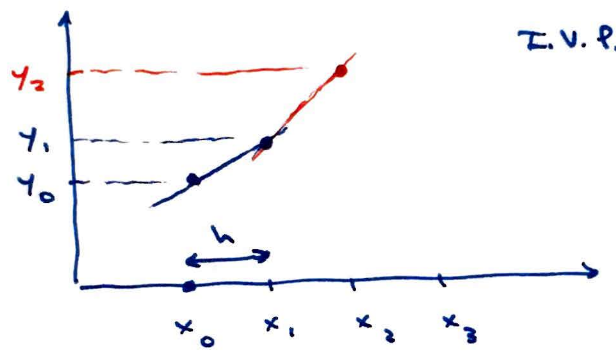
Math 112
Chapter 9.2: Euler's Method

The goal of **Euler's Method** is to approximate the solution of a differential equation at at a discrete set of points.

IDEA: We can follow the direction field for a short time in order to appoximate the solution.

Goal: Sequence of points $(x_1, y_1), (x_2, y_2), \ldots$ that approximates solution $y(x_1), y(x_2), \ldots$



I.V.P. $\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$

Start at I.C. (x_0, y_0). Let $\Delta x = h$

Set $\Delta y = y' \cdot \Delta x$ with $y' = f(x_0, y_0)$

Then $\Delta y = h f(x_0, y_0)$

So $x_1 = x_0 + h$ $\quad y_1 = y_0 + h f(x_0, y_0)$

Repeat now starting at (x_1, y_1) Same steps

$x_2 = x_0 + 2h$ $\quad y_2 = y_1 + h f(x_1, y_1)$

In general, use (x_n, y_n) and D.E. to advance to (x_{n+1}, y_{n+1})

Euler's Method:

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad \text{where} \quad x_n = x_0 + nh$$

EXAMPLES:

Use Euler's Method with $h = 0.1$ to approximate $y(0.3)$ where $y(x)$ is a solution to the initial value problem

$$\begin{cases} y' = y + x \\ y(0) = 1 \end{cases}$$

Need 3 steps size $h = 0.1$

Organize in table

x_n	y_n	$f(x_n, y_n)$
0	1	1
0.1	$1 + (0.1)(1)$ $= 1.1$	1.2
0.2	$1.1 + (0.1)(1.2)$ $= 1.22$	1.42
0.3	$1.22 + (0.1)(1.42)$ $= \boxed{1.362}$	

$y(0.3) \approx y_3 = 1.362$

Use Euler's Method with $h = 0.02$ to approximate $y(1.1)$ where $y(x)$ is a solution to the initial value problem

$$\begin{cases} y' = y\cos(x) \\ y(1) = 2 \end{cases}$$

$x_0 = 1$, $h = 0.02$, Need 5 steps to get to $x = 1.1$

x_n	y_n	$f(x_n, y_n)$
1	2	$2\cos(1) \doteq 1.08061$
1.02	$2 + 0.02(1.08\ldots)$ $= 2.021612\ldots$	1.058043
1.04	2.042773	1.034093
1.06	$\vdots$	$\vdots$
1.08	$\vdots$	$\vdots$
1.1	$\boxed{2.1033}$	

$y(1.1) \approx y_5 = 2.103272\ldots$

Chapter 9.3: Separable Differential Equations

A differential equation is called **separable** if can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

Solutions can be found by "separating" the variables and then integrating.

$$\frac{1}{f(y)}\frac{dy}{dx} = g(x)$$ Rewrite

$$\int \frac{1}{f(y)}\frac{dy}{dx}\,dx = \int g(x)\,dx$$ Integrate w.r.t. x

$$\int \frac{1}{f(u)}\,du = \int g(x)\,dx$$ Let $u = y$ $\quad du = \frac{dy}{dx}\,dx$

(In end same as $\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$)

EXAMPLES:

$$\frac{dy}{dx} = ky$$

Separate: $\frac{dy}{y} = k\,dx$ (If $y \neq 0$!)

Integrate: $\int \frac{dy}{y} = \int k\,dx$

$\ln|y| = kx + C$

(Only one constant.)

Solve for y?

$|y| = Be^{kx}$ $\quad (B > 0)$

$y = \pm Be^{kx}$

OR $y = Ae^{kx}$ $\quad (A \neq 0)$

Now notice that $y = 0$ is also a solution.

So $A = 0$ is also solution.

$y = Ae^{kx}$ for any A.

$$\frac{dy}{dx} = 3x^2y$$

$$\frac{dy}{y} = 3x^2\,dx$$

$$\int \frac{dy}{y} = \int 3x^2\,dx$$

$$\ln|y| = x^3 + C \longrightarrow y = e^{x^3}$$

(same algebra as above)

Solve the initial value problems:

$$\begin{cases} \dfrac{dy}{dt} = y^2 \sin t \\ y(0) = 4 \end{cases}$$

$$\frac{dy}{y^2} = \sin t \, dt$$

$$\int \frac{dy}{y^2} = \int \sin t \, dt$$

$$\frac{-1}{y} = -\cos t + C$$

$$y = \frac{-1}{C - \cos t}$$

Apply I.C.

$$4 = \frac{-1}{C-1}$$

$$C = 3/4$$

Solution to IVP:

$$y = \frac{1}{\cos t - 3/4}$$

$$\begin{cases} \dfrac{dy}{dt} = 3 - 4y \\ y(0) = 10 \end{cases}$$

$$\frac{dy}{3-4y} = dt$$

$$\int \frac{dy}{3-4y} = \int dt$$

$(u = 3-4y)$

$$-\frac{1}{4} \ln|3-4y| = t + C$$

$$\ln|3-4y| = -4t - 4C$$

$$3 - 4y = Ae^{-4t} \quad (A = \pm e^{-4C})$$

$$-4y = -3 + Ae^{-4t}$$

$$y = \frac{3}{4} + Be^{-4t} \quad \left(B = -\frac{A}{4}\right)$$

Apply I.C.

$$10 = \frac{3}{4} + B \qquad B = \frac{37}{4}$$

Soln to I.V.P

$$y = \frac{1}{4}\left(3 + 37e^{-4t}\right)$$

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{-x}{y} \\ y(1) = -3 \end{cases}$$

$$\int y \, dy = \int -x \, dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$x^2 + y^2 = A \quad (A = 2C)$$

integral curves are circles

Apply I.C.

$$1^2 + (-3)^2 = A$$

So $x^2 + y^2 = 10$

And can solve for y

$$y^2 = 10 - x^2$$

$$y = -\sqrt{10 - x^2}$$

(Choose (−) root to match I.C.)

NOTE: It may not always be possible to "solve" explicity for $y(x)$.

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

$$\int 1 - y^2 \, dy = \int x^2 \, dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + C$$

Cannot solve for y, but equation still defines relation between x & y.

$$x^3 + y^3 - 3y = A \quad \text{(See Desmos to plot)}$$

APPLICATIONS:

Solve the initial value problem for the velocity of a falling object that is subject to a drag force

$$\begin{cases} m\dfrac{dv}{dt} = kv - mg \\ v(0) = v_0 \end{cases}$$

Assume $v, v_0 < 0$ and $k < 0$ so drag opposes gravity

$$\frac{dv}{dt} = \frac{k}{m} v - g$$

$$\frac{dv}{dt} = \frac{k}{m}\left(v - \frac{mg}{k}\right)$$

$$\int \frac{dv}{v - \frac{mg}{k}} = \int \frac{k}{m} \, dt$$

$$\ln \left| v - \frac{mg}{k} \right| = \frac{kt}{m} + C$$

$$v = \frac{mg}{k} + Ae^{\frac{kt}{m}}$$

Apply I.C.

$$v_0 = \frac{mg}{k} + A$$

$$A = v_0 - \frac{mg}{k}$$

$$v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right) e^{kt/m}$$

As $t \to \infty$ $e^{kt/m} \to 0$ $(k < 0)$

and $v \to \frac{mg}{k}$ (terminal velocity)

A tank contains 200 L of water with 0.8 kg of salt. At $t = 0$, a solution with concentration 3.5 g/L begins flowing into the tank at a rate of 2 L/min. If the tank is kept well-mixed and is drained at a rate of 2 L/min, find the amount of salt in the tank as a function of time.

Let $m(t)$ = amount of salt in tank in g

$$\frac{dm}{dt} = \underbrace{(2\,\text{L/min})(3.5\,\text{g/L})}_{\text{rate in}} - \underbrace{(2\,\text{L/min})\left(\frac{m\text{ g}}{200\text{ L}}\right)}_{\text{rate out}}$$

I.V.P $\begin{cases} \dfrac{dm}{dt} = 7 - \dfrac{m}{100} \\ m(0) = 800 \end{cases}$

$$\frac{dm}{dt} = \frac{1}{100}(700 - m)$$

$$\int \frac{dm}{700-m} = \int \frac{1}{100}\,dt$$

$$-\ln|700 - m| = \frac{t}{100} + C$$

$$700 - m = Ae^{-t/100}$$

$$m = 700 - Ae^{-t/100}$$

I.C.: $800 = 700 - A$

$$A = -100$$

$$m(t) = 700 + 100e^{-t/100}$$

As $t \to \infty$ $m \to 700$ g

$\dfrac{700\text{ g}}{200\text{ L}} = 3.5$ g/L equilibrium concentration.

Find the family of curves that is orthogonal to every member of the family $y = \dfrac{k}{x}$.

Want slopes that are negative reciprocals at intersections

$\dfrac{dy}{dx} = -\dfrac{k}{x^2}$ so we want y such that $\dfrac{dy}{dx} = \dfrac{x^2}{k}$

where $k = xy$

Require that $\dfrac{dy}{dx} = \dfrac{x}{y}$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$x^2 - y^2 = A$ for any A is orthogonal to $y = \dfrac{k}{x}$ for any k.

(See Desmos.)