mathmatics

NAME:-..

SUBJECT:-..

INSTITUTE:-...

Linear Algebra [Matrices]

* Properties of Determinants :-

i) If two rows or columns of a matrix are identical then the determinant = 0

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

ii) If two rows or columns of a matrix are interchanged, then the sign of determinant changes

$$-\Delta = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

iii) If three rows or columns of matrix are intechanged, then the sign of determinant is unalterned.

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

iv) In the determinant of a matrix, if any column containing the sum or difference of two elements then it can be split into sum or difference of two determinants.

$$\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Delta = ad - bc$$

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* Lower Triangular Matrix :- If all elements above the principal digonals are zeroes.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 5 \end{bmatrix} \qquad \Delta = 15$$

* Upper Triangular Matrix :- If all elements below the principal digonal are zeroes, then it is said to be an upper triangular Matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} \qquad \Delta = -28$$

NOTE :-
If a matrix is either lower triangular or upper triangular then the determinant is the product of the principle digonal elements.

Q. Find the determinant of Matrix

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \qquad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\, 1\, [\cancel{b}+c - a-\cancel{b}]$$

$$= (a-b)(b-c)(c-a)$$

* $\boxed{|A| = |A^T|}$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$|A| = |A^T| = (a-b)(b-c)(c-a)$$

Q. Imp Result

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}$$ $R_2 - R_1$, $R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix} = \underline{\underline{ab}}$$

Q.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 6 \end{vmatrix} = ?$$

(a) 20 (b) 30 (c) 40 (d) 0

Q. Imp Result

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix} = abc$$

Q.

Imp Result

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= abc \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix} \qquad R_1 \to R_1 + (R_2+R_3).$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix} \qquad C_2-C_1,\ C_3-C_1$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Q.

Imp Result

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}.$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Q. $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = ?$

(a) 4 (b) 5 (c) 6 (d) 7

Q. $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 0(\ ) - 1(1-9) + 2(1-6) = 8-10 = -2.$

* SHORTCUT to determinant of Matrix:-

[Applicable only for 3×3 Matrix].

Q. $\left[\begin{array}{ccc|cc} 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \end{array}\right]$

$0 + 9 + 2 - 12 - 0 - 1$

$= -2.$

Q. $\left[\begin{array}{ccc|cc} 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 \end{array}\right]$

$\Delta = 1 + 8 + 8 - 4 - 4 - 4$

$\Delta = 17 - 12 = 5$

Q. $\left[\begin{array}{ccc|cc} 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \end{array}\right]$

$\Delta = 3 + 0 - 4 - 0 - 0 + 2$

$\Delta = 1$

Q. $\left[\begin{array}{ccc|cc} 1 & 2 & 5 & 1 & 2 \\ 3 & 1 & 4 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{array}\right]$

$\Delta = 2 + 8 + 15 - 5 - 4 - 12$

$\Delta = 25 - 21$

$\Delta = 4$

INVERSE OF MATRIX :-

$$\boxed{A^{-1} = \frac{\text{adj } A}{\Delta}}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

→ Interchange Principle digonal element

→ & sign change in second digonal element.

Q. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{\underset{-2}{\cancel{4-6}}} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{\underset{6}{\cancel{10-4}}} \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

Q. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^{-1} = \frac{1}{\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1}} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

For 3X3 Inverse matrix :-

$$A^{-1} = \frac{\text{adj } A}{\Delta}$$

adj A = Transpose of cofactors Matrix

Cofactor of elements = $(-1)^{i+j}$ minor

↳ Just to check sign for even or odd (row+column) no

Minor of an Element = determinant of sq. sub matrix in which the row & column of the particular element lines to be deleted.

ex:- $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

minor of 1 → $\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-2 = -1$.

Cofactor of 1 $= (-1)^{2+1}(-1) = 1$

* Shortcut to find Cofactor of 3×3 Matrix:-

Q. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

$\Delta = 0 + 9 + 2 - 12 - 0 - 1 = -2$

$$\begin{array}{c|cccc} 0 & 1 & 2 & 0 & 1 \\ \hline 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Q. $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$\Delta = 1 + 8 + 8 - 4 - 4 - 4 = 17 - 12 = 5$.

$$\begin{array}{c|cccc} 1 & 2 & 2 & 1 & 2 \\ \hline 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \end{array}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

Q. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$\Delta = \cancel{0+0+(-2)-0} \cdot 3+0-4-0-0+2 = 1.$

$$\begin{array}{c|cccc} 1 & 2 & -2 & 1 & 2 \\ \hline -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \end{array}$$

$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

Q. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$

$\Delta = 2+8+15-5-4-12 = 25-21 = 4$

$$\begin{array}{c|cccc} 1 & 2 & 5 & 1 & 2 \\ \hline 3 & 1 & 4 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 5 & 1 & 2 \\ 3 & 1 & 4 & 3 & 1 \end{array}$$

$A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$

Q.

Q/ $\begin{bmatrix} \cancel{\cos\theta} & \cancel{-\sin\theta} & \cancel{0} \\ \cancel{\sin\theta} & \cancel{\cos\theta} & 0 \end{bmatrix}$

Q $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Delta = \cos^2\theta + \sin^2\theta = 1$$

$A^{-1} =$

$$\begin{array}{c|cccc} \cos\theta & -\sin\theta & 0 & \cos\theta & -\sin\theta \\ \hline \sin\theta & \cos\theta & 0 & \sin\theta & \cos\theta \\ 0 & 0 & 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0 & \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta & 0 & \sin\theta & \cos\theta \end{array}$$

Q. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

RANK OF MATRIX :-

If all minors of order $(r+1)$ are zeroes, but there is atleast one non zero minor of order 'r'. if exists, It is called the rank of matrix and is denoted by $\boxed{\rho(A) = r}$

* Properties of Rank :-

i). If A is a null matrix or zero matrix then rank of $A = 0$

ex :- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

ii). If A is a non zero matrix, then rank of $A \geq 1$

iii) If 'I' be the unit matrix or identity matrix of order $n \times n$, then rank of $I_n = n$

ex :- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ rank = 2

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rank = 3

$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ rank = 4.

iv). If A is a matrix of order $m \times n$, then rank of $A \leq \min\{m, n\}$

ex :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ rank ≤ 2.

* If all determinants of $2\times 2 = 0$ rank < 2
* If at least one determinant of 2×2 is non zero, then rank $= 2$.

Q. Find rank of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}_{2\times 4}$

$\rho(A) \leq 2$

By checking determinant of any one of $2\times 2 \neq 0$ here

$\rho(A) = 2$

** Determinant checking
1^{st} D check $=0$
2^{nd} D check $=0$
3^{rd} D check $=0$
4^{th} D check (no need)
of course There must be rows identical with some common factors.
$\therefore$ rank will $<$ min$\{m,n\}$
$\neq$ min$\{m,n\}$.

Q. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

* purpose $\rightarrow$ make at least one row $=0$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ $\rho(A) \leq 3$.

$R_3 - (R_1 + R_2)$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\neq 0 \Rightarrow \rho(A) = 2$.

Q. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$ ③ $\neq 0 \Rightarrow \rho(A) = 2$.

$\rho(A) \leq 3$

① Can be made '0'

②

Q. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}_{3\times 3}$ $\neq 0 \Rightarrow \rho(A) = 2$.

$\rho(A) \leq 3$

Q. $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 2 \\ 0 & 5 & 6 \end{bmatrix}_{3\times 3}$

$\rho(A) \leq 3$

$\rho(A) = 3$.

Q. $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}_{3\times 4}$

$\rho(A) \leq 3$.

$R_3 - 3R_1$

$\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -10 & -4 & 0 & 2 \end{bmatrix}$

$= 0 \Rightarrow \rho(A) < 3$

$\rho(A) = 2$.

Q. $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 8 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}_{3\times 4}$

$\rho(A) \leq 3$.

$R_2 + R_1$

$R_3 - R_1$

$\begin{bmatrix} 1 & -1 & 3 & 6 \\ 2 & 7 & 0 & 2 \\ 4 & 4 & 0 & 5 \end{bmatrix}$

$\neq 0 \Rightarrow \rho(A) = 3$

Q. $$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ \hline 1 & 3 & 4 & 1 \end{bmatrix}$$

$=0 \quad =0 \quad \neq 0$

$\rho(A) = 2$

$\Rightarrow \rho(A) \leq 3$

$\rho(A) < 3 \quad \{\because R_2, R_1 \text{ identical}\}$

Q. $$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times 4}$$

$\rho(A) \leq 4$

$R_4 - (R_1 + R_2 + R_3)$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) < 4$

$R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\neq 0 \Rightarrow \rho(A) = 3.$

Q. $$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$\rho(A) \leq 4$

R_{12}

$$\begin{bmatrix} 1 & -1 & -2 & 4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 - 2R_1, \quad R_3 - 3R_1$

$R_4 - (R_1 + R_2 + R_3)$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) < 4$$

$\neq 0$

$\neq 0 \Rightarrow \rho(A) = 3.$

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix} \qquad \rho(A) \leq 4$$

$R_4 - (R_1 + R_3)$

$R_3 - (R_1 + R_2)$

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) < 4$

$\rho(A) < 3$

$\rho(A) = 2$

The consistency (→ have solⁿ) & Inconsistency (→ have no solⁿ) of the system of eqⁿ:-

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Coefficient Matrix   Const. Matrix.   Variable Matrix.

$$AB = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

Augmented Matrix.

Eqⁿ form → $\boxed{AX = B}$

* Any System is said to be a consistent System, if it has a solution.

* Inconsistent System has no solution.

→ $\boxed{\text{If } \rho(AB) = \rho(A) = \text{No. of Unknowns}}$, then the system is said to be consistent and having $\boxed{\text{unique solution}}$.

→ $\boxed{\text{If } \rho(AB) = \rho(A) < \text{No. of Unknowns}}$ then the system is said to be consistent and having more than one solution, or $\boxed{\infty \text{ no. of solutions.}}$

* Remember → either 1 solⁿ or ∞ solⁿ exist for System
To confuse, in exam option will be 2 solⁿ or 3 solⁿ ✗

→ $\boxed{\text{If } \rho(AB) \neq \rho(A)}$, then the System is said to Inconsistent and have $\boxed{\text{no solution}}$.

Q. $x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + 9z = 6$

(a) 0 (b) 1 (c) ~~2~~ (d) ∞

NOTE!-

→ If AB is rectangle matrix, then last column must be excluded.

→ If AB is a square matrix, then last column must be included

→ If last column is excluded, then it represents both the matrice's rank.

→ If last column included, then it represents only AB matrix rank.

Now,

$$[AB] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]_{3\times4} \quad R_2 - R_1, \; R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

$\neq 0 \Rightarrow \rho(AB) = \rho(A) = 3 = 3$ (no. of Unknown)

$\therefore$ System has unique soln.

Q. Check the consistency of system –

$$x - 2y + 3z = 2$$
$$2x - 3z = 3$$
$$x + y + z = 0.$$

$\rightarrow$
$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{3\times 4}$$ $R_1 + 2R_3$

$$\begin{bmatrix} 3 & 0 & 5 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$\neq 0 \Rightarrow \rho(AB) = \rho(A) = 3$ = no. of unknowns 3

$\therefore$ System having consistent and having unique sol^n.

Q.
$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

$$[AB] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}_{4\times 4}$$ $R_2 - 3R_1$, $R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & 3 \end{bmatrix}$$ $R_2 + 2R_4$, $R_3 + 3R_4$

Note:-
If 0 present → make adjacent '0'

If no '0' present, then check if 1 or −1 → make adjacent of 1 or −1 = 0

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -17 & 0 & -17 \\ 0 & -11 & 0 & -11 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

take common

take common

then $R_3 - R_2$ and interchange R_4 & R_3

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -7 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\neq 0 \Rightarrow \rho(A) = \rho(AB) = 3 =$ no. of unknowns

$\therefore$ System is consistent & having unique solution.

Q. $4x - 2y + 6z = 8$

$x + y - 3z = -1$

$15x - 3y + 9z = 21$

$$[AB] = \begin{bmatrix} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{bmatrix}_{3\times 4}$$

$R_2 + R_1$

$R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 3 & 0 & 0 & 3 \\ 6 & 0 & 0 & 6 \end{bmatrix}$$

$= 0 \Rightarrow$

$\therefore \rho(AB) = \rho(A) = 2 < 3.$

System – Consistent But ∞ Solution.

We can write system again as—

$$x + y - 3z = -1$$

$$3x = 3$$

$$\Rightarrow x = 1.$$

$$1 + y - 3z = -1$$

$$\Rightarrow y - 3z = -2 \quad \text{—} \quad (1)$$

Since, we are discussing the consistency of only real system of equations, therefore solutions always real numbers.

Let $z = k$ where $k \in R$

$$y - 3k = -2$$

$$\Rightarrow \boxed{\begin{aligned} y &= 3k - 2 \\ z &= k \\ x &= 1. \end{aligned}}$$

For different values of k, we will have diff. solutions and these solutions are infinite.

Q. $2x - y + z = 4$

$3x - y + z = 6$

$4x - y + 2z = 7$ Check the consistency of System.

$-x + y - z = 9$

→ $$[AB] = \begin{bmatrix} -1 & 1 & -1 & 9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{bmatrix}_{4\times4}$$

$R_2 + R_1$
$R_3 + R_1$
$R_4 + R_1$

$$= \begin{bmatrix} -1 & 1 & -1 & 9 \\ 1 & 0 & 0 & 13 \\ 2 & 0 & 0 & 15 \\ 3 & 0 & 1 & 16 \end{bmatrix}$$

C_{12}

$$\left[\begin{array}{c|ccc} 1 & -1 & -1 & 9 \\ \hline 0 & 1 & 0 & 13 \\ 0 & 2 & 0 & 15 \\ 0 & 3 & 1 & 16 \end{array}\right]$$

$\neq 0 \Rightarrow \rho(AB) = 4$

But $\rho(A) = 3$

$\therefore \rho(AB) \neq \rho(A)$

Hence, System is inconsistent and having no solution.

Q. for what values of λ and μ, does the system of equations

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

has —

(a) no solution

(b) Unique solⁿ

(c) More than one solution

→ $[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}_{3\times 4}$ $\quad R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ \hline 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

$\neq 0 \Rightarrow \rho(AB) = \rho(A) = 3$ if $\lambda - 3 \neq 0$

< 3 if $\lambda - 3 = 0$

Case - I

Case - II

Case - III.

Q.

$$\left.\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu \end{aligned}\right\}$$

For what values of λ & μ System has —

(a) no solⁿ

(B) unique solⁿ.

(C) more than one solⁿ.

Eigen Values and Eigen Vectors :-

Characteristic equation :- Let A be the sq. matrix of order $n \times n$ and I be the unit matrix of order $n \times n$ then $|A-\lambda I|=0$ is called the characteristic eq^n where λ is a parameter.

Ther roots of characteristic eq^n are called characterestic roots / latent / eigen / proper values.

• $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ or $[x_1 \ x_2 \ldots x_n]^T$ which satisfies the matrix eq^n $[A-\lambda I]X=0$ is called the corresponding eigen vector of the matrix.

Note!-

i) The sum of eigen values of any matrix is equal to sum of the elements of its principal digonals.

Ex!- $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (a) 1 (b) 3 (c) 5 (d) 7.

Trace of A $= 1+5+1 = 7$.

ii) The Product of eigen values of any matrix is equal to its determinant.

Ex!- The characteristic roots of matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

(a) 2, 5 ✗
(b) 3, 4 ✗
(c) 1, 6 ✓
(d) 2, 3 ✗

$\therefore$ → Sum = sum of principle digonal
$6+1=7$
↳ Product = |determinant value|

iii) The eigen values of a symmetric matrix $(A^T=A)$ are purely real.

Ex!- $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = A^T$ $\therefore$ Eigen values are 1 & 5 which are purely real

Cross check → $1+5=6$ ↓ equal to sum of Principle digonal

$1 \times 5 = 5$ equal to ↳ determinant value

iv) The eigen value of skew symmetric matrix are $(A^T = -A)$ either purely imaginary or zeroes

Ex:- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Eigen Values ?

ⓐ $1, -1$ ✗ ⓑ $i, -i$ ✓ ⓒ $-1, 0$ ✗ ⓓ $0, 1$ ✗

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

v) If a matrix is either lower triangular or upper triangular, then principle digonal elements are called eigen values.

* If Given Matrix only is L.T.M or U.T.M (without operation only original)

↓

Then only follow this rule.

Ex:- $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

$\boxed{\lambda = 1, -4, 7.}$ $\boxed{\lambda = 1, 3, 6}$

Don't Use this Property to find eigen value when L.T.M or U.T.M is made by some operation.

vi) If λ is an eigen value of A

then λ^2 ———————— A^2

λ^3 ———————— A^3

λ^n ———————— A^n

$\frac{1}{\lambda}$ ———————— A^{-1}

Ex:- if $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ and $\lambda = 1, 5$

then for S^2 what will be eigen value → After getting result λ, verify it by sum & product rule.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

then eigen values of A^3 are —

a) 5, 4

b). 9, 1

c) 16, -1

d) 27, -1

Q. Find the eigen values and corresponding eigen vectors of Matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$\rightarrow$ $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

$\Rightarrow$ $10 - 7\lambda + \lambda^2 - 4 = 0$

$\Rightarrow$ $\lambda^2 - 7\lambda \cancel{+} + 6 = 0$

$\Rightarrow$ $(\lambda - 1)(\lambda - 6) = 0$

$\lambda = 1, 6.$

Eigen Vector —

$[A - \lambda I] X = 0.$

Substract λ from matrix and multiply ~~vector~~ X Matrix

for $\lambda = 1$ $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow$ $4x_1 + 4x_2 = 0$

$x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1}$ eigen vector = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $[1 \ -1]^T$.

for $\lambda = 6$ —

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$-x_1 + 4x_2 = 0$

$\frac{x_1}{4} = \frac{x_2}{1}$ eigen vector = $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ or $[4, 1]^T$.

Q. Find the ch. roots of matrix or Eigen Value.

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

1st step

(a) 3, 7, 8 ✓
(b) 2, 2, 14 ✓
(c) 0, 3, 15 ✓
(d) 1, 4, 9 ✗

} By sum of diagonal rule = sum of eigen value.

2nd step

check determinant.

$\Delta = 0$.

so, one root must be 0 so that to make product rule of eigen values correct

Q. Find the latent root of matrix —

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

	1st step	2nd step
(a) 1, 2, 3	✓	✓
(b) 0, 2, 4	✓	✗
(c) 1, 1, 4	✓	✗
(d) 2, 2, 2.	✓	✗

$4 + 2 = 6$.

Q. Find Eigen Values and Corresponding eigen vectors of

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

$\lambda = 1, -4, 7.$ $\{\because$ It is U.T.M$\}$

for $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$2x_2 + 3x_3 = 0.$

$-5x_2 + 2x_3 = 0$

$6x_3 = 0. \Rightarrow x_3 = 0 \Rightarrow x_2 = 0 \rightarrow$ not able to identify any infⁿ abt x_1.

Remember

* for any eigen values if in eigen vectors set, two values are $= 0$ then third must be only non zero

$\therefore$ eigen vector set $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = -4$
$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$5x_1 + 2x_2 + 3x_3 = 0. \Rightarrow 5x_1 + 2x_2 = 0.$

$2x_3 = 0 \Rightarrow x_3 = 0 \qquad \Rightarrow 5x_1 = -2x_2$

$11x_3 = 0. \Rightarrow x_3 = 0. \qquad \Rightarrow \frac{x_1}{2} = \frac{x_2}{-5}$

Eigen Vector: $\begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$.

for $\lambda = 7$.
$$\begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$-6x_1 + 2x_2 + 3x_3 = 0.$

$-11x_2 + 2x_3 = 0 \qquad \Rightarrow 6x_1 = 37 \Rightarrow x_1 = \frac{37}{6}..$

$\Rightarrow \frac{x_2}{2} = \frac{x_3}{11}$

eigen vector: $\begin{bmatrix} \frac{37}{6} \\ 2 \\ 11 \end{bmatrix}$ or $\begin{bmatrix} 37 \\ 12 \\ 66 \end{bmatrix}$

Q. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ one of the eigen value is -2. which of the following is an eigen vector.

ⓐ $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ ⓑ $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$ ⓒ $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ ⓓ $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

→ for $\lambda = -2$ $\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

& $5x_1 - 2x_2 + 2x_3 = 0. \Rightarrow 5x_1 - 2x_2 = 0.$

$\frac{x_1}{2} = \frac{x_2}{5}.$

$\left.\begin{array}{l} x_3 = 0. \\ 3x_3 = 0 \end{array}\right\} x_3 = 0$

eigen vector $= \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

Q. for the matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ which one of the following is an eigen vector.

a) $[1 \ -1 \ 1]^T$

b) $[1 \ 2 \ 1]^T$

c) $[2 \ -1 \ 1]^T$

d) $[1, 1, -1]^T$

Give 1st pref to least eigen value.
Check only by substracting $-\lambda$ in 1st row check eigen value.

If option not found: Again 2nd pref to 2nd least eigen value same substract λ from 2nd row check eigen value

Again if option not found: Again 3rd pref to 3rd least eigen value but do not substract to 3rd row substract from either 1st or 2nd row (no effect) check eigen value.

Cayley - Hamilton Theorem :-

Every sq. matrix satisfies its own characteristic eqⁿ.

Ex:-

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

$\Rightarrow \quad 10 - 7\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda^2 - 7\lambda + 6 = 0$

Means —

$A^2 - 7A + 6I = 0 \rightarrow$ A/c to Cayle Hamilton Theorem

$\downarrow$

For Inverse finding without knowing adj of A & determinant.

$\Rightarrow \quad 6I = 7A - A^2$

$\Rightarrow \quad 6IA^{-1} = 7AA^{-1} - A^2A^{-1}$

$\Rightarrow \quad 6A^{-1} = 7I - A.$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}.$$

Q. Let M be 3×3 matrix with ch. eqⁿ $x^3+1=0$ then the inverse of M = —

→ A/c to Cayle Hamilton theorem —

$M^3 + I = 0.$

$I = -M^3$

$IM^{-1} = -M^3M^{-1}$

$$\boxed{M^{-1} = -M^2}$$

Q. Let P be a 3×3 matrix with ch. eqⁿ $\lambda^3+\lambda^2+2\lambda+1=0$ then inverse of P = ____

(a) P^2+P+I

(b) P^2+P+2I

✓ (c) $-(P^2+P+2I)$

(d) $-(P^2+P+I)$.

→ $P^3+P^2+2P+I=0$

$\Rightarrow I = -(P^3+P^2+2P)$

$\Rightarrow \boxed{P^{-1} = -(P^2+P+2I)}$

Q. find the value of λ for which the following system of eqⁿs

$\lambda x+3y+5z=0$

$2x-4\lambda y+\lambda z=0$

$-4x+18y+7z=0$

have a non trivial solution.

Note:-

$\boxed{AX=0}$ is called Homogeneous.

$X=0$ will always be a solution.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

⇓

This is called Trivial equation.

Note:-

for the System of Eqⁿ $AX=0$

$X=0$ will always a solution

and this solⁿ is called a trivial solⁿ.

for a non trivial solution determinant of coefficient matrix = 0 (always)

$$\begin{bmatrix} \lambda & 3 & 5 \\ 2 & -4\lambda & \lambda \\ -4 & 18 & 7 \end{bmatrix} \begin{matrix} \lambda & 3 \\ 2 & -4\lambda \\ -4 & 18 \end{matrix}$$

$-28\lambda^2-12\lambda+180-80\lambda-18\lambda^2-42=0$

$\Rightarrow -46\lambda^2-92\lambda+138=0$

$\Rightarrow \lambda^2+2\lambda-3=0$

$\lambda = -3$ or 1.

Q. Rank of $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

(a) 1 (b) 2 (c) 3 (d) 4.

$\rightarrow$ $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{3\times 4}$ $\quad \rho(A) \le 3$

$R_2 - 4R_1$

$\begin{bmatrix} 4 & 2 & 1 & 3 \\ -10 & -5 & 0 & +5 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

$= 0 \Rightarrow \rho(A) < 3$

$\rho(A) = 2$.

Q. Rank and nullity of matrix

$\begin{bmatrix} 6 & 1 & 8 & 3 \\ 2 & 3 & 0 & 2 \\ 4 & -1 & -8 & -3 \end{bmatrix}$

*** $\boxed{\text{Nullity of } A = \text{no. of columns} - \text{Rank}}$

$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 2 & 3 & 0 & 2 \\ 4 & -1 & -8 & -3 \end{bmatrix}$ $\quad \rho(A) \le 3$

$\neq 0 \Rightarrow \rho(A) = 3$

$\therefore$ nullity of $A = 4 - 3$

$= 1$ Ans

Q. The rank of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & K \end{bmatrix}$ is 2

then $K = \underline{5}$

Q. The inverse of $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$

****.

Inverse of Digonal matrix is inverse of each element in that digonal.

$\rightarrow \begin{bmatrix} \cancel{\frac{1}{.5}}\,2 & 0 & 0 \\ 0 & \cancel{\frac{1}{1}}\,1 & 0 \\ 0 & 0 & \cancel{\frac{1}{0.2}}\,5 \end{bmatrix}$

Q. The rank of $\begin{bmatrix} 3 & & & & & & & & \\ & 0 & & & & & & 0 & \\ & & 0 & & & & & & \\ & & & -3 & & & & & \\ & & & & 0 & & & & \\ & & & & & 4 & & & \\ & 0 & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & -4 \end{bmatrix}_{9 \times 9}$

**'*.*.

Rank of digonal matrix = no. of non zero elements in that digonal

rank = 4

Q. Let A, B, C, D be $n \times n$ matrices each with non zero determinant and if $ABCD = I$ then $B^{-1} = \underline{\quad}$

$(ABCD)^{-1} = I^{-1}$

$D^{-1} C^{-1} \cdot B^{-1} A^{-1} = I$

$\cancel{D}D^{-1} C^{-1} B^{-1} A^{-1} = ID$

$\cancel{C}C^{-1} B^{-1} A^{-1} = IDC$

$B^{-1} \cancel{A^{-1}}A = IDCA = DCA$

Q. The eigen vector pair of matrix $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\rightarrow$ $\cancel{[3-\lambda}$ $\Rightarrow$ $-9 - \cancel{3\lambda} + \cancel{3\lambda} + \lambda^2 - 16 = 0$

$\Rightarrow \lambda^2 \pm 25$

$\lambda = 5, -5$.

for $\lambda = 5$

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$-2x_1 + 4x_2 = 0$

$4x_1 - 8x_2 = 0.$

$\frac{x_1}{\cancel{8}\ 2} = \frac{x_2}{\cancel{4}\ 1}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

for $\lambda = -5$.

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$8x_1 + 4x_2 = 0$

$\frac{x_1}{\cancel{4}\ 1} = \frac{x_2}{-\cancel{8}\ -2}$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Q. How many of the following matrices ~~of~~ have an eigen value = 1

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$\lambda = 1, 0$ $\qquad \lambda = 0, 0$ $\qquad (1-\lambda)^2 + 1 = 0$ $\qquad (-1-\lambda)^2 = 0$

(a) 1 $\quad$ (b) 2 $\quad$ (c) 3 $\quad$ (d) 4.

Q. If the following represents eqⁿ of line then line passes through the point

$$\begin{bmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 0$$

a) (0, 0)

b) (3, 4)

c) (4, 3)

d) (4, 4)

→

$$\begin{array}{ccccc} x & 2 & 4 & x & 2 \\ y & 8 & 0 & y & 8 \\ 1 & 1 & 1 & 1 & 0 \end{array}$$

$8x + 4y - 32 - 2y = 0$

$\Rightarrow \quad 8x + 2y = 32$

$\cancel{-8x + 0 + 2y - 8x - 0 + 0 = 0}$

$\cancel{\Rightarrow -8x + 2y + 32 = 0}$

$\cancel{\Rightarrow 8x + 2y = 32.}$

Q. A is 3×4 real matrix and $Ax = B$ is an inconsistent System of eqⁿs then the highest possible rank of A —

(a) 1 (b) 2 ✓ (c) 3 (d) 4 ..

$A = 3 \times 4$

$[AB] = 3 \times 5 \Rightarrow \rho(A) \leq 3$

for inconsistent System $\rho(A) \neq \rho[AB]$ $\left\{ \because \rho(A) \text{ can't be greater than } \rho(AB) \right\}$

$\therefore \rho(A) < 3$

$\therefore \rho(A) = 2$ ✓

Q. Given an orthogonal matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ then $(AA^T)^{-1} = $ I

NOTE :- A matrix is said to be orthogonal matrix if $AA^T = I$.

Q. $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$

then $a+b =$ ____

$A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$

$\therefore a = \frac{0.1}{6}$

$b = \frac{2}{6}$

$\therefore a+b = \frac{2.1}{6} = \frac{21}{60} = \frac{7}{20}$ Ans

Q. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}$ and $B = 2A^2$ then $|B| =$ ____

(a) 16 (b) 32 (c) 64 (d) 128.

$B = 2A^2$

$|B| = |2A^2| \longrightarrow$ order $\doteq \underline{\underline{3\times 3}}$

$= 2^{\cancel{n}\,3} |A^2|$

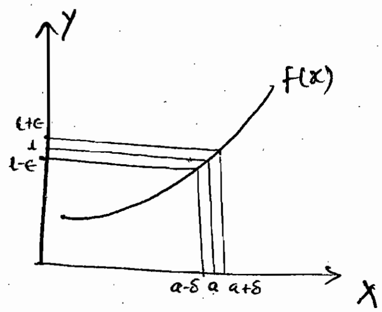
$= 8(16)$

$= 128.$

DIFFERENTIAL CALCULUS

Limit:- A no l is said to be the limit of the fn $f(x)$ as $x \to a$ if $\forall\, \epsilon > 0$ (however small) $\exists$ a $\delta > 0$ such that $|f(x) - l| < \epsilon$ $\forall\, |x-a| < \delta$

$$\boxed{\lim_{x \to a} f(x) = l}$$



$\therefore \quad l - \epsilon < f(x) < l + \epsilon$

$\forall \quad a - \delta < x < a + \delta$

* Left limit :- $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h), \quad (a - \delta < x < a)$.

* Right limit :- $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h), \quad (a < x < a+\delta)$

* Existence of Limit :- The limit of a function exists if both the left and right limits are existed and are equal.

ex:- $\lim\limits_{x \to a} \dfrac{1}{x-a}$.

$\therefore \quad \lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h) = \dfrac{1}{a-h-a} = \dfrac{-1}{h}$

$\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h) = \dfrac{1}{\cancel{a}+h-\cancel{a}} = \dfrac{1}{h}$.

$\because$ LH limit $\neq$ R.H limit

$\therefore$ Limit not exists

Formulae :-

$$\lim_{x\to a} \frac{x^n-a^n}{x-a} = n a^{n-1}$$

$$\lim_{x\to a} \frac{x^m-a^m}{x^n-a^n} = \frac{m}{n} a^{m-n}$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \frac{e^{mx}-1}{x} = m.$$

$$\lim_{x\to 0} \frac{a^x-1}{x} = \log a$$

$$\lim_{x\to 0} \frac{a^x-b^x}{x} = \log\left(\frac{a}{b}\right)$$

- $$\lim_{x\to 0} (1+x)^{1/x} = e$$
- $$\lim_{x\to 0} (1+ax)^{1/x} = e^a.$$

$$\lim_{x\to 0} (1+ax)^{b/x} = e^{ab}$$

$$\therefore \lim_{x\to 0} (1+ax)^{1/bx} = e^{a/b}$$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = e^{-1}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$$

$$\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx} = \frac{m^2}{n^2}$$

$$\lim_{x\to 0} \frac{\cos ax-\cos bx}{x^2} = \frac{b^2-a^2}{2}$$

$\frac{0}{0}, \frac{\infty}{\infty}, \frac{\infty}{0}, 0\times\infty; 1^\infty$ etc are indeterminate form.

If $\frac{0}{0}, \frac{\infty}{\infty}$ then apply L.H. rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \text{------}$$

$$\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$$

$\lim_{x\to\infty} \frac{f(x)}{g(x)}$, if $f(x)$ & $g(x)$ are algebric fn.

Case-1 if degree of $f(x) >$ $g(x)$ degree then result $=\infty$

ex:- $$\lim_{x\to\infty} \frac{2x^3+3x+5}{5x^2+6x+7} = \infty.$$

Case-2 if degree of $f(x) <$ $g(x)$ degree then result $= 0$

ex:- $$\lim_{x\to\infty} \frac{2x^2+3x+4}{x^3+8} = 0$$

Case-3 If $f(x)$ and $g(x)$ have equal degrees then -

result $= \frac{\text{co.eff of } N^r}{\text{coeff of } D^r}$

ex:- $$\lim_{x\to\infty} \frac{2x^2+3x+4}{5x^2+5x+8} = \frac{2}{5}$$

* $|x| = x$ if $x>0$
* $|x| = -x$ if $x<0$.
* $[x]$ = greatest integer not greater than x.
* **Examples :-**

① $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

② $\lim_{x\to 0} \frac{1-\cos x}{x \sin x} = \frac{1-\cos x}{x^2} = \frac{\sin x}{2x} = \frac{\cos x}{2} = \frac{1}{2}$

In Trigonometric fn, avoid LH rule Try alternates

③ $\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x} = \frac{1-\cos x}{(1-\cos x)(1+\cos x)} \frac{1}{\cos x(1+\cos x)} = \frac{1}{1\cdot(1+1)} = \frac{1}{2}$

④ $\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{1-\sqrt{x}} = \frac{-\frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)}{-\frac{1}{2\sqrt{x}}} = \frac{\pi/2}{1/2} = \pi$

⑤ $\lim_{x\to 0} \frac{2^{8\cos x}}{8} \left(\frac{\sin^8\left(\frac{\pi}{6}+x\right) - \sin^8\frac{\pi}{6}}{x} \right)$

$= \frac{2^{8\cdot 1}}{8} \left[\frac{8 \sin^7\left(\frac{\pi}{6}+x\right) \cos\left(\frac{\pi}{6}+x\right)}{1} - 0 \right]$

$= \frac{2^8}{8} \left[8 \times \frac{1}{128} \cdot \frac{\sqrt{3}}{2} \right)$

$= \sqrt{3}$ Ans

⑥ $\lim_{x\to 0} \frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} = -1$

⑦ $\lim_{x\to 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x} = 2/3$

⑧ $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$.

(9) ~~to~~. $\lim_{x\to0}(1+\sin x)^{\operatorname{cosec} x} = e^{\lim_{x\to0}\operatorname{cosec}x[\cancel{1}+\sin x-\cancel{1}]}$ $\quad \overset{\to 0}{}$

$= 1$

(10) $\lim_{x\to0}(1+2x)^{1/3x} = e^{\lim_{x\to0}\frac{1}{3\cancel{x}}[\cancel{1}+2\cancel{x}-\cancel{1}]} = e^{\frac{2}{3}}$

(11) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n} = e^{\lim_{n\to\infty}2n[\cancel{1}-\frac{1}{\cancel{n}}-\cancel{1}]} = \cancel{e^{\lim_{n\to\infty}[2n}} \quad e^{-2}$

(12) $\lim_{x\to\infty}\left(\frac{x-1}{x-2}\right)^{x} = e^{\lim_{x\to\infty}x\left[\frac{\cancel{x}-\cancel{1}-\cancel{x}+\cancel{2}^{1}}{x-2}\right]} = e$

(13) $\lim_{n\to\infty}\frac{1+2+3+\dots+n}{n^2} = \frac{\cancel{n}(n+1)}{2n^{\cancel{2}}} = \frac{1}{2}$

(14) $\lim_{n\to\infty}\frac{1+4+9+\dots+n^2}{n^3} = \frac{\cancel{n}(n+1)(2n+1)}{6n^{\cancel{3}2}} = \frac{2n^2+3n+1}{6n^2} = \frac{\cancel{2}}{\cancel{6}3} = \frac{1}{3}$

(15) $\lim_{n\to\infty}\frac{n[1^3+2^3+\dots+n^3]^2}{(1^2+2^2+\dots+n^2)^3} = \frac{\cancel{n}[\cancel{n^4}^{\,n^2}(n+1)^{\cancel{4}}]}{4^2\,\cancel{n^3}\,(n+1)^{\cancel{3}}(2n+1)^3}\times 6^3$

$= \frac{6^3}{4^2}\,\frac{n^3(n+1)^3}{(2n+1)^3} = \frac{6^3}{4^2\times8} = \frac{27}{16}$

Trick for Partial fraction for two terms बड़ा भाग - छोटा भाग ✓

(16) $\lim_{n\to\infty}\left[\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{n(n+1)}\right] = \sum\frac{1}{n(n+1)} = \sum\frac{1}{n}-\frac{1}{n+1}$

$= \frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}\dots\frac{1}{n+1} = 1$

(17) $\lim_{x\to0}\frac{1-\cos2x}{1-\cos3x} = \frac{+2\sin2x}{3\sin3x} = \frac{4\cos2x}{9\cos3x} = \frac{4}{9}$

(18) $\lim_{x\to0}\frac{\cos2x-\cos3x}{x^2} = \frac{-2\sin2x+3\sin3x}{2x}$

$= \frac{-4\cos2x+9\cos3x}{2} = \frac{5}{2}$

Continuity of function :-

A function f is said to be continuous at $x = a$, if $\underset{x\to a}{Lt}\, f(x) = f(a)$, otherwise it is said to be a discontinuous function.

* Types of Discontinuous fn :-

i) Discontinuity of 1st Type (Jumped Discontinuity)

A fn. f is said to have discontinuity of 1st type if

$$\boxed{\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)}$$

ii) Discontinuity of 2nd Type

A fn. f is said to have discontinuity of 2nd Type if either the left limit or the right limit are both does not exist.

iii) Discontinuity of 3rd Type (Removal Discontinuity)

A fn 'f' is said to have discontinuity of 3rd type

$$\boxed{\lim_{x\to a^-} f(x) = \underset{x\to a^+}{Lt} f(x) \neq f(a)}$$

* ~~Certain Standard functions :-~~

* Standard Continuous function :-

① x^n is a continuous fn $\forall x$ when $n > 0$

$(x, x^2, x^3, x^{2/3}, x^{3/2} \ldots)$

② x^n is a continuous fn $\forall x$ except $x = 0$ when $n < 0$

$\left(x^{-2} = \frac{1}{x^2},\ x^{-3} = \frac{1}{x^3}, \ldots\right)$

③ $|x|$ is a continuous fn $\forall x$.

iv) $\log x$ is a continuous fn $\forall x > 0$.

v) Every exponential function (e^x or a^x) is continuous.

vi) $\sin x$ and $\cos x$ are continuous fn $\forall x$

vii) $\tan x$ and $\sec x$ are continuous fn $\forall x$ except $x = (2n+1)\frac{\pi}{2}$

viii) $\cot x$, $\operatorname{cosec} x$ are continuous fn $\forall x$ except $x = n\pi$

Q. $f(x) = \dfrac{1}{1+2^{1/x}}$ at $x = 0$

$$\lim_{h \to 0^-} f(0-h) = \frac{1}{1+2^{\frac{1}{0-h}}} = 1$$

$$\lim_{h \to 0^+} f(0+h) = \frac{1}{1+2^{\frac{1}{0+h}}} = 0.$$

$\because$ L.H limit $\neq$ RH limit

$\therefore$ discontinuous at $x = 0$.

Q $f(x) = \dfrac{x(e^{1/x} - 1)}{e^{\frac{1}{x}} + 1}$, $x \neq 0$

$= 0$, $x = 0$.

$$\lim_{h \to 0^-} f(0-h) = \frac{-h\left(e^{\frac{1}{-h}} - 1\right)}{e^{\frac{1}{-h}} + 1}$$

$$= \frac{-h \times -1}{1} = 0$$

$$\lim_{h \to 0^+} f(0+h) = \frac{h\left(e^{\frac{1}{h}} - 1\right)}{e^{\frac{1}{h}} + 1} = \frac{h\left(1 - \frac{1}{e^{1/h}}\right)}{\left(1 + \frac{1}{e^{1/h}}\right)} = h = 0$$

$f(0) = 0$

$\therefore$ LH limit $=$ RH limit $= f(0)$ $\Rightarrow$ Continuous at $x = 0$

Q. A fn $f(x)$ is defined by —

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2} - x, & 0 < x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ \frac{3}{2} - x, & \frac{1}{2} < x < 1 \\ 1, & x \geq 1 \end{cases}$$

Which of the following is true —

(a) f is continuous at $x = 0$ ×

(b) f is discontinuous at $x = \frac{1}{2}$ ✓

(c) Continuous at $x = 1$ ×

(d) All are true.

Q.

25.9.14.

Q. A fn $f(x)$ is given by —

$$f(x) = \begin{cases} 0, & x \leq 0 \\ 5x-4, & 0<x\leq 1 \\ 4x^2-3x, & 1\leq x\leq 2 \\ 3x+4, & x>2 \end{cases}$$

Check continuity @ $x = 0, 1, 2$ —

→. At $x=0$ → discontinuous

At $x=1$ → continuous.

At $x=2$ → continuous.

DIFFERENTIABILITY

— A fn f is said to be differentiable at $x=a$ if both ~~bot limit~~

$$L\cdot H\cdot D = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

$$R\cdot H\cdot D = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists and are equal. They are equal to $f'(a)$.

$d(x^n) = nx^{n-1}$

$d\sqrt{x} = \frac{1}{2\sqrt{x}}$

$d\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$

$d\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

$d\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$

$d(e^{ax}) = ae^{ax}$

$d(a^x) = a^x \log a$

$d(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$d(\sin^{-1}hx) = \frac{1}{\sqrt{1+x^2}}$

$d(\tan^{-1}x) = \frac{1}{1+x^2}$

*** $\frac{d}{dx}(x^x) = x^x(1+\log x)$

*** $\frac{d}{dx}(x^{\frac{1}{x}}) = x^{\frac{1}{x}}\cdot\frac{1}{x^2}(1-\log x)$

$\frac{d}{dx}(u\pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

$\frac{d}{dx}(u\cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

then $\frac{dy}{dx} = \frac{2}{1+x^2}$

Q. If $y = x^{x^{x^{\cdots}}}$ then $\frac{dy}{dx} = ?$

$$y = x^y$$

$$\log y = y \log x .$$

$$\frac{dy}{y} = y\left(\frac{1}{x}dx\right) + \log x \, dy$$

$$dy\left(\frac{1}{y} - \log x\right) = \frac{y}{x}dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{(y/x)}{(1/y - \log x)} = \frac{y^2}{x - xy\log x}$$

Q. $x^y = e^{(x-y)}$ in terms of x $\frac{dy}{dx} = ?$

2 marks

$$y \log x = x - y$$

$$y(\log x + 1) = x$$

$$dy = \frac{(\log x+1) - \cancel{x}\left(\frac{1}{\cancel{x}}\right)}{(\log x+1)^2} dx .$$

$$\frac{dy}{dx} = \frac{\log x}{(\log x+1)^2}$$

Q. If $x^m . y^n = a^{m+n}$ then $\frac{dy}{dx}$

$$m \log x + n \log y = m+n \log a$$

$$\frac{m}{x} dx + \frac{n}{y} dy = 0$$

$$\cancel{\frac{dx}{dy}} = \cancel{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{m y}{x n} = -\frac{m}{n} \cdot \frac{y}{x}$$

Q. $x^m . y^n = (x+y)^{m+n} \Rightarrow \frac{dy}{dx} = ?$

$$m \log x + n \log y = m+n \log(x+y)$$

$$\frac{m}{x} dx + \frac{n}{y} dy = \frac{m+n}{x+y} . (dx + dy)$$

$$\Rightarrow \left(\frac{m}{x} - \frac{m+n}{x+y}\right) dx = \left(\frac{m+n}{x+y} - \frac{n}{y}\right) dy$$

$$\frac{my - nx}{x(x+y)} \qquad \frac{my - nx}{y(x+y)}$$

Imp Result.

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Q. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$ then $\frac{dy}{dx} = ?$

$$y = \sqrt{x+y}$$

$$\Rightarrow y^2 = x+y$$

$$\Rightarrow 2y\, dy = dx + dy$$

$$\Rightarrow dy(2y-1) = dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

Imp Result

$$\boxed{\text{If } y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots}}} \quad \text{then } \frac{dy}{dx} = \frac{f'(x)}{2y-1}}$$

Q. $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots}}}$ then $\frac{dy}{dx} = ?$ $\qquad \frac{\sec^2 x}{2y-1}$

Q. $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots}}}$ then $\frac{dy}{dx} = ?$ $\qquad \frac{1/x}{2y-1}$

Q. $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots}}}$ then $\frac{dy}{dx} = ?$ $\qquad \frac{\cos x}{2y-1}$

Q. $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ then $(1-x^2)y_1 - xy =$ ____

(a) 0 (b) 1 (c) y (d) $-y$.

To save time → uv form is easy than $\frac{u}{v}$

$$(y\sqrt{1-x^2})^2 = (\sin^{-1}x)^2$$

$$\Rightarrow y^2(1-x^2) = (\sin^{-1}x)^2$$

$\Rightarrow$ ~~$2y \times dy \times$~~

$$\Rightarrow (1-x^2)\,2y\,dy + y^2(-2x\,dx) = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow (1-x^2)\,2y\,dy = dx\left(\frac{2\sin^{-1}x}{\sqrt{1-x^2}} + 2y^2 x\right)$$

$$\Rightarrow 2y\,[(1-x^2)y_1 - xy] = 2y$$

$$\Rightarrow (1-x^2)y_1 - xy = 1.$$

Q. $y = a\cos(\log x) + b\sin(\log x)$ then $x^2y_2 + xy_1 =$ ____

$$y_1 = \frac{-a\sin(\log x) + b\cos(\log x)}{x}$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x).$$

$$xy_2 + y_1 \cdot 1 = \frac{-a\cos(\log x) - b\sin(\log x)}{x}$$

$$x^2y_2 + xy_1 = y.$$

Q. If $y = \tan^{-1}\left[\dfrac{(3-x)\sqrt{x}}{1-3x}\right] \Rightarrow \left.\dfrac{dy}{dx}\right|_{x=1} = ?$

General differentiation consumes time ∴ replacement substitute.

Let $x = \tan^2\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$.

$$\therefore\ y = \tan^{-1}\left(\frac{(3-\tan^2\theta)\tan\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$y = 3\tan^{-1}\sqrt{x}.$$

$$\Rightarrow \frac{dy}{dx} = 3\cdot\frac{1}{1+(\sqrt{x})^2}\times\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{1+1}\times\frac{1}{2} = \frac{3}{4}\ \text{Ans}$$

Q. $y = \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) \Rightarrow \dfrac{dy}{dx} = ?$

1 + sin x we don't have formula

But 1 + cos x we have formula

That's why this approach

$$= \tan^{-1}\left(\frac{\sin(90-x)}{1+\cos(90-x)}\right)$$

$$= \tan^{-1}\left(\frac{\sin(90-x)}{2\cos^2(45-\frac{x}{2})}\right)$$

$$= \tan^{-1}\left(\frac{\cancel{2}\sin(45-\frac{x}{2})\cdot\cancel{\cos(45-\frac{x}{2})}}{\cancel{2}\cos^{\cancel{2}}(45-x/2)}\right)$$

$$= 45-\frac{x}{2}.$$

$$y = \frac{\pi}{4}-\frac{x}{2}.$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}\ \text{Ans}$$

Q. If $x = a\cos^3\theta,\ y = a\sin^3\theta \Rightarrow \dfrac{dy}{dx} = ?$

$$\cancel{dx = -3a\cos^2\theta \times 2\cos\theta \times \sin\theta}$$

$$\cancel{dy = 3a\sin^2\theta \times 2\sin\theta \times \cos\theta}$$

$$\cancel{\Rightarrow \frac{dy}{dx} = \frac{-\sin^2\theta}{\cos^2\theta} = -\tan^2\theta}$$

$$\frac{dy}{dx} = \frac{3\cancel{a}\sin^{\cancel{2}}\theta \times \cancel{\cos}\theta}{-3\cancel{a}\cos^{\cancel{2}}\theta \times \cancel{\sin\theta}} = -\tan\theta.$$

Q. $x = 3\cos\theta - \cos^3\theta$

$y = 3\sin\theta - \sin^3\theta$

$$dy = 3\cos\theta - 3\sin^2\theta\cos\theta = 3\cos\theta(1-\sin^2\theta)$$

$$dx = -3\sin\theta + 3\cos^2\theta\cdot\sin\theta = -3\sin\theta(1-\cos^2\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\cancel{3}\cos\theta\cdot\cos^2\theta}{-\cancel{3}\sin\theta\cdot\sin^2\theta} = -\cot^3\theta \quad \underline{\text{Ans}}.$$

Q. $x = a(\theta + \sin\theta)$

$y = a(1-\cos\theta)$

$$\frac{dy}{dx} = \frac{\cancel{a}(\theta + \sin\theta)}{\cancel{a}(1+\cos\theta)} = \frac{\cancel{2}\sin\theta/2\cdot\cancel{\cos}\,\theta/2}{\cancel{2}\cos^{\cancel{2}}\theta/2} = \tan\frac{\theta}{2} \quad \underline{\text{Ans}}$$

Q. $x = a[\theta\sin\theta + \cos\theta]$

$y = a[\sin\theta - \theta\cos\theta]$

$$\frac{dy}{dx} = \tan\theta \quad \underline{\text{Ans}}$$

MEAN VALUE THEOREM :-

(1). Rolle's mean value theorem :-

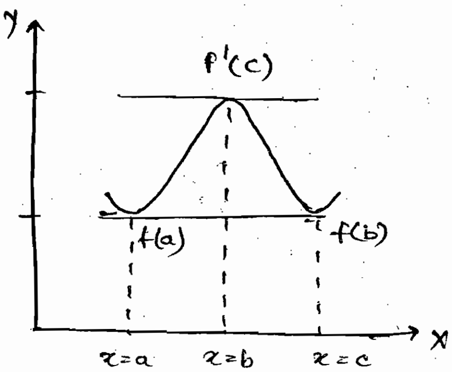
Let $f(x)$ be a function defined such that —

i) $f(x)$ is continuous in $[a, b]$

ii) $f(x)$ is differentiable in (a, b)

iii) $f(a) = f(b)$

then $\exists$ at least one value $c \in (a, b)$ such that $f'(c) = 0$



Proof :-

$f'(c)$ = slope of tangent line.

$= \text{slope of } \overline{AB}$ $\{\because$ two parallel line have equal slopes$\}$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(b) - f(a)}{b - a}$$

$$= \frac{0}{b-a} \qquad \{\because f(a) = f(b)\}$$

$$\boxed{f'(c) = 0}$$

Q. Find c ~~of~~ for the Rolle's mean value theorem for -

$f(x) = x^3 - 4x$ in $[-2, 2]$. (a) $\frac{2}{\sqrt{3}}$ (b) $-\frac{2}{\sqrt{3}}$ (c) $\pm\frac{2}{\sqrt{3}}$ (d) none

→ $f(-2) = -8+8 = 0$

$f(2) = 8-8 = 0$

$\therefore f'(x) = 3x^2 - 4$

$f'(c) = 3c^2 - 4 = 0$

$\Rightarrow c^2 = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$ Ans.

But Problem is not over.

Cross-verify the points in case of $\pm\sqrt{}$

Q. → are points inside interval or not ?

Here $\pm\frac{2}{\sqrt{3}}$ both are inside so ans is $\pm\frac{2}{\sqrt{3}}$ Ans.

Q. $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$.

$f(0) = \frac{0}{1} = 0$

$F(\pi) = \frac{0}{e^\pi} = 0$.

$\therefore f'(x) = \frac{e^x \cos x - \sin x \, e^x}{e^{2x}}$.

$f'(c) = e^c \cos c - \sin c \cdot e^c = 0$

$\Rightarrow \cancel{e^c} \cos c = \cancel{e^c} \sin c$

$\Rightarrow \tan c = 1$

$\left\{ \begin{array}{l} \therefore c = \frac{\pi}{4} \text{ Ans} \\ c \in [0, \pi] \text{ must satisfy this} \end{array} \right.$

Q. $f(x) = \dfrac{x^2+ab}{x(a+b)}$, $[a,b]$, $a>0,\ b>0$.

ⓐ $-\sqrt{ab}$ ⓑ $\sqrt{ab}$ ⓒ $\pm\sqrt{ab}$ ⓓ none.

$$f'(x) = x(a+b)[2x] - [x^2+ab][a+b] = 0$$

$$\Rightarrow 2x^2(\cancel{a}+b) = (x^2+ab)(\cancel{a}+b)$$

$$\Rightarrow x^2 = ab$$

$$\Rightarrow x = \pm\sqrt{ab}$$

But $x = -\sqrt{ab}$ should not $\in [a,b]$ when $a>0,\ b>0$

$$\therefore x = \sqrt{ab}$$

$$\Rightarrow \boxed{c = \sqrt{ab}}$$

Lagrange's mean value theorem :-

Let $f(x)$ be a function defined such that –

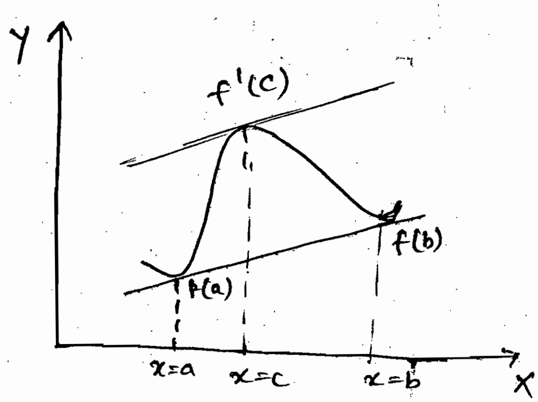
i) $f(x)$ is continuous in $[a,b]$

ii) $f(x)$ is differentiable in (a,b)

iii) $f(a) \neq f(b)$

Then $\exists$ at least one value $c \in (a,b)$ such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$



Q. $f(x) = \log x$ in $[1, e]$.

$f(e) = \log e$

$f(1) = \log 1 = 0.$

$\therefore f'(c) = \dfrac{\log e - \log 1}{e-1} = \dfrac{\log e}{e-1}$

$\Rightarrow \dfrac{1}{c} = \dfrac{\log e}{e-1}$

$\Rightarrow c = \dfrac{e-1}{\cancel{\log e}_{\,1}}$ Ans..

$\Rightarrow c = e-1 \in (1, e)$ $\{\because e \approx 2.71\}$

Q. $f(x) = x^3 - 6x^2 + 11x - 6$ in $[0, 4]$..

$3x^2 - 12x + 11 = \dfrac{4^3 - 6(4)^2 + 11(4) \cancel{-6} - \cancel{(-6)}}{4-0}$

$\Rightarrow 3x^2 - 12x + 11 = \dfrac{64 - 96 + 44}{4} = \dfrac{108 - 96}{4} = \dfrac{\cancel{12}\,3}{\cancel{4}} = 3$

$\Rightarrow 3x^2 - 12x + 9 = 0$

$\Rightarrow 3c^2 - 12c + 9 = 0$

$\Rightarrow c = \dfrac{+12 \pm \sqrt{144 - 96}}{6}$

$= \dfrac{12 \pm \sqrt{48}}{6}$

$= 2 \pm \dfrac{\cancel{4}^{\,2}\sqrt{3}}{\cancel{6}\,3}$

$= 2 \pm \dfrac{2}{\sqrt{3}} \in (0, 4)$

$\therefore c = 2 \pm \dfrac{2}{\sqrt{3}}$ Ans

CAUCHY'S MEAN VALUE THEOREM :-

Let $f(x)$ and $g(x)$ be two functions defined such that

i) f and g are continuous in the closed interval $[a,b]$

ii) f and g are differentiable in (a,b)

iii) $g'(x) \neq 0$ $\forall x \in (a,b)$ then there exists at least one value $c \in (a,b)$ such that —

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

Q. $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$

(a) $\frac{ab}{2}$ (b) $\frac{a+b}{2}$ (c) $\sqrt{ab}$ (d) $\frac{2ab}{a+b}$..

$$\frac{\frac{1}{\cancel{2}} x^{-\frac{1}{2}}}{-\frac{1}{\cancel{2}} x^{-\frac{3}{2}}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}.$$

$$\Rightarrow - x^{-\frac{1}{2}+\frac{3}{2}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$\Rightarrow \cancel{-} c = \cancel{-} \sqrt{ab}$$

$$\therefore \boxed{c = \sqrt{ab}} \quad \text{Ans}$$

which $\in (a,b)$

Q. $f(x) = e^{x}$, $g(x) = e^{-x}$ in $[a,b]$.

$$\frac{e^{c}}{-e^{-c}} = \frac{e^{b}-e^{a}}{e^{-b}-e^{-a}}$$

$$\Rightarrow - e^{2c} = \frac{e^{b}-e^{a}}{\frac{e^{a}-e^{b}}{e^{b+a}}} \Rightarrow e^{2c} = e^{b+a}$$

$$\Rightarrow c = \frac{b+a}{2} \quad \text{Ans}$$

Q. $f(x) = \frac{1}{x^2}$ $\quad g(x) = \frac{1}{x}$ in $[a, b]$.

$$\rightarrow \frac{\cancel{-}2c^{-3}}{\cancel{-}c^{-2}} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\Rightarrow 2c^{-1} = \frac{\overset{a+b}{\cancel{a^2 - b^2}}}{\cancel{a-b}} \times \frac{a\cancel{b}}{a^2 b^{\cancel{2}}}$$

$$\Rightarrow c = \frac{2ab}{a+b}.$$

Q.. $f(x) = \sin x$, $\quad g(x) = \cos x$ in $[0, \frac{\pi}{2}]$.

$$\Rightarrow \frac{\cos x}{-\sin x} = \frac{1-0}{0-1}$$

$$\Rightarrow -\cot x = -1$$

$$\Rightarrow \tan x = 1.$$

$$\therefore c = \frac{\pi}{4}.$$

#

Taylor's mean value Theorem :-

Let $f(x)$ be a function defined such that —

i) $f(x)$ is continuous in $[a, a+h]$

ii) $f^{n-1}(x)$ is differentiable in $[a, a+h]$

then there exists at least one value $\theta \in (0,1)$ Such that —

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$$\cdots + \frac{h^{n-1}}{n-1!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a)$$

Let $a+h = x \Rightarrow h = x-a$.

** $$\therefore \boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots}$$

Maclerin's formula :- $\boxed{a=0}$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

① $f(x) = \sin x$ about $x = 0$

$f(0) = 0$

$f'(0) = \cos 0 = 1$

$f''(0) = -\sin 0 = 0$

$f'''(0) = -\cos 0 = -1$

** $$\therefore \boxed{f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Imp Result

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

Q. $f(x) = e^x$ about $x=0$

$f(0) = 1$

$f'(0) = e^0 = 1$

$f''(0) = e^0 = 1$

$f'''(0) = e^0 = 1$

Imp Result $\therefore$ $\boxed{f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$

Q. $f(x) = \log(1+x)$ about $x=0$.

$f(0) = 0$

$f'(0) = \frac{1}{1+x} = 1$

$f''(0) = -(1+x)^{-2} = -1$

$f'''(0) = +2(1+x)^{-3} = 2$

$f''''(0) = -6(1+x)^{-4} = -6$

$\therefore F(x) = \log(1+x) = x + \frac{x^2}{2!} + \frac{x^3}{3!}2 + \frac{x^4}{4!}(-6)$

Imp Result $\therefore$ $\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}$

Q. $f(x) = \frac{x}{1+x}$ about $x=0$.

$f(x) = \frac{x}{1+x} = 1 - \frac{1}{1+x} = 0$ $\qquad x+1)\,x\,(1 - \frac{1}{x+1}$ $\qquad \frac{x+1}{-1}$ $\qquad \frac{-1}{x}$

$f'(x) = \frac{1}{(1+x)^2} = 1$

$f''(x) = \frac{-2}{(1+x)^3} = -2$

$f'''(x) = \frac{6}{(1+x)^4} = 6$

$$\therefore \quad \frac{x}{1+x} = 0 + x(1) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(6) + \cdots$$

Imp Result

$$\boxed{\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \cdots}$$

Alternate Method

$$f(x) = \frac{x}{1+x}$$

$$= x(1+x)^{-1}$$

$$= x(1 - x + x^2 - x^3 + \cdots) \quad \{\text{Binomial Expansion}\}.$$

$$= x - x^2 + x^3 - x^4 + \cdots$$

Q. find the coeff. of x^2 in the expansion of $f(x) = \cos^2 x$ about $x = 0$.

$\rightarrow$

$$f(x) = \cos^2 x$$

$$= \cos x \cdot \cos x$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= -\frac{x^2}{2!} - \frac{x^2}{2!}$$

$$= -\frac{\cancel{2}x^2}{\cancel{2!}}$$

$$= -x^2 \; = -1$$

Allternate method.

$\because$ coeff. of x^2 asked $\therefore$

$$\hookrightarrow \; = \frac{f''(0)}{2!}$$

$$\therefore \; f(x) = \cos^2 x =$$

$$f'(x) = 2\cos x(-\sin x) = -\sin 2x$$

$$f''(x) = -2\cos 2x$$

$$\therefore \frac{f''(0)}{2!} = \frac{-2}{2!} = -1$$

Q. find the coeff of $(x-2)^4$ in the expansion of e^x about $x=2$

$\rightarrow$ Coeff of $(x-2)^4 = \dfrac{f^{IV}(a)}{4!}$

$f(x) = e^x$

$f''''(\underset{2}{x}) = e^{x^2} = e^2$

$\therefore \dfrac{f^{IV}(a)}{4!} = \dfrac{e^2}{4!}$ Ans

Q. find expansion of x^2 about $x=1$

(a) $1-x+x^2$

(b) $1+x^2$

(c) x^2 $\left\{ \because f(x)=x^2 \text{ is pure algebric fn } \therefore \text{ expansion not possible} \right\}$

(d) $1+x+\frac{x^2}{2}+\frac{x^3}{3}$

Q. $f(x) = \log\left(\frac{1+x}{1-x}\right)$ about $x=0$.

$f(x) = \log(1+x) - \log(1-x)$

$\log(1+x) = x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} \cdots$

$\log(1-x) = -x - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} - \cancel{\frac{x^4}{4}} \cdots$

$\quad + \quad + \quad + \quad +$

$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$

PARTIAL AND TOTAL DERIVATIVES :-

O.D	P.D
$y = f(x)$	$z = f(x,y)$
$x = g(y)$	$u = f(x,y,z)$

Q. $z = x^2 - xy + y^2$

$$\frac{\partial z}{\partial x} = 2x - y \qquad\qquad \frac{\partial z}{\partial y} = -x + 2y$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = -1 \qquad\qquad \frac{\partial^2 z}{\partial x \cdot \partial y} = -1$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

* HOMOGENEOUS FUNCTION :-

A function $f(x,y)$ is said to be homogeneous of degree n in x and y if —

$$\boxed{f(kx, ky) = k^n f(x,y)}$$

or else check overall degree of each term must be same.

ex:- $f(x,y) = \overset{2}{x^2} - \overset{1+1=2}{x^1 y^1} + \overset{2}{y^2}$ → Homogenous (Directly by degree)

$f(kx, ky) = k^2x^2 - k^2 x y + k^2 y^2$

$= k^2(x^2 - xy + y^2)$

Note:-
The product of two homogeneous fn is again a homogeneous function.

ex:- $f(x,y) = (x^3+y^3)(x^2-y^2)$

$n = 3+2 = 5$

If a fn is in rational form, if both numerator and Denominator are homogeneous function, therefore the given Function is also homogeneous.

ex:- $f(x,y) = \dfrac{x^3+y^3}{x^2-y^2}$

$n = 3-2 = 1$

$f(x,y) = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}-y^{1/5}}$

$n = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

Euler's Theorem:-

If z is a homogeneous fn of degree n in x and y then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n \cdot z$

Note:-

This formula is applicable directly if z is an algebric fn

ex:- Q. $z = (x^2+y^2)^{1/3}$ then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \underline{\frac{2}{3}z}$

Q. $z = \dfrac{xy}{x+y} \Rightarrow xz_x + yz_y = \underline{1 \cdot z}$

Q. If $z = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_ny^n$ then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \underline{nz}$

Q. If $z = \log(x^2+y^2)$ then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \_\_$

Shortcut

Note:-

If z is not algebric and let $\phi(z)$ is algebric and homogeneous of degree n in x and y then $\boxed{x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n\dfrac{\phi(z)}{\phi'(z)}}$

so, given $z = \log(x^2+y^2)$

$\Rightarrow e^z = x^2+y^2 \rightarrow$ homogeneous of degree 2

$\therefore \cancel{x\frac{\partial x}{\partial x}} + \cancel{y\frac{\partial y}{\partial}}$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2 \cdot \frac{\cancel{e^z}}{\cancel{e^z}} = 2 \text{ Ans}$$

Q. If $z = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$

$$\Rightarrow x z_x + y z_y = 1 \cdot \frac{\sin z}{\cos z} = \tan z \text{ Ans}$$

Q. If $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = ?$

$\Rightarrow \tan z = \frac{x^3+y^3}{x-y}$ homogeneous of order 2.

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2 \cdot \frac{\tan z}{\sec^2 z} = \sin 2z \text{ Ans}$$

Q. If $z = x^2 \sin^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underline{2z}$

$\because$ at $\frac{y}{x} \neq 0$ $y=0$ & $x=0$
Inverse fn not exists
so it becomes $z = x^2-y^2$

Q. If $z = (x^3+y^3)e^{-x/y} \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = \underline{3 \cdot z}$

$\because$ when $x=y=0$
the $e^{-x/y}$ does not exist.
$\therefore z = x^3+y^3$
homogeneous of order 3

Application of Euler's Theorem :-

If z is an homogeneous fn. of degree n in x and y then —

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + \frac{y^2 \partial z^2}{\partial y^2} = n(n-1) z$$

Note:-

This formula is applicable directly if z is an algebric fn.

Ex:

Q. If $z = (x^2+y^2)^{1/3}$ then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + \frac{y^2\partial^2 z}{\partial y^2} = \frac{2}{3}\left(\frac{2}{3}-1\right)z$

$$= -\frac{2}{9} z$$

Q If $z = \frac{x^3y^3}{x^2+y^2}$ then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 4(4-1)z$

$\hookrightarrow 6-2=4$

$$= 12z.$$

Q. If $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} =$ ______

Shortcut

Note :-

If z is not algebic then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{n\,\phi(z)}{\phi'(z)} = F(z)$$

and $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = F(z)\left[F'(z)-1\right].$

$\therefore F(z) = 2\cdot\frac{\tan z}{\sec^2 z} = \sin 2z$

$\therefore F(z)\left[F'(z)-1\right] = \sin 2z\left[2\cos 2z - 1\right]$

$= \sin 4z - \sin 2z$ Ans

Q. If $\log\left(\frac{x^4+y^4}{x+y}\right)$ then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = ?$

$\rightarrow$ $F(z) = 3 \cdot \frac{e^z}{e^z} = 3$

$F(z)[F'(z)-1] = 3[0-1] = -3$ Ans

Q.

Explicit Function :-

Any function which we can express in the form of either $y = f(x)$ or $x = g(y)$ etc is called an explicit fn.

means $\left.\begin{array}{l} y = \text{purely fn of } x \\ x = \text{purely fn of } y \end{array}\right\} \Rightarrow$ Explicit fn.

ex:- $y = ax^2 + bx + c$

Ordinary Derivatives upto n^{th} order can be easily evaluated.

Implicit function:-

Any function which is not explicit is said to be an implicit function.

ex:- $y = ax^2 + bxy + c$

Ordinary Derivatives upto n^{th} order evaluation becomes complex.

$\therefore$ Partial Derivative is used.

Shortcut

$$\boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-[q^2r - 2pqs + p^2t]}{q^3}}$$

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y}$ & $t = \frac{\partial^2 z}{\partial y^2}$

* If $ax^2 + 2hxy + by^2 = 1$.

Shortcut

$$\boxed{\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}}$$

Ex:-

Q. $2x^2 + 4xy + 3y^2 = 1$

$a = 2$

$b = 3$

$h = 2$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2^2 - 2\times 3}{(2x + 3y)^3} = \frac{-2}{(2x+3y)^3}$$ Ans

Q. $x^2 + xy + 2y^2 = 1$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{1}{2}\right)^2 - 2}{\left(\frac{1}{2}x + 2y\right)^3} = \frac{-14}{(x+4y)^3}$$ Ans

* If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Shortcut

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

ex:-

Q. $2x^2 + 4xy + 3y^2 + 2x + 4y + 1 = 0$

$a = 2$

$b = 3$

$c = 1$

$h = 2$

$g = 1$

$f = 2$

$$\therefore \frac{d^2y}{dx^2} = \frac{6 + \cancel{8} - \cancel{8} - 3 - 4}{(2x + 3y + 2)^3} = \frac{-1}{(2x+3y+2)^3}$$

Q. $y^2 - 5x + 4x^2 = 8$

$a = 1$
$b = 1$
$c = -8$
$h = 0$
$g = -\frac{5}{2}$
$f = 0.$

$$\frac{d^2y}{dx^2} = \frac{-32 + \frac{25}{4}}{(1\,y)^3} = -\frac{153}{4y^3}$$

Alternate method (By Explicit)

$y^2 = 8 + 5x - 4x^2$

$y = \sqrt{8 + 5x - 4x^2}$

$\frac{d^2y}{dx^2}$ will become complex.

$\therefore$ Implicit shortcut is very Easy to use.

COMPOSITE FUNCTION

→ $(g \circ f)\, x = g[f(x)]$.

→ Any function in function is said to be a composite fn.

or

→ If z is a fn in x and y and x, y are functions in t then z is called a composite fn in 't'.

$$z = f(x, y)$$
$$t$$

* Total Derivative of a composite fn :-

If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ then the total derivative of z w.r.to 't' is denoted by —

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

↳ Total Derivative means combination of ordinary + partial derivative.

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}}$$

Q: $z = e^x \sin y$

$x = \log t$

$y = t^2$

$\frac{dz}{dt} = ?$

$\rightarrow$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= e^x \sin y \\ \frac{\partial z}{\partial y} &= e^x \cos y \\ \frac{dx}{dt} &= \frac{1}{t} \\ \frac{dy}{dt} &= 2t \end{aligned}\right\} \quad \therefore \frac{dz}{dt} = e^x \sin y\left(\frac{1}{t}\right) + e^x \cos y\,(2t)$$

$$= \frac{e^x}{t}\left[\sin y + 2t^2 \cos y\right].$$

Q. $u = x^2 + y^2 + z^2$

$x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$

then $\frac{du}{dt} = ?$

$\frac{dy}{dt}$ & $\frac{dz}{dt}$ will take time as in uv form.

Alternate Method:

$$u = (e^{2t})^2 + (e^{2t}\cos 3t)^2 + (e^{2t}\sin 3t)^2$$

$$= e^{4t}\left[1 + \cos^2 3t + \sin^2 3t\right]$$

$$u = 2e^{4t}$$

$$\frac{du}{dt} = 8e^{4t}$$

Q. $u = x^2 - y^2$

$x = e^t \cos t \;,\; y = e^t \sin t$

$\left.\dfrac{du}{dt}\right|_{t=0} = ?$

$\rightarrow u \quad e^{2t}\cos^2 t - e^{2t}\sin^2 t$

$= e^{2t}(\cos^2 t - \sin^2 t)$

$u = e^{2t} \cdot \cos 2t \cdot$

$\Rightarrow \left.\dfrac{du}{dt}\right|_{t=0} = 2e^{2t}(-\sin 2t) + 2\cos 2t \cdot e^{2t} \cdot$

$= 2 \cdot 1 =$

$= 2$ Ans

Q. $u = x^3 y e^z$

$x = t, \quad y = t^2, \quad z = \log t \cdot$

$\left.\dfrac{du}{dt}\right|_{t=2} = ?$

$u = t^3 t^2 e^{\log t} \cdot$

$u = t^6$

$\left.\dfrac{du}{dt}\right|_{t=2} = 6t^5 = 6 \times 2^5 = 192$ Ans

Q. If $x = r\cos\theta \;,\; y = r\sin\theta$..

then ① $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = \_\_\_\_$

② $\dfrac{\partial^2\theta}{\partial x \partial y} = \_\_\_\_$

$$\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \frac{y}{x}$$

$$\Rightarrow \tan\theta = \frac{y}{x}.$$

$$\theta = \tan^{-1}\frac{y}{x}.$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{-y \times -1}{(x^2+y^2)^2} \times 2x.$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

Similarly $\dfrac{\partial^2\theta}{\partial y^2} = \dfrac{-2xy}{(x^2+y^2)^2}$

$$\therefore \boxed{\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.}$$

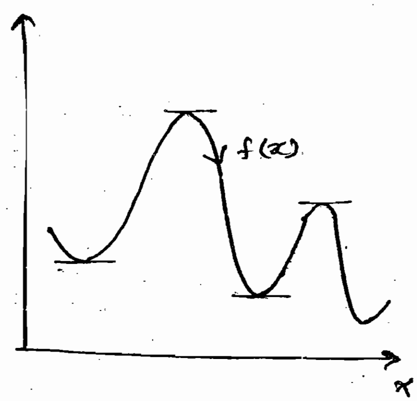
$$\because \frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial x\partial y}$$

Maxima and Minima.

Maximum Value :-

If a continuous fn $f(x)$ increases to a certain value and then decreases that value is called maximum value of the function.



Minimum Value :-

If a continuous fn $f(x)$ decreases to a certain value and then increases, that value is called minimum value of the fn.

Note :-

i) The maxima and minima occurs alternatively.

ii) A fn can have several maximum and several minimum values.

iii) The min^m value may be greater than the max^m value.

iv). The least min^m value is called the global minimum or universal minimum

and the highest max^m value is called the global max^m or universal max^m.

→ For a maxima or minima —

$$f'(x) = 0$$

$$x = \alpha, \beta, \lambda \ldots\ldots$$

$x = \alpha$, $f''(\alpha) = 0$, f is minimum at $x = \alpha$. Min. value $= f(\alpha)$

$x = \beta$, $f''(\beta) < 0$, f is maximum at $x = \beta$. Max^m Value $= f(\beta)$

$x = \lambda$; $f''(\lambda) = 0$, f is stationary..

if $f'''(\lambda) \neq 0$ then 'λ' is said to be inflation point.

Q. $f(x) = x^x$

$f'(x) = x^x(1+\log x)$

$f''(x) = x^x\left(\frac{1}{x}\right) + (1+\log x)\, x^x(1+\log x)$

$f''(x) = x^{x-1} + x^x(1+\log x)^2$

$f'(x) = 0$

$\Rightarrow x^x(1+\log x) = 0$

$\hookrightarrow$ can't be never 0

$\therefore 1 + \log_e x = 0$

$\Rightarrow \log_e x = -1$

$\Rightarrow x = e^{-1} = \frac{1}{e}.$

$f''(x) = f''\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}-1} > 0$ $\left\{ \begin{array}{l} 2^2 = 4 > 0 \\ 2^{-2} = \frac{1}{4} > 0. \end{array} \right\}$

for $f(x)=x^x$, F is min^m at $x = \frac{1}{e}$

Imp Result

$\therefore$ Min^m value $= f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} = e^{-\frac{1}{e}}$ Ans.

Q. $f(x) = x^{\frac{1}{x}}$ has its max^m at $x = e$.

Max^m value $= f(e) = e^{\frac{1}{e}}$.

Imp Result

****

fn	Min value	Max Value.	At.
x^x	$e^{-\frac{1}{e}}$	—	$x = \frac{1}{e}$.
$x^{\frac{1}{x}}$	—	$e^{\frac{1}{e}}$	$x = e$.

f

Q. $f(x) = \frac{\log x}{x} = \frac{1}{x}\log x$

$f'(x) = \frac{1}{x}\left(\frac{1}{x}\right) + \log x\left(-\frac{1}{x^2}\right)$

$f'(x) = \frac{1}{x^2}(1 - \log x)$

$f''(x) = -\frac{2}{x^3}(1-\log x) + \frac{1}{x^2}\left(-\frac{1}{x}\right)$

$f'(x) = 0.$

$\Rightarrow \frac{1}{x^2}(1-\log x) = 0.$

$\Rightarrow \log x = 1$

$\therefore x = e.$

$f''(x) = f''(e) = 0 - \frac{1}{e^3} < 0$

$\therefore$ f is max^m at $x = e$.

Max^m value $= f(e) = \frac{\log e}{e} = \frac{1}{e}.$

Q. $f(x) = x^3 + \frac{3}{x}$

$f'(x) = 3x^2 - \frac{3}{x^2}$

$f''(x) = 6x + 3\times 2x^{-3} = 6x + \frac{6}{x^3}$

$3x^2 - \frac{3}{x^2} = 0$

$\Rightarrow x^2 = \frac{1}{x^2}$

$\Rightarrow x^4 - 1 = 0$

$\Rightarrow (x^2-1)\cancel{(x^2+1)} = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

Min value = 4.

$f''(1) = 6 + 6 = 12 > 0 \rightarrow$ min^m at $x = 1$

$f''(-1) = -6 - 6 = -12 < 0 \rightarrow$ max^m at $x = -1$

Max^m value $= -4$

Q $f(x) = x^2 e^{-x}$

$f'(x) = e^{-x} \times (2x) + x^2(-e^{-x}) = 0$

$= 2xe^{-x} - x^2 e^{-x} = 0$

$\Rightarrow e^{-x}(2x - x^2) = 0$

$\Rightarrow \cancel{e^{-x}} \times x(2-x) = 0$

$x = 0, 2.$

$f''(x) = -2xe^{-x} + e^{-x} \times 2 - [e^{-x}(2x) + x^2(-e^{-x})]$

$= e^{-x}(2 - 4x + x^2)$

Min value $= 0$

at $x = 0$ $f''(x) = 2 > 0 \rightarrow \text{min}^m$ at $x = 0$

at $x = 2$ $f''(x) = \frac{-2}{e^2} < 0 \rightarrow \text{max}^m$ at $x = 2$

Max value $= 4e^{-2}$

Q. $f(x) = x^1 e^{-x}$ has its max at $x = \underline{1.}$

Trick *** Remember in this type max^m occurs at degree.

Shortcut

Q.

$$\boxed{\begin{aligned} &f(x) = a\cos x + b\sin x + c \\ &\text{Min value} = c - \sqrt{a^2+b^2} \\ &\text{Max value} = c + \sqrt{a^2+b^2} \end{aligned}}$$

Q. $f(x) = 3\cos x + 4\sin x + 2.$

Min value $= 2 - \sqrt{3^2+4^2} = 2 - 5 = -3$

Max value $= 2 + \sqrt{3^2+4^2} = 2+5 = \underline{7}.$

Q. $f(x) = 5\cos x + 3\cos\left(x+\frac{\pi}{3}\right) + 3$

$= 5\cos x + 3\cos x \cdot \cos\frac{\pi}{3} - 3\sin x \cdot \sin\frac{\pi}{3} + 3$

$= \cos x\left(5+\frac{3}{2}\right) - \frac{3\sqrt{3}}{2}\sin x + 3$

$\therefore$ Min value

$= c - \sqrt{a^2+b^2}$

$= 3 - \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$= 3 - \sqrt{\frac{169}{4}+\frac{27}{4}}$

$= 3 - \sqrt{\frac{196}{4}}$

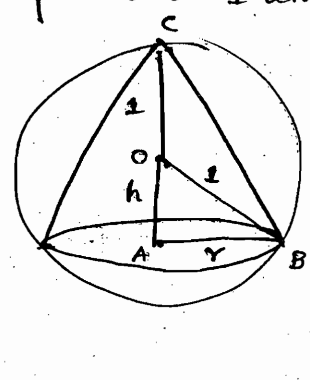
$= 3 - \frac{14}{2}\,7$

$= -4$.

Max value $= 3+7 = 10$

Q. Find the height of the cone of max^M volume that can be inscribed in a sphere of radii 1 unit.

$r^2 = 1-h^2$



$\because$ volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(1-h^2)(1+h)$

$\therefore V = \frac{\pi}{3}(1+h-h^2-h^3)$

$\frac{dV}{dh} = \frac{\pi}{3}(1-2h-3h^2) = 0$

$\Rightarrow 1-2h-3h^2 = 0$

$\Rightarrow 3h^2+2h-1 = 0$

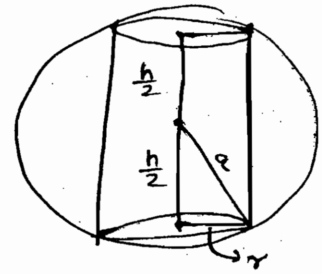
$h = \frac{-2\pm\sqrt{4+12}}{6} = \frac{-2\pm 4}{6} = \frac{-\cancel{8}4}{\cancel{6}3}, \frac{\cancel{2}1}{\cancel{6}3}$

$\therefore h = \frac{1}{3}$

$\therefore$ height $= 1+h = 1+\frac{1}{3} = \frac{4}{3}$

Q. Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radii a units.

$\rightarrow$ $r^2 = a^2 - \frac{h^2}{4}$



volume of cylinder $= \pi r^2 h$

$V = \pi r^2 h$

$= \pi\left(a^2 - \frac{h^2}{4}\right) h$

$V = \pi\left(a^2 h - \frac{h^3}{4}\right)$

$\frac{dv}{dh} = \pi\left(a^2 - \frac{3h^2}{4}\right) = 0$

$\Rightarrow a^2 - \frac{3h^2}{4} = 0$

$\Rightarrow 3h^2 = 4a^2$

$\Rightarrow h^2 = \frac{4a^2}{3}$

Imp Result.

$\boxed{h = \frac{2a}{\sqrt{3}}}$ $\rightarrow$ height for which max^m volume of cylinder occurs in sphere.

If $z = f(x,y)$

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$

$\rightarrow$ for a maxima and minima, Let P and $q = 0$ (Since 1st order derivative are only P & q)

$\rightarrow$ Since, P and Q are fn. in x and y

So, we have two eq^n in x & y.

$\rightarrow$ Solve these eq^ns and get a relation bet^n x and y.

Ex:- $x = y$, $x = -y$, $y = 2x$, $x = 2y$ etc type

$\rightarrow$ Put this relation either in $P = 0$ or in $q = 0$; then the eq^n transformed to either purely in x or in y

$\rightarrow$ Solve the Eq^n and get the roots

Ex – $x = x_0, x_1, x_2 \cdots$ and $y = y_0, y_1, y_2 \cdots$ etc

(x_0, y_0) $(x_1, y_1) \cdots$ are critical points.

→ Take a point x_0, y_0, find the values of r, s and t

* $\boxed{\text{Case-1}}$ → $\boxed{\text{If } rt - s^2 > 0 \text{ and } r > 0}$, then the fn attains its minimum at that point.

→ Min^m value $= f(x_0, y_0)$.

* $\boxed{\text{Case-2}}$ → $\boxed{\text{If } rt - s^2 > 0 \text{ and } r < 0}$, then the fn attains its maximum at that point.

→ Max^m value $= f(x_1, y_1)$

* $\boxed{\text{Case-3}}$ → $\boxed{\text{If } rt - s^2 < 0}$ then the fn has no extremum.

$\boxed{\text{Case-4}}$ → If $rt - s^2 = 0$ then the case is doubtful and needs further investigation.

Q. $f(x, y) = x^2 + y^2 + 6x + 12$ has

a) min^m at $(-3, 0)$

b) max^m at $(-3, 0)$

c) no extremum

d) none.

→
$$\left.\begin{aligned} p &= 2x + 6 \\ q &= \frac{\partial f}{\partial y} = 2y \\ r &= \frac{\partial^2 f}{\partial x^2} = 2 \\ s &= \frac{\partial^2 f}{\partial x \partial y} = 0 \\ t &= \frac{\partial^2 f}{\partial y^2} = 2. \end{aligned}\right\} \quad rt - s^2 = \underbrace{4 - 0^2 > 0 \text{ and } r > 0}_{\text{minimum}}$$

Q. $f = 1 - x^2 - y^2$ has.

(a) min at $(0,0)$
(b) max at $(0,0)$
(c) no extremum
(d) none.

$\rightarrow$

$$\left.\begin{aligned} p &= -2x \\ q &= -2y \\ r &= -2 \\ s &= 0 \\ t &= -2. \end{aligned}\right\} \quad \underbrace{rt - s^2 = 4 > 0 \quad \text{and} \quad -2 < 0.}_{\text{maximum}}$$

Q. $f(x,y) = x^3 + y^3 - 3xy.$

(a) max at $(1,-1)$
(b) min at $(1,1)$
(c) max at $(1,-1)$
(d) min at $(1,1)$

$\rightarrow$

$$\left.\begin{aligned} p &= 3x^2 - 3y \\ q &= 3y^2 - 3x \\ r &= 6x \\ s &= -3 \\ t &= 6y \end{aligned}\right\}$$

at $1,-1$

$\underbrace{6 \times (-6) - 9 < 0 \quad \text{and} \quad r = 6 > 0.}$

~~maxm.~~ no extremum

or for multiple points check
P and Q must be 0 at that point
then consider max & min

at $1,1$

$$\underbrace{6 \times 6 - 9 > 0 \quad \text{and} \quad r = 6 > 0.}_{\text{minm}}$$

Q. $f(x,y) = x^2 + y^2 + xy + x - 4y + 5$ has ______

(a) min at $(2,-3)$

(b) min at $(-2,3)$

(c) max at $(2,-3)$

(d) max at $(-2,3)$

$\rightarrow$

$$\left.\begin{aligned} p &= 2x + y + 1 \\ q &= 2y + x - 4 \\ r &= 2 \\ s &= 1 \\ t &= 2 \end{aligned}\right\} \quad \underbrace{rt - s^2 = 4 - 1 > 0 \quad \& \quad r = 2 > 0.}_{\text{minimum}}$$

Q. $f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$ has ______

(a) min $\frac{8}{3}$ (b) max $\frac{8}{3}$ (c) min $\frac{10}{3}$ (d) max $\frac{10}{3}$.

$\rightarrow$

$$\left.\begin{aligned} p &= 8x - 8 = 0 \Rightarrow x = 1 \\ q &= 12y - 4 = 0 \Rightarrow y = \tfrac{1}{3} \\ r &= 8 \\ s &= 0 \\ t &= 12 \end{aligned}\right\} \quad \underbrace{rt - s^2 = 96 - 0 > 0 \quad \& \quad r = 8 > 0}_{\text{minimum}}$$

$$\therefore f\left(1, \tfrac{1}{3}\right) = 4 + 6\left(\tfrac{1}{3^2}\right) - 8 - \tfrac{4}{3} + 8$$

$$= 4 + \tfrac{2}{3} - \cancel{8} - \tfrac{4}{3} + \cancel{8}$$

$$= 4 - \tfrac{2}{3}$$

$$= \tfrac{10}{3}$$

Q. $f(x,y) = \sin x + \sin y + \sin(x+y)$ has ______

(a) min $\frac{\sqrt{3}}{2}$ (b) max $\frac{\sqrt{3}}{2}$ (c) min $\frac{3\sqrt{3}}{2}$ (d) max $\frac{3\sqrt{3}}{2}$

$\rightarrow$ $p = \cos x + \cos(x+y)$

$q = \cos y + \cos(x+y)$

$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)\Big|_{\frac{\pi}{3},\frac{\pi}{3}} = -\sqrt{3}$

$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)\Big|_{\frac{\pi}{3},\frac{\pi}{3}} = -\frac{\sqrt{3}}{2}$

$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)\Big|_{\frac{\pi}{3},\frac{\pi}{3}} = -\sqrt{3}$

$\rightarrow$ Make $p = 0$ or $q = 0$.

$\therefore$ $\cos x + \cancel{\cos(x+y)} = 0$

$\cos y \underset{-}{+} \cancel{\cos(x+y)} = 0$

$\cos x - \cos y = 0.$

$\Rightarrow x = y$

$\therefore$ $\cos x + \cos 2x = 0$

$\Rightarrow \cos 2x = -\cos x$

$\Rightarrow \cos 2x = \cos(\pi - x)$

$2x = \pi - x$

$\Rightarrow x = \frac{\pi}{3} = y$

$\therefore$ $\underbrace{rt - s^2 = 3 - \frac{3}{4} > 0 \quad \text{and } r = -\sqrt{3} < 0}_{\text{max}^m \cdot \frac{3\sqrt{3}}{2}}$

Definite and Indefinite Integrals:-

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \quad , \text{ if } f(x) \text{ is even.}$$

$$= 0 \quad , \text{ if } f(x) \text{ is odd.}$$

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \quad \text{if } f(2a-x) = f(x)$$

$$= 0 \quad , \text{ if } f(2a-x) = -f(x)$$

$$\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}\,dx = \int_0^{\pi/2} \frac{f(\tan x)}{f(\tan x)+f(\cot x)}\,dx = \frac{\pi}{4}.$$

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)(n-5)\cdots}{n(n-2)(n-4)\cdots} \times K$$

$$\left\{ \begin{array}{l} K = 1, \text{ if } n \text{ is odd} \\ K = \frac{\pi}{2}, \text{ if } n \text{ is even} \end{array} \right\}$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \frac{(m-1)(m-3)\ldots \quad (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times K$$

$$\left\{ \begin{array}{l} K = 1, \text{ if either } m \text{ or } n \text{ or both are odd} \\ K = \frac{\pi}{2}, \text{ if both } m \text{ and } n \text{ are even} \end{array} \right\}$$

$$\int \cos x \, dx = \sin x$$

$$\int \sec^2 x \, dx = \tan x$$

$$\int \sec x \tan x \, dx = \sec x.$$

$$\int \tan x \, dx = \log \sec x$$

$$\int \cot x \, dx = \log \sin x$$

$$\int \sinh x \, dx = \cosh x$$

$$\int \cosh x \, dx = \sinh x$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$$

$f'(x)$

(3)

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}$$

$$\int \frac{f'(x)}{1+(f(x))^2}\,dx = \tan^{-1}[f(x)]$$

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int e^{f(x)} \cdot f'(x)\,dx = e^{f(x)}$$

$$\int e^x [f(x) + f'(x)]\,dx = e^x \cdot f(x)$$

$$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int f'(x)\int g(x)\,dx\,dx$$

To choose $f(x)$, remember I L A T E

Shortcut *** $\rightarrow$ $\int \text{algebraic} \times e^x\,dx = e^x$ (algebraic fn with derivates of alternate sign)

$$\int x e^x\,dx = e^x(x-1)$$

$$\int x^2 e^x\,dx = e^x(x^2-2x+2)$$

$$\int x^3 e^x\,dx = e^x(x^3-3x^2+6-6)$$

$$\int (x^2+x)e^x\,dx = e^x[(x^2+x)-(2x+1)+2]$$

Shortcut when e^{ax} is present

$$\int x e^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right]$$

$$\int x^2 e^{ax}\,dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$

$$\int \log x \, dx = x \log x - x$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

Q. ① $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx = (\sin^{-1} x)_0^{\frac{1}{2}} = \underbrace{\sin^{-1}\tfrac{1}{2}}_{\frac{\pi}{6}} - \underbrace{\sin^{-1} 0}_{0} = \frac{\pi}{6}.$

② $\displaystyle\int_{\log 2}^{\log 3} \frac{e^x}{1+e^x} \, dx = \log(1+e^x)\Big|_{\log 2}^{\log 3} = \log(1+e^{\log 3}) - \log(1+e^{\log 2})$

$= \log 4 - \log 3$

$= \log \frac{4}{3}$ Ans

③ $\displaystyle\int_0^1 \frac{1}{e^x + e^{-x}} \, dx$

$\displaystyle = \int_0^1 \frac{e^x}{e^{2x}+1} \, dx$

$\displaystyle = \int_0^1 \frac{e^x}{1+(e^x)^2} \, dx = \tan^{-1}(e^x)\Big|_0^1 = \tan^{-1} e - \tan^{-1} 1$

$= \tan^{-1} e - \frac{\pi}{4}$ Ans.

④ $\displaystyle\int_0^1 x e^x = e^x(x-1)\Big|_0^1 = e^1(\cancel{1-1}) - e^0(0-1)$

$= 1.$ Ans

Q $\displaystyle\int_{\cancel{2}\, a}^{\cancel{7}\, b} \frac{\overbrace{\sqrt{\cancel{x}}}^{f(x)}}{\underbrace{\sqrt{\cancel{x}}}_{f(x)} + \underbrace{\sqrt{\cancel{9}-x}}_{f(a+b-x)}} \, dx = \frac{b-a}{2} = \frac{7-2}{2} = \frac{5}{2}$

Q. $\displaystyle\int_0^{20} \frac{f(x)}{f(x)+f(20-x)} \, dx = \frac{20-0}{2} = 10$

Q. $\int_0^2 |1-x|\, dx$ → *** observe fn within limits when modulus present.

$= \int_0^2 |1-x|\, dx = \cancel{x\Big|_0^2 - \frac{x^2}{2}\Big|_0^2}$

$= \int_0^1 1-x\, dx + \int_1^2 x-1\, dx$

$= x - \frac{x^2}{2}\Big|_0^1 + \frac{x^2}{2} - x\Big|_1^2$

$\Rightarrow 1 - \frac{1}{2} + (2-\cancel{2}) - (\frac{1}{2} - 1)$

$= \frac{1}{2} + \frac{1}{2}$

$= 1.$ Ans.

**** When Greatest Integer fn present → observe limit betn limits if integer lies → split

if no integer lie betn limits → no need to split.

Q. $\int_{-1.5}^{1} [x+1]\, dx$

$\int_{-1.5}^{-1} [x+1]\, dx + \int_{-1}^{0} [x+1]\, dx + \int_{0}^{1} [x+1]\, dx$

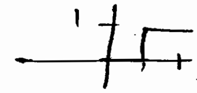
$$[x+1] = \begin{cases} -1, & -1.5 < x < -1 \\ 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

$\int_{-1.5}^{-1} -1\, dx + \int_{-1}^{0} 0\, dx + \int_{0}^{1} 1\, dx$

$= -x\Big|_{-1.5}^{-1} + x\Big|_0^1$

$= -(-1+1.5) + (1-0) = -.5+1 = 0.5$ Ans

Q. $\int_1^2 x[x]\,dx$



$x[x]$ /2

$$\int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = \left(\frac{2^2}{2} - \frac{1^2}{2}\right) = \frac{3}{2} \text{ Ans}$$

*** When limits $-a$ to a check even or odd.

Q. $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

$$= 2\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= 2\tan^{-1}x\Big|_0^1$$

$$= 2\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= 2\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{2} \text{ Ans}$$

Q. $\int_{-a}^{a} \sqrt{\frac{a+x}{a-x}}$

→ no even no odd.

→ Always Try to eliminate sq. root from numerator not from denominator WHEN $\sqrt{F(x)}$ is PRESENT.

$$= \int_{-a}^{a} \frac{a+x}{\sqrt{a^2-x^2}}$$

$$= \int_{-a}^{a} \underbrace{\frac{a}{\sqrt{a^2-x^2}}}_{\text{even}} + \underbrace{\cancel{\frac{x}{\sqrt{a^2-x^2}}}}_{\text{odd}} \xrightarrow{} 0 \; dx$$

$$= 2\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx = 2a\sin^{-1}\left(\frac{x}{a}\right)\Big|_0^a = 2a\left(\sin^{-1}1 - \sin^{-1}0\right)$$

$$= 2a\left(\frac{\pi}{2} - 0\right)$$

$$= a\pi \text{ Ans}$$

Q. $\cancel{\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx}$

Q. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} = a\pi$ (also)

(a) $a\pi$ (b) $\frac{a}{\pi}$ (c) $\frac{\pi}{a}$ (d) $\frac{1}{\pi a}$

When sign changes of x even though answer is same so forget the sign but remember the answer always.

***

Q. $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi^2}{4}$

Imp Result

For Removal of algebric a function always use —

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$$

$$2I = \int_0^{\pi} \frac{(\cancel{x} + \pi - \cancel{x})\sin x}{1+\cos^2 x}\, dx$$

$$2I = \pi \int_0^{\pi} \frac{-\sin x}{1+(\cos x)^2}\, dx$$

$$\int \frac{f'(x)}{1+[f(x)]^2}\, dx = \tan^{-1}[f(x)]$$

$$I = -\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^{\pi} = -\frac{\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = -\frac{\pi}{2} \times -\frac{\pi}{2} = \frac{\pi^2}{4} \text{ Ans}$$

*** $I = \int_0^{\pi/2} \log(\tan x)\,dx = 0$

Imp Results

$I = \int_0^{\pi/2} \log(\cot x)\,dx = 0$

Q. $I = \int_0^{\pi/2} \log(\tan x)\,dx$

$= \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx = \int_0^{\pi/2} \log \cot x \, dx.$

$2I = \int_0^{\pi/2} \left[\log(\tan x) + \log(\cot x)\right] \cdot dx$

$= \int_0^{\pi/2} \log\left(\tan x \cdot \cot x\right) dx$ ($\tan x \cdot \cot x = 1$)

$= 0$ Ans

Whenever limit $-a$ to a given. Check even or odd

Q. $\int_{-\pi}^{\pi} \underbrace{x^4}_{\text{even}} \times \underbrace{\sin^5 x}_{\text{odd}}\, dx$ → odd

even × odd = odd

$= 0.$

Q. $\int_0^{\pi/2} \sin^5 x\, dx$ → If formula forget

for Numerator: if power even → write odd no; if power odd → write even no

for Denominator: if power even → write even; if power odd → write odd.

Sol.

$= \frac{4 \times 2}{5 \times 3 \times 1} = \frac{8}{15}$

If power = odd → keep result as it is

If power = even → multiply result by $\frac{\pi}{2}$.

Q. $\int_0^{\pi/2} \cos^7 x\,dx = \frac{6\times4\times2}{7\times5\times3} = \frac{16}{35}$

Q. $\int_0^{\pi/2} \sin^8 x\,dx = \frac{7\times5\times\cancel{3}\times1}{8\times\cancel{6}\times4\times2} \times \frac{\pi}{2} = \frac{35}{256}\pi$

$\quad 2$

Q. $\int_0^1 \frac{x^6\,dx}{\sqrt{1-x^2}}$

Let $x = \sin\theta$

$dx = \cos\theta\,d\theta$.

if $x \to 0$ $\theta \to 0$

if $x \to 1$ $\theta \to \frac{\pi}{2}$.

$\int_0^{\pi/2} \frac{\sin^6\theta}{\cancel{\cos}\theta}\,\cancel{\cos}\theta\,d\theta$.

$= \frac{5\times3\times1}{6\times4\times2} \times \frac{\pi}{2} = \frac{5\pi}{32}$

Q. $\int_0^{\pi/2} \sin^4 x \cos^5 x\,dx$ → both are even then only multiply by $\frac{\pi}{2}$.

$= \frac{3\times1\times4\times2}{9\times7\times5\times3\times1} =$ → do individually as before in numerator

→ In Denominator — Start from sum of powers and continue.

$= \frac{8}{315}$

Q. $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$.

$$= \frac{5\times3\times1\times3\times1}{10\times8\times6\times4\times2} \times \frac{\pi}{2} = \frac{3\pi}{512}.$$

Q. $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$

$$= \frac{2\times4\times2}{8\times6\times4\times2} = \frac{1}{24}.$$

Q. $\int_0^{\pi} \sin^3 x \, dx$

$\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x) = f(x)$.

$\therefore f(\pi-x) = \sin^3(\pi-x) = \sin^3 x = f(x)$

π Converted in $\frac{\pi}{2}$ To use formula

$\therefore 2\int_0^{\pi/2} \sin^3 x \, dx$

$$= 2 \times \frac{2}{3\times1} = \frac{4}{3}$$

Definite and Improper Integrals :-

An integral $\int_a^b f(x)\,dx$ is said to be an improper integral if ① f becomes infinite in the interval of integration.

② one or both of the limits are infinite.

Ex:- $\int_0^1 \frac{1}{x}\,dx$, $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$, $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$.

$\int_1^{\infty} \frac{1}{x^3}\,dx$, $\int_{-\infty}^{0} \sin hx\,dx$, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Q. $\int_0^1 \frac{1}{x}\,dx = \log x\Big|_0^1 = \log 1 - \log 0 = \log \frac{1}{0} = \log \infty = \infty$

Divergent

Q. $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx.$

A/c to prev. discussion when limit was -1 to 1 ans. was π
Here limit is 0 to 1 ans will be $\frac{\pi}{2}$
But some extra term would be present also

$= \int_0^1 \cancel{1+x} \; \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx.$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^1 \frac{-1}{2}\times\frac{-2x}{\sqrt{1-x^2}}\,dx$

$= \left(\sin^{-1}x - \frac{1}{2}\times 2\sqrt{1-x^2}\right)_0^1$

$= \left(\sin^{-1}1 - \cancel{\sqrt{1-1}}\right) - \left(\cancel{\sin^{-1}0} - \sqrt{1-0}\right).$

$= \frac{\pi}{2} + 1$

Q. $\int_0^3 \frac{1}{(x-1)^{2/3}} dx$

$$= \int_0^3 (x-1)^{-2/3} dx$$

$$= \left[\frac{(x-1)^{-2/3+1}}{-\frac{2}{3}+1} \right]_0^3$$

$$= \frac{(x-1)^{1/3}}{\frac{1}{3}} \Big|_0^3$$

$$= 3 \left[2^{\frac{1}{3}} - (-1)^{1/3} \right]$$

$$= 3 \left[\sqrt[3]{2} + 1 \right] \text{ Ans}$$

Q. $\int_1^\infty \frac{1}{x^3} dx$

$$= \left(\frac{-1}{2x^2} \right)_1^\infty$$

$$= -\frac{1}{2} \left[\frac{1}{\infty} - \frac{1}{1} \right]$$

$$= -\frac{1}{2} [0 - 1]$$

$$= \frac{1}{2} \text{ Ans}$$

Q. $\int_{-\infty}^0 \sinh x \, dx$

$$= \cosh x \, dx \Big|_{-\infty}^0$$

$$= \frac{e^x + e^{-x}}{2} \Big|_{-\infty}^0$$

$$= \frac{e^0 + e^{-0}}{2} - \frac{e^{-\infty} + e^{\infty}}{2} = -\infty \text{ Ans}$$

Q. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 2\int_0^{\infty} \frac{1}{1+x^2}\, dx$

$= 2 \tan^{-1} x \Big|_0^{\infty}$

$= 2\left(\cancel{\tan^{-1}(\infty)}^{\frac{\pi}{2}} - \cancel{\tan^{-1} 0}^{0}\right)$

$= 2 \times \frac{\pi}{2}$

$= \pi$ Ans

Q. $\int_0^{\infty} x e^{-x^2}\, dx$

$= -\frac{1}{2}\int_0^{\infty} e^{-x^2}(-2x)\, dx$ $\left\{ \int e^{f(x)} f'(x) = e^{f(x)} \right\}$

$= -\frac{1}{2}\left(e^{-x^2}\right)_0^{\infty}$

$= -\frac{1}{2}\left[\cancel{e^{-\infty}}^{0} - \cancel{e^{-0}}^{1}\right]$

$= -\frac{1}{2}[0-1) = \frac{1}{2}$

*** Q. $\int_0^{\infty} e^{-x^2}\, dx$ $\left\{ \begin{array}{l} \text{Let } t = x^2 \\ dt = 2x\, dx \\ dx = \frac{dt}{2\sqrt{t}} \\ x \to 0 \to t \to 0 \\ x \to \infty \to t \to \infty. \end{array} \right\}$ $\left\{ \begin{array}{l} \text{donot take } t = -x^2 \\ \Rightarrow x^2 = -t \\ \text{sq never becomes} \\ -\text{ve} \\ \text{Wrong Idea} \end{array} \right.$

Imp Result.

***

$\Gamma \to$ means Γ fn

$= \frac{1}{2}\int_0^{\infty} \frac{e^{t}}{\sqrt{t}}\, dt$ → Definite Improper integral When becomes Complex to evaluate Take Help of Γ function.

$= \frac{1}{2}\int_0^{\infty} e^{t} \cdot t^{-\frac{1}{2}}\, dt$

$= \frac{1}{2}\sqrt{\frac{1}{2}} = \frac{\sqrt{\pi}}{2}$

$\Gamma 1 = 1$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx$

$\Gamma n+1 = n\Gamma n$, if $n > 0$

$\Gamma n+1 = n!$, if $n \in N$

Do clearly Prev. problem here.

—

Q. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy.$

$= \int_0^\infty \int_0^\infty e^{-x^2} . e^{-y^2} dx\, dy.$

$= \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$

$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}.$

Q. $\int_0^\infty \int_0^\infty \int_0^\infty e^{-(x^2+y^2+z^2)} dx\, dy\, dz$

$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{8}$ Ans

Q. $\int_0^\infty x^3 e^{-x}\,dx \xrightarrow{\text{Power will become factorial}} = 3!$

Shortcut

$= \cancel{x^3}\, e^{-x}\left[\frac{x^3}{-1} - \frac{3x^2}{-1^2} + \frac{6x}{-1^3} - \frac{6}{-1^4}\right]_0^\infty = 0 \times \infty$ (indetminate form)

Q. $\int_0^\infty \frac{x}{(x^2+9)^2}\,dx$

Let $t = x^2+9$

$\Rightarrow dt = 2x\,dx \Rightarrow dx = \frac{dt}{2x}$

$x \to 0\ ,\ t \to 9$

$x \to \infty\ ,\ t \to \infty$

$= \int_9^\infty \frac{\cancel{x}}{t^2} \times \frac{dt}{2\cancel{x}}$

$= \frac{1}{2}\left[\frac{-1}{t}\right]_9^\infty$

$= -\frac{1}{2}\left[\frac{1}{\infty} - \frac{1}{9}\right]$

$= -\frac{1}{2}\left[0 - \frac{1}{9}\right]$

$= \frac{1}{18}$ Ans

Q. $\int_{-\infty}^\infty \frac{x}{(x^2+9)^2}\,dx$ → fn is odd $\Rightarrow$ result $= 0$.

Vector Calculus

Gradient of a scalar fn :-

Let $\phi(x,y,z) = c$ be any scalar fn, then the grad of ϕ is denoted by "grad ϕ" or "$\Delta \phi$" and is defined as –

$$\Delta \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

we know that –

$$\bar{r} = xi + yj + zk$$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \quad , \quad \cancel{2}r \cdot \frac{\partial r}{\partial y} = \cancel{2}y \quad , \quad \cancel{2}r\frac{\partial r}{\partial z} = \cancel{2}z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla r = i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}$$

$$= i\left(\frac{x}{r}\right) + j\left(\frac{y}{r}\right) + k\left(\frac{z}{r}\right)$$

$$\nabla r = \frac{xi + yj + zk}{r}$$

$$\therefore \boxed{\text{grad } r = \nabla r = \frac{\bar{r}}{r}}$$

$$\nabla(\log r) = \frac{1}{r}\nabla r = \frac{1}{r}\cdot\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$$

$$\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\nabla r = \frac{-1}{r^2}\cdot\frac{\bar{r}}{r} = \frac{-\bar{r}}{r^3}$$

$$\nabla(r^n) = n r^{n-1}\nabla r = n r^{n-1}\frac{\bar{r}}{r} = n r^{n-2}\bar{r}$$

Tangent vector to a curve :-

Let $\bar{r}(t)$ be the given vector curve, then $\frac{d\bar{r}}{dt}$ is called the tangent vector to the given curve

$\bar{r} = xi + yj + zk$
(fn of x)

$$\boxed{\frac{d\bar{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k}$$

Normal to a surface :-

Let $\phi(x,y,z) = c$ be any surface then $\Delta\phi$ is called the normal to the surface ϕ. and $\frac{\Delta\phi}{|\Delta\phi|}$ is called the unit normal vector to the surface 'ϕ' and is denoted by 'N'

$$\boxed{\begin{array}{l} \text{i.e., } \phi(x,y,z) = c \\ \text{then } N = \dfrac{\nabla\phi}{|\nabla\phi|} \end{array}}$$

Directional Derivative :-

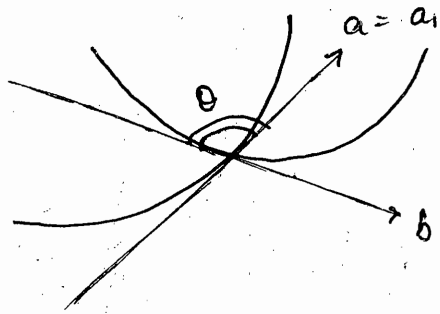
Let $\phi(x, y, z) = c$ be any surface, then $\boxed{\nabla\phi \cdot e}$ is called the directional derivative to the surface ϕ. where e is the unit ~~normal~~ vector in the dir^n of given vector.

i.e $e = \dfrac{\bar{a}}{|a|}$

The $\boxed{\text{max}^m \text{ value of the directional derivative}}$ or the greatest value of directional derivative or the magnitude of a gradient of a scalar fn ϕ is $\boxed{\text{defined by } |\Delta\phi|}$

The angle between the curves :-

The angle bet^n the two curves is the angle bet^n tangents at their point of intersection:.



$a = a_1 i + a_2 j + a_3 K \longrightarrow$ For tangent ①

$b = b_1 i + b_2 j + b_3 K. \longrightarrow$ for tangent ②

Note :-

If the two curves cuts each orthogonally then

$a \cdot b = 0 \quad \left\{ \because \theta \text{ will become } 90^\circ \;\&\; \cos\theta = \cos 90^\circ = 0 \right\}$

The angle betn two surfaces :-

The angle betn two surfaces is the angle betn normals drawn at their surfaces.

If f and g are any two surfaces and 'θ' is the angle betn them then $\boxed{\cos\theta = \dfrac{\nabla f \cdot \nabla g}{|\nabla f|\,|\nabla g|}}$

Note :-

If the two surfaces cuts each orthogonally then $\nabla f \cdot \nabla g = 0$.

Q. Find the unit normal vector to surface at the point $(1, 2, -1)$ where $\phi = x^3 + y^3 + 3xyz = 3$ (surface)

→ $\phi = x^3 + y^3 + 3xyz = 3.$

$$\nabla\phi = (3x^2 + 3yz)\,\hat{i} + (3y^2 + 3xz)\,\hat{j} + (3xy)\,k$$

$$\nabla\phi\,\Big|_{1,2,-1} = -3\hat{i} + 9\hat{j} + 6K.$$

∵ $N = \dfrac{\nabla\phi}{|\nabla\phi|}$ is unit normal to surface

∴ $N = \dfrac{\cancel{3}(-\hat{i} + 3\hat{j} + 2k)}{\cancel{3}\sqrt{1+9+4}} = \dfrac{-\hat{i} + 3\hat{j} + 2k}{\sqrt{14}}$

Q. Unit normal to $x^2y + 2xz = 4$ at $(2,-2,3)$

(a) $\frac{i+2j-2k}{3}$

(b) $\frac{i-2j+2k}{3}$

(c) $\frac{-i+2j+2k}{3}$

(d) $\frac{i+j+2k}{3}$

→ $\nabla\phi\Big|_{2,-2,3} = (2xy + 2z)i + (x^2)j + (2x)K$

$= -2i + 4j + 4K$

$\therefore N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\cancel{2}(-i+j+2k)}{\cancel{2}\sqrt{1+4+4}} = \frac{-i+j+2k}{3}$

Q. U.N to $x^2+y^2+z^2=1$ at $\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right)$

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(a) $\frac{i+j}{\sqrt2}$

(b) $\frac{j+k}{\sqrt2}$

(c) $\frac{i+k}{\sqrt2}$

(d) $\frac{i+j+k}{\sqrt3}$

Type-1

DIRECTIONAL DERIVATIVE (4 Types total)

Q. Find the directional derivative of $f = xy + yz + zx$ at $(1, 2, 0)$ in the dir^n of $i + 2j + 2k$.

→ $\nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial y} + k\dfrac{\partial\phi}{\partial z}$

$\nabla f = i(y+z) + j(x+z) + k(y+x)$

$\nabla f\Big|_{(1,2,0)} = 2i + j + 3k$

$a = i + 2j + 2k$

$e = \dfrac{a}{|a|} = \dfrac{i+2j+2k}{\sqrt{1+4+4}}$

$e = \dfrac{i+2j+2k}{3}$

$D \cdot D = \nabla f \cdot e$

$= 2\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)$

$= \dfrac{10}{3}$

Q. Find D·D of $f = 2xy + z^2$ at $(1, -1, 3)$ in the dir^n of $i + 2j + 3k$.

→ $\nabla f\Big|_{1,-1,3} = (2y)i + (2x)j + (2z)k$

$= -2i + 2j + 6k$

$a = i + 2j + 3k$

$e = \dfrac{i+2j+3k}{\sqrt{1+4+9}}$

$e = \dfrac{i+2j+3k}{\sqrt{14}}$

$D \cdot D = \nabla f \cdot e$

$= -2\left(\frac{1}{\sqrt{14}}\right) + 2\left(\frac{2}{\sqrt{14}}\right) + 6\left(\frac{3}{\sqrt{14}}\right)$

$= \dfrac{20}{\sqrt{14}}$

Type-II.

Q. find the directional derivative of $f = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in the dirn of $\overline{PQ}$ where $Q(5,0,4)$

→ $a = \overline{PQ} = Q - P = 4i - 2j + K$

$$e = \frac{a}{|a|} = \frac{4i - 2j + k}{\sqrt{16+4+1}} = $$

$\sqrt{21}$

dot product

$$\nabla\phi\Big| = (2x)i - (2y)j + (4z)K$$

$$\Big|_{1,2,3} = 2i - 4j + 12K$$

$$\therefore D\cdot D = \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}} \text{ Ans}$$

Q. find the D·D of $f = xy + yz + zx$ at $P(1,2,-1)$ in the dirn of $\overline{PQ}$ where $Q = 1,2,3$.

→ $$\nabla f\Big|_{1,2,-1} = (y+z)i + (x+z)j + (x+y)K$$

$$= i + 3K$$

$a = \overline{PQ} = Q - P = 4K$

$$\therefore e = \frac{a}{|a|} = \frac{\cancel{4}K}{\cancel{4}}$$

$e = K$

$$\therefore \boxed{D\cdot D = 3.}$$

Type-III

Q. find the directional derivative of $f = xy^2 + yz^2 + zx^2$ at (1.1.1) along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$

$$\nabla f \Big|_{1,1,1} = i(y^2 + 2xz) + j(2xy + z^2) + k(2yz + x^2)$$

$$\nabla f \Big|_{1,1,1} = (3i + 3j + 3k)$$

$$a = \frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

$x = t$ | $y = t^2$ | $z = t^3$

$\frac{dx}{dt} = 1$ | $\frac{dy}{dt} = 2t$ | $\frac{dz}{dt} = 3t^2$

→ To take t value take easy relation.

→ If you choose complicated relation to find 't' then after varifying we know that for other ~~points~~ 't' our point varies (Ultimaltely time loss)

$\therefore a = i + 2t j + 3t^2 k$

$= i + 2j + 3k \quad \{\because t = 1\}$

$$\therefore e = \frac{a}{|a|} = \frac{i + 2j + 3k}{\sqrt{14}}$$

$\therefore$ D.D $= \nabla f \cdot e$

$$= \frac{3 + 6 + 9}{\sqrt{14}}$$

$$= \frac{18}{\sqrt{14}}$$

Type - IV

Q. find the directional derivative of surface $f = xyz^2 + xz$ at $(1,1,1)$ in the dirn of normal to surface $3xy^2 + y = z$ at $(0,1,1)$

$$\nabla f = i(yz^2+z) + j(xz^2) + k(2xyz + x)$$

$$\nabla f\Big|_{1,1,1} = 2i + j + 3k$$

$$a = \nabla g\Big|_{0,1,1} = i(3y^2) + j(6xy+1) + k(-1)$$

$$= 3i + j - k$$

$$e = \frac{a}{|a|} = \frac{3i + j - k}{\sqrt{11}}$$

$$\therefore \text{DoD} = \nabla f \cdot e$$

$$= \frac{6+1-3}{\sqrt{11}}$$

$$= \frac{4}{\sqrt{11}}$$

Q. find the maximum value of the directional derivative to the surface $\phi = x^2yz^3$ at $(2,1,-1)$

→ $$\nabla\phi = i(2xyz^3) + j(x^2z^3) + k(3x^2yz^2)$$

$$\nabla\phi\Big|_{2,1,-1} = -4i - 4j + 12k$$

$$|\nabla\phi| = \sqrt{16+16+144}$$

$$= \sqrt{176}$$

$$= \sqrt{16\times 11}$$

$$= 4\sqrt{11}$$

Q. find the greatest value of dirⁿ derivative of $\phi = x^2yz$ at $(1,4,1)$

$\rightarrow$

$$\nabla\phi\Big|_{1,4,1} = (2xyz)i + (x^2z)j + (x^2y)k$$

$$= 8i + j + 4k$$

$$|\nabla\phi| = \sqrt{64+16+1}$$

$$= \sqrt{81}$$

$$= 9$$ ✓

Q. find the magnitude of gradient $u = \frac{x^2}{2} + \frac{y^2}{3}$ at $(1,3)$

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\sqrt{5}$ (d) $\sqrt{10}$

$\rightarrow$

$$\nabla u\Big|_{1,3} = \frac{2x}{2}i + \frac{2y}{3}j$$

$$= i + 2j$$

$$|\nabla u| = \sqrt{1+4} = \sqrt{5}.$$

Q. find the angle betⁿ surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at the point $(2,-1,2)$

$\rightarrow$

$$\nabla f\Big|_{2,-1,2} = 2x\,i + 2y\,j + 2z\,k = 4i - 2j + 4k$$

$$\nabla g = 2x\,i + 2y\,j - k = 4i - 2j - k$$

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f|\,|\nabla g|} = \frac{16 \cancel{+4} - \cancel{4}}{\sqrt{36}\,\sqrt{21}} = \frac{\cancel{16}\,8}{3\cancel{6}\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$ ✓

Q. find acute angle betⁿ the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$.

$\rightarrow$ $$\nabla f\Big|_{1,-2,1} = i(y^2z - 3) + j(2xyz) + k(xy^2 - 2z)$$

$$= i(+1) + j(-4) + k(2)$$

$$\nabla g = 6xi - 2yj + 2k$$

$$(\nabla g)_{1,-2,1} = 6i + 4j + 2k.$$

Due to acute angle we will take only +ve θ.

$$\therefore \cos\theta = \frac{|6 - 16 + 4|}{\sqrt{1+16+4}\sqrt{36+16+4}}$$

$$= \frac{6}{\sqrt{7\times3}\sqrt{7\times4\times2}}$$

$$= \frac{\cancel{6}\,3}{7\times7\times\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$$

Q. find the const. a, b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point $(1, -1, 2)$.

$\rightarrow$ $$\nabla f\Big|_{1,-1,2} = (2ax - a - 2)i - bzj - byk$$

$$= (a-2)i - 2bj + bk$$

$$\nabla g\Big|_{1,-1,2} = (8xy)i + (4x^2)j + (3z^2)k$$

$$= -8i + 4j + 12k$$

A/c to Q –

$$\nabla f \cdot \nabla g = 0 \Rightarrow -8(a-2) - 8b + 12b = 0$$

$$\Rightarrow \boxed{b = 2a - 4} \quad — ①$$

A/c to Q — $1, -1, 2$ lies also on Surface ① —

$$\therefore \quad a(1)^2 - b(-1)(2) = (a+2) \quad \text{—— ①}$$

$$\Rightarrow \quad a + 2b = a + 2$$

$$\Rightarrow \quad 2b = 2$$

$$\Rightarrow \quad b = 1 \quad \underline{\text{Ans}}$$

$\therefore$ from ① —

$$a = \frac{5}{2} \quad \underline{\text{Ans}}$$

Divergence of a vector function :—

Let $F = F_1 i + F_2 j + F_3 k$ be any differentiable vector fn then the divergence of f is denoted by div F or $\nabla \cdot F$ and is defined as —

$$\text{div } F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\boxed{\begin{aligned} &\text{Ex :- } \bar{r} = xi + yj + zk \\ &\text{div } \bar{r} = 1 + 1 + 1 \\ &\text{div } \bar{r} = \nabla \cdot \bar{r} = 3 \end{aligned}}$$

Solenoidal Vector :-

The neccessary and sufficient condn for a vector fn f to be solenoidal is —

$$\text{div } F = 0$$

$$\boxed{\nabla \cdot F = 0.}$$

Note:-

Let ϕ is a scalar and a is vector then—

$$\mathrm{div}(\phi A) = \nabla \cdot (\phi A) = (\mathrm{grad}\ \phi) \cdot A + \phi(\mathrm{div}\ A)$$

$$\boxed{\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi(\nabla \cdot A)}$$

Q. $\nabla^2\left(\frac{1}{r}\right) = \underline{0}$ where $r = |\bar{r}|$ and $\bar{r} = xi + yj + zk$

Imp Result $\rightarrow$

$$\nabla \cdot \nabla\left(\frac{1}{r}\right) = \nabla \cdot \left(-\frac{1}{r^2}\nabla r\right) = \nabla \cdot \left(-\frac{1}{r^2}\frac{\bar{r}}{r}\right) = \nabla \cdot \left(\underbrace{-\frac{1}{r^3}}_{\phi} \cdot \underbrace{\bar{r}}_{A}\right)$$

$$= \nabla\left(-\frac{1}{r^3}\right) \cdot \bar{r} + \left(-\frac{1}{r^3}\right)(\nabla \cdot \bar{r})$$

$$= \frac{3}{r^4}\nabla r \cdot \bar{r} - \frac{3}{r^3}$$

$$= \frac{3}{r^4}\frac{\bar{r}}{r} \cdot \bar{r} - \frac{3}{r^3}$$

$$= \frac{3}{r^4} \times r^2 - \frac{3}{r^3}$$

$$= \frac{3}{r^3} - \frac{3}{r^3}$$

$$= 0$$

****

$$\boxed{\text{SHORTCUT} - \quad \nabla^2[f(r)] = f''(r) + \frac{2}{r}f'(r)}$$

① $\nabla^2(\log r) = -\frac{1}{r^2} + \frac{2}{r} \cdot \frac{1}{r} = \frac{1}{r^2}$

② $\nabla^2(r^n) = n(n-1)r^{n-2} + \frac{2}{r}n\, r^{n-1}$

$$= r^{n-2}[n(n-1) + 2n]$$

$$= r^{n-2}[n(n+1)]$$

Curl of a vector function :-

Let $F = F_1 i + F_2 j + F_3 K$ be any differentiable vector function then the curl of F is denoted by Curl F or $\nabla \times F$ and is defined as —

$$\nabla \times F = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Ex:- $\bar{r} = xi + yj + zK$

$$\text{Curl } \bar{r} = \nabla \times r = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i(0-0) - j(0-0) + K(0-0)$$

$$= 0$$

$$\therefore \boxed{\text{curl } \bar{r} = 0}$$

Irrotational Vector :-

The neccessary and sufficient condn for a vector fn F to be irrotational is —

$$\boxed{\nabla \times F = 0}$$

Note :-

Let F be any non zero vector function then —

$$\text{div}(\text{Curl } F) = 0$$

$$\nabla \cdot (\nabla \times F) = 0.$$

Q. If $F = (x+2y)\,i + (y+2z)\,j + (x+\lambda z)\,K$ is Solenoidal then value of $\lambda = ?$

→ $\nabla \cdot F = 0$ (For Solenoidal)

$\Rightarrow \nabla \cdot \left[(x+2y)\,i + (y+2z)\,j + (x+\lambda z)\,K\right] = 0$

$\Rightarrow 1 + 1 + \lambda = 0$

$\Rightarrow \lambda = -2$

Q. Find the value of λ so that —

$F = (\lambda x^2 y + yz)\,i + (4xy^2 + xz)\,j + 2xyz\,K$ is zero divergence.

→ $\nabla \cdot F = 0$

$\Rightarrow 2\lambda xy + 8xy + 2xy = 0$

$\Rightarrow xy(2\lambda + 8 + 2) = 0$

$\Rightarrow \lambda + 4 + 1 = 0$

$\Rightarrow \lambda = -5$ Ans.

Q. If $f = x^3 + y^3 + z^3 - 3xyz$ then $\text{div}(\text{grad } F) =$ ______

→ $\nabla \cdot (\nabla f) = \nabla^2 f$

$= 6x + 6y + 6z$

$= 6(x+y+z)$

Q. If $F = 2xy\,i - x^2 z\,j$ then $(\text{curl } F)_{(1,1,1)} =$ ______

$$\nabla \times F = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -x^2 z & 0 \end{vmatrix} = i(0+x^2) - j(0-0) + K(-2xz-2x)$$

$= i - 4K$ @ point $(1,1,1)$

Q. If $F = 3x^2 i + 5xy^2 j + xyz^3 k$

then $(\text{div } F)_{1,2,3} =$ ______

→ $\nabla \cdot F = 6x + 10xy + 3xyz^2$

$= 6 + 20 + 54$

$= 80$ Ans

Q. If $\nabla \cdot (r^n \bar{r}) = 0$ then $n =$ ______

→ $\because \nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi (\nabla \cdot A)$

$\Rightarrow (\nabla r^n) \cdot \bar{r} + r^n \left(\cancelto{3}{\nabla \cdot \bar{r}}\right) = 0$

$\Rightarrow n r^{n-1} \nabla r \cdot \bar{r} + 3 r^n = 0$

$\Rightarrow n r^{n-1} \dfrac{\bar{r} \cdot \bar{r}}{r} + 3r^n = 0$

$\Rightarrow n r^{n-2} r^2 + 3r^n = 0$

$\Rightarrow n r^n + 3r^n = 0$

$\Rightarrow n + 3 = 0$

$\Rightarrow n = -3$

Q. Find the constants a, b, c so that the vector fo
$F = (x + 2y + az) i + (bx - 3y - z) j + (4x + cy + 2z) k$
is irrotational.

→ $\nabla \times F = 0 \Rightarrow \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$

$\Rightarrow \quad i(c+1) - j(4-a) + k(b-2) = 0$

$\Rightarrow \quad c+1 = 0 \Rightarrow c = -1$

and $4-a = 0 \Rightarrow a = 4$

and $b-2 = 0 \Rightarrow b = 2$

Ans

Vector Integration :-

① line integral $\int$

② surface integral $\iint$

③ volume integral $\iiint$

(i). Line Integral :-

Any integral which is evaluated along the curve is called the line integral

Let F be any differentiable vector function defined along curve 'c', then the line integral of F is defined by — $\int_C F \cdot d\bar{r}$

If c is a closed curve, it is denoted by — $\oint_C F \cdot d\bar{r}$

Cartesian form :-

Let $F = F_1 i + F_2 j + F_3 k$

$\bar{r} = xi + yj + zk$

$d\bar{r} = dx\, i + dy\, j + dz\, k$

$\int_C F \cdot dr = \int_C (F_1 dx + F_2 dy + F_3 dz)$ is called Cartesian form of line Integral

Note:- If $F(x, y, z)$ are fn. of t, then

$$\int_C F \cdot d\bar{r} = \int \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

Q. Evaluate $\int_C F \cdot d\bar{r}$ where $F = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$

① where c is the curve $x = 2t^2$, $y = t$, $z = t^3$ joining $(0,0,0)$ to $(2,1,1)$

② where c is ~~the curve~~ st. line joining $(0,0,0) \to (0,0,1) \to (0,1,1) \to (2,1,1)$

$\rightarrow$ (1). $\int_C F \cdot d\bar{r} = \int_C \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$

$$= \quad x = 2t^2 \quad \Big| \quad y = t \quad \Big| \quad z = t^3$$

$$\frac{dx}{dt} = 4t \quad \Big| \quad \frac{dy}{dt} = 1 \quad \Big| \quad \frac{dz}{dt} = 3t^2$$

$\because y = t$
$\therefore y \to 0, t \to 0$
$y \to 1, t \to 1$

$$= \int_{t=0}^{1} \left[(2y+3)\frac{dx}{dt} + xz\frac{dy}{dt} + (yz-x)\frac{dz}{dt} \right] dt$$

(with $y = t$, $\frac{dx}{dt} = 4t$, $xz = 2t^2 \cdot t^3 = 2t^5$, $\frac{dy}{dt} = 1$, $yz = t \cdot t^3$, $x = 2t^2$, $\frac{dz}{dt} = 3t^2$)

$$= \int_{t=0}^{1} \left[8t^2 + 12t + 2t^5 + 3t^6 - 6t^4 \right] dt$$

$$= \left[\frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5} \right]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}$$

$$= 9 + \frac{3}{7} - \frac{6}{5}$$

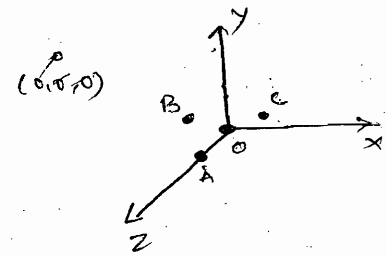
$$= \frac{288}{35}$$

② $\int_C F\cdot dr = \int_C F_1\cdot dx + F_2\,dy + F_3\,dz.$

choose dx, dy, dz such that only that derivative varies, point remains const. others or 1 remains const. & other two varies

① along $\overline{OA}$

$x=0,\ y=0$
$dx=0 \quad dy=0.$
as $z=0$ to $z=1$



$$= \int_{OA} (yz-x)\,dz = 0 \quad \left\{ \because \begin{matrix} x=0 \\ y=0 \end{matrix} \right\}.$$

② along $\overline{AB}$

$x=0,\ z=1$
$dx=0 \quad dz=0.$
as $y=0$ to $1.$

$$= \int_{AB} xz\,dy = 0 \quad \{\because x=0\}.$$

③ along ~~AB~~ ~~BC~~ AB —

$y=1,\ z=1$
$dy=0,\ dz=0.$
as $x=0$ to 2

$$= \int_{x=0}^{2} (2y+3)\,dx = \int_{x=0}^{2} [2(1)+3]\,dx = 5[x]_0^2 = 10$$

$$\therefore \int_C F\cdot dr = 0+0+10 = \underline{\underline{10}}$$

Q. $\int F\cdot dr$ where $F = 3x^2 i + (2xz - y) j + z k$ where c is the st. line joining $(0,0,0)$ to $(2,1,3)$

$\rightarrow$ eq^n of OA $= \dfrac{x-0}{2-0} = \dfrac{y-0}{1-0} = \dfrac{z-0}{3-0} = t$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$\Rightarrow x = 2t, \quad y = t, \quad z = 3t$$

$\boxed{\text{The } eq^n \text{ of a st. line joining } (x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2) \text{ in the symmetric form is } \dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1} = t \text{ where } t \text{ is some scalar.}}$

$$\therefore \int_C F\cdot dr = \int_C \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right] dt$$

$$= \int_{t=0}^{1} [3(4t^2)(2) + (2(2t)(3t) - t)(1) + 3t(3)]\, dt$$

$$= \int_{t=0}^{1} (24t^2 + 12t^2 - t + 9t]\, dt$$

$$= \left| 36\frac{t^3}{3} + \frac{8t^2}{2} \right|_0^1 = 12 + 4 = 16$$

Q. $\int_C F\cdot dr = ?$ where $F = 2xyz\, i + x^2 y\, j + x^2 z\, k$ where C is the st. line joining $(0,0,0)$ to $(1,1,1)$

$\rightarrow$ eq^n of st line —

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$x = t, \quad y = t, \quad z = t$$

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1.$$

$$\therefore \int F\cdot dr = \int_C F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}.$$

$$= \int_{t=0}^{1} \left[2t^3(1) + t^3(1) + t^3(1)\right] dt$$

$$= \int_{t=0}^{1} 4t^3 dt = 4\frac{t^4}{4}\Big|_0^1 = 1-0 = 1$$

Q. $\int_C F\cdot dr$, Where $F = 3xy\,\hat{i} - y^2\hat{j}$ and c is the curve $y = 2x^2$ in the xy plane joining $(0,0)$ to $(1,2)$

$\rightarrow$ $$\int_C F\cdot dr = \int [3xy\,dx - y^2 dy]$$

एक variable के term में change करेंगे limit उसी का लेंगे

$y = 2x^2$

$dy = 4x\,dx$

$x \to 0$ to 1

$$= \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\, 4x\,dx$$

$$= \left| \frac{6x^4}{4} - \frac{16x^6}{6} \right|_0^1 = \frac{3}{2} - \frac{8}{3} = \frac{-7}{6}.$$

Q. $\int_C F\cdot dr = ?$ Where $F = (5xy - 6x^2)\,\hat{i} + (2y - 4x)\,\hat{j}$ and c is the curve $y = x^3$ in the xy plane joining $(1,1)$ to $(2,8)$

$\rightarrow$ $$\int_C F\cdot dr = \int (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

$y = x^3$

$dy = 3x^2 dx$

$x \to 1$ to 2

$$= \int_{x=1}^{2} (5x^4 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2 dx$$

$$= \int_{x=1}^{2} [5x^4 - 6x^2 + 6x^5 - 12x^3]\, dx$$

$$= \left[\frac{\cancel{5}x^5}{\cancel{5}} - \frac{\cancel{6}^{2}x^3}{\cancel{3}} + \frac{\cancel{6}x^6}{\cancel{6}} - \frac{\cancel{12}^{3}x^4}{\cancel{4}} \right]_1^2$$

$$= \cancel{[\ldots]}$$

$$= \cancel{3}.$$

$$= (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3)$$

$$= 32 + 3 = 35 \quad \text{Ans}$$

Q. $\oint_C F \cdot d\bar{r} = ?$ where $F = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x + 2y - 4z)K$

where c is the circle in xy plane having the centre at origin and radius 3 units.

$\rightarrow$ $\int F \cdot dr = \int (2x - y + \cancel{z})\, dx + (x + y - \cancel{z^2})\, dy$

{ Circle is in xy plane $\therefore$ do not consider dz }

Our Relation is —

$$x^2 + y^2 = 9$$

$$x = r\cos\theta = 3\cos\theta$$

$$y = r\sin\theta = 3\sin\theta$$

$$\Rightarrow dx = -3\sin\theta\, d\theta$$

$$dy = 3\cos\theta\, d\theta.$$

as $\theta \to 0$ to 2π to complete a circle

$$= \int_{\theta=0}^{2\pi} 2(3\cos\theta - 3\sin\theta)(-3\sin\theta\, d\theta) + (3\cos\theta + 3\sin\theta) \times (3\cos\theta)\, d\theta$$

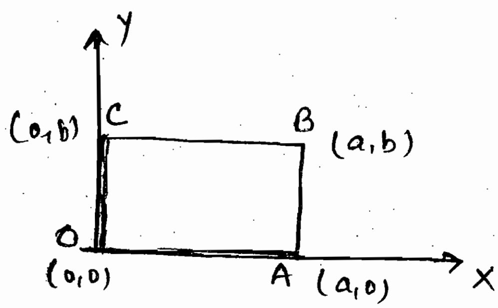
$$= \int_{\theta=0}^{2\pi} (-9(\sin\theta)\cos\theta + 9)\, d\theta$$

$$= \left[-9\,\frac{\sin 2\theta}{2} + 9\theta \right]_0^{2\pi}$$

$$= 18\pi$$

Q. $\oint_C F \cdot d\bar{r}$ where $F = (x^2+y^2)\,\hat{i} - 2xy\,\hat{j}$ where c is the rectangle bounded by $x=0$, $x=a$, $y=0$, $y=b$.

→



$$\int F \cdot dr = \oint (x^2+y^2)\,dx - 2xy\,dy$$

① along OA — $y=0 \Rightarrow dy=0$ not changing
$x \to 0$ to a.

$$\int F \cdot dr = \int_{x=0}^{a} (x^2+0)\,dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

② along AB — $x=a \Rightarrow dx=0$. not changing
$y \to 0$ to b

$$\int F \cdot dr = \int_{y=0}^{b} -2ay\,dy = -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2$$

③ along BC — $y=b \Rightarrow dy=0$ not changing
$x \to a$ to 0

$$\int F \cdot dr = \int_{x=a}^{0} (x^2+b^2)\,dx = 0 - \frac{a^3}{3} - ab^2$$

④ along CO — $x=0 \Rightarrow dx=0$

$$\int F \cdot dr = \int 0 - 0 = 0$$

$$\therefore \int F \cdot dr = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 - 0 = -2ab^2$$

Surface Integral :—

Any integral which is evaluated over surface is called a surface integral.

Let F be any differentiable vector fn. defined over a surface S then the surface integral is defined by $\iint_S F \cdot N \, ds$ where N is the outer unit normal vector to given surface.

$$\iint_S F \cdot N \, ds = \iint_{R_1} F \cdot N \frac{dx\,dy}{|N \cdot k|}$$ where R_1 is projection in xy plane.

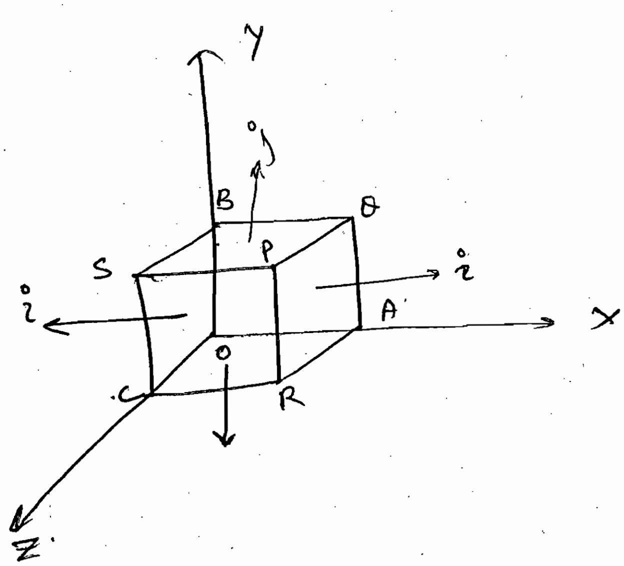
$$= \iint_{R_2} F \cdot N \frac{dy\,dz}{|N \cdot i|} \quad \text{---} \quad R_2 \text{ --- } yz \text{ plane}$$

$$= \iint_{R_3} F \cdot N \frac{dz\,dx}{|N \cdot j|} \quad \text{---} \quad R_3 \text{ --- } zx \text{ plane}$$

Q. Evaluate $\iint_S F \cdot N \, ds$ where $F = zi + xj - 3y^2 z k$ and S is the surface of cylinder $x^2+y^2=16$ included in the I^{st} octant bounded by $\underbrace{z=0 \;\&\; z=5}$;

∵ z is given ∴ either xz plane or yz plane will be choosen for projection

Q. Evaluate $\iint_S F \cdot N \, ds$ where $F = 4xz\,i - y^2 j + yz\,K$ and S is the cube bounding $0 \le x, y, z \le 1$.



→ for POAR —

$N = i$ & $x = 1$ (const)

$F \cdot N = 4\cancel{x}^{1} z = 4z$

$$\iint_S F \cdot N \, ds = \int_{y=0}^{1} \int_{z=0}^{1} 4z \, dy \, dz$$

$$= 4 \left. \frac{z^2}{2} \right|_0^1 \left. y \right|_0^1$$

$$= 4 \times \frac{1}{2} \times 1$$

$$= 2$$

For SBOC — $N = -i$, $x = 0$.

$F \cdot N = -4xz = 0$

$\therefore \iint F \cdot N \, dS = 0$

For PQBS —

$N = j, \quad y = 1.$

$$\int F \cdot N \, ds = \int_{x=0}^{1} \int_{z=0}^{1} -y^2 \, dx \, dz = -x\Big|_0^1 z\Big|_0^1 = -1$$

For OCRA —

$N = j \quad y = 0$

$$\int_S F \cdot N = \underline{0}.$$

For SPRC —

$N = K \qquad z = 1.$

$$\int_S F \cdot N \, ds = \int_{x=0}^{1} \int_{y=0}^{1} y z^2 \, dx \, dy$$

$$= \frac{y^2}{2}\Big|_0^1 x\Big|_0^1$$

$$= \frac{1}{2}$$

For BQOA — $z = 0, \; N = -K$

$$\int F \cdot N \, ds = \cancel{\quad} = 0$$

$$\therefore \iint_S F \cdot N \, ds = 2 + 0 + (-1) + 0 + \frac{1}{2}$$

$$= 1 + \frac{1}{2} = \frac{3}{2} \quad \underline{\text{Ans}}$$

Volume Integral :-

Any integral which is evaluated over a volume is called the volume integral.

Q. $\iiint\limits_V \nabla\cdot F\, dv$ where $F = (2x^2-3z)\hat{i} - 2xy\hat{j} + x\hat{k}$ where V is the region bounded by $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

$$\nabla\cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\rightarrow \quad \nabla\cdot F = 4x - 2x$$

$\cancel{(\text{Put } z=0)}$

$z \rightarrow$ in terms of x & y $\longrightarrow$ 0 to $4-2x-2y$

$y \rightarrow$ $\cancel{(\text{Put } z=0,\ y=0)}$ $\longrightarrow$ 0 to $2-x$

$x \rightarrow$ const $\longrightarrow$ 0 to 2.

$$\iiint \nabla\cdot F\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x \, dx\, dy\, dz$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} x(4-2x-2y)\, dx\, dy$$

$$= 2\int_{x=0}^{2} \cancel{\left[4x - \frac{2x^2}{2} - 2yx\right]_0^{2-x} dx}$$

$$= 2\int_{x=0}^{2} \cancel{\left(8 - 4x - 4 + x^2 - 4x - 4y + 2yx\right) dx}$$

$$\cancel{x^2 - 8x - 4y + 2yx + 4}$$

$$= \cancel{2\left(\left(x^2\right.\right.}$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} \left[4x - 2x^2 - 2xy\right] dx\, dy$$

$$= 2\int_0^2 \left. 4xy - 2x^2y - xy^2 \right|_0^{2-x} dx$$

$$= 2\int_0^2 x(2-x)^2\, dx$$

$$= 2\left[4\frac{x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3}\right]_0^2$$

$$= 2\left[4\times\frac{4}{2} + \frac{16}{4} - \frac{32}{3}\right] = \frac{8}{3} \quad \text{Ans}$$

Vector Transformation

1

Gauss Divergence Theorem :-

→ The gauss divergence theorem use the connection betn surface to volume integrals.

→ Let S be the closed surface enclosed by a volume 'V' and Let F be any differentiable vector fn then

$$\iint_S F\cdot N\, ds = \iiint_V (\text{div } F)\, dv = \iiint_V (\nabla\cdot F)\, dv.$$

* Cartesian form of Divergence Theorem :-

$$\iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz.$$

Q. The value of $\int_S (ax\hat{i} + by\hat{j} + cz\hat{k})\cdot N\, ds$, where s is the surface of sphere $x^2+y^2+z^2=1$.

→ $\nabla\cdot F = a+b+c$

$\therefore \int_S F\cdot N\, ds = \int_V (\text{div } F)\, dv$

$= (a+b+c)\, \frac{4}{3}\pi \cancel{r^3}\, 1$

$= \frac{4\pi}{3}(a+b+c)$ Ans

Q. Find $\int_S (ax\hat{i} + by\hat{j} + cz\hat{k}) \cdot N\, ds$ where s is sphere $x^2+y^2+z^2=1$

Same Problem

or $\int_S (ax^2 + bx^2 + cz^2)\, ds$ where s is sphere $x^2+y^2+z^2=1$

Imp Result

$$= (a+b+c)\frac{4\pi}{3}$$

Q. $\int_S \bar{r} \cdot N\, ds$, where 's' is the closed region enclosed by a volume 'V'.

$\rightarrow$ $\nabla \cdot \bar{r} = 3$.

$$\therefore \int_S \bar{r} \cdot N\, ds = \int_V (\nabla \cdot \bar{r})\, dv = 3V \text{ Ans}$$

Q. $\int_S \bar{r} \cdot N\, ds$ where s is sphere $x^2+y^2+z^2=a^2$

$\rightarrow$ $\nabla \cdot \bar{r} = 3$

$$\therefore \int_S \bar{r} \cdot N\, ds = \int_V (\nabla \cdot \bar{r})\, dv = \frac{4}{\cancel{3}}\pi a^3 \times \cancel{3} = 4\pi a^3 \text{ Ans}$$

Q. $\int_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$ where s is the cube of unit length.

$\rightarrow$ $\nabla \cdot F = 1 + 1 + 1 = 3$

$$\int_S F \cdot N\, ds = \int_V (\text{div } F)\, dv = 3V = 3\cancel{a}^3 = 3 \text{ Ans}$$

(with $a = 1$)

Q. $\int \cancel{F} (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy$ where S is sphere $x^2+y^2+z^2=4$.

$\Rightarrow$ $\nabla \cdot F = 1+1+0 = 2$

$$\therefore \int_S F\cdot N\,ds = \int_V (\nabla\cdot V)\,dV = 2V = 2\times \frac{4}{3}\pi \times 2^3$$

$$= \frac{64\pi}{3} \quad \text{Ans}$$

Q. $\int_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot N\,ds$ where S is cube bounded by $0 \le x, y, z \le 1$.

$\rightarrow$ $\nabla \cdot F = 4z - 2y + y = 4z - y$

$$\int_S F\cdot N\,ds = \int_V (\text{div } F)\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} [4z-y]\,dx\,dy\,dz$$

$$= (x)_0^1 \left[4\left(\frac{z^2}{2}\right)_0^1 (y)_0^1 - \left(\frac{y^2}{2}\right)_0^1 (z)_0^1\right]$$

$$= (1-0)\left[2 - \frac{1}{2}\right]$$

$$= \frac{3}{2} \quad \text{Ans}$$

Q. $\int_S [(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}] \cdot N\,ds$ where S in the cuboid bounded by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

$\rightarrow$ $\nabla \cdot F = 2x + 2y + 2z$

$$\therefore \int_S F\cdot N\,ds = \int_V (\nabla\cdot F)\,dV = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (2x + 2y + 2z)\,dx\,dy\,dz$$

$$= \left[x^2yz + y^2zx + z^2xy\right]_{\substack{x=0\\y=0\\z=0}}^{\substack{y=b\\x=a}}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c). \quad \underline{\underline{\text{Ans}}}$$

Q. A region of volume 10 cubic units is bounded by a closed surface 'S'. and N is the outer unit normal vector to S, then —

$$\int_S (4x\hat{i} + 2y\hat{j} - z\hat{k}) \cdot N\, ds = \_\_\_$$

→ $\nabla \cdot F = 4 + 2 - 1 = 5$

$$\int_V 5\, dv = 5\cancel{V}^{\,10} = \underline{\underline{50.}}$$

*** (ONLY APPLICABLE WHEN R is also in only x-y plane, no z plane)

GREEN'S THEOREM in a plane :- ($\int \rightarrow \iint$)

Let 'R' be the closed region in the xy plane bounded by a closed and non intersecting curve 'c' and Let P and Q are continuous fn of x and y possessing the first order partial derivatives, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

Q. The value of $\oint_C (2x-y)\,dx + (x+y)\,dy$ where c is the circle $x^2+y^2=9$

$\rightarrow$ $$\frac{\partial P}{\partial x} = -1 \quad , \quad \frac{\partial Q}{\partial x} = 1.$$

$\rightarrow$ Const. , so no need to consider limit.

Only for ▭ and □ we will take const. limit of x and y.

for other curves except rectangle & square: else where, we will take one limit const and other as a fn of former one.

$\therefore$ $$\iint_R (1-(-1))\,dx\,dy$$

$$= \int_R 2\,dR = 2R \nearrow \pi r^2 \swarrow 3 = 18\pi \text{ Ans}$$

Q. $\oint_C \overbrace{(x^2+y^2)}^{P}dx - \overbrace{2xy}^{Q}\,dy$, where c is the rectangle bounded by $x=0,\ x=a$
$y=0,\ y=b.$

$\rightarrow$ $$\frac{\partial P}{\partial y} = 2y \quad , \quad \frac{\partial Q}{\partial x} = -2y$$

$$\iint_R (-2y-2y)\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} -4y\,dx\,dy$$

$$= (x)_0^a \left(-4\frac{y^2}{2}\right)_0^b = ax - 2b^2 = -2ab^2$$

Q. $\oint_C (\overset{P}{x^2 - xy^3})\,dx + (\overset{Q}{y - 2xy})\,dy$ where c is the square bounded by the points (0,0) (2,0) (2,2) (0,2).

$\rightarrow \quad \dfrac{\partial P}{\partial y} = -3xy^2 \quad , \quad \dfrac{\partial Q}{\partial x} = -2y$

$$\therefore \iint_R -2y - (-3xy^2)\, dx\, dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} -2y + 3xy^2 \, dx\, dy$$

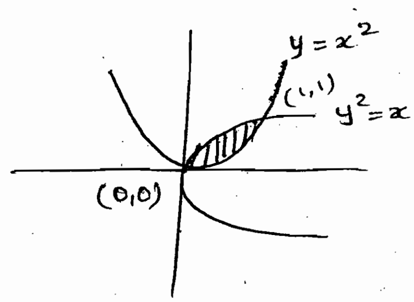
$$= -2\left(\frac{y^2}{2}\right)_0^2 (x)_0^2 + 3\left(\frac{x^2}{2}\right)_0^2 \left(\frac{y^3}{3}\right)_0^2$$

$$= -8 + 16 = 8 \quad \underline{\text{Ans}}$$

Q. $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where 'C' is the region bounded by $y = \sqrt{x}$ and $y = x^2$

$\rightarrow \quad \dfrac{\partial P}{\partial y} = -16y \quad , \quad \dfrac{\partial Q}{\partial x} = -6y$

$$\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (\overset{10y}{\cancel{-6y + 16y}})\, dx\, dy$$



$$= \int_0^1 \overset{5}{\cancel{10}}\left(\frac{y^2}{\cancel{2}}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx.$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = \cancel{5}\left(\frac{3}{\underset{2}{\cancel{10}}}\right) = \frac{3}{2} \quad \underline{\text{Any}}$$

$x^2 = \sqrt{x}$

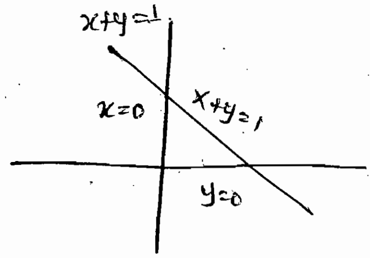
$x^4 = x$

$x(x^3 - 1) = 0$

$x = 0, 1$

$y = 0, 1$

Q. $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$ where c is the region bounded by $x=0,\ y=0,\ x+y=1$



$\rightarrow \quad \dfrac{\partial P}{\partial y} = -16y \quad , \quad \dfrac{\partial Q}{\partial x} = -6y$

$$\int_{x=0}^{1}\int_{y=0}^{1-x} (-6y+16y)\,dx\,dy$$

$$= \int_{0}^{1}\int_{y=0}^{1-x} 10y\,dx\,dy$$

$$= \int_{0}^{1} 10\left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= 5\int_{0}^{1} (1-x)^2 dx$$

$$= 5\left|\frac{(1-x)^3}{-3}\right|_0^1$$

$$= -\frac{5}{3}(0-1).$$

$$= \frac{5}{3} \quad \underline{\text{Ans}}$$

STOKE'S THEOREM :- $\int \to \iint$

Let s be an open surface bounded by a closed and a non interpeecting curve 'c' and F be any differentiable vector function then —

$$\oint F \cdot d\bar{r} = \iint_S (\text{curl } F) \cdot N \, ds = \iint_S (\nabla \times F) \cdot N \, ds$$

Q. Evaluate by the Stoke's theorem

$\oint_C F \cdot d\bar{r}$ where $F = -y^3 i + x^3 j$ and S is the circular disc $x^2+y^2 \leq 1$, $z=0$.

→ $$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(3x^2+3y^2).$$

$$= 3(x^2+y^2)k.$$

$$= 3k \quad \{\because x^2+y^2 \leq 1\}$$

Directly we can say $N = \hat{k}$ $\{\because z=0$ given means xy plane and normal vector will be only $\hat{k}\}$

$$\therefore \int 3k \cdot k \, ds$$

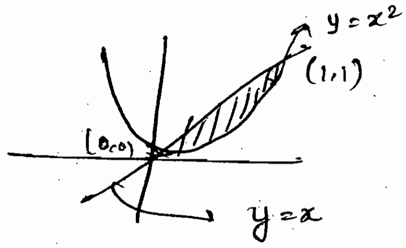
$$= 3\int ds$$

$$= 3s$$

$$= 3\pi(1)^2$$

$$= 3\pi \quad \underline{\text{Ans}}$$

Q. The area bounded by $y = x^2$ and $y = x$ —



$x^2 = x$

$x(x-1) = 0$

$x = 0, 1$

$y = 0, 1$

Imp Trick

from Graph

The area $= \int_0^1 (x - x^2)\,dx$ [Area will be always upper part − lower part.]

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \frac{1}{2} - \frac{1}{3}$

$= \frac{1}{6}$ Ans

Q. The area bounded by $y^2 = 4ax$ & $x^2 = 4ay$ is $\underline{16a^3/3}$

The area bounded by $y^2 = 8x$ & $x^2 = 8y$ is $\underline{64/3}$

I ——————— $y^2 = 4x$ & $x^2 = 4y$ is $\underline{16/3}$

——————— $y^2 = x$ & $x^2 = y$ is $\underline{1/3}$

***Q. The arc length of the curve $y = f(x)$ between $x = a$ & $x = b$ is $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

Imp Result

Q. $y = \frac{2}{3}x^{3/2}$ between $x = 0$ and $x = 1$.

$\frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{\frac{3}{2}-1} = \sqrt{x}$

$\therefore \int_a^b \sqrt{1 + (\sqrt{x})^2}\,dx = \int_a^b \sqrt{1+x}\,dx = \int_a^b (1+x)^{\frac{1}{2}}\,dx$

$= \left.\frac{(1+x)^{3/2}}{3/2}\right|_0^1 = \frac{2}{3}\left[2\sqrt{2} - 1\right]$

Q. A parabolic arc $y = \sqrt{x}$ $1 \le x \le 2$ is revolving around x axis then the volume of the solid revolution.

→ $$\boxed{\begin{aligned} \text{volume} &= \int_a^b \pi y^2 dx \quad (x\text{-axis}) \\ &= \int_a^b \pi x^2 dy \quad (y\text{-axis}). \end{aligned}}$$

$\therefore$ volume $= \int_1^2 \pi (\sqrt{x})^2 dx$

$= \pi \left(\frac{x^2}{2}\right)_1^2 = \frac{\pi}{2}(4-1) = \frac{3\pi}{2}$ Ans

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volume $= \int_a^b \pi x^2 dy$

$y^2 = 8x$, $x = 2$

$y^2 = 8(2)$

$y^2 = 16$

$\Rightarrow y = \pm 4$

$\therefore$ volume $= \int_{-4}^{4} \pi \cancel{x}^2 dy$ $\frac{y^2}{8}$

$= \frac{\pi}{8^2} \int_{-4}^{4} (y^2)^2 dy$

$= \frac{\pi}{64} \, 2 \int_0^4 y^4 dy$

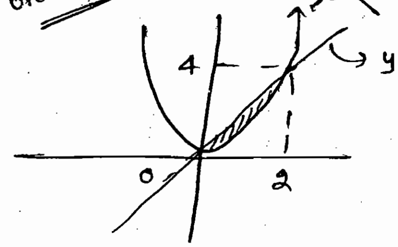
$= \frac{\pi}{32} \times \left. \frac{y^5}{5} \right|_0^4$

$= \frac{\pi}{\cancel{32}} \times \frac{\cancel{4} \times \cancel{4} \times \cancel{4} \times 4 \times 4}{5} = \frac{32\pi}{5}$ Ans

(96)

$$\int_0^2 \int_{y=x^2 \Rightarrow x=\sqrt{y}}^{y=2x \Rightarrow x=\frac{y}{2}} f(x,y)\,dy\,dx$$

For changeng order of integration write $x=\sqrt{y}$

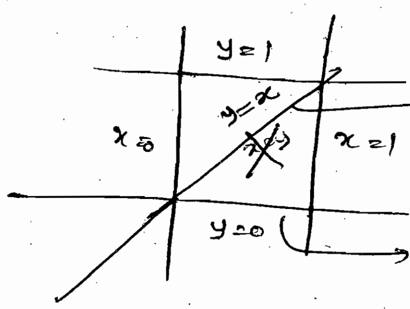


write $x=\frac{2}{y}$

$$\cong \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy \quad \underline{Ans}.$$

(87)

$$\int_0^1 \int_{x=y}^{x=1} xy\sin(xy)\,dx\,dy = \int_0^1 \int_a^b xy\sin xy\,dy\,dx$$



Upper part as upper limit

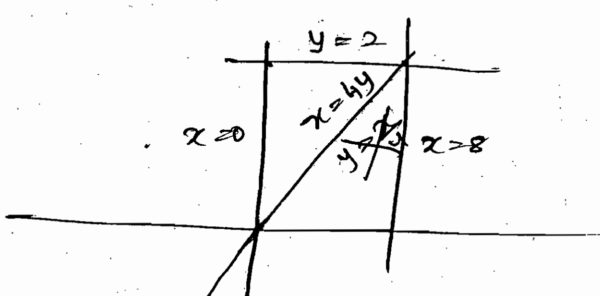
Lower part as lower limit

By changing order of integration.

$$= \int_0^1 \int_0^x xy\sin(xy)\,dy\,dx$$

(88)

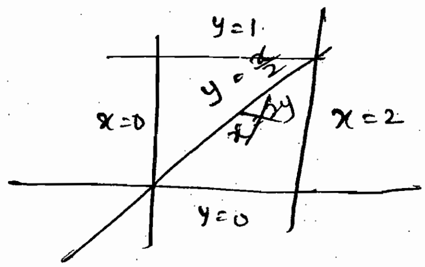
$$\int_0^8 \int_{y=x/4}^{2} f(x,y)\,dy\,dx \simeq \int_r^s \int_p^q f(x,y)\,dx\,dy$$



$$\simeq \int_{y=0}^{2} \int_{x=0}^{4y} f(x,y)\,dx\,dy.$$

(85) $\int_0^1 \int_{x=2y}^{x=2} e^{x^2}\, dx\, dy$

By Changing order of integration —



$$\cong \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx$$

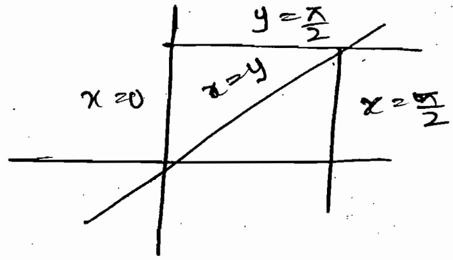
$$= \int_0^2 e^{x^2} (y)_0^{x/2}\, dx.$$

$$= \int_0^2 e^{x^2} \frac{x}{2}\, dx$$

$$= \frac{1}{2\times 2} \int_0^1 e^{x^2}\, 2x\, dx$$

$$= \frac{1}{4}\left(e^{x^2}\right)_0^2 = \frac{e^4 - e^0}{4} = \frac{e^4 - 1}{4} \text{ Ans}$$

(8) $\displaystyle\int_{0}^{\pi/2}\int_{y=x}^{y=\pi/2} \frac{\cos y}{y}\, dy\, dx$



By changing order of integration —

$\cancel{\displaystyle\int_{0}^{y}\int_{0}^{\pi}}$

$$= \int_{0}^{\pi/2}\int_{0}^{y} \frac{\cos y}{y}\, dx\, dy$$

$$= \int_{0}^{\pi/2} \frac{\cos y}{y}\, (x)_{0}^{y}\, dy$$

$$= \int_{0}^{\pi} \cos y\, dy$$

$$= (\sin y)_{0}^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1 - 0$$

$$= 1 \text{ Ans}$$

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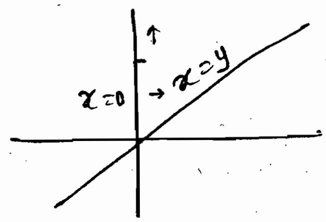
$$\int_{0}^{\infty}\int_{y=x}^{y=\infty} \frac{1}{y} e^{-y/2}\, dy\, dx$$

$=$



$$\simeq \int_{0}^{\infty}\int_{0}^{y} \frac{1}{y} e^{-\frac{y}{2}}\, dx\, dy$$

$$= \int_{0}^{\infty}\int_{0}^{y} \frac{1}{y} e^{-y/2} (x)_0^y\, dy$$

$$= -2\left[e^{-\frac{y}{2}}\right]_0^{\infty} = -2\left[e^{-\infty} - e^{-0}\right] = -2(0-1) = 2$$

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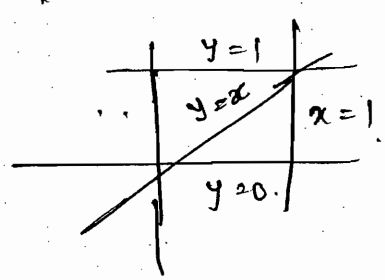
$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} y\sqrt{1+x^3}\, dx\, dy$$



$$\simeq \int_{0}^{1}\int_{y=0}^{x} y\sqrt{1+x^3}\, dy\, dx$$

$$= \int_{0}^{1} \sqrt{1+x^3}\left(\frac{y^2}{2}\right)_0^x\, dx$$

$$= \int_{0}^{1} \sqrt{1+x^3}\, \frac{x^2}{2}\, dx$$

$$= \frac{1}{2}\times\frac{1}{3}\int_{0}^{1} (1+x^3)^{\frac{1}{2}} \cdot 3x^2\, dx$$

$$= \frac{1}{\cancel{6}_3} \left| \frac{(1+x^3)^{3/2}}{\frac{3}{\cancel{2}}} \right|_0^1$$

$$= \frac{1}{9}\left[2\sqrt{2}-1\right].$$

(71) $f(n) = \int_0^{\pi/4} \tan^n x \, dx$ then $f(3)+f(1) = ?$

$\rightarrow$ $f(3)+f(1) = \int_0^{\pi/4} (\tan^3 x + \tan x)\, dx.$

$$= \int_0^{\pi/4} (\tan x)(\sec^2 x)\, dx$$

$$= \left(\frac{\tan^2 x}{2}\right)_0^{\pi/4} = \frac{1}{2} \text{ Ans}$$

(60) $\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x}\, dx = ?$

$\because \int_0^{\frac{\pi}{2}} f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\cos x \sin x}\, dx$$

$$2I = \int_0^{\pi/2} \frac{\cancel{\cos x} - \cancel{\sin x} + \cancel{\sin x} - \cancel{\cos x}}{1+\sin x \cos x}$$

$$= 0$$

$\Rightarrow I = 0$ Ans

(56)

$P(x, y, z)$

$z^2 = 1 + xy$

O

$(0,0,0)$

$$OP = \sqrt{x^2+y^2+z^2}$$

$$= \sqrt{x^2+y^2+1+xy}$$

$$f = x^2+y^2+xy+1$$

$$p = \frac{\partial f}{\partial x} = 2x+y = 0$$

$$q = \frac{\partial f}{\partial y} = 2y+x = 0$$

Put $x = y$.

$$x - y = 0$$

$$x = y$$

$$\cancel{3x = 0} \qquad 3x = 0$$

$$\cancel{x = 0} \quad \Rightarrow x = 0$$

$$y = 0.$$

$$\therefore OP = \sqrt{0+0+0+1}$$

$$= 1$$

(53) $x = 2$ after making $f''(x) = 0$ {Since maximum of slope needed not maximum of curve demanding}

(45) $x = uv$, $y = \frac{v}{u}$.

$$f(x, y)\,dx\,dy = \iint \cancel{f(u,v)} \cdot f\left(uv, \frac{u}{v}\right) \phi(u, v)\,du\,dv$$

$$\phi(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix}$$

$$= \frac{2v}{u} \cdot \text{Ans}$$

Q. A curve c is defined as $x = a\cos^3\theta$
$y = a\sin^3\theta$ in $[0, \frac{\pi}{2}]$

What will be point P on curve C where tangent to curve is parallel to chord joining points $(a,0)$ & $(0,a)$.

$\rightarrow$ $x = a\cos^3\theta$, $y = a\sin^3\theta$.

$$\frac{\cancel{a} \times \cancel{3}\sin^2\theta \times \cos\theta}{\cancel{a} \times \cancel{3}\cos^2\theta \times -\sin\theta} = \frac{a-0}{0-a}$$

$\Rightarrow$ $\cancel{-}\tan\theta = \cancel{-}\frac{\cancel{a}}{a}$

$\Rightarrow$ $\theta = \frac{\pi}{4}$.

$\therefore$ putting values —
we get —
$$\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$$

Q. 19 — Correct $y = a(1-\cos\theta)$.

Q. 8. $\underset{n\to\infty}{Lt} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

$$l = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$$

$$\log l = \frac{1}{n}\left[\log n! - n\log n\right]$$

~~$\log l = \frac{1}{n}\log n! - \log n$~~

$$= \frac{1}{n}\left[\log \frac{1\times2\times3\times\cdots n}{n\times n\times n\cdots n}\right]$$

$$= \frac{1}{n}\left[\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \cdots\right]$$

Imp
Formula

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(\frac{r}{n}\right) = \int_0^1 \log x \, dx$$

$$= \left[x \log x - x\right]_0^1$$

$$\log l = (1 \cancel{\log 1}_{0} - 1]$$

$$l = e^{-1} = \frac{1}{e} \text{ Ans}$$

Q ⑥ $\lim_{n\to\infty} \left[\frac{1}{1+n} + \frac{1}{2+n} + \cdots + \frac{1}{n+n}\right]$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{1+\left(\frac{r}{n}\right)}$$

$$= \int_0^1 \cancel{\log} \frac{1}{1+x} \, dx$$

$$= \log(1+x)\Big|_0^1$$

$$= \log 2 - \log 1$$

$$= \log 2 \text{ Ans}$$

(5) $\underset{n\to\infty}{Lt}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\cdots+\frac{n}{n^2+(n-1)^2}\right]$

$\underset{n\to\infty}{Lt}\ \frac{n}{n^2}\left[\sum_{r=0}^{n-1}\frac{1}{1+\left(\frac{r}{n}\right)^2}\right]$

$= \int_0^1 \frac{1}{1+x^2}\,dx$

$= \left(\tan^{-1}x\right)_0^1$

$= \tan^{-1}1-\tan^{-1}0$

$= \frac{\pi}{4}-0 = \frac{\pi}{4}$ Ans

(7) $\underset{n\to\infty}{Lt}\sum_{K=1}^{n}\frac{1}{3n+K}$

$= \underset{n\to\infty}{Lt}\ \frac{1}{n}\sum_{K=1}^{n}\frac{1}{3+\left(\frac{K}{n}\right)}.$

$= \int_0^1 \frac{1}{3+x}\,dx = \left[\log(3+x)\right]_0^1 = \log 4-\log 3 = \log\frac{4}{3}$ Ans

DIFFERENTIAL EQ^n

An eq^n consisting of Differential coeff. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ... $\frac{d^ny}{dy^n}$ is called a differential eq^n.

* Order - $\frac{d^ny}{dy^n} \to$ is order

* Degree - The degree of diff eq^n is the degree of the highest order

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0 \qquad O \to 2,\ D \to 1$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 5\frac{dy}{dx} + 2y = k\left(\frac{d^3y}{dx^3}\right)^2 \qquad O - 3,\ D - 2$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = 5y \qquad D \to 4,\ O \to 2$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 5\cos\left(\frac{dy}{dx}\right) + 6y = 0 \qquad D \to \text{not } \cancel{\text{defined}} \text{ determined},\ O \to 2.$$

Formation of a Differential Eq^n :-

A diff. eq^n can be formed by eliminating arbitary consts. in the solution.

Ex:-

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$y^{\cancel{2}} = \cancel{2y\frac{dy}{dx}} \cdot x \cdot \cancel{2}\frac{dy}{dx}$$

$$2x\frac{dy}{dx} - y = 0$$

Q. $x^2+y^2=a^2$

$\cancel{2}x + \cancel{2}y\frac{dy}{dx} = 0.$

$$\boxed{x + y\frac{dy}{dx} = 0.}$$

Note :- Differentiate only that much time upto there no. of arbitary constants.

Q. $y = ae^{x} + be^{-x}$

$\frac{dy}{dx} = ae^{x} - be^{-x}.$

$\frac{d^2y}{dx^2} = ae^{x} + be^{-x}$

$$\boxed{\frac{d^2y}{dx} = y}$$

Q. $y = e^{x}(a\cos x + b\sin x)$

$\cancel{\frac{dy}{dx}} =$. $y_1 = e^{x}(a\cos x + b\sin x) + e^{x}(-a\sin x + b\cos x)$

$y_1 - y = e^{x}(-a\sin x + b\cos x)$

$y_2 - y_1 = \underbrace{e^{x}[-a\sin x + b\cos x]}_{y_1 - y} + \underbrace{e^{x}[-a\cos x + b\sin x]}_{-y}$

$y_2 - y_1 = y_1 - y - y$

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Solution of a Differential Eqⁿ :-

* Variable Seperable Method :-

① $$\frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}$$

$$\int \frac{dy}{\sqrt{1+y^2}} = \int -\frac{dx}{\sqrt{1+x^2}}$$

$$\Rightarrow \boxed{\sinh^{-1} y = -\sinh^{-1} x + c}$$

② $$\frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = e^{x} \cdot e^{y}$$

$$\int dy\, e^{-y} = \int e^{x}\, dx$$

$$-e^{-y} = e^{x} + c$$

$$\Rightarrow e^{x} + e^{-y} + c = 0$$

③ $$\frac{dy}{dx} = x^2 e^{x-y}$$

$$\int dy\, e^{y} = \int x^2 e^{x}\, dx$$

$$e^{y} = e^{x}\left[x^2 - 2x + 2\right] + c$$

④ $\frac{dy}{dx} + y^2 = 0$

$\int dy \, y^{-2} = -\int dx$

$\frac{y^{-2+1}}{-1} = -x + c$

$\Rightarrow -y^{-1} = -x + c$

$\Rightarrow y^{-1} + x + c = 0$ or $\boxed{y = \frac{1}{x+c}}$

⑤ $e^x \tan y \, dx + (1+e^x) \sec^2 y \, dy = 0$

$e^x \tan y \, dx = -(1+e^x) \sec^2 y \cdot dy$

$\frac{e^x \, dx}{1+e^x} = -\frac{\sec^2 y \, dy}{\tan y}$

$\int \frac{e^x}{1+e^x} dx + \int \frac{\sec^2 y}{\tan y} dy = \int 0$

$\log(1+e^x) + \log \tan x = \log c$

$\Rightarrow \boxed{(1+e^x) \tan y = c}$

⑥

II^nd Method

* Eq^n Reducable to variable - seperable form :-

If the given diff. eq^n is not directly variable seperable form, by making substitution. again we can transform and under variable seperable form.

Q. $\frac{dy}{dx} = (4x+y+1)^2$

Let $z = 4x+y+1$

$\frac{dz}{dx} = 4 + \frac{dy}{dx}$

$4 + \frac{dy}{dx} = 4 + (4x+y+1)^2$

$\Rightarrow \frac{dz}{dx} = 4 + z^2$

$\Rightarrow \int \frac{dz}{z^2+4} = \int dx$

$\Rightarrow \frac{1}{2} \tan^{-1} \frac{z}{2} = x + c$

$\Rightarrow \boxed{\frac{1}{2} \tan^{-1} \left(\frac{4x+y+1}{2}\right) = x + c}$

Q. $1 + \frac{dy}{dx} = 1 + \sin(x+y) + \cos(x+y)$

$x + y = z$

$1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dz}{dx} = 1 + \sin z + \cos z$

$\Rightarrow \int \frac{dz}{1 + \sin z + \cos z} = \int dx$

Remember

WHENEVER INTEGRAL IS :- in form of $\frac{1}{a + b \sin x} dx$; $\int \frac{1}{a + b \cos x}$, $\int \frac{1}{a \cos x + b \sin x + c} dx$

Put $t = \tan \frac{x}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$$\int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2} = x+c$$

$$\int \frac{2\,dt}{(1+\cancel{t^2}+2t+1-\cancel{t^2})} = x+c$$

$$\int \frac{\cancel{2}}{\cancel{2}(1+t)}\,dt = x+c$$

$$\log(1+t) = x+c.$$

$$\log\left[1+\tan\frac{x+y}{2}\right] = x+c$$

HOMOGENEOUS DIFFERENTIABLE EQUATION :-

A diff. eqⁿ of the form –

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$ is said to be homogeneous differential eqⁿ, if both $f(x,y)$ and $g(x,y)$ are homogeneous function of the same degree.

$y = vx \Rightarrow v = \frac{y}{x}$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

when $\frac{dx}{dy} = \frac{f(x,y)}{g(x,y)}$

$x = vy \Rightarrow v = \frac{x}{y}$

$$\boxed{\frac{dx}{dy} = v + y\frac{dv}{dy}}$$

ILATE

Ex:- $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

$\frac{dx}{dy} = \frac{2xy}{x^2+y}$

→ which form should be selected for substitution?

↳ ~~जिसके~~ numerator में ज्यादा no. of terms होगा

$\therefore \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

$$v + x\frac{dv}{dx} = \frac{\cancel{x^2} + v^2\cancel{x^2}}{2\cancel{x}\cdot v\cancel{x}}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\int \frac{-2v}{1-v^2}\,dv = \int -\frac{dx}{x}$$

$$\Rightarrow \log(1-v^2) = \log\frac{c}{x}$$

$$\Rightarrow 1-\frac{y^2}{x^2} = \frac{c}{x}$$

$$\Rightarrow \frac{x^2-y^2}{x^2} = \frac{c}{x}$$

$$\Rightarrow \boxed{x^2-y^2 = cx}$$

$\frac{1}{x} \to \log cx$

$-\frac{1}{x} \to \log\frac{c}{x}$

$\frac{2}{x} \to \log cx^2$

$-\frac{2}{x} \to \log\frac{c}{x^2}$

$\frac{3}{x} \to \log cx^3$

$-\frac{3}{x} \to \log\frac{c}{x^3}$

Q. $x^2y\,dx - (x^3+y^3)\,dy = 0.$

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x^3+y^3}{x^2y}$$

$$v + y\frac{dv}{dy} = \frac{v^3\cancel{y^3}+\cancel{y^3}}{v^2\cancel{y^3}}$$

$$v + y\frac{dv}{dy} = \frac{v^3+1}{v^2}$$

$$y\frac{dv}{dy} = \frac{1}{v^2}$$

$$\Rightarrow \int\frac{dy}{y} = \int v^2\,dv$$

$$\Rightarrow \log cy = \frac{v^3}{3}.$$

$$\Rightarrow \log cy = \frac{x^3}{3y^3}$$ Ans.

Q. $x\dfrac{dy}{dx} = y + x\tan\left(\dfrac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}.$$

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \tan v$$

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x}$$

$$\log \sin v\ \cancel{\cos v\, dv} = \log cx$$

$$\boxed{\sin\left(\frac{y}{x}\right) = cx} \quad \text{ae}$$

Q. $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0$

Trig → Angle ज्यादा

Algebric. → ज्यादा power numerator में

exponentia $\dfrac{x}{y} - \dfrac{dx}{dy}$

$\dfrac{y}{x} - \dfrac{dy}{dx}$

$$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{\left(1+e^{x/y}\right)}$$

$$v + y\frac{dv}{dy} = \frac{e^{v}(v-1)}{1+e^{v}}$$

$$y\frac{dv}{dy} = \frac{e^{v}(v-1)}{(1+e^{v})} - v$$

$$y\frac{dv}{dy} = \frac{\cancel{ve^{v}} - e^{v} - v - \cancel{ve^{v}}}{(1+e^{v})}$$

$$y\frac{dv}{dy} = \frac{-(v+e^{v})}{(1+e^{v})}$$

$$\int \frac{(1+e^{v})}{v+e^{v}}\,dv = \int -\frac{dy}{y}$$

$$\log(v+e^{v}) = \log\frac{c}{y}$$

$$\Rightarrow \frac{x}{y} + e^{x/y} = \frac{c}{y} \Rightarrow \boxed{x + y\cdot e^{\frac{x}{y}} = c} \quad \text{ae}$$

NON HOMOGENEOUS DIFFERENTIAL FUNCTION :-

A diff. eq^n of the form —

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$ is said to be a non homogeneous differential eq^n

Case-1 :- If $a_1 b_2 - a_2 b_1 \neq 0$, then the substitution is —

$$x = X + h$$
$$y = Y + K$$

where h, K are constants to be determined.

Case-2 :- If $a_1 b_2 - a_2 b_1 = 0$, then ~~the~~ some part of numerator and denominator are equal and that part is considered as substitution.

Q. Find the substitution that transforms a non homogenous diff eq^n $\frac{dy}{dx} = \frac{2x + 2y - 2}{3x + y - 5}$ to homogeneous form.

$\hookrightarrow \neq 0 \rightarrow 1^{st}$ form

$\rightarrow$

$\cancel{\frac{dy}{dx} = \frac{2(X+h) + 2(Y+k) - 2}{3(}}$

h	k	1	h	k	
2	2	-2	2	2	$\rightarrow$ numerator
3	1	-5	3	1	$\rightarrow$ Denominator

$\rightarrow$ write like this only

$$\frac{h}{-10+2} = \frac{k}{-6+10} = \frac{1}{2-6}$$

$$\frac{h}{-8} = \frac{k}{4} = \frac{1}{-4}$$

$h = 2$, $k = -1$ $\Rightarrow$ $\left.\begin{aligned} x &= X + 2 \\ y &= Y + 1 \end{aligned}\right\}$ Ans

Q. find Substitution —

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

$\neq 0 \rightarrow 1^{st}$ form.

h	k	l	h	k
1	2	-3	1	2
2	1	-3	2	1

$$\frac{h}{-6+3} = \frac{k}{-6+3} = \frac{1}{1-4}$$

$$\frac{h}{-3} = \frac{k}{-3} = \frac{1}{-3}$$

$h = 1, \quad k = 1$

$$\boxed{\begin{aligned} x &= X+1 \\ y &= Y+1 \end{aligned}}$$ Ans.

Q. Find the Substitution —

$$\frac{dy}{dx} = \frac{2x+3y+1}{4x+6y+1}$$

$= 0 \rightarrow 2^{nd}$ form

$$\frac{dy}{dx} = \frac{(2x+3y)+1}{2(2x+3y)+1}$$

$$\boxed{z = 2x+3y}$$

Q. $\frac{dy}{dx} = \frac{x+2y+3}{y+2x+5}$

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+y+5}$$

$\neq 0 \rightarrow$ 1st form.

h	k	l	h	k.
1	2	3	1	2
2	1	5	2	1

$$\frac{h}{10-3} = \frac{k}{6-5} = \frac{1}{1-4}$$

$$\frac{h}{7} = \frac{k}{1} = \frac{1}{-3}$$

$$h = -\frac{7}{3}, \quad k = -\frac{1}{3}$$

$$\boxed{\begin{aligned} x &= X + \frac{7}{3} \\ y &= Y - \frac{1}{3} \end{aligned}}$$

Ans

EXACT DIFFERENTIAL EQ^N :-

A diff. eq^n of the form $M\,dx + N\,dy = 0$ is said to be exact if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

$$\boxed{\text{Gen. Sol}^n \rightarrow \int^{x} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c}$$

Ex:- $(x^2 - ay)\,dx + (y^2 - ax)\,dy = 0$

$$\underbrace{\frac{\partial M}{\partial y} = -a \quad , \quad \frac{\partial N}{\partial x} = -a}_{\text{exact}}$$

$$\int (x^2 - ay)\,dx + \int y^2\,dy = c$$

$$\frac{x^3}{3} - axy + \frac{y^3}{3} = c$$

$$\Rightarrow \quad x^3 + y^3 - 3axy = 3c \quad \text{Ans}$$

MORE Efficient Shortcut

GENERALIZATION:-

Ex:- $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$.

$$\frac{\partial M}{\partial y} = h \quad , \quad \frac{\partial N}{\partial x} = h.$$

$$\boxed{\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c} \quad \text{Ans}$$

Q. $e^x \tan y\,dx + (1 + e^x)\sec^2 y\,dy = 0$

$$\underbrace{\frac{\partial M}{\partial y} = e^x \sec^2 y \qquad \frac{\partial N}{\partial x} = e^x \sec^2 y}_{\text{exact}}$$

Sol^n

$\therefore \quad e^x \tan y + \tan y = c$

$$\Rightarrow \boxed{(e^x + 1)\tan y = c}$$

Q. $(x^2+y^2)\,dx + 2xy\,dy = 0$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

exact

Sol^n

$$\boxed{\frac{x^3}{3} + y^2 x = C}$$

Q. $\left(1+e^{\frac{x}{y}}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

exact

$$\frac{\partial M}{\partial y} = e^{x/y} \cdot x \cdot \left(-\frac{1}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\left(\frac{1}{y}\right)$$

$$= e^{x/y}\left(-\frac{1}{y} + \frac{1}{y} - \frac{x}{y^2}\right)$$

Sol^n.

$$\boxed{x + y e^{\frac{x}{y}} = C}$$

Non - Exact Differential Eqⁿ :-

A diff eqⁿ $Mdx + Ndy = 0$ is said to be non.exact, if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

METHOD-I (Method to form a suitable Integrating factor)

IMPORTANT INTEGRATING FACTORS:-

***

① $x\,dy + y\,dx = d(xy)$

② $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

③ $\frac{x\,dy - y\,dx}{xy} = d\left(\log \frac{y}{x}\right)$

④ $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1} \frac{y}{x}\right)$

⑤ $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

⑥ $\frac{y\,dx - x\,dy}{xy} = d\left(\log \frac{x}{y}\right)$

⑦ $\frac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1} \frac{x}{y}\right)$

⑧ $\frac{ye^x\,dx - e^x\,dy}{y^2} = d\left(\frac{e^x}{y}\right)$

⑨ $\frac{2xy\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right)$

⑩ $\frac{x\,dx + y\,dy}{x^2+y^2} = d\left[\frac{1}{2}\log(x^2+y^2)\right]$

Q. $y(1+xy)dx + x(1-xy)\,dy = 0.$

$(y+xy^2)\,dx + (x - x^2y)\,dy = 0.$

$$\underbrace{\frac{\partial M}{\partial y} = 1+2xy \,, \quad \frac{\partial N}{\partial x} = 1-2xy}_{\text{Non exact}}$$

→ we can make it perfect I.F but considering 1st term carefully.

$$\underbrace{y\,dx + xy^2dx + x\,dy}_{\text{Perfect I.F}} - x^2y\,dy = 0$$

$$\frac{d(xy)}{(xy)^2} + \frac{\cancel{xy}[y\,dx - x\,dy]}{(xy)^{\cancel{2}}} = \frac{0}{(xy)^2}$$

$$\Rightarrow \int \frac{1}{(xy)^2}\,dxy + \int d\left(\log \frac{x}{y}\right) = 0$$

$$\Rightarrow \boxed{\frac{-1}{xy} + \log\frac{x}{y} = C} \qquad \left\{\because \frac{dx}{x^2} = -\frac{1}{x}\right\}$$

Q ① $y\,dx - x\,dy + (1+x^2)dx + x^2\sin y\,dy = 0.$

by taking − common it is also integrable

$$\frac{y\,dx - x\,dy}{x^2} + \frac{(1+x^2)\,dx}{x^2} + \frac{\cancel{x^2}\sin y\,dy}{\cancel{x^2}}$$

$$\Rightarrow -\int \frac{(x\,dy - y\,dx)}{x^2} + \int\left(\frac{1}{x^2}+1\right)dx + \int \sin y\,dy = \int 0.$$

$$\Rightarrow -\int d\left(\frac{y}{x}\right) + \int\left(\frac{1}{x^2}+1\right)dx + \int \sin y\,dy = \int 0$$

$$\Rightarrow \boxed{-\frac{y}{x} - \frac{1}{x} + x - \cos y = C.}$$

Q. $\displaystyle\int \frac{ax\cancel{y^2}dx}{\cancel{y^2}} + \int \frac{ye^x dx - e^x dy}{y^2} = \int 0$

$\Rightarrow \int ax\,dx + \int d\left(\frac{e^x}{y}\right) = \int 0$

$\Rightarrow \boxed{a\frac{x^2}{2} + \frac{e^x}{y} = c}$

Q. $(y^2e^x + 2xy)\,dx - x^2dy$.

$\Rightarrow \dfrac{\cancel{y^2}e^x dx}{\cancel{y^2}} + \dfrac{2xy\,dx - x^2dy}{y^2} = 0$

$\Rightarrow \int e^x dx + \int d\left(\frac{x^2}{y}\right) = \int 0.$

$\Rightarrow \boxed{e^x + \frac{x^2}{y} = c}$

Q. $\dfrac{2x^2\cancel{y^2}dx}{\cancel{y^2}} + \dfrac{ye^x dx - e^x dy}{y^2} - \dfrac{y^3 dy}{y^2} = 0$

$\Rightarrow \int 2x^2 dx + \int d\left(\frac{e^x}{y}\right) - \int y\,dy = \int 0$

$\Rightarrow \boxed{2\frac{x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c}$

NON EXACT DIFFERENTIAL EQ^N :-

METHOD -2

If $Mdx + Ndy = 0$ is non exact, but homogeneous and if $~~Md~~$ $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an I.F.

After multiplying in diff. eq^n → non exact becomes exact.

Then for sol^n will follow procedure of Exact diff. eq^n.

→ Q. ① $\underbrace{(x^2+y^2)}_{d=2}dx - \underbrace{2xy}_{d=2}dy = 0$ Homogeneous

$$\frac{\partial M}{\partial y} = +2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

$$\therefore \; I.F = \frac{1}{Mx+Ny} = \frac{1}{x(x^2+y^2)+y(-2xy)} = \frac{1}{x^3+xy^2-2xy^2}$$

$$= \frac{1}{x^3-xy^2}$$

$$= \frac{1}{x(x^2-y^2)}$$

Now multiply by I.F —

$$\int \frac{x^2+y^2}{x(x^2-y^2)}dx - \frac{2xy}{x(x^2-y^2)}dy = 0.$$

$$\Rightarrow \int\left[\frac{2x}{x^2-y^2} - \frac{1}{x}\right]dx = \log c$$

$$\Rightarrow \cancel{\log}\; \frac{x^2-y^2}{x} = \cancel{\log}\, C$$

$$\Rightarrow \boxed{x^2-y^2 = cx}$$

Calculate I·F —

(b)

① $x^2y\,dx - (x^3+y^3)\,dy = 0.$

$\cancel{\frac{\partial M}{\partial y}} = \,^{\circ}\; Mx + Ny \neq 0$

$\because\; \cancel{x^3y} - \cancel{x^3y} - y^4 \neq 0$

$\therefore\; \frac{1}{Mx+Ny} = \frac{-1}{y^4}$

Note:-

In Exam option will be provided as $\frac{1}{y^4}$

coz, (−) and const will not effect integration and ~~by~~ in process of solving '−' or const can be taken out common and can be send to right where 0 is given.

② $(x^2y - xy^2)\,dx - (x^3 - 2x^2y)\,dy$

$Mx + Ny = \cancel{x^3y} - x^2y^2 - \cancel{x^3y} + 2x^2y^2$

$= x^2y^2$

$I.F = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

③ $xy\,dx - (x^2+2y^2)\,dy = 0.$

$Mx + Ny = \cancel{x^2y} - \cancel{x^2y} - 2y^3$

$\therefore\; I.F = \frac{1}{Mx+Ny} = \frac{1}{-2y^3} \simeq \frac{1}{y^3}$ Ans

no importance

NON EXACT DIFFERENTIAL EQ^{n} :-

METHOD - 3 :-

If $Mdx + Ndy = 0$ is non exact, but is in the form of

$$y\,\overset{f(xy)}{f(\cancel{x,y})}\,dx + x\,\overset{g(xy)}{g(\cancel{x,y})}\,dy = 0$$ and if

$$Mx - Ny \neq 0$$

then $\dfrac{1}{Mx-Ny}$ is an I.F.

$\Rightarrow$ $$y(1+xy)dx + x(1-xy) = 0$$

$$Mx - Ny = xy + x^2y^2$$

$$\therefore \frac{1}{Mx-Ny} = \frac{1}{xy(1+xy)}$$

Q ① $$y(\cancel{x^2y^2} + xy + \cancel{1})dx + x(\cancel{x^2y^2} - xy + \cancel{1})dy = 0$$

for fast calculation of $\dfrac{1}{Mx-Ny}$ — cancel कर दो terms having same ~~do~~ order and same const. and increase power or multiply by $\dfrac{1}{\bigcirc} \times \dfrac{1}{xy}$

$$\therefore \frac{1}{Mx-Ny} = \frac{1}{x^2y^2}$$

② $$y(\cancel{x^3y^3} + x^2y^2 + xy + \cancel{1})dx + x(\cancel{x^3y^3} - x^2y^2 - xy + \cancel{1})dy = 0$$

$$\frac{1}{Mx-Ny} = \frac{1}{x^3y^3 + x^2y^2}$$

③ $$y(x^2y^2 + \cancel{1})dx + x(\cancel{1} - 2x^2y^2)dy = 0$$

$$\frac{1}{Mx-Ny} = \frac{1}{x^3y^3}$$

④ $$y[\cancel{xy\sin xy} + \cos xy]dx + x[\cancel{xy\sin xy} - \cos xy]dy = 0$$

$$\frac{1}{xy\cos xy}$$

NON EXACT DIFFERENTIAL EQ^{N} :-

METHOD-4 :-

If $M dx + N dy = 0$ is non exact and if ***

Tip ***
N से devisible तो ∂N पहले नहीं लिखना है

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$$

$$I.F = e^{\int f(x)\,dx}$$

fn	I.F
IF $f(x) = \frac{1}{x}$ →	x
$\frac{-1}{x}$ =	$\frac{1}{x}$
$\frac{2}{x}$	x^2
$-\frac{2}{x}$	$\frac{1}{x^2}$
$\frac{3}{x}$	x^3
$-\frac{3}{x}$	$\frac{1}{x^3}$

Tip
M से devisible तो ∂M पहले नहीं लिखना है

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$$

$$I.F = e^{\int g(y)\,dy}.$$

Q. $(x^2+y^2+1)\,dx - 2xy\,dy = 0.$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = -2y.$$ → 1 term ∴ devisible by N

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$$

$$= \frac{1}{-2xy}(2y + 2y)$$

$$= \frac{4y}{-2xy} = \frac{-2}{x} \rightarrow I.F = \frac{1}{x^2}$$

Q. ① $(y + xy^3)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0.$

$$\frac{\partial M}{\partial y} = 1 + 3xy^2 \qquad \frac{\partial N}{\partial x} = 4xy^2 + 1 + \cancel{y}$$ → 2 terms ∴ devisible by M

$\cancel{1 + 3xy^2 \quad 2xy^2 \quad 2y^4}$

$\cancel{1 + xy^2 + y^4}$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{1}{(y + xy^3)}\left[4xy^2 + 2 + 3xy^2\right] = \frac{1}{y} \rightarrow I.F = y$$

② $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dx = 0$

$$\frac{\partial M}{\partial y} = 4y^3+2 \qquad \frac{\partial N}{\partial x} = y^3-4$$

2 terms
∴ devisible by M
as ~~it~~ M has also
2 terms

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{1}{y^4+2y}\left(\cancel{4y^3+2}\; y^3-4-4y^3-2\right)$$

$$= \frac{1}{y^4+2y}\left(-3y^3-6\right)$$

$$= \frac{-3}{y} \rightarrow I.F = \frac{1}{y^3}$$

③ $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx + \left(\frac{x+xy^2}{4}\right)dy = 0$

$$\frac{\partial M}{\partial y} = 1+\frac{\cancel{3}y^2}{\cancel{3}}, \quad \frac{\partial N}{\partial x} = \frac{1}{4}+\frac{y^2}{4}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= \frac{4}{x+xy^2}\left(1+y^2-\frac{1}{4}-\frac{y^2}{4}\right)$$

$$= \frac{\cancel{4}}{x\cancel{(1+y^2)}}\, 3\left(\frac{\cancel{3}+\cancel{3y^2}}{\cancel{4}}\right)$$

$$= \frac{3}{x} \rightarrow I.F = x^3$$

Q. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0.$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

बीच में + sign आ रहा तो
devisible by M not 'N'

$$\frac{1}{M}\left(\frac{\partial M}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{1}{3x^2y^4 + 2xy}\left(-6x^2y^3 - 4x\right)$$

$$= \frac{-2}{y} \longrightarrow I.F \rightarrow \frac{1}{y^2}$$

Linear Eqⁿ:-

$\frac{dy}{dx} + Py = Q$, Where P and Q are said to be a fn of x, called

Linear eqⁿ.

$I.F = e^{\int P dx}$

General Solⁿ—

$y \underbrace{e^{\int P dx}}_{I.F} = \int \left(Q \cdot \underbrace{e^{\int P dx}}_{I.F}\right) dx + c$

Ex:-

(2)

$\frac{dx}{dy} + Px = Q$, Where P and Q are said to fn. of y

$I.F = e^{\int P dy}$

General Solⁿ—

$x\, e^{\int P dy} = \int \left(Q\, e^{\int P dy}\right) dy + c$

(1) $\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}$.

$P = \frac{1}{x}$

$Q = \frac{\log x}{x}$.

$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = x$

Solⁿ—

$y \cdot x = \int \log x \, dx + c$

$\Rightarrow yx = x \log x - x + c$ Ans

② $(2-x^2)\frac{dy}{dx} - 2xy = x^2$

$$\frac{dy}{dx} - \frac{2xy}{2-x^2} = \frac{x^2}{2-x^2}$$

P Q

$$I.F = e^{\int P\,dx} = e^{\int \frac{2x}{2-x^2}dx} = e^{\log(2-x^2)}$$

$$= 2-x^2$$

G. Solⁿ –

$$y(2-x^2) = \int \frac{x^2}{\cancel{2-x^2}}\cancel{(2-x^2)}\cdot dx + C$$

$$= \frac{x^3}{3} + C.$$

③ $\frac{dy}{dx} + 2xy = e^{-x^2}$, $y(0) = 1$

P Q

$$I.F = e^{\int P\,dx} = e^{\int 2x\,dx} = e^{\frac{2x^2}{2}} = e^{x^2}$$

G. Solⁿ –

$$y e^{x^2} = \int \cancel{e^{-x^2}}\cdot\cancel{e^{x^2}}\,dx + C$$

$$\boxed{y e^{x^2} = x + C}$$

When $y(0) = 1$

$$1\cdot e^0 = C \Rightarrow C = 1$$

$$\therefore \boxed{y e^{x^2} = x + 1}$$

④ $\frac{dy}{dx} + \overset{P}{\frac{1}{x\log x}}\, y = \overset{Q}{\frac{2\log x}{x\log x}}$

$$I.F = e^{\int \frac{1}{x\log x}dx} = e^{\int \frac{\frac{1}{x}}{\log x}dx} = e^{\log(\log x)} = \log x.$$

G. solⁿ

$$y\log x = \int \frac{2\log x}{x\cancel{\log x}} \times \cancel{\log x}\; dx + C$$

$$y\log x = \cancel{2}\frac{(\log x)^2}{\cancel{2}} + c$$

~~$y\log x = 2f.$~~

$$\boxed{y = \log x + \frac{c}{\log x}}$$

⑤ $(x+2y^3)\frac{dy}{dx} = y.$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{2y^{\cancel{3}\,2}}{\cancel{y}}$$

$$\frac{dx}{dy} - \overset{P}{\frac{1}{y}}x = \overset{Q}{2y^2}$$

$$I.F = e^{\int -\frac{1}{y}dy} = \frac{1}{y}$$

G. Solⁿ

$$x\cdot\frac{1}{y} = \int 2y^{\cancel{2}} \times \frac{1}{\cancel{y}}\,dy + C.$$

$$\frac{x}{y} = y^2 + c$$

$$\Rightarrow \boxed{x = y^3 + cy}$$

Q ⑤ $(1+y^2)dx + (x - e^{\tan^{-1}y})\,dy = 0$

$$(1+y^2)\,dx = (e^{\tan^{-1}y} - x)\,dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

P Q

$$\therefore I.F = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

General Solⁿ —

$$x\, e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}\,dy + c.$$

$$= \frac{1}{2}\int \frac{e^{2\tan^{-1}y}}{1+y^2} \times 2\,dy + c$$

$$\boxed{x\, e^{\tan^{-1}y} = \frac{1}{2} e^{2\tan^{-1}y} + c.}$$

BERNOULLI'S EQ^N :-

A diff. eq^n of the form $\frac{dy}{dx} + Py = Q y^n$, where P and Q are fn. of x is said to be a bernaulli's eq^n

$$\therefore \quad y^{-n}\frac{dy}{dx} + P y^{1-n} = Q$$

$$\Rightarrow \quad z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n) y^{-n} \frac{dy}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}.$$

$$\frac{1}{(1-n)}\frac{dz}{dx} + Pz = Q.$$

$$\Rightarrow \frac{dz}{dx} + P(1-n)z = Q(1-n).$$

* $\quad I.F = e^{\int P(1-n)dx}$

Gen. sol^n.

* $\quad y^{1-n}\, e^{\int P(1-n)dx} = \int\left(Q\,(1-n)\, e^{\int P(1-n)dx}\right)dx + C$

I.F $\qquad\qquad$ I.F

$$\frac{dx}{dy} + Px = Q^n$$

$$I.F = e^{\int P(1-n)\,dy}$$

Gen sol^n

$$x^{1-n} \times I.F = \int\left(Q(1-n) \times I.F\right)dy + C$$

Find I.F.

① $\frac{dy}{dx} + \frac{y}{x} = x^2y^3$

$$I.F = e^{\int \frac{1}{x}(1-n)\,dx} = e^{(1-n)\log x} = x^{1-3} = x^{-2} = \frac{1}{x^2}$$

② $\frac{dy}{dx}(xy + x^2y^3) = 1$

$\cancel{\Rightarrow \frac{dx}{dy} - xy = x^2y^3}$

$\cancel{I.F = e^{\int \frac{x}{2}(1-n)\,dx} = e^{\frac{x^2}{2}} = e^{-x^2}}$

$$\Rightarrow \frac{dx}{dy} - xy = y^3x^{2 \to n}$$

$$I.F = e^{\int -y(1-n)\,dy} = e^{\int -y(1-2)\,dy} = e^{\int +y\,dy} = e^{\frac{y^2}{2}}$$

Differential Eqⁿ with constant coefficient :-

A diff. eqⁿ of the form -

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots a_ny = Q.$$

Where $a_0, a_1, a_2 \cdots a_n$ are constants and Q be the function of x is said to be differential equation with constant coefficient.

Let $\frac{d}{dx} = D$

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = Q.$$

$$\boxed{f(D)\,y = Q}$$

This eqⁿ may contain two types of solⁿ
- → one is called complementary fn
- → other is called particular integral.

Note:-

If $Q = 0$, then complementary fn is called the general solⁿ.
if $Q \neq 0$, then only particular integral must be existed.

\# To Find Complimentary fn (C.F) :-

Replace 'D' by m , $f(m) = 0$ is called auxilary eqⁿ

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0.$$

Case-1.

Roots are real and Unequal.

Let $m = \alpha, \beta, \gamma \dots$

$$\therefore \; y_c = c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x} + \dots$$

Case-2

Roots are real and equal.

Let $m = \alpha, \alpha, \beta$

$$y_c = (c_1 + c_2 x) e^{\alpha x} + c_3 e^{\beta x}$$

Let $m = \alpha, \alpha, \alpha, \beta.$

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{\alpha x} + c_4 e^{\beta x}$$

Let $m = \alpha, \alpha, \beta, \beta$

$$y_c = (c_1 + c_2 x) e^{\alpha x} + (c_3 + c_4 x) e^{\beta x}$$

Ex:-

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

$$m^2 - 4m + 4 = 0.$$

$$(m-2)^2 = 0.$$

$$m = 2, 2$$

$$y = (c_1 + c_2 \cdot x) e^{2x}$$

Case-3

Roots are complex —

Let $m = \alpha \pm i\beta$

$$y_c = \left[c_1 \cos\beta x + c_2 \sin\beta x \right] e^{\alpha x}$$

Let $m = \alpha \pm i\beta,\ \alpha \pm i\beta.$

$$y_c = \left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x \right] e^{\alpha x}$$

Ex:-

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 4y = 0$$

$$m^2 - 2m + 4 = 0$$

$$m = \frac{+2 \pm \sqrt{4-16}}{2}$$

$$= \frac{+2 \pm 2\sqrt{-3}}{2}$$

$$= 1 \pm i\sqrt{3}$$

$$= 1 \pm \sqrt{3}\, i$$

$$\therefore\ y = \left[c_1 \cos \overset{\beta}{\sqrt{3}} x + c_2 \sin \overset{\beta}{\sqrt{3}} x \right] e^{\overset{\alpha}{1} x}$$

To Find the roots of Eqn:-

For 3rd degree

i) Every 3rd degree eqn must contain at least 1 real root.

ii) If ~~some~~ sum of all the coeff = 0, then $m = 1$ satisfies eqn
↳ one root

or $(m-1)$ is a factor.

iii) If sum of coeff. of even powers = sum of coeff of odd powers then $m = -1$ satisfies the eqn. or $(m+1)$ will be factor.
↳ one root

① $(D^3 - 3D + 2)y = 0$

$$\underbrace{m^3 - 3m + 2}_{\text{sum} = 0 \;\to\; m = 1 \text{ is root}} = 0$$

coeff.

root

0 always

sum

$$\begin{array}{c|ccc|c|l}
m=1 & 1 & 0 & -3 & 2 & \longrightarrow m^3 \\
 & 0 & 1 & 1 & -2 & \\
\hline
 & 1 & 1 & -2 & 0 & \longrightarrow m^2 \\
m=1 & 0 & 1 & 2 & & \\
\hline
 & 1 & 2 & 0 & & \longrightarrow m
\end{array}$$

$m + 2 = 0$

$m = -2$

$\therefore \quad m = 1, 1 \;\&\; -2$ are roots.

② $[D^3 - 4D^2 + 5D - 2]y = 0.$

$m^3 - 4m^2 + 5m - 2 = 0$

$1 - 4 + 5 - 2 = 0.$

$$\begin{array}{c|cccc}
m=1 & 1 & -4 & 5 & -2 \\
 & 0 & 1 & -3 & 2 \\
\hline
 & 1 & -3 & 2 & 0 \\
m=1 & 0 & 1 & -2 & \\
\hline
 & 1 & -2 & 0 &
\end{array}$$

$m - 2 = 0$

$m = 2$

$\therefore \quad m = 1, 1, 2$ Ans

② $[D^4 - 2D^3 + 2D - 1]\, y = 0.$

$m^4 - 2m^3 + 2m - 1 = 0$

$\cancel{1} - \cancel{2} + \cancel{2} - \cancel{1} = 0.$

$\therefore\ m = 1$

$m = 1$	1	-2	0	2	-1
	0	1	-1	-1	1
$m = 1$	1	-1	-1	1	0
	0	1	0	-1	
$m = -1$	1	0	-1	0	
	0	1	1		
	1	1	0		

$m + 1 = 0$

$m = -1.$

③ $[D^3 + 6D^2 + 11D + 6]\, y = 0$

$m^3 + 6m^2 + 11m + 6 = 0.$

12   12

$12 = 12 \rightarrow m = -1$

$m = -1$	1	6	11	6
	0	-1	-5	-6
	1	5	6	0

$m^2 + 5m + 6 = 0.$

$\cancel{m^2 + 3m}\ (m+3)(m+2) = -3, -2$

$M = -1, -2, -3$

④ $(D^4 - D^3 - 9D^2 - 11D - 4)\, y = 0$

$m^4 - m^3 - 9m^2 - 11m - 4 = 0$

$1 - 1 - 9 - 11 - 4$ $\Rightarrow m = -1$

-12 $\quad -12$

$$\begin{array}{l|rrrr|r}
m=-1 & 1 & -1 & -9 & -11 & -4 \\
 & 0 & -1 & 2 & 7 & 4 \\
\hline
 & 1 & -2 & -7 & -4 & \\
 & & -6 & -6 & & \\
m=-1 & 0 & -1 & 3 & 4 & \\
\hline
 & 1 & -3 & -4 & & \\
 & & -3 & -3 & & \\
m=-1 & 0 & -1 & 4 & & \\
\hline
 & 1 & -4 & & &
\end{array}$$

$m - 4 = 0$

$m = 4$

$\therefore\ m = -1, -1, -1, 4$

$$\left.\right\}\ y = (c_1 + c_2 x + c_3 x^2) e^{-x} + c_4 e^{4x}$$

⑤ $(D^2 - 3D + 4)\, y = 0$

$m^2 - 3m + 4 = 0$

$\cancel{(m - 4)(m-}$ $\quad m = \dfrac{3 \pm \sqrt{9 - 16}}{2}$

$= \dfrac{3 \pm \sqrt{7}\, i}{2}$

$$\left.\right\}\ y = \left[\left(c_1 \cos\frac{\sqrt{7}}{2}x\right) + \left(c_2 \sin\frac{\sqrt{7}}{2}x\right)\right] e^{\frac{3}{2}x}$$

⑥ $[D^2 - (a+b)D + ab]\, y$

$m^2 - (a+b) + ab = 0$

$(m - a)(m - b) = 0$

$m = a, b.$

$\therefore\ \boxed{y = c_1 e^{ax} + c_2 e^{bx}}$

To find the particular Integral:-

$$F(D)y = Q$$

then $y_p = PI = \frac{1}{F(D)} Q$

Case-III. To find P.I of the form $\frac{1}{f(D)}\sin ax$, $\frac{1}{f(D)}\cos ax$, replace D^2 by $-a^2$ if $\phi(-a^2) \neq 0$.

$$\frac{1}{D^2-2D+5}\sin 2x = \frac{1}{-4-2D+5}\sin 2x = \frac{1}{1-2D} \times \frac{1+2D}{1+2D}\sin 2x$$

$$= \frac{1+2D}{1-4D^2}\sin 2x = \frac{(1+2D)}{1-4(-4)}\sin 2x =$$

$$= \frac{\sin 2x + 4\cos 2x}{17}.$$

Case-IV. To find P.I of the form $\frac{1}{D^2+a^2}\sin ax$, $\frac{1}{D^2+a^2}\cos ax$,

$$\frac{1}{D^2+a^2}\sin ax = \frac{x}{2D}\sin ax = \frac{x}{2}\left(\frac{-\cos ax}{a}\right)$$

$$\boxed{\begin{aligned} \frac{1}{D^2+a^2}\sin ax &= \frac{-x}{2a}\cos ax \\ \frac{1}{D^2+a^2}\cos ax &= \frac{x}{2a}\sin ax \end{aligned}}$$

$$\frac{1}{D^2+4}\sin 2x = \frac{-x}{4}\cos 2x$$

$$\frac{1}{D^2+9}\cos 3x = \frac{x}{6}\sin 3x$$

$$\frac{1}{D^2+16}\sin 4x = -\frac{x}{8}\cos 4x$$

** $\frac{1}{D^2+1}\cos 3x = \frac{1}{-9+6}\cos 3x = \frac{1}{8}\cos 3x$

① $(D^2-D-2)y = \cos 2x$

② $(D^2-4)y = \sin^2 x$

③ $(D^2+1)y = \sin 2x \cdot \cos x$

→ ① $\frac{1}{D^2-D-2}\cos 2x = \frac{1}{-4-D-2}\cos 2x = \frac{1}{-6+D} \times \frac{-6+D}{-6+D}\cos 2x$

$= \frac{-6+D}{36-D^2}\cos 2x$

$= \left[\frac{(-6+D)}{36-(-4)}\cos 2x\right]$

$= \frac{-6\cos 2x - 2\sin 2x}{40}$

$= -\frac{\cancel{2}}{\cancel{40}_{20}}(3\cos 2x + \sin 2x)$

$= -\frac{1}{20}(3\cos 2x + \sin 2x).$

② $\frac{1}{D^2-4}\sin^2 x = \frac{1}{D^2-4}\left[\frac{1-\cos 2x}{2}\right] = \frac{1}{2}\left[\frac{1}{D^2-4}(1) - \frac{1}{D^2-4}(\cos 2x)\right]$

$= \frac{1}{D^2-4}e^{0x}$

$= \frac{1}{D^2-D}(5) = \frac{x}{2D-1}(5)$

$= \frac{1}{2}\left[\frac{1}{0-4}\cdot 1 - \frac{1}{-4-4}\cos 2x\right]$

$= -\frac{1}{8} + \frac{1}{16}\cos 2x$

③ $(D^2+1)y = \sin 2x \cdot \cos x$

$$= \frac{1}{2(D^2+1)} 2\sin 2x \cdot \cos x$$

$$= \frac{1}{2}\left[\frac{1}{(D^2+1)}\sin 3x + \frac{1}{D^2+1}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-9+1}\sin 3x - \frac{x}{2(1)}\cos x\right]$$

$$= -\frac{1}{16}\sin 3x - \frac{x}{4}\cos x.$$

SHORTCUT

→ If $(D^2+a^2)y = \tan ax$

$$y_p = \frac{-1}{a^2}\cos ax \cdot \log\left(\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right)$$

→ If $(D^2+a^2)y = \sec ax$

$$y_p = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log(\cos ax)$$

ex:- $(D^2+4)y = \tan 2x$

$$y_p = -\frac{1}{4}\cos 2x \cdot \log\left[\tan\left(\frac{\pi}{4}+x\right)\right]$$

ex:- $(D^2+9)y = \sec 3x$

$$y_p = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x \cdot \log(\cos 3x)$$

Case-V.

To find the P.I of the form $\frac{1}{f(D)}(x^m)$.

$(1+D)^{-1} = 1-D+D^2-D^3+ \dots$

$(1-D)^{-1} = 1+D+D^2+D^3+ \dots$

$(1+D)^{-2} = 1-2D+3D^2-4D^3+ \dots$

$(1-D)^{-2} = 1+2D+3D^2+4D^3+ \dots$

Remember

$D^3(x^2) = 0$

$$\boxed{\begin{array}{c} D^n(x^n) = n! \\ D^{n+1}(x^n) = 0 \end{array}}$$

① $(D^2-4)y = x^2$

$\frac{1}{D^2-4}x^2$

$= \frac{1}{-4\left[1-\frac{D^2}{4}\right]}x^2$

$= -\frac{1}{4}\left[1-\frac{D^2}{4}\right]^{-1}x^2$

$= -\frac{1}{4}\left[1+\frac{D^2}{4}+\cancel{\left(\frac{D^2}{4}\right)^2}+\underbrace{\dots}_{0}\right]x^2$

$= -\frac{1}{4}\left[x^2+\frac{\cancel{2}^{\,1}}{\cancel{4}\,2}\right]$

$= -\frac{1}{4}\left[x^2+\frac{1}{2}\right]$

② $[D^3+8]y = x^4+2x+1$

$= \frac{1}{D^3+8}(x^4+2x+1)$

$= \frac{1}{8\left[1+\frac{D^3}{8}\right]}(x^4+2x+1)$

$= \frac{1}{8}\left[1+\frac{D^3}{8}\right]^{-1}(x^4+2x+1)$

$= \frac{1}{8}\left[1-\frac{D^3}{8}+\cancel{\frac{D^6}{8}}+\dots\right](x^4+2x+1)$

$= \frac{1}{8}\left[(x^4+2x+1)-\frac{1}{8}(24x+0+0)\right]$

$= \frac{1}{8}(x^4-x+1)$

③ $(D^2+2D+1)y = x^2+x$

$$\Rightarrow \frac{1}{(D+1)^2}(x^2+x)$$

$$= [1+D]^{-2}(x^2+x)$$

$$= [1-2D+3D^2](x^2+x)$$

$$= (x^2+x) - 2(2x+1) + 3(2+0)$$

$$= x^2+x-4x-2+6$$

$$= x^2-3x+4$$

<u>Case – VI.</u> To find PI of the form $\frac{1}{f(D)}e^{ax}\cdot v$, where v is a fn. of x.

$$\boxed{\frac{1}{f(D)}e^{ax}\cdot v = e^{ax}\frac{1}{f(D+a)}\cdot v}$$

① $(D^2-2D+1)y = x^2e^x$

② $(D^2-2D+5)y = e^x\sin x$

③ $(D^2-4D+4)y = e^{2x}\cos 3x$

④ $(D^3+3D^2+3D+1)y = x^2e^{-x}$

→ ① $\frac{1}{(D-1)^2}e^xx^2 = e^x\cdot\frac{1}{(D+\cancel{1}-\cancel{1})^2}x^2 = e^x\left[\frac{1}{D^2}x^2\right]$

represents integration

$$= e^x\frac{x^4}{12}$$

(2) $\dfrac{1}{D^2-2D+5}(e^x \sin x)$

$$= e^x \left[\frac{1}{(D+1)^2-2(D+1)+5}\sin x\right]$$

$$= e^x \left[\frac{1}{D^2+1+\cancel{2D}-\cancel{2D}-2+5}\sin x\right]$$

$$= e^x \left[\frac{1}{D^2+4}\sin x\right]$$

$$= e^x \left[\frac{1}{-1+4}\sin x\right]$$

$$= \frac{e^x \sin x}{3}.$$

(3). $\dfrac{1}{(D^2-4D+4)} e^{2x}\cos 3x$

$$= e^{2x}\left[\frac{1}{(D\cancel{+2-2})^2}\cos 3x\right]$$

$$= e^{2x}\left[\frac{1}{D^2}\cos 3x\right]$$

$$= e^{2x}\frac{1}{-9}\cos 3x$$

$$= -\frac{e^{2x}}{9}\cos 3x$$

(f) $\frac{1}{(D^3+3D^2+3D+1)} e^{-x} x^2$

$= \frac{1}{(D+1)^3} e^{-x} x^2$

$= e^{-x} \frac{1}{(D-1+1)^3} x^2$

$= e^{-x} \left[\frac{1}{D^3} x^2 \right]$

$$\boxed{\iiint x^2 \quad \frac{x^3}{3} \quad \frac{x^4}{12} \quad \frac{x^5}{60}}$$

$= e^{-x} \left(\frac{x^5}{60} \right)$ Ans

Case-VII. To find the P.I of the form $\frac{1}{f(D)} x \cdot V$, where V is a fn of x (Trigonometric fn)

$$\boxed{\frac{1}{f(D)} x \cdot V = x \cdot \frac{1}{f(D)} \cdot V - \frac{f'(D)}{[f(D)]^2} \cdot V}$$

① $(D^2+4)y = x\cos x$

② $(D^2-2D+1)y = xe^x \sin x$

① $\frac{1}{D^2+4} (x\cos x)$

$= x \cdot \frac{1}{(D^2+4)} \cos x - \frac{2D}{(D^2+4)^2} \cdot \cos x$

$= x \cdot \frac{1}{-1+4} \cos x - \frac{2(-\sin x)}{(-1+4)^2}$

$= \frac{x}{3} \cos x + \frac{2}{9} \sin x.$

(2) $\dfrac{1}{(D-1)^2} e^x \cdot x \cdot \sin x$

$$= e^x \left[\frac{1}{(D+1-1)^2} x \sin x \right]$$

$$= e^x \left[\frac{1}{D^2} x \sin x \right]$$

$$= e^x \left[x \frac{1}{\cancel{D^2}_{-a^2}} \sin x - \frac{2D}{\cancel{(D^2)^2}_{-a^2}} \cdot \sin x \right]$$

$$= e^x \left[x \frac{1}{-1} \sin x - \frac{2 \cos x}{(-1)^2} \right]$$

$$= e^x (-x \sin x - 2\cos x)$$

Differential Eq^n with variable co-efficients :-

or

Cauchy Euler form :-

→ A diff eq^n is of the form —

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q$$

where $a_0, a_1, a_2 \cdots a_n$ are constants

and Q be the function of x.

is said to be Diff^n Eq^n with Variable Coefficients.

$x = e^z \Rightarrow z = \log x$.

$$\frac{dz}{dx} = \frac{1}{x} \quad \& \quad \frac{dx}{dz} = x$$

$$\frac{d}{dz} = \frac{d}{dx} \cdot \frac{dx}{dz} = x\frac{d}{dx} = xD = \frac{d}{dz} = \theta$$

Now—

$$\frac{d}{dz}\left(x \frac{d}{dx}\right) = \frac{d^2}{dz^2}$$

$$\frac{d}{dx}\left(x \frac{d}{dx}\right) \cdot \frac{dx}{dz} = \theta^2$$

$$x\left(x\frac{d^2}{dx^2} + \frac{d}{dx} \cdot 1\right) = \theta^2$$

$$\therefore x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} = \theta^2$$

$$x^2 \frac{d^2}{dx^2} + \theta = \theta^2$$

$$x^2 \frac{d^2}{dx^2} = \theta(\theta - 1)$$

$$\therefore x^2 \frac{d^2}{dx^2} = x^2 D^2 = \theta(\theta - 1)$$

****

Imp Substitution to Remember

Let $x\dfrac{d}{dx} = xD = \theta$.

$$x^2\frac{d^2}{dx^2} = x^2D^2 = \theta(\theta-1) = \theta^2-\theta$$

$$x^3\frac{d^3}{dx^3} = x^3D^3 = \theta(\theta-1)(\theta-2) = \theta^3-3\theta^2+2\theta$$

Variable Coeff.

observe the diffⁿ betⁿ const coeff approach

$D^2-D+2=0$

$m^2-m+2=0$

① $(x^2D^2 - xD - 3]y = 0$

$= [\theta^2-\theta-\theta-3]y = 0$

$= m^2-2m-3 = 0$

$= (m-3)(m+1) = 0$

$m = 3, -1$

$y = c_1e^{3z} + c_2e^{-z}$

$= c_1(e^z)^3 + c_2(e^z)^{-1}$

$$\boxed{y = c_1x^3 + c_2x^{-1}}$$

② $[x^3D^3 + 3x^2D^2 - 2xD + 2]y = 0.$

$[\theta^3 - \cancel{3\theta^2} + 2\theta + \cancel{3\theta^2} - 3\theta - 2\theta + 2]y = 0$

$[\theta^3 - 3\theta + 2]y = 0$

$m^3 - 3m + 2 = 0$

$1 - 3 + 2 = 0$

$\therefore m = 1$ satisfies.

	1	0	-3	2
m=1	0	1	1	-2
	1	1	-2	0
m=1	0	1	2	
	1	2	0	

$m+2=0 \Rightarrow m=-2.$

$$\therefore y = (c_1 + c_2 z)e^{z} + c_3 e^{-2z}$$

$$\boxed{y = (c_1 + c_2 \log x)\, x + c_3 x^{-2}}$$

③ $[x^2D^2 + xD - 4]\, y = 0$, $\quad x(0) = 0$, $y(1) = 1$

$$[\theta^2 - \cancel{\theta} + \cancel{\theta} - 4]\, y = 0$$

$$m^2 - 4 = 0$$

$$m = \pm\sqrt{4} = \pm 2 = 2, -2$$

$$\therefore y = c_1 e^{2z} + c_2 e^{-2z}$$

$$\Rightarrow y = c_1 x^2 + c_2 x^{-2}$$

$$\Rightarrow y = c_1 x^2 + \frac{c_2}{x^2}$$

$$\Rightarrow x^2 y = c_1 x^4 + c_2$$

$\because x(0) = 0$

$\Rightarrow$ $\cancel{0\,0 = c_1\, 0^4 + c_2}$

$0 \cdot 0 = c_1(0) + c_2$

$\Rightarrow \boxed{c_2 = 0}$

$y(1) = 1$

$\Rightarrow 1^2 \cdot 1 = c_1(1)^4 + c_2$

$1 = c_1 + 0$

$\boxed{c_1 = 1}$

$$\therefore y = 1(x^2) + \frac{0}{x^2}$$

$$\boxed{y = x^2}$$

Q. $\left[x^4D^3 + 2x^3D^2 - x^2D + x\right]y = 1$

$\Rightarrow \left(x^3D^3 + 2x^2D^2 - xD + 1\right)y = \frac{1}{x}$

$\Rightarrow \left(\theta^3 - 3\theta^2 + \cancel{2\theta} + 2\theta^2 - \cancel{2\theta} - \theta + 1\right)y = e^{-z}$

$\Rightarrow \left(\theta^3 - \theta^2 - \theta + 1\right)y = e^{-z}$

$m^3 - m^2 - m + 1 = 0$

$m^2(m-1) - 1(m-1) = 0$

$(m-1)(m^2-1) = 0$

$(m-1)(m-1)(m+1) = 0$

$m = 1, 1, -1$

$\therefore\ y_c = (c_1 + c_2 z)e^{z} + c_3 e^{-z}$

$\Rightarrow y_c = (c_1 + c_2 \log x)\,x + c_3 x^{-1}$

$\Rightarrow y_p = \frac{1}{\theta^3 - \theta^2 - \theta + 1} e^{-z}$

$= \frac{z}{3\theta^2 - 2\theta - 1} e^{-z}$

$= \frac{z}{3(-1)^2 - 2(-1) - 1} e^{-z} = \frac{z}{4} e^{-z}$

$= \frac{\log x}{4} \times \frac{1}{x} = \frac{1}{4x} \log x$

⑤ $(x^2D^2 - xD + 1)y = 0.$

$\theta^2 - \theta - \theta + 1 = 0$

$m^2 - 2m + 1 = 0$

$m = +1, +1$

$= (c_1 + c_2 z) e^{z}$ $\quad z \to \log x,\ e^z \to x$

$= (c_1 + c_2 \log x)\, x$

⑥ $x^2 \dfrac{d^3y}{dx^3} + 2x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0$

$x^3 \dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} = 0.$

$\Rightarrow (x^3D^3 + 2x^2D^2 - 2xD)y = 0$

$\Rightarrow [\theta^3 - 3\theta^2 + \cancel{2\theta} + 2\theta^2 - \cancel{2\theta} - 2\theta]\, y = 0$

$\Rightarrow m^3 - m^2 - 2m = 0$

$\Rightarrow m(m^2 - m - 2) = 0$

$\Rightarrow m(m-2)(m+1) = 0$

$m = 0, 2, -1$

$\therefore y = c_1 e^{0z} + c_2 e^{2z} + c_3 e^{-z}$

$\boxed{y = c_1 + c_2 x^2 + c_3 x^{-1}}$

(7) $4x^2y'' + 12xy' + 3y = 0$

$(4x^2D^2 + 12xD + 3)y = 0$

$\Rightarrow (4\theta^2 - 4\theta + 12\theta + 3)y = 0$

$\Rightarrow 4m^2 + 8m + 3 = 0.$

$\cancel{m = \frac{-8 \pm \sqrt{64-48}}{2} = \frac{8 \pm \sqrt{16}}{2} = \frac{8+16}{2}, \frac{8-16}{2} = 12, -4}$

$m = -\frac{1}{2}, -\frac{3}{2}$

$\therefore y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{3}{2}}$

(8) $x^2y'' + xy' - y = 0$

$[\theta^2 - \cancel{\theta} + \cancel{\theta} - 1]\, y = 0$

$m^2 - 1 = 0$

$m = \pm 1$

$y = c_1 e^{z} + c_2 e^{-z}$

$\boxed{y = c_1 x + c_2 x^{-1}}$

Orthogonal Trajectories :-

→ The family of curves which cuts every member of the given family orthogonally is called the orthogonal Trajectory of the given family.

→ $f(x, y, c) = 0$ —— ①

~~replace~~

→ $F\left(x, y, \frac{dy}{dx}\right) = 0$ —— ②

replace $\left(\frac{dy}{dx}\right)$ by $\left(-\frac{dx}{dy}\right)$

→ $g\left(x, y, -\frac{dx}{dy}\right) = 0$ —— ③

The solⁿ of eqⁿ ③ is called the orthogonal trajectory of the given family.

Note :-

If eqⁿ ② and eqⁿ ③ are identical, then the curve is said to be self orthogonal.

Q. Find the Orthogonal Trajectory of family of lines passing through the origin —

i) family of lines

ii) family of Circles having the centre at the origin.

iii) family of Parabolas

iv) family of ellipses.

*** If fn given, then Steps —

find $\frac{dy}{dx}$

replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

and integrate & solve.

That will represent orthogonal of that fn

① → $y = mx$ —— ①

$\frac{dy}{dx} = m$

$y = x\frac{dy}{dx}$ —— ②

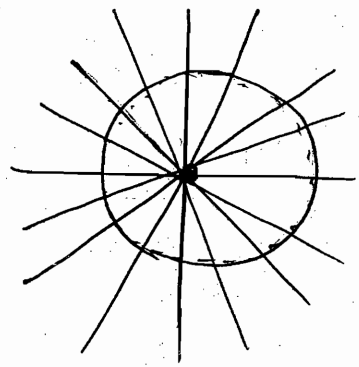
$y = -x\frac{dx}{dy}$ —— ③

$\int y\,dy + \int x\,dx = \int 0$

$\frac{y^2}{2} + \frac{x^2}{2} = c$

$x^2 + y^2 = 2c$

$\boxed{x^2+y^2 = r^2}$ → Circle



② $y^2 = 4ax$ —— ①

$2y\frac{dy}{dx} = 4a$

$y^{\cancel{2}} = 2\cancel{y}\frac{dy}{dx}\,x$

$\cancel{2\int\frac{dy}{y} = \int\frac{dx}{x}}$

$2x\frac{dy}{dx} = y$ —— ②

$y = -2x\frac{dx}{dy}$ —— ③.

$\int y\,dy = \int -2x\,dx$

$\frac{y^2}{2} = -\cancel{2}\frac{x^2}{\cancel{2}} + c$

$\Rightarrow \boxed{x^2 + \frac{y^2}{2} = C}$ → Ellipse...

$\Rightarrow \quad x^2 - y^2 = a^2 \quad \text{——} ①$

$$\cancel{2}x - \cancel{2}y \frac{dy}{dx} = 0$$

$$x + y\frac{dx}{dy} = 0$$

$$x\,dy + y\,dx = 0$$

$$\int d(xy) = \int 0$$

$$\boxed{xy = c}$$

Conclusion – Orthogonal Trajectories of family of —

circle $\xrightarrow{\text{is}}$ st line.

st line $\xrightarrow{\text{is}}$ circle

Parabola $\xrightarrow{\text{is}}$ ellipse

Hyperbola $\xrightarrow{\text{is}}$ Rectangular Hyperbola.

② $\quad y^2 = 4a(x+a) \quad \text{——} ①$

$$2yy_1 = 4a \cdot 1 \Rightarrow a = \frac{yy_1}{2}$$

$$y^2 = 2yy_1\left(x + \frac{yy_1}{2}\right)$$

$$y = 2xy_1 + yy_1^2 \quad \text{——} ②$$

replace y_1 by $-\frac{1}{y_1}$

$$y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$y y_1^2 = -2xy_1 + y$

$y y_1^2 + 2xy_1 = y$ ——— (3).

$\because$ eq^n (2) = eq^n(3).

$\therefore$ Curve is said to be self orthogonal.

\# $\boxed{\dfrac{x^2}{a^2+\lambda^2} + \dfrac{y^2}{b^2+\lambda^2} = 1}$ where λ is a parameter

Imp Result → is also Self-Orthogonal.

\# $x^2+y^2+2gx+c=0$.

Imp Concept $x^2+y^2+2fy+c=0$.

O.T of $x^2+y^2+2gx-5=0$ is —

$x^2+y^2-2fy+5=0$ → always take opposite coefficient sign.

\#

$f(r,\theta,c)=0$ ——— (1)

$F\left(r,\theta,\dfrac{dr}{d\theta}\right)=0$ ——— (2).

replace $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$

$g\left(r,\theta,-r^2\dfrac{d\theta}{dr}\right)=0$ ——— (3)

The sol^n of eq^n (3) is called the Orthogonal of the given family.

(1) $r = a\theta$

$\log r = \log a + \log\theta$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\theta}$

$\frac{1}{\cancel{r}} \times -r^{\cancel{2}}\frac{d\theta}{dr} = \frac{1}{\theta}$

$\int \frac{dr}{r} = -\int \theta\, d\theta$

$\log r = -\frac{\theta^2}{2} + c$

$r = e^{-\frac{\theta^2}{2} + c}$

$r = e^{-\frac{\theta^2}{2}} \cdot e^{c}$

$\boxed{r = k\, e^{-\frac{\theta^2}{2}}}$

(2) $r = \frac{2a}{1-\cos\theta}$

$\log r = \log 2a - \log(1-\cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{1-\cos\theta}\sin\theta$

$\frac{1}{\cancel{r}} \times -r^{\cancel{2}}\frac{d\theta}{dr} = -\frac{\sin\theta}{1-\cos\theta}$

$r\frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta}$

$2\sin^2\frac{\theta}{2}$

$\Rightarrow \frac{dr}{r} = \frac{\cancel{1-\cos\theta}}{\cancel{\sin\theta}}\, d\theta$

$2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$

$\Rightarrow \int \frac{dr}{r} = -2\int \frac{-\frac{1}{2}\sin\theta/2}{\cos\theta/2}\, d\theta$

$\Rightarrow \log r = -2\log\cos\frac{\theta}{2} + \log c.$

$$\Rightarrow \log r = \log \frac{c}{\cos^2\frac{\theta}{2}}$$

$$\Rightarrow r = \frac{c}{\cos^2\frac{\theta}{2}}$$

$$\Rightarrow r = \frac{c}{\frac{1+\cos\theta}{2}}$$

$$\Rightarrow \boxed{r = \frac{2c}{1+\cos\theta}}$$

i.e, $\boxed{r = \frac{2a}{1-\cos\theta}} \xleftrightarrow{\text{Orthogonal}} \boxed{r = \frac{2c}{1+\cos\theta}}$

③ $r^n = a^n \sin n\theta$

$$n \log r = n \log a + \log \sin n\theta.$$

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin\theta}$$

$$\frac{n}{\cancel{r}} - r^{\cancel{2}}\frac{d\theta}{dr} = \frac{n \cos n\theta}{\sin n\theta}.$$

$$-nr\frac{d\theta}{dr} = \frac{n \cos n\theta}{\sin n\theta}$$

$$\int \frac{dr}{r} = \frac{1}{n}\int -n\frac{\sin n\theta}{\cos n\theta}\, d\theta\ldots$$

$$\log r = \frac{1}{n}\log \cos n\theta + \log c$$

$$n \log r = \log \cos n\theta + n \log c$$

$$\log r^n = \log \cos n\theta + \log c^n$$

$$\boxed{r^n = c^n \cos n\theta}$$

i.e, $\boxed{r^n = a^n \sin n\theta} \xleftrightarrow{\text{Orthogonal}} \boxed{r^n = c^n \cos n\theta}$

Q. The diff. eqⁿ of family of curves is given by $\frac{dy}{dx} = \frac{x}{y}$ then the system of orthogonal Trajectories of the given family of Curves —

→ $\frac{dy}{dx} = \frac{x}{y}$

$-\frac{dx}{dy} = \frac{x}{y}$

$-\frac{dx}{x} = \frac{dy}{y}$

$-\log x = \log y + \log c$

$\Rightarrow \boxed{xy = c}$

This is the orthogonal Trajectory of family $\frac{dy}{dx} = \frac{x}{y}$

Differential Eqⁿ of First Order but not first Degree :-

- A differential eqⁿ of the form —

$$a_0\left(\frac{dy}{dx}\right)^n + a_1\left(\frac{dy}{dx}\right)^{n-1} + a_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + a_n = 0.$$

where $a_0, a_1, a_2 \cdots a_n$ are fn. of x and y or constants is said to be differential eqⁿ of 1st order but not first degree.

Let $\frac{dy}{dx} \equiv P$ ∴

$$\boxed{a_0 P^n + a_1 P^{n-1} + a_2 P^{n-2} + \cdots + a_n = 0}$$

$f(x, y, P) = 0$

- This eqⁿ containing 3 variables x, y and P, therefore it may contain 3 types of solution:

→ It may be solvable for P, may be solvable for y, may be solvable for x.

* The eqⁿ solvable for P —

→ If it is solvable for P, then the eqⁿ must be factorized.

→ $[P - f_1(x,y)]\,[P - f_2(x,y)]\,[P - f_3(x,y)] \ldots [P - f_n(x,y)] = 0$

then $\cancel{P} - f_1(x,y) = 0$

replace by $\frac{dy}{dx}$ $\boxed{\frac{dy}{dx} = f_1(x,y)}$

ex:- ① $p^2 - 5p + 6 = 0$ where $p = \frac{dy}{dx}$.

$(P-3)(P-2) = 0.$

$P - 2 = 0$

$\frac{dy}{dx} - 2 = 0.$

$\int dy - \int 2dx = \int 0$

$y - 2x - C_1 = 0$ —— ①

$y - 3x - C_2 = 0$ —— ②

// $\boxed{(y - 2x - C_1)(y - 3x - C_2) = 0}$

② $x^2p^2 + xyP - 6y^2 = 0$

$\Rightarrow x^2p^2 + 3xyp - 2xyP - 6y^2 = 0$

$\Rightarrow xP(xp + 3y) - 2y(xp + 3y) = 0$

$\Rightarrow (xp + 3y)(xp - 2y) = 0.$

$xp + 3y = 0$

$x\frac{dy}{dx} + 3y = 0$

$\frac{dy}{dx} - \frac{3}{x}y = 0$

$y = e^{\int P dx} = \int \cancel{Q}^{\,0} e^{\int P dx} + C$

$y(x^3) - c_1 = 0$ —— ①

$x\frac{dy}{dx} - 2y = 0.$

$\frac{dy}{dx} - \frac{2}{x}y = 0.$

$y\left(\frac{1}{x^2}\right) - c_2 = 0$ —— ②.

Solⁿ obtained $\therefore \boxed{(x^3y - c_1)(y - c_2x^2) = 0}$

② $xyp^2 - (x^2+y^2)p + xy = 0.$

$xyp^2 - x^2p - y^2p + xy = 0.$

$xp(yp - x) - y(yp - x) = 0$

$(xp - y)(yp - x) = 0.$

$xp - y = 0.$

$x\frac{dy}{dx} - y = 0$

$\frac{dy}{dx} - \frac{1}{x}y = 0.$

$y\left(\frac{1}{x}\right) - c_1 = 0$ —— ①

Trick

When to apply linear approach

When to apply variable seperable?

→ $x\frac{dy}{dx}$ or $y\frac{dx}{dy}$ → linear

→ $x\frac{dx}{dy}$ or $y\frac{dy}{dx}$ → Variable seperable

$\therefore yp - x = 0.$

$y\frac{dy}{dx} - x = 0$

$y\,dy - x\,dx = 0$

$\frac{y^2}{2} - \frac{x^2}{2} = c_2$

$\Rightarrow y^2 - x^2 - 2c_2 = 0$ —— ②

$\therefore \boxed{(y - c_1x)(y^2 - x^2 - x_2) = 0}$

* when factorization is not possible, then solvable for P is not possible

two case arises —

When solvable for y ?	When solvable for x ?
① The degree of y must be 1.	① The degree of x must be 1
② Seperate y from x & P $y = f(x, P)$	② Seperate x from y & P $x = f(y, P)$
③ diff. both side w.r.to 'x'	③ Diff. both side w.r.to 'y'.
④ Replace $\frac{dy}{dx}$ by P	④ Replace $\frac{dx}{dy}$ by $\frac{1}{P}$.
⑤ neglect the terms or cancel the terms which doesnot contain $\frac{dP}{dx}$ on both sides.	⑥ Neglect the term or Cancel the term which does not contain $\frac{dP}{dy}$, on both sides.

ex:- $y + px = p^2x^4$ —— ①

∵ degree of $y = 1$ → solvable for y ✓

∵ all coeff are 1, 1, 1 ⇒ means factorization not possible ⇒ not solvable for P ✗

$$y = p^2x^4 - px$$

$$\frac{dy}{dx} = p^2(4x^3) + x^4\left(2p\frac{dp}{dx}\right) - p(1) - x\left(\frac{dp}{dx}\right)$$

$$p - 4p^2x^3 + p = (2px^4 - x)\frac{dp}{dx}$$

$$2p(1 - \cancel{2}px^3) = -x(1 - \cancel{2}px^3)\frac{dp}{dx}$$

$$2p = -x\frac{dp}{dx}$$

$$\Rightarrow \int \frac{dP}{P} = \int -2\frac{dx}{x}$$

$$\Rightarrow \log P = \log \frac{c}{x^2}$$

$$\boxed{P = \frac{c}{x^2}}$$ 2 methods now to get sol^n of diff^n eq^n in Q.

either put $P = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{c}{x^2}$$

and integrate and solve.

$$dy - \frac{c\,dx}{x^2} = 0$$

$$\boxed{y + \frac{c}{x} = c}$$

sol^n

Put in eq^n of Question.

$$\because y + \cancel{p}x = \cancel{p^2}x^4 \cdot$$

$$\frac{c}{x^2} \quad \left(\frac{c}{x^2}\right)^2$$

$$\Rightarrow \boxed{y + \frac{c}{x} = c}$$

sol^n

Ans

(2)

$$y = 2px + y^2p^3 \cdot$$

$$2px = y - y^2p^3$$

$$x = \frac{y}{2p} - \frac{y^2 p^{\cancel{3}\,2}}{2\cancel{p}}$$

$\frac{dx}{dy}$ @

$$2x = \frac{1}{p} \cdot y - y^2p^2$$

$$2\frac{dx}{dy} = \frac{1}{p} \cdot 1 + y\left(-\frac{1}{p^2}\frac{dp}{dy}\right) - 2yp^2 - y^2\left(2p\frac{dp}{dy}\right)$$

$$2 \cdot \frac{1}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2py^2\frac{dp}{dy}$$

$$\frac{2}{p} - \frac{1}{p} + 2yp^2 = \left(-\frac{y}{p^2} - 2py^2\right)\frac{dp}{dy}$$

$$\frac{(1 + \cancel{2}yp^3)}{\cancel{p}} = -y\left(\frac{1 + \cancel{2}yp^3}{p^{\cancel{2}}}\right)\frac{dp}{dy}$$

$$1 = -\frac{y}{P}\frac{dP}{dy}$$

$$\int \frac{dP}{P} = \int -\frac{dy}{y}$$

$$\log P = \log \frac{c}{y}$$

$$\boxed{P = \frac{c}{y}}$$ Put in Q.

$$\because \quad y = 2x \cdot \frac{c}{y} + y^2 \frac{c^3}{y^3}$$

$$\boxed{y^2 = 2cx + c^3}$$

Shortcut

$$\boxed{y = Px + \phi(P)}$$

replace P by c

$$\boxed{y = cx + \phi(c)}$$

ex:- (1) $y = px - p^2$

gen soln $= \boxed{y = cx - c^2}$

(2) $\log(px - y) = p$

$\therefore \ px - y = e^p$

$$y = px - e^p$$

$$\boxed{y = cx - e^c}$$

Numerical Methods

I: Sol^n of non-linear equation :-

***

non linear means ?
linear → degree 1.
non-linar → degree $\neq 1$

Let $f(x) = 0$ be any non linear equation to find the approximate root of $f(x)=0$ choose any two values of $x = a, b$, particularly adjacent values in such a way that their functional values must have opposite signs.

ex:- $f(a) < 0$, $f(b) > 0$

Then, we can say that root lies bet^n a and b. This approximate root can be find by —

1. method of bisection —
2. Regular Falsi method
3. Newton - Raphson .

* Method of Bisection:-

$x = a, b$

$f(a) < 0$, $f(b) > 0$.

$f(a)$ 0 $f(b)$

$$x_1 = \frac{a+b}{2}$$

$f(x_1) < 0$

$f(a)$, $f(x_1)$, $f(b)$
$<$ $<$ $>$

$$x_2 = \frac{x_1 + b}{2}$$

$f(x_2) > 0$

x.

$$x_3 = \frac{x_1 + x_2}{2}$$

upto 13^{th} – 14^{th} step

The rate of convergence in bisection method is very - very slow.

→ Comparitively, the Bisection method, Regula - Falsi method has a little fast & rate of convergence.

* **Regula - Falsi Method :-**

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

* **Newton - Raphson method :-**

Imp 1-marks

→ has a very fast rate of convergence.

→ and the convergence of Newton Raphson method is 2^{nd} order convergence or Quadratic convergence.

initial guess (99% chance given in Q.)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

1% chance not given then how to get x_0

↳ take $f(1) \to x_0$, $f(2) \to x_0$

whatever is nearer to origin that value will be taken.

when $f(1)$ and $f(2)$ must have opposite signs.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

① $f(x) = x^3+2x-5=0$

$x_0 = 1$ find x_1:

$f'(x) = 3x^2+2$

$\longrightarrow$ $x_1 = 1 - \dfrac{x^3+2x-5}{3x^2+2}$

$= 1 - \left[\dfrac{1+2-5}{5}\right]$

$= 1+\dfrac{2}{5}$

$x_1 = 1.4$ Ans

② $f(x) = x^3+3x-7=0$

$x_0 = 1.$

$x_1 = ?.$

$\longrightarrow$ $x_1 = 1 - \dfrac{1^3+3-7}{3\cdot1+3}$

$= 1 - \dfrac{-\cancel{3}}{\cancel{6}\,2}$

$= 1+\dfrac{1}{2}$

$= 1.5$ Ans

③ $f(x) = x^3-2x-5=0.$

$f(0) = -5<0$

$f(1) = 1-2-5 = -6<0$

$f(2) = 8-4-5 = -1<0$ $\longrightarrow$ $x_0 = 2$ is choosen

$f(3) = 27-6-5 = 16>0.$

$x_0 = 2$

$x_1 = 2 - \dfrac{(-1)}{3(2)^2-2} = 2+\dfrac{1}{10} = 2.1$

④ $f(x) = e^x - 1 = 0$

$x_0 = -1$

$x_1 = ?$

ⓐ 0.71828

ⓑ 0.34567

ⓒ 0.12326

ⓓ 0.00000.

$\rightarrow$ $f'(x) = e^x$..

$$x_1 = -1 - \frac{e^{-1} - 1}{e^{-1}}$$

$= -1 - 1 + e$

$= e - 2$

$= 2.71 - 2$

$= 0.71.$

⑤ $f(x) = x^2 - 2x - 1$

$x_0 = 2$

$x_2 = \_\_\_\_$.

ⓐ 2.425 ⓑ 2.423 ⓒ 2.419 ⓓ 2.417.

$\rightarrow$ $f'(x) = 2x - 2$.

$$x_1 = 2 - \frac{-1}{2} = 2 + .5 = 2.5$$

$$x_2 = 2.5 - \frac{6.25 - 5 - 1}{5 - 2}$$

$$= 2.5 - \frac{.25}{3}$$

$= 2.5 - .0833$

$= 2.417$

Q. $f(x) = x^4 - 3x + 1 = 0$

$x_0 = 0$

$x_1 =$ ____

(a) $-\frac{1}{3}$ (b) $\frac{1}{3}$ (c) 3 (d) -3

→ $f'(x) = 4x^3 - 3$

$x_1 = 0 - \frac{1}{-3}$

$= \frac{1}{3}$ Ans

Q. The approximate root of eq^n $f(x) = (x-1)^2 + x - 3 = 0$ is to be solved by the Newton Raphson method. The method will fail in the very first iteration if the initial guess $x_0 =$ ____

(a) 0 (b) 0.5 (c) 1.0 (d) 1.5

→ $f(x) = x^2 - 2x + 1 + x - 3$

$= x^2 - x - 2$

$f'(x) = 2x - 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= x_0 - \frac{x_0^2 - x_0 - 2}{2x_0 - 1}$ $\to 0$ method will fail.

i.e. method will fail if $2x_0 - 1 = 0$

$\Rightarrow x_0 = 0.5$

Q The Newton Raphson iteration $x_{n+1} = \frac{1}{2}\left[x_n + \frac{R}{x_n}\right]$ can be used to compute the

Imp Concept

a) sq. root of R

b) sq. of R

c) logrithmic of R

d) Reciprocal of R.

$\rightarrow$ $x_{n+1} = x_n = \alpha$.

$\therefore \alpha = \frac{1}{2}\left[\alpha + \frac{2}{\alpha}\right]$.

$\Rightarrow 2\alpha - \alpha = \frac{R}{\alpha}$

$\Rightarrow \alpha^2 = R$

$\Rightarrow \alpha = \sqrt{R}$

$\hookrightarrow$ used to compute sq. root of no.

ex:- $x_{n+1} = \frac{1}{2}\left[x_n + \frac{117}{x_n}\right]$ Converges to $\sqrt{117}$.

Q $f(x) = x^3 - x^2 + 4x - 4 = 0$

$x_0 = 2$

$x_1 =$ ______

a) $\frac{2}{3}$, b) $4/3$ c) 1 d) $\frac{3}{2}$.

$\rightarrow$ $f'(x) = 3x^2 - 2x + 4$

$x_1 = 2 - \left[\frac{\cancel{8}^{2}}{\cancel{12}_{3}}\right]$

$= \frac{4}{3}$

$=$

Q. The recursion relation $x = e^{-x}$ is to be solved by Newton Raphson method, then which of the following is true?

a) $x_{n+1} = e^{-x_n}$.

b) $x_{n+1} = x_n - e^{-x_n}$

c) $x_{n+1} = \dfrac{(1+x_n)e^{-x_n}}{(1+e^{-x_n})}$

d) $x_{n+1} = \dfrac{x_n^2 - e^{-x_n}(1+x_n) - 1}{x_n - e^{-x_n}}$

$\rightarrow$ $f(x) = x - e^{-x}$.

$f'(x) = 1 + e^{-x}$

$f'(x_n) = 1 + e^{-x_n}$ → This will come into denominator.

$\therefore$ option c

Q. Consider the series —

$x_{n+1} = \dfrac{x_n}{2} + \dfrac{9}{8x_n}$

$x_0 = 0.5$ is obtained from the Newton Raphson method.

The series converges to —

$\rightarrow$ $x_{n+1} = x_n = \alpha$. (Take it always when given Series Converges)

$\Rightarrow \alpha = \dfrac{\alpha}{2} + \dfrac{9}{8\alpha}$

$\Rightarrow \alpha = \dfrac{4\alpha^2 + 9}{8\alpha}$

$\Rightarrow 4\alpha^2 = 9$

$\Rightarrow \alpha^2 = \dfrac{9}{4}$

$\Rightarrow \alpha = \sqrt{9/4} = \dfrac{3}{2} = 1.5$ Ans

Q. The sol^n of the variables x_1 and x_2 for the equations –

$$u = 10x_2 \cdot \sin x_1 - 0.8 = 0 \quad \text{——— ①}$$

$$V = 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0 \quad \text{——— ②}$$

is to be solved by the Newton Raphson method.

Assuming the initial value, $x_1 = 0.0$

$x_2 = 1.0$.

Jacobian Matrix = ___

→ Jacobian Matrix –
$$\begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix}$$

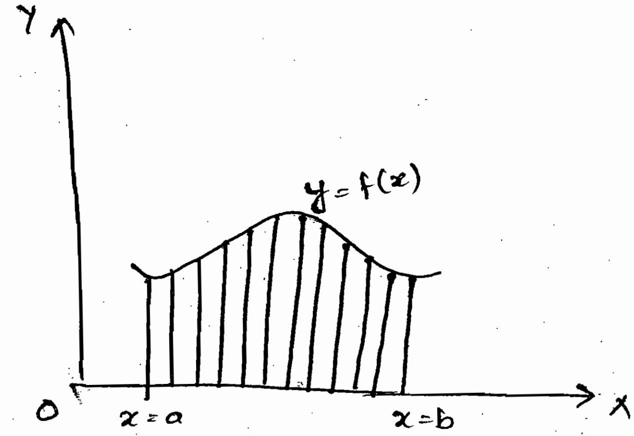
$$\frac{\partial u}{\partial x_1} = 10x_2 \cos x_1 = 10(1)(1) = 10$$

$$\frac{\partial u}{\partial x_2} = 10 \sin x_1 = 10(0) = 0.$$

$$\frac{\partial v}{\partial x_1} = 10x_2 \sin x_1 = 10(1)(0) = 0.$$

$$\frac{\partial v}{\partial x_2} = 20x_2 - 10\cos x_1 = 20(1) - 10(1) = 10.$$

Numerical Integration :-



$$\boxed{h = \frac{b-a}{n}}$$

① Trapezoidal Rule :-

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

② Simpson's $\frac{1}{3}$rd Rule :-

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\right]$$

③ Simpson's $\frac{3}{8}$th Rule :-

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) + 2(y_3 + y_8 + y_9 + \cdots)\right]$$

Note :-

The accuracy of Trapezoidal rule or order of integration of Trapezoidal rule is $O(h^2)$ → accuracy at 2nd point of decimal

The accuracy of Simpson's $\frac{1}{4}$ rule = $O(h^4)$ → accuracy upto 4th place of decimal

The accuracy of Simpson's $\frac{3}{8}$th rule is $O(h^5)$ → accuracy upto 5th place of decimal

Q. $\int_0^1 \frac{1}{1+x^2}dx$.

By Trapezoidal & Simpson's rule and deviding the interval into 4 equal parts. and Derieve the value of π.

$\rightarrow \int_0^1 \frac{1}{1+x^2} dx$

$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$

$y = \frac{1}{1+x^2}$

x	0	0.25	0.50	0.75	1.
y	1.0000	0.9412	0.8000	0.6400	0.5000.

$\frac{1}{1+\frac{1}{16}}$ $\frac{1}{1+\frac{1}{4}}$ $\frac{1}{1+\frac{9}{16}}$

$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{2}\left[(1.0000+0.5000) + 2(0.9412+0.8000+0.6400)\right]$

$\tan^{-1}x \Big|_0^1 = 0.7828$

$\tan^{-1}1 - \cancel{\tan^{-1}0}^{0} = 0.7828$

$\Rightarrow \frac{\pi}{4} = 0.7828$

$\Rightarrow \pi = 3.1312$ we know $\pi = 3.1415$

difference occured at 3rd place.

Simpson $\frac{1}{3^{rd}}$ Rule —

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{3}\left[(1.0000+0.5000) + 4(0.9412+0.6400) + 2(0.8000)\right]$$

$$\tan^{-1}\Big|_0^1 = 0.7854$$

$$\Rightarrow \tan^{-1} - \cancel{\tan^{-1} 0}^{\,0} = 0.7854$$

$$\Rightarrow \frac{\pi}{4} = 0.7854$$

$$\Rightarrow \pi = 0.7854 \times 4$$

$\pi = 3.1416$ we know $\pi = 3.1415$

diff. occur at 4th place.

By Shortcut methods.

Q. $\int_0^6 \frac{1}{1+x^2}\,dx$ by Trapezoidal rule by dividing the interval into 6 equal parts.

a) 1.4132 b) 1.3756 c) 1.2326 d) 1.0986.

→ ~~tan~~

$$\int_0^6 \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1} 6 - \cancel{\tan^{-1} 0}^{\,0}$$

$\therefore = 1.40\ldots$

Compare with option → diff^n will be at 2^{nd} place.

∴ option (a) is correct.

② $\int_1^3 \log_e x \, dx$, $n=2$ = ______ By Simpson $\frac{1}{4}$th rule.

ⓐ 0.50
ⓑ 0.80
ⓒ 1.00
ⓓ 1.29

→ Directly integrate and Compare ~~the~~, value at 4th place will differ.

③ $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} dx$, $n=4$, By Simpsons $\frac{3}{8}$ rule.

ⓐ 0.3883
ⓑ 0.4344
ⓒ 0.5026.
ⓓ 0.5589

→ $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} dx = \int_0^{\pi/4} \tan x \cdot \sec^2 x \, dx .$

$= \left(\frac{\tan^2 x}{2}\right)_0^{\pi/4}$

$= 0.5000.$

④ $\int_1^3 \frac{1}{x} dx$, $n=2$, By Simpson $\frac{1}{3rd}$ rule.

a). 1.000
b). 1.098
c). 1.111
d). 1.120

→ Comparison ⇒ Use only Numeric method or Shortcut method
Do not ever integrate

→ $y = \frac{1}{x}$

x	1	2	3
y	1	0.50	0.333

$\Rightarrow \int_1^3 \frac{1}{x} dx = \frac{1}{3}\left[1.333 + 4(0.5)\right]$

$= \frac{3.333}{3} = 1.111$

Q. Torque Exerted on a flywheel over a cycle is listed in the table —

Angle (°C)	0°	60°	120°	180°	240°	300°	360°
Torque (Nm)	0	1066	323	0	−323	−355	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6.

Estimate the value of the fn by Simpson $\frac{1}{3}$rd rule.

(a) 542 (b) 993 (c) 1444 (d) 1986

→ $\int_0^{2\pi} f(x)\,dx$

$= \frac{60^\circ}{3}\left[(0+0) + 4(1066+0-355) + 2(323-323)\right)$

$= \frac{\frac{\pi}{3}}{3} \times 4(711)$

$= \frac{\pi}{9} \times 2844$

$= 992.7$.

$\simeq 993$.

Numerical Sol^n of Ordinary Differential Eq^n:-

$$\frac{dy}{dx} = f(x,y), \quad x = x_0, \quad y = y_0, \quad y(x_1) = \_\_\_\_$$

Methods –

① Taylor's series method

② Pi Card's method

③ Runge – kutta method.

$\boxed{\text{Taylor Series –}}$

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \dots \infty.$$

Q. $\frac{dy}{dx} = -xy$, $y(0) = 1$, $y(0.1) = \_\_\_\_$

$x + y = 1$
$y = 1 - 0.1$
$= 0.9$ ans

→

$y' = -xy$

$y'' = -xy' - y.1$

$y''' = -xy'' - 2y'$

$y^{iv} = -xy''' - 3y''$

$$y = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(3) + \dots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} \dots\dots \longrightarrow$$ Consider only upto 4th order rest can be neglected

$$\therefore y(0.1) = 1 - \frac{0.01}{2} + \cancel{\frac{0.0001}{8}}$$ neglected

$= 1 - 0.005$

$= 0.995$ Ans

② Picard's method :-

$$\frac{dy}{dx} = f(x,y) \quad , \quad x = x_0 \; , \; y = y_0$$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x_0} f(x,y)\, dx.$$

$$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x,y)\, dx.$$

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\, dx$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\, dx$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x,y_2)\, dx.$$

— — — —

— — —

$$y = (\quad) + \cdots$$

Q. $$\frac{dy}{dx} = \overset{f(x,y)}{\cancel{-xy}} \; , \quad y(0) = 1, \; y(0.1) = \_\_\_$$

(x_0, y_0)

$$y_1 = 1 + \int_0^x -x \overset{y_0}{\cancel{(1)}}\, dx$$

$$y_1 = \left(1 - \frac{x^2}{2}\right)$$

$$y_2 = 1 + \int_0^x -x \overset{y_1}{\cancel{\left(1 - \frac{x^2}{2}\right)}}\, dx$$

$$y_2 = 1 - \frac{x^2}{2} + \frac{x^4}{8}$$

$$\therefore y = 1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots$$

Q. $\frac{dy}{dx} = \frac{x^2}{1+y^2}$, $y(0)=0$, $y(0.25) =$ ____

When derivatives are convinient Taylor Series

When ~~Taylor~~ Ser $\int$ is convinient Pica Card's method

$\rightarrow$ $y_1 = 0 + \int \frac{x^2}{1+0^2} = \frac{x^3}{3}$.

$y_2 = 0 + \int \frac{x^2 \, dx}{1+\left(\frac{x^3}{3}\right)^2}$

$= \tan^{-1}\left(\frac{x^3}{3}\right)$

$y_3 = 0 + \int \frac{x^2}{1+\left(\tan^{-1}\frac{x^3}{3}\right)^2} \rightarrow \int$ becomes difficult.

$\therefore$ Take prev. one as approx $\underline{\underline{sol^n}}$.

$\therefore \boxed{y = \tan^{-1}\frac{x^3}{3}}$

$\boxed{y(0.25) = \tan^{-1}\frac{(0.25)^3}{3}}$

Q. $\frac{dy}{dx} = -xy$, $y(0)=1$, & $y(0.1) =$ ____

$\frac{dy}{dx} + xy = 0$

$I.F = e^{\int P dx} = e^{\int x dx} = e^{\frac{x^2}{2}}$.

$y \cdot e^{x^2/2} = c$.

$1 \cdot e^0 = c \Rightarrow c = 1$

$y e^{\frac{x^2}{2}} = 1$

$y = \frac{1}{e^{\frac{x^2}{2}}} = e^{-\frac{x^2}{2}} = e^{-\frac{(0.1)^2}{2}} = e^{-\frac{0.01}{2}} = e^{-0.005}$

$= \underline{\underline{0.995}}$

Runge - Kutta Method:-

$$y = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h \cdot f(x_0, y_0)$ $\longrightarrow$ where $h = x_1 - x_0$.

$$k_2 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f\left(x_0 + h,\ y_0 + k_3\right)$$

Q. ① $\dfrac{dy}{dx} = \underbrace{x+y}_{f(x,y)}$ $\quad y(\underset{x_0}{0}) = \underset{y_0}{1}$, $y(\underset{x_1}{0.1}) =$ ______

$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

$$k_1 = h\, f(x_0, y_0) = 0.1(0+1) = 0.1$$

$$k_2 = h f\left(\underset{0}{x_0} + \underset{0.05}{\frac{h}{2}},\ \underset{1}{y_0} + \underset{0.05}{\frac{k_1}{2}}\right) = 0.1\left[0.05 + 1.05\right] = 0.11$$

$$k_3 = h f\left(\underset{0}{x_0} + \underset{0.05}{\frac{h}{2}},\ \underset{1}{y_0} + \underset{0.055}{\frac{k_2}{2}}\right) = 0.1\left[0.05 + 1.055\right] = 0.1105$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3) = 0.1\left[0.1 + 1.1105\right] = 0.12105$$

Q Match the following —

E: Newton Raphson → 1) Sol^n of non linear eq^n

F: Runge Kutta → 2) Sol^n of linear algebric

G: Simpson rule → 3) Sol^n of O. D. eq^n.

H: Gauss Elimination. → 4) ~~Sol of Linear Algebraic eq^n~~ Numeric Integration.

Complex Variable

① $z = x+iy$

$\bar{z} = x-iy$

$z\cdot\bar{z} = |z|^2 = x^2+y^2$

② $x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is called the modulus-Amplitude form or Polar form

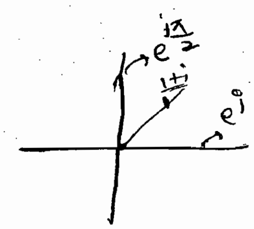
$r = |z| = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

③ $\cos\theta + i\sin\theta = e^{i\theta}$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

$$\boxed{\begin{aligned} 1 &= \cos 0^\circ + i\sin 0^\circ = e^{i0} \\ i &= \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2} = e^{i\pi/2} \\ 1+i &= \sqrt{2}\left[\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right] = \sqrt{2}\,e^{i\pi/4} \end{aligned}}$$



Q. $(i)^i =$ ____ where $i = \sqrt{-1}$

$(x)^x =$ ____ where $x = \sqrt{-1}$

$(i)^i = \left(e^{i\pi/2}\right)^i = e^{-\frac{\pi}{2}}.$

Q. ① $(1+i)(2-5i) =$ ____

② $\left|\dfrac{3+4i}{1-2i}\right| =$ ____

③ $\dfrac{i+3}{i+1} =$ ____

④ $\dfrac{1+2i}{i-2} =$ ____

⑤ $\dfrac{-5+i10}{3+i4} =$ ____

① $(1+i)(2-5i)$

$= 2+2i-5i-\cancel{5i^2}$ $+5$

$= 7-3i$

② $\left|\frac{3+4i}{1-2i}\right| = \frac{\sqrt{9+16}}{\sqrt{1+4}} = \sqrt{\frac{25}{5}} = \sqrt{5}$ //

③ $\frac{i+3}{i+1} \times \frac{i-1}{i-1}$

$= \frac{\cancel{i^2}^{-1}+3i-i-3}{-1-1}$

$= \frac{-4+2i}{-2}$

$= 2-i$

④ $\frac{1+2i}{i-2} = ?$

$= \frac{1+2i}{i-2} \times \frac{i+2}{i+2}$

$= \frac{i+2i+2+4i^2}{-1-4} = -i$

⑤ $\frac{-5+i10}{3+i4}$

$= \frac{-5+i10}{3+i4} \times \frac{3-i4}{3-i4}$

$= \frac{-15+30i+20i-40i^2}{9+16}$

$= \frac{-15+50i+40}{25} = 1+2i$

Q. The value of $(1+i)^8 =$ ______

ⓐ 2 ⓑ 4 ⓒ 8 ⓓ = 16.

$\rightarrow$ $(1+i)^8$

$= \left[(1+i)^2\right]^4$

$= \left[\cancel{1}\cancel{+1}+2i\right]^4$

$= 16\, i^4$

$= 16$ Ans

Q. The Polar form of $2+2i$ is ______

$r = |z| = \sqrt{4+4} = 2\sqrt{2}.$

$\theta = \tan^{-1} 1 = \frac{\pi}{4}.$

$\therefore \quad 2+2i = 2\sqrt{2}\, e^{i\frac{\pi}{4}}$

Q.

Analytic function:-

A single valued fn which is defined and differentiable at each point of domain D is said to be an analytic fn in that domain.

Note:-

The necessary and sufficient condn for a function $\boxed{f(z) = u(x,y) + i\,v(x,y)}$ to be analytic is it should satisfy the $\boxed{\text{Cauchy-Riemann eqn.}}$

Condn to be Analytic

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

Q. $f(z) = (x^2 - y^2) + i(2xy)$

$u = x^2 - y^2$, $v = 2xy$.

$\frac{\partial u}{\partial x} = 2x$ | $\frac{\partial v}{\partial y} = 2x$

$\frac{\partial u}{\partial y} = -2y$ | $\frac{\partial v}{\partial x} = 2y$

Satisfying C.R eqn

$\therefore$ Analytic in nature

Harmonic function:-

— Any function of x and y satisfies the Laplace's eqⁿ is said to be an harmonic function.

represented by $\boxed{\nabla^2 u = 0}$

ex:- ① $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \quad \therefore \text{Harmonic}$$

Note:-

Real and Img part of an Analytic eqⁿ satisfies the Laplacian Equation.

i.e, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

If $f(z) = u(x,y) + iv(x,y)$ If $u(x,y)$ → Real Part given

then $f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci$

If Real part Given then asked directly $f(z)$ then how to get

Q. $u = x^2 - y^2$

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2i}\right)^2$$

$$= \frac{z^2}{4} + \frac{z^2}{4}$$

$$= \frac{z^2}{2}$$

$u(0,0) = 0 - 0 = 0.$

$$f(z) = 2 \times \frac{z^2}{2} - 0 + ci = z^2 + ci$$

$u = x^3 - 3xy^2 + 3x + 1.$

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{z^3}{8} - \frac{3z}{2}\left(\frac{z^2}{4i^2}\right) + \frac{3z}{2} + 1$$

$$= \frac{z^3}{8} + \frac{3z^3}{8} + \frac{3z}{2} + 1$$

$$= \frac{\cancel{4}^{2}\, z^3}{\cancel{8}_{2}} + \frac{3z}{2} + 1$$

$u(0,0) = 0 - 0 + 0 + 1 = 1.$

$$f(z) = 2\left(\frac{z^3}{\cancel{8}} + \frac{3z}{2} + 1\right) - 1 + ci$$

$$f(z) = z^3 + 3z + 1 + ci$$

$\longrightarrow$

Q. $f(z) = z^2$

$= (x+iy)^2$

$= x^2 + i^2y^2 + 2ixy.$

$= x^2 - y^2 + 2ixy..$

If real part given, how to get Img part.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y} \cdot dy$$

$$\boxed{dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy}$$

which is exact.

Let $u = x^2 - y^2$.

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$dv = 2y\,dx + 2x\,dy$

$\hookrightarrow$ exact no need to check

Gen. Solⁿ —

$$v = \int 2y\,dx + \int 0\,dy$$

$$v = 2xy \quad \underline{\text{Ans}}$$

Q. $u = x^3 - 3xy^2 + 3x + 1$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 3$$

$$\frac{\partial u}{\partial y} = -6xy$$

$$dv = \int 6xy\,dx + \int (\cancel{3x^2} - 3y^2 + 3)\,dy$$

$=$

Q. $u = e^x \cos y$.

$$\frac{\partial u}{\partial x} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y.$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\int dv = \int e^x \sin y\,dx + \int \cancel{e^x} \cos y\,dy$$

$$\boxed{V = e^x \sin y.}$$

* If img part given and asked real part then how to get?

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

or

$$\boxed{du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy}$$

which is exact . .

ex:- ① $v = -xy$

$$\frac{\partial v}{\partial x} = -y, \quad \frac{\partial v}{\partial y} = -x.$$

$$\int du = \int -x dx + \int y dy.$$

$$u = -\frac{x^2}{2} + \frac{y^2}{2} = \frac{1}{2}(y^2 - x^2).$$

② Given $v = y^3 - 3x^2 y$, find u where $f(z)$ is harmonic

$$\frac{\partial v}{\partial x} = -6xy \qquad \frac{\partial v}{\partial y} = 3y^2 - 3x^2$$

$$\int du = \int (3y^2 - 3x^2) dx + \int 6 \cancel{xy} dy$$

$$\boxed{u = 3xy^2 - x^3}$$

Singular Points :- or Pole

Singular points are those at which the given function $f(z)$ is not analytic.

These points are also called Singularities or poles.

find Singular Points.

Q. ① $f(z) = \dfrac{1-2z}{z(z-1)(z-2)}$

$z = 0, 1, 2$.

Q. $f(z) = \dfrac{z}{z^2-z-2}$

$f(z) = \dfrac{z}{(z-2)(z+1)}$

$\Rightarrow z = 2, -1$.

Q. $f(z) = \dfrac{z}{z^4-1}$

$z = \pm 1, \pm i$

Cauchy's Theorem:-

If $f(z)$ is an analytic fn and its derivative $f'(z)$ is continuous at all points inside and on a simple and closed curve C, then-

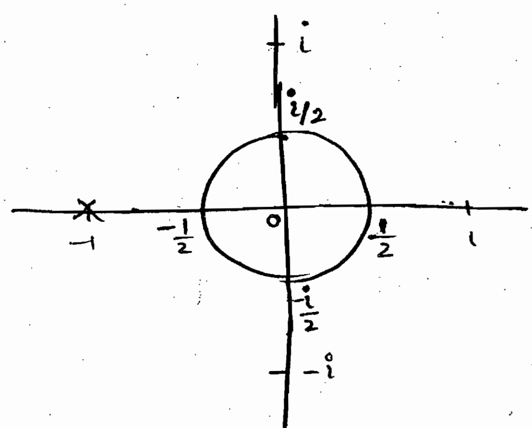
$$\boxed{\oint_C f(z)\,dz = 0}$$

means

if pole outside the limit, then $\int = 0$.

Q. $\oint_C \frac{3z^2+7z+1}{z+1}\,dz$, where C is $|z| = \frac{1}{2}$

→

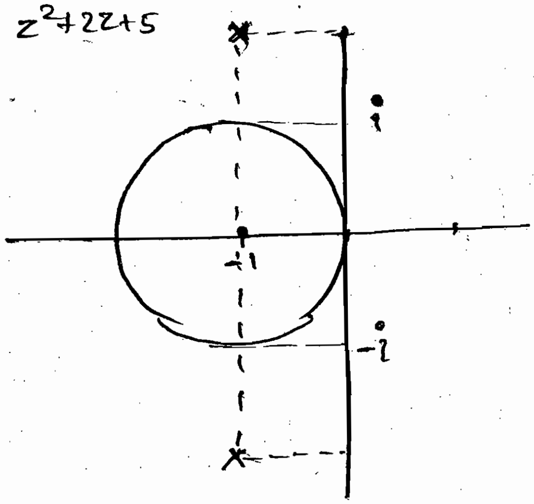


∵ pole is outside curve.

∴ Ans is 0.

Q. $\oint_C \frac{z+4}{z^2+2z+5}\,dz$ where C is $|z+1| = 1$

→

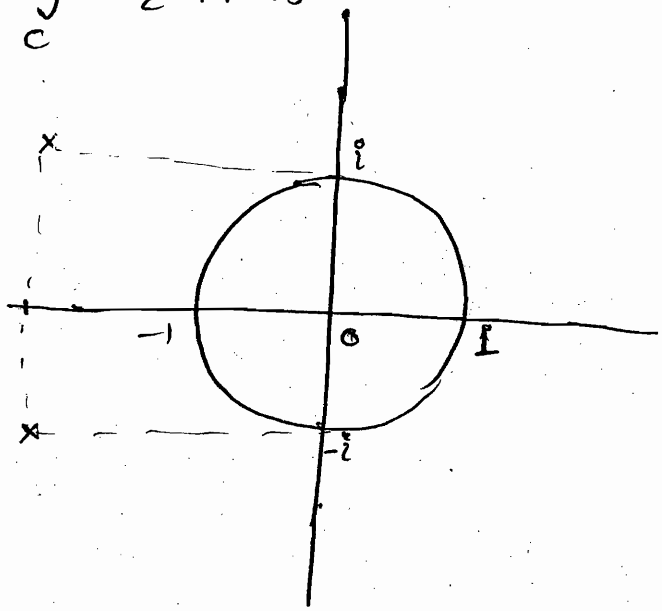


Pole ⇒ $z^2+2z+5=0$

$\Rightarrow \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$,

Pole $= -1+2i,\ -1-2i$ ⇒ outside the Curve ⇒ ∴ Ans $= 0$

Q. $\oint_C \frac{-3z+4}{z^2+4z+5}\, dz$ where c is $|z|=1$



$z^2+4z+5=0$

$\frac{-4 \pm \sqrt{16-20}}{2}$

$= \frac{-4 \pm 2i}{2}$

$= -2 \pm i$

↳ Pole outside the curve.

∴ Answer is '0'.

Cauchy's Integral formula

V.V.Imp 2 marks

If $f(z)$ is analytic for inside and on a simple and closed curve 'c' and if a is any point inside c then,

$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

$$\oint_C \frac{f(z)}{(z-a)^2}\, dz = 2\pi i\, f'(a)$$

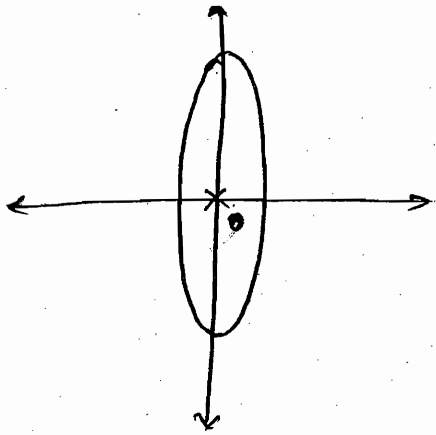
$$\oint_C \frac{f(z)}{(z-a)^3}\, dz = \frac{2\pi i}{2!} f''(a)$$

$$\boxed{\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{n}(a)}$$

Q. ① The value of $\oint_C \frac{\cos z}{z} dz$ where c is an ellipse $9x^2+4y^2=1$

→ $9x^2+4y^2=1$

$$\frac{x^2}{\left(\frac{1}{3}\right)^2}+\frac{y^2}{\left(\frac{1}{2}\right)^2}=1$$



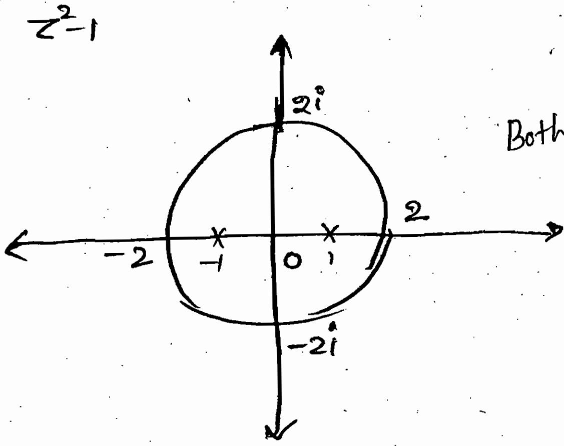
∴ Pole inside
∴ Cauchy Integral formula applicable

$$\therefore \oint \frac{\cos z}{z} dz = 2\pi i \cos z\Big|_{z=0}$$

$$= 2\pi i \text{ Ans}$$

Q. $\oint_C \frac{1}{z^2-1} dz$ where c is the circle $x^2+y^2=4$.

→



Both Pole inside

Partial fraction

Method – 1

Partial fraction –

$$\oint_C \frac{1}{(z-1)(z+1)}\,dz = \frac{1}{2}\oint\left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz$$

$$= \frac{1}{2}\left(\cancel{2\pi i}(1) - \cancel{2\pi i}(1)\right)$$

$$= \underline{\underline{0}}.$$

Observation :-

Sometimes even also pole inside then $\int = 0$

Method – 2.

Shortcut

$$\oint_C \frac{1}{z^2-1}\,dz = \oint_C \frac{1}{(z-1)(z+1)}\,dz.$$

$$= \oint \frac{\left(\frac{1}{z+1}\right)}{(z-1)} + \frac{\left(\frac{1}{z-1}\right)}{(z+1)}\,dz$$

$$= 2\pi i\left(\frac{1}{z+1}\right)'_{z=1} + 2\pi i\left(\frac{1}{z-1}\right)_{z=-1}$$

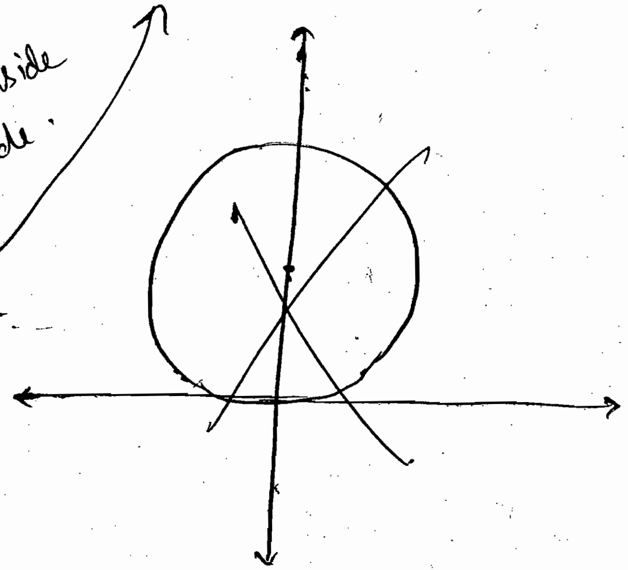
$$= \cancel{2\pi i}\left(\frac{1}{2}\right) + \cancel{2\pi i}\left(\frac{-1}{2}\right)$$

$$= \underline{0}.$$

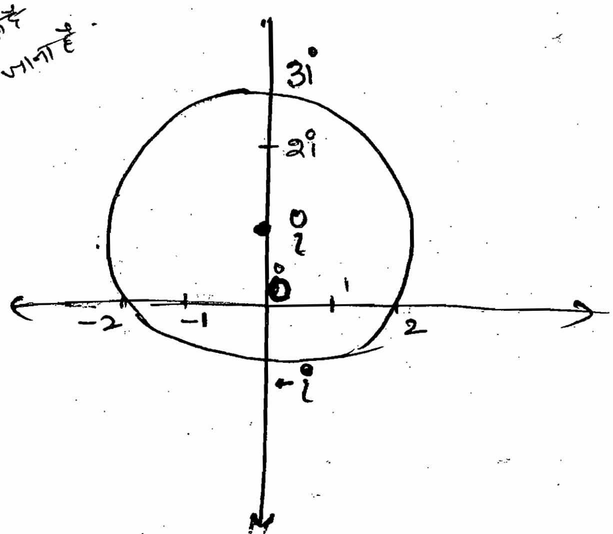
Q. Repeated 3 times

$$\oint_C \frac{z-1}{(z+1)^2(z-2)}\,dz \quad \text{where } C \text{ is } |z-i| = 2$$

If one Pole inside another outside then?



जो बाहर है उसको ऊपर ले जाते हैं
जो inside है उसको नीचे ले जाना है



$$\therefore \oint_C \frac{z-1}{(z+1)^2(z-2)}\,dz$$

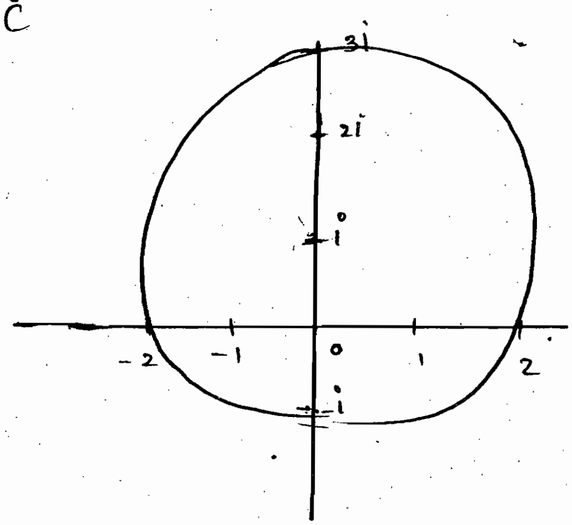
$$= \oint_C \frac{\overbrace{\left(\frac{z-1}{z-2}\right)}^{f(z)}}{(z+1)^2}\,dz \qquad f'(z) = \frac{-1}{(z-2)^2}$$

$$= 2\pi i \left[\frac{-1}{(z-2)^2}\right]_{z=-1}$$

$$= \frac{-2\pi i}{(-1-2)^2}$$

$$= \frac{-2\pi i}{9}$$

Q. $\oint_C \frac{z^2-4}{z^2+4}\,dz$ where C is the circle $|z-i|=2$



$$\cancel{\oint \frac{\frac{z^2-4}{z-2}}{z+2} + \frac{\frac{z^2-4}{z+2}}{z-2}}$$

$$= \cancel{2\pi i\left(4-\right.}$$

$$\oint \frac{z^2-4}{(z-2i)(z+2i)}\,dz$$

✓ ×

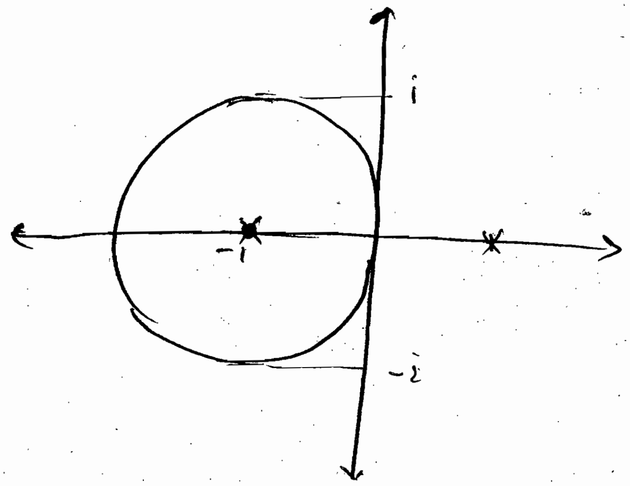
$$= \oint \frac{\frac{z^2-4}{z+2i}}{z-2i}\,dz$$

$$= 2\pi i\left(\frac{z^2-4}{z+2i}\right)_{z=2i}$$

$$= 2\pi i\left(\frac{-4-4}{2i+2i}\right) = 2\pi i\left(-\frac{8}{4i}\right)$$

$$= -4\pi.$$

Q. $\oint_c \frac{z^2}{z^4-1}\,dz \qquad c \text{ is } |z+1| = 1.$



$$\oint \frac{z^2}{(z^2-1)(z^2+1)} = \oint \frac{z^2}{\underbrace{(z-1)}_{\times}\underbrace{(z+1)}_{\checkmark}\underbrace{(z-i)(z+i)}_{\times\ \times}}$$

outside pole

{ inside नीचे रखो
outside वाला ऊपर ले जाओ

$$\oint \frac{\frac{z^2}{(z-1)(z^2+1)}}{z+1}\,dz.$$

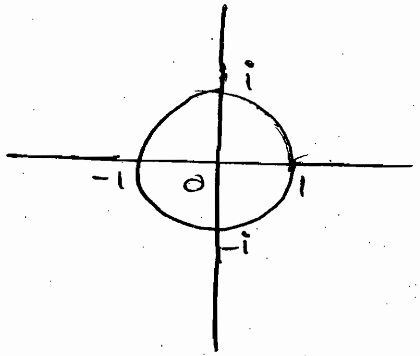
$$= 2\pi i \left(\frac{z^2}{(z-1)(z^2+1)}\right)_{z=-1}$$

$$= \cancel{2}\pi i \left(\frac{1}{-2\times\cancel{2}}\right)$$

$$= -\frac{\pi i}{2}$$

Q. $\oint \frac{5z^2+7z+3}{(z-1)^3}$ where c is $|z|=1$

→



$$\oint \frac{5z^2+7z+3}{(z-1)^3}.$$

$$f(z) = 5z^2+7z+3.$$
$$f'(z) = 10z+7.$$
$$f''(z) = 10.$$

$$\therefore \oint \frac{5z^2+7z+3}{(z-1)^3} = \frac{\cancel{2}\pi i}{\cancel{2}!}(10)$$
$$= 10\pi i$$

***

Residue at $z=a$ where a is the pole of order n.

$$\lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n \cdot f(z)\right].$$

Q. find residue of fn $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

at $z=0$, $\frac{1-0}{(0-1)(0-2)} = \frac{1}{2}$

at $z=1$, $\frac{1-2}{1\times -1} = 1$

at $z=2$, $\frac{1-4}{2\times 1} = -\frac{3}{2}$

Q① find residue of fn. $f(z) = \dfrac{z^2}{(z-1)(z+1)(z-2)}$

at $z=1$ $\quad \dfrac{1}{2\times -1} = -\dfrac{1}{2}$

$z=-1$ $\quad \dfrac{1}{-2\times -3} = \dfrac{1}{6}.$

$z=2,$ $\quad \dfrac{4}{1\times 3} = \dfrac{4}{3}$

Q.② $\quad f(z) = \dfrac{1}{z(z+2)^3}$

$$\frac{1}{z(z+2)^3} = \frac{A}{z} + \frac{a_0}{(z+2)^3} + \frac{a_1}{(z+2)^2} + \frac{a_2}{(z+2)}$$

$A \to \frac{1}{8}$, $a_0 \to -\frac{1}{2}$

for $(z+2)^3 \longrightarrow \lim_{z\to -2} \dfrac{1}{2!}\dfrac{d^2}{dz^2}\left[\cancel{(z+2)^3}\cdot\dfrac{1}{z\cancel{(z+2)^3}}\right]$

$$= \frac{1}{2}\frac{d^2}{dz}\left(\frac{1}{z}\right)$$

$$= \frac{1}{\cancel{2}}\times\frac{\cancel{2}}{z^3}$$

$$= \frac{1}{(-2)^3} = \frac{1}{8}$$

Workbook

① $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\Delta = 1$ $\qquad$ $\Delta = -1$

$\Delta = \pm 1$

③ $XY = Y$

$YX = X$

$X^2 + Y^2 = ?$

$= XX + YY$

$\cancel{= (YX)X + (XY)}$

$= (YX)(YX) + (XY)(XY)$

$= Y(XY) + X(XY)$

$= YX + XY$

$= X + Y$ .. Ans

⑦ ⓐ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Total 8

⑧ ⓑ $A = [a_{ij}]_{m \times n}$.

$a_{ij} = i + j$

$A_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

$\hookrightarrow$ 12 Check with options

(9) 4^4

3×4^4

(10) wrong Q.

(11) 10 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

(12) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(13) (b) $5! = \underline{\underline{120}}$

(16) a_{ij} $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$= \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}$ $\because \Delta = 0$

$\therefore$ inverse not possible.

(15)

(26) Correct $2x + 3y + z = \lambda z \longrightarrow 3^{rd}$

(27)

(38)

$$\begin{bmatrix} \cancel{\frac{1}{1\cdot 2}} & 0 & \cdots \\ & & \\ & & \end{bmatrix}$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \frac{1}{n(n+1)} \cdots$$

$$S_n = \sum T_n$$

$$= \sum \frac{1}{n(n+1)}$$

$$= \sum \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \quad \underline{\underline{A}}$$

(39) $1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdots \frac{1}{n}$

$$= \frac{1}{n!} \quad \underline{\underline{A}}$$

(40)

$$A = \begin{bmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{bmatrix}$$

$\lambda_3 = 0 \quad \because D = 0$

$\lambda_1, \lambda_2 = ?$

$$\lambda_1 + \lambda_2 + \cancel{\lambda_3} = 20$$

$$\lambda_2 = 20 - \lambda \cdot \quad \underline{\underline{A}}$$

(45) 3, 2, −1

$B = A^2 - A$.

$= 9 - 3 = 6$.

$= 2^2 - 2 = 2$

$= (-1)^2 - (-1) = 2$.

$|B| = 6 \times 2 \times 2 = 24$.

(46)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$$

Trace ?

Trace is Σ principal digonals

$$(A)^{102} = (1)^{102}, \left(\frac{-1+i\sqrt{3}}{2}\right)^{102}, \left(\frac{-1-i\sqrt{3}}{2}\right)^{102}$$

$$= 1, \left(e^{-\frac{i\pi}{3}}\right)^{102}, \left(e^{\frac{i\pi}{3}}\right)^{102}$$

$$= 1, e^{-34\pi i}, e^{34\pi i}$$

$$= 1 + 1 + 1 = \underline{\underline{3}}$$

(48) $\lambda = -1, 1, 0$

$|A^{100} + I|$

2, 2, 1

$|A^{100} + I| = 2 \times 2 \times 1 = \underline{\underline{4}}$

(49) $\lambda_1 = 1$, $\lambda_2 = 4$

$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a) $\begin{bmatrix} -4 & -8 \\ 5 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 9 & -8 \\ 5 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

(51) $A = 1, -1, \lambda$

$A + 3I = 1+3, -1+3, \lambda+3.$

$A + 3I = 4, 2, \lambda + 3.$

$A^2 + 3I = 1(4), -1(2), \lambda(\lambda+3).$

$= 4, -2, \underbrace{\lambda^2 + 3\lambda}_{=18}.$

$\lambda^2 + 3\lambda = 18$

$\hookrightarrow \lambda \neq 0$

$\therefore \Delta \neq 0$

$\therefore$ Inverse exists.

Similarly inverse exists for A

Correction → 2×2 not 3×3

(62) $A^{-1} = I - 2A$.

$I = A - 2A^2$

$1 = \lambda - 2\lambda^2$

$2\lambda^2 - \lambda + 1 = 0$

$\Delta = \lambda_1 \lambda_2$

$= \frac{1}{2}$ ✓

(63) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a + d = ad - bc = 1$

then $A^3 = ?$

$\rightarrow \quad \lambda_1 + \lambda_2 = 1$

$\lambda_1 \lambda_2 = 1$

$\lambda_1 + \frac{1}{\lambda_1} = 1$

$\cancel{\lambda_1^2 + 1 - \lambda_1 = 0}$

$\cancel{\lambda_1^2 - \lambda_1 + 1 = 0}$

$\cancel{\lambda_1 =}$

$\lambda_1^2 + 1 = \lambda_1$

$A^2 = A - I$

$A^3 = A^2 - A$

$= (A - I)A$

$= A^2 - A$

$= \cancel{A} - I - \cancel{A}$

$= -I$

Laplace Transform:

→ Let $f(t)$ be any fn. of t defined $\forall$ positive values of T, then the ~~ta~~ Laplace Transform of $f(t)$ is denoted by $\mathcal{L}[f(t)]$ and is defined as —

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = F(s)$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}[F(s)]$$

is called the Inverse Laplace Transform of $F(s)$.

① $\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}\,dt$

Let $x = st \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{dx}{s}$

$t \to \infty \Rightarrow x \to \infty$
$t \to 0 \Rightarrow x \to 0$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \times \frac{dx}{s}$$

$$= \int_0^\infty e^{-x}\,\frac{x^n}{s^n}\cdot\frac{dx}{s}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\,dx$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^{\overline{n+1-1}}\,dx$$

$$\therefore \quad = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\therefore \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \text{ if } n \in N$$

$$\mathcal{L}[t^n] = \frac{n\Gamma n}{s^{n+1}}, \text{ if } n > 0.$$

$\therefore$ $L(1) = \frac{1}{s}$

~~$L(t) = \frac{1}{s^2}$~~

L

Improper Integral.

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$$

$\Gamma(n+1) = n\Gamma n$, if $n > 0$.

$\Gamma(n+1) = n!$, if $n \in N$

$\Gamma 1 = 1$

$\Gamma \frac{1}{2} = \sqrt{\pi}$

$$L(1) = \frac{1}{s}$$

$$L(t) = \frac{1}{s^2}$$

$$L(t^2) = \frac{2}{s^3}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(e^{-at}) = \frac{1}{s+a}$$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

$$L(\cos at) = \frac{s}{s^2+a^2}.$$

$$L(\sinh at) = \frac{a}{s^2-a^2}$$

$$L(\cosh at) = \frac{s}{s^2-a^2}..$$

Q. find $L(t^2+6t+8)$, $L(\sqrt{t})$, $L\left(\frac{1}{\sqrt{t}}\right)$, $L(\sin^2 t)$, $L(\sin^3 t)$, $L(\sin 2t \cdot \cos t)$.

& $f(t) = \begin{cases} 0, & 0<t<2 \\ 3, & t \geq 2 \end{cases}$

find $L[f(t)]$..

→ ① $L(t^2+6t+8) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{8}{s}$

② $L(t^{1/2}) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ +ve no. $= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

③ $L(t^{-\frac{1}{2}}) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$

④ $L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2s} - \frac{1}{2}\frac{s}{(s^2+4)} = \frac{1}{2s} - \frac{s}{2(s^2+4)}$

$\sin 3t = 3\sin t - 4\sin^3 t$

⑤ $L(\sin^3 t) = L\left(\frac{3\sin t - \sin 3t}{4}\right) = \frac{1}{4}\left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right] = \frac{3}{4}\left[\frac{8}{(s^2+1)(s^2+9)}\right]$

$$= \frac{6}{(s^2+1)(s^2+9)}$$

⑥ $L(\sin 2t \cdot \cos t) = \frac{1}{2} L[\sin 3t + \sin t]$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$$

⑦ $f(t) = \begin{cases} 0, & 0<t<2 \\ 3, & t \geq 2 \end{cases}$

$$L[f(t)] = \int_0^\infty f(t) e^{-st}\, dt$$

$$= \int_0^2 0\, dt + \int_2^\infty 3e^{-st}\, dt$$

$$= 3 \left.\frac{e^{-st}}{-s}\right|_2^\infty$$

$$= \frac{3}{-s}\left[0 - e^{-2s}\right]$$

$$= \frac{3e^{-2s}}{s}$$

Shifting Theorem:-

If $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

then $L[e^{at} f(t)] = F(s-a)$

$$L\cdot H\cdot S = L[e^{at} f(t)] = \int_0^\infty f(t)\, e^{-t(s-a)}\, dt$$

$$= F(s-a)$$

Formulae:-

$L(t^n) = \frac{n!}{s^{n+1}}$

$L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$

$L[e^{-at} t^n] = \frac{n!}{(s+a)^{n+1}}$

$L[e^{at} \sin bt] = \frac{b}{(s-a)^2+b^2}$

$L[e^{at} \cdot \cos bt] = \frac{(s-a)}{(s-a)^2+b^2}$

$L[e^{-at} \sin bt] = \frac{b}{(s+a)^2+b^2}$

$L[e^{-at} \cos bt] = \frac{s+a}{(s+a)^2+b^2}$

$L[e^{at} \sinh bt] = \frac{b}{(s-a)^2-b^2}$

$L[e^{-at} \sinh bt] = \frac{b}{(s+a)^2-b^2}$

$L(t \sin at)$

$= I.P \text{ of } L(t e^{ait})$ $\qquad L(t) = \frac{1}{s^2}$

$= I.P \text{ of } \left[\frac{1}{(s-ai)^2} \times \frac{(s+ai)^2}{(s+ai)^2} \right]$

$= I.P \text{ of } \left[\frac{s^2-a^2+2ias}{(s^2+a^2)^2} \right]$

$L(t \sin at) = \frac{2as}{(s^2+a^2)^2}$ $\qquad L(t \sin t) = \frac{2s}{(s^2+1)^2}$

$L(t \cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$ $\qquad L(t \cos t) = \frac{s^2-1}{(s^2+1)^2}$

4 marks Q.

Inverse Laplace Transform :-

$L(t^n) = \frac{n!}{s^{n+1}}$

$L\left(\frac{t^n}{n!}\right) = \frac{1}{s^{n+1}}$

$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

$\therefore L^{-1}\left[\frac{1}{s}\right] = 1$

$L^{-1}\left[\frac{1}{s^2}\right] = t$

$L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2}$

$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$

$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a} \sinh at$

$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

Q. Find $L^{-1}\left[\frac{1}{2s-5}\right]$, $L^{-1}\left[\frac{1}{s(s+1)}\right]$, $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

$L^{-1}\left[\frac{s}{(s+a)^2}\right]$, $L^{-1}\left[\frac{s+23}{s^2 4s+13}\right]$, $L^{-1}\left[\frac{s-2}{s^2+2s+2}\right]$

$\rightarrow$ ① $L^{-1}\left[\frac{1}{2s-5}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s-\frac{5}{2}}\right] = \frac{1}{2}e^{\frac{5}{2}t}$

② $L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}$

③ $L^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t}$

④ $L^{-1}\left(\frac{s}{(s+a)^2}\right] = L^{-1}\left[\frac{s+a-a}{(s+a)^2}\right] = e^{-at} - ate^{-at}$

$= e^{-at}(1-at)$

⑤ $L^{-1}\left[\frac{s+23}{s^2-4s+13}\right]$

$= L^{-1}\left[\frac{s+23}{s^2-2\times 2s+4+13-4}\right]$

$= L^{-1}\left[\frac{s-2+25}{(s-2)^2+3^2}\right]$

$= e^{2t}\cos 3t + \frac{25}{3}e^{2t}\sin 3t$

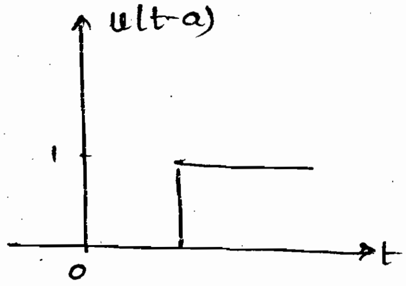
⑥ $L^{-1}\left[\frac{s-2}{s^2+2\times s\times 1+1+2-1}\right]$

$= L^{-1}\left[\frac{s+1-3}{(s+1)^2+1}\right]$

$= e^{-t}\cos t - 3e^{-t}\sin t$

$= e^{-t}[\cos t - 3\sin t]$

Unit Step function :-



The unit step function $u(t-a)$ is defined by —

$$u(t-a) = \begin{cases} 0, & 0 < t < a \\ 1, & t \geq a \end{cases}$$

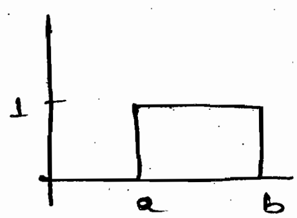
then $\mathcal{L}[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt$

$$= \int_0^a 0\, dt + \int_a^\infty 1\, e^{-st} dt$$

$$= -\frac{1}{s}\left[e^{-\infty s} - e^{-as}\right]$$

$$\boxed{\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}}$$

①



$= u(t-a) - u(t-b)$

$= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{1}{s}\left[e^{-as} - e^{-bs}\right]$.

Periodic function:-

→ A fn. $f(t)$ is said to be periodic with the period 'T', if $f(t+T) = f(t)$, where T is the least +ve number.

ex:- $f(t) = \sin t$

$$f(t+2\pi) = \sin(t+2\pi)$$
$$= \sin t.$$
$$f(t+2\pi) = f(t)$$

Laplace Transform of Periodic function:-

Let $f(t)$ be a periodic fn with period T, then the laplace Transform of the periodic fn is defined by —

$$\boxed{L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt}$$

ex:- $f(t) = \begin{cases} 3, & 0<t<2 \\ 0, & 2<t<4 \end{cases}$

and $f(t+4) = f(t)$, then find $L[f(t)]$.

→ $$L[f(t)] = \frac{1}{1-e^{-s\cancel{T}4}} \left[\int_0^2 e^{-st} \times 3\, dt + \int_3^{\infty} 0 \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\frac{-3}{s} \left[e^{-2t} - 1 \right] \right]$$

$$= \frac{3\,(\cancel{1-e^{-2s}})}{s\,\cancel{(1-e^{-4s})}} \rightarrow (\cancel{1-e^{-2s}})(1+e^{-2s})$$

$$= \frac{3}{s(1+e^{-2s})}$$

Q. $f(t) = \begin{cases} 1, & 0<t<b \\ -1, & b<t<2b \end{cases}$, $f(t+2b) = f(t)$, find $L[f(t)]$

Imp Result

$$\rightarrow \quad L[f(t)] = \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}\,dt - \int_b^{2b} e^{-st}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1-e^{-bs}}{s} + \frac{e^{-2bs} - e^{-bs}}{s}\right]$$

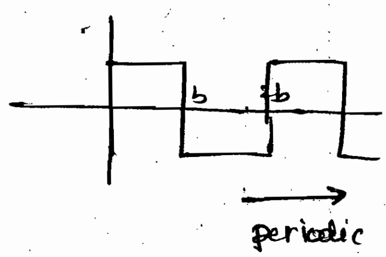
$$= \frac{1}{\left[1-e^{-2bs}\right]} \cdot \frac{1-e^{-bs} + e^{-2bs} - e^{-bs}}{s}$$

$$= \frac{1}{\cancel{(1-e^{-bs})}(1+e^{-bs})}\left[\frac{\overbrace{1-2e^{-bs}+e^{-2bs}}^{(1-e^{-bs})^2}}{s}\right]$$

$$= \frac{1-e^{-bs}}{s(1+e^{-bs})}$$

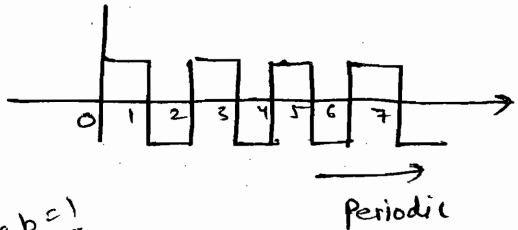
Remember i.e,

Imp Result



$$\longleftrightarrow \quad \frac{1-e^{-bs}}{s(1+e^{-bs})}$$

ex:-



$\mathcal{L}[f(t)] = ?$

Here b = 1

$$\mathcal{L}[f(t)] = \frac{1 - e^{-s}}{s(1+e^{-s})}$$

Q.

$$L[f(t)] = F(s) = \frac{5s^2 + 23s + 6}{s(s^2+2s+2)}$$

then $\lim_{t \to \infty} f(t) =$ ______

(a) 3 (b) 5 (c) $\frac{17}{2}$ (d) ∞.

Q. $f(t) = L^{-1}\left[\dfrac{3s+1}{s^3 + 4s^2 + (K-3)s}\right]$

$\lim_{t \to \infty} f(t) = 1$

then K = ______

→ A/c to F.V.T — $\frac{1}{K-3} = 1 \Rightarrow K = 4$

Q. The Laplace Transform —

$$L[f(t)] = \frac{1}{s^2(s+1)}$$

$f(t) = ?$

$$\rightarrow \frac{1}{s^2(s+1)} = \frac{\cancel{a_0}\,1}{s^2} + \frac{a_1\,(-1)}{s} + \frac{\cancel{A}\,1}{s+1}$$

$$\cancel{\tfrac{1}{2}} = 1 + a_1 + \cancel{\tfrac{1}{2}}$$

$a_1 = -1$

$= t - 1 + e^{-t}$

Probability

→ If an experiment is conducted under essentially given conditions upto n times, Let m cases are favourable to an event E then the Probability of E is denoted by $P(E)$ and is defined by –

$$\boxed{P(E) = \frac{m}{n}}$$

$$P(\bar{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E).$$

$$\therefore \boxed{P(E) + P(\bar{E}) = 1}$$

Sum of success & failure is always 1.

Axioms of Probability :-

① Axiom of Positivity :- $0 \le P(E) \le 1$.

② Certainity :- $P(S) = 1$.

③ Union :- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Note:- If E_1 and E_2 are mutually Exclusive or Disjoint events i.e, $E_1 \cap E_2 = \phi$

then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Sample Space :-

The set of all possible outcomes in an experiment is said to be the sample space.

2 coins tossed $S = \{HH, HT, TH, TT\}$.

1 dice thrown $S = \{1, 2, 3, 4, 5, 6\}$.

2 dice thrown $S = \{(1,1)(1,2)(1,3) \cdots (6,6)\} \longrightarrow$ Total 36

Spade club diamond Heart

♤ ♧ ♢ ♡

A
2
3
⋮
10
J
Q
K

$\underline{\underline{13}}$

Conditional Probability :-

Let S be the sample space, and Event $E_1, E_2 \subset S$ then the conditional Probability of E_1 after the occurance of E_2 is denoted by—

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}, \quad (P(E_2) \neq 0)$$

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}, \quad (P(E_1) \neq 0).$$

Random Variable :-

Mean :- $\Sigma x_i P(x = x_i)$ if exists, it is called the mean of R.V.

$$E = \mu = \Sigma x_i P(x = x_i).$$

Variance of R.V $\Sigma (x_i - \mu)^2 P(x = x_i)$ if exists, it is called the variance of R.V

$$\sigma^2 = \Sigma (x_i - \mu)^2 P(x = x_i)$$

$$= \Sigma (x_i^2 + \mu^2 - 2x_i\mu)\, P(x = x_i)$$

$$= \Sigma x_i^2 P(x = x_i) + \Sigma \mu^2 \cdot P(x = x_i) - \Sigma 2x_i \mu P(x = x_i)$$

$$= \Sigma x_i^2 P(x = x_i) + \mu^2 \underbrace{\Sigma P(x = x_i)}_{1} - 2\mu \underbrace{\Sigma x_i P(x = x_i)}_{\mu}$$

$$= \Sigma x_i^2 [P(x = x_i)] + \mu^2 - 2\mu^2$$

$$\boxed{\sigma^2 = \Sigma x_i^2 P(x = x_i) - \mu^2}$$

③ Standard Deviation:-

The standard deviation is the sq. root of variance and is denoted by σ.

ABOVE WERE FOR GENERAL DISTRIBUTION :

Binomial Distribution:-

n = no. of trials
p = prob. of success.
q = prob of failures.

$$\boxed{p+q=1}$$

$$\boxed{P(X=k) = {}^nC_k \, q^{n-k} p^k}$$

$$\boxed{\begin{aligned} \text{Mean} &= n \cdot p \\ \text{Variance} &= npq \\ S.D &= \sqrt{npq} \end{aligned}}$$

If $p=q$

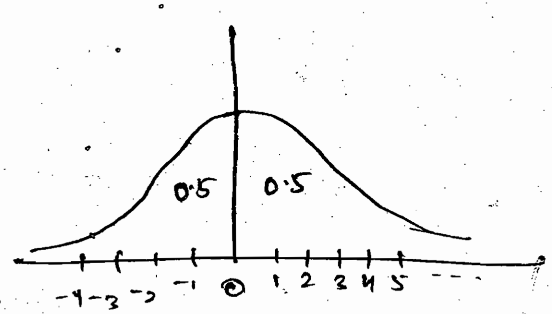
$$\boxed{P(X=K) = {}^nC_K \, p^K}$$

Poisson Distribution:-

$$P(X=K) = \frac{\lambda^K}{K!} e^{-\lambda}$$, where λ is a parameter.

$$\boxed{\begin{aligned} \text{Mean} &= \lambda \\ \text{Variance} &= \lambda \\ S.D &= \sqrt{\lambda} \end{aligned}}$$

Normal Distribution:-



$$\boxed{z = \frac{x-\mu}{\sigma}}$$

μ = mean.
σ = S.D

Q. What is the chance that a Leap year will have 53 Sundays.

→ $52 \times 7 = 364$

for rest 2 days —

S M →
M T
T W
W Th
Th F
F Sat
Sat Sun →

$\left.\right\} \frac{2}{7}$

for Non leap year — $P(53 \text{ Sunday}) = \frac{1}{7}$

Q. from a deck of playing Cards - 2 Cards are drawn at Random what will be the probab. that both Cards will be kings?

if -

①. the first Card is not replaced

② Replacement is allowed.

→ ① $\frac{^4C_1}{^{52}C_1} \times \frac{3}{51}$

$= \frac{\cancel{4}}{\cancel{52}_{13}} \times \frac{\cancel{3}}{\cancel{51}_{17}} = \frac{1}{221}$.

② $\frac{\cancel{4}}{\cancel{52}_{13}} \times \frac{\cancel{4}}{\cancel{52}_{13}} = \frac{1}{169}$

Q. A Bag Contain 3 red 6 white and 7 black balls, what is the Prob that 2 balls drawn are white and Black.

→ $\boxed{\begin{matrix} 6 & 3 & 7 \\ W & R & B \end{matrix}}$ $\frac{^6C_1 \times {^7C_1}}{^{16}C_2} = \frac{6 \times 7}{\frac{16 \times 15}{2}} = \frac{7}{20}$.

Q. A problem in electronics is given to 3 students A, B, C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. What is the Prob. that the prob. will be solved.

→ $P(A) = \frac{1}{2}$

$P(B) = \frac{1}{3}$

$P(C) = \frac{1}{4}$

indivisual

then

$1 - P(\bar{A} \cap \bar{B} \cap \bar{C}) \rightarrow P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$

$= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})$

$= 1 - \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{4}$

$= \frac{3}{4}$

$=$ 75% chances.

Q. A husband & wife appear in interview for 2 vacancies for the same post. The prob. of husband selection — $\frac{1}{7}$ & that of wife selection is $\frac{1}{5}$.

What is the Prob. that — ① both are selected.

② only one of them is selected

③ None of them is selected.

→ $P(A) = \frac{1}{7}$

$P(B) = \frac{1}{5}$

① $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$.

②

$P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= \frac{1}{7} \times \frac{4}{5} + \frac{6}{7} \times \frac{1}{5}$

$= \frac{\cancel{10}\, 2}{\cancel{35}\, 7} = \frac{2}{7}$

③ $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}) \rightarrow$ Don't use $1 - \frac{1}{35}$ Coz indivisual selection नहीं होता है

$= \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$

Q. An integer is selected at random ~~from~~ from the first 200 digits What is the prob. that the integer choosen is devisible by either 6 or 8 ?

→ 1, 2, 3, ... 200

$\div 6 \rightarrow 6, 12, 18 \ldots 198$

$\div 8 \rightarrow 8, 16, 24 \ldots 200.$

$\div 6 \,\&\, \div 8 \rightarrow 24, 48 \ldots, 192.$

no. of integers $\div 6 = \frac{\cancel{198}^{33}}{\cancel{6}} = 33 = n(E_1)$

no. of integer $\div 8 = \frac{\cancel{200}^{25}}{\cancel{8}} = 25 = n(E_2)$

no. of integers $\div 6 \,\&\, \div 8 = \frac{\cancel{192}^{8}}{\cancel{24}} = 8 = n(E_1 \cap E_2)$

$$n(E_1 \cup E_2) = n(E_1) + n(E_2) - n(E_1 \cap E_2)$$
$$= 33 + 25 - 8$$
$$= 50.$$

$\therefore P(E_1 \cup E_2) = \frac{\cancel{50}}{\cancel{200}_4} = \frac{1}{4}$

Q. A hits a target 4 times in 5 shots → B → 3 times in 4 shots

C → 2 in 3 shots.

The fire a volley. What is Prob. that 2 shots at least hit the target?

volley means A, B, C then again ABC -- continuous

→ A → 4 in 5

B → 3 in 4

C → 2 in 3

for at least 2 shots —

$P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$

$= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} + \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$

$= \frac{12}{60} + \frac{8}{60} + \frac{6}{60} + \frac{24}{60} = \frac{\cancel{50}}{\cancel{60}} = \frac{5}{6}$

For at least 1 shot → $1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

Q In a college 25% of student fail in Math, 15% failed in Chemistry, 10% failed in both Maths and chemistry if a student is selected at random —

① If he failed in chem. then what is the probablity that he fails in Math

② vice-versa

③ What is the Prob. that he failed neither Mathematics, nor chemistry?

→

→ $P(M) = 0.25$

$P(C) = 0.15$

$P(M \cap C) = 0.10$

① $P\left(\frac{M}{C}\right) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{2}{3}.$

② $P\left(\frac{C}{M}\right) = \frac{P(C \cap M)}{P(M)} = \frac{0.10}{0.25} = \frac{2}{5}$

③ $P(\overline{M \cup C}) = 1 - P(M \cup C)$

$= 1 - [P(M) + P(C) - P(M \cap C)]$

$= 1 - [0.25 + 0.15 - 0.10]$

$= 1 - 0.3$

$= 0.7.$

Q. A single dice is thrown twice, what is the probability that their sum is neither 8 nor ~~nor~~ 9.

→ ~~P(~~ $n(S) = 36$

$n(E_1) = \{(2,6)(3,5),(4,4)(5,3),(6,2)\}$

$\therefore P(E_1) = 5/36.$

$n(E_2) = \{(3,6),(6,3),(5,4)(4,5)\}$

$P(E_2) = \frac{4}{36}.$

$\because$ These are mutually Exclusive → eg. 8 आने से 9 नहीं आएगा
9 आने से 8 नहीं आएगा

$\therefore P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2)$

$= 1 - [P(E_1) + P(E_2)]$

$= 1 - \left[\frac{5}{36} + \frac{4}{36}\right]$

$= \frac{\cancel{27}\,3}{\cancel{36}\,4} = \frac{3}{4}$ //.

Q. A pair of fair dice is thrown, if the two numbers appearing are different, find the Prob that –

1. The sum is 6
2. an ace appears
3. The sum is 4 or less

→ $n(S) = 36 - \underbrace{\cancel{6}}_{\text{Same no appearing}} = 30$

for Sum is 6 –

$n(E_1) = \{(1,5),(2,4),(5,1),(4,2)\} \Rightarrow P(E_1) = \frac{\cancel{4}\,2}{\cancel{30}\,15} = \frac{2}{15}$

for Ace Appears — $P(E) = 0$ if option available

But Here (Ace means 1)

$n(E_2) = \{(1,2),(1,3),(1,4),(1,5),(1,6), .. \times 2\} = 10$

$\therefore P(E_2) = \frac{\cancel{10}}{\cancel{30}} = \frac{1}{3}$

③ The sum is 4 or less

$n(E) = \{(1,2)\ (1,3),\ (2,1)(3,1)\} = 4$

$P(E) = \frac{\cancel{4}\,2}{\cancel{30}\,15} = \frac{2}{15}$

Binomial Type-2

Q. Team A has probability $\frac{2}{3}$ of winning whenever it plays. Suppose A plays 4 games. find the prob. that A wins more than half of its games.

$\rightarrow$ Team A $\rightarrow \frac{2}{3}$.

$n = 4$

$p = \frac{2}{3}$

$q = \frac{1}{3}$

$P(X>2) = P(X=3) + P(X=4)$

$= \cancel{{}^{n}C_{k}}\ \cancel{q^{n-k}}\ \cancel{p^{k}}$

$= {}^4C_3 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 + {}^4C_4 \left(\frac{2}{3}\right)^4$

$= \frac{4\times 8}{81} + \frac{1\times 16}{81} = \frac{48}{81} = \frac{16}{27}$.

<u>How to know Binomial?</u>

n choose करने के बाद k

Given not imp

asked imp, a/c to that p is selected

Q. In a bank 60% of all customers applied for a loan are rejected. If 4 new loans applications recieved. find the prob that 3 of them are accepted.

$\rightarrow$ $n = 4$

$p = 0.4$

$q = 0.6$

$P(X=3) = {}^4C_3 (0.6)^1 (0.4)^3$

$= 4\,(0.6)(0.4)^3$

$= 0.1536$.

Q. A player tosses 2 fair coins. He wins \$2 if 2 heads occurs and wins \$1 if 1 head occurs. On the other hand he looses \$3, if no head occurs. find the expected value 'E' of the game.

→ \$2 → 2H

\$1 → 1H

− \$3 → 0H

fair game means — no loss no gain //

fairness of game ∝ mean value of game.

if mean = +ve → favourable to player

= −ve → Unfavourable to player.

Mean = 0 → fair Game.

$S = \{(H,H),(H,T),(T,H),(TT)\}$.

x_i	\$2	\$1	−\$3
P_i	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$E(X) = \sum x_i \, P(X = x_i).$$

$$= 2\left(\frac{1}{4}\right) + 1\times\left(\frac{2}{4}\right) - 3\left(\frac{1}{4}\right)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{3}{4}$$

$$= \frac{1}{4} > 0 \longrightarrow \text{favourable to player.}$$

Q. 4 fair coins are tossed simultaneously. find the prob. that at least one head and one tail turn up.

→ $n = 4$

$p = 1/2$

$$P(X=1) + P(X=2) + P(X=3)$$

$$= \left({}^4C_1 + {}^4C_2 + {}^4C_3\right)\left(\frac{1}{2}\right)$$

$$= \frac{4+6+4}{16} = \frac{\cancel{14}^{7}}{\cancel{16}_{8}} = \frac{7}{8}$$

$$= 1 - \left[P(X=4) + P(X=0)\right]$$

$$= 1 - \left[\frac{1+1}{16}\right]$$

$$= 1 - \frac{1}{8} = \frac{7}{8}$$

Q. Five fair coins are tossed simultaneously. Find the prob. that at least one head turn up.

→ $1 - P(X=0)$

$= 1 - {}^5C_0 \left(\frac{1}{2}\right)^5$

$= 1 - \frac{1}{32} = \frac{31}{32}$

Q. 4 fair coins are tossed simultaneously. Find the Prob. of event the no. of times heads show up is more than the no. of times Tails show up.

→ $P(X=3) + P(X=4)$

$= {}^4C_3 + {}^4C_4 + \left(\frac{1}{2}\right)^4$

$= \frac{4+1}{16} = \frac{5}{16}$

Q. There are 5 duplicate and 10 original items in an automobile shop. 3 items are bought up by customer at random. Find the Prob that none of the item is duplicate.

→ Means all original.

$= \frac{{}^{10}C_3}{{}^{15}C_3} = \frac{10 \times 9 \times 8}{15 \times 14 \times 13} = \frac{24}{91}$

Q. Out of 800 families with 4 children each how many families would be expected to have —

① 2 boys and 2 girls.

② At least one boy

③ At most 2 girls.

→ for 1 family —

$n = 4$

$p = q = \frac{1}{2}$

Now for 800 family

$\therefore P(X=2) = {}^4C_2 \left(\frac{1}{2}\right)^4 = 6 \times \frac{1}{16} \times 800 = 300$

Poission distribution
बहुत ज्यादा distribution
n is large (very large),
But discrimination
individual नहीं है
∴ Not Applicable

② $P(X \geq 1) = 1 - P(X=0)$

$$= 1 - {}^4C_0 \left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{16}$$

$$= \frac{15}{16} \times 800$$

$$= 750$$

③ $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= \left({}^4C_0 + {}^4C_1 + {}^4C_2\right) \left(\frac{1}{2}\right)^4$$

$$= \frac{1+4+6}{16}$$

$$= \frac{11}{16} \times 800$$

$$= 550.$$

Q. A book of 300 pages containing 30 printing mistakes. Assuming that these errors are randomly distributed throughout the book and X is the no. of errors per page has a poission's distribution. what is the prob. that 20 pages selected at random will be free of erross.

→ error in page $P = \frac{30}{300} = \frac{1}{10}$

Given $n = 20$

$\lambda = np$

$= 20 \times \frac{1}{10}$

$\boxed{\lambda = 2}$

$P(X=0) = \frac{\lambda^0}{0!} e^{-2} = e^{-2} = 0.135$

Imp
When in Poission distribution λ is not mentioned. we will find it by Binomial help. i.e $\boxed{\lambda = np}$

$= 20 \times \frac{1}{10} = 2$

$\therefore \boxed{\lambda = 2}$

Q. 1000 Students had written an examination with the mean of test = 35 ⟶ and S.D = 5. Assuming the distribution to be normal. find how many student's marks —

ⓐ lie between 25 & 40.

ⓑ How many got more than 40.

ⓒ How many get below 20.

Given $P(0 \le z \le 1) = 0.3415$

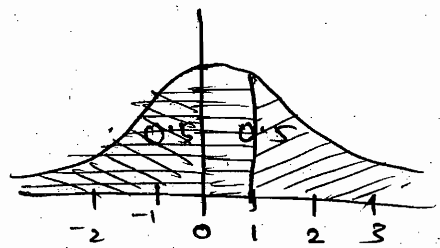
$P(0 \le z \le 2) = 0.4772$

$P(0 \le z \le 3) = 0.449$

→ $\mu = 35$

$\sigma = 5.$

$z = \dfrac{x-\mu}{\sigma}$



$P(\underset{x_1}{25} \le x \le \underset{x_2}{40})$

for $x_1 = 25$ — $z_1 = \dfrac{25-35}{5} = -2$

for $x_2 = 40$ — $z_2 = \dfrac{40-35}{5} = 1.$

$\therefore P(25 \le x \le 40) = P(-2 \le z \le 1)$

$= P(-2 \le z \le 0) + P(0 \le z \le 1)$

$= P(0 \le z \le 2) + P(0 \le z \le 1)$

$= 0.4772 + 0.3415$

$= 0.8187$

$= 0.819 \times \underline{1000}$ Total Students

$= 819.$

$P(x \ge 40) = P(z \ge 1)$

$= 0.5 - P(0 \le z \le 1)$

$= 0.5 - 0.3415$

$= 0.1585$

$= 0.159 \times 1000$

$= \underline{159}$

© $P(X \leq 20)$

$$z = \frac{20-35}{5} = \frac{-15}{5} = -3.$$

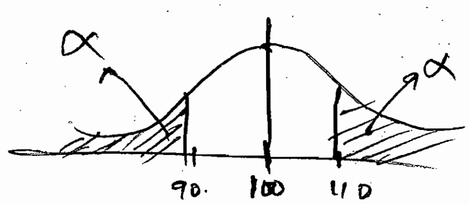
$P(z<-3) = P(X < 20) = 0.5 - P(0 \leq z \leq 3)$

$= 0.5 - 0.499$

$= 0.001 \times 1000$

$= 1$ student.

Q. for a Random Variable —
$(-\infty < x < \infty)$ following a normal distribution with the mean is 100. If the prob. of $P(X \geq 110) = \alpha$ then $P(90 \leq x \leq 110) =$ ____

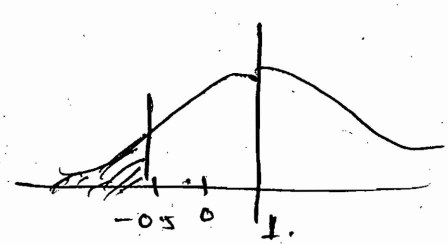
→ 

$P(X \geq 140) = \alpha$

$P(90 \leq x \leq 110) = 1 - 2\alpha.$

Q. Let x be the normal variant with Mean $= 1$ & $\sigma^2 = 4$ then the Prob. of $P(X<0)$ —

ⓐ 0.5 ⓑ >0 and <0.5 ⓒ >0.5 and <1 ⓓ 1.

→ 

$\mu = 1$

$\sigma^2 = 4$

$\sigma = 2$

$$z = \frac{x-\mu}{\sigma} = \frac{-1}{2} = -0.5$$

$P(X<0)$

$= P(Z < -0.5)$

$= 0.5 - P(0 \leq 0.5)$

Q. A box contain 3 blue and 4 red ball. Another identical Box contain 2 blue ball and 5 red ball. One ball is selected at Random from one of the two boxes and it is red. The Prob. that it came from the first box —

→

R-4	R-5
B-3	B-2
E_1	E_2

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P\left(\frac{A}{E_1}\right) = \frac{4}{7}, \quad P\left(\frac{A}{E_2}\right) = \frac{5}{7}$$

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)\cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\left(\frac{4}{7}\right)}{\frac{1}{2}\left(\frac{4}{7}\right)+\frac{1}{2}\left(\frac{5}{7}\right)}$$

$$= \frac{\frac{4}{7}}{9/7} = \frac{4}{9}.$$

Q. Box P has 2 Red & 3 blue balls. Box Q has 3 Red & 1 Blue ball. The Prob. of selecting boxes P and Q are $\frac{1}{3}$ and $\frac{2}{3}$ respectively. One ball is selected from one of two boxes and it is red. the Prob that it came from box P.

→ $P(P) = \frac{1}{3}$, $P(Q) = \frac{2}{3}$, $P\left(\frac{A}{P}\right) = \frac{2}{5}$, $P\left(\frac{A}{Q}\right) = \frac{3}{4}$.

$$P\left(\frac{P}{A}\right) = \frac{P(P)\cdot P\left(\frac{A}{P}\right)}{P(P)\cdot P\left(\frac{A}{P}\right) + P(Q)\cdot P\left(\frac{A}{Q}\right)} = \frac{4}{19}$$

Q. The Probability density fn of the form $P(x) = k e^{-\alpha|x|}$, $x \in (-\infty, \infty)$, then the value of k —

→ $\int_{-\infty}^{\infty} P(x) = 1 \Rightarrow \int_{-\infty}^{\infty} k\, e^{-\alpha|x|} dx = 1 \Rightarrow 2\int_0^{\infty} k e^{-\alpha x} dx = 1.$

$$\Rightarrow \frac{2k}{\alpha}\left[e^{-\infty} - e^{-0}\right] = 1$$

$$\Rightarrow -\frac{2k}{\alpha}(0-1) = 1$$

$$\Rightarrow k = \frac{\alpha}{2} = 0.5\alpha.$$

Q. GATE. Two Bag Contain 10 coins each and the coins in each bag are numbered from 1 to 10. One coin is drawn at random from each bag. What is the prob that one of the coin has value (1, 2, 3, 4) while the other has value (7, 8, 9 or 10)...

Differential Eq^n :-

- Formation of Diff^n eq^n → diff upto no. of constants.
- Solution of Diff. eq^n
  - Variable - Seperable
    - normal
    - reducable