Math 112
Chapter 11.10: Taylor Series

THEOREM:

If f has a power series representation at a,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad\qquad |x-a| < R$$

then the coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

EXAMPLE:

$f(x) = e^x$ and $a = 0$

When we write the series with this formula, we call it a **Taylor series**.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

and when $a = 0$ we call it a **Maclaurin series**.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

When we use only a finite number of terms, we call the sum a **Taylor polynomial**.

We can use the Maclaurin series for e^x to construct series of related functions.

EXAMPLES:

Find the Maclaurin series for the following functions.

$g(x) = e^{-x}$

$h(x) = x^2 e^{2x}$

$f(x) = e^{-(x-2)}$

We can make different Taylor series for the same function *centered* at different a.

EXAMPLE:

$f(x) = e^x$ at $a = 4$.

THEOREM:

Let $R_N = f(x) - T_N(x)$, and let M be number such that $|f^{n+1}(x)| \leq M$ for $|x - a| \leq d$. Then the remainder $R_N(x)$ satisfies

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1} \qquad \text{for} |x-a| < d$$

EXAMPLE:

$f(x) = e^x$ and $a = 0$

MORE EXAMPLES:

Find the Maclaurin series for $\sin x$.

Find the Maclaurin series for $\cos x$.

Find the Taylor series for $\cos x$ at $a = \pi/2$.

Find the Taylor series for 2^x at $a = 3$.

Find the Taylor series for $x^3 + 2x - 1$ at $a = 1$.

Find the Maclaurin series for $\ln(x+1)$.
(*Note we've found this series already using the geometric series.*)

As before, we can find other series by manipulating series we know.

Find the first few terms in the Maclaurin series for $e^x \sin x$.

Find the first few terms in the Maclaurin series for $\tan x$.