Math 112
Chapter 9.3: Population Models

In this section, we consider a dynamic population $P(t)$, and examine differential equations that might be used to describe how the population changes with time.

$$\frac{dP}{dt} = rP$$

Where r is the *per capita growth rate*. If r is constant, the population grows exponentially, but we want to consider also what happens if r is not constant.

EXAMPLE: Seasonal growth $r(t) = 0.05\cos(t)$

EXAMPLE: Diminishing resources $r(t) = t^{-k}$

$\dfrac{dP}{dt} = \dfrac{P}{t}$

$$\frac{dP}{dt} = \frac{P}{t^2}$$

It is also reasonable to assume that the per capita growth is a function of P itself

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where $k > 0$ and $M > 0$ are constants is called the **logistic equation**.

Solution of initial value problem

$$
\begin{cases}
\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right) \\
P(0) = P_0
\end{cases}
$$

.

EXAMPLES:

Suppose a population follows the logistic growth with a carrying capacity of $M = 5000$ and $k = 0.05$/year.

1. If the population starts at 200, what is the population 3 years later?
2. How long does it take for the population to reach size 4000?
3. Compare these numbers to a population that grows at a *constant* 0.05/year and starts at the same size.

Another population experiences logistic growth with $M = 9500$. If the initial population is 1100 and five years later the population as grown to 2300, find the size of the population ten years from the beginning.

Let $a(t)$, and $b(t)$ be two populations of bacteria that experience constant relative growth at rates α and β (so a and b grow exponentially.) Suppose that both populations live in the same evironment and that samples of the populations can only detect the *proportion* of population a. Let $p(t)$ be the proportion.

$$p(t) = \frac{a(t)}{a(t) + b(t)}$$

1. Show that p follows the logistic equation.

2. Find the solution of the logistic equation by using the formulas for a and b.

An contagious disease is spreading among a population. Let $S(t)$ be the number of individuals who are susceptible to the disease, and $I(t)$ be the number who are infected. One possible model for the spread of the disease is known as the SI-model.

$$\begin{aligned}\frac{dS}{dt} &= -kSI \\ \frac{dI}{dt} &= kSI\end{aligned}$$

The per capital change of each population is proportional to the other. Determine a formula for $I(t)$ by showing that it follows logistic growth.

A model for a population that experiences a constant removal rate H can be found by making a modification to the logistic equation. (*The removal may represent fish or plants being harvested from the population.*)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - H$$

If $M = 20000$, $k = 0.03/\text{year}$, and $H = 150$ individuals/year, what is the lowest population that could survive such harvesting?