Math 112

Chapter 9.5: Linear Differential Equations

A differential equation is called **linear** if can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on some interval.

Solution method is based on the Product Rule

$$\frac{d}{dx}[xy] = y + x\frac{dy}{dx}$$

EXAMPLE:

$$x\frac{dy}{dx} + y = x$$

$$\frac{d}{dx}[xy] = x$$

$$\int \frac{d}{dx}[xy]\,dx = \int x\,dx$$

$$xy = \frac{x^2}{2} + C$$

$$y = \frac{x}{2} + \frac{C}{x}$$

More generally, we will need to multiply by an **integrating factor** ~~$u(x)$~~ $u(t)$ in order to use the Product Rule.

$$y' - 2y = 3e^t$$

$$uy' - 2uy = 3ue^t$$

want $uy' + u'y$

Set $\frac{du}{dt} = -2u$

Separable $u = Ce^{-2t}$

Any u should work so take $C = 1$

$$y'e^{-2t} - 2e^{-2t}y = 3e^{-2t}e^{t}$$

$$\frac{d}{dt}\left[ye^{-2t}\right] = 3e^{-t}$$

$$\int \frac{d}{dt}\left[ye^{-2t}\right]dt = \int 3e^{-t}\,dt$$

$$ye^{-2t} = -3e^{-t} + C$$

$$y = Ce^{2t} - 3e^{t}$$

Solve the initial value problem:

$$\begin{cases} xy' + 2y = 4x^2 \\ y(1) = 2 \end{cases}$$

$$y' + \frac{2}{x} y = 4x$$

$$u y' + \frac{2}{x} u y = 4 x u$$

Need $\frac{du}{dx} = \frac{2u}{x}$

$$\frac{du}{2u} = \frac{dx}{x}$$

$$\frac{1}{2} \ln|u| = \ln|x|$$

$$\ln|u| = 2 \ln|x|$$

$$\ln|u| = \ln|x|^2$$

$$u = e^{\ln|x|^2} = x^2$$

$$x^2 y' + 2x y = 4x^3$$

$$\int \frac{d}{dx}[x^2 y]\, dx = \int 4x^3\, dx$$

$$x^2 y = x^4 + C$$

IC: $(1)(2) = 1 + C \qquad C = 1$

$$y = x^2 + \frac{1}{x^2}$$

We can write down the formula for a solution in terms of P and Q

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$u \frac{dy}{dx} + u P y = u Q$$

Need $\frac{du}{dx} = u P$

$$\frac{du}{u} = P\, dx$$

$$\ln u = \int P\, dx$$

So $u = e^{\int P\, dx}$

Then $\int \frac{d}{dx}[u y]\, dx = \int u Q\, dx$

$$u y = \int u Q\, dx$$

So

$$y(x) = \frac{1}{u(x)} \left[\int u(x) Q(x)\, dx + C \right]$$

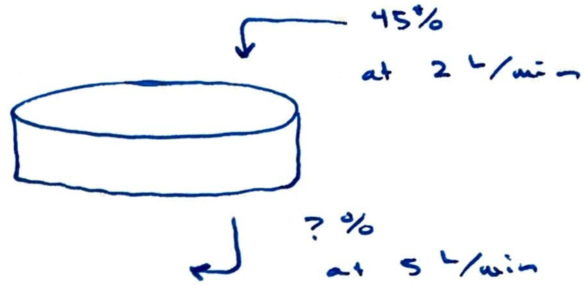
where $u(x) = e^{\int P(x)\, dx}$

APPLICATIONS:

A tank contains 1000 L of whiskey 35% alcohol. At $t = 0$, a whiskey with concentration 45% alcohol begins flowing into the tank at a rate of 2 L/min. If the tank is kept well-stirred and is drained at a rate of 5 L/min, find the concentration of the whiskey blend in the tank as a function of time.

Let $V(t)$ = vol of wh. (L)

$A(t)$ = vol of alc. (L)



IVP for V:

$$\begin{cases} \frac{dV}{dt} = -3 \\ V(0) = 1000 \end{cases}$$

So $V(t) = 1000 - 3t$

DE for A

$$\frac{dA}{dt} = \left(\frac{2 \text{ L wh.}}{\text{min}}\right)\left(\frac{45 \text{ L alc}}{100 \text{ L wh.}}\right) - \left(\frac{5 \text{ L wh}}{\text{min}}\right)\left(\frac{A \text{ L alc}}{(1000-3t) \text{ L wh}}\right)$$

IVP for A

$$\begin{cases} \frac{dA}{dt} = 0.9 - \frac{5A}{1000-3t} \\ A(0) = 350 \end{cases}$$

Linear DE $\quad A' + \underbrace{\left(\frac{5}{1000-3t}\right)}_{P} A = \underbrace{0.9}_{Q}$

$$\int P\,dt = -\frac{5}{3}\ln(1000-3t) = \ln(1000-3t)^{-5/3}$$

$$u = (1000-3t)^{-5/3}$$

$$\int uQ\,dt = \int 0.9(1000-3t)^{-5/3}\,dt$$

$$= \left(\frac{0.9}{-3}\right)\left(-\frac{3}{2}\right)(1000-3t)^{-2/3} + C$$

So $A(t) = (1000-3t)^{5/3}\left[0.45(1000-3t)^{-2/3} + C\right]$

$$\frac{1}{u}\left[\int uQ\,dt + C\right]$$

IC

$$350 = 1000^{5/3}\left[0.45(1000)^{-2/3} + C\right]$$

$$C = \frac{350}{1000^{5/3}} - \frac{0.45}{1000^{2/3}}$$

and

$$A(t) = 0.45(1000-3t) + C(1000-3t)^{5/3}$$

$$\text{Concentration} = \frac{A}{V}$$

$$= 0.45 + C(1000-3t)^{2/3}$$

Note as $3t \to 1000$

concentration $\to 0.45$

(tank goes empty)

The following modification of Newton's Law of Cooling can be used in the case that the ambient temperature varies in time.

$$\begin{cases} \dfrac{dT}{dt} = k(A(t) - T) \\ T(0) = T_0 \end{cases}$$

Suppose for example that $A(t)$ represents the outside temperature and $T(t)$ is the interior temperature of a building with no climate control. If we measure T in hours, it might be reasonable to assume A is a periodic function such as $A(t) = 14 + 5\cos\dfrac{\pi t}{12}$.

Let's take $T_0 = 19$, $k = 0.15/\text{hr}$, and see what the model predicts.

Linear DE $\quad T' + kT = kA \qquad \mu = e^{\int k\,dt} = e^{kt}$

Need $\displaystyle\int kA\,dt = \int ke^{kt}\left[14 + 5\cos\left(\frac{\pi t}{12}\right)\right]dt$

IBP, algebra

$$= 14e^{kt} + \frac{720ke^{kt}}{\pi^2 + 144k^2}\underbrace{\left(k\cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right)\right)}_{f(t)} + C$$

$$T(t) = 14 + Ce^{-kt} + \frac{720k}{\pi^2 + 144k^2}f(t)$$

I.C. : $\quad C = T_0 - 14 - \dfrac{720k^2}{\pi^2 + 144k^2}$

Since $Ce^{-kt} \to 0$ as $t \to \infty$

$$T \to \underbrace{14 + \frac{720k}{\pi^2 + 144k^2}f(t)}$$

Periodic function with same period as $A(t)$.

Amplitude and phase depends on k.

For large k this function close to A

For small k this function has smaller amplitude and lags behind $A(t)$.