**Abstract**

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**Recommender Systems**

Nowadays, Recommender Systems are widely recognized as one of the most beneficial applications of Machine Learning, and there is no doubt that they drive nearly every aspect of our lives. The fundamental objective of these Machine Learning-driven Recommenders is to filter, prioritize and efficiently deliver the necessary information to the consumers in the midst of overwhelmingly numerous choices on the internet. It is also described as:

*“Any system that produces individualized recommendations as output or has the effect of guiding the user in a personalized way to interesting or useful objects in a large space of possible options.”* (Burke, 2002)

Therefore, many companies utilize the recommender systems for the purpose of helping the consumers discover new and relevant items such as movies, musics, jobs, etc. They use the consumer data which may be in explicit or implicit form (e.g. likes, clicks), in order to comprehensively assess consumers’ preferences and then recommend the relevant items to them. Due to the various criterias, there are multiple techniques of recommender systems, each of which differs in how a single recommendation is generated. The most common types of recommender systems are described in the following section.

**Types of recommender systems**

Although there exists a number of different recommender systems in the literature, we will focus on the three most common ones below:

Diagram

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* **Collaborative Filtering recommender systems**

CF recommender systems are one of the most widely used systems next to the content-based recommender systems. Essentially, these systems create a user profile based on the ratings of various items and then aims to compare these against a wider user group[12]. As the word “collaborative” from the name implies, multiple users come together as group – taste of one user will be similar to the other users of group. Therefore, by utilizing the users data which contains their historical preferences on a set of items, system deploys an assumption that the users who have previously agreed are more likely to agree again in the future. So, the system creates the new recommendations by taking the similarities between users based on the ratings into consideration.

Diagram

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Although, CF recommender systems have been used in the industry for many years [13], they still have a limitation such that they can not address the cold start problem –they are not able to recommend items which are not rated by any users (e.g, new items). As a result, it is possible that only famous items may get recommended. Furthermore, traditional CF systems are also memory-wise and computationally expensive and suffers from scalability problems.

* **Content Based recommender systems**

While CF recommender systems, as discussed above, recommend the products or items according to the similarities of user preferences which means that recommendation relies on the user-item interactions, Content Based recommender systems, on the other hand, aims to recommend products or items similar to those a given user has rated positive or liked in the past. So, CB systems generate recommendations based on the comparison between the content of the items and the user profile which was created according to the historical user data[[FIGUREBELOW]]. Note that the content of items is described by terms, tags, features or even plots in case if the items are movies.

An algorithm used to recommend the movies on the Netflix platform is the prominent example which resembles this recommender systems. If a certain user watches and comedy movie and rates it positive via votes or comments, then the new movie recommendations with the same label of that liked movie will be suggested to the user. In other words, based on the content of the consumed item, these recommender systems finds other similar items and recommends them. Note that the such website platforms often keeps the techniques of how the content is actually labeled and matched against each other as secret [12]. Contrary to the CF systems, CB system doesn’t suffer cold-start problem and they can suggest not only famous or older items, but also the unpopular or new items. In addition to this, they are memory-wise and computationally cheap because there is no need for the data of other users in order to be able to compute the recommendation for a specific user.

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* **Hybrid recommender systems**

Hybrid recommender systems combine two or more types of traditional recommender systems in order to have better performance by benefiting from complementary advantages of subsystems. Hybrid systems which combine collaborative filtering and content based approaches, achieves state-of-the-art results in many cases and are used in many large scale recommender systems nowadays. Detailed comparison on advantages and disadvantages of Hybrid Recommenders along with Collaborative Filtering and Content Based Recommenders are shown in table below**[[[Insert table]:**

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Collaborative Filtering** | **Content Based** | **Hybrid** |
| **Number of users** | * Recommendation based on many users having similar interest | * Recommendation based on single user | * Combination of collaborative and content based filtering |
| **Disadvantages** | * Cold start problem * Data sparsity * Scalability * Memory-wise and computationally expensive | * Limited content analysis * Over-specialization | * Increased complexity * Increased expense of implementation |
| **Advantages** | * Serendipitous recommendation * User and item features are not required * Quality may improve over time as more users interact with items * - Minimal domain knowledge required | * User independent * No cold start problem * Interpretable results * Memory-wise and computationally cheap | * Avoids most of the shortcomings of other approaches. |

**Probabilistic Models for Recommender Systems**

The application of probabilistic modeling to the recommendation problem has a rich history which dates back to decades (Breese et al. 1998, Hofmann 2004, Marlin 2004). Many authors incorporated the probabilistic approaches into models which explained the dataset. Initial approaches were probabilistic graphical models such as Bayesian networks and Dependency networks which eventually left their place with subsequent novel topic models such as Latent Dirichles Allocation (LDA) [3] and Probabilistic Latent Semantic Analysis (pLSA) [22]. The term “latent” is used in their name, because both of them are considered probabilistic topic models and the topics they aim to find from the corpus are treated as latent or hidden variables. Detailed explanation of LDA has been discussed **in its own section on later pages**. Note that, asboth models can suggest items which have similar content to other items that a user likes, they have been extensively used for Content Based recommender systems.Furhtermore**,** when it comes to the field of Collaborative Filtering recommender systems, the matrix factorization technique had gained a decent popularity, especially after combined with probabilistic approach [18], [19], [20], [21].

Lately, there has been a lot of interest in combining probabilistic topic modelling with matrix factorization in the field of hybrid recommender systems. One of the major for this is that when a content of item is represented by topic models, the models benefit from interpretable semantics of the latent space characterized by the topic mixtures and this leads to more an interpretable semantics of the item latent factor[\*\*CTMP\*\*]. Initially, Agarwal and Chen proposed probabilistic topic modelling in matrix factorization with fLDA [23]. where the item latent factor took the role of topic proportion in the LDA representation. Despite of being an accurate and interpretable model which handles both cold-start and warm-start scenarios, fLDA still had a limitation in dealing with distinguish items where we have an identical topic mixture, but content details that topic mixture can not cover are of concern to different groups of people. To elaborate on this limitation more, consider that we have two articles; A and B, and both of the articles are about an application of machine learning to social networks. Because both articles are identical in terms of their contents, they will also possesses same topic proportions. Now let’s consider that these two articles are of interest to different kind of users: Article A provides a prominent machine learning algorithm which is applied to social network applications, wheares article B implements a standard machine learning algorithm, but provides a crucial data analysis on social network data. As a result, users who work in machine learning will prefer article A and will hardly be interested in article B, wheares users who work in social networks will be more interested in article B instead of A. However, as the topic proportions of both articles are same, both of them will be recommended to both groups of users [24].

To tackle the limitation mentioned above, a novel approach called Collaborative Topic Regression (CTR) [24] has been proposed by David M. Blei and Chong Wang. The way CTR addresses that limitation is by allowing the item latent factor be an offset from topic proportion. So, by this way, an offset may help explain, for instance, an article A is more important to researchers interested in machine learning than it is to those interested in social network analysis. Therefore, CTR allows the item latent factor to also account for user ratings.

Fundamentally, CTR incorporates techniques of both collaborative filtering based on latent factor models and content analysis based on probabilistic topic modelling. According to CTR model, items are generated by a topic model while users are represented with topic interests [24]. Therefore, CTR is considered as one of the excellent hybrid models which shows that the combination of the content modelling with the matrix factorization methods produces more promising results compared to traditional recommender systems. The graphical model of CTR along with its algorithm is shown below.

**CTR Graphical Model**

Diagram

Description automatically generated

**CTR Algorithm**

1. For each user *u*,draw user latent vector
2. For each item *j*,
   1. Draw topic proportions
   2. Draw item latent offset and set the item latent vector as .
   3. For the *n*-th word of item *j*,
      1. Draw topic index
      2. Draw word
3. For each user-item pair (*u, j*), draw the rating

where is the confidence parameter for . For instance, we trust more if is large.

Despite its advantages, CTR model has significant computational limitations as well. The reason is that the model considers user ratings to have a Gaussian distribution which leads to iterating over all of the entries in rating matrix during training. Because of this, CTR is highly inefficient considering that real-world datasets are very big and sparse.

In order to address CTR’s inefficiency mentioned above, a newer hybrid model called Collaborative Topic Poisson Factorization (CTPF) has been proposed [25]. Fundamentally, CTPF model makes an assumption such that both ratings and items of dataset have a Poisson distribution. By this way, CTPF is only concerned with non-zero ratings during training, and therefore it is much more efficient and scalable. Furthermore, for the purpose of making sure that the model is conditionally conjugate and has closed-form updates, CTPF tries to model the content generation by standard mixtures of Gamma. The graphical model of CTPF along with its algorithm is demonstrated below.

**CTPF Graphical Model**

Diagram

Description automatically generated

**CTPF Algorithm**

1. **Item model:**
   1. Draw topics
   2. Draw item topic intensities
   3. Draw word count
2. **Recommendation model:**
   1. Draw user preferences
   2. Draw item topic offsets
   3. Draw

All the hybrid models mentioned so far (i.e. fLDA, CTR and CTPF) benefit from the interpretable semantics of the item latent factor. However, they still have some limits in terms of computational cost or predictive performance. Therefore, in this thesis, we will explore and implement recent hybrid model called ***Collaborative Topic Model for Poisson distributed ratings (CTMP)*** model which covers the limitation of CTR by considering ratings in Poisson distribution as CTPF does, while modelling contents with LDA [\*\*CTMP\*\*]. Details of CTMP formalization, graphical model and algorithm is shown **in its own section on later pages**. However, it is worthwhile mentioning that CTMP makes the following contributions to the previously mentioned approaches [\*\*CTMP\*\*]:

1. CMTP has been tested in variety of fields where real-world recommendation is the most challenging one. It has been seen that CTMP outperforms the other existing models significantly and its main competency is in recommending scientific articles and commercial product recommendation. Recommending movies are also amongst these, and indeed, this is what we will test on this thesis.
2. CTMP implements fast and scalable coordinate ascent algorithm because it is non-conjugate model. An implemented algorithm is fast and scalable.
3. According to the empirical studies we conducted on different real-world datasets in this thesis, we observe that we can achieve the sparse estimates of topic mixtures via learning in spite of the fact that the model specification does not encourage so. Note that the sparsity is very critical property as it leads to an efficient storage of a data by offering compact content representation.

**Variational Inference**

Variational Bayesian Methods (i.e. Variational Inference) are a group of widely used techniques in a field of statistical Machine Learning. The reason behind their popularity is that VB methods let us reformulate statistical inference problems as optimisation problems. Here, statistical inference problems are the key algorithmic problems where we infer the value of a random variable given the value of another random variable, wheares optimization problems imply to find the values of parameters which minimize the given objective function [\*\*16\*\*].

The most common VB method is the **Mean-Field Approximation** and in the following section we will explain its optimization objective which is known as **Variational Lower Bound or Evidence Lower Bound (ELBO).**

In this paper, we only used Mean-Field variational method for both CTMP and LDA

Put specifications about mean-field vm: q(asdsa|asdas)….

Put somewhere [In a typical application, mean-field methods approximately compute the posterior with a coordinate-ascent optimization algorithm.] [39]

For mean-field !! Define the generic class of conditionally conjugate models [Blei-PPT], choose exponential family

Suppose the following probabilistic model which is a joint distribution of the observed variables ***X*** and the hidden variables ***Z***:

Inference of the hidden variable *Z* is obtained through its posterior distribution which is written as follows using the *Bayes’ Theorem*:

**KL divergence method**

For many interesting models, the posterior is intractable which means that the exact inference is not possible. Therefore, by considering the Mean-Field Approximation, we adopt the approximate posterior inference technique for these models.

The key idea behind it is to find an approximation distribution that are as **closed** as possible to the true posterior distribution . Note that are variational parameters of approximation distribution and we fit the variational parameters in order to make these both distributions closest [\*\*ShortShot\*\*].

TODO – Later : Diagram, schematic

Description automatically generated

TODO – Later: Mean-field approx - <https://www.cs.cmu.edu/~epxing/Class/10708-17/notes-17/10708-scribe-lecture13.pdf>

First, we define this closeness by using Kullback–Leibler divergence (i.e. KL divergence) between these approximate and true distributions [\*\*15\*\*]:

where *L* above is called **the variational lower bound or ELBO**. We reformulate the equation above as follow:

Because the *KL* divergence is always positive (i.e., we get . This proves that *L* is the lower bound of the log probability of observations.

So, our main goal is to **maximize this lower bound *L* i.e. minimizing *KL* divergence with respect to variational parameters** Because *L* reaches to *iff* the approximation distribution is perfectly closed to the true posterior distribution [\*\*ShortShot\*\*]. Note that in the formula above is fixed against all variational parameter .

Diagram

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[\*\*Blei-PPT\*\*]

**Jensen’s inequality method**

Apart from the derivation mentioned above, we also have an alternative way to arrive at the similar conclusions using the Jensen’s inequality which claims for the *concave log function*:

So, the last term in equation above is the **variational lower bound or ELBO**. Note that in the equation belongs to the Shannon entropy:

where . Essentially, we again show that *L* is the lower bound of the log probability of the observations, and our goal is to maximize this lower bound if we want to maximize the marginal probability.

**LDA**

In machine learning, topic modeling is a statistical model for discovering a set of topics that occur in a collection of documents [1]. It is also considered as a probabilistic model which offers an interpretable low-dimensional representation of the documents. For many years, implementation of topic models for the purpose of document classification, corpus exploration and information retrieval has been of interest.

There are many topic modeling algorithms, among which Latent Dirichlet allocation (LDA) is the most popular one. LDA is a three-level hierarchical Bayesian model, and its basic idea is that documents are represented as random mixtures over underlying set of topics, where each topic is characterized by a distribution over words which are biased around those associated under a single theme [\*\*original-lda-paper\*\*]. Therefore, topic probabilities express an explicit representation of each document. This can also be be explained as:

* **Each document is a mixture of topics**.

We consider that each document contains terms/words from some topics in specific proportions. For instance, if we consider that there are 2 topics in the whole corpus, then we might state that some document could be 75% topic A, and 25% topic B, while another document might be consisted of 30% topic A, and 70% topic B.

* **Each topic is a mixture of words**.

We consider that each topic is expressed by the words that explain it most. For example, if we consider that there are 2 topics, namely, “sports” and “education”, in the whole corpus, then the most used words for the sports topic could be “teammate”, “win”, and “play”, while the education topic could contain the words such as “lecture“, “book” and “class”. It is necessary to note that the same words can appear on the multiple topics. For example, the word “time” could participate in both sports and education topics.

Illustrating the geometry of the latent space is a another good way for grasping the concept of LDA:

Diagram

Description automatically generated

By this way, documents can overlap with each other with relationship to their contents, rather than being seperated into different individual groups. Generative process and graphical model of LDA for each document in the whole corpus is described below.

**LDA Terminology**

* A *word* is a term of the vocabulary and it is indexed by .
* A *document* is a series of words given by where is the *n*th word inside the document.
* A *corpus* is a collection of a total *M* documents and it is given by

**LDA Algorithm**

1. Draw topic proportions .
2. For the *n*-th word of document *j*,
   1. Draw topic index
   2. Draw word

**LDA Graphical Model**

Diagram

Description automatically generated

In the algorithm above, is the topic proportion of particular document, while is the topic representation of particular topic and is the Dirichlet prior parameter. As seen above, the topics algorithm tries to find from the whole corpus are treated as latent or hidden variables. Additionally, each document of the corpus is represented in terms of topic proportions or latent themes which are also hidden variables.

Topic proportion is a *k*-dimensional Dirichlet random variable and its domain is in the (*k* − 1)-simplex. In other words, *k*-vector is in the (*k* − 1)-simplex, therefore, . The Dirichlet is a exponential family distribution on the simplex. One of its important properties is that it is conjugate to the multinomial distribution. [4]. Note that, all of these aspects help the development of the inference and parameter estimation for LDA. Additionally, the probability density of ’s simplex is as following:

where is a *k*-vector with components and is the Gamma function:

In the graphical model above, the outer box illustrates the documents, and the inner box illustrates repeated choice of topics and words inside documents. The parameters that are considered as corpus-level areand , and they are supposed to be sampled during the process of generating the corpus [26]. The variables given as are considered as document-level, and they are sampled once per document. Lastly, the variable given as and are considered as word-level, and they are sampled once for each word inside every document. During the LDA process, topics are sampled repeatedly within each document.

Note that LDA possesses so called hidden generative process and according to this process, the model is assumed to generate the observed data (i.e. items, users or ratings). Obviously, this was just a generative assumption in order to facilitate the algorithm and it does not illustrate the true process of the real data [17].

So, the joint distribution of topic proportions topics set of topics *z*, and set of words *w* is as following:

Given a corpus of documents, we can use variational EM to learn the topics and decompose the documents according to them [26]. Further, given a new document, we can use variational inference to situate its content in terms of the topics.

**Inference and Parameter Estimation**

Computing the posterior distribution over the hidden variables given some document is the the main problem we have to overcome in order to be able to use LDA:

Unfortunately, the posterior distribution derived above is intractable for an exact inference. As conjugate prior to the multinomial distribution, LDA uses a Dirichlet prior to simplify posterior inference [\*paraphrase\*] so we can employ Variational inference to solve this posterior inference along with relevant Variational EM algorithm [\*paraphrase\*], Therefore, we can use Variational Inference and relevant Variational EM algorithm in order to learn the topics and then decompose each document of the corpus according to these learnt topics [26].

**Collaborative Topic Model for Poisson distributed ratings**

In this section, we describe learning, prediction phases and key properties of CTMP – hybrid, scalable and interpretable probabilistic content-based collaborative filtering model.

CTMP.1 - Formalization

Before diving into technical parts, let’s provide some notations:

* *U:* represents the number of users inside the dataset
* *J:* representsthe number of items inside the dasaset
* describes the bag-of-word representation for each item *j* where expresses the frequency of term/word in item *j.*
* represents the vocabulary size of the corpus.
* describes the dataset where is a rating provided by user *u* to item *j,* while is the bag-of-word representation of item *j* as already explained above. represents the ratings given to movies by users. Every rating is expressed as binary 0 or 1. If user *u* liked an item *j,* then . On the contrary, if the user *u* do not know about the item *j* or do not like it, then .
* *K:* represents the number of topics inside corpus.
* describes the topic representation. More precisely, every topic *k* is a distribution over the vocabulary. It is described as and . Note that, lies in the (*k* – 1)-simplex.
* describes the topic proportion of the items. is the vector of the distribution on topics for item *j,* and . Note that, lies in the (*k* – 1)-simplex.

In order to learn the topics , we use the Latent Dirichlet allocation (LDA) and its Expectation-Maximization (EM) approach which was described in the respective section of LDA. Furthermore, by learning the topic proportion of each item we describe each item and user in the *K*-dimensional space. Note that these learning procedures will be explained in the further sections below.

Now, we present **latent factors** for each user and item in terms of *K*-dimensional vectors and , respectively. As discussed in [[[Related Work Section]]], the reason why we consider rather than as the latent factor for item is that in order to have better recommendation system, we allowed an offset between and which accounts for the user-specific preference on the item content that alone can not capture. Therefore we denote that where is an offset term which has Gaussian distribution. Note that in the formula above represents an *K*-dimensional identity matrix , and is a regularization parameter. So, we have .

Furthermore, as shown below, the ratings and users’ latent factors are modeled by Poisson and Gamma distributions, respectively. To put everything together, the generative process and graphical model of CTMP is as follow:

**CTMP Algorithm**

1. For each user *u,* draw where
2. For each item *j*:
3. Draw topic proportion
4. For the *n-*th word of item *j*:
   1. Draw topic index
   2. Draw word
5. Draw latent factor
6. For each user-item pair (*u, j*), draw

Note that steps 2(a-b) corresponds to LDA.

**CTMP Graphical Model**

Diagram

Description automatically generated

**Learning CTMP**

Full posterior of latent variables is given as follow:

(1)

The problem with this posterior is that it is intractable, and therefore exact inference is impossible. In order to tackle this problem, we have two methods:

1. Maximum A Posteriori (MAP) for point estimation
2. Bayesian Learning such as MCMC Sampling or Variational Methods for approximate inference

As the prior and posterior distributions of hidden variables and are not conjugate in CTMP model, using Variational Methods of Bayesian Learning in order to infer these hidden variables does not get us closed-form solution. Therefore, we will carry out the point estimates of and using MAP – coordinate ascent algorithm developed by authors of original paper of CTMP [\*\*CTMP\*\*].

Furthermore, in order to facilitate the learning, authors added **a** **new auxiliary variable *y****,* where Poisson( and . Note that we approximate the posterior of and via mean-field variational inference [\*\*5\*\*]. The mean-field variational inference is a type of Variational Bayesian Method which allows to re-write a statistical inference problems as an optimization problem[[\*\*\*insertsomething\*\*]]. Therefore, we can convert the inference problem of CTMP into a full optimization problem where the single objective function which needs to be maximized is as follow:

(2)

As shown in (2), the term of integration and summation over the whole space causes optimization to be intractable. However, Variational method [\*\*5\*\*] also tackles this problem which will be discussed in detail below.

Note that has Poission distribution, and the *K*-dimensional vector follows multinomial distribution: [\*\*6\*\*] So, we get the variational distribution as follow:

(3)

where such that is variational parameter of , and are variational parameters of . Note that . Now we get the **evidence lower bound (*l*)** by applying Jensen’s inequality[[\*\*insertpaperofJensen\*\*]]:

(4)

Note that before learning the hidden parameters, are considered as fixed parameters in the model.

Next, the lower bound *l*( shp, rte, ) is maximized with respect to shp, rte, . According to Appendix A, we express the terms in detail as follow:

(5)

**Learning Parameters**

Equation (5) is the optimization problem and as mentioned before we solve it by coordinate ascent algorithm. CTMP algorithm for learning and is demonstrated below in Algorithm 1

Algorithm 1.

**Input:** Observed data *w, r,*Bernoulli parameter

**Output:** Estimates

**1. init** Initialize

**Learning** . In order to find the point estimate of local topic proportion where

(6)

we use Bernoulli randomness for Online Maximum a Posteriori Estimation (BOPE) algorithm [\*\*7\*\*]. Note that in original paper of CTMP [\*\*CTMP\*\*], authors used a simple Online Maximum a Posteriori Estimation (OPE) algorithm and this difference is the most important one between this and original CTMP paper. By using Bernoulli randomness, BOPE has a faster convergence rate, is more general and flexible compared to OPE. Furthermore, “BOPE implicitely employs a prior which plays a regularization role”[\*\*7\*\*]. Comparison of BOPE with respect to OPE has been carried out in BOPE section in detail. Include this in comparison::: such properties are not found in the common approximate posterior inference methods for topic models, such as Gibbs sampling and variational Bayes. [[[or discuss here???]]] Note that both algorithms tries to lead the solution of the optimization to the closed neighbours of the vertices in the convex hull of compact input domain and they have a fast convergence rate of along with proven quality bound [\*\*8\*\*]. Furthermore, as mentioned earlier too, every topic proportion holds and lies in the (*k* – 1)-simplex. BOPE algorithm for learning is described in Algorithm 2.

Algorithm 2

**Input:** Bernoulli parameter

**Output:** which maximizes over the compact domain }

**1. init** Initialize

**Learning** If we know the estimates of other hidden variables, then solving analytically is possible because the objective function regarding the is a *concave* function.

(7)

The partial derivative of function with respect to , i.e. for all *k,* is the estimate of . This is also so called the stationary point of . Because is the quadratic function in terms of we can Vieta’s formula for the analytical derivation of the function’s root as follow:

(8)

where

(9)

**Learning** . We use the mean-field variational inference for approximating the conditional posterior of and as in [\*\*9\*\*]. So, in order to solve for the variational parameters of and which are , we solve for the stationary point of with respect to each variational parameter, while holding the others same. The update expression of variational parameters is given in Table/Figure?? [\*\*x\*\*] below. The detailed derivation of these expressions is described in Appedix B and Appendix C. One of the biggest advantages of CTMP algorithm is that whenever , we get and and therefore, we only have to update over non-zero ratings (. This property of our model diminishes the training time significantly, so the total training time is much lower than of other models such as CTR, especially whenever the rating dataset is highly sparse. Because, during each epoch of training, we only consider the positive ratings for updating the expression of and skip all *zero* ratings.

Variational parameter updates

Note that the function in the update expression of denotes the digamma function [\*\*10\*\*]:

where and it denotes the gamma function.

(10)

**Learning** . So far, we have provided the update expression of the variables regarding both documents and users such as . Now, we have to do the remaining task which is to solve for First, we express the log likelihood of the whole items corpus *C* as in [\*\*10\*\*]:

(11)

By using Jensen’s inequality, the last term is derived, due to the fact that Next, the lower bound of is maximized with respect to as in [\*\*10\*\*]:

(12)

where Note that the each is separable from each other inside the objective function of . So, we can solve solve each individually. This is carried out by considering the Lagrange function and setting its derivatives to 0 which results in formula of as follow:

(13)

**Prediction**

We rank the items in order to generate recommendations for each user *u* based on their predictive score after we have learned all the parameters. Because the ratings in the dataset are discrete Poisson variables, can be the expectation of the rate parameter given the observed data i.e. as in CTPF[\*\*CTPF\*\*]. However, the derivation in CTMP is a bit different because CTMP neither aims to approximate solely by point estimate nor require **conjugacy** between the complete **conditional distributions** for the inference as CTPF does [\*\*CTMP\*\*] :

Note that only is the MAP estimation of complete conditional distribution. Furthermore, is nearly the expectation over the respective variational distributions of ’s:

Note that both and are the estimation of variational parameters that we learned in above section.

**CTMP key properties**

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Add 3rd contribution of CTMP in Related work

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**Interpretable user profiles**

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**Evaluation**

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Two most common tasks in recommender systems are predicting the score the user might give for a product (the rating prediction task), and recommending a ranked list of most relevant items (the top-N recommendation task)

[[Take definitions of “in-matrix” and “out-of-matrix’ from CTR paper]]

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**BOPE vs OPE**

Maximum a posteriori probability (MAP) estimation has a significant impact on doing posterior inference (i.e. estimating hidden parameters) in many probabilistic models. Especially, many interesting MAP problems are continuous, non-convex and intractable. In the field of non-convex optimization, there have been a variety of different techniques such as Frank–Wolfe [29], Natasha2 [30], Stochastic Majorization-Minimization [31], Concave-Convex procedure [32] which aim to solve the MAP problem [27]. However, non-convex optimization is NP-hard, and techniques mentioned above may not provide viable solution for MAP problem, because they disregard its special underlying structure. Therefore, for solving non-convex MAP problems with state-of-the-art convergence rate, we will explore two efficient algorithms **Online Maximum a Posteriori Estimation (OPE)** [27] and its regularized, general and more flexible version **Bernoulli randomness in Online maximum a Posteriori Estimation (BOPE)** [28]. First, we introduce MAP estimation as following task:

(1)

where we denote as hidden variable, D as the observed data and denotes domain. Note that there also have been proposed many algorithms which directly tries to estimate a full posterior distribution mentioned above, i.e., Variational Bayesian Methods (VBM) [33], Collapsed Gibbs Sampling (CGS) [34], Hessian Approximated Markov Chain Monte Carlo (HAMCMC) [35]. However, these methods provided suboptimal solutions along with slow convergence rate. Therefore, we continue by using *Bayes’ Theorem*:

(2)

where we denote as likelihood of *D,* as x’s prior, and as s marginal probability. Using (2), we rewrite (1) as following:

(3)

We will focus on the conditions where MAP problem is continuous and non-convex, hence intractable, i.e., is non-convex over the continuous compact domain [28]. As previously mentioned, MAP problem (3) will be treated as an optimization problem. Therefore, objective function defines the complexity of this optimization problem where and . So, our problem (3) becomes as a non-convex constrained optimization problem as follow:

(4)

So, in the following sections we will discuss OPE and BOPE algorithms for solving the optimization problem shown above.

**OPE for solving**

Online Maximum a Posteriori Estimation (OPE) is considered as a type of iterative optimization algorithm, which is the stochastic version of Frank–Wolfe algorithm. The biggest advantage of OPE is that it has provably faster convergence rate of to local maximal point compared to the existing stochastic algorithms for nonconvex problems, where signifies the number of iterations during training of its following algorithm [27]:

**OPE Algorithm**

**Output**: which maximizes the objective function over the compact domain .

Initialize arbitrary in .

1. **for** **do**
2. Pick uniformly from


6. **end for**

As illustrated above, the OPE algorithm solves a linear program at each iteration, i.e. directing the optimization solution to the good vertex in the convex hull of compact input domain. In more detail, what OPE does is to develop a sequence of stochastic functions that approximates to by alternatively selecting an from uniformly randomly at each iteration *t*. As proved in its original paper [27], converges to as .

Despite of fast convergence rate, OPE still has a limitation. As stated in algorithm, either likelihood or prior is being used while we are building an approximation function of However, when dealing with new samples, we can rely on likelihood if we have seen enough data, or rely on prior if there is a lack of data.

**BOPE for solving**

In order to overcome the OPE’s limitation mentioned above, new approximation technique to OPE has been proposed as BOPE which retains all theoretical guarantees of OPE’s convergence while being more general and flexible by using Bernoulli distribution and two stochastic bounds [28]. BOPE solves problem (4) by employing Bernoulli distribution with parameter which is supposed to replace the uniform distribution of OPE on likelihood and prior. Furthermore, two stochastic sequences are constructed and they converge to objective function : the lower sequence , the upper sequence . It is worth noting that the Bernoulli parameter determines an impact of likelihood and prior on and . So, during each iteration, using both and stochastic sequences provides further information about , so that we increase our chances of reaching more quickly [28]. Both lower and upper sequences are guaranteed to converge to as BOPE algorithm is described in detail below:

**BOPE Algorithm**

**Input:** Bernoulli parameter

**Output:** which maximizes over the compact domain .

**1. init** Initialize

It’s important to note that one of the reasons why BOPE outperforms OPE is that we can create variants of BOPE by altering the Bernoulli parameter . In addition to this, another property of BOPE is that in order to prevent overfitting of learning process which is the widespread issue that affects all machine learning techniques, BOPE employs implicit regularization. Specifically, according to original paper [28], Bernoulli randomness operates as a regularizer and BOPE uses an implicit prior that is stochastically vanishing with respect to iterations *T*, and this implicit prior is not same as the prior used in MAP estimation. This implicit regularization is very critical, especially in recommender systems where most of the datasets are considered as *sparse*. Therefore, using BOPE in Collaborative Topic Model for Poisson distributed ratings (CTMP) will help us a lot to estimate the unknown parameters without overfitting, because we will be using sparse datasets such as MovieLens and Netflix.

**Experimentation**

Talk about closed-form solutions.

Add description of datasets.

Mention that we run topic modelling LDA on **plots** not other features.

Discussion on my experiments while implementation on Python

Put top 20 words for each topic from pure or CTMPs LDA

Train Time + Machine Specs of Google Cloud (maybe make table on time spent for each hyperparameters)

Sparsity plots

**Theoretical lacks**

Put gamma distribution explanation with formula, somewhere

**Conjugate Priors**

In exact Bayesian inference, conjugacy is a crucial property. Let’s remember *Bayes’ Theorem*:

where,

* Prior **–** is the probability of a set of parameter values before having seen a data. In other words, prior is the probability distribution which expresses one's beliefs about an event before some data is considered.
* Likelihood **–** is the probability of some observed data given a set of parameter values.
* Posterior – is the probability of set of parameter values given the observed data.
* Normalization constant  **–** is a marginal probability of a data that does not depend on the parameters since it contains integral over all possible set of parameter values. It is called normalization constant is because it makes sure that posterior density integrates to one.

For some problems, the posterior inference is intractable, thus exact inference is not possible. Note that problems are considered as intractable if they can not be solved in terms of a closed-form expression [40]. Usually, this intractability of posterior inference stems from computing integrals which are involved, for example, in normalization constant of denominator. These integrals cause intractability either because they themselves are computationally intractable [38] or they have no closed-form solutions [40]. This is where conjugate prior concept helps us to avoid having a model with intractable posterior inference. Idea is that given a likelihood distribution, we need to select a family of prior distributions that allows us to compute integrals e.g. normalization constant in a tractable manner and this must hold for every prior distributions in the family. Furthermore, choice of the family has to be made such that computed posterior distribution is also included in this family. By this way, when we build model with conjugate prior and posterior, the posterior solution can be tractibly derived and in closed form[proportionality of posterior is guaranteed to be tractable and closed form, but exact posterior depends on whether integral of denominator is handled, e.g, Dirichlet-Multi handles, Gamma-Poisson not sure]. Gamma-poisson wasn’t in closedform?[39].

Exponential family of distribution is the best example for this. The reason is that for this family, the likelihood is a standarized function of the parameter, and we can make conjugate priors by simulating the likelihood's form. Moreover, when a likelihood and a prior with same exponential form are multipled, the posterior maintains the same form, which is a crucial property. It is also worth to mention that exponential forms are easily integrated for the most used exponential families [36]. Some pairs of conjugate distributions of exponential family are shown below with the details.

Intractibility- <https://stats.stackexchange.com/questions/4417/what-are-the-factors-that-cause-the-posterior-distributions-to-be-intractable> everythin here

If conjugate 🡪 has closed-form updates or easy calculation (CTMP)

When the model is conditionally conjugate, the coordinate updates are easily derived and in closed form[39]

**Multinomial distribution and Dirichlet priors**

Remembering that the multinomial distribution is the probability distribution where outcomes from experiments are discrete and they involve two or more variables. Note that multinomial distribution is considered as a multivariate generalization of binomial distribution which involves only two outcomes. Mathematically, it is defined as follow:

where, indicates the number of times outcome *i* occurs out of *n* trials, while signifies the probability that outcome *i* occurs.

Now, let’s remember the Dirichlet distribution which is a continuous multivariate probability distribution. It is also considered as a multivariate generalization of beta distribution and defined as follow:

where is a *k*-vector with components is a *k*-dimensional random variable which is in -simplex, therefore Additionally, denotes the gamma function [37], where

According to conjugate Bayesian analysis, the Dirichlet distribution is considered as a conjugate prior for the multinomial distribution. Therefore, when we multiply the likelihood expressed in multinomial form with the prior expressed in dirichlet form, we get the posterior distribution as follow:

which we can confirm that it has the form of Dirichlet distribution. So, the constant of proportionality can be determined from the form of Dirichlet distribution, and therefore we get the **exact posterior inference in closed-form expression** as follow:

*As seen above, choosing Dirichlet conjugate prior and multiplying it to multinomial likelihood yielded an exact posterior inference which is Dirichlet distribution.*

*Links:*

[*https://people.eecs.berkeley.edu/~jordan/courses/260-spring10/other-readings/chapter9.pdf*](https://people.eecs.berkeley.edu/~jordan/courses/260-spring10/other-readings/chapter9.pdf)

[*http://www.inf.ed.ac.uk/teaching/courses/mlpr/assignments/multinomial.pdf*](http://www.inf.ed.ac.uk/teaching/courses/mlpr/assignments/multinomial.pdf)

[*http://www.cs.columbia.edu/~jebara/4771/tutorials/lecture12.pdf*](http://www.cs.columbia.edu/~jebara/4771/tutorials/lecture12.pdf)

*http://gregorygundersen.com/blog/2019/03/16/conjugacy/*

**Poisson distribution and gamma priors**

**https://vioshyvo.github.io/Bayesian\_inference/conjugate-distributions.html**

Let’s now take into account the Poisson distribution from discrete exponential family distributions:

where conjugate prior to this Poisson likelihood has also form of Poisson distribution:

This conjugate prior can be easily expressed as *gamma distribution*:

where

denotes the gamma function above.

Asdasdahfhsaajdfgjhasdgfjhasd----

which we can confirm that it has the form of gamma distribution which is tractable are in closed form, but it is not exact posterior, just a proportionality…. Find somewhere with exact posterior solution.

Also,

A picture containing table

Description automatically generated

**From**

**https://people.eecs.berkeley.edu/~jordan/courses/260-spring10/other-readings/chapter9.pdf**

Basically, we can compute exact posterior distribution if prior distribution is conjugate prior to the likelihood distribution.

We can obtain exact posterior distribution if our prior distribution (in case of the example above, it will be the distribution of weights) is conjugate prior for the likelihood function.

**~~The reason why conjugate priors are very important concept in Bayesian inference is that when we try to compute Bayesian inference through posterior, conjugacy guarantees that even after multiplying likelihood by the prior, the posterior is tractable.~~**

It is now noted that if the posterior has the same distribution as the prior , then they both are referred to as ***conjugate distributions***, and the prior is considered as ***conjugate prior*** to the likelihood . Basically, conjugacy requires and to be of same distribution.

**~~Conjugacy is mutual~~**

**….**

[**http://www.cs.columbia.edu/~jebara/4771/tutorials/lecture12.pdf**](http://www.cs.columbia.edu/~jebara/4771/tutorials/lecture12.pdf)

**Mainly**

**http://www.inf.ed.ac.uk/teaching/courses/mlpr/assignments/multinomial.pdf**

**Visit -** [**https://towardsdatascience.com/understanding-conjugate-priors-21b2824cddae**](https://towardsdatascience.com/understanding-conjugate-priors-21b2824cddae)

[**https://stats.stackexchange.com/questions/44494/why-is-the-dirichlet-distribution-the-prior-for-the-multinomial-distribution**](https://stats.stackexchange.com/questions/44494/why-is-the-dirichlet-distribution-the-prior-for-the-multinomial-distribution)

[**https://en.wikipedia.org/wiki/Dirichlet\_distribution**](https://en.wikipedia.org/wiki/Dirichlet_distribution)

Distribution-Families.pdf

Discrete Distributions -> Poisson distribution

Continuous Distributions -> Gamma Distribution

Exponential Families of Distributions -> They include the continuous families— Dirichlet, Normal, Gamma, and Beta, and the discrete families— Binomial, Poisson, and Negative binomial.

Conjugate priors, conjugacy 🡪 http://www.cs.columbia.edu/~jebara/4771/tutorials/lecture12.pdf

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2?? <https://www.tidytextmining.com/topicmodeling.html>

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[6] Univariate Discrete Distributions, vol. 444 [[[[take from CTMP]]]]

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[15] <https://link.springer.com/referenceworkentry/10.1007%2F978-3-642-04898-2_327>

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[19] <https://pubmed.ncbi.nlm.nih.gov/24467759/>

[20] <https://papers.nips.cc/paper/2007/file/d7322ed717dedf1eb4e6e52a37ea7bcd-Paper.pdf>

[21] <https://www-users.cs.umn.edu/~baner029/papers/10/gpmf.pdf>

[22] <https://arxiv.org/pdf/1301.6705.pdf>

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[24] CTR -

[25] CTPF -

[26] LDA - <https://www.jmlr.org/papers/volume3/blei03a/blei03a.pdf>

[27] OPE - https://arxiv.org/abs/1512.03308

[28] BOPE - <https://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=9138369>

[29] Frank–Wolfe – take from BOPE

[30] Natasha2 – take from BOPE

[31] Stochastic Majorization-Minimization – take from BOPE

[32] Concave-Convex procedure – taken from BOPE

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In particular, they make a link between this lower bound and parameter estimation via the EM algorithm

However, in our model, the prior and [posterior distributions](https://www.sciencedirect.com/topics/computer-science/posterior-distribution) of the hidden variables θ and μ are not conjugate, therefore using variational method to infer them does not obtain [closed-form solution](https://www.sciencedirect.com/topics/engineering/closed-form-solution) [[15]](https://www.sciencedirect.com/science/article/pii/S0888613X17303390" \l "br0150).

Put [15] of original paper, here too.