# Recitation 3

1. (Estimating uniform parameters)

   (a) Suppose we have data $1.2, 2.1, 1.3, 10.5, 5$ which we know is drawn independently from a `uniform(a,b)` distribution. Give the maximum likelihood estimate for the parameters a and b.(Hint: in this case you should not try to find the MLE by differentiating the likelihood function).

   (b) Suppose we have data $x_1, x_2, \ldots, x_n$ which we know is drawn independently from a `uniform(a,b)` distribution. Give the maximum likelihood estimate for the parameters $a$ and $b$.

   **Solution:**

   (a) The pdf for uniform (a, b) one data value is $f\left(x_i \mid a, b\right) = \frac{1}{b-a}$ if $x_i$ is in the interval[a, b] and 0 if it is not. So the likelihood function for our 5 data values is

   $$f(\text{ data } \mid a, b) = \begin{cases} \frac{1}{(b-a)^5} & \text{if all data is in [a, b]} \\ 0 & \text{otherwise} \end{cases}$$

   This is maximized when $(b-a)$ is as small as possible. Since all the data has to be in the interval $[a, b]$ we minimize $(b-a)$ by taking a = minimum of data and b = maximum of data. So $a = 1.2, b = 10.5$.

   (b) The same logic as in part (a) shows a $= \min\left(x_1, \ldots, x_n\right)$ and b $= \max\left(x_1, \ldots, x_n\right)$.

2. (Customer waiting time) Let $R$ be the rate at which customers are served in a queue. Suppose that R is exponential with pdf $f(r) = 2e^{-2r}$ on $[0, \infty)$.

   (a) What's the probability that R is in $[1, 2]$

   (b) Find the cdf of $R$

   (c) Find the pdf of the waiting time per customer $T = 1/R$.

   **Solution**:

   (a)

   $$P(1 \leq r \leq 2) = \int_1^2 2e^{-2r} dr = -\left. e^{-2r} \right|_1^2 = e^{-2} - e^{-4}$$

   (b)

   $$P(R \leq r) = \int_0^r 2e^{-2u} du = -\left. e^{-2u} \right|_0^r = 1 - e^{-2r}$$

   So the cdf for R is

   $$F_R(r) = \begin{cases} 1 - e^{-2r} & \text{for r} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) First, we find the cdf of T. T takes value in $(0, \infty)$
for $0 <$ t :

$$F_T(t) = P(T \leq t) = P\left(\frac{1}{R} < t\right) = P\left(\frac{1}{t} < R\right) = 1 - F_R\left(\frac{1}{t}\right) = e^{-2/t}$$

We then differentiate to get

$$f_T(t) = \frac{d}{dt}\left(e^{-2/t}\right) = \frac{2}{t^2}e^{-2/t}$$

3. (Bus waiting time) You show up at a bus stop at a random time. There are three buses running on a periodic schedule. They all go to the same destination (which is where you want to go). Each of the three buses arrives at your bus stop once every 10 minutes, but at different offsets from each other. The offsets of the three buses are uniformly random and independent. You will get on the first bus that arrives at your bus stop. Let the random variable T denote the number of minutes you have to wait until the first bus arrives. Compute the pdf and the cdf for T.

**Solution:**

$$f_i(x) = \frac{1}{10}$$

$$F_i(x) = \int_0^x \frac{1}{10} du = \frac{x}{10}$$

For t $\in [0, 10]$, the cdf is:

$$\begin{aligned} G(t) = P(T \leq t) &= 1 - P(T > t) \\ &= 1 - P(\min(X_1, X_2, X_3) > t) \\ &= 1 - P((X_1 > t) \cap (X_2 > t) \cap (X_3 > t)) \\ &= 1 - P(X_1 > t) * P(X_2 > t) * P(X_3 > t) \\ &= 1 - \left(1 - \frac{t}{10}\right)^3 \end{aligned}$$

The pdf is the derivative of cdf:

$$g(t) = \frac{dG(t)}{dt} = \frac{3}{10}\left(1 - \frac{t}{10}\right)^2$$

4. (Empirical CDF, Kernel Density Estimation) Notebook