Math 112
Chapter 7.7: Numerical Integration.

The goal of this section is to approximate the value of definite integrals by using finite sums.

EXAMPLE:

$\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\,dt$ Suppose we want to know erf(1).

Estimate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using L_4, R_4, and M_4

It would be even more useful if we could bound the difference $\left| L_4 - \displaystyle\int_0^1 e^{-x^2}\,dx \right|$

We could then ask what size of N do we need to make L_N "good enough"

If we want a more accurate approximation, we might try using shapes other than rectangles. Estimate $\int_0^1 e^{-x^2}\,dx$ using 4 trapezoids. Call this T_4.

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Trapezoid Rule T_N

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{N-1}) + f(x_N)\right]$$

COMPARISON OF APPROXIMATIONS

We can compare the different approximations by looking at the errors. Define the error in using the Midpoint rule as

$$E_M = \int_a^b f(x)\,dx - M_N$$

We can define E_L, E_R, and E_T similarly.

To get a sense of the errors, we can approximate the value of a known integral, $\int_1^3 \frac{1}{x}\,dx$.

Several calculations by computer gives us...

N	E_L	E_R	E_T	E_M
5	-0.1449	0.121678	-0.01165	0.005755
10	-0.06962	0.063716	-0.00295	0.00147
20	-0.03407	0.032593	-0.00074	0.00037

We notice some trends...

1.

2.

3.

4.

ERROR BOUNDS

For the Midpoint Rule and Trapezoid rule we have the following formula to estimate the error

$$|E_M| \leq \frac{K(b-a)^3}{24N^2} \qquad |E_T| \leq \frac{K(b-a)^3}{12N^2}$$

where K is a number such that $|f''(x)| < K$ when x is in the interval $[a,b]$.

EXAMPLES:

Again consider the integral $\int_1^3 \frac{1}{x}\,dx$.

1. If $N = 20$, what is the largest that $|E_T|$ could be? How about $|E_M|$?
2. How large would N need to be if we want $|E_M| < 0.0001$?

Estimate the error E_T in the approximation T_8 for the following integral.

$$\int_0^1 \cos x^2 \, dx$$

EXERCISE: Consider the following integral:

$$\int_0^{\pi} \sin x \, dx$$

1. Find M_{10}, and T_{10}, and the corresponding errors E_M, and E_T.
2. Compute the error bounds using the formulas and then compare them with the true errors E_M, and E_T.
3. How large do we have to choose N so that $|E_T| < 0.00001$? Use the error bound formula.

If we want to make a more accurate approximation for a fixed N, we might approximate area with different shapes. **Simpson's Rule** uses parabolas to approximate a function f, and then estimates the integral of f by using integrals of quadratic functions.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_N = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\right]$$

where N **is an even integer.**

For Simpson's Rule we have a similar formula to estimate the error

$$|E_S| \le \frac{K(b-a)^5}{180N^4}$$

where K is a number such that $|f''''(x)| < K$ when x is in the interval $[a, b]$.

EXAMPLE:

For the integral $\displaystyle\int_1^3 \frac{1}{x}\,dx$ how large do we have to choose N so that $|E_S| < 0.0001$?

EXERCISE: Consider the following integral:

$$\int_0^\pi \sin x\,dx$$

1. Find S_{10} and the corresponding error E_S.
2. Compute the error bound using the formula and then compare them with the true error E_S.
3. How large do we have to choose N so that $|E_S| < 0.00001$? Use the error bound formula.