Homework 0

Solutions

1. (Sets)

   (a) True. For $i \neq j$ we have $(S_i \cap A) \cap (S_j \cap A) = \emptyset$ because if it is not empty, then there exists an element that is both in S_i and S_j, and this would contradict $S_i \cap S_j = \emptyset$. We still need to establish that $\bigcup_i (S_i \cap A) = A$. To prove this we show that the sets contain each other. $\bigcup_i (S_i \cap A) \subseteq A$ because every element in $S_i \cap A$ is in A. Now assume that there is an element in A that is not in $\bigcup_i (S_i \cap A)$. Then the element would not belong to any S_i, but this would imply that it does not belong to Ω, which is impossible because $A \subseteq \Omega$. This implies $A \subseteq \bigcup_i (S_i \cap A)$ and concludes the proof.
   Alternatively, we can apply the rules of set algebra. For $i \neq j$, we have

   $$(S_i \cap A) \cap (S_j \cap A) = (S_i \cap S_j) \cap A = \emptyset \cap A = \emptyset$$

   and

   $$\bigcup_i (S_i \cap A) = \left(\bigcup_i S_i \right) \cap A = \Omega \cap A = A.$$

   (b) False. Let the universe $\Omega := \{1, 2\}$, $A := \{1\}$, and $B := \{2\}$. Then $A^c \cup B^c = \{2\} \cup \{1\} = \Omega$, but $(A \cup B)^c = \Omega^c = \emptyset$.

   (c) False. Let A and B be nonempty, $A = B$, and $A \cap C = \emptyset$. Then $(A \cup B) \cap C = A \cap C = \emptyset$ but $A \cup (B \cap C) = A \cup \emptyset = A$.

2. (Series)

   (a)

   $$\sum_{i=m}^{n} r^i = \frac{1}{1-r} \sum_{i=m}^{n} \left(r^i - r^{i+1} \right) \tag{1}$$

   $$= \frac{1}{1-r} \left(\sum_{i=m}^{n} r^i - \sum_{j=m+1}^{n+1} r^j \right) \tag{2}$$

   $$= \frac{r^m - r^{n+1}}{1-r}. \tag{3}$$

   (b) We need r^{n+1} to remain bounded, so $|r| < 1$.

   (c) *Base step:*
   For $n = 2$

   $$\sum_{i=1}^{n} i = 3 = \frac{n(n+1)}{2}. \tag{4}$$

Induction step:
Assume, for any fixed $n \geq 2$, that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{5}$$

Then

$$\begin{align}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \tag{6}\\
&= \frac{n(n+1)}{2} + (n+1) \tag{7}\\
&= \frac{n(n+1) + 2(n+1)}{2} \tag{8}\\
&= \frac{(n+2)(n+1)}{2}. \tag{9}
\end{align}$$

3. (Derivatives)

   (a) The change in the function over an interval of length h is given by $f(x+h) - f(x)$. Dividing by the length of the interval yields the rate of change. Taking the limit as $h \to 0$ makes it instantaneous.

   (b)

$$\begin{align}
f'(x) &:= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{10}\\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \tag{11}\\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \tag{12}\\
&= \lim_{h \to 0} 2x + h \tag{13}\\
&= 2x. \tag{14}
\end{align}$$

   (c) To ensure the derivative is the same we need $a := f'(y)$, since

$$L_y'(y) := a. \tag{15}$$

   To ensure that the value is the same, we need $b := f(y) - f'(y)y$ because

$$\begin{align}
L_y(y) &:= f'(y)y + b \tag{16}\\
&= f(y). \tag{17}
\end{align}$$

   This yields

$$L_y(x) := f'(y)(x - y) + f(y). \tag{18}$$

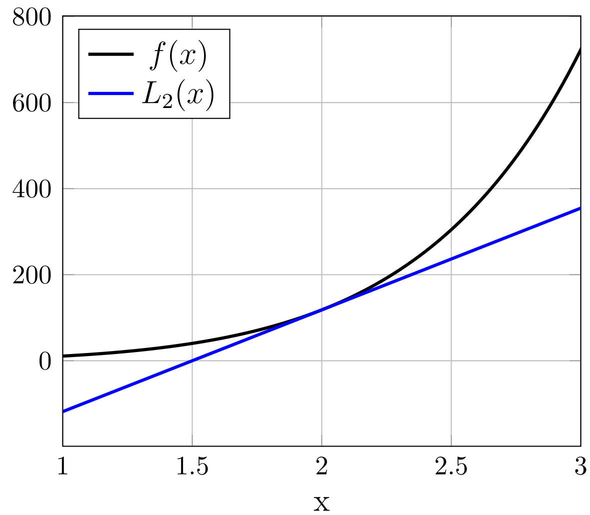
(d) By the product rule, we have

$$f'(x) = (4x^2 + 8x)e^x, \tag{19}$$

so

$$\begin{aligned} L_2(x) &= 32e^2(x-2) + 16e^2 && (20)\\ &= 16e^2(2x-3). && (21) \end{aligned}$$

Here is the plot



4. (Integrals) We will use integrals to compute probabilities and expectations related to continuous quantities.

(a) The quadratic on the upper right quadrant must equal $(x-1)^2$ to go through (0,1), (1/2,1/4), and (1,0). By symmetry, the area below this curve and above the horizontal axis between 0 and 1 is equal to one fourth of the area of the shape. As a result, we have that the area equals,

$$\begin{aligned} 4\int_0^1 (x-1)^2 \, \mathrm{d}x &= 4\int_0^1 x^2 - 2x + 1 \, \mathrm{d}x && (22)\\ &= 4\left(\frac{1}{3} - 1 + 1\right) && (23)\\ &= \frac{4}{3}. && (24) \end{aligned}$$

(b) Let $u = x^2$, then

$$\begin{aligned} f(t) &:= \int_0^t \frac{x}{1+x^2} \, \mathrm{d}x && (25)\\ &= \frac{1}{2}\int_0^{t^2} \frac{1}{1+u} \, \mathrm{d}u && (26)\\ &= \frac{\ln(1+t^2)}{2}. && (27) \end{aligned}$$