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Social Networks 00 (2015) 1–7
Social Networks

Hierarchical Structure in Social Networks

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Abstract

Keywords: Hierarchy, Network, Power

1. Introduction

Hierarchy is an important feature of many organizations, such as firms, social clubs, and military units. Formally, we can define a hierarchy as a system where people or groups are ranked according to status or authority. Yet it is difficult to operationalize this definition for measurement and comparison. There has been a great deal of research on power and status in groups and organizations, but most of this research relies on measurements defined over domain specific rankings, such as job titles. At the same time, networks scholars have defined a number of broadly applicable hierarchy metrics based on network structure, but these metrics are not necessarily grounded in meaningful sociological concepts of status and authority. Contrastingly, social theorists like Michael Mann have noted the messiness of society and that a network-oriented perspective of the "sociospatial and organizational model [of a network]" can explicate the "sources of social power," [20] but they have generally not delved into the methodologies through which to fully explore such power dynamics. In this paper, we seek to bring together these two areas of research, and to develop a framework for measuring hierarchy in social networks that is both generally applicable and exhibits a high degree of construct validity.

Without statistical models/mathematical measurements for hierarchy which are theoretically based, and vice versa; theory that can be statistical/mathematically quantified and verified, the conceptual idea of hierarchy cannot be fully understood. We do not suggest that this project will achieve an overreaching theory and methods, but we strive to take the first step. At the very least, we will try to demonstrate the need for a united theory and corresponding methods. As an interdisciplinary team, we are in the unique position to accomplish our goals.

2. Sociological Theories of Hierarchy

We are still working through evaluating a few different datasets to best suit our purposes. However, at present, this is a little difficult because we really want our measure to be theoretically-grounded, but we haven't yet developed a solid theoretical conception for hierarchy. Thus far, theory-wise, the Mann (1986) definition seems closest to the

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Liu-Driver measures discussed in the Mones et al. (2012) article: i.e., hierarchical networks are those in which the actions of a few nodes are needed to take control of the graph. Another potential definition, also implied, is hierarchy means the mechanisms of collective actions (i.e., the ability of different nodes to connect with one another) hinges on a small number.

3. Measuring Hierarchy

A number of analytical measures of hierarchy have been proposed for directed networks. Most proposed measures return a scalar value that is meant to capture the "hierarchical-ness" of a given network [2, 7, 13, 14, 24]. This theoretically facilitates comparison between measures calculated on the same network, as well as comparison on the same measure across multiple networks. However some approaches to measuring hierarchy only provide a local measure of importance or position in the hierarchy for each node in the network [1]. For these measures, one can calculate a Gini coefficient [37] from the individual level scores and use these coefficients as a proxy for a global measure.

In this section, we introduce and describe ten candidate measures of network hierarchy that have been previously used in the networks literature. It is important to note that most of the measures we consider are only defined for directed networks, and thus for the remained of this paper we assume all networks under consideration are directed. We begin by introducing some terminology that will be common across all measures. For a given network $G = (V, E)$, let $V = \{v_i\}_{i=1}^{N}$ be the set of N vertices (nodes) associated with G, and $E = \{e_j\}_{j=1}^{M}$ be the set of M edges associated with G. Furthermore, for a given edge e_j, let $e_j^{(s)} \in 1 : N$ be the index of the sender of the edge and $e_j^{(r)} \in 1 : N$ be the index of the recipient.

One other important point is that most measures of network hierarchy are meant to be applied to networks where the edge sets capture relations other than "has power over". In this special case, it is theoretically easier to construct a measure of network hierarchy since the network must be directed and acyclic (preventing circular chains of command). However, obtaining such information is usually impossible in most cases (with military personell networks being an obvious exception). Furthermore, if the researcher has collected such an edge-set, then the need for summary measures of the "hierarchical-ness" of the network is likely obviated, as deeper insights could be gained from applying inferential network analysis tools to the raw network. Therefore, we focus or attention on the measurement of hierarchy on networks where edges are not explicitly power relations.

3.1. Analytical Measures of Hierarchy

The most basic measures of hierarchy or differential importance of a nodes in a network can all be derived from basic node-level measures of network centrality [34]. To aggregate from these node-level measures up to a singel measure on a network, one can calculate C, the centralization of the network. For given local measures $\mathbf{c} = \{c_i\}_{i=1}^{N}$, the corresponding centralization measure is defined as:

$$C = \sum_{i=1}^{N} (\max\{c_i\} - c_i) \tag{1}$$

Centralization captures the degree of inequality in the distribution of a given centrality measure over the network. In general, we should then expect that networks that are more centralized are also likely to be more hierarchical. However, this measure attains its maximal value for any centrality measure when one node has a maximal value of the given centrality measure and all the rest of the nodes have the minimal possible value. This will tend to give star networks maximal centralization scores. This implicit assumption in measurement is important to consider when evaluating the validity of centralization based measures of hierarchy. The four centrality measures we consider are: degree centrality, closeness centrality, betweenness centrality, and eigenvector centrality.

The **degree centrality** of node v_i is simply the number of outgoing or incoming edges incident to it. Formally, this can be calculated as:

$$\text{indegree centrality}_i = \text{In}_i = \sum_{j=1}^{M} \mathbb{1}(e_j^{(r)} = i) \tag{2}$$

$$\text{outdegree centrality}_i = \text{Out}_i = \sum_{j=1}^{M} \mathbb{1}(e_j^{(s)} = i) \tag{3}$$

Degree centrality captures the number of friends or connections a node has, and intuitively, we should expect more powerful nodes will have more incoming and outgoing connections. However, degree centrality does not account for the identity of a node's alters. This can lead to difficulties when it is used to asses a node's position in a social hierarchy. For example, in a large company, we might expect that an administrative assistant may have a higher degree centrality than the CEO of a company if the edges being measured are work-related interaction. Furthermore, if there are many administrative assistants at the bottom of the power structure in an organization we might qualitatively consider to be extremely hierarchically structured, the degree centralization of the interaction network might be lower than in a comparably sized "organizationally flat" organization where a handfull of people serve as coordinators. Thus we must take great care in interpreting this statistic, depending on the type of edges it is defined over.

The **betweenness centrality** of node v_i is a measure of the amount of influence a node has on the information transversed through it [7]. Define $D_{i,j}$ as the number of shortest paths in G between v_i and v_j, and $D_{i,j}(k)$ as the number of these shortest paths that pass through v_k, then the betweenness centrality of node k is:

$$\sum_{i \neq k \neq j} \left(\frac{D_{i,j}(k)}{D_{i,j}}\right) \tag{4}$$

Betweenness centrality captures how in-the-middle-of-things a node is and when edges involve sharing information, how important the node is to information flowing quickly across the network, Intuitively, we should expect the nodes with higher betweenness centrality will be more powerful, and that greater inequality on this measure should signal a greater degree of hierarchical structure in a network. Similarly, the **closeness centrality** of node v_i is a measure of how few intermediate edges a given node must traverse to reach all other nodes in the network. Define $d(i, j)$ as the length of the shortest path between v_i and v_j, then closeness centrality of node i is:

$$\sum_{i \neq j} \left(\frac{1}{d(i, j)}\right) \tag{5}$$

Again, intuitively, powerful members of a hierarchy will tend to be able to reach others in the network more easily, and inequality in this measure (as captured by the closeness centralization of the network) should theoretically signal the degree of hierarchy in the network.

The last of the classical centrality based measures is **eigenvector centrality**, which is meant to capture the degree to which a node is connected to other well connected nodes [2]. Let the adjacency matrix of the network be defined as $\mathbf{A}$ and the vector of local eigenvector centrality scores for each node be defined as $\mathbf{w} = \{w(v_1), ..., w(v_V)\}$. Then to calculate the eigenvalue centralities $\mathbf{w}$, one must solve the following eigenvector equation:

$$\mathbf{A}\mathbf{w} = \lambda \mathbf{w} \tag{6}$$

where λ is a vector of positive eigenvalues. The challenge is to find the *dominant eigenvector*, as only the largest eigenvalue results in the desired centrality measure for each node. Intuitively, higher eigenvector centrality should be associated with greater social or organizational importance. Interestingly, the eigenvector centralization of a network can still be large even if there are a relatively large proportion of higher degree nodes, as long as the edges are organized such that only a few nodes are connected to all of these higher degree nodes.

Landau's h is used to compare a directed network to a perfect linear hierarchy (a strict dominance-ordering of nodes) [14, 30]. This measure is defined as follows:

$$h = \frac{12}{N^3 - N} \sum_{i=1}^{N} \left[\text{Out}_i - \frac{N-1}{2} \right]^2 \tag{7}$$

where $h \in [0, 1]$. Note that Landau's h does not provide an individual level metric of importance or relative power. This measure was specifically designed to operate on networks of "has power over" edges, and thus may be difficult to interpret when the network is not explicitly defined on power relations. Because of a preference for chain-like structures, this measure will likely provide poor performance when edges measure social relations. **Kendall's K** is also designed to compare a directed network to a perfect linear hierarchy [13]. As noted in [30], it often gives an identical value to Landau's h, but is theoretically distinct. Begin by defining the number of *cyclic triads* (CyT) in the network as:

$$CyT = \frac{N(N-1)(2N-1)}{12} - \frac{1}{2} \sum \text{Out}_i^2 \tag{8}$$

Then we can define Kendal's K as

$$K = 1 - \frac{d}{d_{max}} \tag{9}$$

where the value d_{max} is defined as follows:

$$d_{max} = \begin{cases} \frac{1}{24}(N^3 - N) & \text{if } N \text{ is odd} \\ \frac{1}{24}(N^3 - 4N) & \text{if } N \text{ is even} \end{cases} \tag{10}$$

Kendal's $K \in [0, 1]$ is also only defined as a global measure, and no individual level analogue exists. It was also specifically designed to operate on networks of "has power over" edges, and is therefore likely a poor choice for networks defined over social interactions.

M-reach degree was developed to identify 'key' players in a network [1]. It is defined as a measure for each node v_i as the number of alters that are reachable from v_i. If G is directed then the reachable alters must lie along an outgoing path from v_i. A closely related measure, **M-reach closeness** is defined as the M-reach degree of a node, but with the contribution of each alter j that is reachable, weighted by the inverse of the shortest path length between i and j. Both of these measures are only defined for individual nodes, so to calculate a global measure for a network, we take the Gini coefficient of the measures calculated for each node i. The Gini coefficient [37] is a measure of inequality originally developed to measure income inequality in a society. For a vector of values $x : \{x_i\}_{i=1}^{N}$ the Gini coefficient G of that vector is:

$$G = \frac{\sum_i \sum_j |x_i - x_j|}{2 \sum_i \sum_j x_i} \tag{11}$$

In words, it is half of the *average absolute difference* of all pairs of entries in the vector, divided by the average, which acts as a normalizing constant. We take the Gini coefficient of the M-reach degree and M-reach closeness values for each node in the network and treat this as our global measure of network hierarchy. Intuitively, this measure is capturing the degree of inequality in access to other nodes in the network. We should expect that a higher degree of inequality in this metric would be associated with a more hierarchical network. One potential shortcoming of these measures is that they only account for out-degree, when incoming ties may be a better signal of power relations in many real-world social networks.

Global reaching centrality (GRC) is a generalization of M-reach degree centrality measure [24], and was designed specifically to measure on any network. When the network is unweighted and directed, let $C_R(i)$ be the local

M-reach degree centrality of node i, then the global reaching centrality of the network can be defined as:

$$GRC = \frac{\sum_{i=1}^{N} [\max(C_R) - C_R(i)]}{N-1} \tag{12}$$

When the graph is weighted and directed, the authors in [24] propose an alternative formulation of $C_R(i)$:

$$C_R^{'}(i) = \frac{1}{N-1} \sum_{j:0<d_{(i,j)}^{out}<\infty} \left(\frac{\sum_{k=1}^{d^{out}(i,j)} w_i^{(k)}(j)}{d^{out}(i,j)}\right) \tag{13}$$

where $d_{i,j}^{out}$ is the (directed) path length from node i to node j, and $w_i^{(k)}$ is the weight of the kth edge along this path. This alternative formulation can then be plugged into equation 12 to calculate the GRC for a weighted,directed network. Finally, when the network is unweighted and undirected, the authors in [24] propose an additional alternative formulation of $C_R(i)$:

$$C_R^{''}(i) = \frac{1}{N-1} \sum_{j:0<d(i,j)<\infty} \frac{1}{d(i,j)} \tag{14}$$

where $d_{i,j}$ is the (undirected) path length from node i to node j. The intuition of for the interpretation of GRC as a measure of network hierarchy is almost identical to the interpretation of the Gini coefficient of the M-reach degree and M-reach closeness values for a network, with only difference being the way the individual measures are aggregated, and its extension to weighted and undirected networks. Again, a potential major shortcoming is that this measure only makes theoretical sense when the outgoing edges in the network signal a power relation where the sender has power over the recipient.

The last measure we consider, **rooted depth** [31], is only defined for networks where a root (a node that has only incoming edges) exists. Let N_r be the number of node-root pairs in the network. Then the rooted depth of the network can be defined as:

$$D = \frac{1}{N_r} \sum_{i=1}^{N_r} l_{ri} \tag{15}$$

where l is the length of the shortest path between root r and node i. This measure can only be calculated for the network as a whole. However, given R roots in a network, we can calculate a local root depth for each node that is equal to the average length of the shortest path between itself and all roots. One of the major problems with this measure is that it is undefined for networks without a root (a node with only incoming edges). This makes it very difficult to apply to most networks (we were only able to calculate it for a small fraction of networks in our sample). Furthermore, since this measure relies on finding nodes with no outgoing ties, it can be very sensitive to the way the network is measured. In general, we do not find rooted depth to be a useful measure in our empirical analysis.

4. Data

A list of references for the network datasets used in this study can be found in Appendix A.

5. Analysis

6. Conclusions

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Appendix A. Dataset References

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