Math 112
Chapter 7.8: Improper Integrals

The goal of this section is to extend the concept of the integral to include some additional cases. These **improper integrals** are of two types.

TYPE I: INFINITE INTERVALS

If $\displaystyle\int_a^t f(x)\,dx$ exists for all $t > a$, then we define $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$.

If the limit exists, we say that $\displaystyle\int_a^\infty f(x)\,dx$ **converges**.

If the limit does not exist, we say that $\displaystyle\int_a^\infty f(x)\,dx$ **diverges**.

EXAMPLES:

$$\int_1^\infty \frac{1}{x^2}\,dx$$

$$\int_1^\infty \frac{1}{x}\,dx$$

For what positive values of p does the integral converge?

$$\int_1^\infty \frac{1}{x^p}\,dx$$

We can also define $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$.

$$\int_{-\infty}^1 e^{3x}\,dx$$

MORE EXAMPLES:

$$\int_0^\infty xe^{-x}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

$$\int_{4}^{\infty} \frac{1}{x^2-5x+6}\, dx$$

TYPE II: DISCONTINUOUS FUNCTIONS

If f is continuous on $[a, b)$ but discontinuous at $x = b$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$.

If f is continuous on $(a, b]$ but discontinuous at $x = a$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$.

EXAMPLES:

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx$$

$$\int_0^1 \frac{1}{x}\,dx$$

$$\int_0^1 \ln x \, dx$$

$$\int_0^5 \frac{y}{y-2} \, dy$$

$$\int_0^1 \frac{\ln z}{\sqrt{z}} \, dz$$

COMPARISON THEOREM

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

EXAMPLES:

$$\int_1^\infty e^{-x^2}\,dx$$

$$\int_0^\infty \frac{x^2}{x^6+9}\,dx$$

EXERCISES:

Determine if each of the integrals converges or diverges.

$$\int_0^\infty \frac{1}{(7x+3)^4}\,dx$$

$$\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$$

$$\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$$

$$\int_{-2}^1 \frac{1}{x^3}\,dx$$

$$\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$$