Recitation 1

1. (Conditioning) A sack contains 12 coins: 4 have heads on both sides, 4 have tails on both sides, and 4 are standard. A coin is randomly drawn from the sack, and flipped. You are shown the result of the flip, and it is a head. What is the probability the coin was double-headed?

   Solution: Let A denote the event of getting a double-headed coin, and let B denote the event of a head being shown. Then we have $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{3}$. Thus the answer is

$$P(A \mid B) = \frac{1/3}{1/2} = \frac{2}{3}$$

2. (Conditioning and Independence) Suppose a can has 3 coins with head-probabilities $1/3, 1/2, 2/3$, respectively. We randomly pick out one coin, and flip it three times. Let H_i be the event the i-th flip is heads.

   (a) Are the H_i independent?
   Solution: No.

$$P(H_i) = \frac{1}{3^2} + \frac{1}{3 \cdot 2} + \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{2}, \forall i;$$
$$P(H_1 H_2) = \frac{1}{3} \cdot \frac{1}{3^2} + \frac{1}{3} \cdot \frac{1}{2^2} + \frac{1}{3} \cdot \frac{2^2}{3^2} = \frac{29}{108}$$

   (b) Are the H_i independent if we condition on which coin we chose?
   Solution: Yes,

$$P(H_1 H_2 \mid C_1) = \frac{1}{3^2} = P(H_1 \mid C_1) P(H_2 \mid C_1)$$
$$P(H_1 H_2 \mid C_2) = \frac{1}{2^2} = P(H_1 \mid C_2) P(H_2 \mid C_2)$$
$$P(H_1 H_2 \mid C_3) = \frac{2^2}{3^2} = P(H_1 \mid C_3) P(H_2 \mid C_3)$$

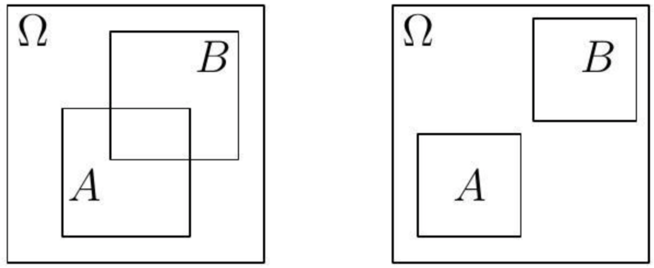
   (c) What is the probability of getting 3 heads?
   Solution:

$$\begin{aligned} P(H_1 H_2 H_3) &= P(H_1 H_2 H_3 \mid C_1) P(C_1) + P(H_1 H_2 H_3 \mid C_2) P(C_2) + P(H_1 H_2 H_3 \mid C_3) P(C_3) \\ &= \frac{1}{3}\left(\frac{1}{3^3} + \frac{1}{2^3} + \frac{2^3}{3^3}\right) = \frac{11}{72} \end{aligned}$$

   (d) Given that we get 3 heads, what is the probability we chose the coin with $2/3$ probability of getting heads?
   Solution:

$$P(C_3 \mid H_1 H_2 H_3) = \frac{P(H_1 H_2 H_3 \mid C_3) P(C_3)}{P(H_1 H_2 H_3)} = \frac{\frac{1}{3}\left(\frac{2^3}{3^3}\right)}{\frac{1}{3}\left(\frac{1}{3^3} + \frac{1}{2^3} + \frac{2^3}{3^3}\right)} = \frac{\frac{2^3}{3^3}}{\frac{1}{3^3} + \frac{1}{2^3} + \frac{2^3}{3^3}} = \frac{64}{99}$$

3. (Independence)



(a) Assuming the area of a set is roughly proportional to its probability in the following diagrams, which picture most closely depicts independent events A, B ?

Solution: The left one.

(b) A box contains 20 red balls, 30 white balls, and 50 blue balls. Suppose that 10 balls are selected at random one at a time, with replacement; that is, each selected ball is replaced in the box before the next selection is made. Determine the probability that at least one color will be missing from the 10 selected balls.

Solution:

$$\begin{aligned}
&P(R \cup W \cup B) \\
&= P(R) + P(W) + P(B) - P(R \cap W) - P(R \cap B) - P(W \cap B) + P(R \cap W \cap B) \\
&= (0.8)^{10} + (0.7)^{10} + (0.5)^{10} - (0.5)^{10} - (0.3)^{10} - (0.2)^{10} + 0 \\
&= (0.8)^{10} + (0.7)^{10} - (0.2)^{10} - (0.3)^{10} \\
&= 0.135616.
\end{aligned}$$

4. (Monte Carlo) Basketball game simulation notebook.