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Homework 3

1) (1.10.1) Show $\delta(ax) = \frac{\delta(x)}{|a|}$ using $\int_{-\infty}^{\infty} \delta(ax)\, d(ax)$

$\int \delta(ax)\, d(ax) = \int \delta(ax)\, a\, dx = a \int \delta(ax)\, dx \qquad y = ax \quad dx = d\left(\frac{y}{a}\right) = \frac{dy}{a}$

$\int_{-\infty}^{\infty} \delta(ax)\, d(ax) = a \int_{-\infty}^{\infty} \delta(y) \frac{dy}{a} = \frac{a}{a} \int_{-\infty}^{\infty} \delta(y)\, dy = 1$ if $y = ax = 0$

Therefore $a\,\delta(ax) = \delta(x) \rightarrow \delta(ax) = \frac{\delta(x)}{a}$

Since $\delta(-ax) = \delta(ax)$ and $\delta(-x) = \delta(x)$ we have to use the absolute value of a to ensure the equality holds

$\therefore \boxed{\delta(ax) = \frac{\delta(x)}{|a|}}$

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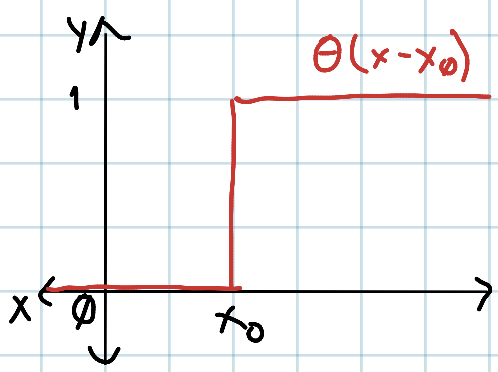
2) (1.10.3) Show that $\delta(x - x_0) = \frac{d}{dx} \Theta(x - x_0)$



- $x < x_0$: $\Theta(x - x_0) = 0 \qquad \frac{d}{dx}(\Theta(x - x_0)) = \frac{d}{dx}(0) = 0$
- $x > x_0$: $\Theta(x - x_0) = 1 \qquad \frac{d}{dx}(\Theta(x - x_0)) = \frac{d}{dx}(1) = 0$
- At $x = x_0$ $(x - x_0)$ is a vertical line going from $y = 0$ to $y = 1$

$\frac{d}{dx} \Theta(x - x_0) = \lim_{h \to 0} \frac{\Theta((x+h) - x_0) - \Theta(x - x_0)}{h} = \lim_{h \to 0} \frac{1 - 0}{h} = +\infty$

Therefore $\boxed{\frac{d}{dx} \Theta(x - x_0) = \begin{cases} 0 & \text{if } x \neq x_0 \\ +\infty & \text{if } x = x_0 \end{cases} = \delta(x - x_0)}$

3) (1.10.4)

$$\Psi(x,0) = \begin{cases} \frac{2xh}{L} & 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x) & \frac{L}{2} \le x \le L \end{cases}$$

$$\Psi(x,t) = \sum_{m=1}^{\infty} \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{m\pi x}{L}\right) \cos\omega_m t \langle m|\Psi(0)\rangle$$

$$\langle m|\Psi(0)\rangle = \left(\frac{2}{L}\right)^{1/2}\left[\int_0^{\frac{L}{2}} \sin\left(\frac{m\pi x}{L}\right) \frac{2xh}{L}\,dx + \int_{\frac{L}{2}}^{L} \sin\left(\frac{m\pi x}{L}\right) \frac{2h}{L}(L-x)\,dx\right]$$

$$= \left(\frac{2}{L}\right)^{1/2}\left[\frac{2h}{L} \underbrace{\int_0^{\frac{L}{2}} x \sin\left(\frac{m\pi x}{L}\right) dx}_{1} + \frac{2h}{L} \underbrace{\int_{\frac{L}{2}}^{L} (L-x)\sin\left(\frac{m\pi x}{L}\right) dx}_{2}\right]$$

$\alpha = \frac{m\pi}{L}$ 1 $u = x$ $dv = \sin(\alpha x)$ $du = dx$ $v = \frac{-1}{\alpha}\cos(\alpha x)$

$$\int_0^{\frac{L}{2}} x\sin(\alpha x)\,dx = \frac{-x}{\alpha}\cos(\alpha x)\Big|_0^{\frac{L}{2}} + \int_0^{\frac{L}{2}} \frac{+1}{\alpha}\cos(\alpha x)\,dx =$$

$$-\frac{x\cos(\alpha x)}{\alpha} + \frac{\sin(\alpha x)}{\alpha^2}\Big|_0^{L/2} = \frac{\sin(\alpha x) - \alpha x\cos(\alpha x)}{\alpha^2}\Big|_0^{L/2} =$$

$$= \frac{1}{\alpha^2}\left[\sin\left(\frac{\alpha L}{2}\right) - \frac{\alpha L}{2}\cos\left(\frac{\alpha L}{2}\right)\right]$$

2 $$\int_{\frac{L}{2}}^{L} (L-x)\sin(\alpha x)\,dx = \int_{L/2}^{L} L\sin(\alpha x)\,dx - \int_{L/2}^{L} x\sin(\alpha x)\,dx$$

$$= \frac{-L}{\alpha}\cos(\alpha x) - \frac{\sin(\alpha x) - \alpha x\cos(\alpha x)}{\alpha^2}\Big|_{L/2}^{L} =$$

$$= \frac{-\alpha(L-x)\cos\alpha x - \sin(\alpha x)}{\alpha^2}\Big|_{L/2}^{L} = \frac{1}{\alpha^2}\left[-\sin(\alpha L) + \frac{\alpha L}{2}\cos\left(\frac{\alpha L}{2}\right) + \sin\left(\frac{\alpha L}{2}\right)\right]$$

$$= \frac{1}{\alpha^2}\left[-\sin(\alpha L) + \frac{\alpha L}{2}\cos\left(\frac{\alpha L}{2}\right) + \sin\left(\frac{\alpha L}{2}\right)\right] = \frac{-2\sin(\alpha L) + 2\sin\left(\frac{\alpha L}{2}\right) + \alpha L\cos\left(\frac{\alpha L}{2}\right)}{2\alpha^2}$$

1+2 $$\frac{1}{2\alpha^2}\left[2\sin\left(\frac{\alpha L}{2}\right) - \alpha L\cos\left(\frac{\alpha L}{2}\right) + 2\sin\left(\frac{\alpha L}{2}\right) - 2\sin(\alpha L) + \alpha L\cos\left(\frac{\alpha L}{2}\right)\right]$$

$$= \frac{1}{2\alpha^2}\left[4\sin\left(\frac{\alpha L}{2}\right) - 2\sin(\alpha L)\right] = \frac{L^2}{m^2\pi^2}\left[2\sin\left(\frac{m\pi}{2}\right) - \overbrace{\sin(m\pi)}^{0}\right]$$

$$\langle m|\Psi(0)\rangle = \left(\frac{2}{L}\right)^{1/2} \frac{2h}{L}\left[\frac{L^2}{m^2\pi^2} \cdot 2\sin\left(\frac{m\pi}{2}\right)\right] = \left(\frac{2}{L}\right)^{1/2} \frac{4hL}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{m\pi x}{L}\right) \cos\omega_m t \, \langle m|\Psi(0)\rangle$$

$$= \sum_{m=1}^{\infty} \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{m\pi x}{L}\right) \cos\omega_m t \cdot \left(\frac{2}{L}\right)^{1/2} \frac{4hL}{m^2\pi^2} \sin\left(\frac{m\pi}{2}\right)$$

$$\boxed{\Psi(x,t) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right) \cos(\omega_m t) \left(\frac{8h}{m^2\pi^2}\right) \sin\left(\frac{m\pi}{2}\right)}$$

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4) (4.2.1, $\overset{a-e}{1-5}$) $\quad L_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad L_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow L_z$ basis

a) Since L_z is diagonalized, the values on the diagonal are its eigenvalues, which are the possible measurable values. Therefore

$\boxed{\text{possible values of } L_z \text{ are } L_z = 1, 0, -1}$

b) $|L_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \langle L_x \rangle = [1 \; 0 \; 0] \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [1 \; 0 \; 0] \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \boxed{0}$

$$L_x^2 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\langle L_x^2 \rangle = [1 \; 0 \; 0] \cdot \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}[1 \; 0 \; 0]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \boxed{\frac{1}{2}}$$

$$\Delta L_x = \left[\langle L_z{=}1|(L_x - \overset{0}{\langle L_x\rangle})^2|L_z{=}1\rangle\right]^{1/2} = \left[\langle L_z{=}1|L_x^2|L_z{=}1\rangle\right]^{1/2} = \sqrt{\langle L_x^2\rangle} = \boxed{\frac{1}{\sqrt{2}}}$$

c) $|L_x - \omega I| = \begin{vmatrix} -\omega & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\omega & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\omega \end{vmatrix} = -\omega\left(\omega^2 - \frac{1}{2}\right) + \frac{1}{2}\omega = -\omega(\omega^2 - 1) = 0 \qquad \boxed{\omega = 0, \pm 1}$

$\omega = 0: \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_2 = 0 \\ x_1 = -x_3 \end{matrix} \qquad \boxed{|\omega = 0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$

$\omega = 1: \begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \sqrt{2}x_1 = x_2 \\ x_2 = \sqrt{2}x_1 + \sqrt{2}x_2 \\ x_2 = \sqrt{2}x_3 \end{matrix} \qquad \boxed{|\omega = 1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}}$

$$\omega = -1: \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -\sqrt{2}\, x_1 = x_2 \\ -x_2 = \sqrt{2}x_1 + \sqrt{2}x_2 \\ -x_2 = \sqrt{2}\, x_3 \end{array} \qquad \boxed{|\omega = 1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}}$$

In L_z basis let $|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $|2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $|3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$|1\rangle = |L_z = 1\rangle$
$|2\rangle = |L_z = 0\rangle$
$|3\rangle = |L_z = -1\rangle$

$$\boxed{|\omega = 0\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|3\rangle \quad |\omega = 1\rangle = \frac{1}{2}|1\rangle + \frac{\sqrt{2}}{2}|2\rangle + \frac{1}{2}|3\rangle \quad |\omega = -1\rangle = \frac{1}{2}|1\rangle - \frac{\sqrt{2}}{2}|2\rangle + \frac{1}{2}|3\rangle}$$

d) $|L_z = -1\rangle = |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad P(L_x = 0) = |\langle \omega = 0 | 3\rangle|^2 = \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{1}{2}}$

$$P(L_x = 1) = |\langle \omega = 1 | 3\rangle|^2 = \left(\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^2 = \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$$

e) $P(L_x = -1) = |\langle \omega = -1 | 3\rangle|^2 = \left(\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)^2 = \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$

$L_z^2 = 1 \quad |\psi\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad L_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Eigenvector for $L_z^2 = 1$ is a linear combination of $|L_z = 1\rangle$ and $|L_z = -1\rangle$

$|\psi_{after}\rangle = L_z^2 |\psi\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ Normalize: $\boxed{|\psi_{after}\rangle = \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}}$

$$P(L_z^2 = 1) = |\langle \psi_{after} | \psi \rangle|^2 = \left(\frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right)^2 = \left(\frac{2}{\sqrt{3}} \left(\frac{1}{4} + \frac{1}{2} \right) \right)^2 = \left(\frac{\sqrt{3}}{2} \right)^2 = \boxed{\frac{3}{4}}$$

$$L_z |\psi_{after}\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \frac{2}{\sqrt{3}} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -\sqrt{\frac{2}{3}} \end{bmatrix} = \frac{1}{\sqrt{3}} |L_z = 1\rangle - \sqrt{\frac{2}{3}} |L_z = -1\rangle$$

$$P(L_z = 1 \text{ after}) = |\langle L_z = 1 | \psi_{after} \rangle|^2 = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right)^2 = \boxed{\frac{1}{3}}$$

$$P(L_z = 0 \text{ after}) = |\langle L_z = 0 | \psi_{after} \rangle|^2 = \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right)^2 = \boxed{0}$$

$$P(L_z = -1 \text{ after}) = |\langle L_z = -1 | \psi_{after} \rangle|^2 = \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right)^2 = \left(\sqrt{\frac{2}{3}} \right)^2 = \boxed{\frac{2}{3}}$$

5) (4.2.3) If $\psi(x)$ has $\langle P \rangle$ mean momentum, $\exp\left(\frac{ip_0x}{\hbar}\right)\psi(x) \to \langle P \rangle + p_0$

$$\langle P \rangle = \langle \psi | P | \psi \rangle = \int_{-\infty}^{\infty} \langle \psi | x \rangle \langle x | P | \psi \rangle dx = \int \psi^* \left(-i\hbar \frac{d\psi}{dx}\right)$$

Let $\psi' = \exp\left(\frac{ip_0x}{\hbar}\right)\psi(x)$

$$\langle P' \rangle = \int_{-\infty}^{\infty} \psi'^* \left(-i\hbar \frac{d}{dx}\psi'\right)dx = \int_{\infty}^{\infty} \exp\left(\frac{-ip_0x}{\hbar}\right)\psi^* \cdot -i\hbar\frac{d}{dx}\left(\exp\left(\frac{ip_0x}{\hbar}\right)\psi\right)dx$$

$$= \int_{-\infty}^{\infty} \exp\left(\frac{-ip_0x}{\hbar}\right)\psi^* \cdot \left[-i\hbar\left(\exp\left(\frac{ip_0x}{\hbar}\right)\frac{d\psi}{dx} + \frac{ip_0}{\hbar}\exp\left(\frac{ip_0x}{\hbar}\right)\psi\right)\right]dx$$

$$= \int_{-\infty}^{\infty} \psi^* \cdot -i\hbar\frac{d\psi}{dx} + p_0(\psi^*\psi)dx = \boxed{\langle P \rangle + p_0}$$

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6) $|\psi\rangle = \frac{1}{\sqrt{2}}|\omega_1\rangle + \frac{1}{2}|\omega_2\rangle + \frac{1}{2}|\omega_3\rangle$

a) $$\langle \Omega \rangle = \sum_{i=1}^{3} |\langle \omega_i|\psi\rangle|^2 \omega_i = |\langle \omega_1|\psi\rangle|^2\omega_1 + |\langle \omega_2|\psi\rangle|^2\omega_2 + |\langle \omega_3|\psi\rangle|^2\omega_3$$

$$\langle \Omega \rangle = \left(\frac{1}{\sqrt{2}}\right)^2\omega_1 + \left(\frac{1}{2}\right)^2\omega_2 + \left(\frac{1}{2}\right)^2\omega_3 = \boxed{\frac{2\omega_1 + \omega_2 + \omega_3}{4}}$$

b) Write ψ in λ_i eigenstates

Project ψ onto λ eigenbasis; $\langle \lambda_i | \omega_j \rangle = \frac{1}{\sqrt{2}}\delta_{ij}$

$$|\psi\rangle = \sum_{i=1}^{3} |\lambda_i\rangle\langle\lambda_i|\psi\rangle = |\lambda_1\rangle\langle\lambda_1|\omega_1\rangle + |\lambda_2\rangle\langle\lambda_2|\omega_2\rangle + |\lambda_3\rangle\langle\lambda_3|\omega_3\rangle$$

$$= \frac{1}{\sqrt{2}}|\lambda_1\rangle + \frac{1}{\sqrt{2}}|\lambda_2\rangle + \frac{1}{\sqrt{2}}|\lambda_3\rangle \to \text{Unnormalized}$$

$$|\psi| = \sqrt{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$|\psi\rangle = \sqrt{\frac{2}{3}} \cdot \left[\frac{1}{\sqrt{2}}|\lambda_1\rangle + \frac{1}{\sqrt{2}}|\lambda_2\rangle + \frac{1}{\sqrt{2}}|\lambda_3\rangle\right] = \boxed{\frac{1}{\sqrt{3}}|\lambda_1\rangle + \frac{1}{\sqrt{3}}|\lambda_2\rangle + \frac{1}{\sqrt{3}}|\lambda_3\rangle}$$