresponding sociomatrix if the $[i, j]$ entry of the original sociomatrix is greater than the $[j, i]$ entry of the original sociomatrix, 0.5 in the $[i, j]$ entry of of the corresponding sociomatrix if the $[i, j]$ entry of the original sociomatrix is equal to the $[j, i]$ entry of the original sociomatrix, and zero otherwise.

Landau's h is used to compare a directed network to a perfect linear hierarchy (a strict dominance-ordering of nodes) [13, 9]. This measure is defined as follows:

$$h = \frac{12}{N^3 - N} \sum_{i=1}^{N} \left[\text{Out}_i - \frac{N-1}{2} \right]^2 \tag{10}$$

where $h \in [0, 1]$. Note that Landau's h does not provide an individual level metric of importance or relative power. This measure was specifically designed to operate on networks of "has power over" edges, and thus may be difficult to interpret when the network is not explicitly defined on power relations. Because of a preference for chain-like structures, this measure will likely provide poor performance when edges measure social relations. **Kendall's K** is also designed to compare a directed network to a perfect linear hierarchy [12]. As noted in [9], it often gives an identical value to Landau's h, but is theoretically distinct. Begin by defining the number of *cyclic triads* (CyT) in the network as:

$$CyT = \frac{N(N-1)(2N-1)}{12} - \frac{1}{2} \sum \text{Out}_i^2 \tag{11}$$

Then we can define Kendal's K as

$$K = 1 - \frac{d}{d_{max}} \tag{12}$$

where the value d_{max} is defined as follows:

$$d_{max} = \begin{cases} \frac{1}{24}(N^3 - N) & \text{if } N \text{ is odd} \\ \frac{1}{24}(N^3 - 4N) & \text{if } N \text{ is even} \end{cases} \tag{13}$$

Kendal's $K \in [0, 1]$ is also only defined as a global measure, and no individual level analogue exists. It was also specifically designed to operate on networks of "has power over" edges, and is therefore likely a poor choice for networks defined over social interactions.

Triangle transitivity (t_{tri}) was proposed as an improvement over Landau's h and Kendal's K in [9]. This measure captures the degree to which triads in a network are not cyclic, with the intuition that more hierarchical networks will contain a lower proportion of cyclic triads relative to the total number of triads. To calculate this measure, we begin by calculating the proportion of triangles that are not cycles.

$$P_t = \frac{N_{transitive}}{N_{transitive} + N_{cycle}} \tag{14}$$

The authors show that in a random network $P_t = 0.75$ in expectation so they normalize their statistic as follows:

$$t_{tri} = 4(P_t - 0.75) \tag{15}$$

where $t_{tri} \in [-3, 1]$ can be negative if there is a particularly high proportion of cyclic triads in the network. This is likely a better measure of hierarchy than Landau's h and Kendal's K when the network is weighted or ties do not represent dominance relationships. However, if there are violations of a linear hierarchy that involve larger cycles (spanning more than three nodes) this measure will fail to pick them up, so care should be used in interpreting this measure for complex networks.

M-reach degree was developed to identify 'key' players in a network [15]. It is defined as a measure for each node v_i as the number of alters that are reachable from v_i. If G is directed then the reachable alters must lie along an outgoing path from v_i. A closely related measure, **M-reach closeness** is defined as the M-reach degree of a node, but with the contribution of each alter j that is reachable, weighted by the inverse of the shortest path length between i and j. Both of these measures are only defined for individual nodes, so to calculate a global measure for a network, we take the Gini coefficient of the measures calculated for each node i. The Gini coefficient [16] is a measure of inequality originally developed to measure income inequality in a society. For a vector of values $x : \{x_i\}_{i=1}^N$ the Gini coefficient G of that vector is:

$$G = \frac{\sum_i \sum_j |x_i - x_j|}{2 \sum_i \sum_j x_i} \tag{16}$$

In words, it is half of the *average absolute difference* of all pairs of entries in the vector, divided by the average, which acts as a normalizing constant. We take the Gini coefficient of the M-reach degree and M-reach closeness values for each node in the network and treat this as our global measure of network hierarchy. Intuitively, this measure is capturing the degree of inequality in access to other nodes in the network. We should expect that a higher degree of inequality in this metric would be associated with a more hierarchical network. One potential shortcoming of these measures is that they only account for out-degree, when incoming ties may be a better signal of power relations in many real-world social networks.

Global reaching centrality (GRC) is a generalization of M-reach degree centrality measure [8], and was designed specifically to measure on any network. When the network is unweighted and directed, let $C_R(i)$ be the local M-reach degree centrality of node i, then the global reaching centrality of the network can be defined as:

$$GRC = \frac{\sum_{i=1}^{N} [\max(C_R) - C_R(i)]}{N - 1} \tag{17}$$

When the graph is weighted and directed, the authors in [8] propose an alternative formulation of $C_R(i)$:

$$C_R^{'}(i) = \frac{1}{N-1} \sum_{j:0<d_{(i,j)}^{out}<\infty} \left(\frac{\sum_{k=1}^{d^{out}(i,j)} w_i^{(k)}(j)}{d^{out}(i,j)}\right) \tag{18}$$

where $d_{i,j}^{out}$ is the (directed) path length from node i to node j, and $w_i^{(k)}$ is the weight of the kth edge along this path. This alternative formulation can then be plugged into equation 17 to calculate the GRC for a weighted,directed network. Finally, when the network is unweighted and undirected, the authors in [8] propose an additional alternative formulation of $C_R(i)$:

$$C''_R(i) = \frac{1}{N-1} \sum_{j:0<d(i,j)<\infty} \frac{1}{d(i,j)} \tag{19}$$

where $d_{i,j}$ is the (undirected) path length from node i to node j. The intuition of for the interpretation of GRC as a measure of network hierarchy is almost identical to the interpretation of the Gini coefficient of the M-reach degree and M-reach closeness values for a network, with only difference being the way the individual measures are aggregated, and its extension to weighted and undirected networks. Again, a potential major shortcoming is that this measure only makes theoretical sense when the outgoing edges in the network signal a power relation where the sender has power over the recipient.

rooted depth [14] is only defined for networks where a root (a node that has only incoming edges) exists. Let N_r be the number of node-root pairs in the network. Then the rooted depth of the network can be defined as:

$$D = \frac{1}{N_r} \sum_{i=1}^{N_r} l_{ri} \tag{20}$$

where l is the length of the shortest path between root r and node i. This measure can only be calculated for the network as a whole. However, given R roots in a network, we can calculate a local root depth for each node that is equal to the average length of the shortest path between itself and all roots. One of the major problems with this measure is that it is undefined for networks without a root (a node with only incoming edges). This makes it very difficult to apply to most networks (we were only able to calculate it for a small fraction of networks in our sample). Furthermore, since this measure relies on finding nodes with no outgoing ties, it can be very sensitive to the way the network is measured. In general, we do not find rooted depth to be a useful measure in our empirical analysis.

The final analytical measure we consider is the **Krackhardt** hierarchy score of a network. [18]. The Krackhardt hierarchy score is only defined for directed networks, as the proportion of un-reciprocated dyads in the reachability matrix of a network. More formally let R be the reachability matrix of G, where R is an $N \times N$ matrix with a one in the $[i, j]$ entry if node j can be reached via an outgoing path from node i. Then the Krackhardt hierarchy score can be defined as:

$$\text{Krackhardt} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{1}(R_{i,j} == 1 \& R_{j,i} == 0)}{\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{1}(R_{i,j} == 1)} \tag{21}$$

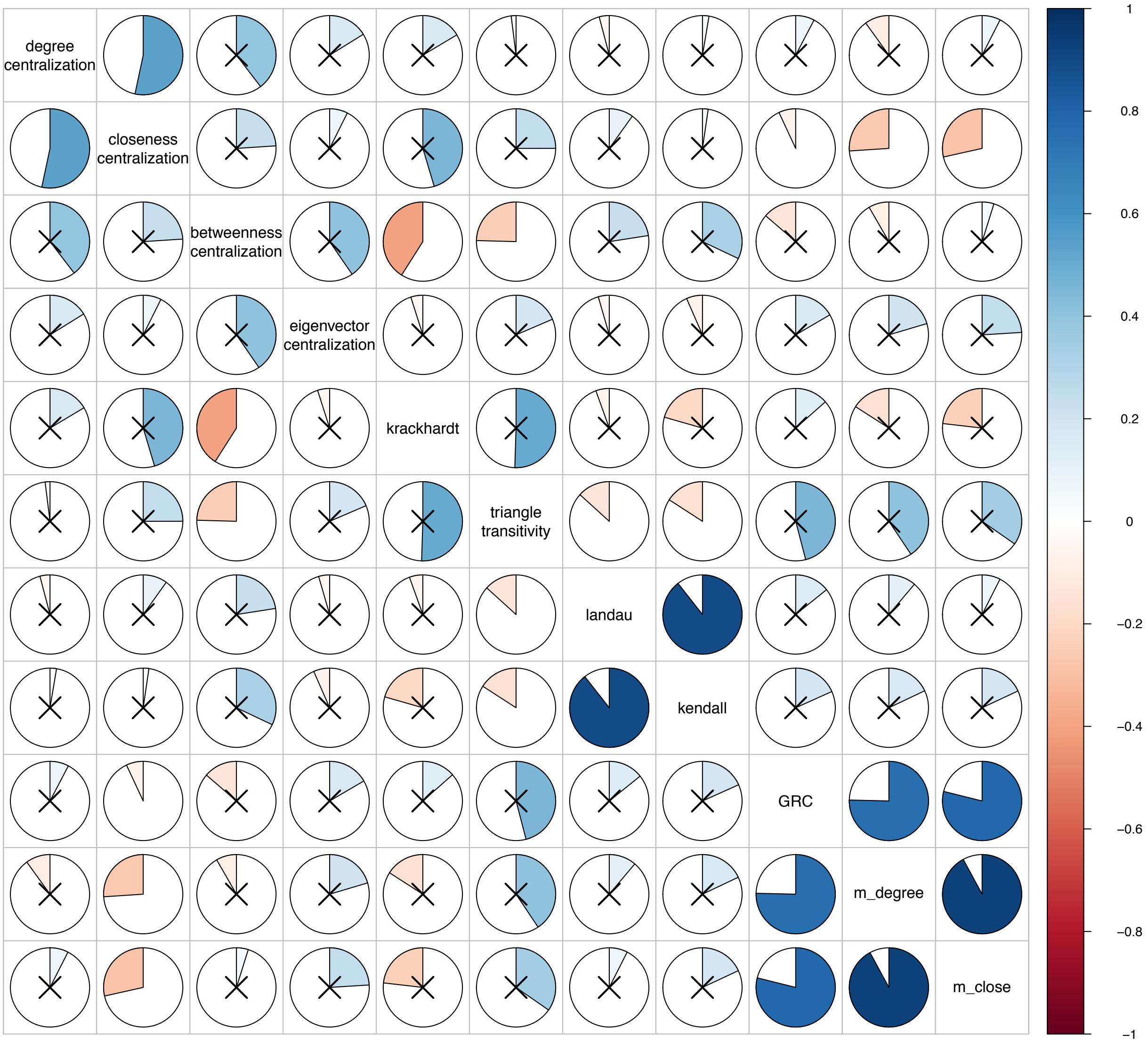
This measure takes a value of one for all directed acyclic graphs, and zero for cycles and cliques. This is a theoretically attractive measure if ties indicate a power relation, but may also be useful for analyzing communication networks, as we might expect that those higher up in the organizational hierarchy will be able to reach a larger part of the network with outgoing messages.

3.2. *Relationships Between Hierarchy Measures*

To begin to understand how these measures are related, we calculate each of these twelve measures on a total of 136 social, organizational, and information networks described in greater detail in Section 4. All of these networks are directed, and for almost all of them, we were not able to calculate the rooted depth of the network, and we therefore exclude it from our further analysis[3]. Figure 1 illustrates the correlation coefficients between the eleven hierarchy measures we were able to calculate on these networks. As we can see, Landau's h and Kendal's K are highly correlated, as are M-reach degree and M-reach closeness Gini coefficients. This makes these two pairs of measures largely redundant, but we include them in the remainder of our analysis for completeness. Landau's h and Kendal's K are also significantly correlated with the simple degree centralization of the network, and M-reach degree and M-reach closeness Gini coefficients are also significantly correlated with GRC (by construction). Also of interest, Krackhardt's hierarchy measure and triangle transitivity are both significantly negatively correlated with betweenness centralization.

[3]This is because the Rooted Depth measure can only be calculated for networks where at-least one node only has incoming ties.

Figure 1. Pearson correlation coefficients between nine measures of network hierarchy. A black **X** in a cell indicates that there was not a significant correlation between measures at the α = 0.05 level of significance. The color of the pie in each cell indicates the sign and magnitude of the correlation with dark blue indicating a large positive correlation and dark red indicating a large negative correlation. Those cells with relatively small, lightly colored pie slices have a correlation coefficient approaching zero.



Perhaps surprisingly, most of the rest of the correlations depicted in Figure 1 are relatively small and statistically insignificant, indicating that if all of these measures are valid, they are likely capturing different dimensions of hierarchy in a network. This finding suggests that either there are multiple dimensions to hierarchy in a network [19], some of these measures do not measure hierarchy, or some combination of both. To investigate this finding further, we perform several statistical and qualitative comparisons between these measures in Section 5.

4. Data

The data used in this study comprise 136 social networks, collected from three primary sources. The first of these is a set of seventeen email communication networks among department managers in North Carolina county governments. These data were collected as part of a field experiment described in Ben Aaron et al. (forthcoming), and comprise all department manager to department manager email communications over a three month period in 2013. In total, these networks include 17,863 emails sent between 362 department managers. The second set of networks record cosponsorship patterns in the United States Senate between 1973 and 2009. There are a total of eighteen networks, each recording the number of times Senator i cosponsors (officially records support for) a piece of legislation sponsored by (introduced by) legislator j during a two year session of Congress. The third primary source from which we obtain network data is the UCI-Net online network data repository[4]. We were able to obtain a total of 101 network datasets from this website, comprising a wide range of social, communication, and economic networks. Links to source data, and references for all networks used in this study will be made available in an appendix[5].

Theoretically, we might expect different types of networks (social, information, biological, etc.) to display distinct relational structures, and perhaps exhibit common hierarchical structures. We therefore decided to hand code each of the 136 networks in our sample into one of eight broad categories. Descriptive statistics for networks in each of these categories are provided in Table 1. We can see that most networks have between twenty five and forty nodes on average, with the notable exception being the cosponsorship networks, which average roughly one hundred nodes. Note that the networks classified as *unknown* have yet to be categorized due to the difficulty in navigating the UCI-Net data archives, but will be classified into one of the eight categories before publication.

Table 1. Network Descriptive statistics for all 136 networks in our sample, aggregated by the network type. All columns are averages over networks of that type.

Type	# of Networks	Nodes	Edges	Density	Clustering Coefficient
friendship	123.00	49.13	866.71	1.54	0.49
association	29.00	56.59	4202.48	2.43	0.58
exchange	17.00	48.53	1369.41	1.44	0.50
support	75.00	71.21	4428.68	1.16	0.60
ecological	11.00	33.64	349.65	0.30	0.61
kinship	5.00	24.00	81.40	0.14	0.28
perception	39.00	44.00	96.46	0.05	0.26
transportation	2.00	1374.00	14826.50	0.01	0.21
biological	3.00	790.33	3682.33	0.01	0.04

5. Analysis

5.1. Principal Components Analysis

In order to find patterns in the various measures of hierarchy presented above, we rely on Principal Components Analysis (PCA). PCA is a dimension reduction technique commonly used in the social sciences. This works by finding

[4]Data are freely available online here: https://sites.google.com/site/ucinetsoftware/datasets

[5]We did not had time to compile this appendix before the end of the semester, but it will be included in the fina version of the paper.

Table 2. PCA Rotation for Components with Eigenvalues above 1

	P.C. 1	P.C. 2	P.C. 3	P.C. 4
Degree Centralization	0.327	−0.290	0.301	−0.233
Closeness Centralization	0.319	−0.427	0.204	0.034
Betweenness Centralization	0.265	−0.100	0.356	−0.486
Eigenvector Centralization	−0.220	−0.222	0.098	−0.472
Landau	0.274	0.447	0.336	0.131
Kendall	0.281	0.437	0.348	0.114
GRC	−0.239	−0.013	0.483	0.298
M–loseness	−0.413	0.076	0.361	−0.104
M–degree	−0.447	0.080	0.281	−0.073
Krackhardt	0.167	−0.377	0.104	0.523
Triangle Transitivity	−0.251	−0.360	0.210	0.278

the eigenvalues and eigenvectors of a set of variables, such that the components identified by the model maximize the variance accounted for. For the purposes of this analysis, we have eleven hierarchy measures calculated across the 136 networks described above. PCA allows us to detect the patterns behind this eleven-dimension data, and constructs principal components such that most of the variance in the eleven dimension problem can be accounted by a fewer number of components.

Of course, there will be as many principal components as there are dimensions in the original data. PCA component eigenvalues are illustrated in Figure 2. As is clear from this figure, while there are nominally 11 principal components, a vast majority of the variance present in the original problem is captured by the first four principal components. This four dimensional problem is much easier to interpret than the original 11-dimensional problem. A graphical comparison of components one and two is provided in Figure 3.[6]

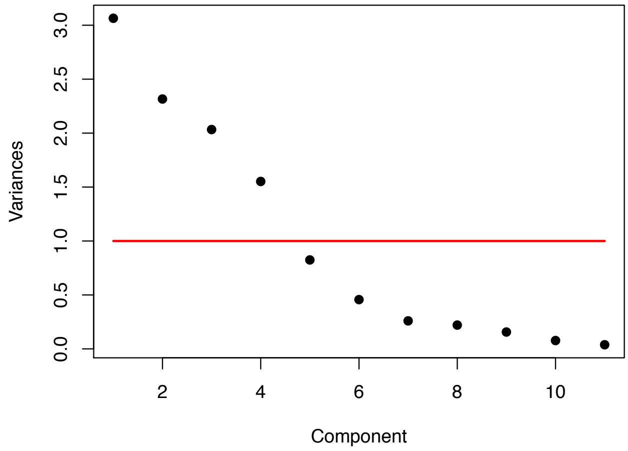
Table 3.

	PC1	PC2	PC3	PC4
Degree Centralization	0.326	−0.305	0.300	−0.184
Closeness Centralization	0.317	−0.433	0.182	0.066
Betweenness Centralization	0.263	−0.129	0.384	−0.442
Eigenvector Centralization	−0.207	−0.261	0.155	−0.470
Landau's h	0.275	0.434	0.352	0.138
Kendall's K	0.282	0.424	0.363	0.126
GRC	−0.242	−0.020	0.454	0.336
M–closeness	−0.415	0.062	0.363	−0.070
M–degree	−0.450	0.068	0.284	−0.051
Krackhardt	0.169	−0.361	0.049	0.544
Triangle Transitivity	−0.253	−0.357	0.171	0.306

Table 2 presents the rotations, or factor loadings, which describe to what extent the original variable relates to the principal components. The first principal component, which accounts for 34.8% of the variance, is most highly correlated with *m*–reach degree and *m*–reach closeness. It is not surprising that these measures load together, as *m*–reach closeness relies on *m*–reach degree for calculation. As discussed in Section 3.1, this family of hierarchy

[6] A graphical comparison of components one and three is provided in Figure A.4. A graphical comparison of components two and three is provided in Figure A.6. A graphical comparison of components one and four, two and four, and three and four are included in Figure A.5, Figure A.7, and Figure A.8, respectively.

Figure 2. Eigenvalues for 9 largest principle components in our analysis indicate that we should examine the first three components, which all have eigenvalues greater than one.



measures describes how many nodes in a dominance network can be reached from a given node. In other words, these measures describe a type of hierarchy in which diffusion of authority is recognized by higher values.

The second principal component is most correlated with Landau's h, Kendall's K, closeness centrality, and triangle transitivity. This component most closely maps onto a conception of linear hierarchy, or chain-like structures and non-cyclical triads which represents one dominant nodes providence over a greater number of subordinate nodes. The third principal component is most correlated with GRC. GRC represents the inequality among m–reach measures, or, in other words, to what extent networks have an equitable distribution of key players, or whether they are dominated by a few, highly influential nodes. This component is also highly correlated with the other M-reach measures and betweenness centrality, although not to the extent of other components.

Finally, the fourth component is most correlated with eigenvector and betweenness centrality. Betweenness centrality describes the extent to which certain nodes exercise a disproportionately central position, best in position to facilitate flows across the network. Eigenvector centrality is a measure of both the importance of the given node as well as the importance of adjacent nodes, echoing the logic that power is knowing powerful people.

While focusing on which specific measures are most correlated with each component, that does not tell the whole story. The linear hierarchy measures (Landau's h, Kendall's K, and triangle transitivity), are all at least moderately correlated with all four of the components described above. The same is true for degree, betweenness, and eigenvector centralization. This points to basic features of hierarchical systems that cut across all of the analytical measures. Indeed, it is difficult to imagine a type of hierarchical system that does not contain at least some tree-like structures, or have certain nodes with a disproportionately central role. Despite these similarities, the fact that it requires over four components to explain at least 80% of the variance in the indicators means that these measures are too dissimilar to use interchangeably.

5.2. Ground Truth Rankings

Seven of the twelve measures of hierarchy discussed above have a local analogue to the global measures we discuss in the previous sections. To further understand the relative performance of these seven measures, we compare their ability to correctly recover the rank of the highest ranking manager in the seventeen county government email networks we analyze. Using our ground-truth knowledge of who should be be ranked at the top of the organizational hierarchy, Table 4 presents average ranking scores for each of the measures. These scores theoretically range between zero and one, with a score of one indicating that the measure ranks the county manager at the top of the hierarchy in every single county, and a score of zero indicating that the county manager was ranked at the bottom of the hierarchy in every single county. The measure with the best performance is closeness centrality (the local analogue to closeness centralization), which provided 82% of optimal performance in ranking the county managers. This finding is not very surprising, because the county manager is the only member of the county government with incentives to

Figure 3. Principle components plot for components one and two.

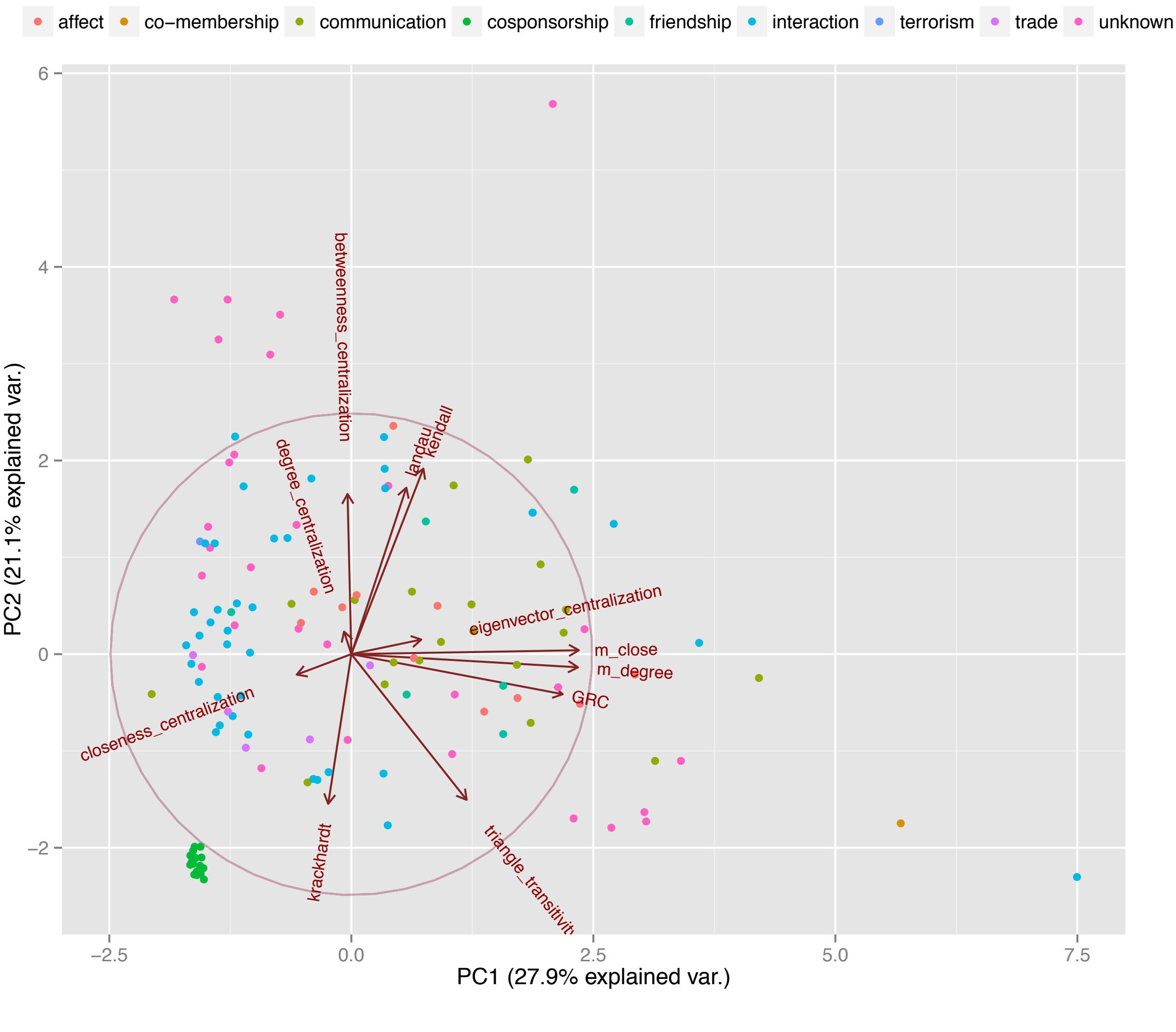


Table 4. Average Ranking index scores for each of the seven measures with a local analogue. A score closer to one indicates that a measure ranked the county manager in each county closer to the top of the hierarchy across the seventeen counties in our sample.

	Average Rank Score
Degree centrality	0.75
Closeness centrality	0.82
Betweenness centrality	0.78
Eigenvector centrality	0.78
m-degree centrality	0.81
m-close	0.63
GRC	0.36

interact with all of the other county managers directly. Generalized Reaching Centrality consistently provided the lowest performance across the board in this task, indicating it may not be a good measure for applications involving communication network data, but we generally see quite a bit of heterogeneity in the rankings across counties (see Figure A.9). It is notable that many of the measures are able to accurately rank the county manager a high proportion of the time, indicating that they are at least partially capturing the concept of hierarchy in these networks.

One of the key issues with this approach is that we cannot compare four of the measures of hierarchy because they lack a local analogue. Therefore we cannot say whether closeness centralization is the best measure of hierarchy in organizational communication networks, because we lack a valid means of comparison. Also, as noted in the previous subsection, theory indicates that there are multiple dimensions to the concept of hierarchy, something that PCA confirms. Therefore we can only cautiously note that many of these measures provide relatively strong performance, and that future research could further investigate the ground-truth performance of these measures on a wider array of networks where the hierarchical ordering is know a-priori.

6. Conclusions

Inferring and measuring the power dynamics underlying human social behavior is a fundamental problem for all social scientists. We began this paper by discussing the importance of networks in reaching a rigorous understand of social dynamics, and how the concept of power is inherently relational, and therefore ripe for a network–oriented approach. However, power can be a nebulous concept to define theoretically, let alone measure empirically. In order to guide our understanding of the latter, we make use of Michael Mann's typology of hierarchies, and provide alternative definitions to guide researchers.

Second, we survey the wide variety of analytical measures currently employed by network researchers. While there are similarities, these measures clearly focus on different aspects of network structure, and, when applied, will lead the researcher to draw different inferences. While this variety is useful to the researcher who wishes to find the exact measure to apply to her specific problem, if the measure of choice is simply *assumed* to be representative of hierarchy, this increases researcher degrees–of–freedom and could lead to uninformed applications that do not adequately square the theoretical concept in question with the analytical technique being used.

Finally, we contrasted the measures performance on a wide variety of observed social networks. First, we applied PCA to reduce the dimensionality of the various hierarchy measures. From this, the following conclusions can be drawn: *m*–reach degree, *m*–reach closeness, and GRC measure a type of diffusion hierarchy where spread of information is critical to control; Degree and closeness centralization measure the extent, rather than specific type, of hierarchy; and the remaining measures map on to Mann's authoritative power type with one capturing rigid, tree–like hierarchies, and the other capturing authoritative hierarchies where power is concentrated in an elite group. Second, for the measures where we could calculate a local score, we applied them to hierarchies with a known power structure, and found strong performance across a number of measures in recovering the rank of the most important node in the network.

This study is only a first step in a larger project of integrating sociological conceptions of hierarchy with graph-theoretic measures of power and prestige inequality in networks. Future work could consider a broader range of networks where the underlying hierarchical structure is better understood to provide a more robust characterization of the different measures of hierarchy used in this study. Future work could also incorporate statistical measures of power and hierarchy and compare them with the analytical measures used in this study. Finally, while the principal components analysis conducted in this study identified a few key underlying dimensions to the measures we consider, other clustering techniques could be applied that do not make such strong orthogonality assumptions between the the different components.

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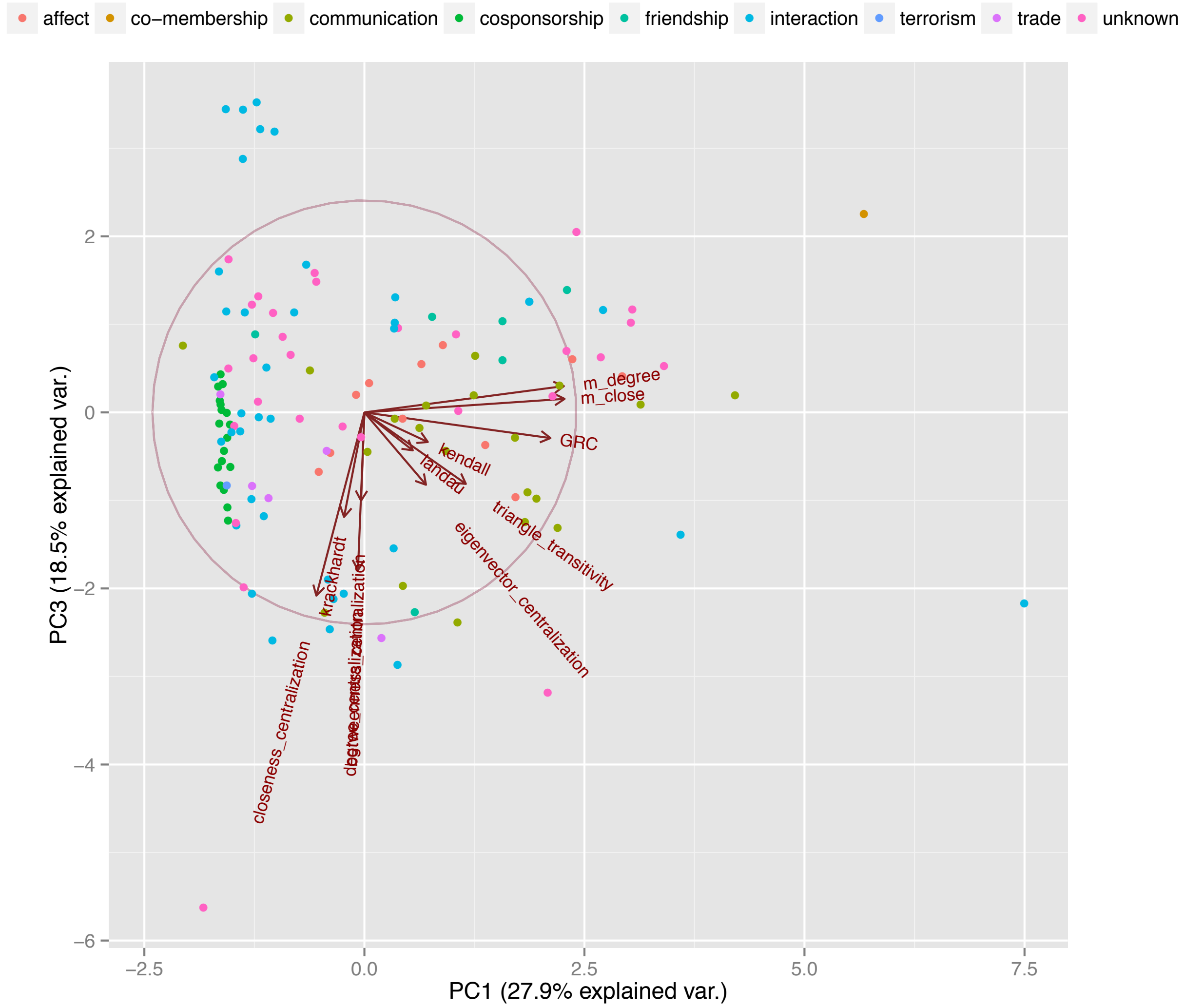
Figure A.4. Principle components plot for components one and three.



Appendix A. Additional Plots

Figure A.5. Principle components plot for components one and four.

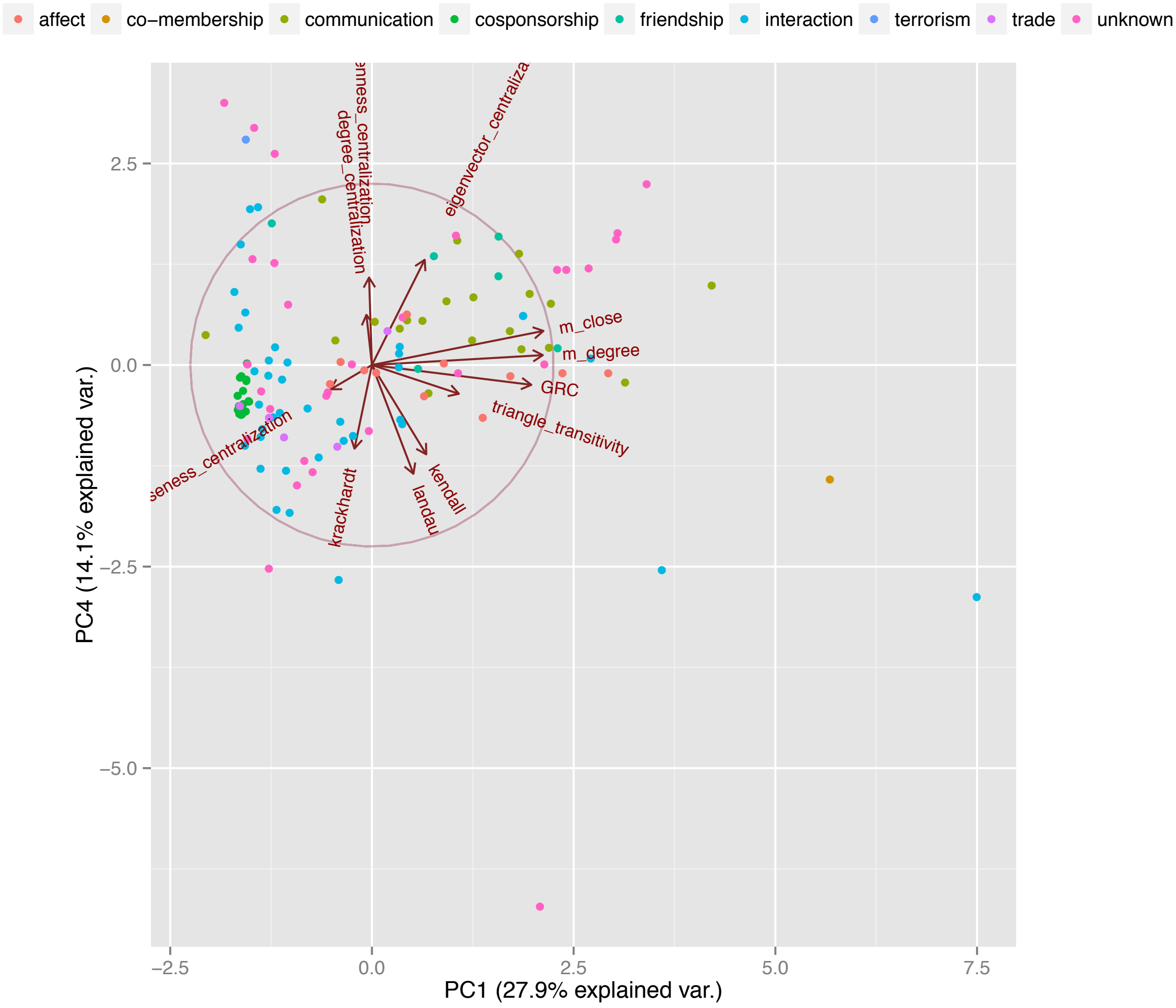


Figure A.6. Principle components plot for components two and three.

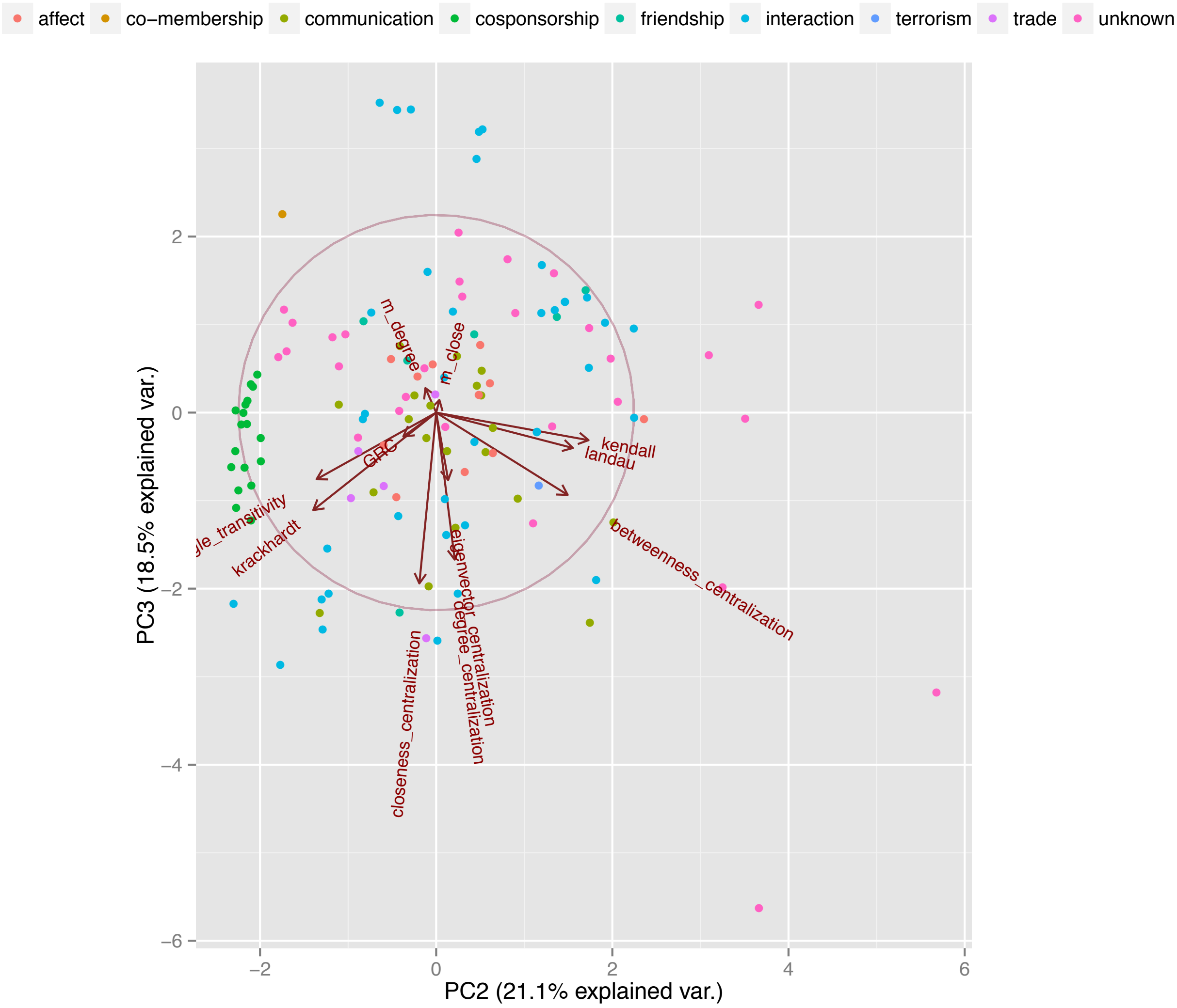


Figure A.7. Principle components plot for components two and four.

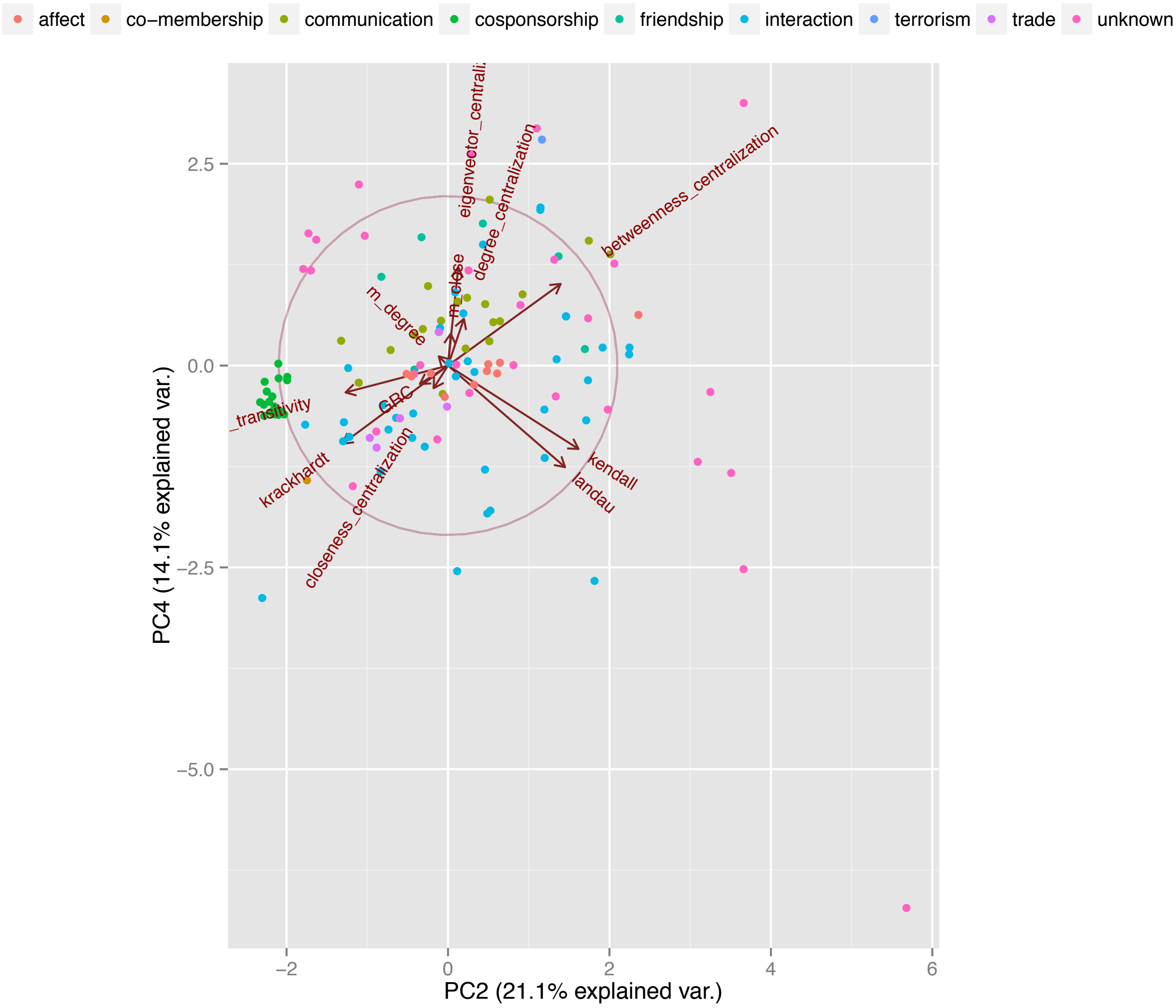


Figure A.8. Principle components plot for components three and four.

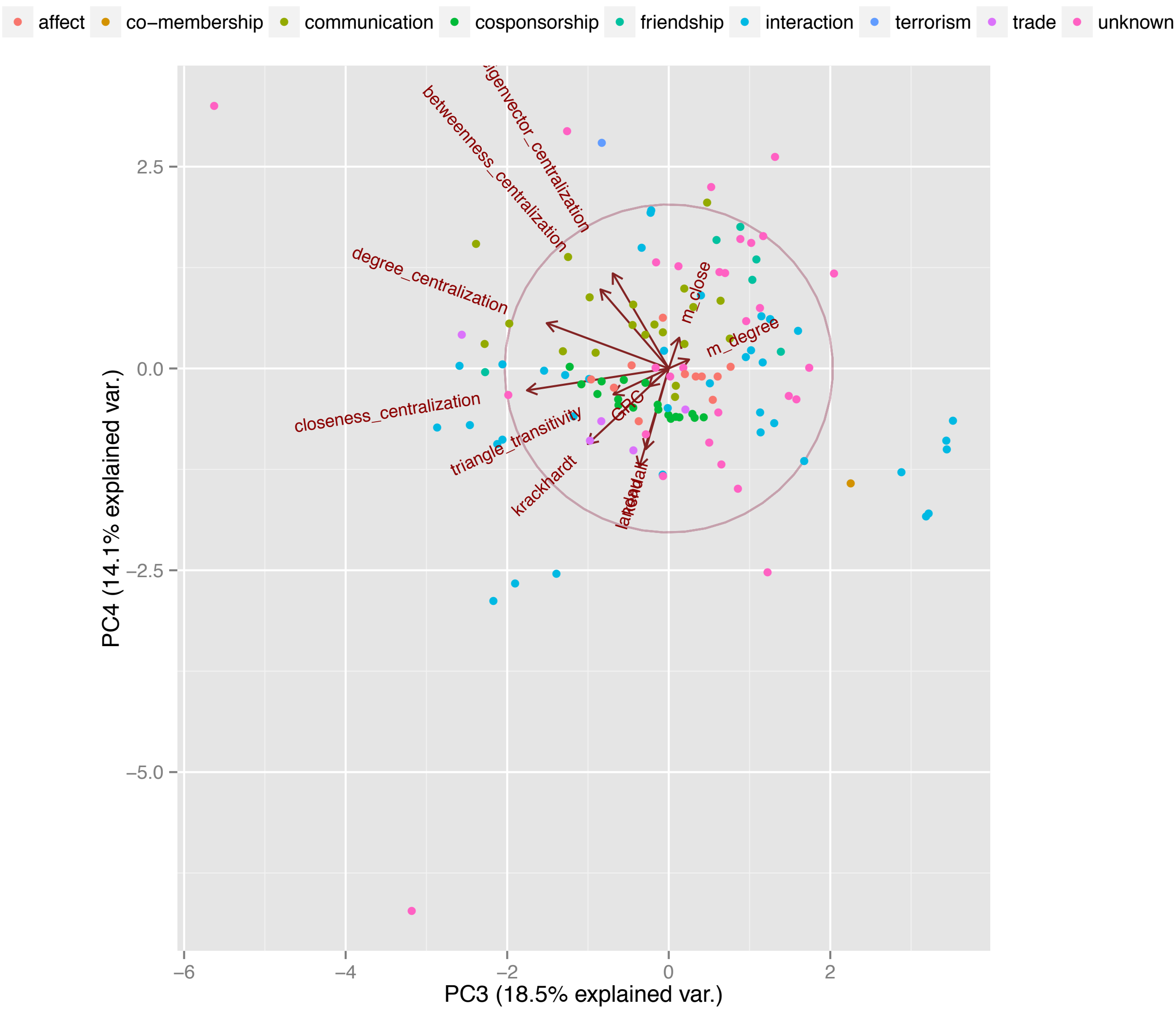


Figure A.9. Rankings scores for each county. The x-axis records how close to the top of the hierarchy a measure placed the county manager, with a higher score indicating better performance. Each column represents a different county.