Math 112
Chapter 9.3: Population Models

In this section, we consider a dynamic population $P(t)$, and examine differential equations that might be used to describe how the population changes with time.

$$\frac{dP}{dt} = rP$$

Where r is the *per capita growth rate.* If r is constant, the population grows exponentially, but we want to consider also what happens if r is not constant.

EXAMPLE: Seasonal growth $r(t) = 0.05\cos(t)$

$$\begin{cases} \frac{dP}{dt} = (0.05\cos t)P \\ P(0) = P_0 \end{cases}$$

$$\int \frac{dP}{P} = \int 0.05 \cos t$$

$$\ln|P| = 0.05 \sin t + C$$

$$P(t) = Ae^{0.05 \sin t}$$

I.C. $\longrightarrow$ $A = P_0$

$$P(t) = P_0 e^{0.05 \sin t}$$

periodic population with amplitude dependent on P_0

EXAMPLE: Diminishing resources $r(t) = t^{-k}$ (Here $r \to 0$ as $t \to \infty$) $k > 0$

$$\frac{dP}{dt} = \frac{P}{t}$$

$$\begin{cases} \frac{dP}{dt} = \frac{P}{t} \\ P(1) = P_0 \end{cases}$$

$$\int \frac{dP}{P} = \int \frac{dt}{t}$$

$$\ln|P| = \ln|t| + C$$

$$P = At$$

I.C $\longrightarrow$ $A = P_0$ Model predicts linear growth

Case $k = 2$

$$\frac{dP}{dt} = \frac{P}{t^2}$$

$$\begin{cases} \dfrac{dP}{dt} = \dfrac{P}{t^2} \\ P(1) = P_0 \end{cases}$$

$$\int \frac{dP}{P} = \int \frac{dt}{t^2}$$

$$\ln|P| = -\frac{1}{t} + C$$

$$P = Ae^{-1/t}$$

I.C. $\longrightarrow$ $A = P_0 e$

$$P(t) = P_0 e^{1 - 1/t}$$

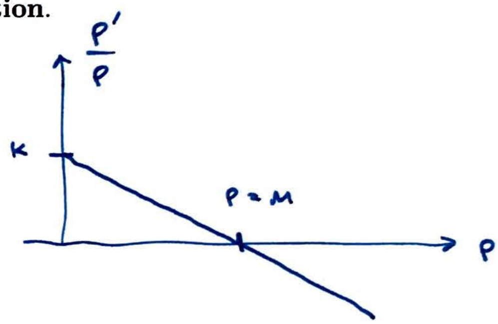
As $t \to \infty$ $P \longrightarrow P_0 e$

It is also reasonable to assume that the per capita growth is a function of P itself

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where $k > 0$ and $M > 0$ are constants is called the **logistic equation**.

per capita growth $\dfrac{\frac{dP}{dt}}{P} = k\left(1 - \frac{P}{M}\right)$



Notes : 1.) $P' > 0$ if $0 < P < M$
$P' < 0$ if $P > M$

(population incr if $P < M$, decrease if $P > M$)

2.) If $P = M$ (or $P = 0$) then population is constant.
M is called the carrying capacity.

3.) If P close to zero logistic equation close to

$$\frac{dP}{dt} = kP$$

4.) See software for direction field. Note that for
$0 < P < M$, max growth rate occurs when
$P = M/2$.

Solution of initial value problem

$$\begin{cases} \dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right) \\ P(0) = P_0 \end{cases}$$

$$\frac{dP}{dt} = \frac{kP}{M}(M-P)$$

$$\int \frac{M\,dP}{M-P} = \int k\,dt \qquad \left(\text{Partial fraction } \frac{M}{M-P} = \frac{1}{P} + \frac{1}{M-P}\right)$$

$$\int \frac{1}{P} + \frac{1}{M-P}\,dP = \int kt$$

$$\ln|P| - \ln|M-P| = kt + C \qquad (\text{Now algebra to solve for } P)$$

$$\ln\left|\frac{P}{M-P}\right| = kt + C$$

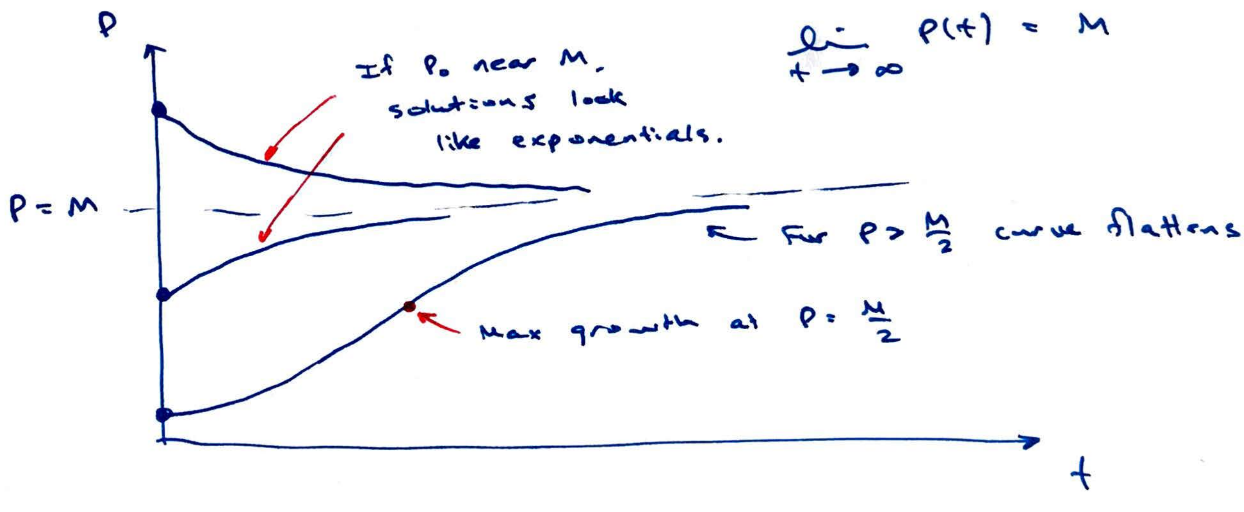
$$\frac{P}{M-P} = Ae^{kt} \qquad \left(\text{Apply I.C. } A = \frac{P_0}{M-P_0}\right)$$

Solve for P, fill in A and get

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kt}}$$

For all $P_0 \neq 0$

$$\lim_{t\to\infty} P(t) = M$$



EXAMPLES:

Suppose a population follows the logistic growth with a carrying capacity of $M = 5000$ and $k = 0.05$/year.

1. If the population starts at 200, what is the population 3 years later?
2. How long does it take for the population to reach size 4000?
3. Compare these numbers to a population that grows at a *constant* 0.05/year and starts at the same size.

$$P(t) = \frac{200(5000)}{200 + 4800e^{-kt}}$$

1.) $P(3) = 231$

2.) Find T so that $P(T) = 4000$

Solve $4000 = \dfrac{200(5000)}{200 + 4800e^{-kT}}$ for T

$\vdots$

$T = 91.3$ years

3.) If $\dfrac{dP}{dt} = 0.05P$, $P(0) = 200$

then $P(t) = 200e^{0.05t}$

$P(3) = 232$ (close to logistic prediction)

Find T so that $P(T) = 4000$

$4000 = 200e^{0.05T}$

$T = 59.9$ years.

Another population experiences logistic growth with $M = 9500$. If the initial population is 1100 and five years later the population as grown to 2300, find the size of the population ten years from the beginning.

$$P(t) = \frac{(9500)(1100)}{1100 + 8400e^{-kt}}$$

use $P(5) = 2300$ to find k

Solve

$$2300 = \frac{(9500)(1100)}{1100 + 8400e^{-5k}}$$

Find $k = 0.178$

and

$P(10) = 4165$

Let $a(t)$, and $b(t)$ be two populations of bacteria that experience constant relative growth at rates α and β (so a and b grow exponentially.) Suppose that both populations live in the same evironment and that samples of the populations can only detect the *proportion* of population a. Let $p(t)$ be the proportion.

$$p(t) = \frac{a(t)}{a(t) + b(t)}$$

1. Show that p follows the logistic equation.

2. Find the solution of the logistic equation by using the formulas for a and b.

1.) Assume $\frac{da}{dt} = \alpha a$ and $\frac{db}{dt} = \beta b$

$$\frac{dp}{dt} = \frac{\frac{da}{dt}(a+b) - a\left(\frac{da}{dt} + \frac{db}{dt}\right)}{(a+b)^2}$$

$$\frac{dp}{dt} = \frac{b\frac{da}{dt} - a\frac{db}{dt}}{(a+b)^2} = \frac{b(\alpha a) - a(\beta b)}{(a+b)^2}$$

$$\frac{dp}{dt} = (\alpha - \beta)\left[\frac{a}{a+b}\right]\left[\frac{b}{a+b}\right]$$

$\frac{dp}{dt} = (\alpha - \beta)p(1-p)$ $\leftarrow$ Logistic eqn. with $k = \alpha - \beta$, $M = 1$

2.) $a = a_0 e^{\alpha t}$ $b = b_0 e^{\beta t}$ so $p = \frac{a_0 e^{\alpha t}}{a_0 e^{\alpha t} + b_0 e^{\beta t}}$

Divide by $(a_0 + b_0)e^{\alpha t}$

$$p = \frac{\left(\frac{a_0}{a_0+b_0}\right)}{\left(\frac{a_0}{a_0+b_0}\right) + \left(\frac{b_0}{a_0+b_0}\right)e^{-(\alpha-\beta)t}}$$

$$p = \frac{p_0}{p_0 + (1-p_0)e^{-(\alpha-\beta)t}}$$

Solution of logistic equation is a combination of exponential curves.

An contagious disease is spreading among a population. Let $S(t)$ be the number of individuals who are susceptible to the disease, and $I(t)$ be the number who are infected. One possible model for the spread of the disease is known as the SI-model.

$$\begin{aligned} \frac{dS}{dt} &= -kSI \\ \frac{dI}{dt} &= kSI \end{aligned}$$

The per capital change of each population is proportional to the other. Determine a formula for $I(t)$ by showing that it follows logistic growth.

Add equations $\longrightarrow$ $\frac{d}{dt}(S+I) = 0$

Means $S + I = N$ for some constant N

(N is total population.)

Now $S = N - I$ so:

$$\frac{dI}{dt} = kI(N-I)$$

$$\frac{dI}{dt} = \frac{k}{N} I \left(1 - \frac{I}{N}\right)$$ a logistic equation

Solution is:

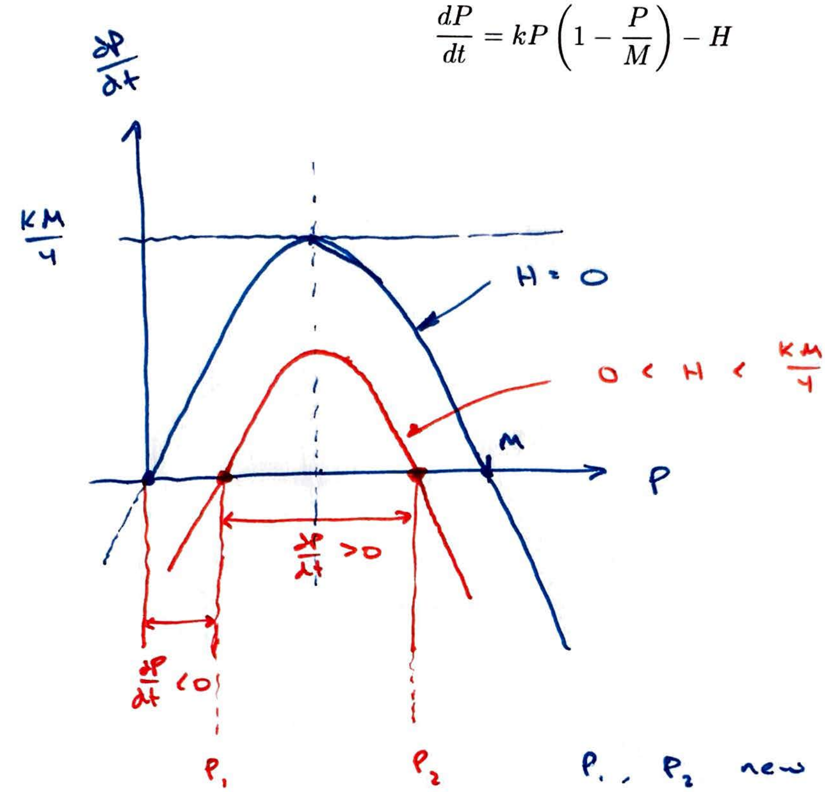
$$I(t) = \frac{N I_0}{I_0 + (N - I_0)e^{-rt}}$$

where $I_0 = I(0)$ and $r = \frac{k}{N}$

As $t \to \infty$ $I \to N$.

A model for a population that experiences a constant removal rate H can be found by making a modification to the logistic equation. (*The removal may represent fish or plants being harvested from the population.*)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - H$$



P_1, P_2 new equilibrium solutions.

If $P_0 > P_1$ $P \to P_2$ as $t \to \infty$

If $P_0 < P_1$ $P \to 0$ as $t \to \infty$

If $M = 20000$, $k = 0.03$/year, and $H = 150$ individuals/year, what is the lowest population that could survive such harvesting?

$$P' = 0.03\,P\left(1 - \frac{P}{20000}\right) - 150$$

P_1 is threshold population

Find P_1, P_2 by setting $P' = 0$

$$0 = (0.03)(20000)\,P - 0.03P^2 - 150(20000)$$

$$0 = P^2 - 20000\,P + 10^8$$

$$P_1 = \frac{20000 - \sqrt{20000^2 - 4(10^8)}}{2} \approx 1340$$