Math 112
Chapter 9: Population Models

Schaefer model
Let $P(t)$ represent a population of fish and consider the following model for the description of harvesting from a population that is growing that undergoes logistic growth.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - EP$$

This model assumes that at a given level of effort, E, the rate at which fish are caught is directly proportional to the current size of the population.

1. If $E < k$, there are two equilibrium solutions, P_1 and P_2. $P_1 = 0$. Find P_2 and predict what will happen to the fish population.
2. Does the survival of the population depend on the initial condition?
3. Calculate the yield $Y = EP_2$. This is the amount of fish that may be harvested indefinitely while the population remains stable.
4. Find the value of E that gives the maximum sustainable yield Y.

Gompertz Model
Let $P(t)$ represent a population of cells and consider the following differential equation.

$$\frac{dP}{dt} = -rP\ln\left(\frac{P}{K}\right)$$

This equation is known as the Gompertz model. Here r and K are positive constants that you will need to interpret.

1. Find any equilibrium solutions.
2. Draw a slope field for the equation. It may be helpful to plot the rate function in terms of P first as we did for the logistic equation. What happens to the rate function as P goes to zero? What happens as P nears an equilibrium?
3. What is the max growth rate and where does it occur? (What value of P?)
4. What is the average growth rate of the population over its range of sizes (zero to equilibrium)?
5. Solve the equation using separation of variables and an initial size of P_0.
6. How is the the Gompertz model similar to the logistic model for population growth?
7. How are the two models different?

Doomsday equation
This one is just for fun. What would happen if the *per capita* growth rate of a population was proportional to the population? This would give the following initial value problem.

$$\begin{cases} \dfrac{dP}{dt} = kP^2 \\ P(0) = P_0 \end{cases}$$

1. Find the solution and show that the model predicts the population going to infinity in a *finite* time.

2. Does the prediction depend on the initial population P_0? Does it depend on k?

Lotka-Volterra model
Let $P(t)$ be the population of a prey species, and $R(t)$ be the population of a predator species that feeds on the prey species. One model for these two interacting species is the following set of "predator-prey" equations

$$\begin{cases} \dfrac{dP}{dt} = 0.08P - 0.001PR \\ \\ \dfrac{dR}{dt} = -0.02R + 0.00002RW \end{cases}$$

1. Describe what happens to the prey population in the absence of the predators.

2. Describe what happens to the predator population in the absence of the prey.

3. What are all of the possible equilibriums of this system of equations? Equilibria are now pairs of values (P, R) that are constant solutions.

4. Determine how each of the populations is changing when there are 500 prey and 50 predators.

5. Determine how each of the populations is changing when there are 3000 prey and 50 predators.

6. Since t is not invoved explicitly, we could sketch a slope field in the PR-plane. At each point (P, R) we can sketch an arrow determined by P' and R'. Try it.