Math 111
Chapter 4.2 Calculus Theorems

Intermediate Value Theorem: If f is continuous on a closed interval $[a, b]$ and N is a number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = N$.

NOTES:

- There may be more than one value of c that satisfies the theorem.

- f may take on values that are not between $f(a)$ and $f(b)$.

(EXAMPLE)
Suppose that the outside temperature this morning at 7:00 was 3°C and at 11:00 the temperature was 10°C. If temperature is a continuous function of time, we can conclude by the Intermediate Value Theorem that at some point the temperature was 8°C.

In fact, we can conclude that for any temperature N, such that $3 \leq N \leq 10$, there was some time when the temperature took on that value.

(COUNTEREXAMPLE) *If the function is not continuous, the conclusion may or may not be true!*

(APPLICATION)
Use the Intermediate Value Theorem to prove that the equation $x^3 + e^x = 0$ has a solution in the interval $[-1, 0]$.

One way to search for the solution in a systematic way is called the **Bisection Method**. In this method, we divide the interval in half and apply the Intermediate Value Theorem to each subinterval.

a_n	b_n	h_n	m_n

Rolle's Theorem: Let f be a function that satisfies the following conditions:

- f is continuous on $[a, b]$.
- f is differentiable on (a, b).
- $f(a) = f(b)$

Then there exists a number c in (a, b) such that $f'(c) = 0$.

(EXAMPLE)
Let $f(x) = x^3 - 4x + 2$ on the interval $[0, 2]$.

(APPLICATION)
Prove that there is *at most* one solution to the equation $e^x + x^3 = 0$ in the interval $[-1, 0]$.

Mean Value Theorem: Let f be a function that satisfies the following conditions:

- f is continuous on $[a, b]$.
- f is differentiable on (a, b).

Then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(EXAMPLES)

1. Let $f(x) = x^2$ on $[-1, 3]$.

2. Let $f(x) = e^{-x}$ on $[0, 2]$.

(APPLICATIONS)

If $s(t)$ represents position, $s'(t)$ is velocity. The *average velocity* over a time interval $a \leq t \leq b$ is

$$v_{avg} = \frac{s(b) - s(a)}{b - a} = \frac{\Delta s}{\Delta t}$$

So if we travel in a car 280 km in 4 hrs, what can we conclude with the Mean Value Theorem?

If cameras that are 5 km apart on the highway take pictures of the same car and the time between the pictures is 2.1 minutes, what can be concluded about the speed of the car?

Suppose that for a given function f, we know that $f'(x) \leq 2$ for all x. If $f(4) = 10$, what is largest that $f(9)$ could be?

Suppose that for a given function g, it is known that $-1 \leq f'(x) \leq 3$ for all x in $[0, 5]$. What is the largest that $f(5) - f(0)$ could be? What is the smallest that $f(5) - f(0)$ could be?

Proof of Mean Value Theorem:

(THEOREM) If $f'(x) = 0$ for all x in $[a, b]$, then $f(x) = C$ on $[a, b]$. (C *is a constant.*)

(COROLLARY) If $f'(x) = g'(x)$ for all x in $[a, b]$, then $f(x) = g(x) + C$ on $[a, b]$. (C *is a constant.*)