Models of Distributed Systems

Lecture 1

Overview

Overview

Contact

- Email:otto@infoiasi.ro
- Office: C211
- Course web page:

  http://www.infoiasi.ro/~otto/msd.html

Evaluation

$$\textit{Final Grade} = 50\% LSA + 50\% T$$

- T - written test (a grade from 1 to 10)
- LSA-seminar and laboratory activity (a grade from 0 to 10):
  - written test (40%)
  - homework (30%)
  - activity during laboratory - (30%)
  - a scientific paper presentation (bonus)
- Minimal conditions: any student should attend at least 7 laboratories, $LSA \geq 5$, $T \geq 5$

Distributed Systems/Computing

- a system which consists of several autonomous computational entities/processes/components
- each entity/process/component can have its own local memory/resources
- the entities communicate with each other by message passing or shared memory
- the entities have a common goal (e.g. solving a large computational problem, execution of a certain task)
- characteristics of a distributed system: parallelism, synchronization, communication

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Distributed Systems

- distributed systems:
  - mobile telephone networks
  - computer networks
  - industrial control systems
  - distributed manufacturing systems
  - distributed software systems

Models for Distributed Systems

The modelling and verification of distributed systems

- Process Calculi
  - CSP (Communicating Sequential Processes)
  - CCS (Calculus of Communicating Processes)
  - π-calculus
- rewriting logic
- automata
- Petri nets - C. A. Petri, 1962

Petri Nets

- Carl Adam Petri, 1962
- formal method used for the modelling and verification of distributed/concurrent systems
- bipartite graphs
- explicit representation of the states and events in a system
- intuitive graphical representation
- formal semantics
- expressiveness (concurrency, nondeterminism, communication,synchronization)
- analysis methods for their properties
- software tools for simulation and analysis of properties

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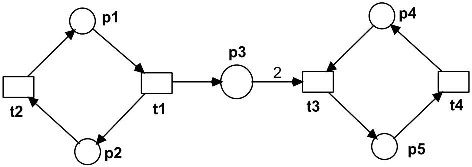
Petri Nets - Definition

Definition 1

A Petri Nets is a 4-uplu $N = (P, T, F, W)$ *such that :*

1. P *- a set of places,* T *- a set of transitions,* $P \cap T = \emptyset$*;*
2. $F \subseteq (P \times T) \cup (T \times P)$ *the flow relation;*
3. $W : (P \times T) \cup (T \times P) \to \mathbb{N}$ *the weight function* *(*$W(x, y) = 0$ *iff* $(x, y) \notin F$*).*



Petri Nets - Definition

If $x \in P \cup T$, then:

- Pre-set of x: $\bullet x = \{y | (y, x) \in F\}$;
- Post-set of x: $x \bullet = \{y | (x, y) \in F\}$.

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Definition 2

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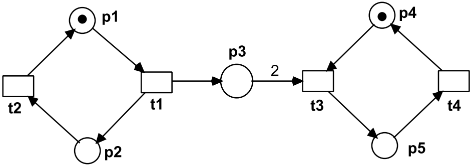
Definition 3

A net does not have isolated elements if for all $x \in P \cup T$, $\bullet x \cup x \bullet \neq \emptyset$

Marking of a Petri Net

Definition 4 (Marking, marked net)

- *Let $N = (P, T, F, W)$ be a Petri net. A marking of N is a function $M : P \to \mathbb{N}$.*
- *Let $N = (P, T, F, W)$ be a Petri net net and $M_0 : P \to \mathbb{N}$. (N, M_0) is called a marked Petri net.*



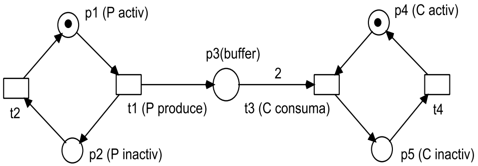
- The distribution of tokens in the places of the net = the marking of the net (the state of the modelled system)

Example

Producer-Consumer System

- A producer (P) - can be in two states: active and inactive;
- A consumer (C) - can be in two states: active and inactive;
- If the producer is active, it can produce a product and place it into a buffer; after producing a product, P becomes inactive;
- If the producer is inactive, it can become active again;
- If the consumer is active, and there are at least two products in the buffer, it can consume two products and become inactive;
- If the consumer is inactive, it can become active again

Example


p1 (P activ)
p4 (C activ)
p3(buffer)
2
t2
t1 (P produce)
t3 (C consuma)
t4
p2 (P inactiv)
p5 (C inactiv)

Firing Rule

Definition 5

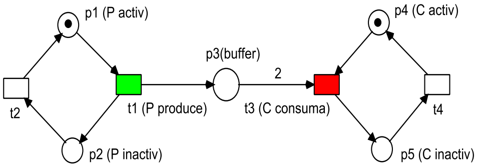
Let $N = (P, T, F, W)$ be a Petri net, M a marking of N and $t \in T$ a transition of N.

- *Transition t is enabled in marking M ($M[t\rangle_N$) if $W(p,t) \leq M(p)$, for all $p \in \bullet t$.*
- *If t is enabled in marking M, then t can fire, producing a new marking M' ($M[t\rangle_N M'$), where*

$$M'(p) = M(p) - W(p,t) + W(t,p),$$

for all $p \in P$.

Example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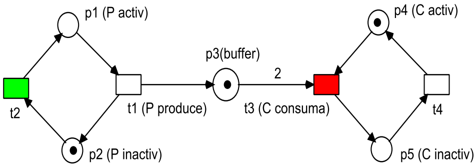
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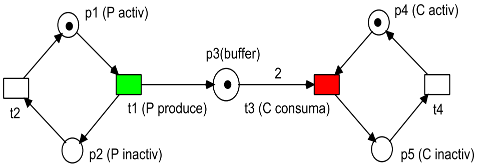
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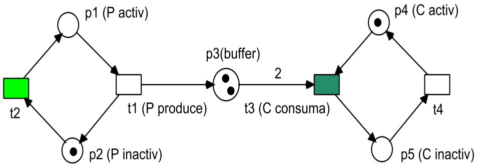
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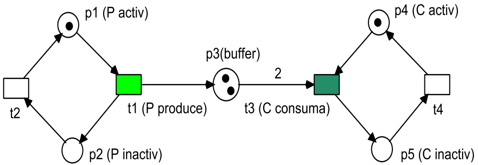
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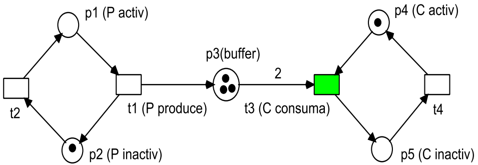
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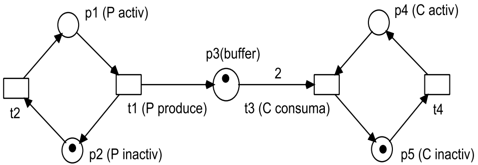
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Occurrence Sequences

- The extension of the firing rule to sequences of transitions
- Let $\sigma \in T^*$ be a sequence of transitions and M a marking.
  - ϵ (the empty sequence) is a transition sequence enabled in M: $M[\epsilon\rangle M$
  - if u is a transition sequence enabled in M such that $M[u\rangle M'$ and $M'[t\rangle M''$, then ut is a transition sequence enabled in M and $M[ut\rangle M''$ ($u \in T^*$, M', M'' markings)
- If $\sigma \in T^*$ and $M[\sigma\rangle$, σ is called an occurrence (firing) sequence from M (or enabled in M).

Occurrence Sequences

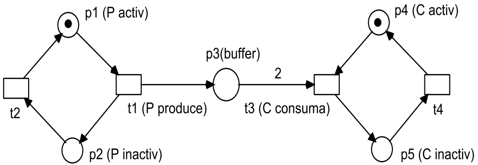
Let $\gamma = (N, M_0)$ be a marked Petri net and the following functions:

- $t^- : P \to \mathbb{N}$, $t^-(p) = W(p,t), \forall p \in P$
- $t^+ : P \to \mathbb{N}$, $t^+(p) = W(t,p), \forall p \in P$
- $\Delta t : P \to \mathbb{Z}$, $\Delta t(p) = W(t,p) - W(p,t)$

If $\sigma \in T^*$ is a transition sequence, then $\Delta\sigma : P \to \mathbb{Z}$, where:

- if $\sigma = \epsilon$, then $\Delta\sigma$ is the function 0.
- if $\sigma = t_1, \ldots, t_n$, then $\Delta\sigma = \sum_{i=1}^{n} \Delta t_i$.

Example



Proposition 1

Let t be a transition, $\sigma \in T^*$ and M, M' markings.

- *If $M[t\rangle M'$, then $M' = M + \Delta t$.*
- *If $M[\sigma\rangle M'$, then $M' = M + \Delta\sigma$*

Properties of Occurrence Sequences

Notation

If σ is a transition sequence and U a set of transitions, $\sigma|_U$ is the sequence of transitions obtained from σ, by keeping only those transitions which are from U.

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Let N be a Petri net, $U, V \subseteq T$ such that $V \bullet \cap \bullet U = \emptyset$. If $\sigma \in (U \cup V)^*$ such that $M[\sigma\rangle M'$, then $M[\sigma|_U \sigma|_V\rangle M'$.

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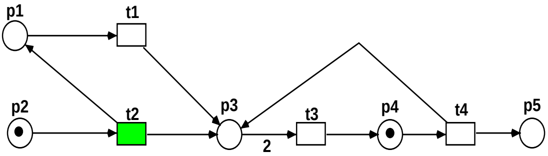
- $U = \{t_1, t_2\}$, $V = \{t_3, t_4\}$ $M = (0, 1, 0, 1, 0)$



$\sigma = t_2 t_4 t_3 t_1 t_4$

Reachable Markings

Definition 6

Let $\gamma = (N, M_0)$ be a marked Petri net. A marking M' is reachable from marking M, if there exists a finite occurrence sequence σ such that : $M[\sigma\rangle M'$.

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Marking M is reachable in γ, if M is reachable from the initial marking M_0.

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Marking M is reachable in γ, if M is reachable from the initial marking M_0.

- The set of reachable markings in γ is denoted by $[M_0\rangle_\gamma$

Overview

Concurrent Transitions

Definition 8

Let $N = (P, T, F, W)$ *be a Petri net. A set of transitions* $U \subseteq T$ *is concurrently enabled in a marking* M *of* N *if:*

$$(\forall p \in P)(\Sigma_{t \in U} W(p, t) \leq M(p)).$$

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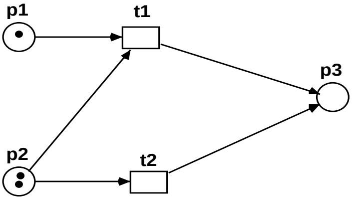

$W(p_1, t_1) + W(p_1, t_2) = 1 \leq M(p_1)$

$W(p_2, t_1) + W(p_2, t_2) = 2 \leq M(p_2)$

Properties

Proposition 2

If U is a set of transitions concurrently enabled in M, then any permutation σ of the transitions from U is enabled in M and always the same marking M' is obtained ($M[\sigma\rangle M'$).

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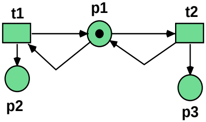
The reverse is not true: $M[t_1t_2\rangle M'$ and $M[t_2t_1\rangle M'$ does not always imply that t_1 and t_2 are concurrently enabled:



$(1,0,0)[t_1\rangle(1,1,0)[t_2\rangle(1,1,1)$

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Properties

Proposition 2

If U is a set of transitions concurrently enabled in M, then any permutation σ of the transitions from U is enabled in M and always the same marking M' is obtained ($M[\sigma\rangle M'$).

Proposition 3

Let N be a pure net, $t_1, t_2 \in T$ and M a marking of N. Then t_1 and t_2 are concurrently enabled in M iff $M[t_1t_2\rangle M'$ and $M[t_2t_1\rangle M'$.

Overview

The Boundness Property

Definition 9

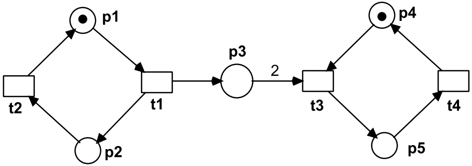
Let $\gamma = (M, M_0)$ be a marked Petri net.

- *A place p is bounded if:*

$$(\exists\, n \in \mathbb{N})(\forall M \in [M_0\rangle)(\, M(p) \leq n)$$

- *The marked net γ is bounded if each place $p \in P$ is bounded.*

Boundness-example



- p_3 unbounded place
- p_1, p_2, p_4, p_5 bounded places

Properties

Proposition 4

A marked Petri net $\gamma = (N, M_0)$ *is bounded iff the set* $[M_0\rangle$ *is finite.*

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A marked Petri net $\gamma = (N, M_0)$ is bounded iff the set $[M_0\rangle$ is finite.

($\Longrightarrow$) Let n such that $(\forall M \in [M_0\rangle)(\forall p \in P)(M(p) \leq n)$. The maximum number of markings is $(n+1)^{|P|}$.

($\Longleftarrow$) Let $n = max\{M(p) | M \in [M_0\rangle,\ p \in P\}$.

Properties

Proposition 4

A marked Petri net $\gamma = (N, M_0)$ *is bounded iff the set* $[M_0\rangle$ *is finite.*

Proposition 5

If $\gamma = (N, M_0)$ *is bounded, there do not exist two markings* $M_1, M_2 \in [M_0\rangle$ *such that* $M_1[*\rangle M_2$ *and* $M_2 > M_1$.

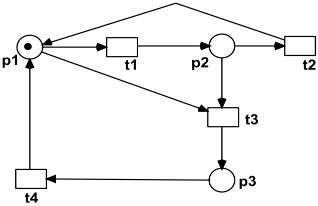
Definition

Definition 10 (quasi-liveness)

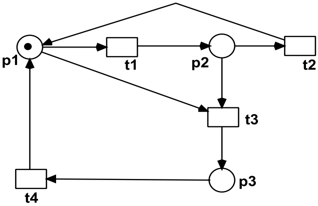
Let $\gamma = (N, M_0)$ be a marked Petri net .

- *A transition $t \in T$ is quasi-live from marking M, if there exists a marking $M' \in [M\rangle$ such that $M'[t\rangle$.*
- *A transition $t \in T$ is quasi-live if it is quasi-live from marking M_0 (there exists a reachable marking $M \in [M_0\rangle$ such that $M[t\rangle$). If a transition is not quasi-live, it is also called a dead transition.*
- *The marked Petri net γ is quasi-live if all its transitions are quasi-live.*

Examples



Examples



- t_3, t_4 dead transitions
- t_1, t_2 quasi-live transitions

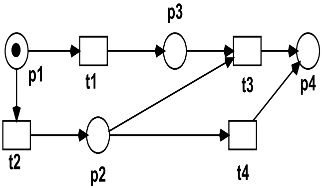
Properties: deadlocks

Definition 11 (deadlock)

Let $\gamma = (N, M_0)$ be a marked Petri net .

- *A marking M of γ is dead if there does not exist any $t \in T$ such that $M[t\rangle$. M is also called a deadlock.*
- *The net γ is deadlock-free, if it does not have any dead reachable markings.*

Example



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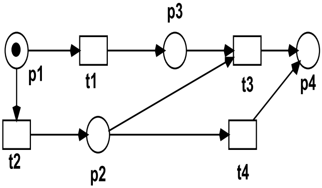


- $(0, 0, 0, 1)$,$(0, 0, 1, 0)$ are dead reachable markings

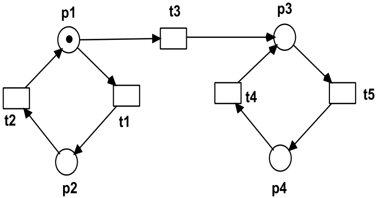
Liveness-definition

Definition 12 (liveness)

Let $N = (P, T, F, W)$ be a Petri net and $\gamma = (N, M_0)$ a marked Petri net .

- *A transition $t \in T$ is live if $\forall M \in [M_0\rangle$, t is quasi-live from M ($\exists M' \in [M\rangle$ such that $M'[t\rangle$).*
- *The marked net γ is live if any transition $t \in T$ is live.*

Example



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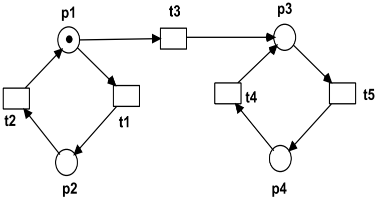


- t_1,t_2,t_3: are not live
- t_4,t_5: live
- the net is quasi-live

Home Markings

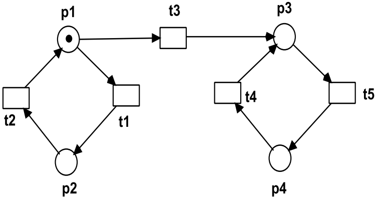
Definition 13

Let $\gamma = (N, M_0)$ *be a marked Petri net and* H *a marking.* H *is home marking iff for all* $M \in [M_0\rangle$, $H \in [M\rangle$.

Home Markings

Definition 13

Let $\gamma = (N, M_0)$ be a marked Petri net and H a marking. H is home marking iff for all $M \in [M_0\rangle$, $H \in [M\rangle$.



$M = (0, 0, 1, 0)$ home marking

Reversibility-definition

Definition 14

The marked net γ is reversible iff its initial marking is a home marking.

Reversibility-definition

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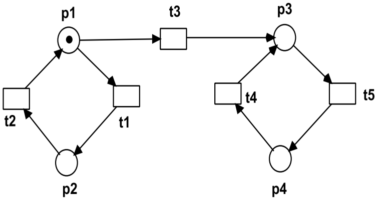

Reversibility-definition

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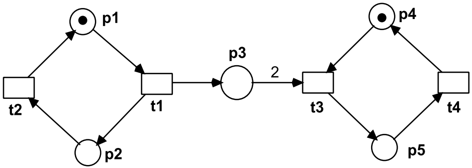

Reversibility-definition

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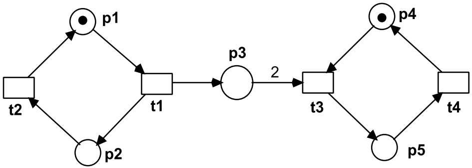


Proposition 6

A Petri net is reversible iff al its reachable markings are home markings

Properties in Live Petri Nets

Let $\gamma = (N, M_0)$ be a marked Petri net .

Proposition 7

If γ is live, then it is quasi-live.

Properties in Live Petri Nets

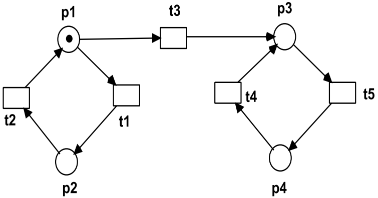
Let $\gamma = (N, M_0)$ be a marked Petri net .

Proposition 7

If γ is live, then it is quasi-live.

Proposition 8

If γ is live and it has at least one transition, then γ is deadlock-free.



- quasi-live net, without deadlocks.
- not live

Properties in Reversible Petri Nets

Proposition 9

A marked reversible net is live iff it is quasi-live.

Properties in Reversible Petri Nets

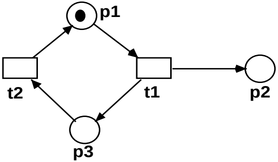
Proposition 9

A marked reversible net is live iff it is quasi-live.

Proposition 10

A marked reversible net is deadlock-free.

Example

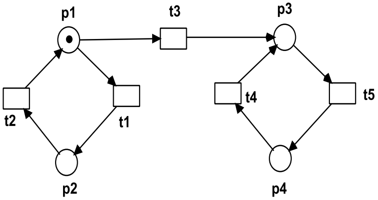


- Live but not reversible net:

  $(1, 0, 0,)[t_1\rangle(0, 1, 1)[t_2\rangle(1, 1, 0)[t_3\rangle.$

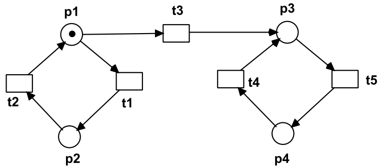
  The initial marking $(1, 0, 0)$ is not reachable from $(1, 1, 0)$.

Example

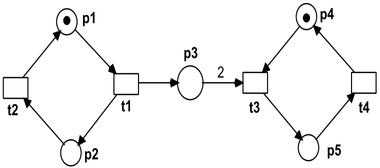


- deadlock-free net, not reversible

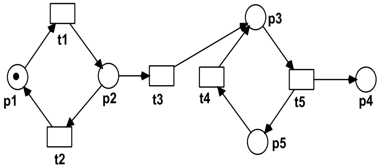
Boundness and Liveness



- bounded net, not live



- unbounded, live net



- unbounded, not live

Boundness and Liveness

Theorem 1

Any connected Petri net (there is an undirected path between any pair of elements) which is bounded and live is strongly connected.

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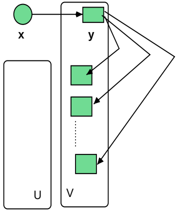
Proof: It can be proven that for any $(x, y) \in F$, there exists a path from y to x.



Case 1: $x \in P, y \in T$.

Let $V = \{t \in T | \text{there exists a path from y to t}\}$ $(y \in V)$

$U = \{t \in T | \text{there does not exist a path from y to t}\}$

$V \bullet \cap \bullet U = \emptyset$.

Boundness and Liveness

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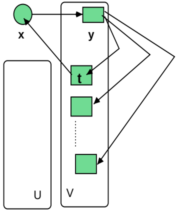

Case 1: $x \in P, y \in T$.

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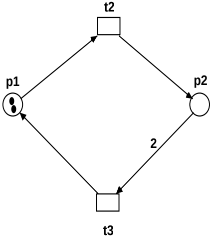
Boundness and Liveness

The reverse does not hold: the net is strongly connected, but it is not live



Models of Distributed Systems

Lecture 2

Overview

1. Properties of Petri Nets
   - Fairness
2. Analysis Methods for Petri Nets
   - Reachability Graph
   - The strongly connected components graph (SCC - graph)
   - Coverability Tree/Graph

Overview

1. Properties of Petri Nets
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Fairness

- Let σ be an infinite occurence sequence:

$$\sigma = M_1[t_1\rangle M_2[t_2\rangle M_3 \dots$$

  - $\mathcal{M}(\sigma)$: the set off all the markings that appear in σ
  - $\mathcal{T}(\sigma)$: the set off all the transitions that appear in σ
  - $OC_t(\sigma)$: the number of occurrences of t in σ
  - $EN_t(\sigma)$: the number of times t is enabled (in the markings of) σ:

$$|\{M|M \in \mathcal{M}(\sigma), M[t\rangle\}|$$

Fairness

Definition 1

Let γ be a marked Petri net and σ an occurrence sequence.

- Transition t is impartial for σ iff it has infinitely many occurrences in σ:

$$OC_t(\sigma) = \infty$$

- Transition t is fair for σ iff an infinite number of enablings implies an infinite number of occurrences:

$$EN_t(\sigma) = \infty \Rightarrow OC_t(\sigma) = \infty$$

- Transition t is just for σ iff a persistent enabling implies an infinite number of occurrences:

$$(\exists i : (\forall k \geq i : M_k[t\rangle)) \Rightarrow (OC_t(\sigma) = \infty)$$

- Transition t is impartial (fair, just) in γ iff it is impartial (fair, just) for all the infinite occurrence sequences in the net.

Fairness

Proposition 1

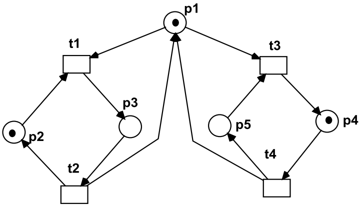
Let γ be a marked Petri net and σ an occurrence sequence.

t is impartial for $\sigma \Rightarrow t$ is fair for $\sigma \Rightarrow t$ is just for σ

Checking fairness properties for a transition t:

- check impartial property: check every infinite occurrence sequence; if t is impartial, it is also fair and just, otherwise check fairness;
- check fairness property: check those infinite occurrence sequences where t is enabled an infinite number of times; if t is fair it is also just, otherwise check for just;
- check just property: check those infinite occurrence sequences where t is continuously enabled from a certain marking on: if $OC_t(\sigma)$ is finite, t has no fairness

Fairness



$$\begin{tikzcd}
{[p1+p2+p4]} \arrow[r, "t1"] \arrow[d, "t3"] & {[p3+p4]} \arrow[r, "t2"] & {[p1+p2+p4]} \arrow[r, "t1"] \arrow[d, "t3"] & {[p3+p4]} \arrow[r, "t2"] & {[p1+p2+p4]} \arrow[r, "t1"] \arrow[d, "t3"] & \cdots\cdots \\
{} & & {} & & {} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[p1+p2+p4]} \arrow[r, "t3"] \arrow[d, "t1"] & {[p2+p5]} \arrow[r, "t4"] & {[p1+p2+p4]} \arrow[r, "t3"] \arrow[d, "t1"] & {[p2+p5]} \arrow[r, "t4"] & {[p1+p2+p4]} \arrow[r, "t3"] \arrow[d, "t1"] & \cdots\cdots \\
{} & & {} & & {} &
\end{tikzcd}$$

Fairness

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{} & & {} & & {} &
\end{tikzcd}$$

- t_1 and t_2 are impartial for the first sequence (also fair and just)
- t_3 is not fair for the sequence
- t_3 is just for the sequence
- t_4 fair

$$\begin{tikzcd}
{[p1+p2+p4]} \arrow[r, "t3"] \arrow[d, "t1"] & {[p2+p5]} \arrow[r, "t4"] & {[p1+p2+p4]} \arrow[r, "t3"] \arrow[d, "t1"] & {[p2+p5]} \arrow[r, "t4"] & {[p1+p2+p4]} \arrow[r, "t3"] \arrow[d, "t1"] & \cdots\cdots \\
{} & & {} & & {} &
\end{tikzcd}$$

- t_3 and t_4 are impartial for the sequence (also fair and just)
- t_1 is not fair for the sequence
- t_1 is just for the sequence
- t_2 fair

t_1 - just,t_2-fair, t_3-just t_4-fair

Overview

1. Properties of Petri Nets
   - Fairness

2. Analysis Methods for Petri Nets
   - Reachability Graph
   - The strongly connected components graph (SCC - graph)
   - Coverability Tree/Graph

Reachability Graph

- the reachability graph describes the state space for Petri nets
- the nodes of the graph are the reachable markings of the net
- the arcs of the graph are labelled by transitions

Definition

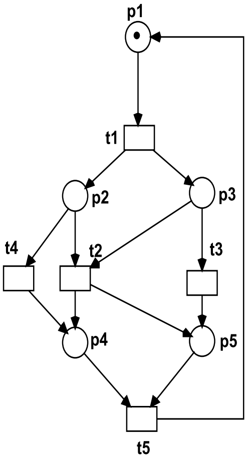
Definition 1 (reachability graph)

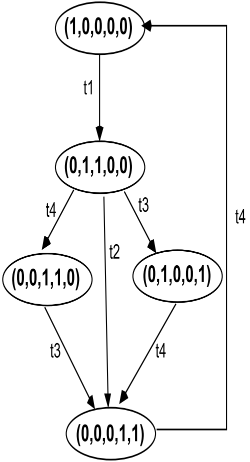
Let $\gamma = (N, M_0)$ *be a marked Petri net*
The reachability graph of γ *is a directed graph with labelled arcs:* $\mathcal{RG}(\gamma) = (V, A, l_A)$*, such that*

- $V = [M_0\rangle$.
- $A = \{(M, M')|\exists t \in T : M[t\rangle M'\}$.
- $l_A : A \to T$, $\forall (M, M') \in A : l_A(M, M') = t$, *if* $M[t\rangle M'$.

A labelled arc $(M, M') \in A, l_A(M, M') = t$ is denoted by (M, t, M').

Reachability Graph





Reachability Graph

Proposition 2

Let $\gamma = (N, M_0)$ *be a marked Petri net ,* $\mathcal{RG}(\gamma)$ *its reachability graph and* $M \in [M_0\rangle$*.*
$M[t_1\rangle M_1[t_2\rangle M_2[t_3\rangle M_3 \ldots$ *is an occurrence sequence enabled in* M *in* γ *iff there exists a path in* $\mathcal{RG}(\gamma)$*:*
$M(M, t_1, M_1)M_1(M_1, t_2, M_2)M_2(M_2, t_3M_3), \ldots$*.*

Reachability Graph

Proposition 2

Let $\gamma = (N, M_0)$ be a marked Petri net , $\mathcal{RG}(\gamma)$ its reachability graph and $M \in [M_0\rangle$.
$M[t_1\rangle M_1[t_2\rangle M_2[t_3\rangle M_3 \ldots$ is an occurrence sequence enabled in M in γ iff there exists a path in $\mathcal{RG}(\gamma)$:
$M(M, t_1, M_1)M_1(M_1, t_2, M_2)M_2(M_2, t_3M_3), \ldots$.

Proposition 3

A marked Petri net $\gamma = (N, M_0)$ is bounded iff its reachablity graph $\mathcal{RG}(\gamma)$ has a finite number of nodes.

Reachability Graph

Proposition 4

A marked Petri net $\gamma = (N, M_0)$ is deadlock free iff its reachability graph $\mathcal{RG}(\gamma)$ does not contain nodes without successors.

Reachability Graph

Proposition 4

A marked Petri net $\gamma = (N, M_0)$ is deadlock free iff its reachability graph $\mathcal{RG}(\gamma)$ does not contain nodes without successors.

Proposition 5

A marked Petri net $\gamma = (N, M_0)$ is live iff for each node M, in its reachability graph $\mathcal{RG}(\gamma)$ there exists a path $M(M, t_1, M_1)M_1(M_1, t_2, M_2)M_2 \ldots M_{k-1}(M_{k-1}, t_k, M_k)M_k$ such that the sequence $t_1, t_2, \ldots t_k$ contains all the transitions in γ.

Reachability Graph

Proposition 4

A marked Petri net $\gamma = (N, M_0)$ is deadlock free iff its reachability graph $\mathcal{RG}(\gamma)$ does not contain nodes without successors.

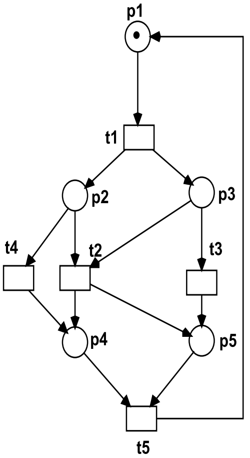
Proposition 5

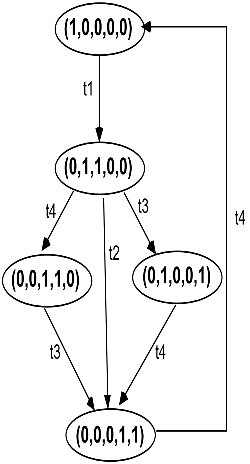
A marked Petri net $\gamma = (N, M_0)$ is live iff for each node M, in its reachability graph $\mathcal{RG}(\gamma)$ there exists a path $M(M, t_1, M_1)M_1(M_1, t_2, M_2)M_2 \ldots M_{k-1}(M_{k-1}, t_k, M_k)M_k$ such that the sequence $t_1, t_2, \ldots t_k$ contains all the transitions in γ.

Proposition 6

A marked Petri net $\gamma = (N, M_0)$ is reversible iff its reachability graph $\mathcal{RG}(\gamma)$ is strongly connected.

Example





- live net
- bounded net
- reversible net

Reachability Graph

Notations:

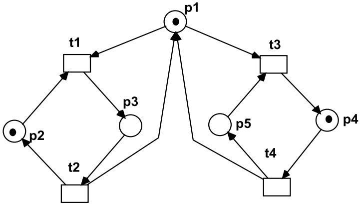
- Let *DSC* be the set of all the directed simple circuits in the graph
- If $sc \in DSC$: $T(sc) = \{t | \exists M, M' \in sc \text{ such that } (M, t, M') \in sc\}$
- If $M \in V$ is a node in the graph, $T(M) = \{t \in T | \exists M' \in V : (M, t, M') \in A\}$

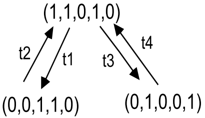
Proposition 7

Let $\gamma = (N, M_0)$ be a bounded marked Petri net, $\mathcal{RG}(\gamma)$ its (finite) reachability graph and t a transition.

- *t is impartial iff $\forall sc \in DSC : [t \in T(sc)]$*
- *t is fair iff $\forall sc \in DSC : [t \in T(sc) \vee \forall M \in sc : t \notin T(M)]$*
- *t is just iff $\forall sc \in DSC : [t \in T(sc) \vee \exists M \in sc : t \notin T(M)]$*

Fairness





t_1 - just, t_2-fair, t_3-just, t_4-fair

Notations

Let $\mathcal{G} = (V, A)$ be a directed graph.

- If $a = (v_1, v_2) \in A$, $s(a) = v_1$, $d(a) = v_2$.
- $\mathcal{DF}$ - the set of finite paths in $\mathcal{G}$.
- If v_1, v_2 are nodes, $DF(v_1, v_2)$: the set of finite paths between v_1 and v_2 in $\mathcal{G}$.
- $\mathcal{DI}$ - the set of infinite paths in $\mathcal{G}$.

Notations

- Let $\mathcal{G} = (V, A)$ be a directed graph. A strongly connected component of $\mathcal{G}$ is a subgraph $\mathcal{G}^*$ induced by the set of nodes $V^* \subseteq V$, where:
  - V^* is a strongly connected set of nodes $(\forall v_1, v_2 \in V^* : DF(v_1, v_2) \neq \emptyset)$
  - $(\forall V' \subseteq V)(V'$ strongly connected $\wedge V^* \subseteq V' \Rightarrow V^* = V')$

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- A strongly connected component c is a terminal strongly connected component iff:

$$(\forall v \in c)(\forall a \in A)(s(a) = v \Rightarrow d(a) \in c)$$

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- The set of strongly connected components $SCC_{\mathcal{G}}$ forms a partition of the nodes of $\mathcal{G}$

Notations

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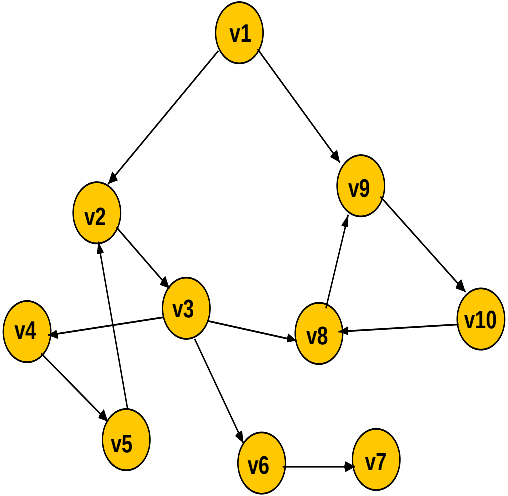
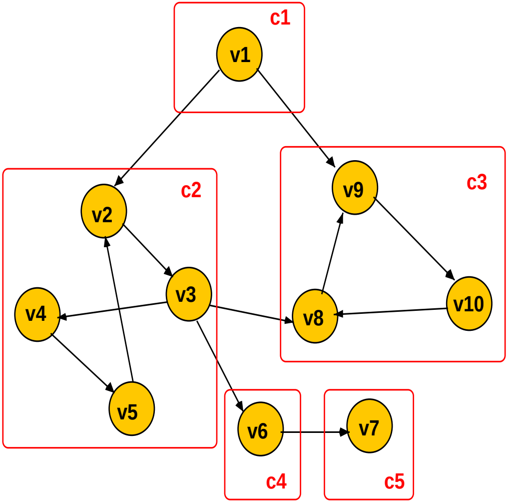
- The set of the strongly connected components of $\mathcal{G}$: $SCC_{\mathcal{G}}$.
- The set of the terminal strongly connected components of $\mathcal{G}$: $SCC_{\mathcal{G}}^{T}$.
- The set of strongly connected components $SCC_{\mathcal{G}}$ forms a partition of the nodes of $\mathcal{G}$
- If $v \in V$, $c_v \in SCC_{\mathcal{G}}$: the component to which the node v belongs.


v1
v2
v9
v3
v4
v8
v10
v5
v6
v7

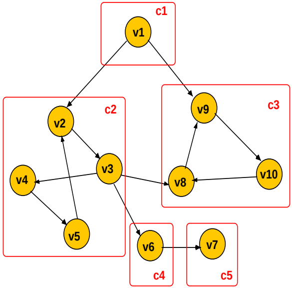

c1
v1
c2
v2
v3
v4
v5
c3
v9
v8
v10
v6
c4
v7
c5

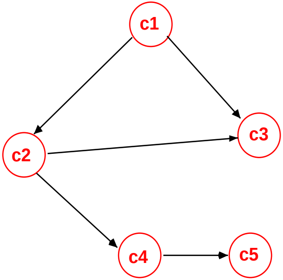
SCC-graph

Definition 2

A directed graph $\mathcal{G}^* = (V^*, A^*)$ *is the SCC-graph corresponding to the directed graph* $\mathcal{G} = (V, A)$*, iff:*

1. $V^* = SCC_{\mathcal{G}}$
2. $A^* = \{a \in A | c_{s(a)} \neq c_{d(a)}\}$





Properties

Proposition 8

Let $\mathcal{G} = (V, A)$ be a directed graph. It holds:

1. *The SCC-graph coresponding to a graph $\mathcal{G}$ is acyclic.*
2. *V is finite $\Longrightarrow$ the set of strongly connected components $SCC_{\mathcal{G}}$ is finite.*
3. *V is finite $\Longrightarrow \forall c_1 \in CTC_{\mathcal{G}}\ \exists c_2 \in CTC_{\mathcal{G}}^{T} : DF(c_1, c_2) \neq \emptyset$.*
4. *$\forall v_1, v_2 \in V : DF(v_1, v_2) \neq \emptyset \Leftrightarrow DF(c_{v_1}, c_{v_2}) \neq \emptyset$.*

Reachability

Proposition 9

Let $\gamma = (N, M_0)$ *be a marked Petri net,* $\mathcal{RG}$ *its reachability graph,* $M_1, M_2 \in [M_0\rangle$*. It holds:*

1. $M_2 \in [M_1\rangle \Leftrightarrow DF(c_{M_1}, c_{M_2}) \neq 0$
2. $M_2 \in [M_1\rangle \Leftarrow |SCC_{\mathcal{RG}}| = 1$

Home markings

Definition 2

Let $\gamma = (N, M_0)$ be a marked Petri net and X a set of markings. X is a home space iff $\forall M' \in [M_0\rangle : X \cap [M'\rangle \neq \emptyset$.

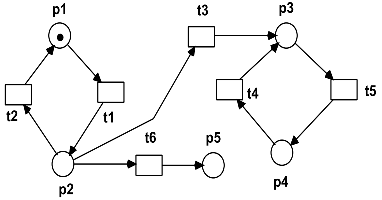
We denote by $\mathcal{HM}_\gamma$ the set of all the home markings of γ and by $\mathcal{H}S_\gamma$ the set of all the home spaces of γ.

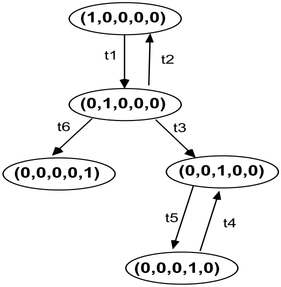
Proposition 10

Let $\gamma = (N, M_0)$ be a marked Petri net, $\mathcal{RG}$ its reachability graph, $X \subseteq [M_0\rangle$ and $M \in [M_0\rangle$. It holds:

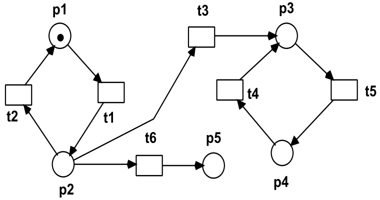
1. $X \in \mathcal{H}S \Leftrightarrow SCC^T_{\mathcal{R}G} \subseteq c_X$.
2. $X \in \mathcal{H}S \Rightarrow |SCC^T_{\mathcal{R}G}| \leq |X|$.
3. $M \in \mathcal{H}M \Leftrightarrow SCC^T_{\mathcal{R}G} = \{c_M\}$.
4. $\mathcal{H}M \neq \emptyset \Leftrightarrow |SCC^T_{\mathcal{R}G}| = 1$
5. $M_0 \in \mathcal{H}M \Leftrightarrow |SCC_{\mathcal{R}G}| = 1$.

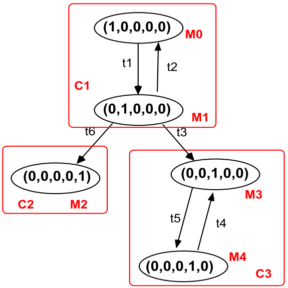
Example


p1
t3
p3
t4
t5
t2
t1
t6
p5
p2
p4


(1,0,0,0,0)
t1
t2
(0,1,0,0,0)
t6
t3
(0,0,0,0,1)
(0,0,1,0,0)
t5
t4
(0,0,0,1,0)

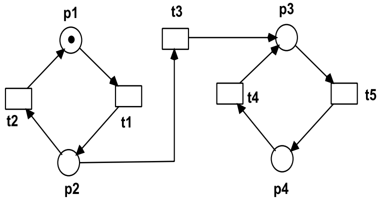
Example

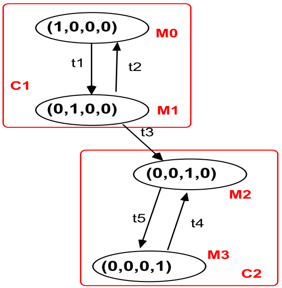




$X = \{M2, M3\}$ is a home space, but there are no home markings!

Example





$M2, M3$ are home markings!

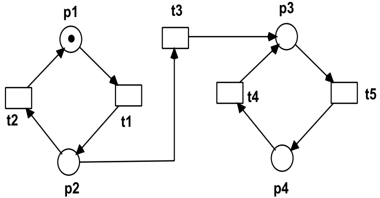
Notations

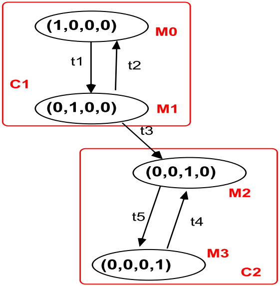
Let $\gamma = (N, M_0)$ be a marked Petri net and $\mathcal{RG} = (V, A, l_A)$ its reachability graph.

- Let $c \in SCC_{\mathcal{RG}}$ a strongly connected component. The set of transitions with the source c:

$$T(c) = \{t \in T | \exists M_1 \in c, \exists M_2 \in V : (M_1, t, M_2) \in A\}$$

Notations





$T(M_1) = \{t_2, t_3\}$, $T(C_1) = \{t_1, t_2, t_3\}$, $T(C_2) = \{t_4, t_5\}$

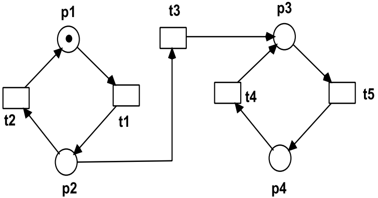
Liveness, quasi-liveness, dealocks

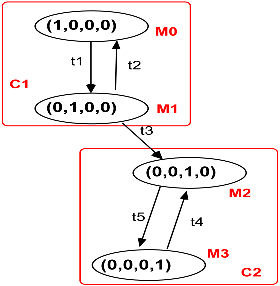
Proposition 11

Let $\gamma = (N, M_0)$ *be a marked Petri net and* $\mathcal{RG}$ *its reachability graph,* $M \in [M_0\rangle$ *and* $t \in T$*. It holds:*

1. M *is a dead marking* $\Leftrightarrow c_M \in SCC^T_{\mathcal{R}G}$ *and* $|c_M| = 1$.
2. t *is quasi-live from* $M \Leftrightarrow (\exists c \in SCC_{\mathcal{R}G})(DF(c_M, c) \neq \emptyset \wedge t \in T(c))$.
3. t *is live* $\Leftrightarrow \forall c \in SCC^T_{\mathcal{R}G} : t \in T(c)$.

Liveness, quasi-liveness, dealocks





$t_1, t2 \notin T(C_2)$, they are not live (just quasi-live); $t_4, t_5 \in T(C_2)$, they are live.

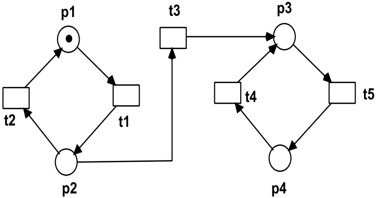
Fairness properties

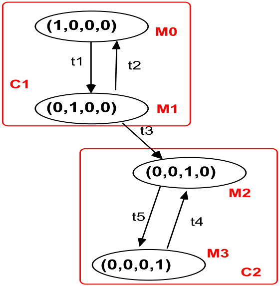
Proposition 12

Let $\gamma = (N, M_0)$ be a bounded marked Petri net and $\mathcal{RG}$ its reachability graph, $M \in [M_0\rangle$ and $t \in T$. It holds:

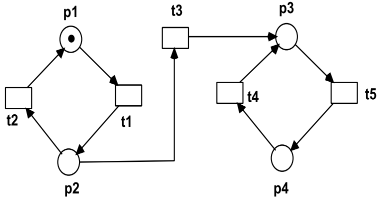
1. *t is impartial $\Leftrightarrow \forall c \in SCC_{\mathcal{R}G\setminus\{t\}}$:$c$ is trivial*
2. *t is fair $\Leftrightarrow \forall c \in SCC_{\mathcal{R}G\setminus\{t\}}$:[$c$ is trivial $\vee\ \forall M \in c : t \notin T(M)$]*
3. *t is just $\Leftrightarrow \forall c \in SCC_{\mathcal{R}G\setminus\{t\}}$:[$c$ is trivial $\vee\ \forall sc \in DSC(c)$: [$\exists M \in sc : t \notin T(M)$]]*

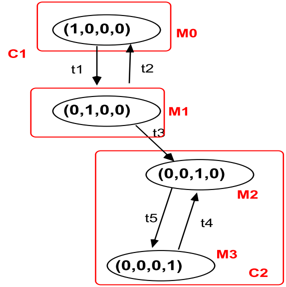
Fairness properties





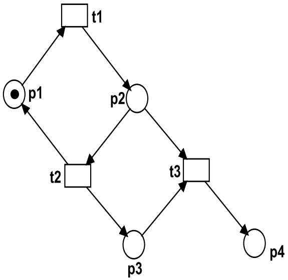
Fairness properties

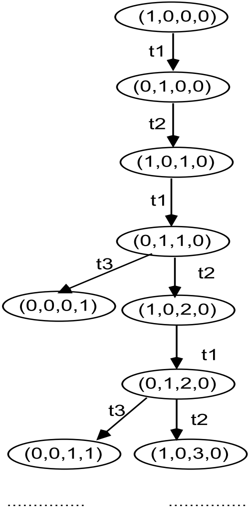




t_1 is not impartial, it is fair

Coverability -Example





Coverability Tree/Graph

- For describing the state space of Petri nets with infinite number of reachable markings: coverability trees and coverability graphs

Coverability Tree/Graph

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Definition 3

Let N be a Petri net, M_1 and M_2 two markings. M_2 covers M_1 if $M_2 \geq M_1$ (i.e. $M_2(p) \geq M_1(p), \forall p \in P$)

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Definition 4

Let $\gamma = (N, M_0)$ a marked Petri net and M a marking of N. M is coverable in γ, if there exists $M' \in [M_0\rangle$ which covers M.

Notations

Let (ω) be a symbol denoting infinity and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$. It holds:

- $\omega + n = \omega - n = \omega$, $\forall n \in \mathbb{N}$.
- $\omega + \omega = \omega - \omega = \omega$.
- $\omega * n = n * \omega = \omega$, $\forall n \in \mathbb{N}$, $n > 0$ and $\omega * 0 = 0$.
- $\omega > n$, $\forall n \in \mathbb{N}$.

Let $\overline{\mathbb{N}}^P = \{M | M : P \to \overline{\mathbb{N}}\}$

Coverability Trees- Construction

- The root of the tree: M_0
- If the successor of a node n (computed as in the case of the reachability graph) is a marking M' and on the path from n to the root there exists a marking M such that $M' \geq M$ and there exists a place p such that $M'(p) > M(p)$, then:
  - $M[\sigma\rangle M'$.
  - It also holds that $M'[\sigma\rangle M''$, $M'' > M'$ and $M''(p) > M'(p)$ (a new marking M'' would appear in the tree).
  - The number of tokens in p is infinite.
- Thus, in the covering tree, the marking M' will be replaced with a marking $\overline{M'}$ such that $\overline{M'}(p) = \omega$
- The transitions enabled in a marking will be computed taking into account the fact that some places contain an infinite number of tokens
- The number of nodes is finite (in the worst case, one can obtain markings with ω in all places)

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Notations

Let $\mathcal{T} = (V, E)$ be a tree.

- If $v \in V$ is a node, v^+ denotes the set of the succesors of v. ($v^+ = \{v' \in V | \exists (v, v') \in E\}$).
- $d(v, v')$ denotes the path from v to v'.

Coverability Tree - Definition

Definition 5

Let $\gamma = (N, M_0)$ be a marked Petri net . A coverability tree for γ is a tree $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ such that :

1. $l_V : V \to \overline{\mathbb{N}}^P$, $l_E : E \to T$;
2. *the root v_0 of $\mathcal{T}_\gamma$ is labelled by M_0: $l_V(v_0) = M_0$;*
3. *for every node v labelled by M ($l_V(v) = M$) it holds:*
   - 3.1 $|v^+| = 0$ *(v is a leaf), if there does not exist any $t \in T$ such that $M[t\rangle$ or there exists $v' \in d(v_0, v)$, $v \neq v'$ labelled by M;*
   - 3.2 $|v^+| = |\{t \in T | M[t\rangle\}|$, *otherwise;*

4. for any $v \in V$ with $|v^+| > 0$, $l_V(v) = M$ and any $t \in T : M[t\rangle$ there exists $v' \in V$, $l_V(v') = M'$ such that :

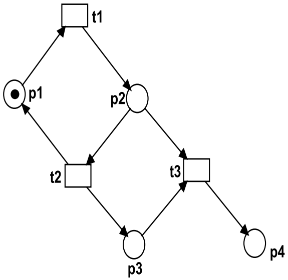
   4.1 $(v, v') \in E$;

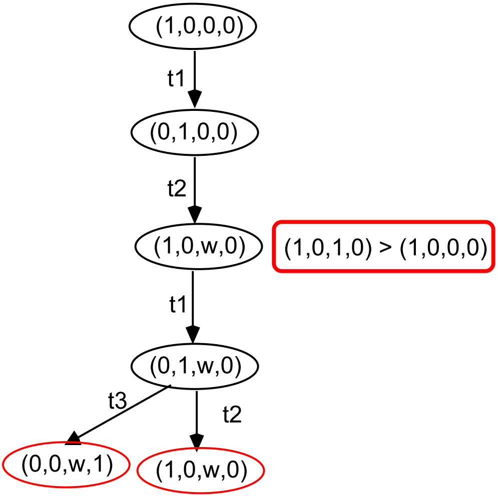
   4.2 $l_E(v, v') = t$;

   4.3 Let $\overline{M} = M + \Delta(t)$. For any $p \in P$ it holds:

   - $M'(p) = \omega$, if there exists $v'' \in d_{\mathcal{T}}(v_0, v)$ such that $l_V(v'') = M''$ $\overline{M} \geq M''$ and $\overline{M}(p) > M''(p)$;
   - $M'(p) = \overline{M}(p)$, otherwise;

Example


t1
p1
p2
t2
t3
p3
p4


(1,0,0,0)
t1
(0,1,0,0)
t2
(1,0,w,0)
(1,0,1,0) > (1,0,0,0)
t1
(0,1,w,0)
t3
t2
(0,0,w,1)
(1,0,w,0)

Coverability Graph

- there are no duplicate nodes in the graph
- $|v^+| = 0$ (node without successors), if there does not exists $t \in T$ such that $M[t\rangle$
- in the definition 5 (4.3) - if there exists $v'' \in d(v_0, v)$ all the paths from v_0 to v should be considered

Coverability Graph

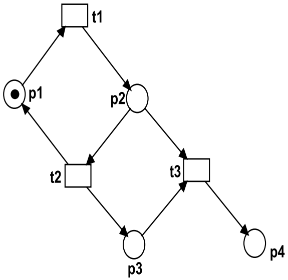
$V = \emptyset$ (the set of nodes in the graph), $done : V \to \{true, false\}$

1. Add a node n_0 to V and label it with M_0, $done(n_0) = false$
2. while (there exists a node $n \in V$ with $done(n) == false$) {
   - Let M be the label of n
   - for(t with $M[t\rangle M'$){
     - let $\overline{M} = M'$
     - for all nodes n'' on the paths from n_0 to n labelled with $M' > M''$, set $\overline{M}(p) = \omega$ for all p with $M'(p) > M''(p)$
     - if a node labelled with $\overline{M}$ does not exists:
       add a new node $\overline{n}$ with label $\overline{M}$
       add an arc from n to $\overline{n}$ labelled with t
       $done(\overline{n}) = false$
     - else add an arc from n to the node labelled with $\overline{M}$

     }
   - $done(n) = true$

   }

Example


t1
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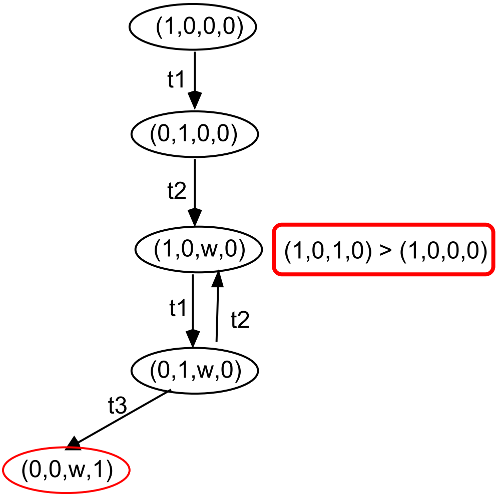
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t1
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t1
t2
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t3
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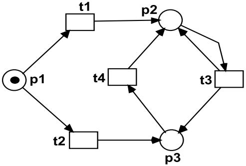
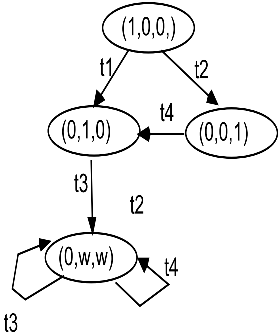
Example

The procedure does not produce a unique graph





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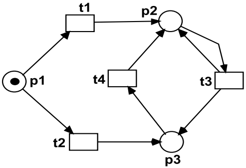

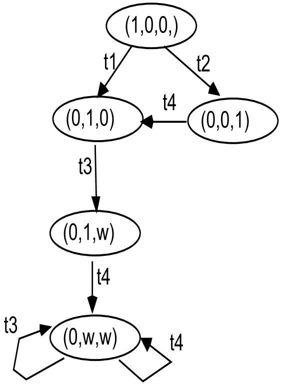

Models of Distributed Systems

Lecture 3

Overview

1. The Coverability Tree/Graph
2. Decision Problems in Petri Nets
3. Linear Algebraic Techniques
   - Place Invariants

Overview

Notations

Let $\gamma = (N, M_0)$ be a marked Petri net and $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ its coverability tree.

- If $(v_1, v_2) \in E$, $l_E(v_1, v_2) = t$, $l_V(v_1) = M_1$ and $l_V(v_2) = M_2$, then we denote

$$v_1 : M_1 \xrightarrow{t} v_2 : M_2.$$

The relation $\xrightarrow{t}$ can be naturally extended to $\xrightarrow{w}$, where $w \in T^*$.

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  $$\Omega(M) = \{p \in P | M(p) = \omega\}.$$

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- Let $v \in V$ be a node with $l_V(v) = M$.

$$\Omega(M) = \{p \in P | M(p) = \omega\}.$$

- $Lab(\gamma)$ is the set of the node labels in the covering tree corresponding to γ: $Lab(\gamma) = \{l_V(v) | v \in V\}$

Coverability Tree Properties

Proposition 1

Let $\gamma = (N, M_0)$ be a marked Petri net and $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ its coverability tree. It holds:

1. $\mathcal{T}_\gamma$ *is finitely branching*
2. *Let $v_{i_0}, v_{i_1}, \ldots, v_{i_m}$ be distinct nodes such that $v_{i_j} \in d(v_0, v_{i_{j+1}})$ for all $0 \leq j \leq m-1$.*
   1. *If $l_V(v_{i_0}) = l_V(v_{i_1}) = \ldots = l_V(v_{i_m})$, then $m \leq 1$;*
   2. *If $l_V(v_{i_0}) < l_V(v_{i_1}) < \ldots < l_V(v_{i_m})$, then $m \leq |P|$;*
3. $\mathcal{T}(\gamma)$ *is finite.*

Proof

1. For every node v: $|v^+| = 0$ or $|v^+| = |\{t \in T | l_V(v)[t\rangle\}|$
2.
   1. $l_V(v_{i_0}) = l_V(v_{i_1})$, so v_{i_1} is a leaf and $v_{i_1} = \ldots v_{i_m}$, hence $m \leq 1$.
   2. $l_V(v_{i_j}) < l_V(v_{i_{j+1}})$, $l_V(v_{i_{j+1}})$ has at least one additional ω-component to $l_V(v_{i_j})$. There may be at most $|P|$ ω-components. Hence $m \leq |P|$.

Proof

3 Assume $\mathcal{T}_\gamma$ is infinite.

Konig's Lemma: Any infinite tree which is finitely branching contains an infinite path.

$\mathcal{T}_\gamma$ finitely branching (1), so there exists an infinite path $v_0, v_1, v_2, \ldots$
Consider the infinite sequence:

$$l_V(v_0), l_V(v_1), \ldots, l_V(v_k), \ldots \ (*)$$

There exists an infinite sub-sequence of (*):

$$l_V(v_{i_0}), l_V(v_{i_1}), \ldots \ (**)$$

such that

$$l_V(v_{i_0}) \leq l_V(v_{i_1}) \leq l_V(v_{i_2}) \ldots$$

Proof

$$l_V(v_{i_0}) \leq l_V(v_{i_1}) \leq l_V(v_{i_2}) \ldots$$

- Assume there does not exist $p \neq q$, $p, q \geq 1$ such that $l_V(v_{i_p}) = l_V(v_{i_q})$
- Consider the first $|P| + 2$ elements in the sequence:

$$l_V(v_{i_0}) < l_V(v_{i_1}) < \ldots < l_V(v_{i_{|P|+1}})$$

From (2(2)), $|P| + 1 \leq |P|$, contradiction!

Coverability Tree Properties

Lemma 1

Let $\gamma = (N, M_0)$ *be a marked Petri net and* $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ *its coverability tree. If* $v_1, v_2 \in V$, $l_V(v_1) = M_1$, $l_V(v_2) = M_2$, $w \in T^*$ *and* $v_1 : M_1 \xrightarrow{w} v_2 : M_2$, *then:*

$$M_2(p) = (M_1 + \triangle w)(p)$$

for all $p \in P \setminus \Omega(M_2)$.

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Definition 1

Let $\gamma = (N, M_0)$ be a marked Petri net , $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ its coverability tree and M a marking. M is coverable in $\mathcal{T}_\gamma$ if $(\exists v \in V : \; M \leq l_V(v))$

Coverability Tree Properties

Lemma 2

Let $\gamma = (N, M_0)$ *be a marked Petri net ,* $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ *its coverability tree and* M *a reachable marking of* γ*. Then there exists a node* $v \in V : l_V(v) \geq M$

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Lemma 3

Let $\gamma = (N, M_0)$ *be a marked Petri net ,* $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ *its coverability tree and* M *a marking of* N*. Then, it holds:* $(\exists\, v \in V \,:\, M \leq l_V(v)) \Rightarrow (\exists\, M' \in [M_0\rangle \,:\, M \leq M')$

Coverability Tree Properties

Proof:

Let M be a marking of N and le v be a node in the tree with $l_V(v) \geq M$.

In $\mathcal{T}_\gamma$ there exists a path $v_0, v_1, \ldots, v_k = v$ and let $t_1, t_2, \ldots t_k$ be the labels of the arcs on this path.

Let $l_V(v_i) = M_i$, for all $i \in \{1, \ldots, k\}$.

I In the marking of v there are no ω-components: $M_0[t_1 t_2 \ldots t_k\rangle M_k \geq M$

II In the marking of v there exist h ω-components.
One can assume:

- $P = \{1, \ldots, n\}$
- If $l_v(v) = M_k$, $M_k(1) = \omega, M_k(2) = \omega, \ldots M_k(h) = \omega$ and $M_k(i) \neq \omega, \forall h+1 \leq i \leq n$
- The ω-components have been introduced in the following order: $1, 2, \ldots, h$.

Proof

Let:

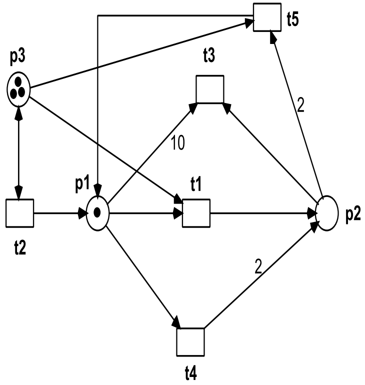
- α_1 the prefix of the sequence $t_1 \ldots t_k$ which caused the introduction of the first ω-component
- $\alpha_1\alpha_2$ the prefix of sequence $t_1 \ldots t_k$ which caused the introduction of the second ω-component
- $\ldots$
- $\alpha_1\alpha_2 \ldots \alpha_h$ the prefix of the sequence $t_1 \ldots t_k$ which caused the introduction of the last (h) ω-component
- α_{h+1} the sequence containing the rest of the transitions.
- $t_1 t_2 \ldots t_k = \alpha_1\alpha_2 \ldots \alpha_h\alpha_{h+1}$

Proof

Let

- u_1 the sufix of α_1 which increases the number of tokens in place 1 (and produces $M_k(1) = \omega$)
- u_2 the sufix of $\alpha_1\alpha_2$ which increases the number of tokens in place 2 (produces $M_k(2) = \omega$)
- ...
- u_h the sufix of $\alpha_1\alpha_2 \dots \alpha_h$ which increases the number of tokens in place h (produces $M_k(h) = \omega$)
- u_i ($1 \leq i \leq h$) causes the increase of the number of tokens in place i with at least one token.
- the sequence $t_1 \dots t_k = \alpha_1 \dots \alpha_h\alpha_{h+1}$ is not necessary enabled in γ at marking M_0

Proof



$$(1,0,3) \xrightarrow{t_1} (0,1,2) \xrightarrow{t_2} \underbrace{(\omega,1,2)}_{(1,1,2)} \xrightarrow{t_3} (\omega,0,2) \xrightarrow{t_4} \underbrace{(\omega,\omega,2)}_{(\omega,2,2)} \xrightarrow{t_5} (\omega,\omega,1)$$

$\alpha_1 = t_1t_2,\ u_1 = t_2$

$\alpha_2 = t_3t_4,\ u_2 = t_4$

$\alpha_3 = t_5$

Proof

A new sequence β will be built with the transitions from $t_1 \ldots t_k$ such that β is enabled M_0 and its firing produces a marking M' such that $M' \geq M$:

$$\beta = \alpha_1(u_1)^{n_1}\alpha_2(u_2)^{n_2}\ldots\alpha_h(u_h)^{n_h}\alpha_{h+1}$$

Let's assume first that $h = 1$ (only one ω-component).
We will build a sequence of transitions $\beta = \alpha_1(u_1)^{n_1}\alpha_2$ enabled in γ: $M_0[\beta\rangle M'$, $M' > M$

- α_1 is the sequence of transitions that introduces the only ω - component
- $M_0 \stackrel{\alpha 1}{\rightarrow} W_1 = (\omega, \ldots) \stackrel{\alpha_2}{\rightarrow} M_k$ ($t_1 \ldots t_k$ = $\alpha_1\alpha_2$)
- $M_0 \stackrel{u_0}{\rightarrow} W \stackrel{u_1}{\rightarrow} W_1 = (\omega, \ldots) \stackrel{\alpha_2}{\rightarrow} M_k$
  - $\alpha_1 = u_0u_1$
  - in γ: $M_0[u_0\rangle W[u_1\rangle\overline{W_1}$, $\overline{W_1} \geq W$, $\overline{W_1}(1) > W(1)$
  - in $\mathcal{T}_\gamma$: $W_1(i) = \overline{W_1}(i)$, for all $i \geq 2$
  - $\overline{W_1}[\alpha_2\rangle_\gamma$??
    - $W_1(i) = \overline{W_1}(i) \geq \alpha_2^-(i)$, for $i \geq 2$
    - $\overline{W_1}(1)$? $\alpha_2{}^-(1)$
  - since $\overline{W_1} > W$ it holds that $\overline{W_1}[u_1u_1\ldots\rangle_\gamma$

Proof

In γ: $M_0[u_0\rangle W[u_1\rangle \overline{W_1}$, $\overline{W_1} \geq W$, $\overline{W_1}(1) > W(1)$

- α_2 would be enabled in $\overline{W_1}$ in γ except for place 1
- since $\overline{W_1} > W$ it holds that:
  - $\Delta(u_1)(i) \geq 0$ (for $i > 1$), $\Delta(u_1)(1) > 0$
  - $\overline{W_1}[u_1 u_1 \ldots\rangle_\gamma$
- produce u_1 until we have enough tokens to produce α_2 and to allow $M'(1) > M(1)$ (where M' is the marking resulted after producing α_2):

$$n_1 \geq {\alpha_2}^-(1) + M(1)$$

- in $\mathcal{T}_\gamma$: $M_0 \overset{\alpha 1}{\rightarrow} W_1 = (\omega, \ldots) \overset{\alpha_2}{\rightarrow} M_k$
  in γ: $M_0[\alpha_1\rangle \overline{W_1}[u_1^{n_1}\rangle \overline{W_1}'[\alpha_2\rangle M' \ > \ M$
  - $\overline{W_1}'(i) \geq \overline{W_1}(i) = W_1(i) \geq \alpha_2^-(i)$, for all $i \in \{2, \ldots, n\}$
  - $\overline{W_1}'(1) = \overline{W_1}(1) + n_1\Delta(u_1) \geq \overline{W_1}(1) + (M(1) + \alpha_2^-(1))\Delta(u_1)(1) \geq \alpha_2^-(1) + M(1) \geq \alpha_2^-(1)$
  - $M' > M$

Proof

Let's assume that $h = 2$ (2 ω-components).
We will build a sequence of transitions $\beta = \alpha_1(u_1)^{n_1}\alpha_2(u_2)^{n_2}\alpha_3$ enabled in γ: $M_0[\beta\rangle M'$, $M' > M$

- α_1 is the sequence of transitions that introduces the first ω - component
- α_2 is the sequence of transitions that introduces the first ω - component
- $M_0 \xrightarrow{\alpha_1} W_1 = (\omega, \ldots) \xrightarrow{\alpha_2} W_2 = (\omega, \omega, \ldots) \xrightarrow{\alpha_3} M_k$ ($t_1 \ldots t_k = \alpha_1\alpha_2\alpha_3$)
- u_1 the suffix of α_1 that introduce the first ω. In γ:
  $M_0[*\rangle W[u_1\rangle \overline{W_1}$, $\overline{W_1} \geq W$, $\overline{W_1}(1) > W(1)$
- u_2 the suffix of $\alpha_1\alpha_2$ that introduces the second ω: in $\mathcal{T}_\gamma$:
  $V \xrightarrow{u_2} \overline{W_2}$, $\overline{W_2} \geq V$, $\overline{W_2}(2) > V(2)$ (and $W_2(i) = \overline{W_2}(i)$, for all $i \geq 3$)

Proof

In $\mathcal{T}_\gamma$: $M_0 \stackrel{\alpha_1}{\rightarrow} W_1 = (\omega, \ldots) \stackrel{\alpha_2}{\rightarrow} W_2 = (\omega, \omega, \ldots) \stackrel{\alpha_3}{\rightarrow} M_k$

In γ: $M_0[\alpha_1\rangle\overline{W_1}$

- we need to be able to produce $(u_1)^{n_1}\alpha_2(u_2)^{n_2}\alpha_3$ from $\overline{W_1}$ in γ
- $\alpha_3^-(i) \leq W_2(i)$, $\forall i \geq 3$ (if we manage to produce a sequence containing $\alpha_1\alpha_2$ in γ, then we get enough tokens in places, 3...n);
- to also ensure enough tokens for α_3 in place 2, fire u_2 (after we make it possible in γ) n_2 times: $n_2 \geq \alpha_3^-(2) + M(2)$
- the sequence $(u_2)^{n_2}\alpha_3$ would have enough tokens in places $2, \ldots, n$ (if we manage to also produce a sequence containing $\alpha_1\alpha_2$)
- we need to also fire α_2 and to ensure enough tokens in the first place for $(u_2)^{n_2}\alpha_3$
- α_2 has enough tokens in places $2, \ldots, n$, if fired after α_1
- in γ we will fire, after α_1, u_1 so that we have enough tokens in place 1 for α_2 and $(u_2)^{n_2}\alpha_3$:

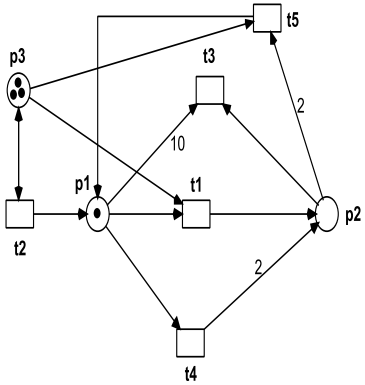
$$n_1 \geq M(1) + \alpha_2^-(1) + n_2u_2^-(1) + \alpha_3^-(1)$$

Proof

$$\beta = \alpha_1(u_1)^{n_1}\alpha_2(u_2)^{n_2}\ldots(u_{h-1})^{n_{h-1}}\alpha_h(u_h)^{n_h}\alpha_{h+1}$$

- the transitions in α_{h+1} ensure enough tokens in places $h+1$, $h+2$, ...,n (it also holds: $M_k(h+1) \geq M(h+1), \ldots, M_k(n) \geq M(n)$)
- the sequence u_h will fire n_h times, where: $n_h \geq M(h) + {\alpha_{h+1}}^{-1}(h)$.
- the sequence u_{h-1} will fire n_{h-1} times, where: $n_{h-1} \geq M(h-1) + {\alpha_{h+1}}^{-}(h-1) + n_h \cdot {u_h}^{-1}(h-1) + {\alpha_h}^{-}(h-1)$
- the sequence u_{h-2} will fire n_{h-2} times, where: $n_{h-2} \geq M(h-2) + {\alpha_{h+1}}^{-}(h-2) + n_h \cdot {u_h}^{-}(h-2) + {\alpha_h}^{-}(h-2) + n_{h-1} \cdot {u_{h-1}}^{-}(h-2) + {\alpha_{h-1}}^{-}(h-2)$
- ....
- $n_1 \geq M(1) + {\alpha_{h+1}}^{-}(1) + n_h \cdot {u_h}^{-}(1) + {\alpha_h}^{-}(1) + \ldots + n_2 \cdot {u_2}^{-}(1) + {\alpha_2}^{-}(1)$

Proof



$(1,0,3) \xrightarrow{t_1} (0,1,2) \xrightarrow{t_2} (\omega,1,2) \xrightarrow{t_3} (\omega,0,2) \xrightarrow{t_4} (\omega,\omega,2) \xrightarrow{t_5} (\omega,\omega,1)$.
$(\omega,\omega,1) \geq M = (4,3,1)$.

Proof

În γ the sequence of transitions:
$\beta = \alpha_1(u_1)^{n_1}\alpha_2(u_2)^{n}_{2} \ldots \alpha_h(u_h)^{n_h}\alpha_{h+1}$.

It holds:

- $M_0[\beta\rangle$
- $M_0[\beta\rangle M^*$, $M^* \geq M$

Coverability Tree Properties

Theorem 1

Let $\gamma = (N, M_0)$ *be a marked Petri net ,* $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ *its coverability tree and* M *a marking of* N. *It holds:* $(\exists v \in V : \ M \leq l_V(v)) \Leftrightarrow (\exists M' \in [M_0\rangle : \ M \leq M')$

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Proposition 2

Let $\gamma = (\Sigma, M_0)$ *be a marked Petri net ,* $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ *its coverability tree and* $t \in T$*. It holds:*

$$t \text{ is quasi-live} \quad \Leftrightarrow \quad (\exists (v, v') \in E : \ l_E(v, v') = t).$$

Coverability Tree Properties

Proposition 3

Let γ be a marked Petri net and $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ its coverability tree. The following relations are equivalent:

(1) $Lab(\gamma) \subseteq \mathbb{N}^P$;

(2) $[M_0\rangle = Lab(\gamma)$;

(3) $[M_0\rangle$ *is finite.*

Overview

Decision Problems

- The boundness problem: given a marked Petri net γ, is γ bounded?
- The place boundness problem: given a marked Petri net γ and p a place, is p bounded?
- The reachability problem: given a marked Petri net γ and a marking M, $M \in [M_0\rangle$?
- The liveness problem: given a marked Petri net γ, is γ live?
- The quasi-liveness problem: given a marked Petri net γ, is γ quasi-live?
- The covering problem: given a marked Petri net γ and a marking M, is M coverable in γ?
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Decision Problems

Let γ be a marked Petri net and $\mathcal{T}_\gamma = (V, E, l_V, l_E)$ its coverability tree.

- A marking M of γ is coverable iff there exists $v \in V$: $l_V(v) = M' \wedge M' \geq M$.
  (Theorem 1)
- A transition t of γ is quasi-live iff $\exists (v, v') \in E : l_E(v, v') = t$.
  (Prop. 2)
- γ is bounded iff $Lab(\gamma) \subseteq \mathbb{N}^P$.
  (Prop. 3)
- A place $p \in P$ is unbounded iff $\exists v \in V$ such that $l_V(v)(p) = \omega$.
  (Prop. 3)

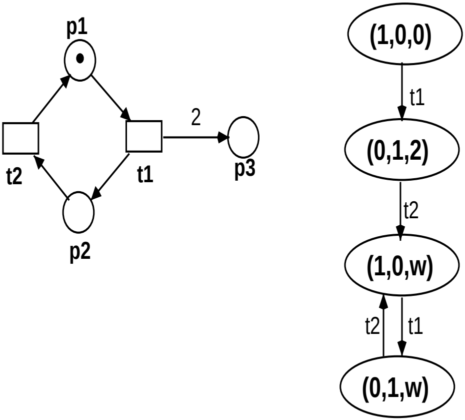
Decidable Problems

Theorem 2

The boundness, covering and quasi-liveness problems are all decidable for marked Petri nets.

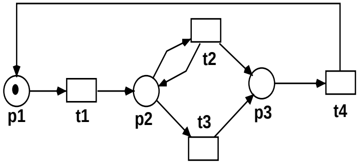
- The reachability problem is decidable.
  - Mayr 1981, Kosaraju 1982, Lambert 1992
  - necessary conditions for reachability can be established, based on the structure on the net
  - there exist special classes of Petri nets for which the reachability problem can be solved in polinomyal time
- The liveness problem is decidable
  - recursive equivalent to the reachability problem (Hack 1975)
  - necessary conditions for liveness can be established, based on the structure on the net

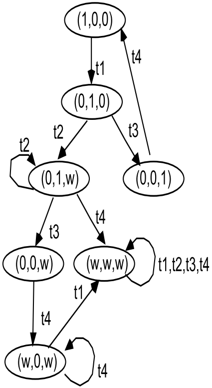
Example



Marking $(1, 0, 3)$ is not reachable.

Example





- Quasi-live net
- All places are unbounded
- Marking $(1, 2, 3)$ is coverable

Decidability of the Home Marking Problem

Definition 2

- *A set of markings $E \subseteq N^P$ is a liniar set if there exists a marking M and a finite set of markings $\{N_1, N_2, \ldots, N_m\}$ such that*

$$(\forall M' \in E)(\exists k_1, \ldots, k_m \in \mathbb{N})(M' = M + \sum_{i=1}^{m} k_i N_i)$$

- *A finite reunion of liniar sets is called a semi-liniar set.*

The following problems are deciable:

- Let E be a semi-liniar set. Does $E \cap [M_0\rangle \neq \emptyset$? (Is there a reachable marking in E ?)
- Let E be a semi-liniar set. Does $[M_0\rangle \subset E$?

Co-initial Part

- Let $(A, \leq)$ be a partial ordered set. A co-initial part of A is a subset $F \subseteq A$ such that:

$$\forall a \in A, \exists f \in F : f \leq a$$

- Given a semi-liniar set of markings E, one can compute a co-initial part of $[M_0\rangle \setminus E$

Algorithm for computing the co-initial part of $[M_0\rangle \setminus E$

Input: E a semi-liniar set of markings

Output: F: the finite co-inital part of $[M_0\rangle \setminus E$

begin
  $F = \emptyset$;
  $A := E$;
  while $[M_0\rangle \not\subseteq A$
    choose $M \in [M_0\rangle \setminus A$;
    $F := F \cup \{M\}$;
    $A := A \cup \{M' \in \mathbb{N}^P | M \leq M'\}$;
  endwhile
  // F is the co-initial part of $[M_0\rangle \setminus E$
end

Algorithm for computing the co-initial part of $[M_0\rangle \setminus E$

- $F \subseteq [M_0\rangle \setminus E$
- $A = E \cup \bigcup_{M \in F} \{M' \in N^P | M \leq M'\}$;
- A is a semi-liniar set of markings
- The condition of the while loop is decidable
- if $[M_0\rangle \not\subseteq A$, M can be chosen effectively by an exhaustive search in the set of markings (it is decidable whether a marking $M \in [M_0\rangle \setminus A$)
- The algorithm terminates and F is finite: otherwise, F would be infinite: $F = \{M_n\}_{n \in \mathbb{N}}$ (M_n the marking added at the n-th itereration). Then $\exists\, n_1 < n_2$ with $M_{n_1} < M_{n_2}$; Contradiction.
- When the algorithm terminates: $[M_0\rangle \subseteq A = E \cup \bigcup_{M \in F} \{M' \in N^P | M \leq M'\}$ and for every $M' \in [M_0\rangle \setminus E$, $\exists M \in F : M \leq M'$

Decidability of the Home Marking Problem

- Let γ be a marked Petri net , P its set of places, M_0 the intitial marking, H a marking.
- Let $m = H(P)$ (the total number of tokens in places, in marking H).
- Let $\mathcal{H}_1 = \{M \in \mathbb{N}^P | M(P) \leq m\} \cap [M_0\rangle$ (a finite set of markings, hence a liniar set of markings)
- Let $\mathcal{H}_2$ be the co-initial part of $[M_0\rangle \setminus \mathcal{H}_1$
  ($\mathcal{H}_2 \subseteq [M_0\rangle \setminus \mathcal{H}_1$ finite such that $\forall M' \in [M_0\rangle \setminus \mathcal{H}_1$, there exists $M \in \mathcal{H}_2$ with $M \leq M'$).
- $\mathcal{H}_2$ can be effectively computed

Decidability of the Home Marking Problem

Theorem 3

H is a home marking w.r.t. M_0 iff H is reachable from any marking in the set $\mathcal{H}_1 \cup \mathcal{H}_2$

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Consequence 1

The home marking problem is decidable for Petri nets.

Overview

Analysis methods for Petri Nets

Analysis methods for the properties of Petri nets:

- reachability tree or covering tress/graphs.
- large dimension of the reachability/covering tree
- there exist analysis methods for the behavioural properties of Petri nets based on their structure
- linear algebraic techniques (invariants)

The Incidence Matrix

- The incidence matrix describes the structure of the net.

The Incidence Matrix

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- Let $N = (P, T, F, W)$ a marked Petri net
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Definition 3

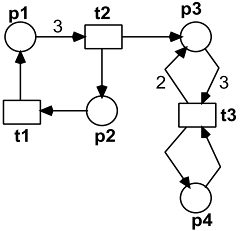
Let $N = (P, T, F, W)$ be a marked Petri net . The $m \times n$ - dimensional matrix, given by:

$$C(i, j) = W(t_j, p_i) - W(p_i, t_j), \forall 1 \leq i \leq m, 1 \leq j \leq n$$

is called the incidence matrix of N.

The Incidence Matrix

- $C(i,j) = \Delta t_j(p_i) \Rightarrow$ the number of tokens by which transition t_j modifies place p_i.



$$C = \begin{array}{c|ccc} & t1 & t_2 & t3 \\ \hline p_1 & & & \\ p_2 & & & \\ p_3 & & & \\ p_4 & & & \end{array} \begin{pmatrix} 1 & -3 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

- The components of C are integers;
- Any matrix or line/column vector having all the components 0 will be denoted by **0**.
- If we represent the function Δt as: $\begin{pmatrix} \Delta t(p_1) \\ \Delta t(p_2) \\ \dots \\ \Delta t(p_m) \end{pmatrix}$ then we can denote

$C = (\Delta t_1, \Delta t_2, \dots, \Delta t_n)$

The state equation

- Let M be a marking and t_j a transition enabled in M, $M[t_j\rangle M'$. If M is regarded as a m - dimensional column vector, then:

$$M' = M + C \cdot f$$

where f is a n - dimensional column vector, with 1 on line j and 0 otherwise.

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- Let $\sigma \in T^*$. *The characteristic function of* σ is $\overrightarrow{\sigma} : \{1, \ldots, n\} \rightarrow \mathbb{N}$, such that $\overrightarrow{\sigma}(i)$ is the number of occurrences of transition t_i in σ.
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Theorem 4 (State equation)

Let $N = (P, T, F, W)$ be a marked Petri net and M, M' two markings. If $M' \in [M\rangle$, then there exists a n-dimensional column vector f such that $M' = M + C \cdot f$

Place Invariants

- A place invariant is a weight vector associated to places of the net.
- Describes how the tokens are conserved in the places of the net, in all the reachable markings.
- They are used to obtain equations/relations that hold in every reachable marking of the net (invariant properties regarding the distribution of tokens in places)
- Based on these relations certain behavioural properties can be deduced, based only on the structure of the net and on its initial marking.

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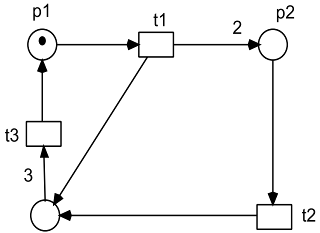
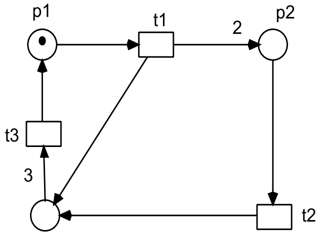
Place Invariants - Example



Reachable markings:

$M_0 = (1,0,0)[t_1\rangle$
$M_1 = (0,2,1)[t_2\rangle$
$M_2 = (0,1,2)[t_2\rangle$
$M_3 = (0,0,3)[t_3\rangle$
$M_0 = (1,0,0)[t_1\rangle \ldots.$

Place Invariants - Example



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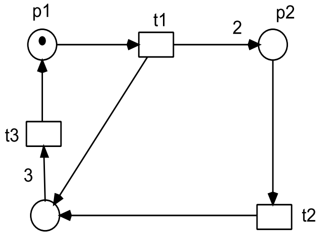
Place Invariants - Example



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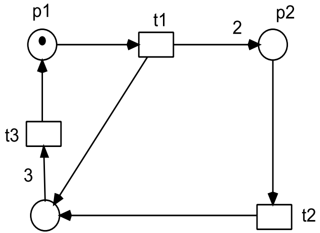
Consider the line vector: $i = (3, 1, 1)$.

For any marking $M \in [M_0\rangle$, $M = \begin{pmatrix} M(p_1) \\ M(p_2) \\ M(p_3) \end{pmatrix}$

$i \cdot M = (3, 1, 1) \cdot M = 3 \cdot M(p_1) + M(p_2) + M(p_3)$.

Place Invariants - Example



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$i \cdot M = (3, 1, 1) \cdot M = 3 \cdot M(p_1) + M(p_2) + M(p_3)$.

Remark: $i \cdot M_0 = i \cdot M_1 = i \cdot M_2 = i \cdot M_3 = 3$.

For all M: $i \cdot M = 3 \Rightarrow 3 \cdot M(p_1) + M(p_2) + M(p_3) = 3$.

Place Invariants - Definition

For a given net, how could a weight vector i be obtained such that for every two reachable markings M, M', it should hold $i \cdot M = i \cdot M'$?

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- If $i \cdot M = i \cdot M'$, it would hold: $i \cdot M_0 + i \cdot C \cdot f = i \cdot M_0 + i \cdot C \cdot f'$, so $i \cdot C \cdot (f - f') = 0$.

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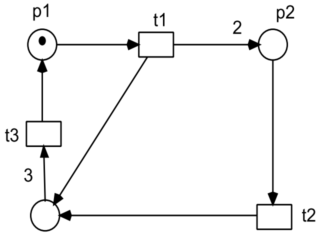
- Hence, one should find i such that $i \cdot C = 0$.

Definition 4

Let $N = (P, T, F.W)$ be a marked Petri net . A place invariant (P-invariant) of N is any m-dimensional vector i with integer components, which verifies: $i \cdot C = 0$.

Example



$$C = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -3 \end{pmatrix}$$

$$i \cdot C = (3,1,1) \cdot \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -3 \end{pmatrix} = (0,0,0) = \mathbf{0}$$

$i = (3,1,1)$ is a P-invariant.

Place Invariants - Definitions

Definition 5

Let $N = (P, T, F, W)$ be a marked Petri net .

- *If i is a P-invariant of N, then the set*

  $$||i|| = \{p \in P \mid i(p) \neq 0\}$$

  is called the support set of i.
- *The P-invariant i is* positive *if $i \geq \mathbf{0}$.*
- *A positive P-invariant $i > \mathbf{0}$ is called* minimal *if there does not exist other P-invariant i' such that $\mathbf{0} < i' < i$.*

Remarks

- Any Petri net has at least one P-invariant, $i = \mathbf{0}$, but only non-null invariants should be considered.
- A net is said to have P-invariants if it has at least one non-null P-invariant.
- If $i_1, i_2, \ldots, i_n$ are P-invariants and $x_1, x_2, \ldots, x_n \in \mathbb{Z}$, then $x_1 \cdot i_1 + x_2 \cdot i_2 + \ldots + x_n \cdot i_n$ is also a P-invariant.

Properties of P-invariants

Theorem 5

Let $\gamma = (N, M_0)$ *be a marked Petri net . If* i *is a non-null P-invariant, then, for any* $M \in [M_0\rangle$*, it holds:*

$$i \cdot M = i \cdot M_0.$$

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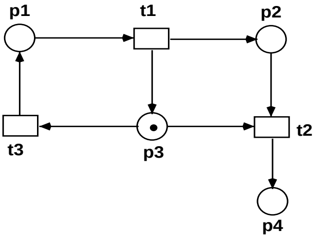
Let $\gamma = (N, M_0)$ be a marked Petri net . If i is a non-null P-invariant, then, for any $M \in [M_0\rangle$, it holds:

$$i \cdot M = i \cdot M_0.$$

- The theorem gives a neccessary conditions for reachability:
  if M is a marking and there exists a P-invariant i such that $i \cdot M \neq i \cdot M_0$, then $M \notin [M_0\rangle$

Place Invariants

The reverse of the theorem is not true: there exists M such that : $i \cdot M = i \cdot M_0$, but M is not reachable:



- $M = (0, 0, 1, 0)$, $M_0 = (1, 0, 0, 0)$
- $i = (\alpha, 0, \alpha, \alpha)$
- $i \cdot M = \alpha = i \cdot M_0$
- $M \notin [M_0\rangle$!

Models of Distributed Systems

Lecture 4

Overview

Overview

Place Invariants

Definition 1

Let $N = (P, T, F, W)$ *a marked Petri net .* N *is covered by P-invariants iff there exists a P-invariant* i *such that* $||i|| = P$.

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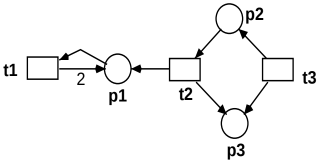

$$(x, y, z) \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \mathbf{0} \Longrightarrow$$

P-invariants: $(0, \alpha, \alpha)$

The net is not covered by P-invariants (there does not exists a P-invariant $i > \mathbf{0}$ with $i(p_1) \neq 0$)

Place Invariants

Theorem 1

Let $\gamma = (N, M_0)$ be a marked Petri net .

(1) *If $i > \mathbf{0}$ is a P-invariant of N, then any place $p \in ||i||$ is bounded.*

(2) *If γ is covered by P-invariants, then γ is bounded.*

- The reverse of (1) and (2) is not true.

Theorem 2

Let N be a Petri Net and $i > \mathbf{0}$ is a P-invariant of N. It holds that:

$$\bullet||i|| = ||i||\bullet$$

Overview

Transition Invariants

- Decribe sequences of transitions which lead from a marking to the same marking.

Definition 2

A marking M of a Petri net N is reproducible if there exists a non-empty sequence of transitions w such that $M\,[w\rangle\,M$.

Transition Invariants - Definitions

Definition 3

Let $N = (P, T, F, W)$ be a marked Petri net . A T-invariant *of N is any n-dimensional column vector $j \in \mathbb{Z}^n$ for which $C \cdot j = \mathbf{0}$, where C is the incidence matrix of N.*

- A net N is said to have T-invariants if it has at least non-null T-invariant.

T-invariants and Reproducible Markings

Theorem 3

A Petri net N has positive T-invariants iff N has reproducible markings.

T-invariants and Reproducible Markings

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A Petri net N has positive T-invariants iff N has reproducible markings.

($\Longrightarrow$): Let $j > \mathbf{0}$ a T-invariant of N. Let M be a marking given by

$$M(p) = \sum_{t_k \in p^\bullet} j(k) \cdot W(p, t_k)\ ,$$

for any $p \in P$.
Consider the sequence of transitions:

$$w = t_1^{j(1)} \ldots t_n^{j(n)}$$

It holds: $M\ [w\rangle\ M'$.
Marking M is reproducible.

($\Longleftarrow$) Let $M[w\rangle M$. Then $M = M + C \cdot \overrightarrow{w}$. Hence $C \cdot \overrightarrow{w} = \mathbf{0}$ and $\overrightarrow{w}$ is a T-invariant.

T-invariants Covering

Definition 4

Let $N = (P, T, F, W)$ *be a Petri net.* N *is* covered by T-invariants *iff there exists a T-invariant* $j > 0$ *with* $||j|| = T$

T-invariants, Liveness and Boundness

Theorem 4

Any live and bounded marked Petri net is covered by T-invariants.

T-invariants, Liveness and Boundness

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Any live and bounded marked Petri net is covered by T-invariants.

In a live and bounded Petri net, there exists a reachable marking $M \in [M_0\rangle$ and a transition sequence σ such that

- σ contains all the transitions in T
- $M[\sigma\rangle M$

$j = \overrightarrow{\sigma}$ is a T-invariant with $||j|| = T$.

T-invariants, Liveness and Boundness

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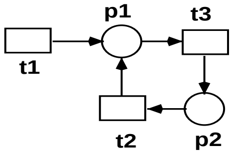
Any live and bounded marked Petri net is covered by T-invariants.



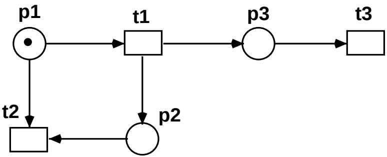
$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Longrightarrow$$

$$\begin{cases} x + y - z = 0 \\ -y + z = 0 \end{cases}$$

- T-invariants: $(0, \alpha, \alpha)$.
- The net is not live or it is not bounded.

Transition Invariants

There exist Petri nets covered by T-invariants which are not live/bounded:



$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

- T-invariants: $(\alpha, \alpha, \alpha)^T$
- The net is covered by T-invariants but it is not bounded.

Overview

Siphons - definition

Definition 5

Let $N = (P, T, F, W)$ *be a Petri net and* $R \subseteq P$ *a set of places.* R *is called a siphon if* $\bullet R \subseteq R\bullet$. *A siphon is proper if* $R \neq \emptyset$.

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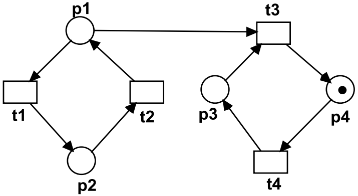

$\{p_1, p_2\}$ is a siphon.

Fundamental Property of Siphons

- Notations: let $R \subseteq P$ be a set of places and M a marking. $M(R) = \sum_{p \in R} M(p)$

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Let $N = (P, T, F, W)$ be a Petri net, $R \subseteq P$ a proper siphon and M a marking of N. R is marked at M, if $M(R) \neq 0$.

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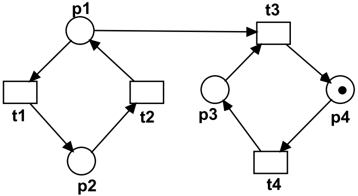
Proposition 1 (Fundamental property of siphons)

Let $N = (P, T, F, W)$ be a Petri net and $R \subseteq P$ a proper siphon. Let M be a marking of the net such that $M(R) = 0$. Then, $\forall M' \in [M\rangle$, $M'(R) = 0$.

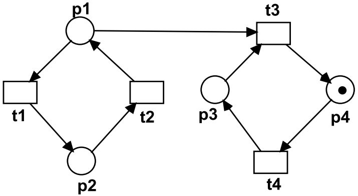
Example



$[M_0\rangle = \{(0,0,1,0),(0,0,0,1)\}$, $\{p_1,p_2\}$ are never marked.

Fundamental Property of Siphons

- A necessary condition for reachability is obtained
- If R is a siphon with $M_0(R) = 0$ and $M(R) \neq 0$, then $M \notin [M_0\rangle$.



$R = \{p_1, p_2\}$ siphon, $M_0(R) = 0 \Rightarrow$ marking $M = (1, 0, 0, 1)$ is not reachable.

Example

Proposition 2

Let $\gamma = (N, M_0)$ *be a marked Petri net with* $W(f) = 1, \forall f \in F$ *and* M *a marking. If* M *is a dead marking, then the set of places* $R = \{p \in P | M(p) = 0\}$ *is a proper siphon.*

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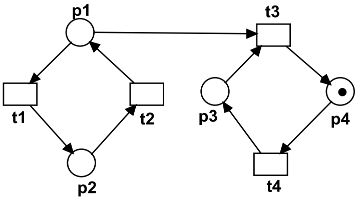

- $M = (0, 0, 1, 0)$ is a dead marking, so $\{p_1, p_2, p_4\}$ is a siphon .

Example

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Proposition 3

Let $N = (P, T, F, W)$ be a Petri net i a P-invariant for N. The support set, $||i||$, is a siphon.

Overview

Traps-definition

Definition 7

Let $N = (P, T, F, W)$ *be a net and* $R \subseteq P$ *a set of places.* R *is called a trap if* $R\bullet \subseteq \bullet R$*. A trap is proper if* $R \neq \emptyset$*.*

Traps-definition

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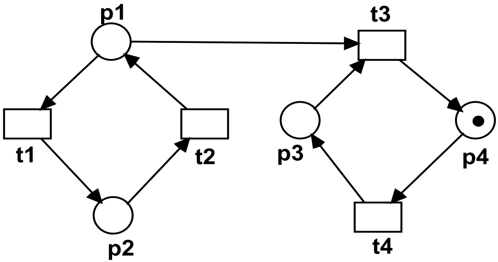

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Fundamental Property of Traps

Proposition 5

Let $N = (P, T, F, W)$ be a Petri net and $R \subseteq P$ a proper trap. Let M be a marking of the net such that $M(R) \neq 0$. Then, $\forall M' \in [M\rangle$, $M'(R) \neq 0$.

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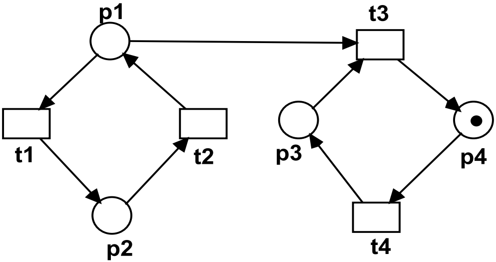
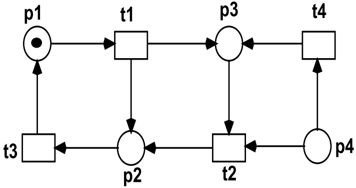


$\{p_3, p_4\}$ is a trap.

Places p_3, p_4 remain marked in any reachable marking.

Fundamental Property of Traps

- A necessary condition for reachability is obtained
- Given a marking M and R a trap with $M_0(R) \neq 0$, if $M(R) = 0$, then $M \notin [M_0\rangle$



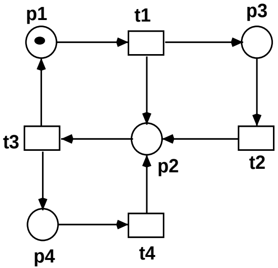
$R = \{p_1, p_2, p_3\}$ trap, $M_0(R) \neq 0$.
$M = (0, 0, 0, 1) \notin [M_0\rangle$ $(M(R) = 0)$

Characterization of Traps

Proposition 6

A set R of places is a trap iff for any transition t:

$$|\bullet t| \cdot |R \cap t\bullet| \geq |R \cap \bullet t|$$



$R = \{p_1, p_3, p_4\}$ is not a trap:

$|\bullet t_2| \cdot |R \cap t_2\bullet| = 0 < |R \cap \bullet t_2| = 1$

Overview

A necessary condition for liveness

Proposition 7

Let $\gamma = (N, M_0)$ be a live marked Petri net . Any siphon R of N is marked at M_0.

A necessary condition for liveness

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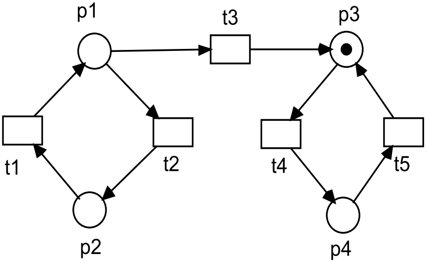
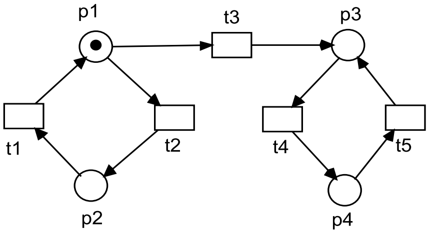


$\{p_1, p_2\}$ is not marked at the initial marking, hence the net is not live.

A necessary condition for liveness



The reverse is not true:

All siphons are marked in the initial marking:

- $\{p_1, p_2\}$

The net is not live.

Deadlocks

Proposition 8

Let $\gamma = (N, M_0)$ be a marked Petri net with $W(f) = 1, \forall f \in F$. If any proper siphon of N includes a trap marked at M_0, then γ is deadlock-free.

Deadlocks

Proposition 8

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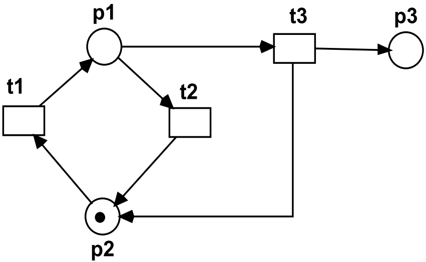
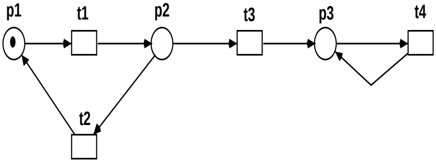


- Siphons: $\{p_1, p_2\}$, $\{p_1, p_2, p_3\}$,
- Traps: $\{p_1, p_2\}$, $\{p_3\}$, $\{p_1, p_2, p_3\}$

The net is deadlock-free.

Deadlocks

The reverse is not true:



- Deadlock-free net
- Proper siphons: $\{p_1, p_2\}$
- Proper traps: $\{p_3\}$

Overview

Notations

Let (N, M_0) be a marked Petri net and p a place of N.

- $min_{p\bullet} = min_{t \in p\bullet} W(p, t)$
- $max_{p\bullet} = max_{t \in p\bullet} W(p, t)$

Min-controlled/ max-controlled siphons

Definition 8

Let (N, M_0) *be a marked Petri net . A siphon S is min-controlled (min-cs) if:*

$$\forall M \in [M_0\rangle, \exists p \in S : M(p) \geq min_{p\bullet}.$$

(N, M_0) *satisfies the min-cs property if all its siphons satisfy the min-cs property.*

Definition 9

A siphon S is max-controlled (max-cs) if:

$$\forall M \in [M_0\rangle, \exists p \in S : M(p) \geq max_{p\bullet}.$$

(N, M_0) *satisfies the max-cs property if all its siphons satisfy the max-cs property.*

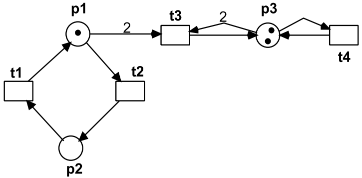
The cs-property

Definition 10

A Petri net $(N; M_0)$ is said to be satisfying the controlled-siphon property (cs-property) if and only if each minimal siphon of $(N; M_0)$ is min or max controlled.

- a max-controlled siphon is also a min-controlled siphon
- If $\forall t, t' \in p\bullet : W(p,t) = W(p,t')$, then a min-controlled siphon is also a max-controlled siphon.

Example



$\{p_1, p_2\}$ min-cs, not max-cs

$\{p_3\}$ max-cs,min-cs

The net has the min-cs property and the cs-property

Properties

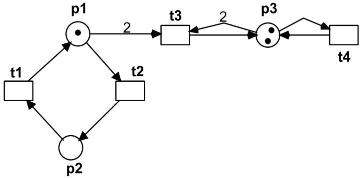
Proposition 9

If a marked Petri net (N, M_0) is live, then it satisfies the min-cs property.

Proposition 10

If a marked Petri net (N, M_0) satisfies the max-cs property, then it is deadlock-free.

Example



- $\{p_1, p_2\}$ min-cs, not max-cs
- $\{p_3\}$ max-cs,min-cs
- min-cs, not live
- deadlock-free, not max-cs

Overview

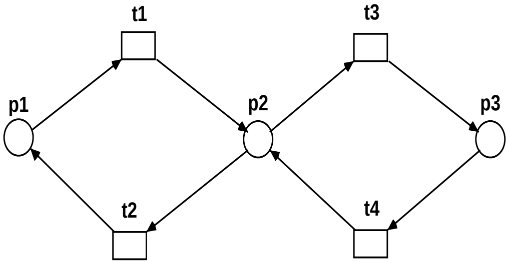
Definition of S-systems

Definition 11

A S-system (state machine) is a marked Petri net (N, M_0), *where* $N = (S, T, F, W)$ *such that:*

- $W(x, y) = 1, \forall (x, y) \in F$
- $| \bullet t| = |t \bullet | = 1 \; \forall t \in T.$

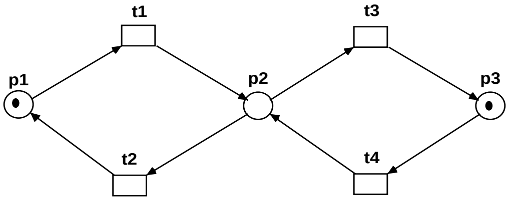
The weight function is usually omitted: $N = (S, T, F)$.



Properties of S-systems

Proposition 11 (fundamental property of S-systems)

Let (N, M_0) be a S-system. If $M \in [M_0\rangle$, then $M_0(S) = M(S)$.



- A necessary condition for reachability in S-systems: $M = (0, 1, 0)$, $M(S) \neq M_0(S)$, hence $M \notin [M_0\rangle$

Liveness and Boundness in S-systems

Theorem 5 (boundness)

A S-system (N, M_0) is bounded

Liveness and Boundness in S-systems

Theorem 5 (boundness)

A S-system (N, M_0) is bounded

Theorem 6 (liveness)

A S-system (N, M_0) is live iff it is strongly connected and $M_0(S) \neq 0$.

Liveness and Boundness in S-systems

Theorem 5 (boundness)

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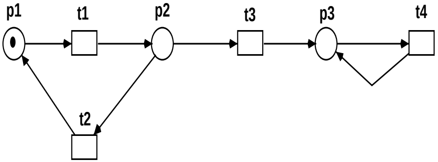

Reachability in S-systems

Lemma 1

Let (N, M_0) a strongly connected system and M, M' markings such that $M(S) = M'(S)$. Then $M[*\rangle M'$.

Reachability in S-systems

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Let (N, M_0) *a strongly connected system and* M, M' *markings such that* $M(S) = M'(S)$. *Then* $M[*\rangle M'$.

Consequence 1

Let (N, M_0) *a strongly connected system and* M *a marking such that* $M(S) = M_0(S)$. *Then* $M \in [M_0\rangle$.

Reachability in live S-systems

Theorem 7 (reachability)

Let (N, M_0) be a strongly connected S-system and M one of its markings. M is reachable iff $M(S) = M_0(S)$

Reachability in live S-systems

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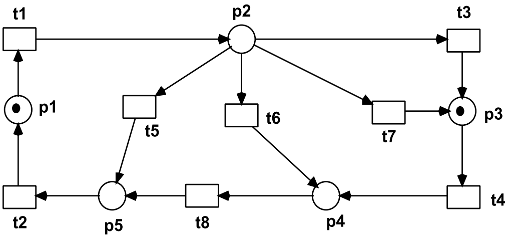
Let (N, M_0) be a strongly connected S-system and M one of its markings. M is reachable iff $M(S) = M_0(S)$



$(0, 0, 0, 1, 1)$ reachable from the initial marking.

Properties of S-systems

Lemma 1

Let (N, M_0) be a connected S-system. A vector $i : S \to \mathbb{Z}$ is a P-invariant of N iff $i = (x, x, \ldots, x)$.

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Theorem 8

Let (N, M_0) *be a strongly connected S-system and* M *a marking.* $M \in [M_0\rangle \Leftrightarrow$ *for every P-invariant* i*:* $i \cdot M = i \cdot M_0$

Properties of S-systems

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Let (N, M_0) be a connected S-system. A vector $i : S \to \mathbb{Z}$ is a P-invariant of N iff $i = (x, x, \ldots, x)$.

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Let (N, M_0) be a strongly connected S-system and M a marking. $M \in [M_0\rangle \Leftrightarrow$ for every P-invariant i: $i \cdot M = i \cdot M_0$

Theorem 9 (reversibility in live S-systems)

A strongly connected S-system is reversible.

Models of Distributed Systems

Lecture 5

Overview

Overview

1. Special Classes of Petri Nets
   - T-systems
2. Free Choice Petri Nets

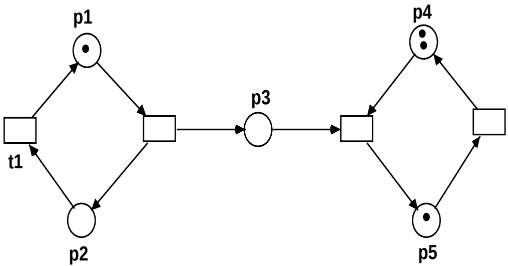
Definition of T-systems

Definition 1

A T-system (marked graph) is any marked Petri net (N, M_0), where $N = (P, T, F, W)$ such that :

- $W(x, y) = 1, \; \forall (x, y) \in F$
- $|\bullet p| = |p \bullet| = 1$, *for all* $p \in P$.

The weight function is usually omitted: $N = (P, T, F)$.



T-system: (N, M_0)

The Fundamental Property of T-systems

Definition 2

Let γ be a circuit of a Petri net, M a marking, R the set of places from the circuit γ.

- *The number of tokens on the circuit, in marking M, is: $M(\gamma) = M(R)$;*
- *Circuit γ is called marked (in marking M), if $M(\gamma) > 0$;*
- *Circuit γ is called initially marked, if $M_0(\gamma) > 0$.*

The Fundamental Property of T-systems

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Proposition 1 (The Fundamental Property of T-systems)

Let (N, M_0) be a T-system and γ a circuit of the T-system. Then:

$$\forall M \in [M_0\rangle : \ M(\gamma) = M_0(\gamma)$$

Liveness

Theorem 1

A T-system (N, M_0) *is live iff all of its circuits are initially marked.*

Liveness

Theorem 1

A T-system (N, M_0) is live iff all of its circuits are initially marked.

Proof:

($\Longrightarrow$) Assume there exists a circuit γ such that $M_0(\gamma) = 0$. $\forall M \in [M_0\rangle$ $M(\gamma) = 0$. It contradicts the liveness of the system.

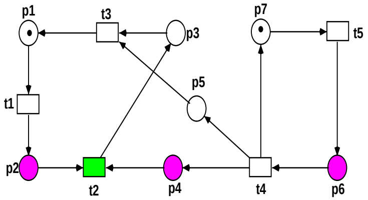
Liveness

Theorem 1

A T-system (N, M_0) is live iff all of its circuits are initially marked.

Proof:
$(\Longrightarrow)$ Assume there exists a circuit γ such that $M_0(\gamma) = 0$. $\forall M \in [M_0\rangle\ M(\gamma) = 0$. It contradicts the liveness of the system.

$(\Longleftarrow)$ Let $t \in T$ and $M \in [M_0\rangle$.
Let $P_M = \{p | \exists$a path from p to t and for all the places s on the path:$M(s) = 0\}$.



$t = t_2$, $P_M = \{p_2, p_4, p_6\}$.

Liveness

$P_M = \{p | \exists$a path from p to t and for all the places s on the path:$M(s) = 0\}$
one can prove $P(k)$, $\forall k \geq 0$:

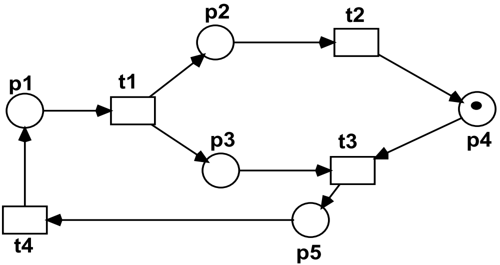
$$P(k) : \forall M \in [M_0\rangle \; with |P_M| = k, \exists M_t : M[*\rangle M_t[t\rangle$$

(Induction on $|P_M|$)

- $P(0)$: Let $M \in [M_0\rangle$ with$|P_M| = 0$, $\forall p \in \bullet t : M(p) > 0$, so $M[t\rangle$.
- Assume $P(k)$ for all $k < n$. We prove $P(n)$:
  Let $M \in [M_0\rangle$ with $|P_M| = n$.
  There exists a maximal path $\pi = t_1 p_1 t_2 \dots p_{m-1} t_m = t$, with $p_1, \dots, p_{m-1} \in P_M$
  It holds: $M[t_1\rangle M'$.
  It can be proven that:
  - $P_{M'} \subseteq P_M$
  - $P_{M'} \neq P_M$

$P_{M'} \subset P_M \Rightarrow |P_{M'}| = k < |P_M| \Rightarrow \; \exists M'_t$ with $M'[*\rangle M'_t[t\rangle$. Since $M[t_1\rangle M'$:
$M[t_1\rangle M'[\sigma'\rangle M'_t[t\rangle$.

Liveness



- not live

Boundness

Proposition 2

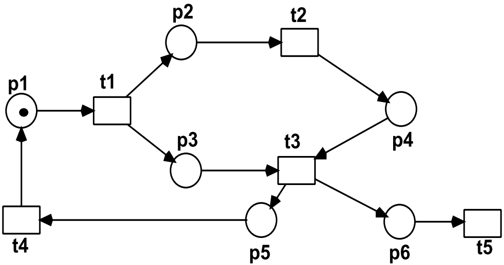
Let (N, M_0) be a strongly connected T-system. Then (N, M_0) is bounded.

(N, M_0) strongly connected, so any place p is on a circuit γ. In any reachable marking M, $M(p) \leq M(\gamma) = M_0(\gamma)$.

Theorem 2

A live T-system is bounded iff it is strongly connected.

Boundness



- live
- unbounded

Reachability

Theorem 3

Let (N, M_0) be a live T-system. M is reachable iff, for any place invariant i, it holds: $i \cdot M = i \cdot M_0$

Lemma 1

Let (N, M_0) be a connected T-system. A vector $j : T \to \mathbb{Z}$ is a T-invariant of N iff $j = (x, x, \dots, x)^T$.

Liveness in Strongly Connected T-systems

Theorem 4

Let (N, M_0) be a strongly connected T-system. The following statements are equivalent:

1. *(N, M_0) is live.*
2. *(N, M_0) is deadlock free.*
3. *(N, M_0) contains an infinite occurrence sequence.*

Liveness in Strongly Connected T-systems

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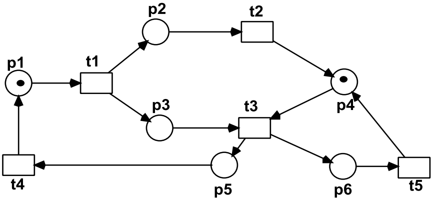


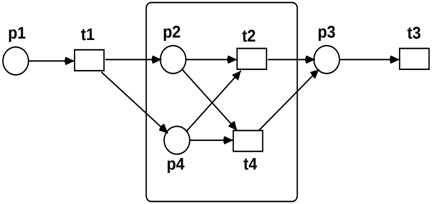
Overview

1. Special Classes of Petri Nets
   - T-systems
2. Free Choice Petri Nets

Definition 3

A Petri net is free-choice iff for any two transitions t_1 and t_2:

$$\bullet t_1 \cap \bullet t_2 \neq \phi \implies \bullet t_1 = \bullet t_2$$



Proposition 3

A Petri net is free-choice iff for any two places p_1 and p_2:

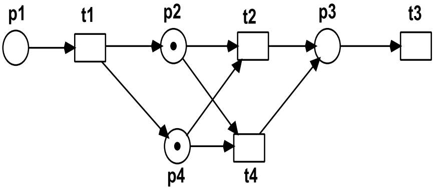
$$p_1 \bullet \cap p_2 \bullet \neq \emptyset \implies p_1 \bullet = p_2 \bullet$$

.

Properties in Free-choice Nets

Proposition 4

Let N be a free-choice Petri net, p be a place and M a marking of N. If $\exists t \in p\bullet : M[t\rangle$, then $(\forall t \in p\bullet)(M[t\rangle)$.



Clusters

Definition 4 (Clusters)

Let x be a node in a Petri net. The cluster of x, denoted by $[x]$, is the minimal set of nodes such that :

- $x \in [x]$
- *if a place* $p \in [x]$, *then* $p\bullet \subseteq [x]$
- *if a transition* $t \in [x]$, *then* $\bullet t \subseteq [x]$

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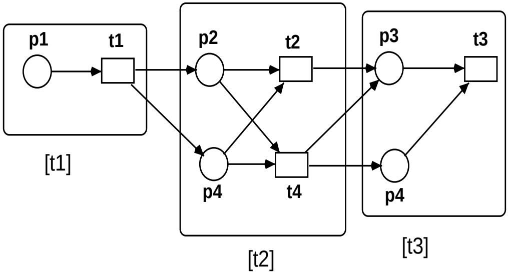

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- *if a place $p \in [x]$, then $p\bullet \subseteq [x]$*
- *if a transition $t \in [x]$, then $\bullet t \subseteq [x]$*

Proposition 5

Let N be a free-choice net and M a marking of N. If t is a transition and $M[t\rangle$, then $M[t'\rangle, \forall t' \in [t]$.

Place liveness in Free Choice Petri Nets

Definition 5

Let $\gamma = (N, M_0)$ *be a Petri Net. A place p is quasi-live in a marking M iff there exists a marking* $M_p \in [M\rangle$ *such that* $M_p(p) \neq 0$. *If place p is not quasi-live in marking M, then p is dead in M.*

Definition 6

Let $\gamma = (N, M_0)$ *be a Petri Net. A place p is live from marking M iff it is quasi-live from every reachable marking* $M' \in [M\rangle$. *Place p is live in* γ *iff it is live in* M_0 $(\forall M \in [M_0\rangle$ *there exists a marking* $M_p \in [M\rangle$ *such that* $M_p(p) \neq 0)$

Definition 7

Let $\gamma = (N, M_0)$ *be a Petri Net.* γ *is place-live iff all its places are live.*

Proposition 6

A live Petri net is place-live.

Place liveness in Free Choice Petri Nets

Proposition 7

Let N be a free-choice Petri net and M a marking. If $\forall p \in \bullet t$, p is live from M, then t is quasi-live from M.

Proof

- let $\{p_1, \ldots, p_n\} = \bullet t$
- $p_1, \ldots, p_n$ live from $M \Rightarrow M[*\rangle M_1[*\rangle M_2 \ldots M_{n-1}[*\rangle M_n$ with $M_i(p_i) > 0$
- the only transitions that remove tokens from $p_1, \ldots, p_n$ are the transitions in $[t]$
- if a transition in $[t]$ is enabled in the sequence, t is also is enabled in a marking reachable from M, otherwise $M_n(p_i) > 0$ for all i, hence $M_n[t\rangle$

Lemma 1

If a free-choice Petri net is place-live, then it is live.

Consequence 1

A free-choice Petri net is live iff it is place-live.

Liveness in Free Choice Petri Nets

Proposition 8

Let N be a non-live free-choice Petri net. There exists a siphon R and a reachable marking M such that $M(R) = 0$.

Liveness in Free Choice Petri Nets

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Let N be a non-live free-choice Petri net. There exists a siphon R and a reachable marking M such that $M(R) = 0$.

Proof: if N is not live, it is not place-live, so there exists a place p and a reachable marking L such that p is dead in L.

- Let $Dead(M)$ denote the set of dead places in a marking M.
- Let $M \in [L\rangle$ be a marking such that for all $M' \in [M\rangle$: p' is dead in M iff p' is dead in M' (i.e. $Dead(M) = Dead(M')$).
- Let $R = Dead(M)$. It holds:
  1. $R \neq \emptyset$ ($p \in Dead(M)$)
  2. if $t \in \bullet R$, then t is dead in M
  3. $\bullet R \subseteq R \bullet$ (from 2 and proposition 7)

Liveness in Free Choice Petri Nets

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  1. $R \neq \emptyset$ ($p \in Dead(M)$)
  2. if $t \in \bullet R$, then t is dead in M
  3. $\bullet R \subseteq R \bullet$ (from 2 and proposition 7)

Lemma 2

Let N be a free-choice Petri net. If every proper siphon includes an initially marked trap, then N is live.

Liveness: Commoner's Theorem

Theorem 5 (Commoner)

A marked free-choice Petri net (N, M_0) *is live iff any proper siphon includes an initially marked trap.*

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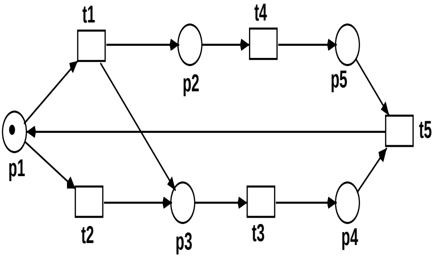
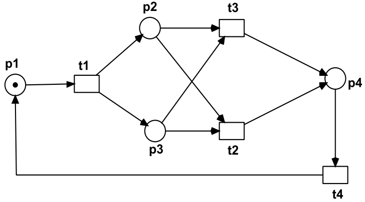


$\bullet\{p_1, p_2, p_5\} = \{t_1, t_4, t_5\}$, $\{p_1, p_2, p_5\}\bullet = \{t_1, t_2, t_4, t_5\}$.
$\{p_1, p_2, p_5\}$ siphon, does not include a trap marked at M_0.

Liveness: Commoner's Theorem

Theorem 5 (Commoner)

A marked free-choice Petri net (N, M_0) is live iff any proper siphon includes an initially marked trap.



$\bullet\{p_1, p_2, p_4\} = \{t_1, t_2, t_3, t_4\}$, $\{p_1, p_2, p_4\}\bullet = \{t_1, t_2, t_3, t_4\}$.

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every siphon includes a trap marked at M_0.

Liveness and Deadlocks

Theorem 6

Let (N, M_0) be a bounded, strongly connected marked free-choice Petri net. (N, M_0) is live iff it is deadlock-free.

Liveness and Deadlocks

Theorem 6

Let (N, M_0) be a bounded, strongly connected marked free-choice Petri net. (N, M_0) is live iff it is deadlock-free.

Proof: $(\Leftarrow)$. Let M be a reachable marking. We show that every transition is quasi-live in M.

- Let M be the marking with the maximum number of dead transitions in it. We show this number is 0.
- M is not dead, there exists a transition t such that $M[t\rangle$
- The net is strongly connected, there exists a path from t containing all the transitions of the net
- One can prove that for every two consecutive transitions u, v on this path:

  u is not dead in $M \Rightarrow v$ is not dead in M:
  - there exists $M[\sigma\rangle$ such that u appears infinitely often in σ
  - let $s \in u\bullet$, $s \in \bullet v$: place s can get an infinite number of tokens, unless there exists some transition $v' \in \bullet s$ enabled in some marking $M' \in [M\rangle \Rightarrow M'[v\rangle \Rightarrow v$ is not dead in M

Definition 8

A Petri net N is well-formed if there exists a marking M such that (N, M) is live and bounded.

The Rank Theorem

Theorem 7 (Rank Theorem)

Let N be a free-choice net, C its incidence matrix and $\mathcal{C}_N$ the set of clusters of N. N is well-formed iff:

1. *N is connected and it has at least a place and a transition.*
2. *N is covered by place invariants*
3. *N is covered by transition invariants*
4. *$Rang(C) = |\mathcal{C}_N| - 1$.*

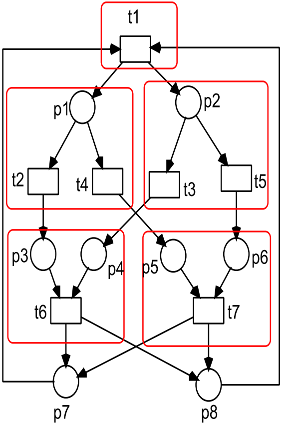
Proposition 9

A well-formed marked free-choice Petri net (N, M_0) is bounded.

Proposition 10

In a well-formed free-choice Petri net, any minimal siphon is a trap.

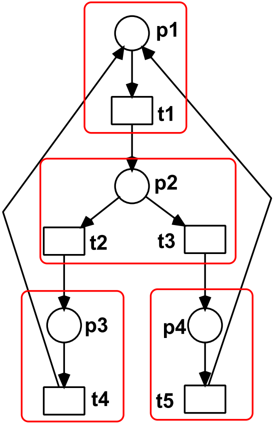
Example



$$C = \begin{pmatrix} -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

- place invariant: $(1, 1, 1, 1, 1, 1, 1, 1)$, the net is covered by place invariants
- transition invariant: $(2, 1, 1, 1, 1, 1, 1)^t$, the net is covered by transition invariants
- Rank(C)=6, number of clusters: 5.
- the net is not well-formed

Example



$$C = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

- place invariants: $(\alpha, \alpha, \alpha, \alpha)$, the net is covered by place invariants.
- transition invariants: $(\alpha + \beta, \alpha, \beta, \alpha, \beta)$, the net is covered by transition invariants.
- Rank(C)=3, the number of clusters: 4.
- well-formed net

Liveness and Boundness

Theorem 8

A free-choice marked Petri net (N, M_0) is live and bounded iff:

1. *N is well-formed.*
2. *M_0 marks any proper siphon of N.*

Proof:

$\Leftarrow$:

- boundness results from Prop. 9
- liveness: Since in well-formed free-choice Petri nets every minimal siphon is also a trap (Prop. 10), then the condition in the Commoner Theorem holds.

Liveness and Boundness

Theorem 9

A free-choice marked Petri net (N, M_0) is live and bounded iff:

1. *N is well-formed.*
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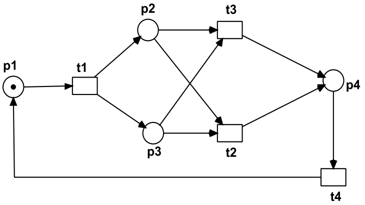
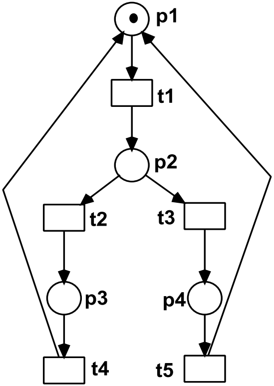


$$C = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

- place invariants: $(\alpha, \beta, \alpha - \beta, \alpha)$, the net is covered by place invariants.
- transition invariants: $(\alpha, \beta, \alpha - \beta, \alpha)$, the net is covered by transition invariants.
- Rank(C)=2, the number of clusters: 3.
- Siphons: $\{p_1, p_2, p_4\}$, $\{p_1, p_3, p_4\}$, $\{p_1, p_2, p_3, p_4\}$ (marked at M_0)

Example



- well-formed net
- the only siphon, $\{p_1, p_2, p_3, p_4\}$ is marked at M_0
- the net is live and bounded

Algorithm for checking condition 2 in the liveness and boundness theorem

Input: A Petri net (P, T, F) and a set R of places

Output: The maximal siphon S included in R ($S \subseteq R$)

begin

$S := R$

while there exists $p \in S$ and $t \in \bullet p$ such that $t \notin S\bullet$

$S := S \setminus \{p\}$

end

- Check if every siphon is marked at M_0:
  - Let R be the set of places not marked at M_0
  - Let S be the maximal siphon included in R
  - Every siphon is marked at M_0 iff $S = \emptyset$
- Both condition 1 and condition 2 from the liveness and boundness theorem can be checked in polynomial time

Home Markings

Proposition 11

Let (N, M_0) be a marked live free-choice net. If M is a home marking of (N, M_0), then M marks any proper trap of N.

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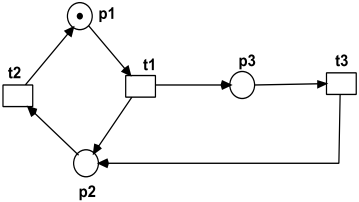

- siphons: $\{p_1, p_2, p_3\}$
- traps: $\{p_1, p_2, p_3\}$, $\{p_1, p_2\}$
- live net
- $(0, 0, 1)$ is not a home marking
- the reverse is not true: $(1, 0, 0)$ marks every trap but it is not a home marking

Home Markings

Theorem 10 (Existence of Home Markings)

Any live and bounded marked free-choice Petri net has home markings.

Theorem 11 (Home Marking Theorem)

Let (N, M_0) be a live and bounded marked free-choice Petri net. A reachable marking $M \in [M_0\rangle$ is home-markging iff it marks any proper trap of N.

Home Markings

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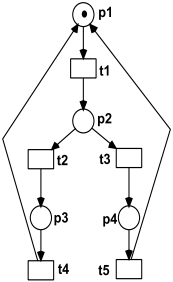


- well-formed net
- the only siphon, $\{p_1, p_2, p_3, p_4\}$ is marked at M_0
- the net is live and bounded
- the only trap: $\{p_1, p_2, p_3, p_4\}$
- every reachable marking is a home-marking (the net is reversible)

Models of Distributed Systems

Lecture 6

Overview

Overview

1 Asymmetric Choice Petri Nets

2 Workflow Modelling: Workflow Nets

- Workflow Nets
- The Soundness Property for Workflow Nets
- Soundness in Special Classes of Workflow Nets

3 Other notions of soundness

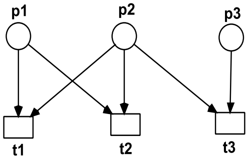
Definition

- weaker conditions than the conditions in the definition of free-choice Petri nets ($s \bullet \cap p\bullet = \emptyset$ or $s\bullet = p\bullet$)

Definition 1

A Petri net $N = (P, T, F)$ is a asymmetric choice Petri net, iff for every two place s and p it holds: either $p \bullet \cap s\bullet = \emptyset$, or $s\bullet \subseteq p\bullet$ or $p\bullet \subseteq s\bullet$

Example



- the net is not free-choice ($p_1\bullet \neq p_2\bullet$, $p_3\bullet \neq p_2\bullet$)
- the net is asymmetric choice: $p_1\bullet \subseteq p_2\bullet$, $p_3\bullet \subseteq p_2\bullet$

Properties

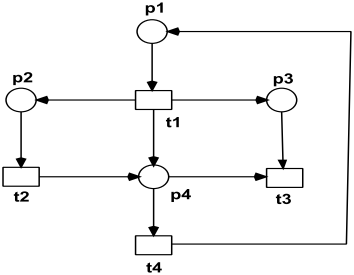
A sufficient condition for liveness:

Proposition 1

Let (N, M_0) be a marked asymmetric choice net. If every proper siphon of N includes an initially marked trap, than (N, M_0) is live.

The reverse is not true (the Commoner's theorem does not hold in the case of asymmetric choice nets).

Example



- the set of all places is a siphon
- there is no proper trap
- the net is live

Properties

A sufficient condition for well-formedness:

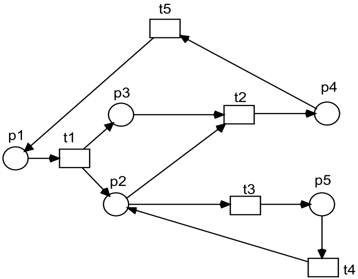
Theorem 1

Let N be an asymmetric choice net, C its incidence matrix and $\mathcal{C}_N$ the set of clusters of N. If the following conditions hold, then N is well-formed:

1. *N is connected and it has at least a place and a transition.*
2. *N is covered by place invariants*
3. *N is covered by transition invariants*
4. $Rank(C) = |\mathcal{C}_N| - 1$.

Properties

The reverse of Theorem 1 does not hold:



$Rank(C) = 4$, $|\mathcal{C}_N| = 4$ (condition 4 does not hold), but the net is well-formed

$M = (1, 0, 0, 0, 0)$

Proposition 2

Let N be an asymmetric choice net satisfying Conditions (1) to (4) of Theorem 1 and M be a marking of N. (N, M) is live and bounded iff M marks all proper siphons of M.

Overview

Workflow Processes

- **Workflow process:** a complex process that executes inside an organization:
  - set of tasks executed in a specific order
  - specific data: used, modified, produced during task execution
  - resources: necessary for executing the tasks of the process

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  - resources: necessary for executing the tasks of the process

- **Workflow Management Systems (WFMS)**: permit the definition of workflow processes and ensure their execution

Workflow Perspectives

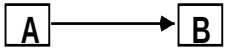
- Process perspective: tasks, their order of execution;
- Resource perspective: resources, resource organization, the way in which resources are assigned for the execution of tasks;
- Data perspective:
  - control data (used for controlling the execution of the process)
  - production data (created/used by tasks)

Workflow Components

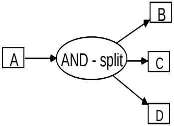
- **Case:** workflow instance, the subject of the tasks and operations inside the workflow process
- **Task**: elementary operation in the workflow process
- **Resource**: executes the task
- **Work item**: task + case (a task that is being executed for a specific case)
- **Activity**: task + case + resource (a task that is executed for a specific case, by a resource)
- **Execution control structures (routing constructs)**: describe the logical dependence between tasks

Execution Control Structures

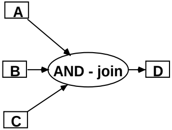
- Sequence:



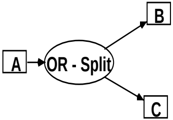
- AND-split



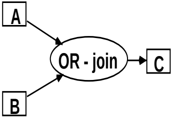
- AND-join



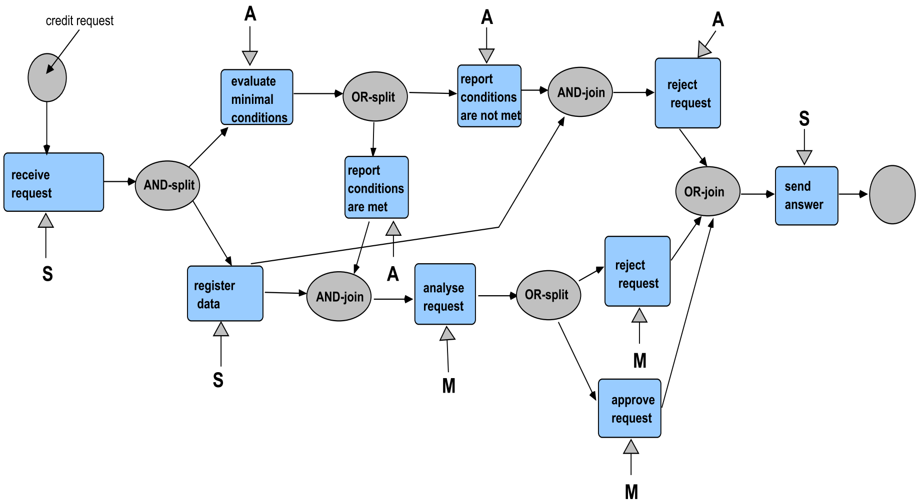
- OR-split



- OR-join



Example



credit request
receive request
S
AND-split
evaluate minimal conditions
A
register data
S
OR-split
report conditions are met
A
report conditions are not met
A
AND-join
AND-join
analyse request
M
OR-split
reject request
A
reject request
M
approve request
M
OR-join
send answer
S

Workflow Specification Languages

- Workflow Management Systems use definitions of workflow processes, expressed in a certain specification language:
- Approaches:
  - Product specific language
  - UML Activity Diagrams
  - Workflow Graphs
  - BPMN
  - XML-based languages: BPEL, XPDL
  - Process algebra
  - Petri nets

Petri Nets in Workflow Modelling

- Workflow nets:
  - model the process perspective (abstracts from data and resources)
  - model the execution of one case
- High level Petri nets for modelling the other perspectives of the workflow, besides the process perspective

Petri Nets in Workflow Modelling

- tasks $\longrightarrow$ transitions

Petri Nets in Workflow Modelling

- tasks $\longrightarrow$ transitions
- case $\longrightarrow$ a token in the net

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Petri Nets in Workflow Modelling

- tasks $\longrightarrow$ transitions
- case $\longrightarrow$ a token in the net
- pre-conditions and post-conditions for the execution of tasks $\longrightarrow$ places
- work item $\longrightarrow$ transition enabled in a certain marking
- activity $\longrightarrow$ the execution of a transition
- execution control structures $\longrightarrow$ places or transitions

Workflow Nets Definition

Definition 2

A workflow net (WF-net) is a Petri net $PN = (P, T, F)$ such that :

1. *P contains an input place i and an output place o such that $\bullet i = \emptyset$ and $o\bullet = \emptyset$.*
2. *For every $n \in P \cup T$, there exists a path in PN from i to n and from n to o.*

Remarks

- $W(x, y) = 1$, for every $(x, y) \in F$.
- If a transition t^* is added to PN such that $\bullet t^* = \{o\}$ and $t^* \bullet = \{i\}$, then the resulting Petri net is strongly connected.

Notations:

- The initial marking of a WF-net, M_0:

$$M_0(i) = 1, M_0(p) = 0, \forall p \neq i.$$

  We write $M_0 = i$

- The final marking of a WF-net, M_f:

$$M_f(o) = 1, M_f(p) = 0, \forall p \neq o.$$

  We write $M_f = o$

Soundness

- In a workflow the execution/processing of a case should always terminate;

Soundness

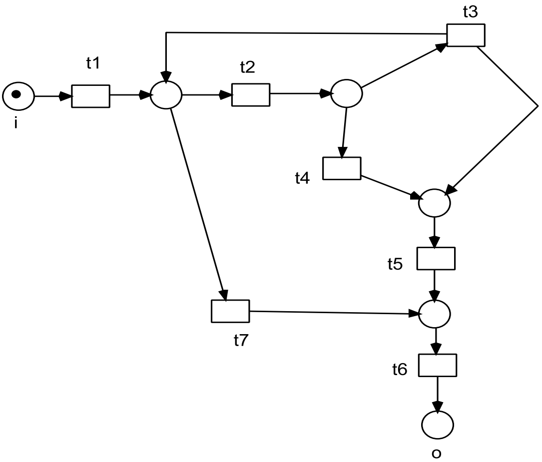
- In a workflow the execution/processing of a case should always terminate;
- There do not exist useless tasks (every task should be able to execute at a certain point)

Definition 3

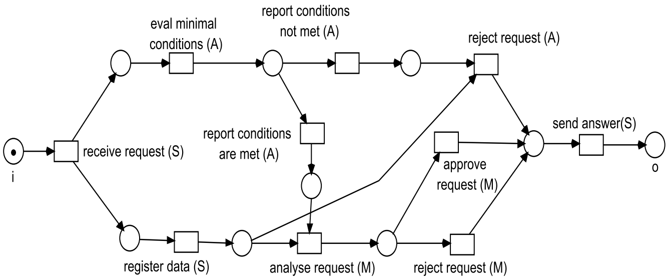
A workflow net PN = (P, T, F) is sound iff:

1. $\forall M \in [i\rangle, o \in [M\rangle$ *(the termination condition)*
2. $\forall t \in T$, *t is quasi-live*

Soundness Definition



Example

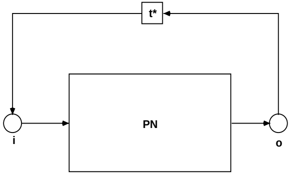

eval minimal conditions (A)
report conditions not met (A)
reject request (A)
receive request (S)
report conditions are met (A)
send answer(S)
approve request (M)
i
o
register data (S)
analyse request (M)
reject request (M)

Closure of a Workflow Net

Definition 4

Let $PN = (P, T, F)$ *be a workflow net. The closure of* PN *is the net* $\overline{PN} = (\overline{P}, \overline{T}, \overline{F})$*, such that :*

- $\overline{P} = P$
- $\overline{T} = T \cup \{t^*\}$
- $\overline{F} = F \cup \{(o, t^*), (t^*, i)\}$



Properties

Lemma 1

Let $PN = (P, T, F)$ be a workflow net for which the termination condition in the definition of soundness holds. Then:

1. $(\forall M \in [i\rangle)(M \geq o \Rightarrow M = o)$
2. (PN, i) *is bounded.*
3. *the set of reachable markings of* (PN, i) *is the same as the set of reachable markings of* $(\overline{PN}, i)$.
4. (PN, i) *is quasi-live iff* $(\overline{PN}, i)$ *is quasi-live.*

A Characterization of Soundness

Lemma 2

Let $PN = (P, T, F)$ be a sound WF-net. Then, $(\overline{PN}, i)$ is live and bounded.

Lemma 3

Let $PN = (P, T, F)$ be a WF-net. If $(\overline{PN}, i)$ is live and bounded, then PN is sound.

A Characterization of Soundness

Theorem 2

A WF-net PN is sound iff $(\overline{PN}, i)$ is live and bound.

Proposition 3

A WF-net PN is sound iff o is a home marking in (PN, i) and (PN, i) is quasi-live.

Consequence 1

The soundness problem is decidable for WF-nets.

Free-choice Workflow Nets

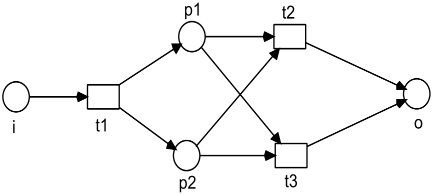
Definition 5

A worfklow net is free choice iff for every two transitions t_1 *and* t_2, $\bullet t_1 \cap \bullet t_2 \neq \emptyset \implies \bullet t_1 = \bullet t_2$.

Free-choice Workflow Nets

Definition 5

A worfklow net is free choice iff for every two transitions t_1 and t_2, $\bullet t_1 \cap \bullet t_2 \neq \emptyset \Longrightarrow \bullet t_1 = \bullet t_2$.



Soundness in free-choice Petri nets

- The following problem can be solved in polynomial time: given a free-choice Petri net, decide if it is live and bounded (Desel & Esparza,1995).
- If PN is a free-choice Petri net, then $\overline{PN}$ is a free-choice Petri net.

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- If PN is a free-choice Petri net, then $\overline{PN}$ is a free-choice Petri net.

Consequence 2

The following problem can be solved in polynomial time: given a free-choice workflow net, decide if it is sound.

Lemma 1

A sound free-choice workflow net is safe.

Soundness in free-choice Petri nets

Boundness & Liveness Theorem

A free-choice marked Petri net (N, M_0) is live and bounded iff:

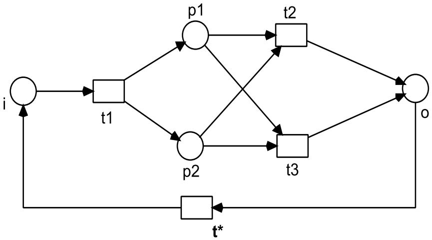
1. N is well-formed.
2. M_0 marks any proper siphon of N.

Rank Theorem

Let N be a free-choice net, C its incidence matrix and $\mathcal{C}_N$ the set of clusters of N. N is well-formed iff:

1. N is connected and it has at least a place and a transition.
2. N is covered by place invariants
3. N is covered by transition invariants
4. $Rang(C) = |\mathcal{C}_N| - 1$.

Free-choice Workflow Nets



- place invariants: $(\alpha, \beta, \alpha - \beta, \alpha)$, the net is covered by place invariants.
- transition invariants: $(\alpha, \beta, \alpha - \beta, \alpha)$, the net is covered by transition invariants.
- Rank(C)=2, the number of clusters: 3.
- well-formed net
- siphons: $\{i, p_1, o\}, \{i, p_2, o\}, \{i, p_1, p_2, o\}$ (marked at M_0)
- live and bounded

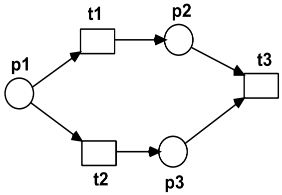
Well-structured workflow nets

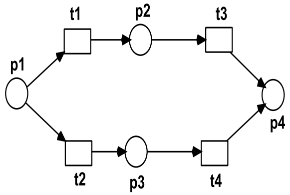
- A path C from an element n_1 to an element n_k, in a Petri net, is any sequence $n_1, n_2, \ldots, n_k$, such that $(n_i, n_{i+1}) \in F$, for any $1 \leq i \leq k-1$.
  $\alpha(C) = \{n_1, n_2, \ldots, n_k\}$.
- C is an elementary path, iff for every n_i and n_j in C, $i \neq j \Rightarrow n_i \neq n_j$.

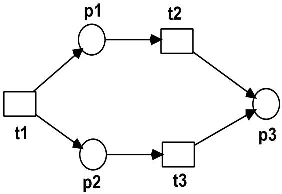
Definition 6

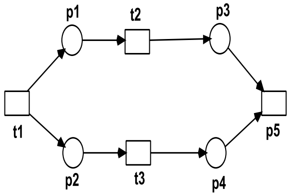
A Petri net PN is well - handled iff for every pair of elements x and y such that one of the elements is a place and the other is a transition, there do not exist two distinct elementary paths, C_1 and C_2 from x to y such that $\alpha(C_1) \cap \alpha(C_2) = \{x, y\}$.

Example


t1
p2
p1
t3
t2
p3


t1
p2
t3
p1
p4
t2
p3
t4


p1
t2
t1
p3
p2
t3


p1
t2
p3
t1
p5
p2
t3
p4

Well-structured Petri nets

Lemma 4 (Esparza & Silva 1990)

A well-handled Petri net is bounded.

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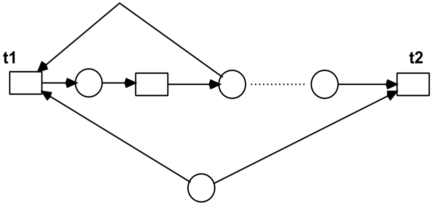
Definition 7

A WF-net PN is well-structured, iff its closure $\overline{PN}$ is well-handled.

Elementary extended non-self controlling Petri nets

Definition 8

A Petri net is elementary extended non-self controlling (ENSeC) iff for any two transitions t_1 and t_2 such that $\bullet t_1 \cap \bullet t2 \neq \emptyset$, there does not exist an elementary path from t_1 to t_2 such that $\bullet t_1 \cap \alpha(C) = \emptyset$.



Well-structured Petri nets

Theorem 3 (van der Aalst 1996)

Let PN be a WF - net. If PN is well-structured, than $\overline{PN}$ is ENSeC.

Theorem 4 (Barkaoui 1995)

The liveness problem can be solved in polynomial time for bounded ENSeC.

Consequence 3

The following problem can be solved in polynomial time: given a well-structured workflow net, decide if it is sound.

$\overline{PN}$ well-handled $\Rightarrow$ bounded. Liveness can be decided in polynomial time for $\overline{PN}$, which is ENSeC and bounded.

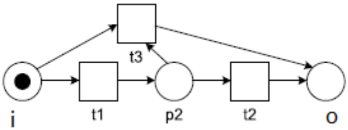
Overview

Weak soundness

- A WF-net for which holds only the first condition in the definition of soundness (termination) is called weak sound.

Definition 9

A WF-net is weak sound iff $\forall M \in [i\rangle, o \in [M\rangle$



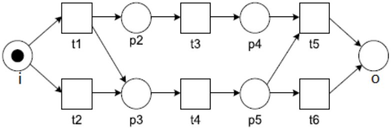
- The net is not sound (it is not quasi-live), but it is weak sound.

Relaxed soundness

Definition 10

Let N be a WF-net. N is relaxed sound if and only if for each transition t:

$$\exists M, M' \in [i\rangle_N : M[t\rangle M' \wedge o \in [M'\rangle$$

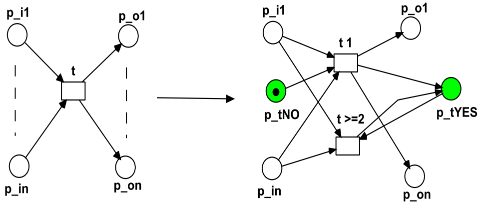


(a) a relaxed sound WF-net

- The net is not sound (termination property does not hold), but it is relaxed sound.

Relaxed soundness

The following Petri net PN' is built starting from PN:



- The initial marking of PN': $i + p_{t1NO} + \ldots + p_{tnNO}$, where $t1, \ldots, tn$ are all the transitions in PN
- PN is relaxed sound iff for every transition ti, a marking $p_{tiYES} + o + M$ is reachable in PN'. (M is a marking in which only places $p_{tkYES}, p_{tjNO}, k, j \in \{1, \ldots, n\}$ can contain one token. The set of all such markings is finite.)

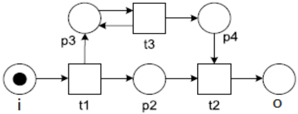
Relaxed soundness is decidable for workflow nets.

Lazy soundness

Definition 11

Let N be a WF-net. N is lazy sound if and only if:

1. $\forall M \in [i\rangle_N \; \exists M' \in [M\rangle : M'(o) = 1$ *(option to complete)*
2. $\forall M \in [i\rangle_N : \; M(o) \leq 1$ *(proper completion)*



(b) a lazy sound net

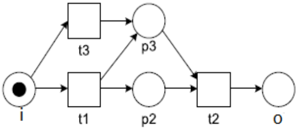
- The net is not sound, but it is lazy sound.

Lazy soundness is decidable for workflow nets (using the coverabilty tree).

Easy soundness

Definition 12

Let N be a WF-net. N is easy sound if and only if $o \in [i\rangle_N$



- The net is not sound, but it is easy sound.

Easy soundness is decidable for workflow nets (reachability of marking *o* is decidable).

Relations between the soundness notions

Proposition 4

A sound WF-net is weak sound.

Proposition 5

A sound WF-net is relaxed sound.

Proposition 6

A relaxed sound WF-net is easy sound.

Proposition 7

A weak sound WF-net is easy sound and lazy sound.

Relations between types of soundness