Math 111
Chapter 3.8: Natural Growth/Decay

Law of Natural Growth/Decay

$$\frac{dy}{dt} = ky$$

This differential equation is often given together with an initial condition $y(0) = y_0$. Together, the equation and initial condition are known as an *initial value problem.*

$$\begin{cases} \dfrac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}$$

(THEOREM) The only solution to the initial value problem is $y(t) = y_0 e^{kt}$.

(EXAMPLE)
Find the solution to the initial value problem.

$$\begin{cases} \dfrac{dy}{dt} = 2y \\ y(0) = 10 \end{cases}$$

APPLICATION: POPULATION GROWTH

Suppose that a colony of bacteria grows at a rate proportional to its size (law of natural growth) and that the population of the colony doubles in 8.5 hours. Find the population of the colony at time t if the initial population is 1200 bacteria.

A city population is 120000 and experiences a continuous relative growth of 3% in the population over 10 years. What is the populaton at the end of the 10 year period?

The following statements are equivalent:

- y has a growth rate proportional to its size.
- y has a constant *relative* growth rate.
- y grows exponentially.
- y has a fixed "doubling time".

APPLICATIONS: EXPONENTIAL DECAY

1. Suppose that the amount of a drug in the bloodstream decays exponentially with a half-life of 0.5 hours. If the amount of the drug starts at 180 mg, how much of the drug remains 2 hours later? How much remains 15 minutes later? At what time T will there be exactly 50 mg of the drug in the bloodstream?

2. The level of carbon-14 decays exponentially and is used to date old organic objects. The half-life of carbon-14 is known to be 5730 years. If a wooden object contains 17.1% of the level of carbon-14 as a living tree does, how old is the object?

3. A pond has water volume 200000 m^3. The pond is fed by a stream at a rate of 1000 $\mathrm{m}^3/\mathrm{day}$, and is drained at the same rate by a stream on the opposite side so that the volume remains constant. Suppose that at time $t = 0$, 4kg of a pollutant is spilled into the pond. Assume that the pollutant mixes uniformly around the pond and drains with the outflowing stream.

   (a) Write down $P(t)$, the amount of pollution remaining after t days.

   (b) How many days before the level of pollution in the pond is reduced to 0.1kg?

LAW OF COOLING

Consider an object with temperature $T(t)$ that is exposed to an environment with contant temperature A. The temperature may be modeled with Newton's Law of Cooling.

$$\begin{cases} \dfrac{dT}{dt} = k(A - T) \\ T(0) = T_0 \end{cases}$$

(EXAMPLES)

1. A hot cake is removed from the oven and placed in a room of temperature 70°F. If the temperature of the cake follows $\dfrac{dT}{dt} = 0.92(70 - T)$ and the inital temperature of the cake is 200°F, find the temperaure at time $t = 2$ hours.

2. A hot coffee at 80°C is left in a room at 20°C. After 10 minutes the coffee is 76°C. How long until the coffee cools to 72°C?

(EXERCISES)

1. It is determined that an invasive species of fish grows with a constant relative growth rate and that the population doubles in 3 years
   (a) How long does it take to increase by a factor of 10?
   (b) What is the value of the relative growth rate? (Units will be fish/year/fish.)
2. A certain radioactive material has half-life of 2.5 years.
   (a) How long until a mass of 300 grams of this material decays to 5 gram?
   (b) What is the growth rate at the time it reaches mass 5 gram? (Units will be grams/year.)
3. Let $v(t)$ be the velocity of a falling object of mass 1 kg. A simple model for the velocity is the equation $v' = 9.8 - 0.1v$.
   (In this model, 9.8 is the acceleration due to gravity, and $0.1v$ is the drag force that the object experiences as it falls.)
   (a) Find a formula for $v(t)$ if $v(0) = 0$. (*Hint: Change variables. Let* $u(t) = 9.8 - 0.1v(t)$.)
   (b) Determine what happens to the velocity as $t \to \infty$.