Math 111
Chapter 2.2: Limits

(DEFINITION) Suppose f is a function defined near a number a. Then

$$\lim_{x \to a} f(x) = L$$

means $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a. We say that L is the **limit** of f as x goes to a.

(EXAMPLES)

1. $$\lim_{x \to 0} x^2 - 3 =$$

   How close to 0 does x need to be for $x^2 - 3$ to be within 0.1 of the limit?

2. $$\lim_{x \to 2} 2^x =$$

   How close to 2 does x need to be for 2^x to be within 0.05 of the limit?

3. $$\lim_{x \to 0} \frac{4x + 2x^2}{x} =$$

4. Suppose that

$$g(x) = \begin{cases} \dfrac{4x + 2x^2}{x} & x \neq 0 \\ 3 & x = 0 \end{cases}$$

Find $\lim\limits_{x \to 0} = g(x)$.

5. $$\lim_{x \to 2} \frac{x - 2}{x^2 + x - 6} =$$

6. $$\lim_{x \to 4} \frac{\frac{1}{4} - \frac{1}{x}}{4 - x} =$$

7. $$\lim_{x \to 0} \frac{\sin x}{x} =$$

8. $$\lim_{x \to 0} \frac{\sqrt{x^4+16}-4}{x^4} =$$

One sided limits

All functions do not have limits in all circumstances. Consider such a function:

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

We say that $\lim_{x \to 0} h(x)$ does not exist.

(DEFINITION) Suppose f is a function defined near a number a. Then

$$\lim_{x \to a^+} f(x) = L$$

means $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a with $x > a$. We say that L is the **limit** of f as x goes to a **from the right**.

(THEOREM) $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

(EXAMPLES)

1. $$f(x) = \begin{cases} 2 - 2x & x \leq 0 \\ \sin x & x > 0 \end{cases}$$

   Find $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

2. $$p(x) = \begin{cases} |x| & x \neq 0 \\ 4 & x = 0 \end{cases}$$

   Find $\lim_{x \to 0^+} p(x)$, $\lim_{x \to 0^-} p(x)$, and $\lim_{x \to 0} p(x)$.

3. $$g(t) = \begin{cases} e^t & t < 3 \\ \ln t & t > 3 \end{cases}$$

   Find $\lim_{t \to 3^+} g(t)$, $\lim_{t \to 3^-} g(t)$, and $\lim_{t \to 0} g(t)$.

Infinite Limits

(EXAMPLE) $\lim_{x\to 0} \frac{1}{x^2} = \infty$

(DEFINITION) Suppose f is a function defined near a number a. Then

$$\lim_{x\to a} f(x) = \infty$$

means $f(x)$ can be made arbitrarily large by taking x sufficiently close to a. We say that the limit of f is **infinity**.

(EXAMPLES)

1. $$\lim_{x\to 0} -\frac{1}{x^2} =$$

2. $$\lim_{x\to 2^+} \frac{x}{x-2} =$$

3. $$\lim_{x\to 2^-} \frac{x}{x-2} =$$

Vertical Asymptotes

(DEFINITION) The graph of a function f has a **vertical asymptote** at $x = a$ if any of the following are true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(EXAMPLES)

$\tan x$

$\ln x$

$$f(x) = \frac{3+x}{e^x - e}$$

$$g(t) = \begin{cases} 2 & t \leq -4 \\ \dfrac{1}{t+4} & t > -4 \end{cases}$$