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Homework 4

1) (5.1.2) Solve the eigenvalue equation $\hat{H}|E\rangle = \frac{\hat{p}^2}{2m}|E\rangle = E|E\rangle$ in the X basis to regain $|E\rangle = \beta|p=(2mE)^{1/2}\rangle + \gamma|p=-(2mE)^{1/2}\rangle$

In X basis, $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ $\qquad \frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ $\qquad |E\rangle = \psi_E(x)$

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x}\psi = E\psi_E \qquad \frac{\partial^2\psi_E}{\partial x^2} = \frac{-2mE}{\hbar^2}\psi_E \qquad \text{Let } \omega^2 = \frac{2mE}{\hbar^2} \qquad \omega = \pm\frac{\sqrt{2mE}}{\hbar}$$

$$\frac{\partial^2\psi_E}{\partial x^2} = -\omega^2\psi_E \rightarrow \psi_E(x) = a\exp[i\omega x] + b\exp[-i\omega x]$$

Let $a = \frac{\beta}{\sqrt{2\pi\hbar}}$ and $b = \frac{\gamma}{\sqrt{2\pi\hbar}}$ $\rightarrow$ $\boxed{\psi_E(x) = \frac{\beta}{(2\pi\hbar)^{1/2}}\exp\left(\frac{i\sqrt{2mE}\,x}{\hbar}\right) + \frac{\gamma}{(2\pi\hbar)^{1/2}}\exp\left(\frac{-i\sqrt{2mE}\,x}{\hbar}\right)}$

If $E<0$, then $\psi_{E<0}(x) = \frac{\beta}{(2\pi\hbar)^{1/2}}\exp\left(\frac{i\sqrt{2m(-E)}\,x}{\hbar}\right) + \frac{\gamma}{(2\pi\hbar)^{1/2}}\exp\left(\frac{-i\sqrt{2m(-E)}\,x}{\hbar}\right) =$

$$\psi_{E<0}(x) = \frac{\beta}{(2\pi\hbar)^{1/2}}\exp\left(\frac{-\sqrt{2m|E|}\,x}{\hbar}\right) + \frac{\gamma}{(2\pi\hbar)^{1/2}}\exp\left(\frac{\sqrt{2m|E|}\,x}{\hbar}\right) \qquad \psi_{E<0}\rightarrow\infty \text{ as } x\rightarrow\pm\infty$$

Since $\psi_{E<0}$ blows up at $\pm\infty$, it can't be normalized to unity or the Dirac delta function and therefore doesn't belong to the Hilbert space

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2) (5.1.3) $U(t) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(\frac{i\hbar t}{2m}\right)^\ell\frac{d^\ell}{dx^\ell}$, $\psi(x,0) = \frac{\exp(-x^2/2)}{\pi^{1/4}} = \pi^{-1/4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(2)^n}$, $\psi(t) = U(t)\psi(0)$

$$\ell=0:\quad 1\cdot\pi^{-1/4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(2)^n} = \pi^{-1/4}\sum_n\frac{(-1)^n x^{2n}}{n!(2)^n} =$$

$$\ell=1:\quad \left(\frac{i\hbar t}{2m}\right)\frac{d^2}{dx^2}\left[\pi^{-1/4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(2)^n}\right] = \left(\frac{i\hbar t}{2m}\right)\pi^{-1/4}\sum_n\frac{(-1)^n\overbrace{2n(2n-1)}^{0\text{ for } n<1}x^{2n-2}}{n!(2)^n} = \pi^{-1/4}\left(\frac{i\hbar t}{2m}\right)\sum_{n=1}\frac{(-1)^n(2n)!\,x^{2n-2}}{n!\,2^n(2n-2)!}$$

$$\ell=2:\quad \frac{1}{2!}\left(\frac{i\hbar t}{2m}\right)^2\frac{d^4}{dx^4}\left[\pi^{-1/4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(2)^n}\right] = \frac{1}{2!}\left(\frac{i\hbar t}{2m}\right)^2\pi^{-1/4}\sum_n\frac{(-1)^n\overbrace{(2n)(2n-1)(2n-2)(2n-3)}^{0\text{ for } n<2}x^{2n-4}}{n!(2)^n}$$

$$= \pi^{-1/4}\frac{1}{2!}\left(\frac{i\hbar t}{2m}\right)^2\sum_{n=2}\frac{(-1)^n(2n)!\,x^{2n-4}}{n!\,2^n(2n-4)!} = \pi^{-1/4}\frac{1}{\ell!}\left(\frac{i\hbar t}{2m}\right)^\ell\sum_{n=\ell}^{\infty}\frac{(-1)^n(2n)!\,x^{2n-2\ell}}{n!\,2^n(2n-2\ell)!}$$

Let $r=n-\ell$
$n=r+\ell$

$$U(t)\psi(0) = \pi^{-1/4}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{\ell!}\left(\frac{i\hbar t}{2m}\right)^\ell\left(\frac{-1}{2}\right)^{r+\ell}\frac{(2r+2\ell)!\,x^{2r}}{(r+\ell)!\,(2r)!}$$

$$\frac{(2k)!}{k!} = \frac{2k \cdot (2k-1)(2k-2)\cdots}{k \cdot \qquad (k-1)\cdot\cdots} = 2^k (2k-1)(2k-3)\cdots$$

For $n=\ell$ term: $\pi^{-1/4} \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\frac{i\hbar t}{2m}\right)^{\ell} \left(\frac{-1}{2}\right)^{\ell} \frac{(2\ell!)}{(\ell!)} = \pi^{-1/4} \sum_{\ell=0}^{\infty} \frac{(2\ell)!}{(\ell!)(\ell!)(-2)^{\ell}} \Big($

To be continued...

3) (5.2.2) a) Show for any normalized $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$

$$|\psi\rangle = \sum_{i=0}^{\infty} a_i|\psi_i\rangle \qquad H|\psi_i\rangle = E_i|\psi_i\rangle \qquad H|\psi\rangle = \sum_{i=0}^{\infty} E_i a_i |\psi_i\rangle$$

$$\langle\psi|H|\psi\rangle = \sum_{ij}^{\infty} \langle\psi_i|a_i^* H a_j|\psi_j\rangle = \sum_{ij}^{\infty} a_i^* a_j \langle\psi_i|E_j|\psi_j\rangle = \sum_{ij}^{\infty} a_i^* a_j E_j \delta_{ij}$$

$$\langle\psi|H|\psi\rangle = \sum_{i=0}^{\infty}|a_i|^2 E_i = \langle E\rangle$$

$\langle E\rangle$ is the expectation value of the energy, i.e. the average over all the possible energies in an ensemble. $\langle E\rangle = E_0$ if there is only one possible energy state and $\langle E\rangle > E_0$ if multiple energy states are possible.

b) $\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left(\frac{-\alpha x^2}{2}\right) \qquad E(\alpha) = \langle\psi_\alpha|H|\psi_\alpha\rangle \qquad H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} - |V(x)|$

$$H|\psi_\alpha\rangle = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - |V(x)|\right)|\psi_\alpha\rangle = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\left[\left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left(\frac{-\alpha x^2}{2}\right)\right] - |V(x)|\left[\left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left(\frac{-\alpha x^2}{2}\right)\right]$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\alpha}{\pi}\right)^{1/4}\frac{d}{dx}\left(-\alpha x\exp\left[\frac{-\alpha x^2}{2}\right]\right) - |V(x)|\left[\left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left(\frac{-\alpha x^2}{2}\right)\right]$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\alpha}{\pi}\right)^{1/4}\left(\alpha^2 x^2 - \alpha\right)\exp\left(\frac{-\alpha x^2}{2}\right) - |V(x)|\left[\left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left(\frac{-\alpha x^2}{2}\right)\right] = \left[\frac{-\hbar^2}{2m}(\alpha^2x^2 - \alpha) - |V(x)|\right]\psi_\alpha(x)$$

$$E(\alpha) = \langle\psi_\alpha|H|\psi_\alpha\rangle = \int_{-\infty}^{\infty}\left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left(\frac{-\alpha x^2}{2}\right)\left[-\frac{\hbar^2}{2m}(\alpha^2x^2-\alpha) - |V(x)|\right]\left(\frac{\alpha}{\pi}\right)^{1/4}\exp\left(\frac{-\alpha x^2}{2}\right)dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}\left[\frac{-\hbar^2}{2m}(\alpha^2x^2-\alpha) - |V(x)|\right]\exp(-\alpha x^2)\,dx$$

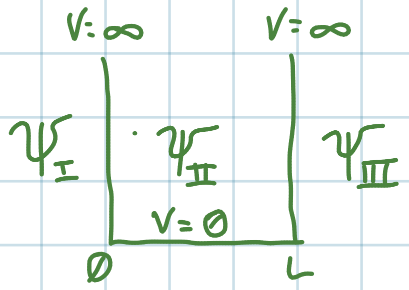
$$= \left(\frac{\alpha}{\pi}\right)^{1/2}\left(\frac{-\hbar^2}{2m}\right)\left[\alpha^2\underbrace{\left(\frac{1}{2\alpha}\right)\left(\frac{\pi}{\alpha}\right)^{1/2}}_{A.2.3} - \alpha\underbrace{\left(\frac{\pi}{\alpha}\right)^{1/2}}_{A.2.2}\right] - \left(\frac{\alpha}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}|V(x)|\exp(-\alpha x^2)\,dx$$

$$E(\alpha) = \frac{-\hbar^2}{2m}\left[\frac{\alpha}{2} - \alpha\right] - \left(\frac{\alpha}{\pi}\right)^{1/2}\int|V(x)|\exp(-\alpha x^2)\,dx = \frac{\hbar^2\alpha}{2m} - \left(\frac{\alpha}{\pi}\right)^{1/2}\int|V(x)|\exp(-\alpha x^2)\,dx$$

As $\alpha \to 0$, $\frac{\hbar^2\alpha}{2m} \to 0$ and $E(\alpha) \approx -\left(\frac{\alpha}{\pi}\right)^{1/2}\int|V(x)|\,dx$

$$\boxed{\text{Since both } \alpha \text{ and } |V(x)| \text{ are positive } E(\alpha\to 0) < 0}$$

4) (5.2.5) Box from $x=0$ to L; show $\psi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right)$ and $E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$



$\psi_I = \psi_{III} = 0$ For region II $\hat{H} = \frac{\hat{p}^2}{2m}$ $(V_{II} = 0)$

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi_{II} = E\psi_{II} \qquad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \qquad \text{Let } k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \rightarrow \psi(x) = A\cos(kx) + B\sin(kx)$$

$$\psi(0) = 0 \rightarrow A = 0 \qquad \psi(L) = 0 \rightarrow \sin(kL) = 0 \rightarrow n\pi = kL \rightarrow k = \frac{n\pi}{L}$$

$$\psi(x) = B\sin\left(\frac{n\pi x}{L}\right) \qquad 1 = \int \psi^*\psi\, dx = B^2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx =$$

$$B^2\left(\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{L}\right)}{4(n\pi/L)}\right)\Bigg|_0^L = B^2\left(\frac{L}{2}\right) = 1 \rightarrow B = \sqrt{\frac{2}{L}} \rightarrow \boxed{\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{L} \rightarrow \frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2} \rightarrow \boxed{E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}}$$

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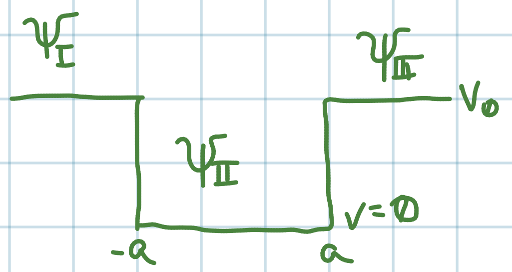
5) (5.2.6) $V(x) = \begin{cases} 0 & x \le |a| \\ V_0 & x \ge |a| \end{cases}$ ① Show that even solutions have energies that satisfy $k\tan ka = \kappa$ and the odd solutions satisfy $k\cot ka = -\kappa$ where k and $i\kappa$ are the real/complex wave numbers inside/outside the well. Note $k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2}$



① I $\hat{H}_I = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_0$ $\quad \left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right)\psi_I = E\psi_I \quad \frac{d^2\psi_I}{dx^2} + \frac{2m(E-V_0)}{\hbar^2}\psi_I = 0$

Let $\kappa = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ $\quad \psi_I(x) = A\exp(-\kappa x) + B\exp(\kappa x)$ (Eq 5.2.4)

$$\psi_I(-\infty) = 0 \rightarrow A = 0 \rightarrow \psi_I(x) = B\exp(\kappa x)$$

III Same $\hat{H}$ as I $\rightarrow \psi_{III}(x) = F\exp(-\kappa x) + G\exp(\kappa x)$ $\quad \psi_{III}(\infty) = 0 \rightarrow G = 0$

$$\psi_{III}(x) = F\exp(-\kappa x)$$

(II) $\psi_{II}(x) = Ce^{-ikx} + De^{ikx}$ $\qquad k=\sqrt{\frac{2mE}{\hbar^2}}$

Continuity at $x=\pm a$ for ψ and ψ'

$\psi_I(-a)=\psi_{II}(-a) \quad \psi_{II}(a)=\psi_{III}(a) \quad \psi_I'(-a)=\psi_{II}'(-a) \quad \psi_I'(a)=\psi_{III}'(a)$

$\boxed{\psi \text{ even} \to B=F \text{ and } C=D}$ $\quad \psi_{II}(x) = C(\exp(-ikx)+\exp(ikx)) = 2C\cos(kx)$

$(x=-a)$ $\quad B\exp(-\kappa a) \overset{①}{=} 2C\cos(-ka) \qquad B\kappa\exp(-\kappa a) \overset{②}{=} -2kC\sin(-ka)$

$②/① \to \kappa = \frac{-k\sin(-ka)}{\cos(-ka)} = k\frac{\sin(ka)}{\cos(ka)} \Rightarrow \boxed{\kappa = k\tan(ka)}$

$\boxed{\psi \text{ odd} \to -B=F \text{ and } C=-D}$ $\quad \psi_{II}(x) = D(\exp(ikx)-\exp(-ikx)) = 2iD\sin(kx)$

$(x=-a)$ $\quad B\exp(-\kappa a) \overset{①}{=} 2iD\sin(-ka) \qquad B\kappa\exp(-\kappa a) \overset{②}{=} 2iDk\cos(-ka)$

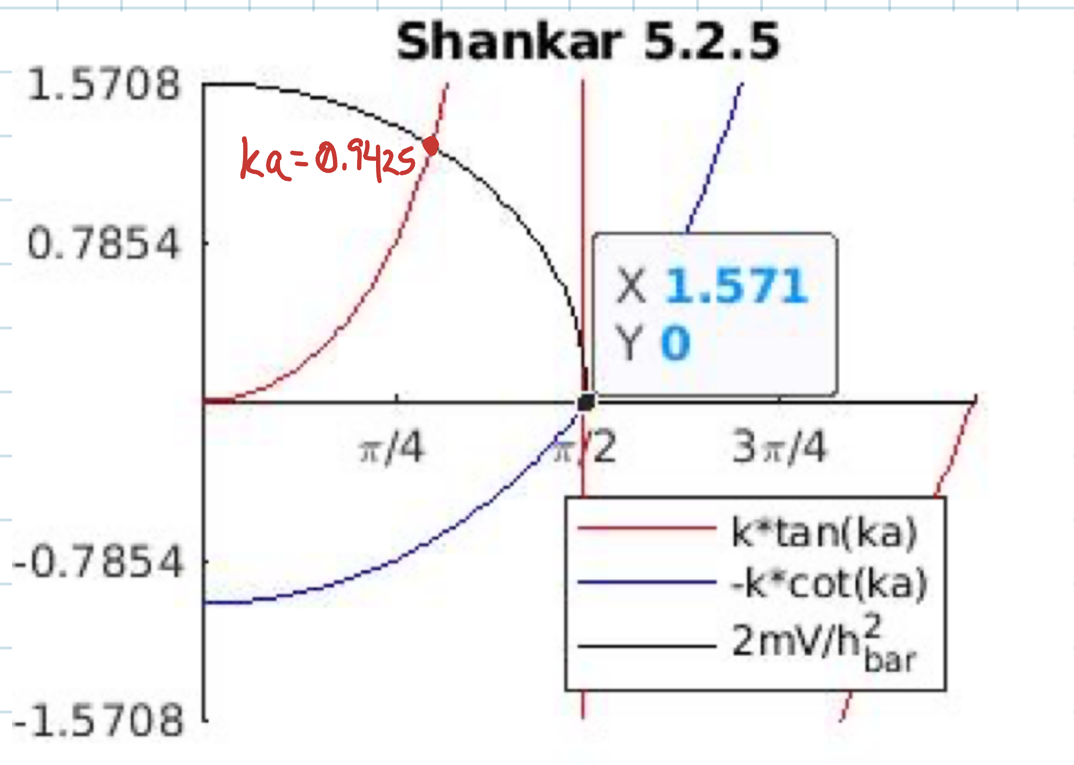
$②/① \to \kappa = \frac{k\cos(-ka)}{\sin(-ka)} = k\frac{\cos(ka)}{-\sin(ka)} \to \boxed{k\cot(ka) = -\kappa}$

As $V_0\to\infty$, $\kappa \to \sqrt{\frac{-2mV_0}{\hbar^2}} \to i(-\infty)$ and $\psi_I = \psi_{III} = e^{-\infty} = 0 \to \psi(|x|>a) = 0$

From above if $\kappa \to -\infty$ $\quad \cos(kx)=0$ for ψ_{even} and $\sin(kx)=0$ for ψ_{odd}

and $\psi(x) = \begin{cases} A\sin(kx) = A\sin\left(\frac{n\pi x}{a}\right) & n \text{ even}\\ A\cos(kx)=A\cos\left(\frac{n\pi x}{a}\right) & n\text{ odd}\end{cases} \qquad A = \left(\frac{2}{L}\right)^{1/2}$

b)



c) $k\tan(ka)$ will intersect a circle of any radius

$-k\cot(ka)$ will not intersect a circle of radius $< \frac{\pi}{2}$

$\cot(ka) < 0 \to ka \le \frac{\pi}{2} \qquad \kappa = 0$

$k^2+\kappa^2 = \frac{2mV_0}{\hbar^2} \to \frac{\pi^2}{4a^2} \le \frac{2mV_0}{\hbar^2} \Rightarrow \boxed{V_0 \ge \frac{\pi^2\hbar^2}{8ma^2}}$

On this circle $\quad ka = 0.9425 = \sqrt{\frac{2mE}{\hbar^2}}\,a$

$E = \frac{\hbar^2}{2ma^2}(0.9425)^2 = \boxed{0.888\frac{\hbar^2}{2ma^2}}$

6) (5.3.1) $V = V_r(x) - iV_i$ $\qquad \hat{H} = \frac{\hat{P}}{2m} + V(\hat{x}) = \frac{\hat{P}}{2m} + V_r(\hat{x}) - iV_i$

$$\hat{H}^* = \frac{\hat{P}^*}{2m} + V_r(\hat{x}^*) - (iV_i)^* = \frac{\hat{P}}{2m} + V_r(\hat{x}) + iV_i \neq \hat{H} \Rightarrow \boxed{\text{Not Hermitian}}$$

$$① \; i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \Psi + (V_r(x) - iV_i)\Psi \xrightarrow{\text{mult by } \Psi^*} i\hbar \Psi^* \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2 \Psi^*}{2m} \nabla^2 \Psi + (V_r(x) - iV_i)\Psi\Psi^*$$

$$② \; -i\hbar \frac{\partial \Psi^*}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \Psi^* + (V_r(x) + iV_i)\Psi^* \xrightarrow{\text{mult by } \Psi} -i\hbar \Psi \frac{\partial \Psi^*}{\partial t} = \frac{-\hbar^2 \Psi}{2m} \nabla^2 \Psi^* + (V_r(x) + iV_i)\Psi\Psi^*$$

$$①-② \Rightarrow i\hbar \frac{\partial}{\partial t}(\Psi\Psi^*) = \frac{-\hbar^2}{2m}(\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) - 2iV_i \Psi\Psi^* \qquad P(r,t) = \Psi\Psi^*$$

$$\frac{\partial P}{\partial t} = \frac{-\hbar}{2mi} \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) - \frac{2V_i}{\hbar} P(\vec{r},t) \qquad \text{Let } \vec{j} = \frac{\hbar}{2mi}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)$$

$$\frac{d}{dt} \int P(\vec{r},t)\, d^3\vec{r} = \int -\nabla \cdot \vec{j} - \frac{2V_i}{\hbar} P(\vec{r},t)\, d^3\vec{r} = -\int_S \vec{j} \cdot d\vec{S} - \frac{2V_i}{\hbar} \int P(\vec{r},t)\, d^3\vec{r}$$

Let $\mathcal{P} = \int P(\vec{r},t)\, d^3\vec{r}$; As $S \to \infty \to \int_S \vec{j} \cdot d\vec{S} = 0 \to \frac{d\mathcal{P}}{dt} = \frac{-2V_i}{\hbar} \mathcal{P}$

$$\boxed{\mathcal{P}(t) = \mathcal{P}_0 \exp\left(\frac{-2V_i t}{\hbar}\right)}$$

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7) (5.3.4) $\Psi = A\exp\left(\frac{ipx}{\hbar}\right) + B\exp\left(\frac{-ipx}{\hbar}\right)$, show $j = (|A|^2 - |B|^2)\frac{p}{m}$ $\qquad \vec{j} = j\hat{x}$

$$\vec{j} = \frac{\hbar}{2mi}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \qquad \nabla \Psi = \frac{Aip}{\hbar} \exp\left(\frac{ipx}{\hbar}\right) - \frac{Bip}{\hbar} \exp\left(\frac{-ipx}{\hbar}\right)$$

$$j = \frac{\hbar}{2mi}\Bigg[\left(A\exp\left(\frac{-ipx}{\hbar}\right) + B\exp\left(\frac{ipx}{\hbar}\right)\right)\left(\frac{Aip}{\hbar}\exp\left(\frac{ipx}{\hbar}\right) - \frac{Bip}{\hbar}\exp\left(\frac{-ipx}{\hbar}\right)\right) -$$

$$\left(A\exp\left(\frac{ipx}{\hbar}\right) + B\exp\left(\frac{-ipx}{\hbar}\right)\right)\left(\frac{-Aip}{\hbar}\exp\left(\frac{-ipx}{\hbar}\right) + \frac{Bip}{\hbar}\exp\left(\frac{ipx}{\hbar}\right)\right)\Bigg] =$$

$$= \frac{\hbar}{2mi}\Bigg[\frac{A^2 ip}{\hbar} - \frac{ABip}{\hbar}\exp\left(\frac{-2ipx}{\hbar}\right) + \frac{ABip}{\hbar}\exp\left(\frac{2ipx}{\hbar}\right) - \frac{B^2 ip}{\hbar} +$$

$$\frac{A^2 ip}{\hbar} - \frac{ABip}{\hbar}\exp\left(\frac{2ipx}{\hbar}\right) + \frac{ABip}{\hbar}\exp\left(\frac{-2ipx}{\hbar}\right) - \frac{B^2 ip}{\hbar}\Bigg] =$$

$$j = \frac{\hbar}{2mi}\left[\frac{2A^2 ip}{\hbar} - \frac{2B^2 ip}{\hbar}\right] = \boxed{(|A|^2 - |B|^2)\frac{p}{m}}$$

8) (7.3.2) $H_3(y) = -12y + 8y^3$ $\qquad H_4(y) = 12 - 48y^2 + 16y^4$

$$C_{n+2} = C_n \frac{(2n+1-2\varepsilon)}{(n+2)(n+1)} \qquad \varepsilon_n = \frac{2n+1}{2}$$

For H_3: $\varepsilon_3 = \frac{7}{2}$ $\quad C_1 = -12$ $\quad C_3 = C_1 \frac{(2(1)+1-7)}{(3)(2)} = \frac{-4}{6} C_1 = \frac{-2}{3}(-12) = \boxed{8}$

For H_4: $\varepsilon_4 = \frac{9}{2}$ $\quad C_0 = 12$ $\quad C_2 = C_0 \frac{(2(0)+1-9)}{(2)(1)} = \frac{-8}{2} C_0 = -4(12) = \boxed{-48}$

$$C_4 = C_2 \frac{(2(2)+1-9)}{(4)(3)} = \frac{-4}{12} C_2 = \frac{-1}{3}(-48) = \boxed{16}$$

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9) (7.3.4) $A_n = \left[\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right]^{1/4}$ $\quad H_n'(y) = 2nH_{n-1}$ $\quad H_{n+1}(y) = 2yH_n - 2nH_{n-1}$

$\psi_n = A_n \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right]$ $\quad$ Let $y = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$ $\quad \psi_n = A_n \exp\left(\frac{-y^2}{2}\right) H_n(y)$

$$\langle n'|X|n\rangle = \int_{-\infty}^{\infty} A_{n'} \exp\left(\frac{-y^2}{2}\right) H_{n'}(y)\, x\, A_n \exp\left(\frac{-y^2}{2}\right) H_n(y)\, dx$$ $\boxed{x = \left(\frac{\hbar}{m\omega}\right)^{1/2} y,\ dx = \left(\frac{\hbar}{m\omega}\right)^{1/2} dy}$

$$= A_{n'} A_n \cdot \left(\frac{\hbar}{m\omega}\right) \int y H_{n'}(y) H_n(y) \exp(-y^2)\, dy$$ $\boxed{yH_n = \frac{1}{2}H_{n+1} + nH_{n-1}}$

$$= A_{n'} A_n \cdot \left(\frac{\hbar}{m\omega}\right) \int H_{n'}(y) \left[\frac{1}{2} H_{n+1}(y) + n H_{n-1}(y)\right] \exp(-y^2)\, dy$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \left(\frac{1}{2^n 2^{n'} (n!)(n'!)}\right)^{1/2} \left(\frac{\hbar}{m\omega}\right) \int \left[\frac{1}{2} H_{n'} H_{n+1} + n H_{n'} H_{n-1}\right] \exp(-y^2)\, dy$$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{\hbar}{m\omega}\right)^{1/2} \left(\frac{1}{2^n 2^{n'} (n!)(n'!)}\right)^{1/2} \left[\frac{1}{2} \delta_{n+1,n'} \left(\pi^{1/2} 2^{n'} (n'!)\right) + n \delta_{n-1,n'} \left(\pi^{1/2} 2^{n'} (n'!)\right)\right]$$

$$= \left(\frac{\hbar}{m\omega}\right)^{1/2} \left[\frac{2^{n'}(n'!)}{2\sqrt{2^n 2^{n'} (n!)(n'!)}} \delta_{n+1,n'} + \frac{n\, 2^{n'}(n'!)}{\sqrt{2^n 2^{n'} (n!)(n'!)}} \delta_{n-1,n'}\right]$$

$$= \left(\frac{\hbar}{m\omega}\right)^{1/2} \left[\sqrt{\frac{2^{n'}(n'!)}{2^{n+2}(n!)}} \delta_{n+1,n'} + n\sqrt{\frac{2^{n'}(n'!)}{2^n (n!)}} \delta_{n-1,n'}\right]$$

$$= \left(\frac{\hbar}{m\omega}\right)^{1/2} \left[\sqrt{\frac{2^{n+1}(n+1)n!}{2^{n+2}(n!)}} \delta_{n+1,n'} + n\sqrt{\frac{2^{n-1}(n-1)!}{2^n (n!)}} \delta_{n-1,n'}\right] = \left(\frac{\hbar}{m\omega}\right)^{1/2} \left[\sqrt{\frac{n+1}{2}} \delta_{n+1,n'} + \sqrt{\frac{n}{2}} \delta_{n-1,n'}\right]$$

$$\boxed{\langle n'|X|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} \left[\sqrt{n+1}\, \delta_{n+1,n'} + \sqrt{n}\, \delta_{n-1,n'}\right]}$$

$$\langle n'|P|n\rangle = \int_{-\infty}^{\infty} A_{n'} \exp\left(\frac{-y^2}{2}\right) H_{n'}(y) \left(-i\hbar \frac{d}{dx}\right) A_n \exp\left(\frac{-y^2}{2}\right) H_n(y)\, dx$$

$$= A_{n'} A_n \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{2}\right) H_{n'}(y) \left(-i\hbar \frac{d}{(\hbar/m\omega)^{1/2} dy}\right) \exp\left(\frac{-y^2}{2}\right) H_n(y) \left(\frac{\hbar}{m\omega}\right)^{1/2} dy$$

$$= -i\hbar A_{n'} A_n \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{2}\right) H_{n'}(y) \exp\left(\frac{-y^2}{2}\right) \left[H'_n(y) - y H_n(y)\right] dy$$

$$= -i\hbar \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{1}{2^n 2^{n'} (n!)(n'!)}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-y^2) H_{n'}(y) \left[2n H_{n-1}(y) - \left(\frac{1}{2} H_{n+1}(y) + n H_{n-1}(y)\right)\right] dy$$

$$= -i\hbar \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{1}{2^n 2^{n'} (n!)(n'!)}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-y^2) H_{n'}(y) \left[n H_{n-1}(y) - \frac{1}{2} H_{n+1}(y)\right] dy$$

$$= -i\hbar \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{1}{2^n 2^{n'} (n!)(n'!)}\right)^{1/2} \left[n \delta_{n',n-1} \sqrt{\pi}\, 2^{n-1} (n-1)! - \frac{1}{2} \delta_{n',n+1} \sqrt{\pi}\, 2^{n+1} (n+1)!\right]$$

$$= -i (\hbar m\omega)^{1/2} \left[\frac{2^{n-1}(n!)}{\sqrt{2^n 2^{n'} (n!)(n'!)}} \delta_{n',n-1} - \frac{2^n (n+1)!}{\sqrt{2^n 2^{n'} (n!)(n'!)}} \delta_{n',n+1}\right]$$

$$= -i (\hbar m\omega)^{1/2} \left[\frac{2^{n-1}(n!)}{\sqrt{2^n 2^{n-1} (n!)(n-1)!}} \delta_{n',n-1} - \frac{2^n (n+1)!}{\sqrt{2^n 2^{n+1} (n!)(n+1)!}} \delta_{n',n+1}\right]$$

$$= -i (\hbar m\omega)^{1/2} \left[\sqrt{\frac{2^{n-1}\, n!}{2^n (n-1)!}} \delta_{n',n-1} - \sqrt{\frac{2^n (n+1)!}{2^{n+1} (n!)}} \delta_{n',n+1}\right] = -i (\hbar m\omega)^{1/2} \left[\sqrt{\frac{n}{2}} \delta_{n',n-1} - \sqrt{\frac{n+1}{2}} \delta_{n',n+1}\right]$$

$$\boxed{\langle n'|P|n\rangle = i \left(\frac{\hbar m\omega}{2}\right)^{1/2} \left[\sqrt{n+1}\, \delta_{n',n+1} - \sqrt{n}\, \delta_{n',n-1}\right]}$$

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10) (7.3.6) $V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2 & x > 0 \\ \infty & x \le 0 \end{cases}$ Let $\psi_+ = \psi(x > 0)$ and $\psi_- = \psi(x \le 0)$

Boundary conditions: $\boxed{\psi(x<0) = 0 \quad \psi(x \to \infty) = 0 \quad \psi(0) = 0}$

For $x > 0$, $\psi_n(x) = A_n \exp\left(\frac{-m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right]$ $\quad A_n = \left[\frac{m\omega}{\pi \hbar 2^{2n} (n!)^2}\right]^{1/4}$

$\psi_n(0) = 0$ only for even values of n (H_n for n odd has constant terms)

$$\boxed{E_n = \left(n + \frac{1}{2}\right) \hbar\omega \quad \text{for } n \text{ odd}}$$

$$\boxed{\psi_n(x) = \left[\frac{m\omega}{\pi \hbar 2^{2n} (n!)^2}\right]^{1/4} \exp\left(\frac{-m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right] \quad \text{for } n \text{ odd and } x \ge 0}$$

11) a) Show by induction $[\hat{X}^n, \hat{P}] = i\hbar n \hat{X}^{n-1}$

$\boxed{\text{Step 1: Prove for } n=1}$ $[\hat{X}, \hat{P}] = i\hbar$ $\boxed{\text{True for } n=1}$

$(\hat{X}\hat{P} - \hat{P}\hat{X})F(x) = x(-i\hbar \frac{d}{dx})F(x) + i\hbar \frac{d}{dx}(xF(x)) = -i\hbar x F' + i\hbar x F' + i\hbar F = i\hbar F(x)$

$\boxed{\text{Step 2: Assuming that } [\hat{X}^n, \hat{P}] = i\hbar n \hat{X}^{n-1} \text{ is true, prove for } n+1.}$

$(\hat{X}^{n+1}\hat{P} - \hat{P}\hat{X}^{n+1})F(x) = x^{n+1}(-i\hbar \frac{d}{dx})F(x) - (-i\hbar \frac{d}{dx})(x^{n+1}F(x)) = -i\hbar x^{n+1}F' + i\hbar[x^{n+1}F' + (n+1)x^n F]$

$= i\hbar(n+1)x^{n+1}F \rightarrow [\hat{X}^{n+1}, \hat{P}] = i\hbar(n+1)\hat{X}^n$ $\boxed{\text{True for } n+1 \text{ if true for } n}$

$\therefore \boxed{[\hat{X}^n, \hat{P}] = i\hbar n \hat{X}^{n-1} \text{ true for all } n}$

b) Prove $[F(\hat{X}), \hat{P}] = i\hbar \frac{\partial F}{\partial \hat{X}}$ Let $F(x) = \sum_n c_n x^n$

$[F(\hat{X}), \hat{P}]\, g(x) = \sum_n c_n x^n (-i\hbar \frac{\partial}{\partial x}) g(x) + i\hbar \frac{\partial}{\partial x}\left(g(x) \sum_n c_n x^n\right) =$

$= -i\hbar\left[g'(x)\left(\sum_n c_n x^n\right)\right] + i\hbar\left[g'(x)\left(\sum_n c_n x^n\right)\right] + i\hbar\left[g(x)\left(\sum_{n+1} c_n x^{n-1}\right)\right] = i\hbar\left[\sum_{n-1} c_n x^{n-1}\right] g(x)$

$[F(\hat{X}), \hat{P}] = i\hbar \sum_{n-1} c_n x^{n-1} \Rightarrow \boxed{[F(\hat{X}), \hat{P}] = i\hbar \frac{\partial F}{\partial \hat{X}}}$

12) (7.4.2) Find $\langle X\rangle$, $\langle P\rangle$, $\langle X^2\rangle$, $\langle P^2\rangle$, $\Delta X\cdot\Delta P$ in the state $|n\rangle$

$$\boxed{\begin{array}{l} \hat{X}=\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a+a^\dagger) \quad \hat{P}=i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^\dagger-a) \quad a|n\rangle=\sqrt{n}|n-1\rangle \quad a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle \\ a^2|n\rangle=\sqrt{n^2-n}|n-2\rangle \quad a^{\dagger 2}|n\rangle=\sqrt{n^2+3n+2}|n+2\rangle \quad a^\dagger a|n\rangle=n|n\rangle \quad aa^\dagger|n\rangle=(n+1)|n\rangle \end{array}}$$

$$\langle\hat{X}\rangle=\langle n|\hat{X}|n\rangle=\langle n|\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a+a^\dagger)|n\rangle=\langle n|\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(a|n\rangle+a^\dagger|n\rangle\right)=$$

$$=\langle n|\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\sqrt{n}|n-1\rangle+\sqrt{n+1}|n+1\rangle\right]=\left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\sqrt{n}\overset{\emptyset}{\langle n|n-1\rangle}+\sqrt{n+1}\overset{\emptyset}{\langle n|n+1\rangle}\right]=\boxed{\emptyset}$$

$$\langle P\rangle=\langle n|\hat{P}|n\rangle=\langle n|i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^\dagger-a)|n\rangle=\langle n|\left[i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(a^\dagger|n\rangle-a|n\rangle\right)\right]=$$

$$=\langle n|\left[i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle\right)\right]=i\left(\frac{m\omega\hbar}{2}\right)^{1/2}\left[\sqrt{n+1}\overset{\emptyset}{\langle n|n+1\rangle}-\sqrt{n}\overset{\emptyset}{\langle n|n-1\rangle}\right]=\boxed{\emptyset}$$

$$\boxed{\hat{X}^2=\frac{\hbar}{2m\omega}\left(a^2+aa^\dagger+a^\dagger a+a^{\dagger 2}\right)} \quad \langle n|\hat{X}^2|n\rangle=\frac{\hbar}{2m\omega}\langle n|a^2+aa^\dagger+a^\dagger a+a^{\dagger 2}|n\rangle$$

$$=\frac{\hbar}{2m\omega}\langle n|\left[\sqrt{n^2+n}\overset{\to\emptyset}{|n-2\rangle}+(n+1)|n\rangle+n|n\rangle+\sqrt{n^2+3n+2}\overset{\to\emptyset}{|n+2\rangle}\right]=\frac{\hbar}{2m\omega}(2n+1)=\boxed{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$$

$$\boxed{\hat{P}^2=\frac{-m\omega\hbar}{2}\left(a^{\dagger 2}-a^\dagger a-aa^\dagger+a^2\right)} \quad \langle n|\hat{P}^2|n\rangle=\frac{-m\omega\hbar}{2}\langle n|a^{\dagger 2}-a^\dagger a-aa^\dagger+a^2|n\rangle$$

$$=\frac{-m\omega\hbar}{2}\langle n|\left[\sqrt{n^2+3n+2}|n+2\rangle-n|n\rangle-(n+1)|n\rangle+\sqrt{n^2+n}|n-2\rangle\right]=\frac{-m\omega\hbar}{2}(-2n-1)$$

$$\boxed{\langle P^2\rangle=m\omega\hbar\left(n+\frac{1}{2}\right)}$$

$$\Delta X=\sqrt{\langle X^2\rangle-\langle X\rangle^2}=\sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)} \qquad \Delta P=\sqrt{\langle P^2\rangle-\langle P\rangle^2}=\sqrt{m\omega\hbar\left(n+\frac{1}{2}\right)}$$

$$\Delta X\,\Delta P=\sqrt{\left(\frac{\hbar}{m\omega}\right)\left(n+\frac{1}{2}\right)}\sqrt{m\omega\hbar\left(n+\frac{1}{2}\right)}=\boxed{\hbar\left(n+\frac{1}{2}\right)}$$

13) (7.4.5) at $t=\varnothing$ $\Psi(\varnothing) = \frac{1}{\sqrt{2}}(|\varnothing\rangle + |1\rangle)$ a) Find $\Psi(t)$

(Eq 4.3.12) $|\Psi(t)\rangle = \sum_E |E\rangle\langle E|\Psi(\varnothing)\rangle \exp\left(\frac{-iEt}{\hbar}\right)$ (Eq 7.4.17) $E_n = (n+\frac{1}{2})\hbar\omega$

$$\Psi(t) = \frac{1}{\sqrt{2}}\left[\exp\left(\frac{-iE_\varnothing t}{\hbar}\right)|\varnothing\rangle + \exp\left(\frac{-iE_1 t}{\hbar}\right)|1\rangle\right] = \boxed{\frac{1}{\sqrt{2}}\left[\exp\left(\frac{-i\omega t}{2}\right)|\varnothing\rangle + \exp\left(\frac{-3i\omega t}{2}\right)|1\rangle\right]}$$

b) $\langle\hat{X}(\varnothing)\rangle$ $\langle\Psi(\varnothing)|\hat{X}|\Psi(\varnothing)\rangle = \langle\Psi(\varnothing)|\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)|\Psi(\varnothing)\rangle$

$$= \langle\Psi(\varnothing)|\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\left[\frac{1}{\sqrt{2}}(|\varnothing\rangle + |1\rangle)\right] = \langle\Psi(\varnothing)|\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}(|\varnothing\rangle + |1\rangle + \sqrt{2}|2\rangle) =$$

$$= \frac{1}{\sqrt{2}}(\langle\varnothing| + \langle 1|)\left[\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}(|\varnothing\rangle + |1\rangle + \sqrt{2}|2\rangle)\right] = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)\underbrace{(2)}_{\langle\varnothing|\varnothing\rangle + \langle 1|1\rangle} = \boxed{\sqrt{\frac{\hbar}{2m\omega}}}$$

$\langle P(\varnothing)\rangle$ $\langle\Psi(\varnothing)|\hat{P}|\Psi(\varnothing)\rangle = \langle\Psi(\varnothing)|i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)\left[\frac{1}{\sqrt{2}}(|\varnothing\rangle + |1\rangle)\right] =$

$$= i\frac{\sqrt{m\omega\hbar}}{2}\langle\Psi(\varnothing)|\left[|1\rangle + \sqrt{2}|2\rangle - |\varnothing\rangle\right] = i\frac{\sqrt{m\omega\hbar}}{2\sqrt{2}}(\langle\varnothing| + \langle 1|)(|1\rangle + \sqrt{2}|2\rangle - |\varnothing\rangle) =$$

$$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}(\langle 1|1\rangle - \langle\varnothing|\varnothing\rangle) = \boxed{\varnothing}$$

$\langle X(t)\rangle$ $\langle\Psi(t)|\hat{X}|\Psi(t)\rangle = \langle\Psi(t)|\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\left[\frac{1}{\sqrt{2}}\left(\exp\left(\frac{-i\omega t}{2}\right)|\varnothing\rangle + \exp\left(\frac{-3i\omega t}{2}\right)|1\rangle\right)\right] =$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\langle\Psi(t)|\left[\exp\left(\frac{-3i\omega t}{2}\right)|\varnothing\rangle + \exp\left(\frac{-i\omega t}{2}\right)|1\rangle + \sqrt{2}\exp\left(\frac{-3i\omega t}{2}\right)|2\rangle\right] =$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left[\langle\varnothing|\exp\left(\frac{i\omega t}{2}\right) + \langle 1|\exp\left(\frac{3i\omega t}{2}\right)\right]\left[\exp\left(\frac{-3i\omega t}{2}\right)|\varnothing\rangle + \exp\left(\frac{-i\omega t}{2}\right)|1\rangle + \sqrt{2}\exp\left(\frac{-3i\omega t}{2}\right)|2\rangle\right] =$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left[\overset{\langle\varnothing|\varnothing\rangle}{\exp(-i\omega t)} + \overset{\langle 1|1\rangle}{\exp(i\omega t)}\right] = \boxed{\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)}$$

$\langle P(t)\rangle$ $\langle\Psi(t)|P|\Psi(t)\rangle = \langle\Psi(t)|i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)\left[\frac{1}{\sqrt{2}}\left(\exp\left(\frac{-i\omega t}{2}\right)|\varnothing\rangle + \exp\left(\frac{-3i\omega t}{2}\right)|1\rangle\right)\right] =$

$$= i\frac{\sqrt{m\omega\hbar}}{2}\langle\Psi(t)|\left[\exp\left(\frac{-i\omega t}{2}\right)|1\rangle + \sqrt{2}\exp\left(\frac{-3i\omega t}{2}\right)|2\rangle - \exp\left(\frac{-3i\omega t}{2}\right)|\varnothing\rangle\right] =$$

$$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left[\langle\varnothing|\exp\left(\frac{i\omega t}{2}\right) + \langle 1|\exp\left(\frac{3i\omega t}{2}\right)\right]\left[\exp\left(\frac{-i\omega t}{2}\right)|1\rangle + \sqrt{2}\exp\left(\frac{-3i\omega t}{2}\right)|2\rangle - \exp\left(\frac{-3i\omega t}{2}\right)|\varnothing\rangle\right] =$$

$$= \frac{-1}{2i}\sqrt{\frac{m\omega\hbar}{2}}\left[\overset{\langle 1|1\rangle}{\exp(i\omega t)} - \overset{\langle\varnothing|\varnothing\rangle}{\exp(-i\omega t)}\right] = \boxed{-\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t)}$$

c) Find $\langle \dot{X}(t) \rangle$ and $\langle \dot{P}(t) \rangle$ using $\frac{d}{dt}\langle \Omega \rangle = \left(\frac{-i}{\hbar}\right)\langle \psi | [\Omega, H] | \psi \rangle$

Eq 1.5.10

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 \qquad \langle \dot{X} \rangle = \frac{-i}{\hbar}\left[\hat{X}, \frac{\hat{P}^2}{2m}\right] = \frac{-i}{2m\hbar}\langle [\hat{X}, \hat{P}^2] \rangle = \frac{-i}{2m\hbar}\langle 2i\hbar \hat{P} \rangle = \frac{\langle P \rangle}{m}$$

$$\langle \dot{X}(t) \rangle = \frac{1}{m}\langle P(t) \rangle = \frac{1}{m}\left[-\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t)\right] = -\sqrt{\frac{\hbar\omega}{2m}}\sin(\omega t)$$

$$\langle \dot{X}(t) \rangle = \frac{d}{dt}\langle X(t) \rangle \rightarrow \langle X(t) \rangle = \int -\sqrt{\frac{\hbar\omega}{2m}}\sin\omega t = \sqrt{\frac{\hbar\omega}{2m}}\frac{1}{\omega}\cos(\omega t)$$

$$\boxed{\langle X(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)}$$

$$\langle \dot{P}(t) \rangle = \frac{1}{i\hbar}\langle [P, \tfrac{1}{2}m\omega^2\hat{X}^2] \rangle = \frac{m\omega^2}{2i\hbar}\langle [P, X^2] \rangle$$

$$[P, X^2]f = -i\hbar\frac{d}{dx}(x^2 f) + x^2\left(i\hbar\frac{d}{dx}\right)f = -i\hbar x^2 f' - i\hbar(2x)f + i\hbar x^2 f' = -i\hbar(2x)f$$

$$\langle \dot{P}(t) \rangle = \frac{m\omega^2}{2i\hbar}(-2i\hbar\langle X \rangle) = -m\omega^2\langle X \rangle = -m\omega^2\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = -\sqrt{\frac{m\omega^3\hbar}{2}}\cos(\omega t)$$

$$\langle P(t) \rangle = \int -\sqrt{\frac{m\omega^3\hbar}{2}}\cos\omega t\, dt = -\sqrt{\frac{m\omega^3\hbar}{2}}\cdot\left(\frac{1}{\omega}\right)\sin(\omega t) = \boxed{\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t)}$$

---

14) (7.4.9) $\hat{X} \to x \quad \hat{P} \to -i\hbar\frac{d}{dx} + f(x)$ a) Prove $[\hat{X}, \hat{P}] = i\hbar$

$$[\hat{X}, \hat{P}]g = x\left(-i\hbar\frac{d}{dx} + f\right)g + \left(i\hbar\frac{d}{dx} - f\right)(xg) = -i\hbar x g' + xfg + i\hbar x g' + i\hbar g - xfg$$

$$[\hat{X}, \hat{P}]g = i\hbar g \Rightarrow \boxed{[X, P] = i\hbar}$$

b) $|\tilde{x}\rangle = \exp\left(\frac{ig(x)}{\hbar}\right)|x\rangle$ where $g(x) = \int^x f(x')dx'$; Verify $\langle \tilde{x} | X | \tilde{x}' \rangle = x\delta(x-x')$

$$\langle \tilde{x} | \hat{X} | \tilde{x}' \rangle = \langle x | \exp\left(\frac{-ig(x)}{\hbar}\right)\cdot x \cdot \exp\left(\frac{ig(x)}{\hbar}\right)|x\rangle = x\exp\left(\frac{i[g(x') - g(x)]}{\hbar}\right)\langle x | x' \rangle$$

if $g(x') \neq g(x) \rightarrow \delta = 0$

$$\langle \tilde{x} | \hat{X} | \tilde{x}' \rangle = x\exp\left(\frac{i[g(x') - g(x)]}{\hbar}\right)\delta(x-x') = \boxed{x\delta(x-x')}$$

Continued on next page

Verify $\langle \tilde{x} | \hat{P} | \tilde{x}' \rangle = \left[-i\hbar \frac{d}{dx} + f(x) \right] \delta(x-x')$ where $g(x) = \int^x f(x')\,dx'$

$$\langle \tilde{x} | \hat{P} | \tilde{x}' \rangle = \langle x | \exp\left(\frac{-ig(x)}{\hbar}\right) \cdot \left[-i\hbar \frac{d}{dx'} \right] \cdot \exp\left(\frac{ig(x')}{\hbar}\right) | x' \rangle =$$

$$= -i\hbar \langle x | \exp\left(\frac{-ig(x)}{\hbar}\right) \left[\left(\frac{i}{\hbar} \overbrace{\frac{dg(x')}{dx'}}^{F(x')} \right) \exp\left(\frac{ig(x')}{\hbar}\right) | x' \rangle + \exp\left(\frac{ig(x')}{\hbar}\right) \frac{d}{dx'} | x' \rangle =$$

$$= -i\hbar \exp\left(\frac{i[g(x') - g(x)]}{\hbar}\right) \left[\langle x | \frac{i}{\hbar} F(x') + \frac{d}{dx'} | x' \rangle \right] = \left(-i\hbar \frac{d}{dx'} + f(x') \right) \exp\left(\frac{i[g(x') - g(x)]}{\hbar}\right) \langle x | x' \rangle$$

$$= \left(-i\hbar \frac{d}{dx'} + f(x') \right) \exp\left(\frac{i[g(x') - g(x)]}{\hbar}\right) \delta(x-x') = \boxed{\left(-i\hbar \frac{d}{dx} + f(x) \right) \delta(x-x')}$$

---

15) (7.5.2) $a|n\rangle = \sqrt{n}\,|n-1\rangle$ in the x-basis, derive $H_n'(y) = 2n H_{n-1}(y)$

Letting $y = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$ $\quad \psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left(\frac{-y^2}{2}\right) H_n(y)$ (Eq 7.3.22)

$$a|n = \frac{1}{\sqrt{2}} \left(y + \frac{d}{dy} \right) |n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\frac{1}{\sqrt{2}} \left(y + \frac{d}{dy} \right) \left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left(\frac{-y^2}{2}\right) H_n(y) \right] = \sqrt{n} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^{n-1}(n-1)!}} \exp\left(\frac{-y^2}{2}\right) H_{n-1}(y)$$

$$\frac{1}{\sqrt{2^{n+1} n!}} \left[y \cdot \exp\left(\frac{-y^2}{2}\right) H_n(y) + \exp\left(\frac{-y^2}{2}\right) H_n'(y) - y \exp\left(\frac{-y^2}{2}\right) H_n(y) \right] = \sqrt{\frac{n}{2^{n-1}(n-1)!}} \exp\left(\frac{-y^2}{2}\right) H_{n-1}(y)$$

$$\frac{1}{\sqrt{2^{n+1} n!}} \left[\exp\left(\frac{-y^2}{2}\right) H_n'(y) \right] = \sqrt{\frac{n}{2^{n+1}(n-1)!}} \exp\left(\frac{y^2}{2}\right) H_{n-1}(y) \Rightarrow H_n'(y) = \sqrt{\frac{2^{n+1} n (n!)}{2^{n-1} (n-1)!}} H_{n-1}(y)$$

$$H_n'(y) = \sqrt{4n^2}\, H_{n-1}(y) \rightarrow \boxed{H_n'(y) = 2n H_{n-1}(y)}$$