Math 112
Chapter 11.8: Power Series

A **power series** is a series that has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where x is a variable, a is a constant, and the sequence of values c_n are constants. The series is considered a function of x.

EXAMPLE:

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

THEOREM: For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

R is called the **radius of convergence**.

EXAMPLES: (For each series, find the radius of convergence and the **interval of convergence**.)

$$\sum_{n=1}^{\infty} \frac{4^n x^n}{n^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{2^{2n}(n!)^2}$$

$$\sum_{n=0}^{\infty} x^n (n!)$$

$$\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^3}$$

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

$$\sum_{n=0}^{\infty} \frac{n^2}{6^{n+2}} (x+3)^n$$

EXERCISES:

For each series, find the radius of convergence and the interval of convergence.

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+4}$$

$$\sum_{n=1}^{\infty} n^n x^n$$

$$\sum_{n=2}^{\infty} \frac{(x-5)^n}{3^n \ln n}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n \qquad (k \text{ is some positive integer})$$

Suppose $\lim_{n\to\infty} \sqrt[n]{|c_n|} = C$. Explain why the radius of convergence for $\sum c_n x^n$ is $\frac{1}{C}$.

Chapter 11.9: Power Series Construction

The goal of this section is to find ways to represent some functions as power series be relating them to the sum of the geometric series.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ... = \frac{1}{1-x} \qquad \text{for } |x| < 1$$

EXAMPLES:

In each case, the goal is to write the function as a power series of the form $\sum c_n x^n$.

$$\frac{1}{1+x}$$

$$\frac{x^3}{1+x^2}$$

$$\frac{1}{x+6}$$

THEOREM:
If the power series $\sum c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x) = c_0 + c_1(x-1) + c_2(x-a)^2 + ... = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable for $|x-a|<R$ and

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2.... = \sum_{n=1}^{\infty} c_n n(x-a)^{n-1} \\ \int f(x)\,dx &= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3}... = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \end{aligned}$$

The radius of convergence for both of these new series is also R.

EXAMPLES:

In each case, the goal is to write the function as a power series of the form $\sum c_n x^n$.

$\dfrac{1}{(1-x)^2}$

$\ln|1+x|$

$\arctan x$