Math 221
Class Exercises: Feb. 9

1. Consider the following linear system

$$\begin{aligned} x_1 - 4x_2 &= 1 \\ 3x_1 + hx_2 &= k \end{aligned}$$

   (a) Give all values of h and k such that there is no solution to the system.
   (b) Give all values of h and k such that there is a unique solution to the system.

2. Is the set of polynomials $\{x^3+1, x^3-x, x^2+1, x+1\}$ a basis for $\mathbb{P}_3$? Explain why or why not.

3. Let U and W be subspaces of a vector space V

   (a) Show that $U \cap W$ is also a subspace.
   ($U \cap W$ is the set of vectors that are in both U and W.)
   (b) Show by giving a counterexample that in general $U \cup W$ is not a subspace.
   ($U \cup W$ is the set of vectors that are in U or W.)

4. Is it possible to determine if the matrix with the following structure must have zero determinant?

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

   (The $*$ entries are not equal, just nonzero.)

5. If B and C are invertible matrices, is it true that $\operatorname{adj}(BC) = (\operatorname{adj} B)(\operatorname{adj} C)$?

6. For what vectors b does there exist a solution to $Ax = b$?

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   (a) Find a basis for the null space $\mathcal{N}(A)$.
   (b) Find a basis for the column space $\mathcal{C}(A)$.
   (c) In the case that b is in $\mathcal{C}(A)$ write the general form of the solution to $Ax = b$.
   (d) Find the rank of A^T.

7. If $x,y \in \mathbb{R}^3$ and $B = xy^T$, what is the rank of B?

8. If $A \in \mathbb{M}_{3\times 5}$ and e_1, e_2 and e_3 (the standard basis vectors for $\mathbb{R}^3$) are in $\mathcal{C}(A)$, does A have a right inverse? Does it have a left inverse?

9. How many 4×4 permutation matrices are there? Does this set span $\mathbb{M}_{4\times 4}$? Is the set linearly independent?

10. Show that an $n \times n$ matrix A is invertible if and only if A^T is invertible. (Can you supply an argument that does not rely on determinants?)