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# Homework 2

1) Prove $|V+W| \leq |V| + |W|$ starting w/ $|V+W|^2$. Use $Re\langle V|W\rangle \leq |\langle V|W\rangle|$ and $|\langle V|W\rangle| \leq |V||W|$

$$|V+W|^2 = \langle V+W|V+W\rangle = (\langle V| + \langle W|)(|V\rangle + |W\rangle) = \langle V|V\rangle + \langle V|W\rangle + \langle W|V\rangle + \langle W|W\rangle$$

$$= |V|^2 + \langle V|W\rangle + \langle V|W\rangle^* + |W|^2 =$$

$$= |V|^2 + (Re\langle V|W\rangle + Im\langle V|W\rangle) + (Re\langle V|W\rangle - Im\langle V|W\rangle) + |W|^2$$

$$= \boxed{|V|^2 + 2Re\langle V|W\rangle + |W|^2}$$

$$(|V| + |W|)^2 = \boxed{|V|^2 + 2|V||W| + W^2}$$

$$\boxed{Re\langle V|W\rangle \leq |\langle V|W\rangle| \leq |V||W|} \Rightarrow \boxed{|V|^2 + 2Re\langle V|W\rangle + |W|^2 \leq |V|^2 + 2|V||W| + |W|^2}$$

$$\therefore |V+W|^2 \leq (|V| + |W|)^2 \Rightarrow |V+W| \leq |V| + |W|$$

Show the inequality becomes an equality if $|V\rangle = a|W\rangle$, $a$ = real positive scalar

$$|V+W| = \sqrt{|aW|^2 + 2Re\langle W|a|W\rangle + |W|^2} = \sqrt{a^2|W|^2 + 2a|W|^2 + |W|^2} = \boxed{(a+1)|W|}$$

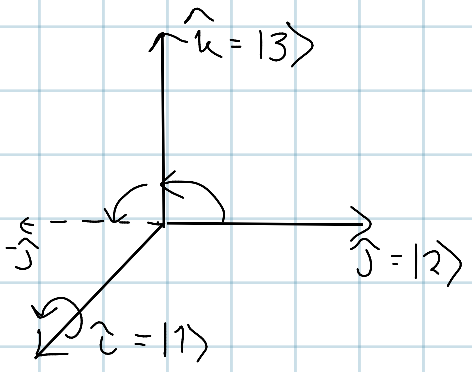
$$|V| + |W| = |aW| + |W| = \boxed{(a+1)|W|} \quad \therefore \boxed{|V+W| = |V| + |W|}$$

2) Show that $\left[R\left(\frac{\pi}{2}\hat{i}\right), R\left(\frac{\pi}{2}\hat{j}\right)\right] \neq \emptyset$ $\quad$ $\left[R\left(\frac{\pi}{2}\hat{i}\right), R\left(\frac{\pi}{2}\hat{j}\right)\right] = R\left(\frac{\pi}{2}\hat{i}\right)\left[R\left(\frac{\pi}{2}\hat{j}\right)\right] - R\left(\frac{\pi}{2}\hat{j}\right)\left[R\left(\frac{\pi}{2}\hat{i}\right)\right]$

$R\left(\frac{\pi}{2}\hat{i}\right)|1\rangle = |1\rangle$

$R\left(\frac{\pi}{2}\hat{i}\right)|2\rangle = |3\rangle$

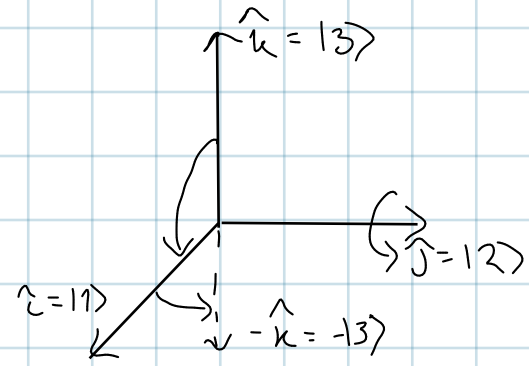
$R\left(\frac{\pi}{2}\hat{i}\right)|3\rangle = -|2\rangle$




$R\left(\frac{\pi}{2}\hat{j}\right)|1\rangle = -|3\rangle$

$R\left(\frac{\pi}{2}\hat{j}\right)|2\rangle = |2\rangle$

$R\left(\frac{\pi}{2}\hat{j}\right)|3\rangle = |1\rangle$




Let $|v\rangle = v_1|1\rangle + v_2|2\rangle + v_3|3\rangle$

$R\left(\frac{\pi}{2}\hat{i}\right)|v\rangle = v_1|1\rangle + v_2|3\rangle - v_3|2\rangle$ $\quad$ $R\left(\frac{\pi}{2}\hat{j}\right)\left[R\left(\frac{\pi}{2}\hat{i}\right)|v\rangle\right] = -v_1|3\rangle + v_2|1\rangle - v_3|2\rangle$

$R\left(\frac{\pi}{2}\hat{j}\right)|v\rangle = -v_1|3\rangle + v_2|2\rangle + v_3|1\rangle$ $\quad$ $R\left(\frac{\pi}{2}\hat{i}\right)\left[R\left(\frac{\pi}{2}\hat{j}\right)|v\rangle\right] = v_1|2\rangle + v_2|3\rangle + v_3|1\rangle$

$$R\left(\tfrac{\pi}{2}\hat{i}\right)\left[R\left(\tfrac{\pi}{2}\hat{j}\right)|v\rangle\right] - R\left(\tfrac{\pi}{2}\hat{j}\right)\left[R\left(\tfrac{\pi}{2}\hat{i}\right)|v\rangle\right] = \left[v_2|1\rangle - v_3|2\rangle - v_1|3\rangle\right] - \left[v_3|1\rangle + v_1|2\rangle + v_2|3\rangle\right]$$

$$= \boxed{(v_2 - v_3)|1\rangle + (-v_3 - v_1)|2\rangle + (-v_1 - v_2)|3\rangle}$$

$\downarrow$ $v_2 = v_3$ $\quad$ $\downarrow$ $v_1 = -v_3$ $\quad$ $\downarrow$ $v_2 = -v_1 = v_3$ $\quad \Rightarrow$ $\boxed{\text{Only } \emptyset \text{ when } v_1 = v_2 = v_3 = \emptyset}$

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3) a) $\Omega\Lambda$ $\quad (\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger = \Lambda\Omega$

$\therefore$ The adjoint of the products of two Hermitian operators is the transpose of the operators, thus $\Omega\Lambda$ is not Hermitian.

b) $\Omega\Lambda + \Lambda\Omega$ $\quad \Omega\Lambda + (\Omega\Lambda)^\dagger = \dfrac{2\left[\Omega\Lambda + (\Lambda\Omega)^\dagger\right]}{2} = 2\left[\text{Hermitian part of } \Omega\Lambda\right]$

According to Eq. 1.6.18 on pg 27 of Shankar, $\Omega\Lambda + \Lambda\Omega$ yields the Hermitian part of the product $\Omega\Lambda$ multiplied by 2.

c) $[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega = \Omega\Lambda - (\Omega\Lambda)^\dagger = 2\left[\text{anti-Hermitian part of } \Omega\Lambda\right]$

According to Eq. 1.6.18, $[\Omega, \Lambda]$ is the anti-Hermitian part of the product $\Omega\Lambda$ multiplied by 2.

3) d) $i[\Omega, \Lambda]$ $\quad i[\Omega\Lambda - (\Omega\Lambda)^\dagger] = i \cdot 2[\text{Anti-Hermitian part of } \Omega\Lambda]$

An anti-Hermitian operator is purely complex, so multiplying by $i$ will make the operator Hermitian (i.e., real).

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4) $UU^\dagger = I \qquad \det(I) = \det(UU^\dagger) = \det(U)\det(U^\dagger) = \det(U)\det((U^T)^*)$

$1 = \det(U) \cdot \det(U^T)^* \Rightarrow \sqrt{\det(U) \cdot \det(U)^*} = 1 \Rightarrow \sqrt{aa^*} = 1$ where $a = \det(U)$

By definition, $\det(U)$ is a complex number of modulus 1.

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5) a) $\mathrm{Tr}(\Omega\Lambda) = \mathrm{Tr}(\Lambda\Omega) \qquad (\Omega\Lambda)_{ij} = \sum_k \Omega_{ik}\Lambda_{kj}$

$\mathrm{Tr}(\Omega\Lambda) = \sum_i (\Omega\Lambda)_{ii} = \sum_i \left(\sum_k \Omega_{ik}\Lambda_{ki}\right) = \sum_k \left(\sum_i \Lambda_{ki}\Omega_{ik}\right) = \mathrm{Tr}(\Lambda\Omega)$

b) $\mathrm{Tr}(\Omega\Lambda\Theta) = \mathrm{Tr}(\Lambda\Theta\Omega) = \mathrm{Tr}(\Theta\Omega\Lambda)$

$(\Omega\Lambda\Theta)_{il} = \sum_k (\Omega\Lambda)_{ik}\Theta_{kl} = \sum_k \left(\sum_j \Omega_{ij}\Lambda_{jk}\right)\Theta_{kl} = \sum_k \sum_j \Omega_{ij}\Lambda_{jk}\Theta_{kl}$

$\mathrm{Tr}(\Omega\Lambda\Theta) = \sum_i (\Omega\Lambda\Theta)_{ii} = \sum_i \left[\sum_j \sum_k \Omega_{ij}\Lambda_{jk}\Theta_{ki}\right] = \underbrace{\sum_k \left[\sum_i \sum_j \Theta_{ki}\Omega_{ij}\Lambda_{jk}\right]}_{\boxed{\mathrm{Tr}(\Theta\Omega\Lambda)}} = \underbrace{\sum_j \left[\sum_k \sum_i \Lambda_{jk}\Theta_{ki}\Omega_{ij}\right]}_{\boxed{\mathrm{Tr}(\Lambda\Theta\Omega)}}$

c) $\mathrm{Tr}(\Omega) = \mathrm{Tr}(U^\dagger \Omega U)$

From part b, $\mathrm{Tr}(U^\dagger\Omega U) = \mathrm{Tr}(UU^\dagger\Omega) = \mathrm{Tr}(I\Omega) = \mathrm{Tr}(\Omega)$

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6) $\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad$ a) $\Omega^\dagger = \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T\right)^* = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \Omega \Rightarrow \boxed{\text{Hermitian}}$

b) $|\Omega - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2) + 1(\lambda) = \lambda - \lambda^3 = \lambda(1-\lambda^2) = 0 \Rightarrow \boxed{\lambda = 0, \pm 1}$

$|0\rangle$: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = 0 \\ 0 = 0 \\ x_1 = 0 \end{matrix}$ $\therefore$ $x_2$ is arbitrary, let $x_2 = 1$ so $||0\rangle| = 1$ $\Rightarrow \boxed{|0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

$|1\rangle$: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x_1 + x_3 = 0 \\ -x_2 = 0 \\ x_1 - x_3 = 0 \end{matrix}$ $\therefore$ $x_1 = x_3$, $x_2 = 0$. Let $x_1 = x_3 = 1$, $\sqrt{1+0+1} = \sqrt{2}$ $\Rightarrow \boxed{|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$

Continued on next page

$$|-1\rangle = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_2 = 0 \\ x_1 + x_3 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = -x_3,\ x_2 = 0 \\ \text{Let } x_1 = 1, \text{ then} \\ \| |-1\rangle \| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2} \end{matrix} \Rightarrow \boxed{|-1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

c) $$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \qquad U^\dagger \Omega = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$U^\dagger \Omega U = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow U^\dagger \Omega U \text{ is a diagonal matrix}$$

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7) $$\Omega = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

a) $$\Omega\Omega^\dagger = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

b) $$|\Omega - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$= \lambda^2 - 2\lambda\cos\theta + 1 = \lambda^2 - \lambda(e^{i\theta} + e^{-i\theta}) + 1 = (\lambda - e^{i\theta})(\lambda - e^{-i\theta}) = 0$$

$$\boxed{\lambda = e^{i\theta}, e^{-i\theta}}$$

c) $$e^{i\theta}: \begin{bmatrix} \cos\theta - e^{i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{i\theta} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[e^{-i\theta} - e^{i\theta}] & \sin\theta \\ -\sin\theta & \frac{1}{2}[e^{-i\theta} - e^{i\theta}] \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} -i\sin\theta\, x_1 + \sin\theta\, x_2 = 0 \\ -\sin\theta\, x_1 - i\sin\theta\, x_2 = 0 \end{matrix} \Rightarrow \begin{matrix} x_2 = i x_1 \\ x_2 = \frac{-x_1}{i} = i x_1 \end{matrix} \Rightarrow \boxed{|e^{i\theta}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}}$$

$$e^{-i\theta}: \begin{bmatrix} \cos\theta - e^{-i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{-i\theta} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[e^{i\theta} - e^{-i\theta}] & \sin\theta \\ -\sin\theta & \frac{1}{2}[e^{i\theta} - e^{-i\theta}] \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$\Rightarrow \begin{matrix} i\sin\theta\, x_1 + \sin\theta\, x_2 = 0 \\ -\sin\theta\, x_1 + i\sin\theta\, x_2 = 0 \end{matrix} \Rightarrow \begin{matrix} x_2 = -i x_1 \\ x_2 = \frac{x_1}{i} = -i x_1 \end{matrix} \quad \boxed{|e^{-i\theta}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

$$\langle e^{i\theta}|e^{-i\theta}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{2}(1 + i^2) = \boxed{0}$$

d) $$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \qquad U^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \qquad U^\dagger\Omega = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\theta + i\sin\theta & \sin\theta - i\cos\theta \\ \cos\theta - i\sin\theta & \sin\theta + i\cos\theta \end{bmatrix}$$

$$U^\dagger\Omega = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\theta} & -ie^{i\theta} \\ e^{-i\theta} & ie^{-i\theta} \end{bmatrix} \qquad U^\dagger\Omega U = \frac{1}{2}\begin{bmatrix} e^{i\theta} & -ie^{i\theta} \\ e^{-i\theta} & ie^{-i\theta} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2e^{i\theta} & 0 \\ 0 & 2e^{-i\theta} \end{bmatrix} = \boxed{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}}$$

8) $M^i M^j + M^j M^i = 2\delta_{ij} I$; if $i=j$ then $M^i M^i = I$

a) $|\omega_i\rangle$ = eigenvector of $M^i$; $M^i|\omega_i\rangle = \lambda_i|\omega_i\rangle$

$M^i M^i |\omega_i\rangle = M^i(\lambda|\omega_i\rangle) = \lambda^2|\omega_i\rangle = I|\omega_i\rangle$, therefore $\lambda^2$ is an eigenvalue of $I$

If $I$ has dimension $n$, then $|I^n - \lambda^2 I^n| = (1-\lambda^2)^n = 0 \Rightarrow \boxed{\lambda = \pm 1}$

b) $\mathrm{Tr}\, M^i M^j \quad = \mathrm{Tr}\, M^i I = \mathrm{Tr}\, M^i$

$$\mathrm{Tr}(M^i M^j M^j) \overset{M^i M^j = -M^j M^i}{=} -\mathrm{Tr}(M^j M^i M^j) = \overbrace{-\mathrm{Tr}(M^i M^j M^j)}^{\mathrm{Tr}(ACB)=\mathrm{Tr}(CBA)} = -\mathrm{Tr}(M^i I) = -\mathrm{Tr}(M^i)$$

$\mathrm{Tr}(M^i) = -\mathrm{Tr}(M^i) \therefore \mathrm{Tr}(M^i) = 0$

c) Since $\lambda = \pm 1$, if $M^i$ is diagonalized all the elements on the diagonal are either 1 or -1. In order for the trace to be zero, there must be a number of 1 elements equal to the number of -1 elements and therefore an even number of elements on the diagonal, which means an even-dimensional matrix.

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9) $\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \qquad \Omega\Lambda|\alpha_i\rangle = \Lambda\Omega|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$

$$[\Omega, \Lambda] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} = 0$$

$$|\Omega - \omega I| = \begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = (1-\omega)(\omega^2 - \omega) + \omega = \omega^2 - \omega - \omega^3 + \omega^2 + \omega = -\omega^3 + 2\omega^2 = 0$$

$\omega = 0, 2 \quad$ (0 is degenerate)

$0$: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad$ $x_1 = -x_3$, $x_2$ arbitrary $\Rightarrow |0_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad |0_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$2$: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad$ $x_1 = x_3$, $x_2 = 0$ $\Rightarrow |2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$|\Lambda-\lambda I|=\begin{vmatrix}2-\lambda & 1 & 1\\ 1 & -\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=-\lambda(2-\lambda)^2+\lambda-2+\lambda-2-2+\lambda=-(\lambda-2)[\lambda(2-\lambda)+3]=0$$

$$=(\lambda-2)(\lambda+1)(\lambda-3) \Rightarrow \lambda=-1,2,3$$

*Since $|\omega=2\rangle$ is a nondegenerate eigenvector of $\Omega$, it must also be an eigenvector of $\Lambda$

$$\begin{bmatrix}2 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 2\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{\lambda}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow 3=\lambda \Rightarrow |2\rangle \text{ is the eigenvector for } \lambda=3$$

$$\lambda=-1: \begin{bmatrix}3 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{matrix}3x_1+x_2+x_3=0\\ x_1+x_2-x_3=0\\ x_1-x_2+3x_3=0\end{matrix} \Rightarrow \begin{matrix}x_2=-2x_1\\ x_1=-x_3\end{matrix} \quad |\lambda=-1\rangle=\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\-1\end{bmatrix}$$

$$\lambda=2: \begin{bmatrix}0 & 1 & 1\\ 1 & -2 & -1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{matrix}x_2=-x_3\\ x_1-2x_2-x_3=0\\ x_1=x_2\end{matrix} \quad |\lambda=2\rangle=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$U=\begin{bmatrix}1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}\\ -2/\sqrt{6} & 1/\sqrt{3} & 0\\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2}\end{bmatrix} \quad U^\dagger=\begin{bmatrix}1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix} \quad U^\dagger\Omega=\begin{bmatrix}1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 2/\sqrt{2} & 0 & 2/\sqrt{2}\end{bmatrix}$$

$$U^\dagger\Omega U=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 2/\sqrt{2} & 0 & 2/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2}\\ 1/\sqrt{3} & -2/\sqrt{6} & 0\\ -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2}\end{bmatrix}=\boxed{\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{bmatrix}}$$

$$U^\dagger\Lambda U=\begin{bmatrix}1/\sqrt{6} & -2/\sqrt{6} & -1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}\\ -2/\sqrt{6} & 1/\sqrt{3} & 0\\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}-1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6}\\ 2/\sqrt{3} & 2/\sqrt{3} & -2/\sqrt{3}\\ 3/\sqrt{2} & 0 & 3/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}\\ -2/\sqrt{6} & 1/\sqrt{3} & 0\\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2}\end{bmatrix}=\boxed{\begin{bmatrix}-1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}}$$

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10)

$$\Omega=\begin{bmatrix}-2k/m & k/m\\ k/m & -2k/m\end{bmatrix} \qquad \Omega|I\rangle=-\omega_I^2|I\rangle \quad \Omega|II\rangle=-\omega_{II}^2|II\rangle$$

$$|\Omega+\omega^2 I|=\begin{vmatrix}-2k/m+\omega^2 & k/m\\ k/m & -2k/m+\omega^2\end{vmatrix}=\left(\frac{-2k}{m}+\omega^2\right)^2-\frac{k^2}{m^2}=\omega^4-\frac{4k}{m}\omega^2+\frac{4k^2}{m^2}-\frac{k^2}{m^2}=\omega^4-\frac{4k}{m}\omega^2+\frac{3k^2}{m^2}=$$

$$=\left(\omega^2-\frac{3k}{m}\right)\left(\omega^2-\frac{k}{m}\right)=0 \Rightarrow \omega^2=\frac{3k}{m},\frac{k}{m} \Rightarrow \boxed{\omega=\sqrt{\frac{3k}{m}},\sqrt{\frac{k}{m}}}$$

$$\omega_I=\sqrt{\frac{k}{m}}: \begin{bmatrix}-k/m & k/m\\ k/m & -k/m\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_1=x_2 \Rightarrow \boxed{|I\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$\omega_{II}=\sqrt{\frac{3k}{m}}: \begin{bmatrix}k/m & k/m\\ k/m & k/m\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_1=-x_2 \Rightarrow \boxed{|II\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}}$$

11) a) $|\ddot{x}\rangle = \Omega|x\rangle \quad |x(t)\rangle = U(t)|x(0)\rangle \qquad \frac{d^2}{dt^2}|x\rangle = \Omega|x(t)\rangle = \Omega\, U(t)|x(0)\rangle = \frac{d^2}{dt^2} U(t)|x(0)\rangle$

$$\left[\frac{d^2}{dt^2} U(t) - \Omega\, U(t)\right]|x(0)\rangle = 0 \quad \Rightarrow \quad \boxed{\frac{d^2}{dt^2} U(t) - \Omega\, U(t) = 0}$$

b) From the previous problem, $\Omega$ is diagonalized in the $|I\rangle, |II\rangle$ basis and therefore $U(t)$ will be diagonalized in the same basis.

In the $|I\rangle|II\rangle$ basis, $\Omega = \begin{bmatrix} -\omega_I^2 & 0 \\ 0 & -\omega_{II}^2 \end{bmatrix}$ and $U(t) = \begin{bmatrix} U_I(t) & 0 \\ 0 & U_{II}(t) \end{bmatrix}$

$$\frac{d^2}{dt^2}\begin{bmatrix} U_I(t) & 0 \\ 0 & U_{II}(t) \end{bmatrix} = \begin{bmatrix} -\omega_I^2 & 0 \\ 0 & -\omega_{II}^2 \end{bmatrix}\begin{bmatrix} U_I(t) & 0 \\ 0 & U_{II}(t) \end{bmatrix} \Rightarrow \begin{matrix} \frac{d^2}{dt^2} U_I(t) = -\omega_I^2 U_I(t) \\ \frac{d^2}{dt^2} U_{II}(t) = -\omega_{II}^2 U_{II}(t) \end{matrix} \Rightarrow \begin{matrix} U_I(t) = \cos(\omega_I t) \\ U_{II}(t) = \cos(\omega_{II} t) \end{matrix}$$

$$\boxed{U(t) = \begin{bmatrix} \cos(\omega_I t) & 0 \\ 0 & \cos(\omega_{II} t) \end{bmatrix}}$$

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12) $U = e^{iH} \qquad U^\dagger = (e^{iH})^\dagger = e^{-iH} \qquad U^\dagger U = e^{-iH} e^{iH} = e^{(-iH + iH)} = e^{0} = 1 \Rightarrow \boxed{\text{Unitary}}$