Math 112
Chapter 9.1: Differential Equations

A **differential equation** is a relationship between and unknown function and its derivatives.

EXAMPLES:

$\frac{dy}{dt} = 4y$ Solution: $y = Ce^{4t}$ for any C.

Check: $\frac{dy}{dt} = (Ce^{4t}) \cdot 4 = 4y$

$\frac{dy}{dt} = 3t^2 y$ Solution: $y = Ce^{t^3}$ for any C.

Check: $\frac{dy}{dt} = (Ce^{t^3}) \cdot 3t^2 = 3t^2 y$

$\frac{dy}{dt} = \cos t$ Solution: $y = \sin t + C$

$\frac{dy}{dt} + y = e^{-t}$ Solution: $y = e^{-t}(t + c)$ for any c

Check: $\frac{dy}{dt} + y = -e^{-t}(t+c) + e^{-t} + e^{-t}(t+c) = e^{-t}$

NOTES:

1.) Solutions easy to check. May be hard to find.

2.) Solution must involve anti-derivative, so the general form of solution contains arbitrary constant.

(Family of possible solutions for single DE.)

If we attach an **initial condition** to the differential equation, we get what is called an **initial value problem**.

DE

I.V.P $\begin{cases} y' = \cos t \\ y(0) = 6 \end{cases}$

I.C.

$y = \sin t + C$ solves DE for any C

but only

$y = \sin t + 6$ solves I.V.P.

A differential equation alone has a family of solutions, a well-posed initial value problem has only one solution.

$$\begin{cases} y' + y = e^{-t} \\ y(0) = 2 \end{cases}$$

$y = e^{-t}(t + c)$ solves D.E.

Apply I.C.

$2 = e^{0}(0 + c)$

$c = 2$

$y = e^{-t}(t + 2)$ solution to I.V.P.

EXAMPLES OF DIFFERENTIAL EQUATIONS AS MODELS:

1. Let $P(t)$ represent the size of a growing population. Determine an equation for P if it is assumed that rate at which the population grows is proportional to its size.

$$\underbrace{\frac{dP}{dt} = kP}_{\text{rate of change of population proportional to population}}$$

(Law of Natural Growth)

Solution is exponential function $P = Ce^{kt}$

where C is initial value of population.

2. An object falls from a given height with only the force of gravity acting upon it. Find a differential equation for $v(t)$, its velocity as a function of time.

Know $F = ma$, F is gravity and equals $-mg$

Know that acceleration $a = \frac{dv}{dt}$

O.E. is $\frac{dv}{dt} = -g$ I.V.P is $\begin{cases} \frac{dv}{dt} = -g \\ v(0) = v_0 \end{cases}$

3. An object falls from a given height with only the force of gravity acting upon it. Find a differential equation for $h(t)$, its *height* as a function of time.

Same physics, now $\frac{dh}{dt} = v$ so $\frac{dv}{dt} = \frac{d^2h}{dt^2}$

O.E. is $\frac{d^2h}{dt^2} = -g$ Involves 2^{nd} derivative, expect 2 arbitrary constants in solution.

I.V.P should have 2 initial condition $\begin{cases} \frac{d^2h}{dt} = -g \\ h(0) = h_0 & \leftarrow \text{initial height} \\ h'(0) = v_0 & \leftarrow \text{initial velocity} \end{cases}$

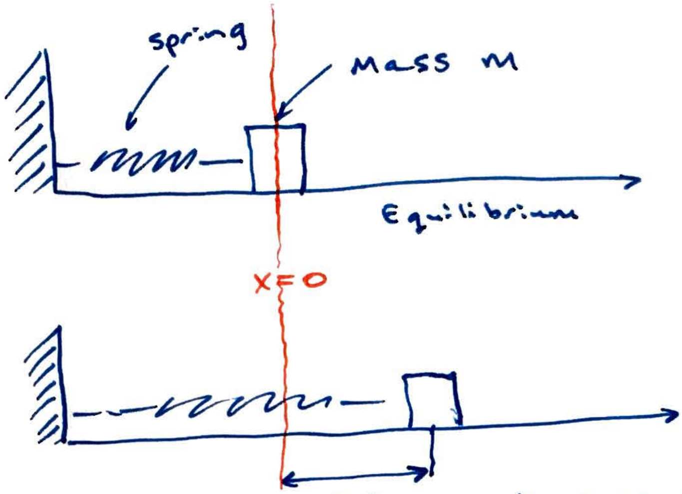
4. An object falls from a given height, and in addition to the force of gravity, it experiences a drag force that is proportional to its velocity. Find a differential equation for the velocity $v(t)$.

Again $F = ma$, now $F = \underbrace{kv}_{\text{drag force proportional to } v} - \overbrace{mg}^{\text{gravity}}$

O.E. $m\frac{dv}{dt} = kv - mg$

or $\frac{dv}{dt} = \frac{k}{m}v - g$

5. A mass is attached to one end of a spring, while the other end is attached to a wall. The force needed to stretch the spring is proportional to the length it is stretched from equilibrium. Let $x(t)$ be the position of the mass and find a differential equation for $x(t)$.



$x(t)$ is displacement from equilibrium.

Again $F = ma$ and $a = \frac{d^2x}{dt^2}$.

Assume only force is due to spring $F = -kx$

where $k > 0$ constant.

Then DE: $\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$ tradition to let $\frac{k}{m} = \omega^2$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

I.V.P has 2 conditions

$$\begin{cases} \frac{d^2x}{dt^2} + \omega^2 x = 0 \\ x(0) = x_0 \\ x'(0) = x_0' \end{cases}$$

Initial position and velocity.

Solns to DE are $\cos(\omega t)$ and $\sin(\omega t)$

Soln to I.V.P is some combination of $\cos(\omega t)$, $\sin(\omega t)$.

Direction Fields

If a differential equation can be written in the form

$$\frac{dy}{dx} = f(x, y)$$

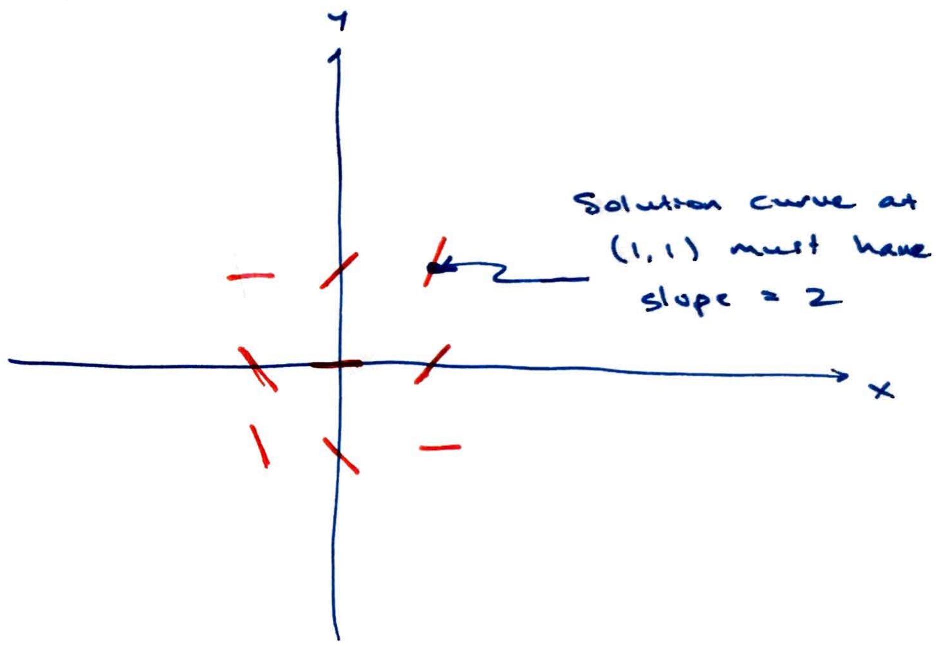
we say that f is the rate function. We can use f to draw a **direction field**.

EXAMPLES:

$$\frac{dy}{dx} = y + x$$

Slope table

x	y	$\frac{dy}{dx}$
0	0	0
1	0	1
1	1	2
1	0	1
-1	0	-1



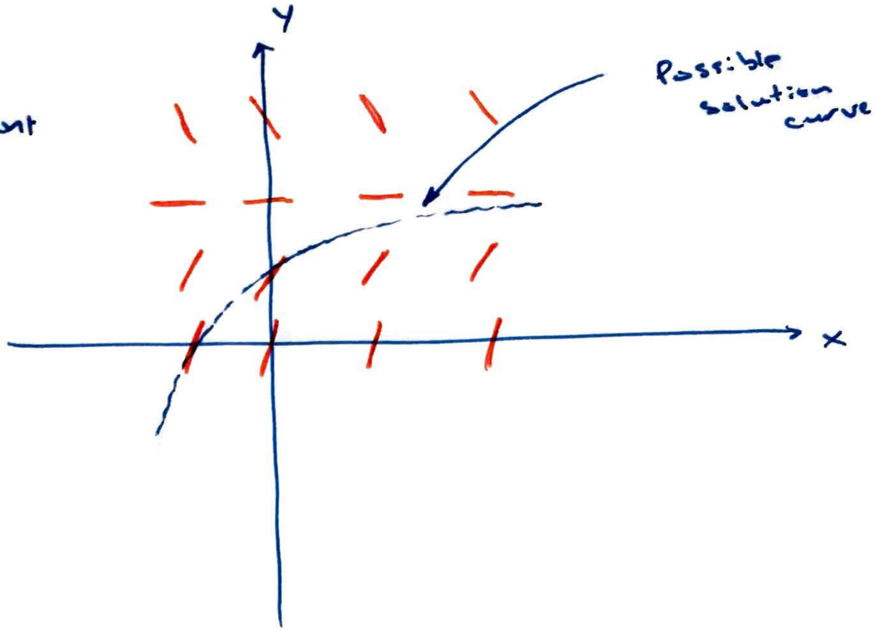
Solution (integral curve) must have slope that matches direction field at all points.

Use software to plot slopes at many points. Can understand nature of solutions by picture.

$$\frac{dy}{dx} = \underbrace{4 - 2y}_{}$$

slope independent of x

y	$\frac{dy}{dx}$
0	4
1	2
2	0
3	-2



$$\frac{dy}{dx} = y \cos x$$