Math 112
Chapter 11.2: Integral Test

Integral Test:
Suppose that f is a continuous, positive, decreasing function on $[1, \infty)$ and that $a_n = f(n)$.

If $\displaystyle\int_1^\infty f(x)\,dx$ converges, then $\displaystyle\sum_1^\infty a_n$ converges.

If $\displaystyle\int_1^\infty f(x)\,dx$ diverges, then $\displaystyle\sum_1^\infty a_n$ diverges.

EXAMPLES:

$\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ (Harmonic Series)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad (p\text{-series})$$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

$$\sum_{n=1}^{\infty} ne^{-n}$$

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{13} + \frac{1}{18} + ...$$

Remainder Estimate for the Integral Test:

Suppose $f(k) = k$, where f is continuous, positive, and decreasing for $x \geq n$ and $\sum a_n$ is convergent. Let $R_n = s - s_n$ be the remainder. Then

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx$$

EXAMPLES:

How close is $\displaystyle\sum_{1}^{15} \frac{1}{n^3}$ to $\displaystyle\sum_{1}^{\infty} \frac{1}{n^3}$?

How large should N be so that $\displaystyle\sum_{1}^{\infty} \frac{1}{n^3} - \sum_{1}^{N} \frac{1}{n^3} < 0.0001$?

How large should N be so that $\displaystyle\sum_{2}^{\infty} \frac{1}{n(\ln n)^2} - \sum_{1}^{N} \frac{1}{n(\ln n)^2} < 0.01$?

Math 112
Chapter 11.4: Comparison Tests

Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ also converge.
2. If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ also diverges.

EXAMPLES:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n + 2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3}$$

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n - 2}$$

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number that is *not zero*, the either both series converge or both diverge.

EXAMPLES:

$$\sum_{n=1}^{\infty} \frac{1}{5^n - 2}$$

$$\sum_{n=0}^{\infty} \frac{1}{4 + \sqrt{n}}$$

$$\sum_{n=0}^{\infty} \frac{2^n}{8 + 3^n}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^4+3n}}{5n^3+2}$$

EXERCISES: (Use a test to determine if the series converge.)

$$\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\sum_{n=0}^{\infty} \frac{1+e^n}{1+ne^n}$$

$$\sum_{n=1}^{\infty} \frac{n+2}{3n+4}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$