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PHYS755 Homework 1

Ex. 1.1.3 (pg 4), Ex. 1.1.4 (pg 5), Ex. 1.1.5 (pg 5), Ex. 1.3.1 (pg 15), Ex. 1.3.2 (pg 16)

1.1.3) f, g, h are vectors; a and b are scalars

	$f(0)=f(L)=0$	$f(0)=f(L)$	$f(0)=4$
$f+g \in \mathbb{V}$	✓	✓	✗
$a(f+g) = af + ag$	✓	✓	✓
$(a+b)f = af + bf$	✓	✓	✓
$a(bf) = abf$	✓	✓	✓
$f+g = g+f$	✓	✓	✓
$f+(g+h) = (f+g)+h$	✓	✓	✓
$f + 0 = f$	✓	✓	✓
$f + (-f) = 0$	✓	✓	✓

Functions satisfying $f(0)=4$ do not have closure.

- $f(0) + g(0) = 4+4 = 8$

  $\therefore F+g \notin \mathbb{V}$

- $a\, f(0) = a(4)$

  $\therefore af \in \mathbb{V}$ iff $a = 1$.

1.1.4) $|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad |3\rangle = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}$ $\qquad a|1\rangle + b|2\rangle + c|3\rangle = |0\rangle$

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b & b \\ 0 & b \end{bmatrix} + \begin{bmatrix} -2c & -c \\ 0 & -2c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$b - 2c = 0 \Rightarrow b = 2c$

$a + b - c = 0$ $\quad a + c = 0 \Rightarrow a = -c \Rightarrow b = 2c = -2a$

Let $b = 2$, $c = 1$, and $a = -1$ $\quad \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

In addition, $|3\rangle$ can be written as $|1\rangle - 2|2\rangle$

$$|1\rangle - 2|2\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} = |3\rangle$$

$\boxed{\text{Therefore } |1\rangle, |2\rangle, \text{ and } |3\rangle \text{ are not linearly independent}}$

1.1.5) $(1,1,0),\ (1,0,1),\ (3,2,1)\ ;\ (1,1,0),\ (1,0,1),\ (0,1,1)$

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$a + b + 3c = 0$ ①

$a + 2c = 0$ ② $\Rightarrow a = -2c$

$b + c = 0$ ③ $\Rightarrow b = -c$

Let $c = 1,\ b = -1,\ a = -2$

$$-2\begin{pmatrix}1\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\0\\1\end{pmatrix} + 1\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$-2 - 1 + 3 = 0$ ✓

$-2 + 2 = 0$ ✓

$-1 + 1 = 0$ ✓

$$\therefore\ \boxed{a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \text{ if } a = 2b = -2c}$$

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$a + b = 0$ ①

$a + c = 0$ ②

$b + c = 0$ ③

$$\boxed{\begin{array}{l} a = -b \\ a = -c = -b \\ b = -c \end{array} \Rightarrow \text{True only if } a = b = c = 0}$$

1.3.1) $\vec{A} = 3\hat{i} + 4\hat{j}$ $\quad \vec{B} = 2\hat{i} - 6\hat{j}$ $\quad |A\rangle = [3\ 4]$ $\quad |B\rangle = [2, -6]$

$$|1\rangle = \frac{|A\rangle}{|A|} = \frac{[3,4]}{\sqrt{3^2+4^2}} = \boxed{\left[\frac{3}{5}, \frac{4}{5}\right]} \qquad \langle 1|B\rangle = \left[\frac{3}{5}, \frac{4}{5}\right] \cdot [2,-6] = \frac{6}{5} - \frac{24}{5} = \frac{-18}{5}$$

$$|2'\rangle = |B\rangle - |1\rangle\langle 1|B\rangle = [2,-6] - \left[\frac{3}{5}, \frac{4}{5}\right]\left(\frac{-18}{5}\right) = \left[2 + \frac{54}{25}, -6 + \frac{72}{25}\right] = \left[\frac{104}{25}, \frac{-78}{25}\right]$$

$$|2'| = \sqrt{\left(\frac{104}{25}\right)^2 + \left(\frac{-78}{25}\right)^2} = \sqrt{\frac{16900}{625}} = \sqrt{\frac{676}{25}} = \frac{26}{5}$$

$$|2\rangle = \frac{|2'\rangle}{|2'|} = \left(\frac{5}{26}\right)\left[\frac{104}{25}, \frac{-78}{25}\right] = \boxed{\left[\frac{4}{5}, \frac{-3}{5}\right]}$$

$$\langle 1|2\rangle = \left[\frac{3}{5}, \frac{4}{5}\right] \cdot \left[\frac{4}{5}, \frac{-3}{5}\right] = 0 \qquad \boxed{\therefore\ |1\rangle \text{ and } |2\rangle \text{ are an orthonormal basis}}$$

$$|3\rangle = \frac{|B\rangle}{|B|} = \frac{[2,-6]}{\sqrt{4+36}} = \boxed{\left[\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right]} \qquad \langle 3|A\rangle = \left[\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right] \cdot [3,4] = \frac{3-12}{\sqrt{10}} = \frac{-9}{\sqrt{10}}$$

$$|4'\rangle = |A\rangle - |3\rangle\langle 3|B\rangle = [3,4] + \left[\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right]\left(\frac{+9}{\sqrt{10}}\right) = \boxed{\left[\frac{39}{10}, \frac{13}{10}\right]}$$

$$|4'| = \sqrt{\left(\frac{39}{10}\right)^2 + \left(\frac{13}{10}\right)^2} = \sqrt{\frac{1690}{100}} = \frac{13}{\sqrt{10}}$$

$$|4\rangle = \frac{|4'\rangle}{|4'\rangle} = \frac{\sqrt{10}}{13}\left[\frac{39}{10}, \frac{13}{10}\right] = \left[\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right]$$

$$\langle 3|4\rangle = \left[\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right] \cdot \left[\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right] = 0 \qquad \boxed{\therefore\ |3\rangle \text{ and } |4\rangle \text{ are an orthonormal basis.}}$$

1.3.2) $|I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$ to $|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \quad |3\rangle = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$$|1\rangle = \frac{|I\rangle}{|I|} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \qquad \langle 1|II\rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0$$

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad |2\rangle = \frac{|2'\rangle}{|2'|} = \frac{1}{\sqrt{1^2+2^2}}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}$$

$$\langle 1|III\rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = 0 \qquad \langle 2|III\rangle = \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = 0 + \frac{2}{\sqrt{5}} + \frac{10}{\sqrt{5}} = \frac{12}{\sqrt{5}}$$

$$|3'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{12}{\sqrt{5}}\begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 12/5 \\ 24/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/5 \\ 1/5 \end{bmatrix}$$

$$|3\rangle = \frac{|3'\rangle}{|3'|} = \frac{1}{\sqrt{\left(\frac{-2}{5}\right)^2 + \left(\frac{1}{5}\right)^2}}\begin{bmatrix} 0 \\ -2/5 \\ 1/5 \end{bmatrix} = \sqrt{5}\begin{bmatrix} 0 \\ -2/5 \\ 1/5 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}$$