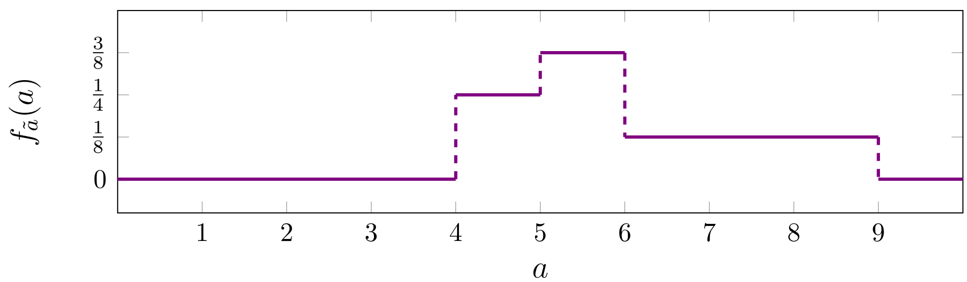
Homework 3

Solutions

1. (Fish)

   (a) The conditional probability is one, because there is a single data point larger than 6 and it is also larger than 7.

   (b) KDE yields the following estimate for the pdf:



   (c) We have

$$\begin{aligned}
\mathrm{P}(\tilde{a} > 7 \,|\, \tilde{a} > 6) &= \frac{\mathrm{P}(\tilde{a} > 7, \tilde{a} > 6)}{\mathrm{P}(\tilde{a} > 6)} && (1) \\
&= \frac{\mathrm{P}(\tilde{a} > 7)}{\mathrm{P}(\tilde{a} > 6)} && (2) \\
&= \frac{\int_7^9 \frac{1}{8}\,\mathrm{d}a}{\int_6^9 \frac{1}{8}\,\mathrm{d}a} && (3) \\
&= \frac{2}{3}. && (4)
\end{aligned}$$

2. (Nuclear power plant)

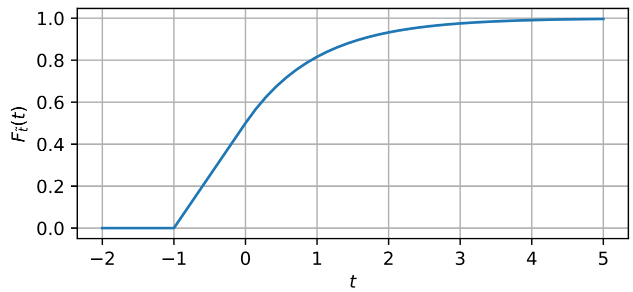
   (a) The pdf should integrate to one. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_{\tilde{t}}(t)\ \mathrm{d}t &= \int_{-1}^{0} \alpha\,\mathrm{d}t + \int_{0}^{\infty} \alpha \exp(t)\ \mathrm{d}t && (5) \\
&= \alpha(0 - (-1)) + \alpha(\exp(0) - \exp -\infty) && (6) \\
&= 2\alpha, && (7)
\end{aligned}$$

   so $\alpha = 1/2$.

   (b) We have

$$F_{\tilde{t}}(t) = \begin{cases} 0 & \text{if } t < -1 \\ \int_{-1}^{t} \frac{1}{2}\mathrm{d}a = \frac{1}{2}(t+1) & \text{if } t \in [-1, 0) \\ \frac{1}{2} + \int_0^t \frac{1}{2} e^{-a}\mathrm{d}a = \frac{1}{2}(2 - e^{-t}) & \text{if } t \geq 0. \end{cases} \qquad (8)$$



(c) We compute the cdf of $\tilde{t}$ conditioned on the event $\left\{\tilde{t} < 0\right\}$:

$$F_{\tilde{t}\,|\,\tilde{t}<0}(t) := \mathrm{P}(\tilde{t} \leq t \,|\, \tilde{t} < 0) \tag{9}$$

$$= \frac{\mathrm{P}(\tilde{t} \leq \min\{0, t\})}{\mathrm{P}(\tilde{t} < 0)} \tag{10}$$

$$= \begin{cases} 0 & \text{if } t < -1, \\ \frac{\int_{a=-1}^{t} f_{\tilde{t}}(a)\,\mathrm{d}a}{\int_{a=-1}^{0} f_{\tilde{t}}(a)\,\mathrm{d}a} & \text{if } -1 \leq t < 0, \\ 1 & \text{if } t \geq 0. \end{cases} \tag{11}$$

For $-1 \leq t < 0$, we have

$$\int_{a=-1}^{t} f_{\tilde{t}}(a)\,\mathrm{d}a = \frac{t+1}{2}, \tag{12}$$

so

$$F_{\tilde{t}\,|\,\tilde{t}<0}(t) = \begin{cases} 0 & \text{if } t < -1, \\ 1+t & \text{if } -1 \leq t < 0, \\ 1 & \text{if } t \geq 0. \end{cases} \tag{13}$$

Differentiating, we obtain

$$f_{\tilde{t}\,|\,\tilde{t}<0}(t) = \begin{cases} 0 & \text{if } t < -1, \\ 1 & \text{if } -1 \leq t < 0, \\ 0 & \text{if } t \geq 0. \end{cases} \tag{14}$$

3. (Measurements)

(a) Let $\tilde{d}$ be the time the particle takes to decay. The pmf of the reading $\tilde{r} = \lceil \tilde{d} \rceil$ is a geometric of parameter $1 - e^{-\lambda}$,

$$\mathrm{P}\left(\tilde{r} = r\right) = \mathrm{P}\left(r - 1 \leq \tilde{d} < r\right) = \int_{r-1}^{r} \lambda\, e^{-\lambda x} \mathrm{d}x = e^{-\lambda(r-1)} - e^{-\lambda r} \tag{15}$$

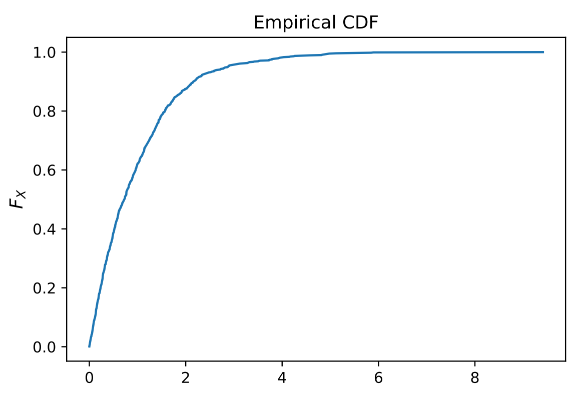
$$= \left(e^{-\lambda}\right)^{r-1}\left(1 - e^{-\lambda}\right) \quad \text{for } r = 1, 2, 3, \ldots \tag{16}$$

(b) Let $\tilde{x}$ be the error, clearly $0 \leq \tilde{x} \leq 1$. Its cdf is

$$\begin{aligned}
F_{\tilde{x}}(x) &= \mathrm{P}(\tilde{x} \leq x) && (17)\\
&= \mathrm{P}\left(\lceil \tilde{d} \rceil - \tilde{d} \leq x\right) && (18)\\
&= \mathrm{P}\left(\cup_{i=1}^{\infty} \left\{i - x \leq \tilde{d} \leq i\right\}\right) \qquad \text{union of disjoint events} && (19)\\
&= \sum_{i=1}^{\infty} \mathrm{P}\left(i - x \leq \tilde{d} \leq i\right) && (20)\\
&= \sum_{i=1}^{\infty} \lambda \int_{i-x}^{i} e^{-\lambda x} \mathrm{d}x && (21)\\
&= \sum_{i=1}^{\infty} e^{-\lambda(i-x)} - e^{-\lambda i} && (22)\\
&= \left(e^{\lambda x} - 1\right) \sum_{i=1}^{\infty} e^{-\lambda i} && (23)\\
&= \frac{e^{-\lambda}\left(e^{\lambda x} - 1\right)}{1 - e^{-\lambda}} = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1}. && (24)
\end{aligned}$$

Differentiating we obtain

$$f_{\tilde{x}}(x) = \begin{cases} \frac{\lambda e^{\lambda x}}{e^{\lambda} - 1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$



4. (a)

(b) Suppose we assume $x_1, x_2, \cdots, x_n$ are sorted, then $y_i = \frac{1}{n}$. It is independent from the data.