Math 221
Class Exercises: Apr. 6

1. The transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ are defined as follows:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ x+y \end{bmatrix} \qquad S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z-x \\ y \end{bmatrix}$$

   (a) Is T injective (one-to-one)?
   (b) Is $S \circ T$ an isomorphism?
   (c) Find $[S]_\alpha^\beta$ where α and β are the standard ordered basis for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

2. Let $T : \mathbb{M}_{2\times 2} \to \mathbb{M}_{2\times 2}$ be defined by $T(B) = B^T$.

   (a) Show that T is linear.
   (b) Find $[T]_\alpha^\alpha$ if

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

3. Decide if each of the following statements is true or false and explain why:

   (a) If $V \perp W$, then $V^\perp \perp W^\perp$.
   (b) If $V \perp W$ and $W \perp Z$, then $V \perp Z$.

4. The equation $x + 2y - 3z = 0$ defines a plane in $\mathbb{R}^3$.

   (a) Find a matrix that has this plane as its null space. Is the matrix unique?
   (b) Find a matrix that has this plane as its row space. Is the matrix unique?

5. (a) Find the matrix P_1 that projects orthogonally onto Span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

   (b) Find the matrix P_2 that projects orthogonally onto Span $\left\{ \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$.

   (c) Is the product $P_1 P_2$ also an orthogonal projection?

6. Let $U = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

   Find a matrix P so that Px is the projection of x onto U along W.

   (a) Check that $P^2 = P$.
   (b) Check that Px is in U for any $x \in \mathbb{R}^3$.
   (c) Check that $Pw = 0$ for any $w \in W$.