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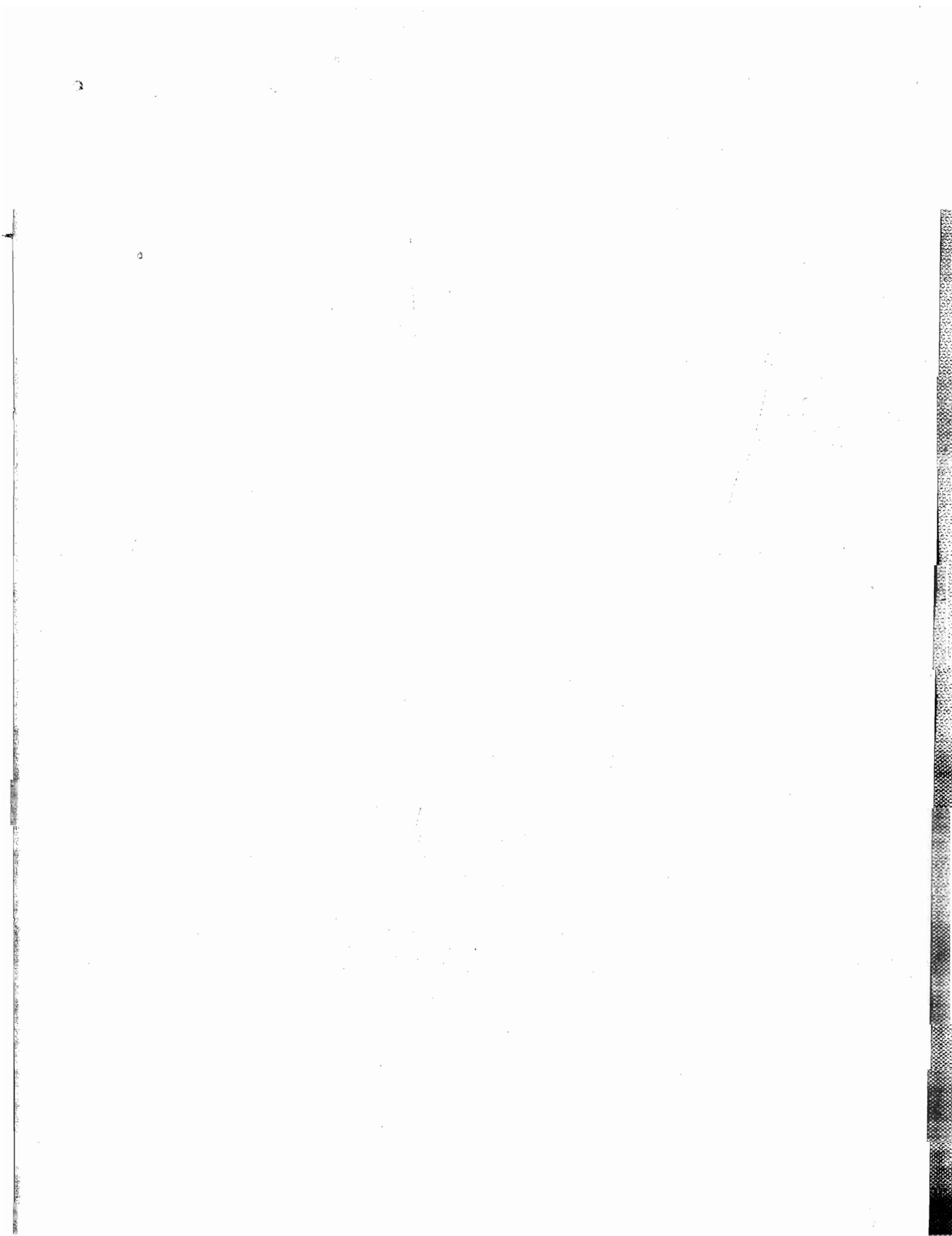
NAME:-.....

SUBJECT:-..... MATHEMATICS —

INSTITUTE:-..... By : (Sriramchau.)

PLEASE CONTACT FOR:-

- PHOTOSTAT (LASER DIGITAL)**
- PRINT OUT FROM COMPUTER**
- SPIRAL BINDING, HARD BINDING**
- TEST PAPER FOR PSU, GATE, IES**
- ALL NOTES AVAILABLE**
- ALL BOOK AVAILABLE**

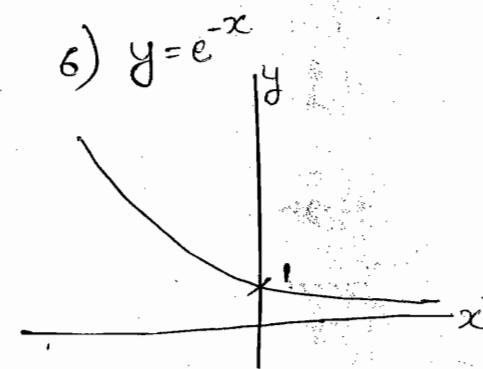
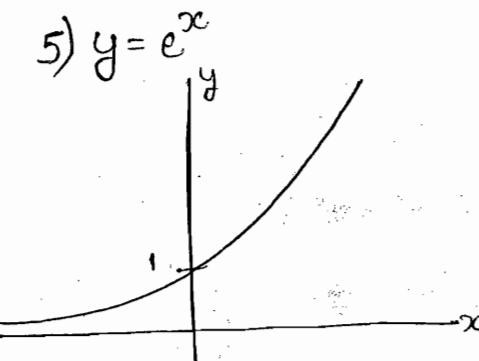
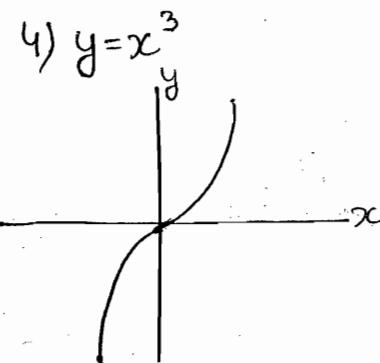
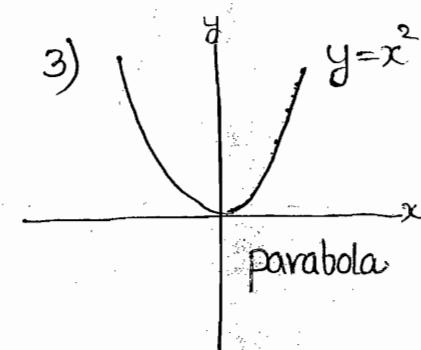
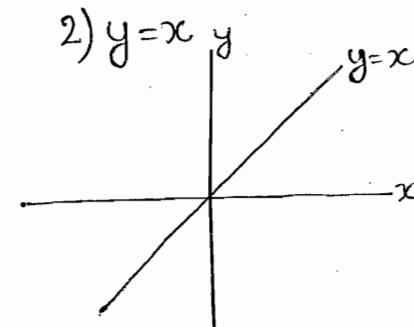
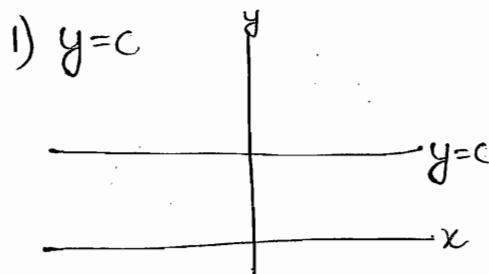


1. Calculus.
2. Diff. Eq's.
3. Linear Algebra
4. Probability
5. Complex Analysis
6. Vector Calculus
7. Numerical Methods
8. Laplace Transforms

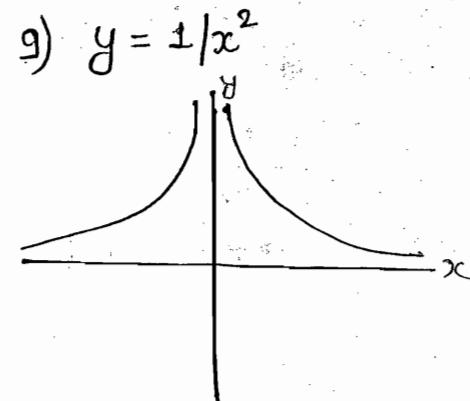
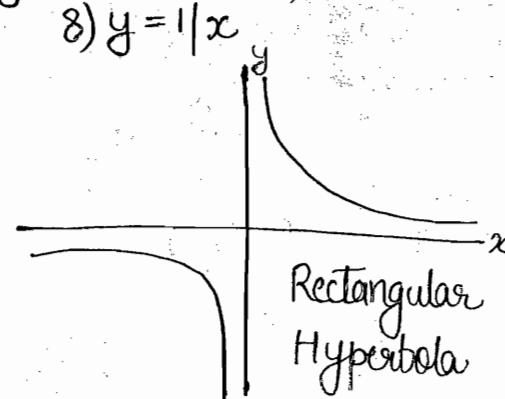
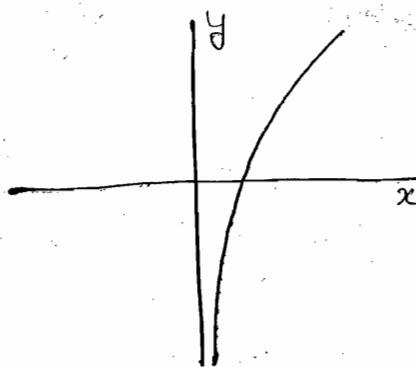
Sridhar
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CALCULUS

Functions :

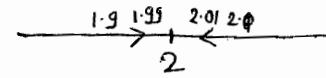


7) $y = \log x$ (In Maths log is base e notation)



* $f(x) = \frac{x^2 - 4}{x - 2}$ $f(2)$ is not defined.

$\underset{x \rightarrow 2}{\text{Lt}} f(x) = \underset{x \rightarrow 2}{\text{Lt}} \frac{x^2 - 4}{x - 2}$



$$\underline{x \rightarrow 2^-}$$

$$x = 1.9 \quad f(x) = 3.9$$

$$x = 1.99 \quad f(x) = 3.99$$

$$x = 1.999 \quad f(x) = 3.999$$

$$\downarrow 4$$

$$\underline{x \rightarrow 2^+}$$

$$f(x) = 4.1 \quad x = 2.1$$

$$f(x) = 4.01 \quad x = 2.01$$

$$f(x) = 4.001 \quad x = 2.001$$

$$\downarrow 4$$

$\underset{x \rightarrow a}{\text{Lt}} f(x) = L$ means $\underset{x \rightarrow a^-}{\text{Lt}} f(x) = \underset{x \rightarrow a^+}{\text{Lt}} f(x) = L$

i.e. $\boxed{L \cdot H \cdot L = R \cdot H \cdot L = L}$

Note - $f(a)$ need not be defined.

* $\underset{x \rightarrow 3}{\text{Lt}} 3x + 4 = 13$ $f(x) = 3x + 4$
 $f(3) = 13$

Continuity :-

Def : $f(x)$ is continuous at $x=a$ means

$$\underset{x \rightarrow a}{\text{Lt}} f(x) = f(a) \quad \text{i.e. } \boxed{RHL = LHL = f(a)}$$

otherwise, $f(x)$ is discontinuous at $x=a$

Removable discontinuity - $L \cdot H \cdot L = R \cdot H \cdot L \neq f(a)$

Ex :- $f(x) = \frac{x^3 - 9}{x - 3}$ has removable discontinuity at $x=3$

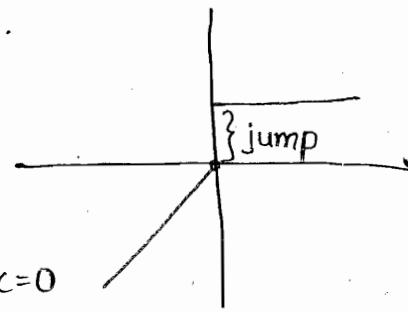
$$\underset{x \rightarrow 3}{\text{Lt}} f(x) = 6 \neq f(3) \quad f(3) \text{ not defined.}$$

Jump discontinuity - (Discont. of first kind)

$$LHL \neq RHL$$

Ex :- $f(x) = \begin{cases} x & x < 0 \\ x+1 & x \geq 0 \end{cases}$

$LHL \neq RHL \Rightarrow$ jump discontinuity at $x=0$.



Ex :- $f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$

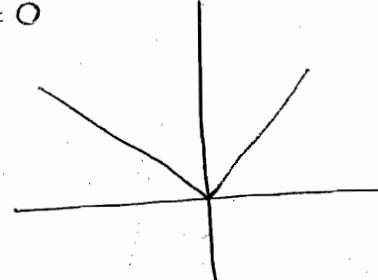
a) Continuous only at $x=0$ c) Discont. only at $x=0$

b) Continuous $\forall x$ d) None

Sol :- At $x=0$; $LHL=0$; $RHL=0$; $f(0)=0$

$$LHL = RHL = f(0)$$

$|x|$ is continuous for $x < 0$, $x > 0$ & $x = 0$
i.e. continuous for all x .



Results :-

1. Polynomial f^n 's, exponential fns, $\sin x$, $\cos x$ are continuous for all x .
2. If f & g are continuous then
 - (i) $f \pm g$ is continuous.
 - (ii) fg is continuous.
 - (iii) f/g is continuous if $g \neq 0$.

Ex :- $f(x) = |x-1| + |x-2|$

$$|x-1| = \begin{cases} -(x-1) & x < 1 \\ x-1 & x \geq 1 \end{cases} \quad |x-2| = \begin{cases} -(x-2) & x < 2 \\ x-2 & x \geq 2 \end{cases}$$

$$f(x) = \begin{cases} -(x-1) - (x-2) & x < 1 \\ (x-1) - (x-2) & 1 \leq x < 2 \\ (x-1) + (x-2) & x \geq 2 \end{cases}$$

$$f(x) = \begin{cases} -2x+3 & x < 1 \\ 1 & 1 \leq x < 2 \\ 2x-3 & x \geq 2 \end{cases}$$

We check continuity at pt. $x=1$
 \therefore it is cont. at all other pts.

continuous at $x=1$

At $x=2$; LHL = 1; RHL = $2(2) - 3 = 1$; $f(2) = 1$

continuous at $x=2$

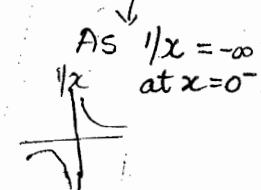
\Rightarrow continuous $\forall x$.

Ex: $f(x) = \begin{cases} \frac{1}{1+2^{\frac{1}{x}}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Soln:- At $x=0$; $f(x)=0$; LHL = $\lim_{x \rightarrow 0^-} \frac{1}{1+2^{\frac{1}{x}}} = \frac{1}{1+2^{-\infty}} = 1$

RHL = $\lim_{x \rightarrow 0^+} \frac{1}{1+2^{\frac{1}{x}}} = \frac{1}{1+2^{\infty}} = \frac{1}{\infty} = 0$; LHL = $\frac{1}{1+0} = 1$

LHL \neq RHL \Rightarrow not cont. at $x=0$.



Ex:- $\lim_{x \rightarrow 0} \frac{1}{x}$ a) ∞ b) $-\infty$ c) Does not exist

Soln:- LHL = $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$; RHL = $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$

LHL \neq RHL \Rightarrow does not exist.

Q:- $f(x) = 2x+1$. $x \leq 1$ This f^n is cont. $\forall x$
 ax^2+b $1 < x < 3$ find a, b .
 $5x+2a$ $x \geq 3$

Soln:- At $x=1$; LHL = $2(1)+1 = 3$; RHL = $a(1)^2+b = a+b$
cont. at $x=1 \Rightarrow a+b = 3$ ————— ①

At $x=3$; LHL = $a(3)^2+b = 9a+b$; RHL = $5(3)+2a = 15+2a$
cont. at $x=2 \Rightarrow 9a+b = 15+2a$ or $7a+b = 15$ ————— ②

On solving; $a=2$ & $b=1$

Differentiability — The derivative of f at x_c is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ provided limit exist.}$$

$f'(a)$ exist means

$$\boxed{LHD = RHD}$$

Left hand Derivative = RH Derivative

$$2. (x^n)' = nx^{n-1}$$

$$3. \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$4. (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$5. (e^{ax})' = ae^{ax}$$

$$6. (\log x)' = \frac{1}{x}$$

$$7. (\sin x)' = \cos x$$

$$8. (\cos x)' = -\sin x$$

$$10. (\cot x)' = -\operatorname{cosec}^2 x$$

$$11. (\sec x)' = \sec x \tan x$$

$$12. (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$$

$$13. (\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$$

$$14. (\tan^{-1} x)' = \frac{1}{1+x^2}$$

$$15. (\sec^{-1} x)' = \frac{1}{|x| \sqrt{x^2-1}}$$

Ex :- $f(x) = |x|$

a) Differentiable for all x b) Diff. only at $x=0$

~~c)~~ Not Diff. at $x=0$ d) None

Sol: - $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$ $f'(x) = \begin{cases} -1 & x < 0 \rightarrow \text{LHD} \\ 1 & x \geq 0 \rightarrow \text{RHD} \end{cases}$

At $x=0$; LHD \neq RHD \Rightarrow not differentiable at $x=0$.

$$* (|x|)' = \frac{|x|}{x} \quad x \neq 0$$

Ex :- $f(x) = |x-1| + |x-2|$

$$\begin{array}{lll} f(x) = -2x+3 & x < 1 & f'(x) = -2 & x < 1 \\ 1 & 1 \leq x < 2 & 0 & 1 \leq x < 2 \\ 2x-3 & x \geq 2 & 2 & x > 2 \end{array}$$

At $x=1$; LHD = -2 & RHD = 0 ; LHD \neq RHD

Not differentiable at $x=1$.

At $x=2$; LHD = 0, RHD = 2 ; LHD \neq RHD

Not Diff. at $x=2$

$$Q:- f(x) = \begin{cases} x^2 + 3x + a & x \leq 1 \\ bx+2 & x > 1 \end{cases}$$

is differentiable for all x . Find a, b.

$$2(1) + 3 = b \Rightarrow b = 5 \quad \text{--- (1)}$$

Differentiable \Rightarrow Continuous

$f(x)$ continuous at $x=1$; LHL = RHL

$$\begin{aligned} (1)^2 + 3(1) + a &= b(1) + 2 \\ a+4 &= b+2 \\ a-b &= -2 \quad \text{--- (2)} \end{aligned}$$

From (1), $a = 3$ & $b = 5$ Ans.

Q:- A real f^n $f(x) = \begin{cases} \alpha x^2 + \beta x & x < 0 \\ \alpha x^3 + \beta x^2 + 5\sin x & x \geq 0 \end{cases}$

is twice differentiable then $\alpha = \frac{5}{2}$ & $\beta = \frac{5}{2}$

Solⁿ:- Cont. at $x=0$; RHL = LHL

$$\text{LHL} = 0 + 0 = 0 ; \text{RHL} = 0 + 0 + 0 ; 0 = 0$$

$$\begin{aligned} f'(x) &= 2\alpha x + \beta \quad x < 0 \rightarrow \text{LHD} = \beta \Rightarrow \beta = 5 \\ &3\alpha x^2 + 2\beta x + 5\cos x \quad x \geq 0 \rightarrow \text{RHD} = 5 \end{aligned}$$

$$\begin{aligned} f''(x) &= 2\alpha \quad x < 0 \rightarrow \text{LHD} = 2\alpha \Rightarrow 2\alpha = 2\beta \\ &6\alpha x + 2\beta - 5\sin x \quad x \geq 0 \rightarrow \text{RHD} = 2\beta \quad \alpha = 5 \end{aligned}$$

$$\Rightarrow \alpha = \beta = 5 \quad \text{Ans.}$$

Indeterminate Form:- $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 1^\infty, 0^0, \infty^0$

L'Hospital's rule : $(\frac{0}{0} \text{ or } \frac{\infty}{\infty})$

$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\text{then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Results - $(\frac{0}{0}, \frac{\infty}{\infty})$

$$1. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$2. \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$3. \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$4. \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \quad (a > 0)$$

$$\textcircled{2} \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} \left(\frac{0}{0} \right) = \lim_{x \rightarrow 0} \frac{\sec^2 x}{1} = 1$$

$$\textcircled{3} \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \left(\frac{0}{0} \right) = \lim_{x \rightarrow 0} \frac{e^x}{1} = 1$$

$$\textcircled{4} \quad \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \left(\frac{0}{0} \right) = \lim_{x \rightarrow a} \frac{nx^{n-1}}{1} = na^{n-1}$$

Examples :-

$$\textcircled{5} \quad \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 5x} = \lim_{x \rightarrow 0} \frac{\tan 3x / 3x}{\sin 5x / 5x} \times \frac{3}{5} = \frac{3}{5} \lim_{x \rightarrow 0} \frac{\tan 3x / 3x}{\sin 5x / 5x} = \frac{3}{5}$$

$$(\text{Or}) \quad \lim_{x \rightarrow 0} \frac{3\sec^2 3x}{5 \cos 5x} = \frac{3}{5}$$

$$\text{Q:-} \quad \lim_{x \rightarrow 0} \frac{\sin hx - \sin x}{x \cdot \sin^2 x}$$

$$\text{Soln:-} \quad \lim_{x \rightarrow 0} \frac{\sin hx - \sin x}{x \cdot x^2} \times \frac{1}{\frac{\sin^2 x}{x^2}}$$

$$\lim_{x \rightarrow 0} \frac{\sin hx - \sin x}{x^3} \left(\frac{0}{0} \right) \quad \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} = 1$$

L'rule:

$$\lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{3x^2} \left(\frac{0}{0} \right)$$

L'rule:

$$\lim_{x \rightarrow 0} \frac{\sin hx + \sin x}{6x} \left(\frac{0}{0} \right)$$

L'rule:

$$\lim_{x \rightarrow 0} \frac{\cosh x + \cos x}{6}$$

$$= \frac{1+1}{6} = \frac{1}{3} \underline{\text{Ans.}}$$

$$\text{Q:-} \quad \lim_{x \rightarrow 0} \frac{2^{8\cos x} [\sin^8(\frac{\pi}{6}+x) - \sin^8 \frac{\pi}{6}]}{8x}$$

$$\lim_{x \rightarrow 0} \frac{2^{8\cos x}}{8} \lim_{x \rightarrow 0} \frac{\sin^8(\frac{\pi}{6}+x) - \sin^8 \frac{\pi}{6}}{x} \left(\frac{0}{0} \right)$$

$$= \frac{2^8}{8} \lim_{x \rightarrow 0} \frac{8\sin^7(\frac{\pi}{6}+x) \cos(\frac{\pi}{6}+x)}{1}$$

$$= 2^8 \sin^7(\frac{\pi}{6}) \cos(\frac{\pi}{6})$$

$$= 2^8 \left(\frac{1}{2}\right)^7 \frac{\sqrt{3}}{2}$$

$$= \sqrt{3} \underline{\text{Ans.}}$$

$$\text{Soln: } \lim_{x \rightarrow 0} \frac{2\cos 2x + a \cos x}{3x^2} = b \Rightarrow \frac{2+a}{0} = b \text{ [finite]}$$

Case (i) : $2+a \neq 0 \rightarrow b=\infty \times$

Case (ii) : $2+a=0 \Rightarrow a=-2$ ✓

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{2\cos 2x - 2\cos x}{3x^2} \left(\frac{0}{0}\right) = \lim_{x \rightarrow 0} \frac{-4\sin 2x + 2\sin x}{6x} \left(\frac{0}{0}\right) \\ & = \lim_{x \rightarrow 0} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -1 = b \end{aligned}$$

$$\therefore a = -2 \text{ & } b = -1 \quad \underline{\text{Ans}}.$$

$\pm\infty, 0^\circ, \infty^\circ$:- No direct form to solve these forms. So we take log.
 $y = f(x)^{g(x)}$ $\Rightarrow \log y = g(x) \log f(x)$

Results :-

$$1. \lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$2. \lim_{x \rightarrow 0} (1+ax)^{1/x} = e^a$$

$$3. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e^\infty$$

$$4. \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

$$5. \lim_{x \rightarrow \infty} (x)^{1/x} = 1$$

$$6. \lim_{x \rightarrow 0^+} x^x = 1$$

$$\textcircled{1} \quad \lim_{x \rightarrow 0} (1+x)^{1/x} \quad (1^\infty) ; \quad \log y = \lim_{x \rightarrow 0} \frac{1}{x} \log(1+x) = \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} \quad (\frac{0}{0})$$

$$\lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{1} \Rightarrow \log y = 1 \Rightarrow y = e$$

$$\textcircled{2} \quad \log y = \lim_{x \rightarrow 0} \frac{1}{x} \log(1+ax) = \lim_{x \rightarrow 0} \frac{\log(1+ax)}{x} \quad (\frac{0}{0})$$

$$\lim_{x \rightarrow 0} \frac{\frac{1}{1+ax} \cdot a}{1} \Rightarrow \log y = a \Rightarrow y = e^a$$

$$\lim_{t \rightarrow 0} (1+t)^{1/t} = e$$

$$⑤ y = \lim_{x \rightarrow \infty} (x)^{1/x} (\infty)^0 \Rightarrow \log y = \lim_{x \rightarrow \infty} \frac{\log x}{x} \left(\frac{\infty}{\infty}\right)$$

$$\log y = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0 \Rightarrow \log y = 0 \Rightarrow y = e^0 = 1$$

$$⑥ y = \lim_{x \rightarrow 0} x^x (0^0) ; \log y = \lim_{x \rightarrow 0} x \log x = \lim_{x \rightarrow 0} \frac{\log x}{1/x} \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = 0 \Rightarrow y = e^0 = 1 \text{ no std form} \quad \text{1st convert in std form}$$

$$\underline{\text{Ex:}} - \lim_{x \rightarrow \infty} \left(\frac{1+x}{1+2x} \right)^x$$

$$\lim_{x \rightarrow \infty} \left(\frac{x(1+1/x)}{x(1+2/x)} \right)^x = \frac{\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x}{\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x} \right)^x} = \frac{e}{e^2} = \frac{1}{e} \underline{\text{Ans.}}$$

$$\underline{\text{Ex:}} - \lim_{x \rightarrow 0} e^x (\cos x)^{1/\sin^2 x}$$

$$y = \lim_{x \rightarrow 0} e^x (\cos x)^{1/\sin^2 x} \Rightarrow \log y = \lim_{x \rightarrow 0} [\log e^x + \log (\cos x)^{1/\sin^2 x}]$$

$$\log y = \lim_{x \rightarrow 0} [x] + \lim_{x \rightarrow 0} \frac{\log \cos x}{\sin^2 x} \left(\frac{0}{0}\right) = 0 + \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} \cdot [-\sin x]}{2\sin x \cos x}$$

$$\log y = \lim_{x \rightarrow 0} \frac{-1}{2\cos^2 x} = -\frac{1}{2} \Rightarrow y = e^{-1/2} \underline{\text{Ans.}}$$

$$\underline{\text{Ex:}} - \lim_{x \rightarrow \infty} \left(\frac{5-x}{3-x} \right)^x = \lim_{x \rightarrow \infty} \left(\frac{1-5/x}{1-3/x} \right)^x = \frac{e^{-5}}{e^{-3}} = \frac{e^3}{e^5} = \frac{1}{e^2} \underline{\text{Ans.}}$$

* $\infty - \infty$ & $0 \cdot \infty$ forms:

$$\underline{\text{Ex:}} - \lim_{x \rightarrow \infty} (e^{1/5x} - 1) \cdot x [0 \cdot \infty] = \lim_{x \rightarrow \infty} \frac{e^{1/5x} - 1}{1/x} \left[\frac{0}{0}\right] = \lim_{x \rightarrow \infty} \frac{e^{1/5x}}{5 \cdot \frac{1}{5x}}$$

$$\text{Let } \frac{1}{5x} = t \quad x \rightarrow \infty \Rightarrow t \rightarrow 0$$

$$\frac{1}{5} \lim_{t \rightarrow 0} \frac{e^t - 1}{t} = \frac{1}{5} \times 1 = \frac{1}{5} \underline{\text{Ans.}}$$

$$= \lim_{x \rightarrow \pi/2} \left[\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right] = \lim_{x \rightarrow \pi/2} \left[\frac{1 - \sin x}{\cos x} \right] \left[\frac{0}{0} \right]$$

L'H rule : $\Rightarrow \lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cos x} = \frac{0}{1} = 0$ Ans.

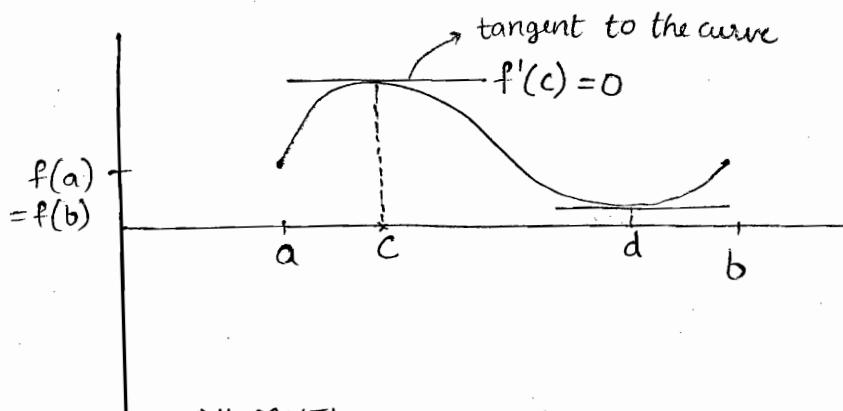
Mean Value Theorems :— (MVT)

Rolle's MVT :— If $f(x)$ is defined in $[a, b]$ such that

- (i) f is continuous in $[a, b]$ $[a, b] : a \leq x \leq b \quad x \in \mathbb{R}$
- (ii) f is differentiable in (a, b) $(a, b) : a < x < b \quad x \in \mathbb{R}$
- (iii) $f(a) = f(b)$

then there exists atleast one $c \in (a, b)$ such that

$$f'(c) = 0$$



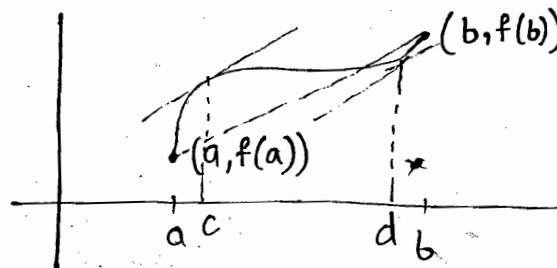
All MVT's are special case of this MVT.

Lagrange MVT (1st MVT) — If $f(x)$ is defined in $[a, b]$ such that

- i) f is continuous in $[a, b]$.
- ii) f is differentiable in (a, b) .

then there exists atleast one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Note :— If $f(a) = f(b)$
Lagrange MVT reduces to
Rolle's MVT.

such that

- (i) f & g are continuous in $[a, b]$
- (ii) f & g are differentiable in (a, b)
- (iii) $g' \neq 0$ in (a, b)

then there exists $c \in (a, b)$ such that

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

Q:- 1. Find c of Rolle's MVT : $f(x) = x(x-1)(x-2)$ in $[1, 2]$

2. $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$ 3. $f(x) = |x|$ in $[-1, 1]$

Sol:- 1. $f(x) = x^3 - 3x^2 + 2x$ in $[1, 2]$

(i) \therefore polynomial are cont. $\forall x$; it is cont in $[1, 2]$.

(ii) $f(x)$ is diff. in $(1, 2)$ [$\because f(x)$ is polynomial]

(iii) $f(1) = 0$; $f(2) = 0$

Three conditions of Rolle's MVT satisfied.

$$f'(x) = 3x^2 - 6x + 2$$

Find c such that $f'(c) = 0 \Rightarrow 3c^2 - 6c + 2 = 0$

$$\begin{aligned} c &= \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6} \\ &= 1 \pm \frac{\sqrt{12}}{6} = 1 \pm \frac{\sqrt{2}}{\sqrt{6}} = 1 \pm \frac{1}{\sqrt{3}} \end{aligned}$$

$\therefore c \in [1, 2]$

$$\Rightarrow c = 1 + \frac{1}{\sqrt{3}} \quad \underline{\text{Ans.}}$$

2. $f(x) = \frac{\sin x}{e^x} = e^{-x} \sin x$

(i) f is continuous $\forall x$ or in $[0, \pi]$

(ii) $f'(x) = e^{-x} \cos x - e^{-x} \sin x$ exists for all x or in $(0, \pi)$

(iii) $f(0) = 0$, $f(\pi) = 0$

Rolle's MVT is satisfied & applicable

$$\begin{aligned} \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \end{aligned}$$

$$f'(c) = 0 ; e^{-c} \cos c - e^{-c} \sin c = 0$$

$$e^{-c} [\cos c - \sin c] = 0$$

$$e^{-c} \neq 0 \Rightarrow \cos c = \sin c \Rightarrow \tan c = 1$$

$$c = \frac{\pi}{4} \in (0, \pi) \quad \underline{\text{Ans.}}$$

$|x|$ is not diff. at $x=0$

\therefore Rolle's MVT is not applicable.

Q:- Find c of Lagrange's MVT :

1) $f(x) = \log x$ in $[1, e]$

3) $f(x) = \sqrt{x^2 - 4}$ in $[2, 4]$

2) $f(x) = lx^2 + mx + n$ in $[a, b]$

Solⁿ:- 1) $f(x) = \log x$ in $[1, e]$

i) $f(x)$ is cont. in $[1, e]$

ii) $f'(x) = \frac{1}{x}$ exists in $(1, e)$

} Lagrange's MVT applicable.

Find $c \rightarrow f'(c) = \frac{f(e) - f(1)}{e-1} \Rightarrow \frac{1}{c} = \frac{1-0}{e-1}$

$C = e-1 \in (1, e)$

Ans.

2) $f(x)$ is cont. in $[a, b] \because$ polynomial f^n .

$f(x)$ is diff. in $(a, b) \because f$ is polynomial f^n

$f'(x) = 2lx + m$ exists in (a, b)

Lagrange's MVT is applicable.

Find $c \rightarrow f'(c) = \frac{f(b) - f(a)}{b-a} \Rightarrow 2lc + m = \frac{2b^2 + mb + n - 2a^2 - ma - n}{b-a}$

$2lc + m = \frac{l[b^2 - a^2] + m[b-a]}{b-a} = l[b+a] + m$

$C = \frac{a+b}{2} \in (a, b)$ Ans.

* 3) $f(x) = \sqrt{x^2 - 4}$ in $[2, 4] \rightarrow$ its continuity starts from 2 afterwards.

i) f is cont. in $[2, 4]$

ii) $f'(x) = \frac{2x}{2\sqrt{x^2 - 4}} = \frac{x}{\sqrt{x^2 - 4}}$ exists in $(2, 4)$

} Lagrange's MVT applicable.

$$4c^2 = 12c^2 - 48 \Rightarrow 8c^2 = 48 \Rightarrow c^2 = 6 \Rightarrow c = \pm\sqrt{6}$$

$$c = \sqrt{6} \in (2, 4) \quad \underline{\text{Ans.}}$$

Q:- Find c of Cauchy MVT :-

- 1) $f(x) = x^2$ $g(x) = x^3$ in $[1, 2]$
 2) $f(x) = e^x$ $g(x) = e^{-x}$ in $[a, b]$

Solⁿ :- 1. (i) f, g cont. in $[1, 2]$

(ii) $f' = 2x$, $g' = 3x^2$ exists in $(1, 2)$

(iii) $g' = 3x^2 \neq 0$ in $(1, 2)$ or $3x^2 = 0$ for $x=0 \notin (1, 2)$

Cauchy MVT is applicable.

$$\text{Find } c \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)} \Rightarrow \frac{2c}{3c^2} = \frac{4-1}{8-1} \Rightarrow \frac{2}{3c} = \frac{3}{7} \Rightarrow \frac{2}{3c} = \frac{3}{7}$$

$$c = \frac{14}{9} \in (1, 2) \quad \underline{\text{Ans.}}$$

2. (i) f, g is cont. in $[a, b]$ or $\forall x$

(ii) $f' = e^x$, $g' = -e^{-x}$ exists (a, b) or $\forall x$

(iii) $g' = -e^{-x} = 0$ for $x \rightarrow \infty \notin (a, b)$

Cauchy MVT is applicable.

$$c: \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$c = \frac{a+b}{2} \in (a, b) \quad \underline{\text{Ans.}}$$

$$e^{2c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \frac{e^b - e^a}{e^a - e^b} \times e^b e^a$$

$$\frac{a+b}{2} = c$$

Taylor's Series :-

If $f(x)$ is continuously differentiable at $x=a$ then $f(x)$ can be expressed as power series in powers of $(x-a)$ [about $x=a$]

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

It is called Taylor's series in powers of $(x-a)$

[abt $x=a$ or with centre at $x=a$]

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

Ex: — $f(x) = \sin x$

$$f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f''''(x) = \sin x$$

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f''''(0) = 0, f^v(0) = 1$$

$$\sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} (-1) + \frac{x^4}{4!} (0) + \frac{x^5}{5!} (1) + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Some power series :—

$$1) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$2) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$3) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$4) \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$5) \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$6) \sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$7) \cosh hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\text{Ex: } \sum_{n=0}^{\infty} \frac{1}{n!} =$$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\text{Put } x = 1 ; \quad \sum_{n=0}^{\infty} \frac{1}{n!} = e^1 = e \quad \underline{\text{Ans.}}$$

$$\underline{\text{Soln}} : - 3 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right] + 2 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]$$

$$= 3 + 2x - 3 \frac{x^2}{2!} - 2 \frac{x^3}{3!} + 3 \frac{x^4}{4!} + 2 \frac{x^5}{5!} + \dots \quad \underline{\text{Ans}}$$

Q:- For $x \ll 1$, $\coth(x)$ can be expressed as

$$\underline{\text{Soln}} : - \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{\left[1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots \right]}{\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right]}$$

For $x \ll 1$;

$$x^2, x^3, x^4 \text{ very small &} \\ \text{can be ignored.} \Rightarrow \coth x = \frac{1}{x} \quad \underline{\text{Ans}}$$

$$\begin{aligned} Q : - \log\left(\frac{1+x}{1-x}\right) &= \log(1+x) - \log(1-x) \\ &= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right] - \left[x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \right] \\ &= 2 \left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right] \quad \underline{\text{Ans}}. \end{aligned}$$

Q:- What is the coefficient $(x-2)^3$ in Taylor's series representation of e^x in powers of $(x-2)$

$$\underline{\text{Soln}} : - x-2=t \Rightarrow x=t+2$$

$$\begin{aligned} e^x &= e^{t+2} = e^2 \cdot e^t = e^2 \left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right] \\ &= e^2 \left[1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \dots \right] \end{aligned}$$

$$\text{Coeff of } (x-2)^3 = \frac{e^2}{3!} = \frac{e^2}{6} \quad \underline{\text{Ans.}}$$

Q:- Expansion of $\frac{\sin x}{x-\pi}$ in powers of $(x-\pi)$.

$$\underline{\text{Soln}} : - \text{Put } x-\pi=t \Rightarrow x=t+\pi$$

$$\begin{aligned} \frac{\sin x}{x-\pi} &= \frac{\sin(t+\pi)}{t} = \frac{-\sin t}{t} = -\frac{1}{t} \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right] \\ &= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \frac{t^6}{7!} - \dots \end{aligned}$$

$$x - \pi - \frac{(-\pi)^3}{3!} + \frac{(-\pi)^5}{5!} - \dots \quad \text{Ans.}$$

Remember —

Silver (sin, cosec)	All
Tea (tan, cot)	Cups (cos, sec)

Ex :- $\sin(\pi + \theta) = -\sin \theta$
 $\cos(270^\circ - \theta) = -\sin \theta$

$$f^n\left(n, \frac{\pi}{2} \pm \theta\right) = \begin{cases} \pm f^n(\theta) & \text{never} \\ \pm \cos f^n(\theta) & \text{odd} \end{cases}$$

f^n	$\cos f^n$
sin	cos
tan	cot
sec	cosec

Partial Derivatives :-

Let $f(x, y) \rightarrow f^n$ of two variables.

Notation : $\frac{\partial f}{\partial x} = f_x = p$; $\frac{\partial f}{\partial y} = f_y = q$; $\frac{\partial^2 f}{\partial x^2} = f_{xx} = r$
 $\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = s$; $\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = t$; $f_{xy} = f_{yx}$

Ex :- $f(x, y) = x^3 + y^3 - 3axy$

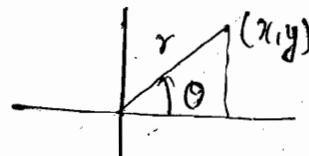
$$f_x = 3x^2 + 0 - 3ay$$

$$f_y = 0 + 3y^2 - 3ax$$

$$f_{xy} = f_{yx} = -3a$$

$$f_{xx} = 6x, \quad f_{yy} = 6y$$

Polar coordinates : $x = r \cos \theta$; $y = r \sin \theta$



$$\frac{\partial x^2}{\partial x^2} \quad \frac{\partial y^2}{\partial y^2}$$

$$\theta = \tan^{-1} \frac{y}{x} \quad \Delta \quad r = \sqrt{x^2 + y^2}$$

$$\theta_{xx} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{x^2 \cdot (-y)}{x^2 + y^2} \cdot \frac{1}{x^2} = \frac{-y}{x^2 + y^2}$$

$$\theta_{xy} = \frac{-(-y) \cdot 2x}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\theta_{yy} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$\theta_{yy} = \frac{-x \cdot 2y}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0}$$

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L-2

Homogeneous function:-

$f(x, y) = x^3 + y^3 \rightarrow$ Homogeneous f^n of degree 3

$f(x, y) = x^3 + x^2y + y^3 \rightarrow H.f^n$ of degree 3

$f(x, y) = \frac{x^4 + y^4}{x^2 + y^2} \rightarrow H.f^n$ of degree $4-2=2$

$f(x, y) = \frac{x^8 + y^8}{\sqrt{x} + \sqrt{y}} \rightarrow H.f^n$ of degree $8 - \frac{1}{2} = \frac{15}{2}$

$f(x, y) = \frac{\sqrt{x} - \sqrt{y}}{x + y} \rightarrow H.f^n$ of degree $\frac{1}{2} - 1 = -\frac{1}{2}$

Ex :- $x^3 + x^2y + 1 \rightarrow$ Not a $H.f^n$.

$$\left. \begin{array}{l} (1) f(kx, ky) = k^n f(x, y) \\ (2) f(x, y) = x^n \phi\left(\frac{y}{x}\right) \end{array} \right\} H.f^n \text{ of degree } n.$$

Ex :— $f(x, y) = x^3 + y^3$

$$f(kx, ky) = (kx)^3 + (ky)^3 = k^3 [x^3 + y^3] = k^3 f(x, y)$$

(or)

$$f(x, y) = x^3 + y^3 = x^3 \left[1 + \left(\frac{y}{x} \right)^3 \right] = x^3 \phi\left(\frac{y}{x}\right)$$

Euler's theorem :—

(I) If $f(x, y)$ is $H.f^n$ of degree n

$$(1) xf_x + yf_y = nf$$

$$(2) x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} = n(n-1)f$$

Ex $f(x, y) = \frac{x^3 + y^3}{x + y}$

$$1) xf_x + yf_y = 2.f(x, y)$$

$$2) x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} = 2(2-1)f = 2f$$

(II) Let $u(x, y)$ is not $H.f^n$ but $f(u)$ is $H.f^n$ of degree "n".

$$(1) xu_x + yu_y = \frac{n f(u)}{f'(u)} = G_1(u)$$

$$(2) x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u) [G'(u) - 1]$$

Ex :— $u = \log\left(\frac{x^3 + y^3}{x + y}\right)$ is not $H.f^n$ but $\frac{x^3 + y^3}{x + y}$ is $H.f^n$

$$(1) xu_x + yu_y = ?$$

$$(2) x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = ?$$

$$xu_x + yu_y = \frac{2e^4}{e^4} = 2 \text{ Ans.}$$

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u) [G'(u) - 1] = 2[0-1] = -2 \text{ Ans.}$$

(III) Let u is not H.fⁿ but it is sum of two fⁿ where
 $U + V + W$

V is H.fⁿ of degree n & W is H.f^m of degree m .

$$1) xu_x + yu_y = nv + mw$$

$$2) x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)v + m(m-1)w$$

$$\text{ex: } u = \frac{x^3+y^3}{x-y} + x \sin\left(\frac{y}{x}\right) \quad 1) xu_x + yu_y = 2 \frac{x^3+y^3}{x-y} + x \sin\left(\frac{y}{x}\right)$$

$$2) x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 2 \frac{x^3+y^3}{x-y} + 0$$

V is H.fⁿ of degree $3-1=2$, W is H.fⁿ of degree 1.

$$\text{ex: } \sin u = \frac{x+2y+3z}{x^8+y^8+z^8}$$

$$u = \tan^{-1} \left(\frac{x^3+y^3}{x-y} \right)$$

$$xu_x + yu_y + zu_z = ?$$

$$xu_x + yu_y = ?$$

$\sin u$ is H.fⁿ of degree $1-8=-7$

$\tan u$ is H.fⁿ of degree $3-1=2$

$$xu_x + yu_y + zu_z = n \frac{f(u)}{f'(u)}$$

$$xu_x + yu_y = n \frac{f(u)}{f'(u)}$$

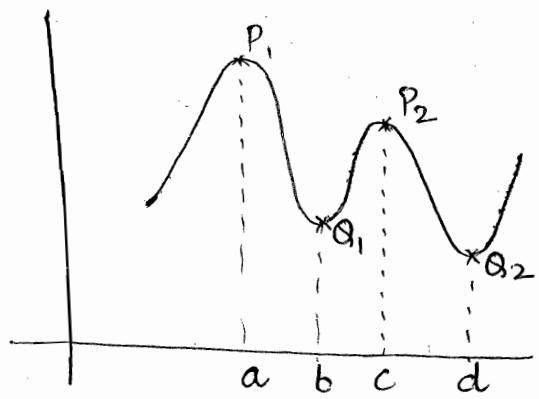
$$= -7 \frac{\sin u}{\cos u}$$

$$= 2 \frac{\tan u}{\sec^2 u}$$

$$= -7 \tan u \text{ Ans.}$$

$$= 2 \frac{\sin u \cos^2 u}{\cos u}$$

$$= \sin 2u \text{ Ans.}$$



$y = f(x)$ is the f^n

Local max^m P_1 & P_2 at $a \& c$ respectively.

Local minimum Q_1 & Q_2 at $b \& d$ respectively.

→ At local maximum & minimum the tangent have zero slope & is parallel to x -axis. i.e. $f'(x) = 0$

→ To find maxima & minima : Idea $\rightarrow f'(x) = 0$

Procedure :— Let $y = f(x)$ be given f^n .

1) Find $\frac{dy}{dx}$

2) Solve $\frac{dy}{dx} = 0$ for stationary pts.

3) Find $\frac{d^2y}{dx^2}$

4) Let "a" be stationary pt.

At $x=a$,

(i) If $\frac{d^2y}{dx^2} < 0$ then $f(x)$ has local max at $x=a$

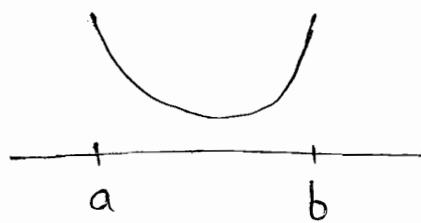
(ii) If $\frac{d^2y}{dx^2} > 0$, then $f(x)$ has ^{local} min at $x=a$

(iii) If $\frac{d^2y}{dx^2} = 0$, test fails.

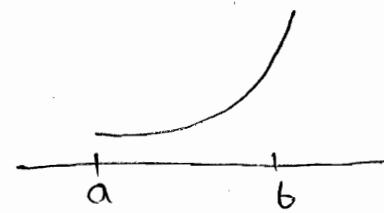
5) At $x=a$, If $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$,

then $f(x)$ has point of inflection at $x=a$.

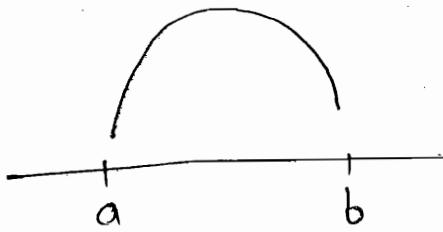
(1) If $\frac{d^2y}{dx^2} > 0$ in $[a, b]$ then $y = f(x)$ is concave up in $[a, b]$



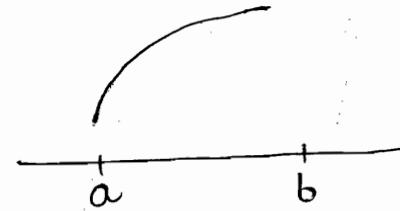
(or)



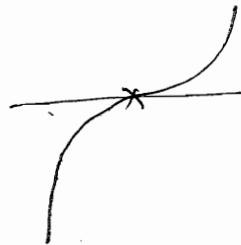
(2) If $\frac{d^2y}{dx^2} < 0$ in $[a, b]$ then $y = f(x)$ is concave down in $[a, b]$



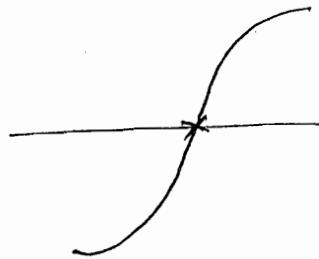
(or)



Def :- The pt. where a curve changes from concave up to concave down or vice versa is called pt. of inflection.



(or)



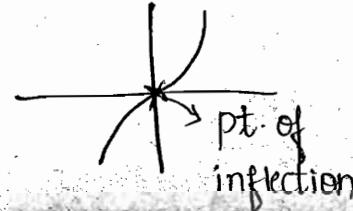
even at pt. of inflection tangent is parallel to x-axis

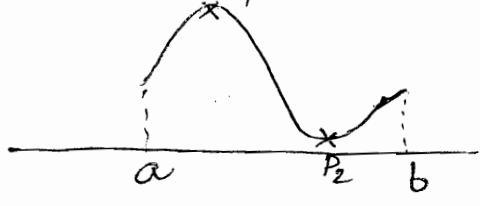
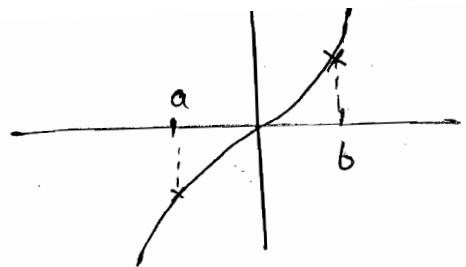
Ex :- $y = x^3$

$$1) \frac{dy}{dx} = 3x^2 \quad 2) \frac{dy}{dx} = 0 \quad 3x^2 = 0 \Rightarrow x=0 \text{ is stationary pt.}$$

$$3) \frac{d^2y}{dx^2} = 6x \quad \text{at } x=0 \quad \frac{d^2y}{dx^2} = 0 \quad \text{so we can't say whether it is maxima or minima.}$$

$$4) \frac{d^3y}{dx^3} = 6 \neq 0 \text{ at } x=0 \quad \therefore y = x^3 \text{ has pt. of inflection (neither maxima nor minima) at } x=0$$





Result :- 1) The absolute max^m of $f(x)$ in the $[a, b]$ is given by $\max^m \{f(a), f(b), \text{all local max}^m \text{ of } f(x) \text{ in } [a, b]\}$

2) The absolute min^m of $f(x)$ in $[a, b]$ is given by $\min^m \{f(a), f(b), \text{all local min}^m \text{ of } f(x) \text{ in } [a, b]\}$.

Ex :- Discuss max, min of $f(x) = x^3 - 9x^2 + 24x + 5$

Solⁿ :- 1) find $f'(x) : f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8)$

2) $f'(x) = 0 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x = 2, 4 \rightarrow \text{stationary pts.}$

3) $f''(x) = 6x - 18$

4) $f''(x) \Big|_{x=2} = -6 < 0 \quad \text{local max}^m$

$f''(4) = 6 > 0 \quad \text{local min}^m$

Local max of $f(x) = f(2) = 2^3 - 9(2)^2 + 24(2) + 5 = 25$ at $x = 2$

Local min of $f(x) = f(4) = (4)^3 - 9(4)^2 + 24(4) + 5 = 21$ is at $x = 4$

Q:- Find absolute max, min of $f(x) = x^3 - 9x^2 + 24x + 5$ in $[1, 6]$

Solⁿ :- 1) $f'(x) = 3x^2 - 18x + 24$

2) $f'(x) = 0 \Rightarrow x = 2, 4 \rightarrow \text{stationary pts.}$

3) find $f(1), f(2), f(4), f(6)$

$f(1) = 21, f(2) = 25, f(4) = 21, f(6) = 41$

Absolute Max^m = $\max^m \{f(1), f(6), f(2), f(4)\}$

$= f(6) = 41 \text{ is at } x = 6$

Q:- Find max, min of $f(x) = \frac{\log x}{x}$ in $(0, \infty)$

$$\text{Sol}^n: - f'(x) = \frac{x \cdot \frac{1}{x} - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

$$f'(x) = \frac{1 - \log x}{x^2} = 0 \Rightarrow 1 - \log x = 0 \Rightarrow \log x = 1 \Rightarrow \boxed{x = e}$$

$$f''(x) = \frac{x^2 \left(\frac{-1}{x}\right) - 2x(\log x + 1)}{x^4} = \frac{-x - 2x(\log x + 1)}{x^4} = \frac{-x - 2x\log x - 2x}{x^4} = \frac{-3x + 2x\log x}{x^4} = \frac{-3 + 2\log x}{x^3}$$

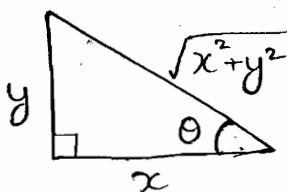
$$f''(e) = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

$f(x)$ has local max^m at $x = e$ & local max^m of $f(x) = f(e)$

$$= \frac{\log e}{e} = \frac{1}{e} \underline{\underline{\text{Ans}}}$$

Q:- For right angle triangle if the sum of the lengths of the hypotenuse & a side kept const in order to have max^m area of the triangle ; the angle b/w the hypotenuse & the side

Solⁿ:



$$\sqrt{x^2 + y^2} + x = \text{const.} = C \quad \text{--- (1)}$$

Area = Max^m

Area of right angle Δ's = $\frac{1}{2}xy$

$$A^2 = \frac{1}{4}x^2y^2 \quad \text{--- (2)}$$

$$(1) \sqrt{x^2 + y^2} = C - x$$

$$\text{Squaring, } x^2 + y^2 = C^2 - 2Cx + x^2 = C^2 - 2Cx$$

$$\text{Let } f(x) = A^2 = \frac{1}{4}x^2y^2$$

$$f(x) = \frac{1}{2}x^2(C^2 - 2Cx)$$

$$f(x) = \frac{1}{4}(C^2x^2 - 2Cx^3)$$

2) $f'(x) = 0 \Rightarrow x = 0, \frac{c}{3} \rightarrow$ stationary pts.

3) $f''(x) = \cancel{0} \frac{1}{4} [2c^2 - 12cx]$

4) $f''(0) = \frac{2c^2}{4} > 0$ local min^m

$f''\left(\frac{c}{3}\right) = \frac{-2c^2}{4} < 0$ local max^m ✓

$x = \frac{c}{3}$; $f(x)$ is max; $f\left(\frac{c}{3}\right) = A^2 = \frac{1}{4}x^2y^2$

$$y^2 = c^2 - 2cx = c^2 - 2c \cdot \frac{c}{3} = \frac{c^2}{3} \Rightarrow y = \pm \frac{c}{\sqrt{3}}$$

$y \rightarrow +ve$ side $\rightarrow + \frac{\sqrt{c}}{\sqrt{3}} \frac{c}{\sqrt{3}}$

$$\tan \theta = \frac{y}{x} = \frac{c/\sqrt{3}}{c/3} = \sqrt{3} \Rightarrow \theta = 60^\circ \text{ or } \frac{\pi}{3}$$

Q:- The right circular cone of largest volume that can be enclosed by a sphere of rad. 1m has a height of —

- a) $\frac{1}{3}$ m b) $\frac{2}{3}$ m c) $\frac{2\sqrt{2}}{3}$ m d) $\frac{4}{3}$ m.

Solⁿ:- Volume of cone,

$$V = \frac{1}{3} \pi r^2 h$$

By Right angle Δ,

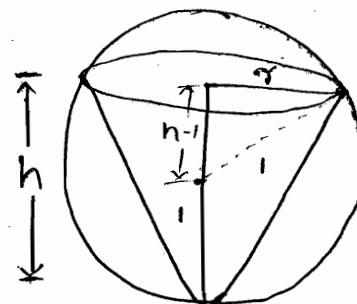
$$(h-1)^2 + r^2 = 1^2$$

$$r^2 = 1^2 - (h-1)^2$$

$$= 1^2 - h^2 + 2h - 1^2 + 2h$$

$$r^2 = 2h - h^2$$

$$V = \frac{1}{3} \pi (2h - h^2) h = \frac{\pi}{3} (2h^2 - h^3)$$



$$V'(h) = \frac{\pi}{3} [4h - 3h^2]$$

$$v''(h) = \frac{\pi}{3}[4-6h] \text{ at } h=0 \Rightarrow v''(0) = \frac{4\pi}{3} > 0 \text{ local min}^m$$

Ans at $h = \frac{4}{3}$; $v''\left(\frac{4}{3}\right) = -\frac{2\pi}{3} < 0$ local max^m

Q:— $f(x) = 2x^3 - 9x^2 + 12x - 3$ in $[0, 3]$

Soln :— $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

$$f'(x) = 0 \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow \begin{aligned} x^2 - 2x - x + 2 &= 0 \\ x(x-2) - 1(x-2) &= 0 \\ x = 2, 1 &\rightarrow \text{st. pts.} \end{aligned}$$

$$f''(x) = 12x - 18$$

$$f''(1) = 12 - 18 = -6 < 0 \rightarrow \text{local max}^m$$

$$f''(2) = 24 - 18 = 6 > 0 \rightarrow \text{local min}^m$$

$$f(0) = -3, f(1) = 2 - 9 + 12 - 3 = 2, f(2) = 16 - 36 + 24 - 3 = 1$$

$$f(3) = 54 - 81 + 36 - 3 = 6$$

Absolute max^m of $f(x) = 6$; Absolute min^m of $f(x) = -3$
 Abs. max^m at $x = 3$; Abs. min^m is at $x = 0$

Function of two variables :—

To find local max, min,

1) Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

2) Solve $f_x = 0$ & $f_y = 0$ for stationary pts. (Critical pts.)

3) Find $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$

4) Let (a, b) be stationary pt. & at (a, b)

(i) If $rt - s^2 > 0$ &

(a) $r > 0$ then $f(x, y)$ has local min^m at (a, b)

(b) $r < 0$ then $f(x, y)$ has local max^m at (a, b)

(ii) If $rt - s^2 = 0$, test fails.

(neither max^m nor minimum) at (a, b)

Ex :- $f(x, y) = 2x^4 + y^2 - x^2 - 2y$

Soln :- 1) Find f_x & f_y ;

$$f_x = 8x^3 - 2x, \quad f_y = 2y - 2$$

2) $f_x = 0 \Rightarrow x = 0, x = \pm \frac{1}{2}$; $f_y = 0 \Rightarrow y = 1$

St. pts $\rightarrow (0, 1); (\pm \frac{1}{2}, 1); (-\frac{1}{2}, 1)$

3) $r = 24x^2 - 2$; $s = 0$; $t = 2$

4)	St. Pts	$r = 24x^2 - 2$	$t = 2$	$s = 0$	$rt - s^2$	so what=?
	(0, 1)	$r = -2$	2	0	-4	Saddle
	($\frac{1}{2}, 1$)	$r = 4$	2	0	8	{ local min ^m
	($-\frac{1}{2}, 1$)	$r = 4$	2	0	8	

Q:- The distance b/w origin & the pt. nearest to it on the surface $z^2 = 1 + xy$ is _____

Soln :- Distance from origin to (x, y, z)

$$d = \sqrt{x^2 + y^2 + z^2}$$

\therefore on the surface $z^2 = 1 + xy$

$$d^2 = x^2 + y^2 + 1 + xy = f(x, y) \text{ (Let)}$$

$$f_x = 2x + y, \quad f_y = 2y + x \Rightarrow 2x + y = 0 \quad \& \quad 2y + x = 0$$

$$x = 0, \quad y = 0 \Rightarrow \boxed{y = 0, x = 0}$$

$$rt - s^2 = 4 - 1 = 3 > 2 \quad \& \quad r > 0 \rightarrow \text{local min}^m.$$

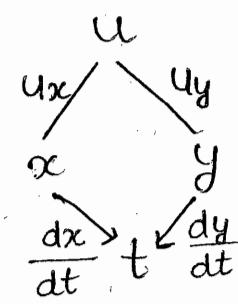
$$\text{Local min}^m \text{ distance} = d = \sqrt{0^2 + 0^2 + z^2}$$

$$d^2 = (1 + xy) \Rightarrow d = 1 \quad \underline{\text{Ans.}}$$

Let $u = f(x, y)$ & $x = f_1(t)$ & $y = f_2(t)$

$$\frac{du}{dt} = \text{Total derivative of } u \text{ w.r.t } t$$

$$= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$



Ex :- $f(x, y) = 2xy$; $x=t$, $y=t^2$

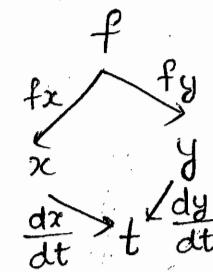
$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$= 2y \cdot 1 + 2x \cdot 2t$$

$$= 2y + 4xt$$

$$= 2t^2 + 4t^2$$

$$= 6t^2 \quad \underline{\text{Ans.}}$$



Explicit Function - $y = f(x)$

Implicit function - $f(x, y) = c$

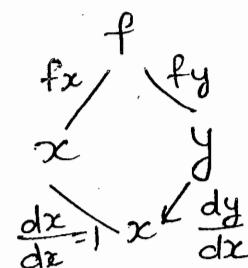
Ex $\rightarrow y^2 + y + \sin x = 0 \rightarrow$ Writing y in terms of x is complex \Rightarrow Implicit function.

Derivative of Implicit F^n :-

$$f(x, y) = c$$

To find $\frac{dy}{dx}$; Total derivative w.r.t x

$$\frac{df}{dx} = 0 \quad ; \quad f_x \cdot 1 + f_y \frac{dy}{dx} = 0$$



$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

Ex :- $x^3 + y^3 + 3axy = 0$; $\frac{dy}{dx} = 0 \Rightarrow f = x^3 + y^3 + 3axy$

$$\frac{dy}{dx} = \frac{-3x^2 - 3ay}{3y^2 + 3ax} = \frac{-(x^2 + ay)}{y^2 + ax} \quad \underline{\text{Ans.}}$$

Solⁿ :- $y = \sqrt{\tan x + y} \Rightarrow y^2 = \tan x + y \Rightarrow y^2 - y - \tan x = 0$
 Even fⁿ given is in explicit form, it is a implicit fⁿ.

$$\frac{dy}{dx} = -\frac{(\sec^2 x)}{2y-1} = \frac{\sec^2 x}{2y-1} \text{ Ans.}$$

Q:- $x^a y^b = (x+y)^{a+b}$ Find dy/dx

Solⁿ :- Take log on both sides \rightarrow

$$\log(x^a) + \log(y^b) = (a+b)\log(x+y)$$

$$a\log x + b\log y - (a+b)\log(x+y) = 0$$

$$f_x = \frac{a}{x} - \frac{(a+b)}{x+y} = \frac{ay - bx}{x(x+y)}$$

$$f_y = \frac{b}{y} - \frac{(a+b)}{x+y} = \frac{bx - ay}{y(x+y)}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(ay - bx)}{bx - ay} \quad \left| \frac{(bx - ay)}{y(x+y)} \right. = \frac{y}{x} \text{ Ans.}$$

Total Differential - Let $u = f(x, y)$

The total diff. of u:

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$

Ex:- 1) $d(xy) = \frac{\partial}{\partial x}(xy)dx + \frac{\partial}{\partial y}(xy)dy = ydx + xdy$

2) $d(x^2y) = y \cdot 2xdx + x^2dy = 2xydx + x^2dy$

3) $d(x^2y^2) = 2xy^2dx + 2yx^2dy$

4) $d\left(\frac{y}{x}\right) = \frac{x \cdot dy - y \cdot dx}{x^2}$

5) $d(\tan^{-1}(x^2+y^2)) = \frac{1}{1+(x^2+y^2)^2} d(x^2+y^2) = \frac{1}{1+(x^2+y^2)^2} [2xdx + 2ydy]$

x^y

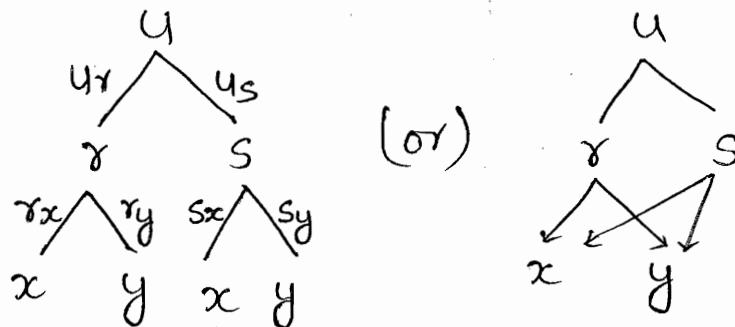
DIFFERENTIAL EQUATION

D.E :-

$$ydx + xdy = 0 \Rightarrow d(xy) = 0 \Rightarrow \boxed{xy = C}$$

Change of variable (chain rule) :-

Let $u = f(r, s)$ & $r = f_1(x, y)$; $s = f_2(x, y)$



$$\frac{\partial u}{\partial x} = u_r \cdot r_x + u_s \cdot s_x$$

$$\frac{\partial u}{\partial y} = u_r \cdot r_y + u_s \cdot s_y$$

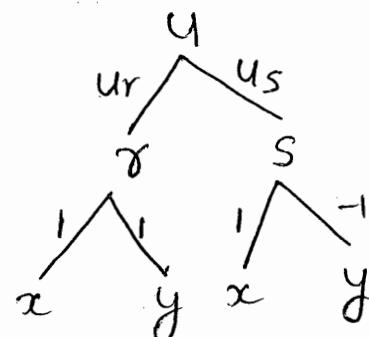
Ex :- $u = f(r, s)$; $r = x+y$ & $s = x-y$; $u_x + u_y = \underline{\hspace{2cm}}$

Soln :- $u_x + u_y = \underline{\hspace{2cm}}$

$$u_x = u_r \cdot 1 + u_s \cdot 1 = u_r + u_s$$

$$u_y = u_r \cdot 1 + u_s \cdot (-1) = u_r - u_s$$

$$u_x + u_y = 2u_r \quad \underline{\text{Ans.}}$$



Integration -

$$1) \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$2) \int \sin x dx = -\cos x$$

$$3) \int \tan x dx = \log \sec x$$

$$4) \int \sec^2 x dx = \tan x$$

$$5) \int \sec x \tan x dx = \sec x$$

$$6) \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$$

$$\int a^2 + x^2 \ dx = a^2 x + \frac{x^3}{3} + C$$

$$9. \int \cot x \ dx = \log \sin x \quad 10. \int [f(x)]^n f'(x) \ dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\underline{\text{Ex}} : - \int \tan x \ dx = - \int \frac{\sin x}{\cos x} \ dx = - \log(\cos x) = \log \sec x$$

$$\underline{\text{Ex}} : - \int (\log x)^2 \cdot \frac{1}{x} \ dx = \frac{(\log x)^3}{3}$$

$$11. \int e^{f(x)} f'(x) \ dx = e^{f(x)}$$

$\underline{\text{Ex}} : - \int e^{x^2} x \ dx = \frac{1}{2} \int e^{x^2} 2x \ dx$

(or) put $x^2 = t \quad 2x \ dx = dt$
 $= \frac{1}{2} \int e^t dt = \frac{1}{2} e^t = \frac{1}{2} e^{x^2}$

$$12. \int e^x [f(x) + f'(x)] \ dx = e^x f(x)$$

$$\underline{\text{Ex}} : - \int e^x (\sin x + \cos x) \ dx = e^x \sin x$$

Integration by parts :-

I	L	A	T	E	\rightarrow	Exponential
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow		
Log						
Inverse						
						Trigonometric
						Algebraic

$$\boxed{\int U \cdot V = u \int v - \int \cancel{du} [u' \int v]}$$

$$\underline{\text{Ex}} : - \int \log x \ dx = \int \log x \cdot 1 \ dx$$

$$= \log x [x] - \int \frac{1}{x} \cdot x \ dx = \log x [x] - x$$

$$= x(\log x - 1) \ \underline{\text{Ans.}}$$

$$\boxed{\int \log x \ dx = x(\log x - 1)}$$

$$Q : - \int \sqrt{x} \cdot \log x \ dx = \int \log x \cdot \sqrt{x} \ dx$$

$$= \log x \left[\frac{x^{3/2}}{3/2} \right] - \int \frac{1}{x} \cdot \frac{x^{3/2}}{3/2} \ dx$$

3 0 3 3 3 L U

Leibniz Rule — Useful when $u = \text{polynomial}$ f^n i.e. x^n ; $n = \text{+ve}$
 (or) $V = e^{ax}$ or $\sin ax$ or $\cos ax$

$$\boxed{\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots}$$

where $v_1, v_2, v_3, v_4, \dots$ are successive integrations &
 u', u'', \dots are successive differentiations.

Ex :- $\int e^{2x} x^2 dx = x^2 \left[\frac{e^{2x}}{2} \right] - 2x \left[\frac{e^{2x}}{4} \right] + 2 \left[\frac{e^{2x}}{8} \right] + 0$

Ex :- $\int x^2 e^{-3x} dx = x^2 \left[\frac{e^{-3x}}{-3} \right] - 2x \left[\frac{e^{-3x}}{9} \right] + 2 \left[\frac{e^{-3x}}{-27} \right]$

Q :- $\int x^3 \sin x dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x(\cos x) - 6 \sin x$

Q :- $\int x^2 \cos 2x dx = x^2 \left(\frac{\sin 2x}{2} \right) - 2x \left(\frac{-\cos 2x}{4} \right) + 2 \left(\frac{-\sin 2x}{8} \right)$

DIFFERENTIAL EQUATIONS

D.E. is an eqⁿ involving derivatives of dependent variables w.r.t
 one or more independent variables.

DE
I

↓
 Ordinary DE (ODE or DE)

→ one independent variable

Ex :- $\frac{d^2y}{dx^2} + 9 \frac{dy}{dx} + y = \sin x$

↓
 Partial DE (PDE)

→ Two or more independent variables.

Ex :- $\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$

Order :- It is order of highest order derivative occurring in D.E.

Degree :- The power of highest order derivative when the
 dependent variables & its derivatives in the DE are free of
 fractional powers.

$$\left[\frac{d^2y}{dx^2} \right] = \frac{dy}{dx^2}$$

Squaring both sides $\rightarrow \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 = \left(\frac{d^2y}{dx^2} \right)^2$

Order = 2 & Degree = 2

Ex: $\left(\frac{d^2y}{dx^2} \right)^3 + 3 \frac{dy}{dx} + y = \sqrt{\sin x}$

Order = 2, degree = 3

Formation of DE :— [By eliminating arbitrary constants (parameters)]

Method:—

1) $f(x, y, c_1, c_2) = 0$ $\xrightarrow{\text{Diff. twice}}$ Two parameter family of curves

Diff. twice & eliminate c_1 & c_2

2) Diff. w.r.t. x : $F'(x, y, c_1, c_2) = 0 \quad \text{--- } ②$

3) Diff. w.r.t. x : $F''(x, y, c_1, c_2) = 0 \quad \text{--- } ③$

4) Eliminate c_1 & c_2 using $①, ②$ & $③$

$f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0 \rightarrow 2^{\text{nd}} \text{ order D.E. of family.}$

Having n constants $\rightarrow n$ times differentiation $\rightarrow n^{\text{th}}$ order DE

Q:— $y = mx$

Sol n :— $m \rightarrow$ arbitrary constant

$$\frac{dy}{dx} = m \Rightarrow y = \frac{dy}{dx} x \quad \text{or} \quad \boxed{x \frac{dy}{dx} - y = 0}$$

Q:— $y = Ax + Bx^2$

$$\left[\frac{dy}{dx} = A + 2Bx ; \frac{d^2y}{dx^2} = 2B \right] \text{ (or)} \quad \frac{y}{x} = A + Bx$$

Diff. w.r.t. x

$$\frac{xy' - y \cdot 1}{x^2} = B$$

Again Diff. w.r.t. x

$$\frac{x^2[xy'' + y' - y] - [xy - y] \cdot 2x}{x^4} = 0 \rightarrow x^2y'' - 2xy' + 2y = 0 \rightarrow \text{D.E. } ①$$

D.E. of the family of st. lines.

$$x^2y'' - 2xy' + 2y = 0$$

$$\frac{dy}{dx} \rightarrow y' \rightarrow D^1y$$

$$\vdots$$

$$\frac{d^n y}{dx^n} \rightarrow y^{(n)} \rightarrow D^n y$$

Result :-

~~Ex :-~~ $y = C_1 e^{ax} + C_2 e^{bx}$ $\rightarrow C_1, C_2$: arbitrary constant.

D.E. $\Rightarrow y'' - (a+b)y' + (ab)y = 0$ \rightarrow For only these type of prblms.

Ex :- $y = C_1 e^{2x} + C_2 e^{3x}$

D.E. $\Rightarrow y'' - 5y' + 6y = 0$

Ex :- $y = C_1 e^{-x} + C_2 e^{3x}$

D.E. $\Rightarrow y'' - 2y' - 3y = 0$

Ex :- Form D.E. of family of circles with radius r & having their centre on x -axis.

Solⁿ :- $r \rightarrow$ fixed \rightarrow its a constant

$a \rightarrow$ varying \rightarrow parameter or arbitrary constant.



Eqⁿ of circle with centre $(a, 0)$ & radius r

$$(x-a)^2 + y^2 = r^2 \quad \text{--- (1)}$$

Diff. w.r.t. $x \rightarrow 2(x-a) + 2yy' = 0 \Rightarrow (x-a) = -yy'$

$$(-yy')^2 + y^2 = r^2 \quad \text{or} \quad y^2 y'^2 + y^2 = r^2$$

$y^2(1+y'^2) = r^2$

Ans. $y' \rightarrow dy/dx$

Solutions of D.E. :-

Ex :- $\frac{dy}{dx} - 2y = 0 \quad \text{--- (1)}$

Solⁿ :- $y = e^{2x}$ $\frac{dy}{dx} = 2e^{2x}$ Put in (1) $\Rightarrow 2e^{2x} - 2e^{2x} = 0$
(Let) $D.E.$ is satisfied

$y = e^{2x}$ is the solⁿ.

satisfying D.E.

Solution is of two type — General solⁿ & Particular solⁿ.

General solⁿ — A solⁿ of n^{th} order D.E. eliminating containing n arbitrary constants (parameters) (or)

A solⁿ of DE containing

$$\boxed{\text{No. of arbitrary constants} = \text{Order of DE}}$$

Particular solⁿ — A solⁿ of DE obtained from general solution

Note — To get particular solⁿ of n^{th} order D.E. we need n distinct conditions.

Initial conditions — Conditions given at same pt.

Boundary conditions — Conditions given at different pt.

Initial Value Prblms (IVP) : DE + Initial Conditions

Boundary Value Prblms (BVP) : DE + Boundary Conditions

Solutions of 1st order DE :—

Methods :— 1) Variable Separable 2) Exact DE
3) Linear DE of 1st order.

Variable Separable :—

$$\underline{\text{Ex:—}} \frac{dy}{dx} = \frac{x}{y} \Rightarrow ydy = xdx \quad \text{Integrating both sides,}$$

$$\int ydy = \int xdx + C$$

$$\boxed{\frac{y^2}{2} = \frac{x^2}{2} + C} \rightarrow \text{gen. Sol}^n. \quad \text{or} \quad \boxed{y^2 - x^2 = C} \rightarrow \text{general Sol}^n$$

$$\frac{dy}{dx} = \frac{x^2}{y}$$

Gen. Solⁿ

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y(0) = 1 \rightarrow x=0 \\ y=1$$

$$C = 1/2$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{1}{2}$$

$$\boxed{y^2 - x^2 = 1}$$

Gen. Solⁿ

$$y^2 - x^2 = C$$

$$y(0) = 1 \quad x=0, y=1$$

$$1 - 0 = C$$

$$C = 1$$

$$\boxed{y^2 - x^2 = 1}$$

Same Solⁿ

$$\text{Ex :- } \frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + \cos y}$$

$$2. e^x \tan y dx + (1+e^x) \sec^2 y dy = 0$$

$$3. \frac{dy}{dx} = 1+x+y+xy$$

$$\text{Sol}^n : -1. (\sin y + \cos y) dy = x(2\log x + 1) dx$$

$$-\cos y + \sin y = 2 \left[(\log x) \int x dx - \left(\int \frac{d}{dx} \log x \int x dx \right) dx \right] + \frac{x^2}{2} + C$$

$$-\cos y + \sin y = 2 \log x \cdot x^2 - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$\boxed{\sin y - \cos y = x^2 \log x + C} \quad \underline{\text{Ans}}$$

$$2. e^x \tan y dx = -(1+e^x) \sec^2 y dy$$

$$\frac{e^x}{1+e^x} dx = -\frac{\sec^2 y}{\tan y} dy$$

$$\log(1+e^x) = -\log \tan y + \log C \Rightarrow \log(1+e^x) \cdot \tan y = \log C$$

$$\boxed{(1+e^x) \tan y = C} \quad \underline{\text{Ans}}$$

dx

$$\frac{dy}{1+y} = (1+x)dx$$

$$\log(1+y) = x + \frac{x^2}{2} + C \quad \underline{\text{Ans}}$$

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L-3

Equations reducible to variable separable :

1) $\frac{dy}{dx} = \sin(x+y)$

Let $x+y=t$; Diff. w.r.t. x $1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$
 $\frac{dt}{dx} - 1 = \sin t \Rightarrow \int \frac{dt}{1+\sin t} = \int dx$

$\int \frac{1-\sin t}{(1-\sin t)(1+\sin t)} dt = x+c \Rightarrow \frac{1-\sin t}{\cos^2 t} dt = x+c$

$\int \sec^2 t dt - \int \sec t \tan t dt = x+c \Rightarrow \tan t - \sec t = x+c$

$$\tan(x+y) - \sec(x+y) = x+c \quad \underline{\text{Ans}}$$

Homogeneous D.E :

$\frac{dy}{dx} = f(x,y)$ where $f(x,y)$ is H.fⁿ of degree '0'.

Put $y = vx$ & solve.

Ex :- $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$ R.H.S is H.fⁿ of degree '0'.

Put $y = vx$, Diff. w.r.t. $x \rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

$$v + x \frac{dv}{dx} = v + \tan v$$

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x} = \log \sin v = \log x + \log c$$

$$\sin v = xc \Rightarrow \sin \frac{y}{x} - xc = 0 \quad \underline{\text{Ans}}$$

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

(i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ we can reduce it to homogeneous form.

Put $x = X+h$ & $y = Y+k$

$$\frac{dy}{dx} = \frac{a_1X + b_1Y}{a_2X + b_2Y} \rightarrow H.f^n$$

where h & k are given by

$$a_1h + b_1k + c_1 = 0$$

$$a_2h + b_2k + c_2 = 0$$

Ex :- $\frac{dy}{dx} = \frac{2x+y+6}{-x+y-3}$

find h & k which reduces this non H. form to H. form.

Put $x = X+h$ & $y = Y+k$

$$\overbrace{2h+k+6=0} \text{ & } \overbrace{-h+k-3=0}$$

$$\begin{array}{r} h \\ | \\ \begin{array}{rrrr} 1 & 6 & k & 2 \\ & -3 & -1 & 1 \end{array} \end{array}$$

$$\frac{h}{-3-6} = \frac{k}{-6+6} = \frac{1}{2+1}$$

$$\Rightarrow h = -3 \text{ & } k = 0 \quad \underline{\text{Ans}}$$

Substitution is $x = X-3$ &

$$y = Y$$

Ex :- $\frac{dy}{dx} = \frac{x-y+2}{x+y-1}$

$$\begin{array}{l} h-k+2=0 \\ h+k-1=0 \end{array}$$

$$2h+1=0$$

$$h = -\frac{1}{2} \text{ & } k = \frac{3}{2} \quad \underline{\text{Ans}}$$

Exact DE :-

Def :- The Differential form $Mdx + Ndy$ is called exact if there exists $U(x, y)$ such that

$$dU = Mdx + Ndy$$

Condition for exactness : $Mdx + Ndy$ exists if & only if (iff)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\text{iff } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

General Solⁿ :-

$$\int M dx + \int N dy = C$$

y const. (terms not containing
x or terms independent
of x)

$$\text{Ex} : - y dx + x dy = 0$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1 \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact}$$

General Solⁿ :- $\int y dx + \int \cancel{x dy} = C$

y const no x

$$xy = C \quad \underline{\text{Ans}}$$

$$\text{Ex} : - 2xy^2 dx + 2x^2 y dy = 0$$

$$\frac{\partial M}{\partial y} = 2xy \quad \frac{\partial N}{\partial x} = 2x^2 y \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact}$$

General Solⁿ :-

$$y^2 \int 2x dx + \int \cancel{2x^2 y dy} = C$$

y const no x

$$x^2 y^2 = C \quad \underline{\text{Ans}}$$

$$\text{Ex} : - (x^4 - 2xy^2 + y^4) dx - (2x^2 y - 4xy^3 + \sin y) dy = 0$$

$$M = x^4 - 2xy^2 + y^4, \quad N = -2x^2 y + 4xy^3 - \sin y$$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3, \quad \frac{\partial N}{\partial x} = -4xy + 4y^3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact}$$

Gen. Solⁿ :- $\frac{x^5}{5} - x^2 y^2 + xy^4 - \int \sin y dy = C$

$$\boxed{\frac{x^5}{5} - x^2 y^2 + xy^4 + \cos y = C} \quad \underline{\text{Ans}}$$

$$\left(\frac{1}{y^2}\right) [ydx - xdy] = 0$$

$\frac{ydx - xdy}{y^2} = 0 \Rightarrow d\left(\frac{x}{y}\right) = 0 \Rightarrow \boxed{\frac{x}{y} = C}$

makes non exact form to exact form. It is known as Integrating Factor (IF).

Integrating Factor - A factor which reduces a non-exact D.E. to exact D.E. [After multiplying IF non-exact becomes exact].

Non exact D.E. reducible to exact D.E. -

Methods of Finding IF :-

Let $Mdx + Ndy = 0$ is non exact DE.

Form I : M & N are homogeneous fn of same degree

$$I.F. = \frac{1}{Mx + Ny}, Mx + Ny \neq 0 \text{ (provided)}$$

Form 2 : $M = y f_1(x, y)$ & $N = x f_2(x, y)$

$$I.F. = \frac{1}{Mx - Ny}, \text{ provided } Mx - Ny \neq 0.$$

$$\underline{Ex} : -(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$$

$$M = x^2y - 2xy^2, N = -x^3 + 3x^2y$$

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \frac{\partial N}{\partial x} = -3x^2 + 6xy \quad \text{Non-exact}$$

But M & N are H.fn of same degree(3),

$$I.F. = \frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

$$\frac{x^2y - 2xy^2}{x^2y^2} dx - \frac{x^3 - 3x^2y}{x^2y^2} dy = 0 \quad \text{exact.}$$

Gen soln :-

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx - \int \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = C \Rightarrow \boxed{\frac{x}{y} - 2\log x + 3\log y = C} \quad \underline{\text{Ans}}$$

y const. no x

$$\begin{aligned}
 \underline{\text{Soln}}:- M &= xy^2 + 2x^2y^3 & N &= x^2y - x^3y^2 \\
 &= y(xy + 2x^2y^2) & &= x(xy - x^2y^2) \\
 &= y f_1(x, y) & &= x f_2(x, y)
 \end{aligned}$$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \neq \frac{\partial N}{\partial x} = 2xy - 3x^2y^2 \rightarrow \text{Non-exact}$$

But $M = y f_1(x, y)$ & $N = x f_2(x, y)$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{\cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3} = \frac{1}{3x^3y^3}$$

$$\frac{xy^2 + 2x^2y^3}{3x^3y^3} dx + \frac{x^2y - x^3y^2}{3x^3y^3} dy = 0 \rightarrow \text{Exact form.}$$

$$\begin{aligned}
 \text{gen. Soln:} &- \int \left(\frac{1}{3x^2y} + \frac{2x}{3x} \right) dx + \int \left(\frac{1}{3xy^2} - \frac{1}{3y} \right) dy = c \\
 &\text{y const.} \quad \text{no } x
 \end{aligned}$$

$$\boxed{\frac{-1}{3xy} + \frac{2}{3} \log x - \frac{1}{3} \log y = c} \quad \underline{\text{Ans.}}$$

$$\begin{aligned}
 \text{Form 3:} \quad & \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \quad [f^n \text{ of } x \text{ alone}]
 \end{aligned}$$

$$\text{I.F.} = e^{\int f(x) dx}$$

$$\begin{aligned}
 \text{Form 4:} \quad & \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) \quad [f^n \text{ of } y \text{ alone}]
 \end{aligned}$$

$$\text{I.F.} = e^{\int g(y) dy}$$

$$\underline{\text{Ex:}} -(1) (x^2 + y^2 + x) dx + xy dy = 0$$

$$\underline{\text{Sol:}} \quad M = x^2 + y^2 + x, \quad N = xy$$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = y \quad \text{Non-exact}$$

N is looking simpler so first check by it.

$$\frac{\partial y}{\partial x} = \frac{-y}{xy} = \frac{1}{x} = f(x)$$

$$I.F. = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$x(x^2+y^2+x)dx + x(xy)dy = 0 \rightarrow \text{Exact eqn}$$

Gen solⁿ :-

$$\int_{y \text{ const}} (x^3 + xy^2 + x^2)dx + \int_{no x} x^2 y dy = C$$

$$\boxed{\frac{x^4}{4} + \frac{x^2 y^2}{2} + \frac{x^3}{3} = C} \quad \underline{\text{Ans.}}$$

$$Ex :- (y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$$

$$Sol^n :- M = y^4 + 2y, N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \frac{\partial N}{\partial x} = y^3 - 4 \rightarrow \text{Non-exact}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \frac{-3y^3 - 6}{y^4 + 2y} = g(y) = -\frac{3}{y}$$

$$I.F. = e^{\int g(y)dy} = \frac{1}{y^3}$$

$$\frac{y^4 + 2y}{y^3} dx + \frac{xy^3 + 2y^4 - 4x}{y^3} dy = 0 \rightarrow \text{Exact}$$

Gen Solⁿ :-

$$\int_{y \text{ const}} \left(y + \frac{2}{y^2} \right) dx + \int_{no x} \left(x + 2y - \frac{4x}{y^3} \right) dy = C$$

$$\boxed{xy + \frac{2x}{y^2} + y^2 = C} \quad \underline{\text{Ans.}}$$

$$Q:- \text{If } x^\alpha y^\beta \text{ is I.F. of } (x^7 y^2 + 3y)dx + (3x^8 y - x)dy = 0$$

$$\alpha = ? \quad \beta = ?$$

$$(x^{\alpha+7}y^{\beta+2} + 3x^\alpha y^{\beta+1})dx + (3x^{\alpha+8}y^{\beta+1} - x^{\alpha+1}y^\beta)dy = 0$$

→ Exact

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ must be satisfied.

$$(\beta+2)x^{\alpha+7}y^{\beta+1} + (\beta+1)3x^\alpha y^\beta = (\alpha+8)3x^{\alpha+7}y^{\beta+1} - (\alpha+1)x^{\alpha+1}y^\beta$$

$$3(\alpha+8) = \beta+2 \quad \& \quad 3(\beta+1) = -(\alpha+1)$$

$$\alpha - \beta = -6$$

$$\cancel{\beta+1} = -\cancel{\alpha+1}$$

$$\cancel{\alpha+\beta} = \cancel{-2}$$

$$\begin{aligned} \cancel{\alpha-\beta} &= -6 \\ \cancel{\alpha+\beta} &= -2 \\ 2\alpha &= -8 \\ \alpha &= -4 \quad \& \quad \beta = 2 \end{aligned}$$

$$3\alpha - \beta = -22$$

$$\alpha + 3\beta = -4$$

$$10\alpha = -70$$

$$\boxed{\alpha = -7 \quad \& \quad \beta = 1} \quad \underline{\text{Ans.}}$$

Linear D.E. :-

Linear in y : $\frac{dy}{dx} + Py = Q$, where P, Q are fns of x

$$\text{I.F.} = e^{\int P dx}$$

General Solⁿ :-

$$\boxed{y \times \text{I.F.} = \int Q \cdot \text{I.F.} dx + C}^*$$

Linear in x : $\frac{dx}{dy} + Px = Q$, where P, Q are fns of y

$$\text{I.F.} = e^{\int P dy}$$

General Solⁿ :-

$$\boxed{x \times \text{I.F.} = \int Q \cdot \text{I.F.} dy + C}$$

$$\text{ex.:- } \boxed{e^{\int \frac{q}{x} dx} = x^a}$$

$$e^{a \log x} = e^{\log x^a}$$

$$Q:- x^2 \frac{dy}{dx} + 2xy = \frac{2 \log x}{x}$$

$$\text{Sol}^n:- \quad \frac{dy}{dx} + \frac{2}{x}y = \frac{2 \log x}{x^3} \quad P = \frac{2}{x}, \quad Q = \frac{2 \log x}{x^3}$$

(Linear in y)

$$\text{Gen. Soln} :- y \cdot x^2 = \int Q \cdot I.F. dx + C = 2 \int \log x \cdot \frac{1}{x} dx + C$$

$$\text{Let } \log x = t ; \frac{1}{x} dx = dt$$

$$y x^2 = \frac{x^2 t^2}{2} + C \Rightarrow \boxed{x^2 y = [\log x]^2 + C} \quad \underline{\text{Ans}}$$

$$Q:- (x+2y^3) \frac{dy}{dx} = y$$

$$\underline{\text{Soln}}:- y \frac{dx}{dy} = x + 2y^3 \Rightarrow \frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{1}{y} x = 2y^2 \quad P = -\frac{1}{y}, Q = 2y^2, \text{ IF} = e^{\int \frac{1}{y} dy} = \frac{1}{y}$$

(Linear in x)

$$G.S:- x \cdot \text{IF} = \int Q \cdot \text{IF} dy + C$$

$$\frac{x}{y} = \int 2y^2 \times \frac{1}{y} dy + C \Rightarrow \boxed{\frac{x}{y} = y^2 + C} \quad \underline{\text{Ans}}$$

$$Q:- \text{Maxm value of } y ; y' + 2y \tan x = \sin x ; y(\pi/3) = 0$$

$$\underline{\text{Soln}}:- \frac{dy}{dx} + (2 \tan x) y = \sin x \rightarrow \text{Linear in } y.$$

$$\text{I.F.} = e^{\int 2 \tan x dx} = e^{2 \log \sec x} = \sec^2 x$$

$$G.S:- y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} dx + C$$

$$y \cdot \sec^2 x = \int \sin x \cdot \sec^2 x dx + C = \int \tan x \sec x dx + C$$

$$y \sec^2 x = \sec x + C \Rightarrow \boxed{y = \cos x + C \cos^3 x}$$

$$y(\frac{\pi}{3}) = 0 \Rightarrow x = \frac{\pi}{3}, y = 0$$

$$0 = \frac{1}{2} + C \frac{1}{4} \Rightarrow \boxed{C = -2} \Rightarrow y = \cos x - 2 \cos^3 x$$

$$1) \frac{dy}{dx} = -\sin x + 4 \cos x \sin x = -\sin x + 2 \sin 2x$$

$$2) y' = 0 \Rightarrow -\sin x + 4 \cos x \sin x = 0 \\ \sin x(4 \cos x - 1) = 0 \Rightarrow x = 0 \text{ or } \cos x = \frac{1}{4}$$

$$3) y'' = -\cos x + 4 \cos 2x = -\cos x + 4[2 \cos^2 x - 1] \Rightarrow y''(0) = 3 > 0 \text{ min } x$$

$$\cos x = \frac{1}{4}$$

$$\text{Max}^m \text{ value of } y = y \Big|_{\cos x = \frac{1}{4}} = \frac{1}{4} - 2\left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8} \text{ Ans.}$$

Non-linear eqn's reducible to linear form :-

$$\underline{\text{Bernoulli's eqn}} : - \frac{dy}{dx} + Py = Qy^n$$

$$\underline{\text{Method}} : - y^n \frac{dy}{dx} + Py^{1-n} = Q \quad \text{Put } [y^{1-n} = t]$$

$$\underline{\text{Ex}} : - \frac{dy}{dx} + \frac{y}{x} = x^2 y^6$$

$$y^{-6} \frac{dy}{dx} + \frac{1}{x} y^{1-6} = x^2$$

$$\text{Put } y^{-5} = t$$

$$-5y^{-6} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow y^{-6} \frac{dy}{dx} = \frac{1}{5} \frac{dt}{dx}$$

$$-\frac{1}{5} \frac{dt}{dx} + \frac{1}{x} t = x^2$$

$$\frac{dt}{dx} - \frac{5}{x} \cdot t = -5x^2$$

$$\text{I.F.} = \frac{1}{x^5}$$

G.S. :-

$$\begin{aligned} t \cdot \frac{1}{x^5} &= \int -5x^2 \frac{1}{x^3} dx + C \\ &= -5 \int \frac{1}{x^3} dx + C \\ &= \frac{-5}{-2} \cdot x^2 + C \end{aligned}$$

$$\frac{x}{x^5} = \frac{5}{2} x^2 + C$$

$$\boxed{\frac{1}{x^5 y^5} = \frac{5}{2} x^2 + C} \text{ Ans.}$$

$$2) \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{x^2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

$$\text{Put } \tan y = t, \sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + 2x \cdot t = x^3$$

$$\text{I.F.} = e^{\int 2x dx} = e^{x^2}$$

G.S. :-

$$t \cdot e^{x^2} = \int x^3 e^{x^2} + C$$

$$\text{Put } x^2 = u \Rightarrow 2x dx = \frac{du}{2}$$

$$t \cdot e^{x^2} = \int \frac{u}{2} du + C$$

$$t \cdot e^{x^2} = \frac{u}{2} \left[u \cdot e^u - 1 \cdot e^u \right] + C$$

$$\tan y \cdot e^{x^2} = \frac{e^{x^2}}{2} [x^2 - 1] + C$$

$$\boxed{e^{x^2} \tan y = \frac{e^{x^2}}{2} [x^2 - 1] + C} \text{ Ans.}$$

which every member of one family cuts every member of other family at right angles are called OT's.

Method :-

- 1) $F(x, y, c) = 0 \rightarrow$ Family of curves
- 2) Eliminating c , we get $f(x, y, \frac{dy}{dx}) = 0 \rightarrow$ D.E. of family.
- 3) Replace $\frac{dy}{dx} = -\frac{dx}{dy}$, we get $f(x, y, -\frac{dx}{dy}) = 0 \rightarrow$ D.E. of OT.
 $(\because m_1 m_2 = -1)$
- 4) Solve: $G(x, y, c) = 0 \rightarrow$ O.T.

Polar coordinates -

Method is same except replace $\frac{dr}{d\theta}$ with $-r^2 \frac{d\theta}{dr}$

Ex :- 1) $y = k(x-1)$ 2) $y^2 = 4a(x+a)$ 3) $r = a(1+\cos\theta)$. Find OT.

Soln :- \downarrow Family

1) $y' = k \Rightarrow y = y'(x-1) \rightarrow$ D.E. of family

Replace $\frac{dy}{dx} = -\frac{1}{y'} \Rightarrow y = -\frac{1}{y'}(x-1) \Rightarrow [yy' = (1-x)] \rightarrow$ D.E. of OT.

$y \frac{dy}{dx} = (1-x) \Rightarrow y dy = (1-x) dx \Rightarrow \boxed{\frac{y^2}{2} = x - \frac{x^2}{2} + C} \rightarrow$ O.T. Ans.

2) $y^2 = 4a(x+a) \rightarrow$ Family

$2yy' = 4a \Rightarrow y^2 = 2yy'(x + \frac{yy'}{2}) \Rightarrow \frac{y}{2y'} = x + \frac{yy'}{2} \rightarrow$ D.E. of family

Replace $y' = \frac{1}{y} \Rightarrow -\frac{yy'}{2} = x - \frac{y}{2y'} \Rightarrow \frac{y}{2y'} = x + \frac{yy'}{2} \rightarrow$ D.E. of OT.

D.E. of family = D.E. of O.T. \Rightarrow Family = OT

These are called as self orthogonal.

\rightarrow It means every member of the family cuts each other.

$$\frac{dr}{d\theta} = -a \sin \theta \Rightarrow \frac{\frac{dr}{d\theta}}{r} = \frac{-a \sin \theta}{\alpha(1+\cos \theta)}$$

$$\boxed{\frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin \theta}{1+\cos \theta}} \rightarrow \text{DE of family}$$

$$\text{Replace } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr} \Rightarrow r \frac{d\theta}{dr} = \frac{\sin \theta}{1+\cos \theta} \rightarrow \text{DE of OT}$$

Var. S. :-

$$\frac{1+\cos \theta}{\sin \theta} d\theta = \frac{dr}{r} \Rightarrow \int \frac{2\cos^2 \theta/2}{2\sin \theta \cos \theta/2} d\theta = \log r + \log c$$

$$2 \int \cot \frac{\theta}{2} d\theta = \log r + \log c = 2 \log \sin \frac{\theta}{2} = \log cr$$

$$\Rightarrow \sin^2 \frac{\theta}{2} = cr \Rightarrow cr = \frac{1-\cos \theta}{2}$$

$$r = \frac{1}{2c} (1-\cos \theta)$$

$$\boxed{r = b(1-\cos \theta)} \rightarrow \text{OT}$$

Ans.

Linear Differential Eqn's (LDE) -

General form of n^{th} order LDE

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = x \rightarrow f^n \text{ of } x \quad \text{--- (1)}$$

$$[a_0 D^n + a_1 D^{n-1} + \dots + a_n] y = x$$

1. $x \equiv 0 \rightarrow$ Homogeneous LDE (HLDE)

2. $x \neq 0 \rightarrow$ Non H. LDE (NHLDE)

$a_0, a_1, a_2, \dots, a_n$ are all constant \rightarrow LDE with const. coefficients

otherwise (1) is L.D.E. with variable coefficients.

~~Ans~~

$$[a_0 D^n + a_1 D^{n-1} + \dots + a_n] y = x$$

$$F(D) \cdot y = x$$

$$\text{HLDE : } F(D) \cdot y = 0$$

$$\text{2nd order HLDE : } [a_0 D^2 + a_1 D + a_2] y = 0$$

$$\text{Auxiliary eqn : } -a_0 m^2 + a_1 m + a_2 = 0, \Rightarrow m_{1,2} = \dots$$

(A.T.)

Roots can be →

Roots $\xrightarrow{\text{(L.I.)}}$

1. Real & distinct

$(b_1 \neq b_2)$

$$e^{b_1 x}, e^{b_2 x}$$

General Solⁿ

$$c_1 e^{b_1 x} + c_2 e^{b_2 x}$$

2. Real & equal

$(b_1 = b_2)$

$$e^{bx}, x e^{bx}$$

$$c_1 e^{bx} + c_2 x e^{bx}$$

$$(c_1 + c_2 x) e^{bx}$$

3. Complex conjugates

$(a \pm i b)$

$$e^{ax} \cos bx, e^{ax} \sin bx$$

$$e^{ax} (c_1 \cos bx + c_2 \sin bx)$$

Ex:-

Roots

Gen. Solⁿ

1) 1, 2

$$c_1 e^x + c_2 e^{2x}$$

2) ± 1

$$c_1 e^x + c_2 e^{-x}$$

3) 2, 2

$$(c_1 + c_2 x) e^{2x}$$

4) 2, 2, 2

$$(c_1 + c_2 x + c_3 x^2) e^{2x}$$

5) $1 \pm i$

$$e^x [c_1 \cos x + c_2 \sin x]$$

6) $\frac{1 \pm i\sqrt{3}}{2}$

$$e^{x/2} \left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right]$$

7) $\pm 2i$

$$c_1 \cos 2x + c_2 \sin 2x$$

8) 1, 2, 3

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

9) 2, 3, 3

$$c_1 e^{2x} + (c_2 + c_3 x) e^{3x}$$

10) 2, 1 $\pm i$

$$c_1 e^{2x} + e^x [c_2 \cos x + c_3 \sin x]$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

$$3. \frac{d^2y}{dx^2} + 9y = 0$$

$$4. \frac{d^2y}{dx^2} - 9y = 0 \quad 5. y''' - y = 0$$

$$6. y'' + y' + y = 0$$

$$7. [D^4 + D^2 + 1]y = 0$$

Sol:-

$$\textcircled{1} \text{ A.E. : } m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$$

$$y = C_1 e^{2x} + C_2 e^{3x}$$

$$\textcircled{2} (D^2 - 4D + 4)y = 0 \Rightarrow \text{A.E. : } m^2 - 4m + 4 = 0 \quad m = 2, 2$$

$$y = (C_1 + C_2 x)e^{2x}$$

$$\textcircled{3} (D^2 + 9)y = 0 \Rightarrow \text{A.E. : } m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$$y = C_1 \cos 3x + C_2 \sin 3x$$

$$\textcircled{4} (D^2 - 9)y = 0 \Rightarrow \text{A.E. : } m^2 - 9 = 0 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3$$

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

$$\textcircled{5} (D^3 - 1)y = 0 \Rightarrow \text{A.E. : } m^3 - 1 = 0 \Rightarrow m^3 = 1 \Rightarrow m = 1, \omega, \omega^2$$

$$y = (C_1 e^{x/2} + C_2 x e^{x/2} + C_3 x^2 e^{x/2}) e^{-x/2} \quad (m+1)(m^2+m+1) = 0$$

$$\textcircled{6} (D^2 + D + 1)y = 0 \Rightarrow \text{A.E. : } m^2 + m + 1 = 0 \Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$y = e^{-x/2} \left[C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right] \quad m = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\textcircled{7} \text{ A.E. : } m^4 + m^2 + 1 = 0 \Rightarrow m^4 + m^2 + m^2 + 1 - m^2 = 0$$

$$y = C_1 e^{x/2} + C_2 e^{-x/2} \quad m^4 + 2m^2 + 1 - m^2 = 0$$

$$e^{x/2} \left[C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right] \quad (m^2 + 1)^2 - m^2 = 0$$

$$+ e^{-x/2} \left[C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x \right] \quad (m^2 + 1 - m)(m^2 + 1 + m) = 0$$

$$m^2 + 1 - m = 0 \quad \& \quad m^2 + 1 + m = 0$$
$$m = \frac{1 \pm \sqrt{1-4}}{2}; \quad m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$m = \frac{1 \pm i\sqrt{3}}{2}; \quad m = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\text{Soln} :- (D^3 - 9D^2 + 23D - 15)y = 0$$

$$A.E. = m^3 - 9m^2 + 23m - 15 = 0$$

→ Roots are factors of constants

$$\begin{array}{|r|rrrrr|} \hline 1 & 1 & -9 & 23 & -15 \\ \hline & +0 & +1 & -8 & +1 & 15 \\ & -1 & -8 & 15 & 0 & \\ \hline \end{array}$$

If zero comes here 1 is factor.
(or any other)

$$(m-1)(m^2 - 8m + 15) = 0 \Rightarrow m=1, 3, 5$$

$$y = C_1 e^x + C_2 e^{3x} + C_3 e^{5x}$$

$$\textcircled{9} \quad x'' + 6x' + 8x = 0 ; x(0) = 1 ; x'(0) = 0$$

Soln :- If D.E. is in x then independent var. is 't'.

$$(D^2 + 6D + 8)x = 0 \Rightarrow A.E.: m^2 + 6m + 8 = 0 \Rightarrow m = -2, -4$$

$$x = C_1 e^{-2t} + C_2 e^{-4t}$$

$$x(0) = 1 \Rightarrow 1 = C_1 + C_2$$

$$x'(0) = 0 \Rightarrow x' = -2C_1 e^{-2t} - 4C_2 e^{-4t}$$

$$0 = -2C_1 - 4C_2 \Rightarrow C_1 + 2C_2 = 0$$

$$C_1 + C_2 = 1$$

$$\underline{C_1 + 2C_2 = 0}$$

$$-C_2 = 1$$

$$C_2 = -1 \quad \& \quad C_1 = 2$$

$$x = 2e^{-2t} - e^{-4t} \quad \underline{\text{Ans}}$$

$$\textcircled{10} \quad x'' + 9x = 0 ; x(0) = 1, x'(0) = 1$$

$$\text{Soln} :- (D^2 + 9)x = 0 \Rightarrow A.E.: m^2 + 9 = 0 \Rightarrow m = \pm i\sqrt{3}$$

$$x = C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t$$

$$x(0) = 1 \Rightarrow 1 = C_1 + 0 \Rightarrow C_1 = 1$$

$$x'(0) = 1 \Rightarrow x' = -C_1 \sin \sqrt{3}t + C_2 \cos \sqrt{3}t$$

$$1 = 0 + 3C_2 \Rightarrow C_2 = 1/3$$

$$x = \cos \sqrt{3}t + \frac{1}{3} \sin \sqrt{3}t \quad \underline{\text{Ans}}$$

$$\text{General Sol}^m : \boxed{y = y_c + y_p}^*$$

where y_c = Complementary f'n : It is sol^m of corresponding
 $\xrightarrow{(C.F.)}$ having constants $F(D)y = 0$.

y_p = Particular Integral or Particular Sol^m.
 $\xrightarrow{(P.I.)}$
 = Sol^m of $F(D)y = x \rightarrow \cancel{\text{not}}$
 → Not having any const.

$$\boxed{y_p = \frac{1}{F(D)} x}$$

Note :- 1) $\frac{1}{D}$ means Integration

$$\text{ex } \frac{1}{D} \sin 2x = -\frac{\cos 2x}{2}$$

$$2) \frac{1}{D-a} x = e^{ax} \int x e^{-ax} dx$$

$$\text{ex} :- \frac{1}{D-2} e^{5x} = e^{2x} \int e^{5x} e^{-2x} dx = e^{2x} \int e^{3x} dx = e^{2x} \frac{e^{3x}}{3}$$

$$= \frac{e^{5x}}{3}$$

$$\text{Shortcut} : \frac{1}{D-2} e^{5x} \xrightarrow{D=5} \frac{e^{5x}}{3}$$

Methods of finding PI :-

$$1) x = e^{ax} \Rightarrow y_p = \frac{1}{F(D)} e^{ax}$$

$$\text{Put } D=a$$

$$y_p = \frac{1}{F(a)} e^{ax}, \text{ if } F(a) \neq 0$$

$$\text{If } F(a)=0 ; y_p = x \left[\frac{1}{F'(D)} e^{ax} \right]$$

$$\text{Put } D=a ; y_p = x \left[\frac{1}{F'(a)} e^{ax} \right], \text{ if } F'(a) \neq 0$$

$$Y_p = \frac{1}{F(D)} \sin ax = \frac{1}{\phi(D^2, D)} \sin ax$$

$$\text{Put } D^2 = -a^2$$

$$Y_p = \frac{1}{\phi(-a^2, D)} \sin ax \quad \text{if } \phi(-a^2, D) \neq 0$$

$$\text{If } \phi(-a^2, D) = 0 \Rightarrow Y_p = x \left[\frac{1}{\phi'(D^2, D)} \sin ax \right]$$

$$\text{Put } D^2 = -a^2 \text{ if } \phi'(D^2, D) \neq 0.$$

& so on.

$$\underline{\text{Ex:}} \quad y'' - 3y' + 2y = 5e^{4x}$$

$$\underline{\text{Soln:}} \quad (D^2 - 3D + 2)y = 5e^{4x}$$

$$\text{gen. soln: } Y_c + Y_p$$

$$\text{For } Y_c: A.E. = m^2 - 3D + 2 = 0 \Rightarrow m = 1, 2$$

$$Y_c = C_1 e^x + C_2 e^{2x}$$

$$\text{For } Y_p: Y_p = \frac{1}{D^2 - 3D + 2} 5e^{4x}; \quad \text{Put } D = 4$$

$$Y_p = \frac{1}{6} 5e^{4x} = \frac{5}{6} e^{4x}.$$

$$\text{gen. Soln: } Y = C_1 e^x + C_2 e^{2x} + \frac{5}{6} e^{4x} \quad \underline{\text{Ans.}}$$

$$2) \quad y'' - 4y' + 4y = 3e^{2x}$$

$$\underline{\text{Soln:}} \quad (D^2 - 4D + 4)y = 3e^{2x}$$

$$Y_c: A.E.: m^2 - 4m + 4 = 0; \quad m = 2, 2$$

$$Y_c = [C_1 + C_2 x] e^{2x}$$

$$Y_p: Y_p = \frac{1}{D^2 - 4D + 4} 3e^{2x}$$

$$\text{Put } D=2, \text{ Denominator}(D_r) = 0$$

$$= 3x \left[\frac{1}{2D-4} e^{2x} \right]$$

$$\text{Put } D=2, D_r = 0$$

$$Y_p = \frac{3x^2 e^{2x}}{2}$$

$$\text{gen sol}^n = Y_c + Y_p = (C_1 + C_2 x) e^{2x} + \frac{3x^2 e^{2x}}{2} \quad \underline{\text{Ans}}$$

$$3. \quad y'' - 5y' + 6y = \log 2$$

$$\text{Sol}^n: [D^2 - 5D + 6]y = \log 2$$

$$Y_p = \frac{1}{D^2 - 5D + 6} \log 2 \cdot e^{0x}$$

$$\text{Put } D=0 ; \quad Y_p = \frac{\log 2}{6}$$

$$Q:- 1. \quad y'' + 2y' + y = \sin 2x$$

$$\text{Sol}^n: [D^2 + 2D + 1]y = \sin 2x$$

$$Y_c: \text{A.E. : } m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$$

$$Y_c = [C_1 + C_2 x] e^{-x}$$

$$Y_p = \frac{1}{[D^2 + 2D + 1]} \sin 2x$$

$$\text{Put } D^2 = -2^2 = -4 \Rightarrow Y_p = \frac{1}{-4 + 2D + 1} \sin 2x = \frac{1}{2D - 3} \sin 2x$$

$$Y_p = \frac{2D + 3}{4D^2 - 9} \sin 2x ; \quad \text{Put } D^2 = -4$$

$$Y_p = \frac{2D + 3}{-16 - 9} \sin 2x = \frac{2D + 3}{-25} \sin 2x = \frac{-1}{25} [2D \sin 2x + 3 \sin 2x]$$

$$Y_p = \frac{-1}{25} [4 \cos 2x + 3 \sin 2x]$$

$$\text{gen. Sol}^n: \quad y = [C_1 + C_2 x] e^{-x} - \frac{1}{25} [4 \cos 2x + 3 \sin 2x] \quad \underline{\text{Ans}}$$

$$2. \quad [D^2 + 4]y = \cos 2x$$

$$\text{Sol}^n: \quad Y_p = m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$Y_c = C_1 \cos 2x + C_2 \sin 2x$$

Put $D^2 = -4$, $Dx = 0$

$$Y_p = x \left[\frac{1}{2D+4} \cos 2x \right] = \frac{x \sin 2x}{4}$$

$$Y = C_1 \cos 2x + C_2 \sin 2x + \frac{x \sin 2x}{4} \quad \underline{\text{Ans}}$$

Formulas :-

$$1) \boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}}$$

$$\underline{\text{Ex}}:- \frac{1}{(D-2)^2} e^{2x} \Rightarrow \frac{x^2}{2!} e^{2x}$$

$$2) \boxed{\frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax}$$

$$3) \boxed{\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax}$$

$$\underline{\text{Ex}}:- \frac{1}{D^2+4} \cos 2x = \frac{x}{4} \sin 2x$$

$$\underline{\text{Ex}}:- D^2 x^2 = 2 \quad D^4 x^2 = 0 \\ D^3 x^2 = 0 \quad |$$

17 Dec 2014
L-4

Binomial Expansions :-

$$(1-x)^{-1} = 1+x+x^2+x^3+x^4+\dots$$

$$(1+x)^{-1} = 1-x+x^2-x^3+x^4-\dots$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

Form 3 : $x = x^n$

$$Y_p = \frac{1}{F(D)} x^n = \frac{1}{(1+\phi(D))^n} x^n = (1+\phi(D))^{-n} x^n$$

$$Y_C: AE = m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$$

$$Y_C = [C_1 + C_2 x] e^{-x}$$

$$Y_P: Y_P = \frac{1}{D^2 + 2D + 1} \cdot x^2 = \frac{1}{[1 + (D^2 + 2D)]^1} x^2$$

$$Y_P = [1 + (D^2 + 2D)^{-1}] x^2 = [1 - (D^2 + 2D) + (D^2 + 2D)^2] x^2$$

$$Y_P = [1 - D^2 - 2D + D^4 + 4D^3 + 4D^2] x^2 \quad [\text{Higher order terms vanish.}]$$

$$= x^2 - 2 - 4x + 8 = x^2 - 4x + 6$$

$$(\text{Or}) \quad Y_P = \frac{1}{(D+1)^2} x^2 = [1 + D]^2 x^2 = [1 - 2D + 3D^2] x^2 \\ = x^2 - 4x + 6$$

Form 4: $x = e^{ax} \cdot V$, V is a f^n of x .

$$Y_P = \boxed{\frac{1}{F(D)} e^{ax} \cdot V = e^{ax} \left[\frac{1}{F(D+a)} \cdot V \right]}$$

Form 5: $x = x \cdot V$, V is a f^n of x .

$$Y_P = \boxed{\frac{1}{F(D)} x \cdot V = x \frac{1}{F(D)} V - \frac{F'(D)}{[F(D)]^2} V}$$

$$\text{Ex: } (D^2 + 2D + 1)y = e^x \cdot \sin 2x$$

$$Y_P = \frac{1}{D^2 + 2D + 1} e^x \cdot \sin 2x = e^x \left[\frac{1}{(D+1)^2 + 2(D+1) + 1} \right] \sin 2x$$

$$= e^x \left[\frac{1}{D^2 + 4D + 4} \sin 2x \right] = e^x \left[\frac{1}{-4 + 4D + 4} \sin 2x \right]$$

$$= -\frac{e^x}{4} \frac{\cos 2x}{2} = -\frac{e^x \cos 2x}{8} \quad \underline{\text{Ans.}}$$

$$(D^2 + g)y = x \cdot \sin x$$

$$y_p = \frac{1}{D^2 + g} [x \sin x] = x \frac{1}{D^2 + g} \sin x - \frac{2D}{(D^2 + g)^2} \sin x$$

$$= x \frac{1}{-1+g} \sin x - \frac{2D}{(-1+g)^2} \sin x = \frac{x}{8} \sin x - \frac{1}{32} \cos x \quad \text{Ans.}$$

Ex:- Find y_p : $y'' - 2y' + y = xe^x \sin x$

Sol":- $(D^2 - 2D + 1)y = xe^x \sin x$

$$y_p = \frac{1}{D^2 - 2D + 1} e^x (x \sin x) = e^x \left[\frac{1}{(D+1)^2 - 2(D+1) + 1} x \sin x \right]$$

$$y_p = e^x \left[\frac{1}{D^2} (x \sin x) \right] = e^x \left[x \frac{1}{D^2} \sin x - \frac{(D^2)'}{(D^2)^2} \sin x \right]$$

$$= e^x \left[-x \sin x - \frac{2D}{(-1)^2} \sin x \right] = e^x \left[-x \sin x - 2 \cos x \right]$$

L.D.E. with variable coefficients — reducible to L.D.E. with constant coefficients.

1) Cauchy - Euler Homogeneous L.D.E. —

Homogeneous

$$\boxed{a_0 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = x}$$

$$[a_0 x^2 D^2 + a_1 x D + a_2] y = x$$

Put $x = e^z$ (or) $z = \log x$

$$xD \equiv D_1 \quad ; \quad D = \frac{d}{dx}, \quad D_1 = \frac{d}{dz}$$

$$x^2 D^2 = D_1(D_1 - 1)$$

$$x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$$

⋮

Lengendre's L.D.E. —

$$C_0(ax+b)^2 \frac{d^2 y}{dx^2} + C_1(ax+b) \frac{dy}{dx} + C_2 y = x$$

Put $ax+b = e^z$ (or) $z = \log(ax+b)$

$$(ax+b)D = aD_1$$

$$(ax+b)^2 D^2 = a^2 D_1(D_1-1)$$

$$(ax+b)^3 D^3 = a^3 D_1(D_1-1)(D_1-2)$$

$$D = \frac{d}{dx} \quad \& \quad D_1 = \frac{d}{dz}$$

$$Q: 1. [x^2 D^2 - 4xD + 6]y = x^2$$

$$x = e^z \text{ or } z = \log x$$

$$xD = D_1 \quad \& \quad x^2 D^2 = D_1(D_1-1); \quad D_1 = \frac{d}{dz}$$

$$[D_1(D_1-1) - 4D_1 + 6]y = (e^z)^2$$

$$[D_1^2 - 5D_1 + 6]y = e^{2z}$$

$$y_c = m^2 - 5m + 6; \quad m = 2, 3$$

$$y_c = C_1 e^{2z} + C_2 e^{3z}$$

$$y_p = \frac{1}{D_1^2 - 5D_1 + 6} e^{2z}$$

$$\text{Put } D_1 = 2, \quad D_2 = 0$$

$$y_p = z \frac{1}{2D_1 - 5} e^{2z}$$

$$\text{Put } D_1 = 2$$

$$y_p = -z e^{2z}$$

$$y = y_c + y_p$$

$$= C_1 e^{2z} + C_2 e^{3z} - z e^{2z}$$

$$= C_1 e^{2 \log x} + C_2 e^{3 \log x} - \log x e^{2 \log x}$$

$$y = C_1 x^2 + C_2 x^3 - x^2 \log x$$

$$2. (2x-1)^2 y'' + (2x-1) y' - 2y = 0$$

$$\text{Put } (2x-1) = e^z \text{ or } z = \log(2x-1)$$

$$(2x-1)D = 2D_1$$

$$(2x-1)^2 D^2 = 4 D_1(D_1-1) \quad D_1 = \frac{d}{dz}$$

$$[4D_1(D_1-1) + 2D_1 - 2]y = 0$$

$$(4D_1^2 - 2D_1 - 2)y = 0$$

$$4m^2 - 2m - 2 = 0$$

$$\text{or} \quad 2m^2 - 1 - 1 = 0$$

$$m = \frac{1 \pm \sqrt{3}}{4} = 1, -\frac{1}{2}$$

$$y = C_1 e^z + C_2 e^{-1/2} z$$

$$= C_1 e^{\log(2x-1)} + C_2 e^{\log(2x-1)^{-1/2}}$$

$$y = C_1 (2x-1) + C_2 (2x-1)^{-1/2}$$

$$\begin{vmatrix} 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

$$\begin{vmatrix} 5 & 2 \end{vmatrix} = 6 - 5 = 1$$

Wronskian :-

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

Result :- If $W(y_1, y_2) \neq 0$ then y_1 & y_2 are linearly Independent (L.I.)

If $W(y_1, y_2) = 0$ then y_1 & y_2 are linearly dependent (L.D.)

Ex :- $y_1 = 2x$ $y_2 = 3x$

$$W = \begin{vmatrix} 2x & 3x \\ 2 & 3 \end{vmatrix} = 6x - 6x = 0 \Rightarrow L.D. \quad y_2 = \frac{3}{2}y_1$$

\hookrightarrow We can express one in terms of other one.

Ex :- $y_1 = e^{2x}$, $y_2 = e^{3x}$

$$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} \neq 0 \quad L.I.$$

Wronskian of three f^n's: $W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}$

Method of variation of parameters :-

$$a_0y'' + a_1y' + a_2y = X$$

Let $y_c = c_1y_1 + c_2y_2$

$$Y_p = y_1 \int \frac{-y_2}{W} \cdot X dx + y_2 \int \frac{y_1}{W} \cdot X dx$$

$$Y_c : AE : m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$Y_c = C_1 \cos 2x + C_2 \sin 2x$$

$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2[\cos^2 2x + \sin^2 2x] = 2$$

By method of variation of parameters :-

$$\begin{aligned} Y_p &= y_1 \int \frac{-y_2}{W} \sec 2x dx + y_2 \int \frac{y_1}{W} \sec 2x dx \\ &= \cos 2x \left[\frac{-\sin 2x}{2} \right] \sec 2x dx + \sin 2x \int \frac{\cos 2x}{2} \sec 2x dx \\ &= \frac{\cos 2x}{2} \int -2 \frac{\sin 2x}{\cos 2x} dx + \frac{\sin 2x}{2} \int dx \end{aligned}$$

$$Y_p = \frac{\cos 2x}{4} \log(\cos 2x) + \frac{x \cdot \sin 2x}{2}$$

LINEAR ALGEBRA

OR MATRICES

Matrix - Rectangular array of no.s.

$$A = [a_{ij}]_{m \times n} ; \quad a_{ij} = i^{\text{th}} \text{ row } j^{\text{th}} \text{ column element}$$

$m \times n \rightarrow$ Order or size

$m \rightarrow$ no. of rows, $n \rightarrow$ no. of columns

Row matrix - $[a_{ij}]_{1 \times n}$
(Row Vector)

$$\underline{\text{Ex}} : - [1 \ 2 \ 3]_{1 \times 3}$$

Column Matrix - $[a_{ij}]_{m \times 1}$
(Column Vector)

$$\underline{\text{Ex}} : - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$$

$$\text{Matrix Multiplication} - A = [a_{ij}]_{m \times n} \quad B = [b_{ij}]_{n \times r}$$

No. of columns of A = No. of rows of B

$$C = AB = [c_{ij}]_{m \times r}$$

$$c_{ij} = [a_{i1} \ a_{i2} \ \dots \ a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

1) No. of multiplications reqd. to find $A_{m \times n} \times B_{n \times r} = m \times r \times n$

2) No. of additions reqd. to find $A_{m \times n} \times B_{m \times n} = m \times r \times (n-1)$

$$A_{m \times n} B_{n \times r} = C_{m \times r}$$

To find 1 element $\rightarrow n$ multiplication
 $\rightarrow \underbrace{\dots}_{m \times r} \ \underbrace{\dots}_{r} \ \rightarrow m \times r \times n$

2) To find 1 element $\rightarrow n-1$ additions
 $\rightarrow \underbrace{\dots}_{m \times r} \ \underbrace{\dots}_{r} \ \rightarrow m * r * (n-1)$

Ex :- No. of multiplications reqd $A_{4 \times 5} * B_{5 \times 6} = 4 \times 6 \times 5 = 120$ Ans.

Ex :- $A = [a_{ij}]_{m \times n}$ where $a_{ij} (i+j) \neq i, j$

Sum of elements of A = _____

$$\text{Sol}^n : A = \begin{bmatrix} 1+1 & 1+2 & \dots & 1+n \\ 2+1 & 2+2 & \dots & 2+n \\ \vdots & \vdots & & \vdots \\ m+1 & m+2 & \dots & m+n \end{bmatrix}$$

$$1^{\text{st}} \text{ row} \rightarrow (1+1+\dots n \text{ times}) + (1+2+3+\dots n) = n + \frac{n(n+1)}{2}$$

$$2^{\text{nd}} \text{ row} \rightarrow (2+2+\dots n \text{ times}) + (1+2+\dots n) = 2n + \frac{n(n+1)}{2}$$

$$m^{\text{th}} \text{ row} \rightarrow (m+m+\dots n \text{ times}) + (1+2+\dots n) = mn + \frac{n(n+1)}{2}$$

$$= n \cdot m \frac{(m+1)}{2} + mn \left(\frac{n+1}{2} \right)$$

$$= \frac{mn}{2} [m+n+2] \quad \underline{\text{Ans}}$$

Ex :- No. of different matrices with 4 elements whose elements can be either 0, 1, 2 or 3.

Soln :- $\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}_{4 \times 4}$ or $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}_{2 \times 2}$ (or) $\begin{bmatrix} \square \\ \square \\ \square \\ \square \end{bmatrix}_{4 \times 1}$

Each position can be filled in 4 ways 4^4 4^4 4^4

$$\rightarrow 4^4 + 4^4 + 4^4 \quad \underline{\text{Ans}}$$

Elementary Row Transformation (Operations) -

- 1) $R_i \leftrightarrow R_j$ (Interchanging two rows)
- 2) $R_i \rightarrow kR_i$ ($k \neq 0$) [Multiply a row with non-zero const.]
- 3) $R_i \rightarrow R_i + kR_j$ [Adding const multiple of other row]

Ex :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix}$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \\ 2 & 4 & 5 \end{bmatrix}$$

$$R_2 \rightarrow 2R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 12 & 14 \\ 2 & 4 & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 2 & 4 & 5 \end{bmatrix}$$

Leading non-zero element in a row - is the 1st non-zero element in the row.

$$\begin{bmatrix} 0 & 2 & 7 & 4 \\ 0 & 1 & 9 & 1 \\ 0 & 3 & 10 & 5 \end{bmatrix}$$

(or row echelon form)

Echelon Form - A matrix 'A' is said to be in echelon form

if 1) Zero rows (if any) must be at the bottom of the matrix.

right of leading non-zero entry in preceding row.

Ex:-

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon form

Reduced row echelon form.
(RREF)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon Form

$$\begin{bmatrix} 5 & 7 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

not echelon
form

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & \xrightarrow{+} 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon Form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Echelon form

$$\begin{bmatrix} 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 7 & 8 & 1 \\ 0 & 5 & 2 & 1 & 3 \end{bmatrix}$$

Not echelon form

Reduced row echelon form — A matrix 'A' is said to be in reduced rowechelon form, if

(1-2) A is echelon form.

3) The leading non-zero entry in any row(if exist) must be 1
(if exist) (Leading 1).

4) Leading 1, is the only non-zero entry in its column.

Ex:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

↳ only non-zero entry

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

↳ only non-zero entry.

Result — Every matrix can be reduced to

1) Echelon form (Gaussian elimination)

& 2) Reduced row echelon form (Gauss-Jordan elimination)

Using elementary row transformations.

1) Gaussian elimination \rightarrow Forward elimination

2) Gauss Jordan \rightarrow \rightarrow \rightarrow + Backward elimination

Note:— Before applying elimination techniques —

1) Transfer all zero rows to the bottom of the matrix.

2) Arrange rows in such a way that the row having more zeroes before leading non-zero element is below the row having less zero rows before leading non-zero element.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ Reduce it to echelon form} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank - The no. of non-zero rows in echelon form of matrix A is rank of A.

Notation - Rank (A) or $P(A)$

Gaussian elimination -

Ex :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

Using these make these to 0

Forward Elimination

$$\begin{array}{ccc|c} 1 & 2 & 3 & \\ 0 & 2 & 4 & \\ 0 & 0 & 6 & \end{array} \rightarrow \begin{array}{ccc|c} 1 & 2 & 3 & \\ 0 & 2 & 4 & \\ 0 & 0 & 6 & \end{array}$$

Moving fwd + below becomes zero.

$$R_2 \rightarrow R_2 - 2R_1 \quad \& \quad R_3 \rightarrow R_3 - 3R_1$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -4 & -5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

\sim

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P(A) = 3$$

Echelon Form

Backward Elimination :-

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 / -1$$

\sim

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

Using these

$$R_1 \rightarrow R_1 - 3R_3 \quad \& \quad R_2 \rightarrow R_2 + 5R_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + R_2 / 2 \quad \& \quad R_2 \rightarrow R_2 / -4$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced Row echelon form

$\left[\begin{array}{|c|c|c|} \hline | & | & | \\ \hline \end{array} \right]$ equal Rows

$$R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - R_1 \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 1$$

$$Q:- A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \quad \rho(A) = ?$$

$R_2 = 2R_1 \text{ & } R_3 = 3R_1$ (Proportional Rows)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 1$$

$$Q:- A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \quad \rho(A)$$

$R_3 = 2R_2, R_4 = 3R_2$

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

$$Q:- A = \begin{bmatrix} 1 & a & a^2 & \dots & a^n \\ 1 & a & a^2 & \dots & a^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a & a^2 & \dots & a^n \end{bmatrix} \quad \rho(A) = 1$$

$$\sim \begin{bmatrix} 1 & a & a^2 & \dots & a^n \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$Q:- A = [a_{ij}]_{m \times n} \quad a_{ij} = 5 \quad \forall i, j, \quad \rho(A) = \underline{\hspace{2cm}}$$

$$A = \begin{bmatrix} 5 & 5 & \dots & 5 \\ 5 & 5 & \dots & 5 \\ \vdots & \vdots & \ddots & \vdots \\ 5 & 5 & \dots & 5 \end{bmatrix} = \begin{bmatrix} 5 & 5 & \dots & 5 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \rho(A) = 1$$

$$Q:- \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 0 & 5 \\ 0 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \rho(A) = \underline{\hspace{2cm}}$$

$$R_4 \rightarrow R_4 - R_1$$

$$\sim \left[\begin{array}{ccccc} 1 & 2 & 1 & 4 \\ 0 & \textcircled{-3} & -2 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 0 \end{array} \right] \sim \left[\begin{array}{ccccc} 1 & 2 & 1 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & \textcircled{+}3 & 3 \\ 0 & 0 & \boxed{2} & 0 \end{array} \right] \sim \left[\begin{array}{ccccc} 1 & 2 & 1 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & -6 \end{array} \right]$$

$$\rho(A) = 4$$

Transpose —

$$A = [a_{ij}]_{m \times n}, A^T = [b_{ij}]_{n \times m} \quad \boxed{b_{ij} = a_{ji}} \quad \forall i, j$$

$$A = \left[\begin{array}{cc} 2 & 3 \\ 4 & 5 \\ 6 & -7 \end{array} \right]_{3 \times 2} \quad A^T = \left[\begin{array}{ccc} 2 & 4 & 6 \\ 3 & 5 & -7 \end{array} \right]_{2 \times 3}$$

Results —

$$1) (A^T)^T = A$$

$$2) (A+B)^T = A^T + B^T$$

$$3) (kA)^T = kA^T$$

$$4) (AB)^T = B^T A^T$$

Square Matrices :-

$$A = [a_{ij}]_{m \times n} \rightarrow \text{Square matrix of order } n.$$

Diagonal —

$$A = [a_{ij}]_{n \times n} \text{ where } a_{ij} = 0 \quad \forall i \neq j$$

$$\text{Ex: } \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{array} \right]$$

Upper Triangular Matrix —

$$A = [a_{ij}]_{n \times n} \text{ where } a_{ij} = 0 \quad \forall i > j$$

$$\text{Ex: } \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{array} \right]$$

$A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0 \forall i < j$

Identity Matrix -

$A = I_n = [a_{ij}]_{n \times n}$ where $a_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result :- $A_{n \times n} I_n = A_{n \times n}$

Invertible (Non-singular) - A square matrix $A_{n \times n}$ is invertible if there exist square matrix $B_{n \times n}$ such that

$$A \cdot B = B \cdot A = I_n$$

We write $B = A^{-1}$

Otherwise then A is non-invertible (singular).

Results :-

- 1) $(A^{-1})^{-1} = A$
- 2) $(kA)^{-1} = \frac{1}{k} A^{-1}$
- 3) $(AB)^{-1} = B^{-1}A^{-1}$
- 4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- 5) $I^{-1} = I$

Ex :- Let A, B, C, D, E, F are invertible matrices such that

$$ABCDEF = I, D^{-1} = \underline{\hspace{1cm}}$$

- a) ABCEF b) EFCBA c) EFABC d) ABCFE

$$\overbrace{ABCDEF}^D = I$$

$$D^{-1} = EFABC$$

Ex :- $ABC = I, B^{-1} = \underline{\hspace{1cm}}$

$$(ABC)^{-1} = I^{-1}$$

$$C^{-1}B^{-1}A^{-1} = I$$

$$C^{-1}B^{-1}A^{-1}A = IA \Rightarrow C^{-1}B^{-1} = IA$$

$$CC^{-1}B^{-1} = IA$$

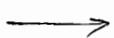
$$B^{-1} = CIA \text{ or } CA \quad \underline{\text{Ans}}$$

Ex :- X & Y are singular matrices such that

$$XY = Y \quad \& \quad YX = X$$

$$X^2 + Y^2 = \underline{\hspace{1cm}}$$

$$= YX + XY = X + Y$$



Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ (1) $x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n$

(2) If $x \cdot y = 0$ then x & y are orthogonal

Ex :- $x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ $x \cdot y = 2 - 2 = 0 \rightarrow$ orthogonal.

(3) Length or norm of a vector :

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Ex :- $x = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\|x\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$

(4) Normalising vector : $\frac{x}{\|x\|} \rightarrow$ Making new vector whose length is 1.

Ex :- $x = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ $\|x\| = \sqrt{4+1^2+4} = \sqrt{9} = 3$

$$y = \frac{x}{\|x\|} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$\|y\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$$

Orthonormal set - Let x_1, x_2, \dots, x_n be n vectors of dimension $n \times 1$.
The set of vectors x_1, x_2, \dots, x_n are said to be orthonormal.

$$x_i \cdot x_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Ex :- $x_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, x_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$x_1, x_2 \rightarrow$ Orthonormal set.

Def - A square matrix A is

(1) Symmetric if $A^T = A$ (2) Skew symmetric if $A^T = -A$

(3) Orthogonal if $A^T = A^{-1}$ (or) $A \cdot A^T = A^T \cdot A = I$

Ex :-) $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is symmetric matrix.

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is skew symmetric matrix

3) $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $A \cdot A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

A is orthogonal matrix. $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Q:- $A = \begin{bmatrix} 1 & 4 & 5 \\ a & 2 & 2 \\ b & c & 3 \end{bmatrix}$ is symm. then $a, b, c = ?$
 $a=4, b=5, c=2$

Q:- No. of symm. matrices of order n whose elements are either 0 or 1 = $\frac{n^2+n}{2}$

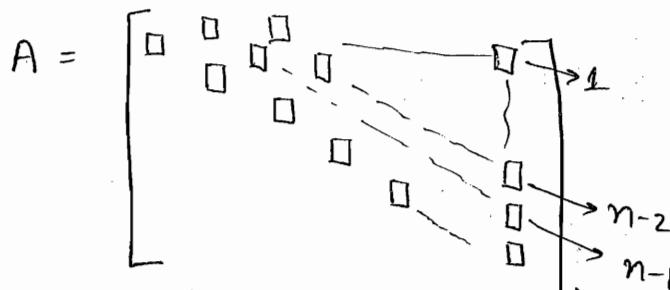
a) 2^{n^2}

b) $2^{\frac{n^2-n}{2}}$

$\checkmark \boxed{\frac{n^2+n}{2}}$

if elements 0, 1, 2

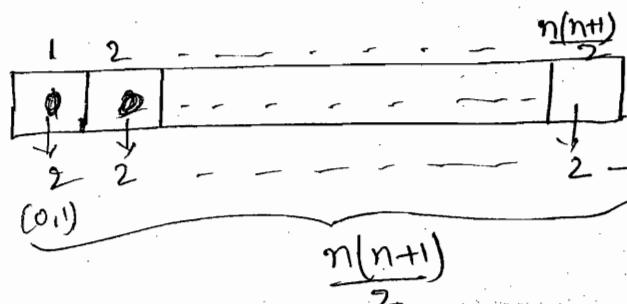
d) 2^n then 3^{ans}



To form a symmetric matrix of order n we need to fix (fill)

$$= n + (n-1) + \dots + 2 + 1$$

$$= \frac{n(n+1)}{2} \text{ elements}$$



$$2^{\frac{n(n+1)}{2}} = 2^{\frac{n^2+n}{2}}$$

Each position can be filled in 2 ways (0 or 1)

be 0.

(2) If A is square matrix :

1. $A + A^T$ is Symmetric Matrix

2. $A - A^T$ is Skew Symm. Matrix

$$(1) (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

$$(2) (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$$

$$3. A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Every matrix can be expressed as sum of symmetric & skew-symmetric matrix.

*4. The rows/columns of orthogonal matrix form orthonormal set

Ex :- $A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}_{n \times n}$ Row 1 is orthogonal.

$$\text{then } R_i \cdot R_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Q:- $A = R \begin{bmatrix} 1/9 & -4/9 & 8/9 \\ 8/9 & 4/9 & 1/9 \\ \alpha/9 & -7/9 & \beta/9 \end{bmatrix}$ is orthogonal $\alpha = ?$, $\beta = ?$

Sol:- A is orthogonal $\Rightarrow R_1 \cdot R_3 = R_2 \cdot R_3 = 0$

$$R_1 \cdot R_3 = 0 \Rightarrow \frac{\alpha + 28 + 8\beta}{81} = 0 \Rightarrow \alpha + 28 + 8\beta = 0$$

$$R_2 \cdot R_3 = 0 \Rightarrow \frac{8\alpha - 28 + \beta}{81} = 0 \Rightarrow 8\alpha - 28 + \beta = 0$$

$$\begin{array}{cccc} \alpha & \beta & 1 & \\ 8 & 28 & 1 & 8 \\ 1 & -28 & 8 & 1 \end{array} \Rightarrow \frac{\alpha}{-28 \times 8 - 28} = \frac{\beta}{28 \times 28 + 28} = \frac{1}{1 - 8 \times 8} = \frac{1}{-63}$$
$$\Rightarrow \alpha = 4 \quad \& \quad \beta = -4$$

Determinant - No. associated with square matrix

2×2 Matrix - $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Ex:- $\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 6 = -4$

Minor of a_{ij} - Determinant of square submatrix obtained by deleting i^{th} row & j^{th} column.

Ex:- $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$ $M_{31} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$ $M_{23} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$

Cofactor of a_{ij} - $C_{ij} = (-1)^{i+j} M_{ij}$

$$C_{31} = (-1)^{3+1} M_{31} = +M_{31} = +5 ; C_{22} = +8 ; C_{12} = -16$$

Cofactor matrix of A = Cofactor $A = \begin{bmatrix} +3 & -16 & +2 \\ -14 & +8 & -(-4) \\ +5 & -0 & +(-10) \end{bmatrix}$

Laplace Expansion :- $n \times n$ matrix

Row Expansion - $a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

Column Expansion - $a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$

Ex:- $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} \Rightarrow |A| = 2(3) + 3(-16) + 1(2) = -40$

Results:-

1) Expansion of determinant of 2×2 matrix contain 2 terms.

2) $n \times n$ of 3×3 contains $3 \cdot 2 = 3!$ terms.

3) $n \times n$ of $n \times n$ contains $n!$ terms.

(5) The matrix A has

(i) Zero row / zero column

(ii) Two proportional rows/columns

then $|A| = 0$.

Ex :- $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 5 & 100 \end{bmatrix} \Rightarrow |A| = 0$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 5 & 2 \end{bmatrix}$ Proportional rows
 $R_2 = 2R_1$, $|A| = 0$

Ex :- $\begin{bmatrix} 4 & 5 & 6 \\ 0 & 7 & 1 \\ 0 & 0 & 5 \end{bmatrix} = |A| = 4 * 7 * 5 = 140$

(6) If A is diagonal or upper triangular or lower triangular matrix, then $|A| = \text{Product of main diagonal entries}$.

* $|I| = 1$

(7) $A \xrightarrow{R_i \leftrightarrow R_j} B, |B| = -|A|$

(8) $A \xrightarrow{\substack{R_i \rightarrow kR_i \\ k \neq 0}} B, |B| = k|A|$

(9) $A \xrightarrow{R_i \rightarrow R_i + kR_j} B, |B| = |A|$

Ex :- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, |A| = 3$ $B = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}, |B| = -3$

$$C = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ 3g & 3h & 3i \end{bmatrix} = 2 * 3 * 3 = 18$$

\downarrow
 $|A|$

Ex :- $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -5 & -2 \\ -6 & 2 & 3 \end{bmatrix}$ $R_1 \rightarrow R_1 + R_2 + R_3, = \begin{bmatrix} 0 & 0 & 0 \\ 4 & -5 & -2 \\ -6 & 2 & 3 \end{bmatrix}$

$$|A| = 0$$

(10) If elements of each row add upto zero, $|A| = 0$

(11) If A is $n \times n$ matrix, $|kA| = k^n |A|$

Ex :- $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \Rightarrow |A|=0 \rightarrow \text{odd order } (3 \times 3)$

(12) Determinant of skew symmetric matrix of odd order is 0.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |A| = 1 \rightarrow \text{even-order } (2 \times 2)$$

$$(13) |AB| = |A||B|$$

Adjoint of A - $\text{Adj } A = [\text{cofactor } A]^T$

Formula :- $A^{-1} = \frac{\text{Adj } A}{|A|}$, $|A| \neq 0$

(1) ~~If~~ A is non singular iff $|A| \neq 0$

(2) A is singular iff $|A|=0$

(3) If A is invertible, $|A^{-1}| = \frac{1}{|A|}$

Ex :- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3 \times 3}, |A|=3$

$$\text{Det}(2A^{-1}) =$$

$$|(2A)^{-1}| = \left| \frac{1}{2} A^{-1} \right| = \left(\frac{1}{2} \right)^3 \frac{1}{|A|} = \frac{1}{24}$$

Ex :- $A = \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$; A is singular then $x =$

$$|A|=0 = -2(48-12x) \Rightarrow x=4 \text{ Ans.}$$

(4) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(5) $\text{Adj } A = |A|A^{-1}$ (6) $(\text{Adj } A)^{-1} = \frac{A}{|A|}$

$$(\text{Adj } A)^{-1} = (|A|A^{-1})^{-1} = \frac{1}{|A|}(A^{-1})^{-1} = \frac{A}{|A|}$$

$$\text{Adj } A = |A| A^{-1}$$

$$|\text{Adj } A| = \left| |A| A^{-1} \right| = |A|^n |A^{-1}| = \frac{|A^n|}{|A|} = |A|^{n-1}$$

$$8. \text{ Adj}(\text{Adj } A) = , A_{n \times n}$$

$$\text{Adj } A = |A| A^{-1}$$

$$\text{Adj}(\text{Adj } A) = |\text{Adj } A| (\text{Adj } A)^{-1} = |A|^{n-1} \cdot \frac{A}{|A|} = |A|^{n-2} A$$

Ex :- A is 3×3 matrix, $|A|=4$

$$|\text{Adj } A| = |A|^{3-1} = 4^2 = 16$$

$$\text{Adj}(\text{Adj } A) = |A|^{3-2} A = 4^1 A = 4A$$

$$Q:- A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix} \quad \text{Adj } A = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & K & 7 \end{bmatrix}$$

$$K = ? , A^{-1} = ?$$

$$\text{Adjoint } A = [\text{cofactor } A]^T$$

$$K = a_{32} \text{ of adj } A = \text{cofactor } a_{23} \text{ of } A = (-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} = -5$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$|A| = -11 \times 1 + (-9) \times 2 + 1 \times 0 = -11 - 18 = -29$$

$$Q:- R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix} \quad \text{Top row of } R^{-1} = ?$$

- a) $\begin{bmatrix} 5 & 6 & 7 \end{bmatrix}$ b) $\begin{bmatrix} 5 & -3 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$

$$\frac{1^{\text{st}} \text{ row of } R^{-1}}{|R|} = \frac{\text{First row of Adj } R}{|R|} = \frac{[\text{First column cofactor } R]^T}{|R|}$$

$$\therefore = \frac{[5 \quad -3 \quad +1]}{|R|}$$

$$|R| = 1(5) + 2(-3) + 2(1) = 5 - 6 + 2 = 1$$

Find 2nd row of R^{-1} & 3rd row of R^{-1}

Rank -

Minor - Determinant of any square submatrix.

$$\text{Ex:-- } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

(I) 1x1 Submatrices $\rightarrow [1] \quad [2] \dots$

Minors of order 1 1 2

(II) 2x2 Submatrices $\rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

Minors of order 2 $\rightarrow \begin{matrix} 1 \\ 1 \end{matrix} \quad \begin{matrix} 1 \\ 1 \end{matrix} \quad 0$

(III) 3x3 Submatrices $\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \dots$

\downarrow

\downarrow

$P(A) = 3$

Minors of order 3 $\rightarrow \begin{matrix} 1 \\ -5 \\ 2 \end{matrix}$

Rank :-- The order of highest order non-zero minor.
(vanishing)

Method :-- The no. of non-zero rows in echelon form is rank of A.

Results -

- 1) A matrix A is said to be of rank r if

4(2) every minor of order higher than r (if exist) must be zero.

Ex :- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 2$

2) $\rho(A) = 0$ iff $A = 0$ (zero matrix)

3) $\rho(A^T) = \rho(A)$

4) $\rho(A_{m \times n}) \leq \min\{m, n\}$

5) $\rho(A+B) \leq \rho(A) + \rho(B)$

Ex :- $A_{3 \times 4}$ $B_{3 \times 4}$ $(A+B)_{3 \times 4}$
Max^m Rank $\downarrow 3$ + $\downarrow 3$ $\geq \downarrow 3$

6) $\rho(A-B) \geq \rho(A) - \rho(B)$

Ex :- $A_{3 \times 4}$ $B_{3 \times 4}$ $(A-B)_{3 \times 4}$
Max^m Rank $\downarrow 3$ - $\downarrow 3$ $\leq \downarrow 3$

* 7) $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

Ex :- $A_{3 \times 4}$ $B_{4 \times 6}$ $(AB)_{3 \times 6}$
Max^m Rank $\downarrow 3$ $\downarrow 4$ $\downarrow 3$

8) Let $A_{n \times 1}$ & $B_{1 \times m}$ non zero matrices

$$\boxed{\rho(AB) = 1}$$

Ex :- $A = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}_{3 \times 1}$ $B = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}_{1 \times 3}$ $AB = \begin{bmatrix} 4 & 2 & 2 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{bmatrix}_{3 \times 3}$

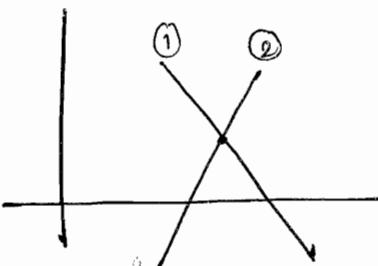
$$= \begin{bmatrix} 4 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3}$$

$\hookrightarrow \text{Rank} = 1$

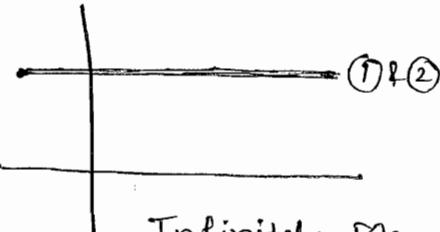
$$= \begin{bmatrix} 2R_1 \\ 4R_1 \\ 3R_1 \end{bmatrix}_{3 \times 1}$$

proportional rows

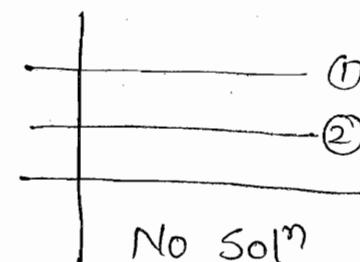
$$\text{Ex : } \begin{aligned} a_1x + b_1y &= c_1 \quad \text{--- (1)} \\ a_2x + b_2y &= c_2 \quad \text{--- (2)} \end{aligned}$$



Unique Solⁿ



Ininitely Many
Solⁿs.



No Solⁿ.

Consistent

Inconsistent

System of m linear equations in n unknowns :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

(i) $m < n$ under determined System.

(ii) $m = n$ Determined System.

(iii) $m > n$ Over determined System.

Matrix form :

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

$$AX = b \quad m \times n \quad m \times 1$$

$A \rightarrow$ Coefficient Matrix

$X \rightarrow$ Solⁿ Vector

$b \rightarrow$ Column Vector

$b = 0$, Homogeneous Linear System (HLS)

$b \neq 0$, Non-Homogeneous Linear System (NHLS)

Augmented Matrix : $[A : B]$

$$A_{m \times n} X_{n \times 1} = b_{m \times 1}$$

\downarrow
 $n = \text{No. of unknowns}$

$P(A : b) \neq P(A)$

No Solⁿ

(Inconsistent)

$P(A) = P(A : b) = r$

Consistent

Unique Solⁿ

$$r=n$$

Infinite Solⁿ

$$r < n$$

In case of infinitely many solⁿ :-

No. of Linearly independent solutions = No. of free variables
 $= n - r$.

Ex :- $x + y + z = 6$

$$x + 2y + 3z = 4$$

$$x + 2y + 3z = 5$$

Matrix form : $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$

$$A \quad x \quad = b$$

Augmented Matrix : $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \end{array} \right]$

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$Q:- x+y+z=6 ; x+2y+3z=4 ; x+2y+4z=5$$

Sol :- Augmented matrix :

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 5 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1 \text{ & } R_3 \rightarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 3 & -1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Echelon form

$\rho(A) = \rho(A:B) = 3 = r \rightarrow \text{consistent}$
 $r=n$
 $\therefore \text{Unique soln.}$

Method 1 - Back elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$R_1 \rightarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Reduced echelon form

$$R_1 \rightarrow R_1 - R_3 \text{ & } R_2 \rightarrow R_2 - 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\begin{aligned} x &= 9 \\ y &= -4 \\ z &= 1 \end{aligned}$$

$$x = \begin{bmatrix} 9 \\ -4 \\ 1 \end{bmatrix}$$

from reduced echelon form

Method 2 - (Back Substitution) [For unique soln].

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Echelon Form

$$x + y + z = 6$$

$$y + 2z = -2$$

$$z = 1$$

$$y + 2(1) = -2$$

$$y = -4$$

$$x + (-4) + 1 = 6 \Rightarrow x = 9$$

Imp

$$Q:- x+y+z=6$$

$$x+2y+3z=4$$

$$x+2y+4z=5$$

$$(A:B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Echelon Form

$\text{S}(A) = \text{S}(A:B) \Rightarrow 2 = r \rightarrow \text{consistent}$
 $n = 3 \quad r < n$

Ininitely Many solution

No. of L.I. solutions. = $n - r = 3 - 2 = 1$

Back elimination :

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_1 \rightarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 8 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Reduced row echelon form.

Basic Variables (Dependent variables) -

Variables corresponding leading 1's in reduced row echelon form

B.V $\rightarrow x, y$.
Basic variables

Other variables are free variables

f.V $\rightarrow z$
free variables

$$\text{Let } Z = t \quad x - z = 8, y + 2z = -2$$

$$x = 8 + t, y = -2 - 2t, z = t$$

$$x = \begin{bmatrix} 8+t \\ -2-2t \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

\downarrow
 1 L.I. solⁿ which create all infinite solutions.

$$\text{Ex: } x + y + z = 60$$

$$x + 4y + 6z = 20$$

$$x + 4y + \lambda z = u$$

For what values of λ & u ,
 system has no solⁿ, unique solⁿ,
 infinite solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & u \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 0 & 3 & 5 & -40 \\ 0 & 0 & \lambda-6 & u-20 \end{array} \right]$$

Echelon form

(2) Infinite Solⁿ

$$\lambda-6=0, u-20=0$$

$$\lambda=6 \quad \& \quad u=20$$

$$\rho(A) = \rho(A:B) = 2 = r$$

$$r < n$$

$$2 < 3$$

Homogeneous Linear System - $AX = 0$

(1) $x = 0$ is trivial solⁿ

$\therefore AX = 0$ is always consistent.

$$A_{m \times n} X_{n \times 1} = 0$$

\downarrow
No. of Unknowns

\downarrow
 $\rho(A) = n$

Unique Solⁿ

Zero Solⁿ

$\boxed{X = 0}$

\downarrow
 $\rho(A) < n$

Infinite Solⁿ (or)

Non-zero Solⁿ

(or) $\boxed{X \neq 0}$

Results -

(I) $AX = 0$ & A is square matrix of order n.

1. Unique Solⁿ iff $\rho(A) = n$; iff $|A| \neq 0$.

(II) An underdetermined HLS always has infinite solution.

$$Am \times n \times n \times 1 = 0 ; n \rightarrow \text{no. of unknowns}$$

Underdetermined $m < n$; $\rho(A) \leq \min\{m, n\}$

$\boxed{\rho(A) \leq n}$ Infinite solⁿ.

Ex :- $x+y+z=0$, $x+2y+3z=0$, $x+2y+4z=0$

- a) Unique solⁿ b) Infinite solⁿ c) Exactly two solⁿ d) None

Solⁿ :- Method 1 :

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right]$$

$$\rho(A) = 3 = n \rightarrow \text{Unique sol}^n, \boxed{x=0}$$

Method 2 :

$$A = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{array} \right]$$

Shortcut for det. of 3×3 matrix only

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 8 + 3 + 2 - 2 - 6 - 4 = 1$$

repeat two columns

$$|A| \neq 0 \Rightarrow \rho(A) = 3$$

Unique solⁿ.

Q:- $x+y+z=0$, $x+2y+3z=0$

→ It is underdetermined HLS, Rank = 2

$$\rho(A) = 2 < n \Rightarrow \text{infinite sol}^n.$$

$$x + 3y + 4z = 0$$

$$x + 3y + 4z = 0$$

Soln :- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{bmatrix} x = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon form
 $P(A) = 2 < 3$
 \Rightarrow Infinite Sol's.

$$R_2 \rightarrow R_2/2$$

$$R_1 \rightarrow R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon
B.V $\rightarrow x, y$
f.V $\rightarrow z$

$$z = 2t$$

$$x - \frac{1}{2}t = 0$$

$$y + \frac{3}{2}t = 0$$

$$x = t$$

$$y = -3t$$

$$z = 2t$$

$$x = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Result :- In case HLS with infinite solutions, No. of L.I. sol's

$$= \text{Nullity} = n - r$$

Ex :- $x + y + z = 0$, Nullity = _____

Soln :- $r = 1$, $n = 3$, Nullity = $3 - 1 = 2$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x = 0$$

↑
Echelon form, reduced row echelon, $P(A) = r = 1$

Ex :- The system $ax + y + z = 0$

$$x + ay + z = 0$$

$$x + y + az = 0$$

has infinite soln.

The set of values of a for this to happen is

- a) $\{1, -1\}$
- b) $\{-1, 2\}$
- c) $\{-2, 2\}$
- d) $\{-2, 1\}$

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}_{3 \times 3}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$(a+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}_{3 \times 3}$$

$$R_2 \rightarrow R_2 - R_1 \text{ & } R_3 \rightarrow R_3 - R_1$$

$$(a+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & a-1 & 0 \\ 0 & a & a-1 \end{vmatrix} = 0$$

$$(a+2)(a-1)^2 = 0 \Rightarrow \boxed{a = -2, 1, 1} \Rightarrow \{1, -2\}$$

Linearly Independent vectors —

Ex :- $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad v_2 = 2v_1 \rightarrow \text{Linearly Dependent}$

Ex :- $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad v_1, v_2 \text{ cannot be expressed one in terms of other,} \quad v_1, v_2 \rightarrow \text{L.I.}$

Result :- Let v_1, v_2, \dots, v_n be n vectors & let $A = [v_1 \ v_2 \ \dots \ v_n]$

(i) $\rho(A) = n \Rightarrow v_1, v_2, \dots, v_n$ are L.I..

(ii) $\rho(A) < n \Rightarrow v_1, v_2, \dots, v_n$ are L.D..

Ex :- $A = [v_1 \ v_2] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2 \times 2} \Rightarrow |A| = 4 - 4 = 0 \rightarrow \text{L.D.} \quad \therefore \rho(A) < 2$

Ex :- $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \Rightarrow |A| = -2 - 2 = -4 \neq 0 \rightarrow v_1, v_2 \text{ are L.I.}$

Ex :- $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) < 3 \rightarrow \text{L.D.}$$

Def :— Let 'A' be square matrix of order n .

A scalar " λ " is eigen value of A

if there exist a non-zero vector X ($X \neq 0$)

such that

$$AX = \lambda X$$

X is eigen vector for λ

Characteristic eqⁿ — $AX = \lambda X \Rightarrow AX = \lambda I X \Rightarrow AX - \lambda I X = 0$

$$(A - \lambda I)X = 0 \rightarrow \text{H.L.S.}$$

$X \neq 0 \Rightarrow \text{Infinite soln.} \Rightarrow |A - \lambda I| = 0 \rightarrow \text{C.E.}$

Method :— Let $A_{n \times n}$ be square matrix

1) Solve $|A - \lambda I| = 0$ for eigen values $\lambda = \lambda_1, \lambda_2, \dots$

2) For each eigen value λ . Solve the HLS $(A - \lambda I)X = 0$ for eigen vectors $X \neq 0$

Q :— $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ Find eigen values & eigen vectors

Soln :— 1) Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$$

$$\lambda_1 = 0 \quad \& \quad \lambda_2 = 5$$

$$\cancel{\lambda - 5\lambda + \lambda^2 - 4} = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5 \rightarrow \text{eigen values}$$

2) Eigen vector for $\lambda = 0$:

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}X = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \text{ reduced row echelon}$$

$$B.V \rightarrow x, f.V = y \Rightarrow y = t, x + 2y = 0$$

$$x = -2t$$

$$X = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $\lambda_1 = 0$, $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a eigen vector

simple integer form, can be $\begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -20 \\ 10 \end{bmatrix}$ etc.

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} X = 0$$

$$R_2 \rightarrow 2R_2 + R_1$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} X = 0$$

$$\Rightarrow A \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \sim$$

$$R_1 \rightarrow -R_1/4 \quad \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

B.V. $\rightarrow x$, f.v. $\rightarrow y$

Echelon form

Reduced row echelon form

$$y = 2t, \quad x - \frac{1}{2}y = 0 \Rightarrow x = t \Rightarrow y = 2t$$

$$X = \begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = 5$, $x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a eigen vector.

Trace :— $A_{n \times n}$

$\text{tr } A = \text{Sum of main diagonal entries} = \text{Sum of eigen values}$

$$\underline{\text{Ex:}} \quad \text{tr } A = 1+4 = 5 = \lambda_1 + \lambda_2$$

~~$|A| = 0 = \lambda_1 \cdot \lambda_2$~~ $\rightarrow \text{Det.} = \text{Product of eigen values.}$

$$Q:— A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\text{Sofn:}} \quad 1) \text{C.E.} \quad |A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \quad \lambda^3 = 0$$

$\lambda = 0 \rightarrow "0"$ is eigen value of algebraic multiplicity 3.

Eigen vector for $\lambda = 0$

$$[A - 0I]x = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{B.V.} \rightarrow \text{leading element.}}$$

Reduced echelon form.

$$x=t \quad y=s$$

$$z=0$$

$$\cancel{x} = \begin{bmatrix} t \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are eigen vectors for $A=0$

"0" is eigen value with geometric multiplicity \rightarrow no. of eigen vectors 2.

Note - Geometric Multiplicity \leq Algebraic Multiplicity.

Q:- $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ then eigen value for eigen vector $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \underline{\quad}$

- a) 2 b) 4 c) 6 d) 1

Solⁿ:- $AX = \lambda X$

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \textcircled{6} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Ans.

Q:- A matrix M has eigen values \downarrow & \downarrow with eigen vector
 $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ resp. $M = \underline{\quad}$ $\lambda_1 \quad \lambda_2$

Solⁿ:- $AX = \lambda X \Rightarrow MX = \lambda X$
 $MX_1 = \lambda_1 X_1 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2} \quad a-b=1, c-d=-1$

$$MX_2 = \lambda_2 X_2 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} 2a+b=8 \\ 2c+d=4 \end{array}$$

$$a=3, b=2, c=1, d=2$$

$$M = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \underline{\text{Ans.}}$$

(1)

$\text{Tr } A = \text{Sum of eigen values.}$

(2) $|A| = \text{Product eigen values.}$

(3) $|A| = 0$ iff $\lambda=0$ is an eigen value.

(4) Eigen vectors corresponding distinct eigen values are L.I.

Ex:- $M = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ $\lambda_1 = 1$, $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $A = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$
 $\lambda_2 = 4$, $x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$|A| = 3 \neq 0 \rightarrow x_1, x_2$ are L.I.

(5) Eigen vectors corresponding to distinct eigen values of symm. matrix are orthogonal.

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \text{Symm. Matrix}$

$$\lambda = 0 \quad x_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \lambda = \lambda, \quad x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_1 \cdot x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \Rightarrow x_1, x_2 \text{ are orthogonal.}$$

Q:- $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \lambda_1 = ?, \lambda_2 = ?$

Sol:- $\lambda_1 + \lambda_2 = \text{tr } A = 1+1=2$

$$\lambda_1 \cdot \lambda_2 = |A| = 0$$

Let $\lambda_1 = 0, \lambda_2 = 2 \Rightarrow$ Eigen values 0,2.

(6) Eigen values of matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2 \times 2}$ are 2,0

————— || ————— $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3 \times 3}$ are 3,0,0

————— || ————— $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4 \times 4}$ are 4,0,0,0

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n} = \underbrace{n, 0, 0, \dots, 0}_{n-1 \text{ times}}$$

(7) If λ is eigen value of A then

- (i) eigen value of kA is $k\lambda$.
- (ii) eigen value of $A + kI$ is $\lambda \pm k$.
- (iii) ——— of A^2 is λ^2 .
- (iv) ——— of A^m is λ^m .

(8) If λ is eigen of an invertible matrix A

- (i) Eigen value of $A^{-1} = 1/\lambda$
- (ii) ——— of $\text{Adj } A = \frac{|A|}{\lambda}$

Q:- Let $1, -2, -1$ are eigen values of $A_{3 \times 3}$.

(1) Then eigen values of $3A = 3, -6, -3$

$$(2) \quad A + 3I = 1+3, -2+3, -1+3 = 4, 1, 2$$

$$(3) \quad A^2 = 1^2, -2^2, -1^2 = 1, 4, 1$$

$$(4) \quad A^5 = 1^5, -2^5, -1^5 = 1, -32, -1$$

$$(5) \quad A^{-1} = 1/1, 1/-2, 1/-1 = 1, -1/2, -1$$

$$(6) \quad \text{Adj } A = 2|1, 2|-2, 2|-1 = 2, -1, -2$$

$$|A| = 1 \times -2 \times -1 = 2$$

$$(7) \quad A - 4I = 1-4, -2-4, -2-1 = -3, -6, -5$$

$$(8) \quad |A^2 + 4A| = |A(A + 4I)| = |A||A + 4I|$$

$$|A| = 1 \times -2, -1 = 2$$

$$|A^2 + 4A| = 2 \times 30 = 60$$

$$|A + 4I| = 5 \times 2 \times 3 = 30$$

Ex:- Let $1, -1, 0$ are eigen value then $|A^{100} + I| =$ _____

Sol:- Eigen values of $A^{100} = 1^{100}, -1^{100}, 0^{100} = 1, 1, 0$

$$A^{100} + I = 1+1, 1+1, 0+1 = 2, 2, 1$$

$$|A^{100} + I| = 2 \times 2 \times 1 = 4 \quad \underline{\text{Ans.}}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix} \rightarrow \text{lower star}$$

Eigen values of $A = 2, 1, 4$.

(9) If A is diagonal matrix or upper triangular or lower triangular matrix, then eigen values are the main diagonal entries.

Q:- $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ $|A| = \underline{\quad}$

Soln:- $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $|A| = |B+I|$
 $A = B + I$ $= 5 * 1 * 1 * 1$
 $= 5$

Eigen values of $B \rightarrow 4, 0, 0, 0$

$B+I \rightarrow 5, 1, 1, 1$

Complex Matrices — Let $A = \begin{bmatrix} 1+i & 2+3i \\ 4-i & 5 \end{bmatrix}$

Conjugate of A : \bar{A}

$$\bar{A} = \begin{bmatrix} 1-i & 2-3i \\ 4+i & 5 \end{bmatrix}$$

Conjugate transpose — $\bar{A}^T = A^0$

$$\bar{A}^T = \begin{bmatrix} 1-i & 4+i \\ 2-3i & 5 \end{bmatrix}$$

Note:- A is real matrix $\boxed{\bar{A}^T = A^T}$

Def:- A complex square matrix A is:

1) Hermitian — if $\boxed{\bar{A}^T = A}$

2) Skew Hermitian — if $\boxed{\bar{A}^T = -A}$

Ex :- $A = \begin{bmatrix} 2 & 2-3i \\ 2+3i & 3 \end{bmatrix}$ is Hermitian

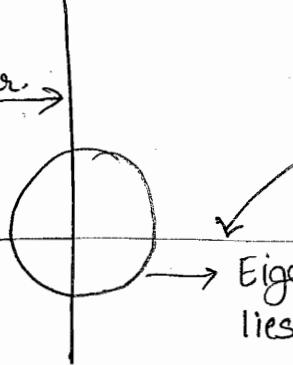
$$\underline{\text{Ex}} : A = \begin{bmatrix} 2i & 2+3i \\ -2+3i & 0 \end{bmatrix} \quad \bar{A}^T = \begin{bmatrix} -2i & -2-3i \\ 2-3i & 0 \end{bmatrix} = - \begin{bmatrix} 2i & 2+3i \\ -2+3i & 0 \end{bmatrix}$$

$$\underline{\text{Ex}} : A = \begin{bmatrix} \pm 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \bar{A}^T = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A\bar{A}^T = I \rightarrow A \text{ is unitary}$$

*

Eigen value of Skew-Her.
(skew sym) lies on
imag. line.



Eigen values of Hermitian (symm.)
lie on real line.

Eigen vector of unitary (orthogonal) matrix
lies on unit circle.

(10) Eigen values of Hermitian (or) Symm. are real no.

$$\underline{\text{Ex}} : A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ e.v.} \rightarrow 0, 5 \rightarrow \text{real no.}$$

(11) Eigen values of Skew-Hermitian or skew-symm. matrix are either pure imaginary or "0".

$$\underline{\text{Ex}} : A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \lambda^2 + 1 = 0 \\ \lambda = \pm i \leftarrow \text{pure imag.}$$

(12) Eigen values of unitary (orthogonal) matrix have absolute value 1.

$$\underline{\text{Ex}} : A = \begin{bmatrix} \pm 1/\sqrt{2} & \pm 1/\sqrt{2} \\ -1/\sqrt{2} & \pm 1/\sqrt{2} \end{bmatrix} \text{ orthogonal} \quad \text{Eigen value} = \frac{1 \pm i}{\sqrt{2}}$$

$$\lambda_1 = \frac{1+i}{\sqrt{2}}, |\lambda_1| = 1; \lambda_2 = \frac{1-i}{\sqrt{2}} \Rightarrow |\lambda_2| = 1$$

Every square matrix satisfies its C.E.

Ex :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ CE $\rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

By Cayley Hamilton Theorem :

$$A^2 - 3A - 4I = 0$$

$$\boxed{\lambda^2 - 3\lambda - 4 = 0} \rightarrow \begin{array}{l} \text{Characteristic} \\ \text{Polynomial eqn.} \\ |A| \rightarrow \text{const. term in } \end{array}$$

Finding A^{-1} :- Cayley Hamilton theorem is used to find A^{-1} .

$$A^2 - 3A - 4I = 0$$

Multiply A^{-1} $\rightarrow A - 3I - 4A^{-1} = 0$

$$\begin{aligned} A^{-1} &= \frac{1}{4} \cancel{-3I} - (A - 3I) \\ &= \frac{1}{4} \begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix} \end{aligned}$$

Finding powers of A - Cayley H. theorem is used to find A^m .

$$A^2 = 3A + 4I$$

Multiply by A :

$$\begin{aligned} A^3 &= 3A^2 + 4A \\ &= 3[3A + 4I] + 4A = \boxed{13A + 12I = A^3} \end{aligned}$$

Multiply by A :

$$A^4 = 3A^3 + 4A^2 = 3[13A + 12I] + 4[3A + 4I]$$

$$\begin{aligned} A^6 &= A^3 A^3 \\ &= (13A + 12I)(13A + 12I) \\ &= 169A^2 + 312A + 144I = 169(3A + 4I) + 312A + 144I \end{aligned}$$

$$\boxed{A^6 = 819A + 820I}$$

then $B = A^4 - 5A^2 + 5I$, $|B| = \underline{\hspace{2cm}}$

Solⁿ:— C.E.

$$(\lambda-1)(\lambda+1)(\lambda-2)(\lambda+2) = 0$$

$$(\lambda^2-1)(\lambda^2-4) = 0$$

$$\boxed{\lambda^4 - 5\lambda^2 + 4 = 0} \rightarrow \text{C.E.}$$

By Cayley H. theorem: $A^4 - 5A^2 + 4I = 0$

$$B = A^4 - 5A^2 + 5I$$

$$B = A^4 - 5A^2 + 4I + I$$

$$B = 0 + I$$

$$B = I$$

$$|B| = 1$$

PROBABILITY

Random experiment (RE) — An experiment in which outcome is not certain.

Sample Space — Set of all possible outcomes of a random experiment.

Event — Any subset of sample space is called event.

Ex :— Let S be a set with n elements.

$$\text{No. of subsets} = 2^n$$

Ex :— R.E.

Sample Space

Event

1) Tossing two coins

$$S = \{HH, HT, TH, TT\}$$

$E_1 = \text{getting exactly one head} = \{HT, TH\}$

$2^4 \rightarrow$ distinct problems
 $\rightarrow 16$ Events

$E_2 = \text{getting no heads}$
 $= \{TT\}$

$E_3 = \text{getting Rajnikant}$
 $= \{ \}$

2) Selecting an integer from 1 to 20 at random.

$$S = \{1, 2, \dots, 20\}$$

 $2^{20} \rightarrow$ distinct problems.

$E_1 = \text{getting multiple of 3}$
 $= \{3, 6, 9, 12, 15, 18\}$
 $|E_1| = 12$ — a

$$|E_2| = \left[\frac{20}{5} \right] = 4$$

$$E_3 = \text{Multiple of } 3 \text{ & } 5$$
$$|E_3| = \left[\frac{20}{3 \times 5} \right] = 1$$

Def :- For an event A,

$$\text{probability of } A = P(A) = \frac{\text{favourable no. of outcomes}}{\text{total no. of outcomes}} = \frac{|A|}{|S|}$$
$$= \frac{m}{n}$$

(1) The odds in favour of A = $m : n-m$

\nearrow fav \nwarrow non-fav.

(2) The odds against A = $n-m : m$

\nearrow non-fav \nwarrow fav

Ex :- odds in favour of A are $2:5$ non-fav.

& odds against B are $3:4$ $P(A) = ?$ $P(B) = ?$

Solⁿ :- Total = $5+2=7$ $P(A) = \frac{2}{7}$

Total = $3+4=7$ $P(B) = \frac{4}{7}$

Axioms :- (Universally accepted)

(1) For any event A of Sample space S, $0 \leq P(A) \leq 1$

(2) $P(S) = 1$

(3) If A & B are mutually exclusive (disjoint) events

$$P(A \cup B) = P(A) + P(B)$$

Results :-

1. Prob. of impossible event, $P(\emptyset) = 0$

2. Prob. of non happening of an event, $P(A^c) = 1 - P(A)$

Ex :- (i) Two dice are rolled.

(ii) $P[\text{Sum of outcomes} = 8]$

$$\text{Sum} = 8 \rightarrow (2,6) (3,5) (4,4) (5,3) (6,2) \rightarrow 5$$

Two dice rolled :

Sum	2	3	4	5	6	7	8	9	10	11	12
Fav. Case	1	2	3	4	5	6	5	4	3	2	1

$$(i) P[\text{Sum } 3 \text{ or } 5] = P[3] + P[5] = \frac{2+4}{36} = \frac{6}{36} = \frac{1}{6}$$

$$(ii) P[\text{Sum} = \text{Prime no.}] = P[3 \text{ or } 5 \text{ or } 7 \text{ or } 11] \\ = \frac{1+2+4+6+2}{36} = \frac{15}{36}$$

$$(iii) P[\text{even no.}] = P[2 \text{ or } 4 \text{ or } 6 \text{ or } 8 \text{ or } 10 \text{ or } 12] \\ = \frac{1+3+5+5+3+1}{36} = \frac{18}{36} = \frac{1}{2}$$

$$(iv) P[\text{Sum} = \text{odd no.}] = 1 - P[\text{even no.}] = 1 - \frac{1}{2} = \frac{1}{2}$$

$$(v) P[\text{atleast } 3 = \text{sum}] = 1 - P[\text{sum} = \text{atmost } 2] \\ = 1 - P[S \leq 2] \\ = 1 - \frac{1}{36} = \frac{35}{36}$$

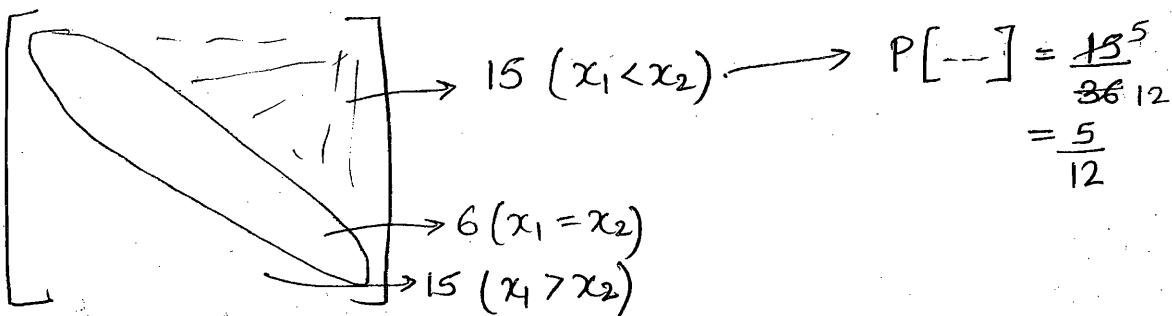
$$(vi) P[\text{unequal outcomes}] = 1 - P[\text{equal outcomes on both dice}] \\ = 1 - \frac{6}{36} = \frac{30}{36} = \frac{5}{6}$$

every outcome has same chance

Q:- A fair dice is rolled twice.

P[outcome on the second toss is greater than the outcome in the first toss] = ?

Sol :-



30% families are having 2 children & remaining families having 1C. A child is selected at random.

(i) P [child is from family having 2C] -

(ii) P [child is from family having 1C] -

Solⁿ:- N → No. of families.

$$\begin{aligned}\text{Total no. of children} &= \frac{50}{100} \times N * 3 + \frac{30}{100} \times N * 2 + \frac{20}{100} * N * 1 \\ &= \frac{23N}{10}\end{aligned}$$

$$P[\text{child is from family having 2C}] = \frac{\text{fav.}}{\text{Total}} = \frac{6N/10}{23N/10} = \frac{6}{23}$$

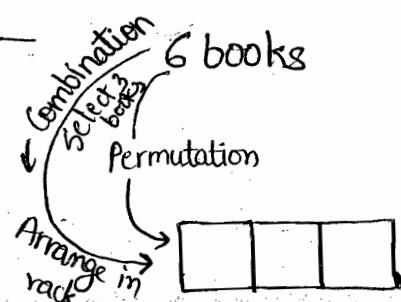
$$P[\text{from 1C}] = \frac{2N/10}{23N/10} = \frac{2}{23}$$

Q:- A coin is tossed N times.

$$P[\text{No. of heads} \neq \text{No. of tails}] = \frac{0}{2^n} = 0$$

<u>Solⁿ</u> :-	No. of H	No. of T	Difference
	n	0	n
	n-1	1	n-2
	n-2	2	n-4
	1	1	
	1	1	
	1	1	
	2	n-2	4-n
	1	n-1	2-n
	0	n	-n

Permutations & Combinations -



1) No. of r -permutations (Arrangements) of n -objects = ${}^n P_r = \frac{n!}{(n-r)!}$

2) No. of permutation (arrangements) of n -objects (in line) = $n!$

3) No. of circular permutations of n -objects = $(n-1)!$

[Fix one object & arrange remaining linearly]

Q:- How many ways 8 people can be arranged in a line
8! ways

a) so that certain pair is always together. $\rightarrow 7! \times 2!$

\boxed{AB}

1 unit + 6 units = 7 units

7 units $\xrightarrow{\text{in line}} 7!$

AB $\longrightarrow 2!$

Q:- How many ways 8 people can be arranged in a circle.

$$(8-1)! = 7!$$

a) so that certain pair is always together. $6! \times 2!$

7 units $\xrightarrow{\text{in circle}} 6!$

AB $\longrightarrow 2!$

Q:- 8 people arranged in a line.

$$(1) P[\text{certain pair always together}] = P[\boxed{AB}] = \frac{AB}{8!} = \frac{2! \times 7!}{8!}$$

$$= \frac{2}{8} = \frac{1}{4}$$

$$(2) P[\text{certain pair never together}] = P[\del{AB}]$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Q:- Above problem for circle.

$$(1) P[\boxed{AB}] = \frac{2! \times 6!}{7!} = \frac{2}{7}$$

$$(2) P[\del{AB}] = 1 - \frac{2}{7} = \frac{5}{7}$$

Q:- 5 men & 5 women are arranged in a line.

- 1) P[all 5M together]
- 2) P[no two men together]

Soln:- Total Possibilities \rightarrow 5M & 5W can be arranged
 $= (5+5)! = 10!$

1) all 5 men together $\rightarrow 5! \times 6!$

5M

1 unit + 5 units = 6 units $\rightarrow 6!$ ways

5M $\longrightarrow 5!$

$$P[\text{all } 5M \text{ together}] = \frac{5! \times 6!}{10!}$$

2) No two men together =

5W $\longrightarrow 5!$ ways \rightarrow No. constraint on W,
consider first W.

$\times W_1 \times W_2 \times W_3 \times W_4 \times W_5 \times$

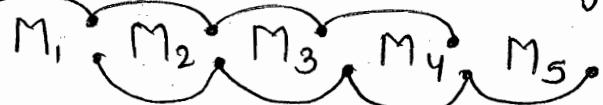
5M in 6 gaps $\longrightarrow 6P_5$ ways.

No two men together = $5! \times 6P_5$

$$P[\text{No 2M together}] = \frac{5! \times 6P_5}{10!}$$

3) M & W arranged alternatively:-

5M $\longrightarrow 5!$ ways



To get alternate arrangement 5W can be arranged $(5! + 5!)$ ways.

Men & women arranged alternatively = $5!(5! + 5!)$

$$P[M \& W \text{ alternatively}] = \frac{5!(5! + 5!)}{10!}$$

Q:- A box contains 3R & 4W balls. Balls are drawn from box one after the other at random & arranged in a line.

P[they are arranged alternatively] = ?

$$\frac{R \& W alternately}{7!} = \frac{4! \times 5!}{7!}$$

$$4W \longrightarrow 4!$$

$$w_1 \times w_2 \times w_3 \times w_4 \rightarrow 3R \text{ in } 3! \text{ ways.}$$

Combinations:

1) No. of r -combinations (r selections) of n objects = ${}^n C_r = \frac{n!}{r!(n-r)!}$

$$2) {}^n P_r = {}^n C_r * r!$$

$$3) {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n$$

$$4) {}^n C_0 + {}^n C_2 + {}^n C_4 + \dots = 2^{n-1}$$

$$5) {}^n C_1 + {}^n C_3 + {}^n C_5 + \dots = 2^{n-1}$$

$$6) 0 \cdot {}^n C_0 + 1 \cdot {}^n C_1 + 2 \cdot {}^n C_2 + \dots + n \cdot {}^n C_n = n \times 2^{n-1}$$

Q:- 10 friends meet in a party & shake hands with each other

$$\text{No. of handshakes} = \text{No. of 2-selections} = {}^{10} C_2 = 45$$

A \leftrightarrow B

2) send greeting cards to each other \rightarrow No. of cards send / posted

$$= 10 P_2 = 90$$

Q:- How many ways 5 members committee can be selected from 6M & 7W.

$$\underline{\text{Sol}}: - 13 C_5$$

2) With exactly 2 women in the committee $\rightarrow 7 C_2 \times 6 C_3$

Q:- How many ways 5 members can be selected from 6M & 7W

M ⁶	W ⁷
5	0 $\rightarrow 6 C_5 \times 7 C_0$
4	1 $\rightarrow 6 C_4 \times 7 C_1$
3	2 $\rightarrow 6 C_3 \times 7 C_2$
2	3 $\rightarrow 6 C_2 \times 7 C_3$
1	4 $\rightarrow 6 C_1 \times 7 C_4$
0	5 $\rightarrow 6 C_0 \times 7 C_5$

$$\text{Sum} = 13 C_5$$

$$\rightarrow {}^6C_3 \times {}^7C_2 + {}^6C_2 \times {}^7C_3 + {}^6C_1 \times {}^7C_4 + {}^6C_0 \times {}^7C_5$$

or Total \rightarrow atmost 1 women. $= {}^{13}C_5 - ({}^6C_5 \times {}^7C_6 + {}^6C_4 \times {}^7C_4)$

Q:- 5 members committee formed from 7W & 6M.

(1) $P[\text{Committee has atleast } 2W] = \frac{{}^7C_2 \times {}^6C_3 + {}^7C_3 \times {}^6C_2 + {}^7C_4 \times {}^6C_1}{{}^{13}C_5}$
 (or)

$$P[\text{atleast } 2W] = 1 - P[\text{atmost } 1W]$$

$$= 1 - \frac{[{}^6C_5 \times {}^7C_0 + {}^6C_4 \times {}^7C_1]}{{}^{13}C_5}$$

Q:- 4 numbers are selected from 6 +ve & 8 -ve nos. What is the prob. that their product is +ve.

Soln:- Total possibilities $\rightarrow {}^{14}C_4$

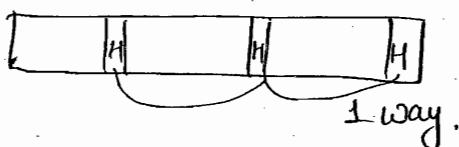
P(6)	N(8)	
4	0	$\rightarrow \checkmark {}^6C_4 \times {}^8C_0$
3	1	
2	2	$\rightarrow \checkmark {}^6C_2 \times {}^8C_2$
1	3	
0	4	$\rightarrow \checkmark {}^6C_0 \times {}^8C_4$

$$P[\text{Product is +ve}] = \frac{{}^6C_4 \times {}^8C_0 + {}^6C_2 \times {}^8C_2 + {}^6C_0 \times {}^8C_4}{{}^{14}C_4}$$

$$= \frac{505}{1001}$$

Q:- A coin is tossed n times. $P[\text{No. of heads} = 3] = ?$
 P[getting 3 heads] = ?

Soln:- Total no. $= 2^n$, fav. 3 heads $= \frac{{}^nC_3 * 1}{\text{Total}}$

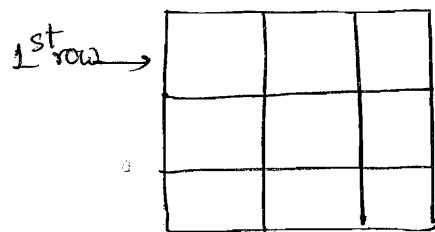


$$P[\text{No. of heads}] = \frac{{}^nC_3}{2^n}$$

$P[\text{the two squares have a side in common}]$

Soln:— Total no. of squares = 64

Two squares can be selected = $64C_2$



3x3

2

8x8

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$$3C_2 * 3C_2 + 3C_3 * 3C_3 \\ = 20 = 1+9+9+1$$

$P[A \text{ & } B \text{ get equal heads}] = \frac{20}{2^3 * 2^3} = \frac{5}{16}$

Permutations with constrained repetitions -

$$P(n; q_1, q_2, \dots, q_t) = n! / q_1! q_2! \dots q_t!$$

[Arrangement of n objects in which q_1 are alike, q_2 are alike, \dots , q_t are alike.]

$$q_1 + q_2 + \dots + q_t = n$$

Ex :- No. of arrangements of letters in the word MISSISSIPPI.

$$\underline{\text{Soln}}: P\left(11, \underset{S}{4}, \underset{P}{4}, \underset{M}{2}, \underset{I}{1}, \underset{I}{1}\right) = \frac{11!}{4!4!2!1!}$$

a) so that all I's are together, 4I's

$$1 \text{ unit} + 7 \text{ unit} = 8 \text{ unit}$$

$$8 \text{ unit's} \rightarrow P\left(8; \underset{S}{4}, \underset{P}{2}, \underset{M}{1}, \underset{I}{1}, \underset{I}{1}\right) = \frac{8!}{4!2!1!1!}$$

4I's can be arranged $\rightarrow 1$

$$4I's = \frac{8!}{4!2!1!1!}$$

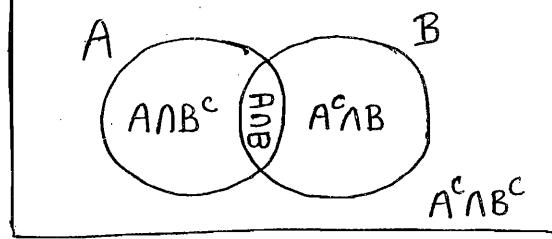
b) The letter of word MISSISSIPPI arranged.

$$P[\text{all 4I's occur together}] = \frac{\frac{8!}{4!2!1!1!}}{\frac{11!}{4!4!2!}}$$

Addition Theorem :-

1) If A & B are any two events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$2) P(A^c \cap B^c) = 1 - P(A \cup B)$$

Ex :- A box contains tickets numbered 1 to 20. A ticket is drawn at random.

$$1) P[\text{Multiple of 5 or multiple of 7}]$$

$$2) P[\text{Multiple of 3 or multiple of 5}]$$

Sol :- $A = \text{Multiple of 5}$, $B = \text{Multiple of 7}$

$$|A \cap B| = \left| \frac{20}{\text{LCM}(5,7)} \right| = 0 \rightarrow \text{Disjoint event}$$

$$P(A \cup B) = P(A) + P(B) = \frac{4}{20} + \frac{2}{20} = \frac{6}{20} = \frac{3}{10}$$

2) $A = \text{mult. of 3}$, $B = \text{Mult. of 5}$

$$|A \cap B| = \left| \frac{20}{\text{LCM}(5,3)} \right| = \left| \frac{20}{15} \right| = \frac{1}{3} \rightarrow \text{Not disjoint}$$

\hookrightarrow have some value in set.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{20}{3} + \frac{20}{5} - \frac{20}{15} = \frac{9}{20} \end{aligned}$$

Cards :- (52 Cards)

♠ ♣ ♦ ♥ → 4 suits

A	-	-
2	-	-
10	-	-
J	-	-
Q	-	-
K	-	-

$$(i) P[\text{King}] = \frac{4C_1}{52C_1} = \frac{4}{52} = \frac{1}{13}$$

$$(ii) P[\text{ACE}] = \frac{4C_1}{52C_1} = \frac{4}{52} = \frac{1}{13}$$

$$(iii) P[\text{King or Ace}] = P[K] + P[A] = \frac{4C_1 + 4C_1}{52C_1} = \frac{8}{52} = \frac{2}{13}$$

$$(iv) P[\text{King or Heart}] = P[K] + P[H] - P[H \cap K] = \frac{4C_1 + 13C_1 - 1C_1}{52C_1} = \frac{4}{13}$$

Q:- Two cards are drawn from pack of 52 cards.

$$P[\text{both are queen or both red}] = P[2Q] + P[2R] - P[2Q \cap 2R]$$

$$P[2Q \cup 2R] = \frac{4C_2 + 26C_2 - 2C_2}{52C_2} = \frac{55}{221}$$

Q:- A box contains 125 bolts & 200 nuts.

$\frac{1}{5}$ th bolts & $\frac{3}{4}$ th of nuts are defective. An item is selected from box. $P[\text{Def. or nut}]$

$$\text{Soln: } \frac{B}{125} \quad \frac{N}{200} \quad \text{Total} = 325$$

$$P[\text{Def. or Nut}] = P[D] + P[N] - P[D \cap N]$$

$$\begin{aligned} \text{Def: } \frac{1}{5} \times 125 &= 25 \\ &\quad \frac{3}{4} \times 200 \\ &= 150 \\ &\quad \text{DNN} \end{aligned} \quad \begin{aligned} &= \frac{175 + 200 - 150}{325} \\ &= \frac{225}{325} \end{aligned}$$

Conditional Probability -

Ex:-

3 red(R)
4 white (W)

1st draw: A ball is drawn at random.

Replaced

$$P[R_2 | R_1] = \frac{3}{7}$$

$$P[R_2 | NR_1] = \frac{3}{7}$$

(R_1 & R_2 are independent)

$$\begin{array}{c} \boxed{\frac{3R}{7}} \rightarrow P[R_2 | R_1] = \frac{2}{6} = \frac{1}{3} \\ \boxed{\frac{3W}{7}} \quad P[R_2 | NR_1] = \frac{3}{6} = \frac{1}{2} \end{array}$$

Dependent events

Not Replace

does not effect happening or non-happening of other event then the events are said to be independent, otherwise they are dependent

Conditional Prob. :-

$P[A|B]$ → Prob. of A given B has already occurred.

$$P[A|B] = \frac{P(A \cap B)}{P[B]}, P(B) \neq 0$$

$$P[B|A] = \frac{P(A \cap B)}{P(A)}, P(A) \neq 0$$

Ex :— A die is rolled once,

$P[\text{multiple of } 3 \text{ given that a no. is greater than } 4 \text{ has occurred}]$

$$= \frac{1}{2}$$

Method 1 :— we know directly as $> 4 = 5 \& 6$ out of which only 6 is multiple of 3 i.e. getting 6. It has prob. $\frac{1}{2}$

(done)

Method 2 :— $A = \text{Multiple of } 3 = \{3, 6\}$, $B = \text{Greater than } 4 = \{5, 6\}$

$A \cap B = \text{Multiple of } 3 \& \text{greater than } 4 = \{6\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{2/6} = \frac{1}{2}$$

Multiplication Theorem :-

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B/A) \\ &= P(B) \cdot P(A|B) \end{aligned}$$

Results —

1) If A & B are independent events,

$$(i) P(A|B) = P(A)$$

$$(ii) P(B|A) = P(B)$$

$$(iii) P(A \cap B) = P(A) \cdot P(B)$$

$$(iv) A^c, B \text{ are also independent ; } P(A^c \cap B) = P(A^c) \cdot P(B)$$

e) A^c, B^c are — — —

Q:- The prob. of solving a prblm by 3 students $x, y & z$ are

$$P[x] = \frac{1}{4}, P[y] = \frac{3}{7}, P[z] = \frac{2}{9}.$$

If all of them try independently then what is the prob. that prblm cannot be solved. ~~&~~ (solved or atleast one solve prblm)

$$\text{Soln: } P[x^c] = \frac{3}{4}, P[y^c] = \frac{4}{7}, P[z^c] = \frac{7}{9}$$

$$\begin{aligned} P[\text{cannot solved}] &= P[x^c \cap y^c \cap z^c] = P[x^c y^c z^c] \\ &= P[x^c] P[y^c] P[z^c] = \frac{3}{4} \times \frac{4}{7} \times \frac{7}{9} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} P[\text{can be solved}] &= P[\text{at least one solve the prblm}] \\ &= P[x \cup y \cup z] = 1 - P[\text{none}] \\ &= 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

→

(Q) Let $P(G_1) = 0.5, P(G_2) = 0.6, P(G_3) = 0.8$ where G_1, G_2, G_3 are targets hit by gun G_1, G_2, G_3 resp. Let $G_1, G_2 \& G_3$ are independent then what is the prob. that atleast two hits registered.

$$\begin{aligned} \text{Soln: } P[\text{atleast two hits registered}] &= P[G_1 G_2 G_3^c] + P[G_1 G_2^c G_3] + \\ &\quad P[G_1^c G_2 G_3] + P[G_1 G_2 G_3] \\ &= 0.5 \times 0.6 \times 0.2 + 0.5 \times 0.4 \times 0.8 + 0.5 \times 0.6 \times 0.8 + 0.5 \times 0.6 \times 0.8 \\ &= 0.7 \end{aligned}$$

Q:- A pair of fair dice rolled twice. What is the prob. that an odd no. will follow an even no.

$$\begin{aligned} \text{Soln: } &\quad \square \quad \square \\ &\quad P_e \quad P_o \end{aligned} \quad = \frac{3}{6} \cdot \frac{3}{6} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P_e \cdot P_o$$

Q:- A fair coin is tossed 10 times. What is the prob. that only 1st two tosses will yield heads.

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^{10}} \text{ Ans.}$$

Q:- A coin is tossed again & again until head appears for 1st time.
 $P[\text{odd no. of tosses reqd.}]$ or $P[\text{No. of tosses reqd. is odd}]$

Solⁿ:— ① or ① ② ③ or ① ② ③ ④ ⑤ ——
H P T H TT TT H

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \left[\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \dots \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - \frac{1}{2}} \right] = \frac{2}{3} \text{ Ans.}$$

$$P[\text{no. of tosses reqd. is even}] = 1 - P[\text{no. of tosses = odd}]$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

Q:- A pair of fair dice rolled again & again till a total of 5 or 7 is obtained. What is the prob. that a total of 5 comes

Solⁿ:— before total of 7.

$P[\text{total 5 comes before total 7}]$

$$P[S=5] = \frac{4}{36}, \quad P[S=7] = \frac{6}{36}$$

$$P[S=5 \text{ or } S=7] = \frac{10}{36}$$

$$P[S \neq 5 \text{ & } S \neq 7] = 1 - P[S=5 \text{ or } S=7]$$

$$= 1 - \frac{10}{36} = \frac{26}{36}$$

To get $S=5$ before $S=7$

① or ① ② or ① ② ③ or —
 $S=5$ $S \neq 5$ $S=5$ $S \neq 5$ $S \neq 5$ $S=5$ $S \neq 5$ $S=5$ or —
 $\Delta S \neq 7$ $\Delta S \neq 7$ $\Delta S \neq 7$

$$\frac{4}{36} + \frac{26}{36} \cdot \frac{4}{36} + \frac{26}{36} \cdot \frac{26}{36} \cdot \frac{4}{36} + \dots$$

$$\frac{4}{36} \left[1 + \frac{26}{36} + \left(\frac{26}{36} \right)^2 + \dots \right] = \frac{4}{36} \cdot \frac{1}{1 - \frac{26}{36}} = \frac{36 \times 4}{26 \times 10} = 0.4$$

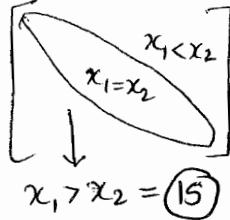
$$\text{or } \frac{2}{5}$$

of getting sum = 7 when it is known that digit in the 1st die is greater than that of 2nd one.

Soln:- Digit on 1st die x_1 ,
Digit on 2nd die x_2 .

$$P[S=7 | (x_1 > x_2)] = \frac{3}{15} = \frac{1}{5}$$

$S=7 \rightarrow (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$



Q:- Bag P contain 3W & 4B balls
Bag Q contain 4W & 3B balls.

A ball is transferred at random from P to Q & then a ball is transferred at random from Q to P. A ball is then taken from P. What is the chance that it is a white ball.

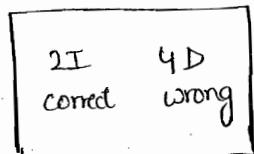
$$\begin{array}{|c|c|} \hline & \begin{matrix} 3W \\ 4B \end{matrix} & \begin{matrix} 4W \\ 3B \end{matrix} \\ \hline P & & Q \\ \hline \end{array}$$

$$\begin{array}{l} \textcircled{1} \textcircled{2} \textcircled{3} \text{ or } \textcircled{1} \textcircled{2} \textcircled{3} \text{ or } \textcircled{1} \textcircled{2} \textcircled{3} \\ \begin{matrix} W & W & W \\ W & B & W \\ \hline 3 & 7 & 5/8 \end{matrix} \begin{matrix} 3/7 \\ 1^{\text{st}} \text{ case} \end{matrix} \begin{matrix} B & W & W \\ W & B & W \\ \hline 3 & 7 & 3/8 \end{matrix} \begin{matrix} 2/7 \\ 2^{\text{nd}} \text{ case} \end{matrix} + \begin{matrix} 4 & 7 & 4/8 \\ 4/7 & 4/8 & 4/7 \end{matrix} \end{array}$$

$$\begin{array}{l} \text{or } \textcircled{1} \textcircled{2} \textcircled{3} \\ \begin{matrix} B & B & W \\ B & B & W \\ \hline 4 & 7 & 4/8 \end{matrix} \\ + \frac{4}{7} \cdot \frac{4}{8} \cdot \frac{3}{7} \\ = \frac{25}{56} \text{ Ans.} \end{array}$$

$$\begin{array}{c} \begin{matrix} 3W \\ 2W \\ 2W \\ 4B \end{matrix} \quad \begin{matrix} 4W \\ 5W \\ 4W \\ 3B \end{matrix} \\ \text{1st case.} \\ = \frac{45+18+64+48}{7 \cdot 8 \cdot 7} = \frac{25}{7 \cdot 8} \end{array}$$

P-38: Q-48



$\textcircled{1}$	$\textcircled{2}$	or	$\textcircled{1}$	$\textcircled{2}$
Correct key is lost	lock is opened		wrong key is lost	lock is opened
			(correct key selected)	(correct key is selected)

$$\frac{2}{6} \cdot \frac{1}{5} + \frac{4}{6} \cdot \frac{2}{5} = \frac{2+8}{30} = \frac{1}{3}$$

Q-30: P-36 : $P(A) = x, P(B) = y$

$$P(A^C) = 1-x, P(B^C) = 1-y$$

A & B both agree = Both true or Both false

$$\begin{aligned} &= AB \text{ or } A^C B^C \\ &= xy + (1-x)(1-y) \end{aligned}$$

$$P(\text{statement true} | \text{Both agree}) = \frac{xy}{xy + (1-x)(1-y)}$$

$xy \rightarrow \text{Statement True \& both agree.}$

Q1:- Parcels from sender S do receiver R pass sequentially through two post offices. Each post office has prob. ~~of~~ $1/5$ of losing an incoming parcel, independently of all other parcels. Given a parcel is lost, prob. it was lost by second post office is

Soln:- $P[\text{Lost by 1st}] = 1/5$

$$P[\text{Lost by 2nd}] = \frac{4}{5} \cdot \frac{1}{5} = \frac{4}{25}$$

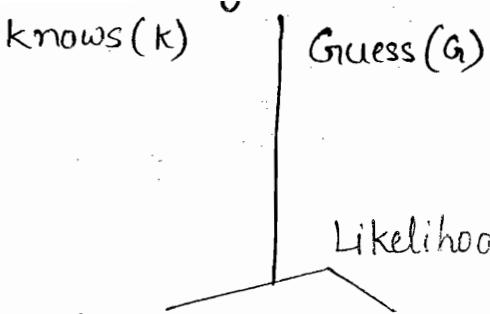
i.e. $P[\text{not lost by 1st}] = 1 - \frac{1}{5} = \frac{4}{5}$

$$P[\text{lost by 2nd}] = \frac{4}{25}$$

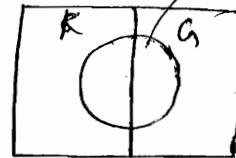
$$P[\text{lost}] = P[\text{Lost by 1st or Lost by 2nd}]$$

$$= \frac{1}{5} + \frac{4}{5} \cdot \frac{1}{5} =$$

$$P[\text{Lost by 2nd} | \text{Lost}] = \frac{\frac{4}{25} \cdot \frac{1}{5}}{\frac{1}{5} + \frac{4}{5} \cdot \frac{1}{5}} = \frac{4}{9}$$



$P(K) = 0.7$ } Prior
 $P(G) = 0.3$ } Probabilities
 Correct (C)



Sol :- $P(C|K) = 1$ $P(C|G) = \frac{1}{4}$ \rightarrow (4 options; one is correct)

Correct] = $P[K \cap C \text{ or } G \cap C]$

Total Prob. $= P[K \cap C] + P[G \cap C] = P(K) P(C|K) + P(G) P(C|G)$
 $= 0.7 \times 1 + 0.3 \times \frac{1}{4} = 0.7 + 0.3 \times 0.25 = 0.775.$

Now, $P[K|C]$ $P[G|C]$ } Posterior Probabilities or Bayes Probabilities

$$P(K|C) = \frac{P[K \cap C]}{P[C]} = \frac{0.7 \times 1}{0.775} = 0.9039$$

$$P[G|C] = \frac{P[G \cap C]}{P[C]} = \frac{0.3 \times \frac{1}{4}}{0.775} = 0.0967$$

Ex :- There are three companies X, Y & Z supply computers to a university

Company	Computers supplied	Prob. of being defective.
X	60%	0.01
Y	30%	0.02
Z	10%	0.03

1) $P[\text{Comp being Defective}] \rightarrow \text{Total Prob.}$

2) $P[X|D]$

$P[Y|D]$
 $P[Z|D]$

$$P(X) = 0.6, P(Y) = 0.3, P(Z) = 0.1$$

$$P(D|X) = 0.01, P(D|Y) = 0.02$$

$$P(D|Z) = 0.03$$

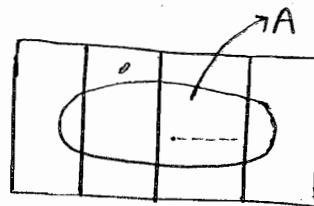
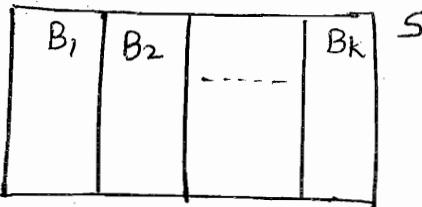
$$1) P[\text{Defected}] = 0.6 \times 0.01 + 0.3 \times 0.02 + 0.1 \times 0.03 = 0.006 + 0.006 + 0.003 = 0.015$$

$$2) P[X|D] = \frac{0.6 \times 0.01}{0.015} = 0.4$$

$$P[Z|D] = \frac{0.1 \times 0.03}{0.015} = 0.2$$

$$P[Y|D] = \frac{0.3 \times 0.02}{0.015} = 0.4$$

Let B_1, B_2, \dots, B_k the events which partition S .



Let A be any event following these events

The total prob. of A :

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + \dots + P(B_k) \cdot P(A|B_k)$$

(or)

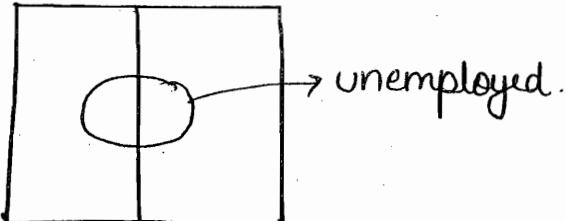
$$P(A) = \sum_{i=1}^k P(B_i) P(A|B_i)$$

Bayes Theorem :- Let $P(A) \neq 0$

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} \Rightarrow P(B_j|A) = \frac{P(B_j) P(A|B_j)}{P(A)}$$

Ex:-

Population	
M	F



$$P(M) = 0.7$$

$$P(U|M) = 0.25$$

$$P(F) = 0.3$$

$$P(U|F) = 0.15$$

$$P(U) = 0.7 \times 0.25 + 0.3 \times 0.15$$

$$P(M|U) = \frac{0.7 \times 0.25}{P(U)} ; P(F|U) = \frac{0.3 \times 0.15}{P(U)}$$

^(RV) Random Variable — (Main purpose \rightarrow outcome as nos)

$$X: S \rightarrow R$$

It associates real no. to outcomes of the sample space.

R.V

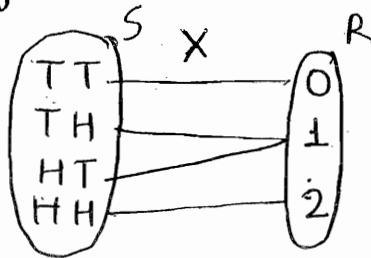
↓
Discrete RV

→ take discrete values

↓
Continuous RV

→ take values in the interval.

x : No. of heads.



$$X = \{0, 1, 2\}$$

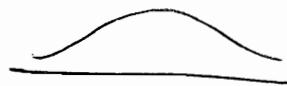
$X = x_1$	x_2	\dots	x_n
$P(x) = P(x_1)$	$P(x_2)$	\dots	$P(x_n)$

Discrete Prob. distribution

Ex :- Working of a bulb ; X : No. of hrs bulb works before failure

$$X: [0, \infty)$$

$P(a \leq X \leq b)$; we can find continuous prob. distributions generally expressed graphs.



Prob. Mass Functions - If X is discrete R.V. then its pmf is (PMF) defined as $P(X=x) = p(x)$ & pmf is such that (1) $p(x) \geq 0$ (2) $\sum_x p(x) = 1$

Ex :- Tossing a coin twice - X : no. of heads

$X:$	0	1	2
$P(x)$	$1/4$	$2/4$	$1/4$

$$P(X=0) = P(0) = 1/4$$

$$P(X=1) = P(1) = 2/4$$

$$P(X=2) = P(2) = 1/4$$

Ex :- A R.V. has the following distribution

$$\begin{array}{ccccccccc} x & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ p & 0 & k & 2k & 3k & 4k^2 & 2k^2 & 7k^2 + k & \end{array} \quad \sum p = 1$$

$$0 + k + 2k + 3k + 4k^2 + 2k^2 + 7k^2 + k = 10k^2 + 9k = 1$$

(i) $k = ?$ (ii) $P(X > 2)$

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - [0 + k + 2k] = 1 - 3k = 1 - \frac{3}{10} = 0.7$$

$$\begin{aligned} & \cancel{10k} \quad 10k^2 + 9k - 1 = 0 \\ & -9 \pm \sqrt{81 + 40} = \frac{-9 \pm 11}{20} \end{aligned}$$

$$= 0.1 \quad \text{(neglected -ve value)}$$

Q:- A die is loaded so that Prob of getting face x is αx .
P[getting odd no. when the die rolled]

Soln:-

$$P[x] \propto x$$

$$P[x] = kx$$

X	1	2	3	4	5	6
$P(x)$	k	$2k$	$3k$	$4k$	$5k$	$6k$

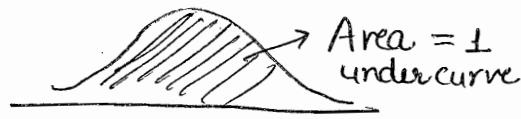
fair(unbiased)	Biased
coin	coin
$P(H) = 1/2$	$P(H) = 3/4$
$P(T) = 1/2$	$P(T) = 1/4$
dice: $P(1) = \frac{P(6)}{6} = 1/6$	

$$P[\text{odd no.}] = P\{1 \text{ or } 3 \text{ or } 5\} = P(1) + P(3) + P(5) = k + 3k + 5k = \frac{9}{21} = \frac{3}{7}$$

Prob. density f^n (pdf) - If X is a cont. R.V. then its prob. density $f^n f(x)$ is defined such that,

$$(i) f(x) \geq 0 \quad (ii) \int_{-\infty}^{\infty} f(x) dx = 1$$

$$(iii) P(a \leq X \leq b) = \int_a^b f(x) dx$$



Q:- A cont. R.V. X has pdf $f(x)$ given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & 0 < w \end{cases} \quad i) \lambda = ? \quad (ii) P(50 < x < 100)$$

$$\underline{\text{Soln}} :-(i) \int_0^{\infty} \lambda e^{-x/100} dx = 1 \Rightarrow \frac{-\lambda}{1/100} [e^{-x}]_0^{\infty} = 1 \Rightarrow \lambda = 1/100$$

$$(ii) \int_{50}^{100} \frac{1}{100} e^{-x/100} dx = \frac{1}{100} \left[\frac{e^{-x/100}}{-1/100} \right]_{50}^{100} = -1 \left[e^{-1} - e^{-1/2} \right] = e^{-1/2} - e^{-1} \quad \underline{\text{Ans.}}$$

Statistics :— characterizes the distribution.

Measures of:

1) Central tendency :— Central value

Mean, Median, Mode

2) Dispersion : Spread of the data.

Variance = σ^2 , Std Deviation = σ

For ungrouped data: x_1, x_2, \dots, x_n

$$\text{Mean} = \bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$$

Median → Sort the data

Median = Mid value or avg. of mid value.

III defined Distribution : $\boxed{\text{Mode} = 3 \text{Median} = 2 \text{Mean}}$

$$\text{Variance}, \sigma^2 = \frac{\sum_{i=1}^n (x_i - \mu)^2}{n}$$

Avg of squared deviations from mean?

Std deviation - S.D. = Positive square root of variance = σ

Q:- 6, 2, 1, 2, 3, 4. Find mean, median, mode, σ^2 , σ

$$\text{Soln: } \text{Mean} = \frac{6+2+1+2+3+4}{6} = 3$$

$$\text{For median: } 1, 2, 2, 3, 4, 6 \rightarrow \frac{2+3}{2} = 2.5$$

Mode: 2

$$\text{Variance: } \sigma^2 = \frac{(6-3)^2 + (2-3)^2 + (1-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2}{6} = \frac{8}{3}$$

$$\sigma = \sqrt{\frac{8}{3}}$$

Mathematical Expectation:- Or expected value \rightarrow gives averages.

$$E[x] = \begin{cases} \sum_x x p(x) & X \text{ is discrete R.V.; } p(x) \text{ is Pmf} \\ \int_{-\infty}^{\infty} x f(x) dx & X \text{ is C.R.V.; } f(x) \text{ is pdf.} \\ & \downarrow \text{Cont.} \end{cases}$$

$$1) \text{ Mean} = \mu = E[x]$$

$$2) E[x^2] = \begin{cases} \sum_x x^2 p(x) & X \text{ is D.R.V.} \\ \int_{-\infty}^{\infty} x^2 f(x) dx & X \text{ is C.R.V.} \end{cases}$$

$$\text{Variance: } \text{Var}(x) = E[(x-\mu)^2]$$

Result -

$$1. E[c] = c \quad 2. E[ax+b] = aE[x] + b \quad 3. E[x+y] = E[x] + E[y]$$

$$4. E[x \cdot y] = E[x] \cdot E[y] \text{ iff } x \& y \text{ are independent R.V.}$$

$$**5. \text{Var}(x) = E[x^2] - \mu^2 \quad 6. \text{Var}(c) = 0$$

$$7. \text{Var}(ax+b) = a^2 \text{Var}(x) \quad 8. \text{Var}(x+y) = \text{Var}(x) + \text{Var}(y)$$

iff $x \& y$ are independent R.V.

Mean of the following distribution.

x	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Sol:- $\mu = E[x] = \sum_x x p(x) = 0 * \frac{1}{4} + 1 * \frac{2}{4} + 2 * \frac{1}{4}$
 $\Rightarrow \mu = 1$

Variance = $E[x^2] - \mu^2$

$$E[x^2] = \sum x^2 p(x) = 0 * \frac{1}{4} + 1 * \frac{2}{4} + 4 * \frac{1}{4} = \frac{3}{2}$$

x^2	0	1	4
$p(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

→ Square only x
 $p(x)$ remains same.

$$\text{Var.}(x) = \frac{3}{2} - 1^2 = \frac{1}{2} = \sigma^2 ; \text{ S.D.}, \sigma = 1/\sqrt{2}$$

Continuous Distribution -

Uniform distribution - A cont. R.V. x is said to follow uniform distribution over the interval $[a, b]$ if its pdf $f(x)$ is given by

$$f(x) = \frac{1}{b-a} \quad a \leq x \leq b$$

Mean = $\mu = E[x] = \int x f(x) dx$

$$\boxed{\text{Mean} = \mu = \frac{a+b}{2}} \quad = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_a^b = \frac{a+b}{2}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \left[\frac{1}{b-a} \cdot \frac{x^3}{3} \right]_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{b^2 + ab + a^2}{3}$$

$$\text{Variance} = \frac{a^2 + ab + a^2}{3} - \left(\frac{a^2 + 2ab + b^2}{4} \right) = \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

$$\boxed{\text{Variance}(x) = \frac{(b-a)^2}{12}}$$

$$, \quad \boxed{\text{S.D.} = \frac{b-a}{\sqrt{12}}}$$

uniformly at random. Then the expected length of shorter stick is.

Soln:— Let x : Length of shorter stick.

x is uniformly distributed b/w 0 to $\frac{1}{2}$

$$\text{pdf, } f(x) = \frac{1}{\frac{1}{2}-0} = [0, \frac{1}{2}]$$

$$f(x) = 2 \quad 0 \leq x \leq \frac{1}{2}$$

$$\mu = E[x] = \frac{a+b}{2} = \frac{1}{4} = \left(\frac{0+\frac{1}{2}}{\frac{1}{2}} \right)$$

Q:— The life of electric bulb is a R.V. with exp. distribution

$$f(t) = \alpha e^{-\alpha t} \quad t \geq 0$$

The prob. that its value lies b/w 100 hrs to 200 hrs.

a) $e^{-100\alpha} - e^{-200\alpha}$

b) $e^{200\alpha} - e^{-100\alpha}$

c) $e^{-100} - e^{-200}$

d) $e^{-200} - e^{-100}$

Soln:— $P(100 < T < 200) = \int_{100}^{200} \alpha e^{-\alpha t} dt = \alpha \frac{e^{-\alpha t}}{-\alpha} \Big|_{100}^{200} = e^{-100\alpha} - e^{-200\alpha}$

Exponential Distribution :-

A cont. R.V. $X \sim E(\lambda)$ [X follows exp. distribution with parameter λ]

If pdf is given by $f(x) = \lambda e^{-\lambda x} \quad x \geq 0$

$$\text{Mean} = \mu = \frac{1}{\lambda}, \quad \text{Variance} = \sigma^2 = \frac{1}{\lambda^2}$$

Normal Distribution :- (Gaussian distribution)

A. cont. R.V. $X \sim N(\mu, \sigma^2)$

↓
Mean

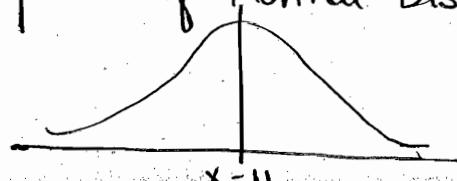
→ Variance

if pdf is given by

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2} \quad -\infty \text{ to } \infty$$

Properties of Normal Distribution

1)

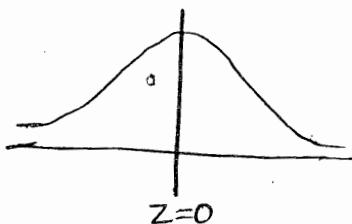


- 1) Normal curve is bell shaped.
- 2) —||— —||— is symmetrical abt. mean ($X = \mu$)
- 3) $P(X > \mu) = 0.5$
- 4) $P(X < \mu) = 0.5$

$\therefore \text{Mean} = \text{Median} = \text{Mode}$

Standard normal variate — $Z = \frac{X - \mu}{\sigma}$

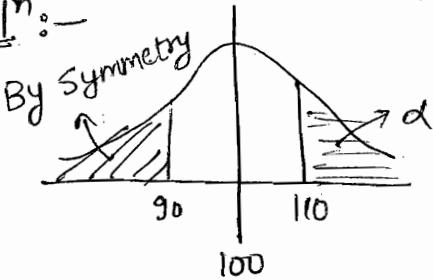
Def :— The std normal variate $Z \sim N$ (Mean = 0, variance = 1)
& its pdf is given by $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty$



- 1) Standard Normal Curve is symmetric abt $z=0$.
- 2) $P(Z < 0) = P(Z > 0) = 0.5$
- 3) The area under the curve [i.e $P(0 < z < z_1)$] is tabulated in std normal tables.

Ex :— X is a normal R.V. with mean 100. $P(X \geq 110) = \alpha$, then
 $P(90 < X < 110)$ a) $1-\alpha$ b) $1-2\alpha$ c) 2α d) $1-\alpha/2$

Solⁿ :—



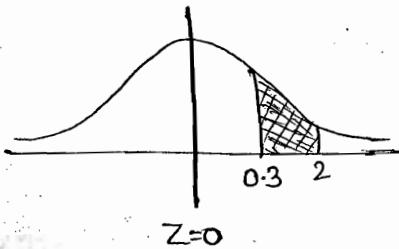
$$\begin{aligned} P(90 < X < 110) &= 1 - 2\alpha \\ &= 1 - P(X < 90) - P(X > 110) \end{aligned}$$

Q :— Suppose that temperature during December is normally distributed with mean $= 20^\circ$ & std deviation, $\sigma = 3.33^\circ$. Find prob. ($21.11^\circ < X < 26.66^\circ$)

$P(21.11^\circ < X < 26.66^\circ)$ • Area under the curve $z=0$ to $z=2$ is 0.4772.
 $z=0$ to $z=0.33$ is 0.1293

Solⁿ :— $Z = \frac{X - \mu}{\sigma}$

$$P\left(\frac{21.11 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{26.66 - \mu}{\sigma}\right) = P(0.3 < Z < 2)$$



$$\begin{aligned} P(0 < Z < 2) - P(0 < Z < 0.3) \\ = 0.4772 - 0.1293 \end{aligned}$$

Bernoulli's Trials :- Independent trials of repetitive nature in which prob. (success or failure) of a particular event does not change from one trial to next trial.

Binomial Distribution :- A discrete R.V. X is said to follow binomial distribution with parameters n, p if its pmf is given by

$$P(X=r) = {}^n C_r P^r q^{n-r} \quad r=0, 1, 2, \dots, n$$

$n \rightarrow$ No. of Bernoulli's trial.

$$\text{Mean} = np$$

$p \rightarrow P(A)$ = Prob. of success of an event.

$$\text{Variance} = \sigma^2 = npq$$

$q \rightarrow P(A^c)$

R.V. $\rightarrow X$: No. of times A happens.

$P(X=r) \rightarrow$ Prob. of event A happens r times (or)
prob. of r successes of event A.

A. R.V. which follows binomial distribution is called binomial R.V.

Q:- A coin is tossed 8 times (i) What is the prob. of getting 3 heads.

(ii) Prob. of getting atleast 2 heads.,

Solⁿ:- A: getting heads ; $P = P(A) = 1/2$; $q = 1/2$

X : No. of times A happens.

$$(i) P(X=3) = {}^8 C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^5 = {}^8 C_3 \left(\frac{1}{2}\right)^8 = \frac{{}^8 C_3}{2^8}$$

$$(ii) P(X \geq 2) = 1 - P[X \leq 1] = 1 - [P(0) + P(1)] = 1 - \left(\frac{{}^8 C_0 + {}^8 C_1}{2^8} \right) \\ = 1 - \frac{(1+8)}{2^8} = 1 - \frac{9}{2^8} = \frac{247}{256}$$

Q:- A coin is loaded in such a way that prob. of getting heads is 3 times prob. of getting tails. If the coin is tossed 4 times, what is the prob. of getting atleast 3 heads.

Solⁿ:- $P(H) = 3 P(T)$, Let $P(T) = x$

$$P(H) = 3x$$

$$x + 3x = 1 \Rightarrow x = 1/4 \quad P = \frac{3}{4} \xrightarrow{\text{getting head}}, q = 1/4$$

$$P(X \geq 3) = P(3) + P(4) = {}^4 C_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right) + {}^4 C_4 \left(\frac{3}{4}\right)^4 = \frac{4 \cdot 3^3}{4 \cdot 4^3} + \frac{3^4}{4^4}$$

Q:- A die has four blank faces & 2 faces marked 3. What is chance of getting 12 in 5 throws.

$$\underline{\text{Soln}}:- P[0] = \frac{4}{6} = \frac{2}{3}; P[3] = \frac{2}{6} = \frac{1}{3}$$

$$P[\text{getting } 12 \text{ in 5 throws}] = P[4 \text{ 3's in 5 throws}]$$

$$\underline{n=5}; p = \text{prob. getting 3} = \frac{1}{3}; q = \frac{2}{3}$$

$$P[X=4] = 5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1 = 5 \cdot \frac{2}{3^5} = \frac{10}{243}$$

Q:- A man takes step fwd with prob. 0.4 & step bckwd with prob 0.6. The prob. that at the end of 11 steps, his one step away from starting pt. is _____

$$\underline{\text{Soln}}:- n=11; p = \text{prob. of taking fwd step} = 0.4 \\ q = \text{prob. of taking bckwd step} = 0.6$$

$$\begin{aligned} P[\text{one step away}] &= P[\text{one step fwd or one step bckwd}] \\ &= P[\text{one step F}] + P[\text{one step B}] \\ &= P[6F, 5B] + P[5F, 6B] \\ &= P[X=6] + P[X=5] \\ &= 11C_6 (0.4)^6 (0.6)^5 + 11C_5 (0.4)^5 (0.6)^6 \end{aligned}$$

$$\therefore {}^n C_r = {}^n C_{n-r}; 11C_6 = 11C_5$$

$$\begin{aligned} &= 11C_6 (0.4)^5 (0.6)^5 \left[\frac{0.4+0.6}{1} \right] \\ &= 11C_6 (0.4)^5 (0.6)^5 \end{aligned}$$

Q:- An Unbiased coin tossed ∞ times. What is the prob. that the 4^{th} head appears at 10^{th} toss.

- a) 0.067 b) 0.073 c) 0.082 d) 0.091

Soln:

9 tosses 3H

(10)
H

$$P(X=3) \cdot \binom{1}{2} = \frac{9}{2} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{9 \cdot 3}{2^{10}} = \frac{27}{256} = 0.082$$

\hookrightarrow at 10th toss

Q:- The prop. of a man hitting a target is $1/4$. If he fires 4 times the prob. of hitting target atleast twice.

Solⁿ:- $n=4$; $p=1/4$, $q=3/4$

$$\begin{aligned} P(X \geq 2) &= 1 - [P(X \leq 1)] = 1 - [P(0) + P(1)] = 1 - \left[4C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^4 + 4C_1 \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^3 \right] \\ &= 1 - \left[\frac{3^4 + 4 \cdot 3^3}{4^4} \right] = \frac{67}{4^4} = 0.26 \end{aligned}$$

Binomial Freq. Distribution - An experiment consists of n bernoulli's trials & the experiment is repeated N times.

The frequency of r successes (Expected value of r successes)

$$= N * nCr p^r q^{n-r}$$

Q:- In 256 sets of 12 tosses of a coin. How many cases one can expect 8 heads.

Solⁿ:- $N=256$; $n=12$, $p=1/2$, $q=1/2$.

Expected value of 8 heads in total 256 = $N * nCr p^r q^{n-r}$.

$$= 256 * P(X=8) = 256 * 12C8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4$$

$$= \frac{256 * 12C8}{2^{12}} = \frac{12C8}{2^4} = \frac{495}{16} = 30.9 \approx 31.$$

Poisson Distribution:- These are the distributions related to events which are extremely rare.

Limiting case of binomial distribution when no. of trials n is (a) very large; & p is very small.

Def:- A discrete R.V $X \sim P(\lambda)$ is said to follow poisson distribution if its pmf is given by

$$P(X=r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

$$r=0,1,2,\dots$$

$$\text{Mean } \mu = \lambda$$

$$\text{Variance } = \lambda$$

$X \sim B(n, p)$ Put $\lambda = np$ when $n \rightarrow \infty$ & p is very small
 $\approx X \sim P(\lambda)$

Q:- The prob. of a bad reaction from certain injection is 0.001. What is the chance that out of 2000 individuals more than two will get a bad reaction.

Sol:- $n = 2000$; $P = 0.001$; $n \rightarrow \infty$ & p is very small
 $\lambda = np = 2$

$$X \sim B(n, p) \approx P(2) \quad P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x = 0, 1, 2, \dots$$

$$\begin{aligned} P[\text{more than two will get bad reaction}] &= P[X \geq 2] = 1 - P[X \leq 2] \\ &= 1 - \{P[X=0] + P[X=1] + P[X=2]\} \\ &= 1 - \left[\frac{e^{-2} 2^0}{0!} + \frac{e^{-2} 2^1}{1!} + \frac{e^{-2} 2^2}{2!} \right] = 1 - \frac{5}{e^2} \text{ Ans.} \end{aligned}$$

Q:- A traffic office imposes on an avg 5 penalties daily on traffic violators. Assume that no. of penalties on different days is independent & follows poisson's distribution. What is the prob. that there will be less than 4 penalties on a day. $\therefore \lambda = 5$

$$\begin{aligned} \text{Sol:- } P[X < 4] &= P(0) + P(1) + P(2) + P(3) \\ &= \frac{e^{-5} 5^0}{0!} + \frac{e^{-5} 5^1}{1!} + \frac{e^{-5} 5^2}{2!} + \frac{e^{-5} 5^3}{3!} = e^{-5} \left[1 + 5 + \frac{25}{2} + \frac{125}{6} \right] \\ &= \underline{0.265} \text{ Ans.} \end{aligned}$$

DEFINITE INTEGRAL

- 1) $\int_a^b f(x) dx = - \int_b^a f(x) dx$
- 2) $\int_0^a f(x) dx = \int_0^a f(a-x) dx$
- 3) $f(-x) = f(x); f(x)$ is even f^n .
- 4) $f(-x) = -f(x); f(x)$ is odd f^n .
- 5) $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ even $f(x)$
 $= 0$ $f(x)$ odd
- 6) $\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$; $f(2a-x) = f(x)$
 $= 0$; $f(2a-x) = -f(x)$

$$\int_0^{\pi} \cos x dx = 0 \quad \therefore \cos(\pi - x) = -\cos x$$

$$Q := -1 \int_{-\pi/2}^{\pi/2} x^{10} \log \left(\frac{1+\sin x}{1-\sin x} \right) dx$$

$$3. \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx$$

$$5. \int_0^{\pi/4} \log(1 + \tan x) dx$$

$$\underline{\text{Soln}}: -1 \cdot f(x) = (x^{10}) \log \left(\frac{1+\sin(-x)}{1-\sin(-x)} \right) = x^{10} \log \left(\frac{1-\sin x}{1+\sin x} \right) \\ = -x^{10} \log \left(\frac{1-\sin x}{1+\sin x} \right) \rightarrow -f(x) \text{ odd } f^n.$$

$$\int f(x) dx = 0 \quad \underline{\text{Ans}}$$

$$2. I = \int_0^{\pi/2} (a^2 \cos^2 x + b^2 \sin^2 x) dx \quad \underline{\text{--- ①}}$$

$$I = \int_0^{\pi/2} \left[a^2 \cos^2 \left(\frac{\pi}{2} - x \right) + b^2 \sin^2 \left(\frac{\pi}{2} - x \right) \right] dx = \int_0^{\pi/2} (a^2 \sin^2 x + b^2 \cos^2 x) dx \quad \underline{\text{--- ②}}$$

$$\textcircled{1} + \textcircled{2} = 2I = \int_0^{\pi/2} (a^2 + b^2) dx = (a^2 + b^2)x = (a^2 + b^2) \frac{\pi}{2}$$

$$I = \frac{\pi}{4} (a^2 + b^2) \quad \underline{\text{Ans}}$$

$$3. I = \int_0^{\pi/2} \frac{\cos \left(\frac{\pi}{2} - x \right) - \sin \left(\frac{\pi}{2} - x \right)}{1 + \cos \left(\frac{\pi}{2} - x \right) \sin \left(\frac{\pi}{2} - x \right)} dx = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \cos x \sin x} dx \quad \underline{\text{--- ③}}$$

$$2I = 0 \Rightarrow I = 0$$

$$4. I = \int_0^{\pi} \frac{1}{\cos^2 x [a^2 + b^2 \tan^2 x]} dx = \int_0^{\pi} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$

$$\text{Rule ⑥ } f(2a-x) = f(x) \quad \sec^2(\pi-x) = (-\sec x)^2 = \sec^2 x \\ \tan^2(\pi-x) = (-\tan x)^2 = \tan^2 x$$

$$\begin{aligned} \text{Put } \tan x &= t & x=0 & t=0 \\ \sec^2 x dx &= dt & x \rightarrow \frac{\pi}{2} & t \rightarrow \infty \end{aligned}$$

$$\int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2} \int_0^{\infty} \frac{dt}{(a/b)^2 + t^2}$$

$$= \frac{2}{b^2} \times \left(\frac{b}{a}\right)^2 \left[\tan^{-1} \frac{ta}{b} \right]_0^{\infty} = \frac{2}{ab} \left[\frac{\pi}{2} - 0 \right]$$

$$I = \frac{\pi}{ab} \quad \underline{\text{Ans}}$$

5. $I = \int_0^{\pi/4} \log(1 + \tan x) dx$ Rule ②, $I = \int_0^{\pi/2} \log(1 + \tan(\frac{\pi}{4} - x)) dx$

$$I = \int_0^{\pi/4} \log \left(1 + \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \right) dx = \int_0^{\pi/4} \log \left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x} \right] dx$$

$$I = \int_0^{\pi/4} \log 2 dx - \int_0^{\pi/4} \log(1 + \tan x) dx$$

$$2I = \log 2 [x]_0^{\pi/2} \Rightarrow I = \frac{\pi}{8} \log 2 \quad \underline{\text{Ans}}$$

6. ~~$I = \int_0^{\pi/4} \frac{1 - \tan(\frac{\pi}{4} - x)}{1 + \tan(\frac{\pi}{4} - x)} dx$~~ = ~~$\int_0^{\pi/4} \frac{1 + \tan(\frac{\pi}{4} - x)}{1 - \tan(\frac{\pi}{4} - x)} dx$~~

$$I = \int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} dx = \left[\log [\cos x + \sin x] \right]_0^{\pi/4}$$

$$= \log \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \log(1+0)$$

$$= \log \sqrt{2} \quad \underline{\text{Ans}}$$

Results :-

$$(1) \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \dots \{ \text{upto } 0 \text{ or -ve term} \}$$

* $\frac{\pi}{2}$; if n is even. Multiply with $\frac{\pi}{2}$

Ex :- $\int_0^{\pi/2} \sin^5 x dx = \frac{4}{5} \cdot \frac{2}{3}$

$\int_0^{\pi} \sin^2 x dx$ Rule ⑥

$$\sin^2(\pi - x) = \sin^2 x$$

$$= 2 \int_0^{\pi/2} \sin^2 x dx = 2 \cdot \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^6 x dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \times \frac{\pi}{2}$$

(2) $\int_0^{\pi/2} \sin^n x \cos^m x dx = \frac{(n-1)(n-3) \dots (m+1)(m-3) \dots}{(m+n)(m+n-2) \dots} \quad \left\{ x \frac{\pi}{2} \text{ if } m \& n \text{ both are even.} \right.$

$$(3) \int_0^{\pi/2} \sin^6 x \cos^4 x dx = \frac{53131}{10 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{3\pi}{512}$$

Ex :- 1) $\int_0^{\pi/2} \frac{x^6}{\sqrt{1-x^2}} dx$ Put $x = \sin \theta$, $dx = \cos \theta d\theta$; $x=0 \rightarrow \theta=0$
 $x=1 \rightarrow \theta=\pi/2$

$$\int_0^{\pi/2} \frac{\sin^6 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta = \int_0^{\pi/2} \sin^6 \theta d\theta = \frac{5!}{6!} \cdot \frac{3!}{4!} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$$

$$2) I = \int_0^{\pi} x \sin^6 x \cos^4 x dx = \int_0^{\pi} x \sin^6 x \cos^4 x dx \quad \text{Rule (2)} \quad I = \int_0^{\pi} (\pi-x) \sin^6 x \cos^4 x dx$$

$$2I = \pi \int_0^{\pi} \sin^6 x \cos^4 x dx = \pi \cancel{\frac{53131}{108642}}$$

$$2I = 2\pi \int_0^{\pi/2} \sin^6 x \cos^4 x dx = \frac{15 \cdot 31}{2 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times \frac{\pi}{2} \times \frac{\pi}{2} = \frac{3\pi^2}{512} \text{ Ans.}$$

Limiting Case Prblm :-

$$(1) \lim_{n \rightarrow \infty} \left[\frac{n}{n^2} + \frac{n}{n^2+1^2} + \dots + \frac{n}{n^2+(n-1)^2} \right] = \lim_{n \rightarrow \infty} \left[\sum_{k=0}^{n-1} \frac{n}{n^2+k^2} \right]$$

As $n \rightarrow \infty$, the summation can be approximated with integration
 $(\therefore \text{limiting case of summation is integration})$.

$$= \lim_{n \rightarrow \infty} \int_{x=0}^{n-1} \frac{n}{n^2+x^2} dx = \lim_{n \rightarrow \infty} n \cdot \frac{1}{n} \tan^{-1} \frac{x}{n} \Big|_0^{n-1} = \lim_{n \rightarrow \infty} \tan^{-1} \frac{n-1}{n} \cdot \tan^{-1} \frac{1}{n}$$

$$= \lim_{n \rightarrow \infty} \tan^{-1} \left(1 - \frac{1}{n} \right) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$(2) \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \right)^{1/n} \quad Y = \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \right)^{1/n} \quad \text{Apply log on both sides.}$$

$$\log Y = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{n!}{n^n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1 \cdot 2 \cdot 3 \dots n}{n \cdot n \cdot n \dots n} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \left(\frac{1}{n} \right) + \log \left(\frac{2}{n} \right) + \dots + \log \left(\frac{n}{n} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \log \left(\frac{k}{n} \right) \quad \text{Approx. with int.}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \int_{x=1}^n \log \left(\frac{x}{n} \right) dx \quad \text{Put } \frac{x}{n} = t \quad \begin{matrix} \text{Limits} \\ x=1 \quad t=1/n \\ x=n \quad t=1 \end{matrix}$$

$$dx = ndt$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log n = \lim_{n \rightarrow \infty} \frac{\log n}{n} = 0$$

$$= t \log t - t \Big|_0^1 = -1 \Rightarrow \log y = -1 \text{ or } y = \frac{1}{e} \text{ Ans.}$$

$$(3) \lim_{a \rightarrow 0} \frac{x^a - 1}{a} \quad (a^x)' = a^x \log a$$

$$\lim_{a \rightarrow 0} \frac{\frac{d}{da}(x^a - 1)}{\frac{d}{da} a} = \lim_{a \rightarrow 0} \frac{x^a \log x}{1} = \log x$$

Complex Analysis

$$z = x + iy$$

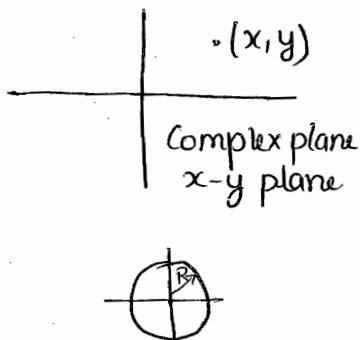
$$(1) \bar{z} = x - iy$$

$$(2) |z| = \sqrt{x^2 + y^2}$$

$$(3) z \cdot \bar{z} = |z|^2$$

(4) $|z| = R$; Circle with centre at $(0,0)$ & rad. R

(5) $|z - z_0| = R$; Circle with centre at z_0 & rad. R .

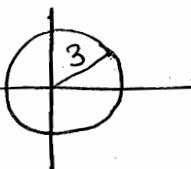


$$\underline{\text{Ex:}} - C: |z| = 3$$

$2+3i$ lies outside C

Distance $(0,0)$ to $(2,3)$

$$= \sqrt{(2-0)^2 + (3-0)^2} = \sqrt{13} > 3$$

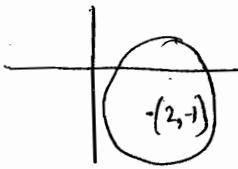


$$\underline{\text{Ex:}} - C: |z - 2+i| = 2; \text{ Centre } (2, -1); \text{ rad} = 2$$

$1+i$ lies inside C

$$\text{Distance } (2, -1) \text{ to } (1, 1) = \sqrt{(1-2)^2 + (1+1)^2}$$

$$= \sqrt{5} < 2 \text{ lies inside } C.$$



(6) Polar & Exponential Forms:

$$\text{Cartesian form: } z = x + iy$$

$$\text{Polar form: } z = r [\cos \theta + i \sin \theta]$$

$$\text{exponential form: } z = re^{i\theta}$$

where $r = \text{Mod } z$

$\theta = \arg z$ or amplitude z

$$r = |z| = \sqrt{x^2 + y^2}; \theta = \tan^{-1} \frac{y}{x}$$

(7) Principle Argument: $\arg z$

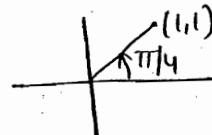
$$-\pi < \theta \leq \pi$$

$$\arg z = \text{Arg } z + 2k\pi \quad k=0, \pm 1, \pm 2$$

Any argument \downarrow principle argument

Ex: - Express $1+i$ in polar & exponential form: $z = 1+i$

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}; \theta = \tan^{-1} \frac{y}{x} = \frac{\pi}{4}$$



$$\text{Polar} = \sqrt{2} \left[\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right] = 1+i$$

$$\text{exponential} = z = \sqrt{2} e^{i\pi/4}$$

$$\text{or} \\ z = \sqrt{2} e^{i(\pi/4 + 2k\pi)}$$

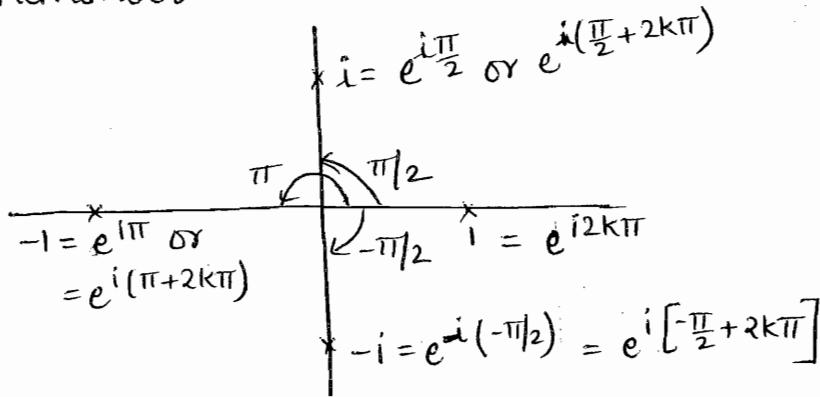
$$k=0, \pm 1, \pm 2, \dots$$

$$\gamma = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}, \quad \theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}1 = -\frac{3\pi}{4}$$

$$z = \sqrt{2} e^{-i\frac{3\pi}{4}} \quad \text{or} \quad z = \sqrt{2} e^{i[-\frac{3\pi}{4}+2k\pi]}$$

Arg.

Remember -



Q:- (i)ⁱ

$$i = e^{i\pi/2}$$

$$(i)^i = (e^{i\frac{\pi}{2}})^i = e^{-\frac{\pi}{2}}$$

$$Q:- \ln \sqrt{i} = \ln(i)^{1/2}$$

$$= \ln(e^{i\pi/2})^{1/2}$$

$$= \ln e^{i\pi/4} = i\frac{\pi}{4}$$

(8) n^{th} root of a complex no. z_0 ; $z_0 = r_0 e^{i\theta_0}$

The n distinct n^{th} roots of z_0 are given

$$(z_0)^{1/n} = (r_0)^{1/n} e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}$$

$k = 0, 1, 2, \dots, n-1$

Ex :- Find square roots of i . find $(i)^{1/2}$

$$\underline{\text{Soln}}: - i = e^{i\left(\frac{\pi}{2} + 2k\pi\right)} \Rightarrow (i)^{1/2} = e^{i\left(\frac{\pi}{4} + \frac{2\pi k}{2}\right)} = e^{i\left(\frac{\pi}{4} + k\pi\right)}$$

$k=0,1$

Distinct roots are:

$$k=0; e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$$

$$k=1; e^{i\left(\frac{\pi}{4}+\pi\right)} = e^{i\frac{5\pi}{4}} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{-1-i}{\sqrt{2}}$$

* Cube roots of unity $\Rightarrow 1, \omega, \omega^2$.

Ex :- $(x-1)^3 + 8 = 0$. Find roots of the eqn.

$$\underline{\text{Soln}}: - (x-1)^3 = -2^3 \quad (x-1)^3 = (-2\omega)^3$$

$$x-1 = -2$$

$$x = -1$$

$$x-1 = -2\omega$$

$$x = 1+2\omega$$

$$(x-1)^3 = (-2\omega^2)^3$$

$$x-1 = -2\omega^2$$

$$x = 1-2\omega^2$$

Complex Function :- If $z = x+iy$ is complex variable.

$f(z) = u(x,y) + iV(x,y)$ is a complex f.

$$\underline{\text{Ex}}: - f(z) = z^2 = (x+iy)(x+iy)$$

$$f(z) = \underbrace{x^2-y^2}_{\text{Real "U"}} + i\underbrace{2xy}_{\text{Imag. "V"}}$$

$$[z^2 \neq |z|^2]$$

Let $f(z) = u+iv$. $f'(z)$ exist at z_0

iff the 1st partial derivatives $u_x, u_y, v_x \& v_y$ exist, continuous at z_0 & satisfy C-R equations.

$$u_x = v_y \quad \& \quad u_y = -v_x$$

at z_0

Also, $f'(z) = u_x + i v_x$ at z_0

Def :— Neighbourhood of z_0 :— nbd of z_0 ; $|z-z_0| < \epsilon$

Def :— $f(z)$ is analytic at z_0 if f is differentiable at every pt. in some nbd of z_0 .

Def :— A pt. " z_0 " where $f^n f(z)$ fails to be analytic is singular pt. or singularity of $f(z)$

Def :— A $f^n f(z)$ which is analytic throughout complex plane is called entire f^n

Milne Thompson Method — If $f(z) = u(x,y) + iv(x,y)$ is analytic f^n , then Put $x=z \& y=0$

in $f(z), f'(z)$... to express in terms of z .

Ex :— $f(z) = (x^2-y^2) + i 2xy$

a) analytic for all (x,y)

b) \rightarrow for ~~all~~ $y=x$

c) \rightarrow for $x>0, y>0$

d) None.

$$\cancel{U} v = x^2 - y^2, V = 2xy$$

$$U_x = 2x \rightarrow \cancel{V}_x = 2y$$

$$U_y = -2y \rightarrow \cancel{V}_y = 2x$$

$$\Rightarrow U_x = V_y \& U_y = -V_x$$

\rightarrow CR eqⁿ satisfies for all (x,y)

$\Rightarrow f$ is differentiable for all z .

$\Rightarrow f$ is analytic for all z .

\Rightarrow entire f^n .

$$f'(z) = U_x + i V_x$$

$$f'(z) = 2x + 2iy$$

$\therefore f(z)$ analytic

Use Milne Thompson method

$$f(z) = z^2, f'(z) = 2z$$

a) analytic $f(x,y)$ for $y=x$

c) nowhere analytic d) None

$$U = x^2, V = y^2, U_x = 2x, V_y = 2y$$

$$U_y = 0, V_x = 0 \Rightarrow \text{CR eqn satisfies}$$

only when $x=y$

$\Rightarrow f$ is diff. on $y=x$, but have to
be diff. on surrounding

\therefore Nowhere analytic

Formula :-

If $f(z) = U+iV$ is analytic

$$1) f'(z) = U_x + iV_x$$

$$2) f'(z) = U_x - iU_y$$

$$3) f'(z) = V_y - iU_y$$

$$4) f'(z) = V_y + iV_x$$

Ex :- $U = x+y+xy$

$$U_x = 1+y \Rightarrow U = \int U_x dx = \int (1+y) dx \quad y \text{ const}$$

$$U_y = 1+x \quad y \text{ const}$$

$$\Rightarrow U = \int U_y dy = \boxed{y + xy + C_2} \quad x \text{ const}$$

\Rightarrow Combining the two :-(comparing)

$$\boxed{U = x+xy+y+C}$$

Ex :- Let $f(z) = U+iV$ be analytic f

$$U = e^x \cos y. (1) \text{ Find } V \quad (2) \text{ find } f(z)$$

Solⁿ :- Find V (Method 1) :

$$U = e^x \cos y; U_x = e^x \cos y; U_y = -e^x \sin y = -V_x$$

(By CR eqns) $\leftarrow \rightarrow$

$$V_x = e^x \sin y; V_y = e^x \cos y$$

$$V = \int e^x \sin y dx \quad V = \int V_y dy = \int e^x \cos y dy \\ y \text{ const} \quad x \text{ const} \quad x \text{ const} \\ = e^x \sin y + C_1 \quad = e^x \cos y + C_2$$

$$\therefore \boxed{V = e^x \sin y + C}$$

Put $x=z$ & $y=0$ [Milne Thompson]

$$\boxed{f(z) = e^z + iC} \text{ or } e^z + C$$

Method 2 :- find $f(z)$

$$U = e^x \cos y; U_x = e^x \cos y; U_y = -e^x \sin y$$

$$f'(z) = U_x - iU_y$$

$$f'(z) = e^x \cos y + ie^x \sin y$$

M-T method: put $x=z$ & $y=0$

$$\boxed{\int f'(z) dz = \boxed{e^z}} \Rightarrow \boxed{f(z) = e^z + C}$$

Ex :- $U = \frac{\sin 2x}{\cosh 2y + \cos 2x}, f(z) = U+iV$ is analytic

Find $f(z)$.

$$\text{Sol}^n : - U_x = \frac{(\cosh 2y + \cos 2x) 2 \cos 2x}{(\cosh 2y + \cos 2x)^2} - (P)$$

$$P = \sin 2x 2(-\sin 2x)$$

$$\text{Put } x=z \text{ & } y=0$$

$$U_x = \frac{(1+\cos 2z) 2 \cos 2z + 2 \sin^2 2z}{(1+\cos 2z)^2}$$

$$= \frac{2 \cos 2z + 2(\cos^2 2z + \sin^2 2z)}{(1+\cos 2z)^2}$$

$$= \frac{2(1+\cos 2z)}{(1+\cos 2z)^2} = \frac{2}{1+\cos 2z}$$

$$U_y = \frac{-\sin 2x}{(\cosh 2y + \cos 2x)^2} \cdot 2 \sinh 2y$$

$$x=z \text{ & } y=0$$

$$U_y = 0$$

$$f'(z) = U_x - iU_y = \frac{2}{1+\cos 2z}$$

$$f(z) = \int \frac{2}{2 \cos^2 z} dz + C$$

$$= \int \sec^2 z dz + C$$

$$\boxed{f(z) = \tan z + C}$$

$$U-V = (x-y)(x^2+4xy+y^2)$$

$$\text{Sofn} :- U-V = x^3-y^3+3x^2y-3xy^2$$

$$\text{If } f(z) = iU - V \quad \text{--- (2)}$$

$$(1) + (2) : (1+i)f(z) = (U-V) + i(U+V)$$

$$\text{Let } (1+i)f(z) = F(z) = U + iV$$

$$U = x^3 - y^3 + 3x^2y - 3xy^2$$

$$U_x = 3x^2 + 6xy - 3y^2 \quad \text{Put } x=z, y=0$$

$$U_x = 3z^2$$

$$U_y = -3y^2 + 3x^2 - 6xy \quad \text{Put } x=z, y=0$$

$$U_y = 3z^2$$

$$F'(z) = U_x - iU_y = 3z^2 - i3z^2$$

$$F'(z) = (1-i)3z^2$$

$$F(z) = (1-i)z^3 + C$$

$$(1+i)f(z) = \frac{1-i}{1+i} z^3 + \frac{C}{1+i}$$

$$f(z) = -\frac{2i}{2} z^3 + C$$

$$f(z) = -iz^3 + C \quad \underline{\text{Ans.}}$$

Result :- If $f(z) = U+iV$ is analytic then U & V are harmonic fns.

$$\text{i.e. } \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

(2) V is called harmonic conjugate of U .

Taylor's Series -(T.S.)

If $f(z)$ is analytic at $z=a$, then $f(z)$ can be expressed in ~~some~~ powers of $(z-a)$ & this power series is called T.S. of $f(z)$ abt $z=a$ (or) T.S. of $f(z)$

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

If $a=0$, it is McLaurin's Series

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \dots$$

$$\underline{\text{Ex}} : - 1) e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots, |z| < \infty$$

$$2) \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots, |z| < \infty$$

$$3) \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots, |z| < \infty$$

$$4) \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots, |z| < 1$$

Laurent's Series :-

Power Series of $f(z)$ in positive & negative powers of $(z-a)$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=-\infty}^{\infty} \frac{b_n}{(z-a)^n}$$

Ex :- Expand $f(z) = \frac{1}{(z-2)(z-1)}$ in powers of z .

$$(i) |z| < 1 \quad (\text{ii}) 1 < |z| < 2$$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$(i) |z| < 1 ; \left| \frac{z}{2} \right| < 1$$

$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} - \frac{1}{-1(1-z)}$$

$$= -\frac{1}{2} \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \dots \right] + \left[1 + z + z^2 + \dots \right]$$

$$(ii) 1 < |z| < 2$$

$$|z| < 2 \Rightarrow \left| \frac{z}{2} \right| < 1 ; 1 < |z| \Rightarrow \left| \frac{1}{z} \right| < 1$$

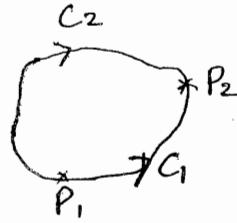
$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}$$

$$= -\frac{1}{2} \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \dots \right] - \frac{1}{z} \left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \right]$$

- (1) Polynomial $f^n : P(z)$
 - entire f^n
 - Zeros : $p(z) = 0$

(Contour Integration) -

$$\int_C f(z) dz$$

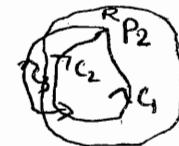


In general,

$$\int_{C_2} f(z) dz \neq \int_{C_1} f(z) dz$$

Complex integration depends on path of integration.

Result - If $f(z)$ is analytic in a region R then



where C_1 & C_2 lies in \mathbb{R} .

Cauchy Integral Theorem -

If $f(z)$ is analytic inside & on closed curve C then $\oint_C f(z) dz = 0$

$$\underline{\text{Ex:--}} \quad \oint_C \frac{z^2 + 3z + 4}{(z-2)(z-3)} dz \quad C \text{ is circle} \quad |z|=1$$

Solⁿ:- Singularities, $z=2,3$ both lies outside C.

By CIT, $\oint_C = 0$



(5) $\sinh z \cosh z$

→ entire f^n

→ Zeros:

$$\sinh z = 0 \text{ iff } z = n\pi i ; n=0,\pm 1,\pm 2\dots$$

$$\cosh z = 0 \text{ iff } z = (2n+1)\frac{\pi i}{2} ; n=0,\pm 1,\pm 2$$

→ Periodicity

$$\begin{array}{l} \rightarrow 2\pi i \\ \rightarrow \pi i \end{array}$$

(2) Singularities of $\tan z$

$$\tan z = \frac{\sin z}{\cos z}$$

Singularities : $\cos z = 0$ iff

$$z = (2n+1) \frac{\pi}{2} \quad n = 0, \pm 1, \pm 2$$

Singularities: $z = (2n+1)\pi/2$

$n = 0, \pm 1, \pm 2$

$-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

$-1.57, 1.57$

all lies outside C $\therefore \oint_C \tan z dz = 0$

Cauchy Integral Formula -

If $f(z)$ is analytic inside & on C & "a" is any point inside "C".

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

$$\Rightarrow \oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Result :-

$$1) f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

$$2) f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$$

$$3) f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz$$

$$4) f'''(a) = \frac{3!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^4} dz$$

$$5) f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$6) \oint_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$Q:- \oint_C \frac{z^2+3z+2}{(z-1)(z-3)} dz \quad C: |z|=2$$

Solⁿ :- Singularities 1, 3

Only $z=1$ lies inside C .

$$\oint_C \frac{z^2+3z+2}{(z-1)(z-3)} dz = 2\pi i f(1)$$

$$= 2\pi i \times (-3) = -6\pi i$$

$$f(1) = \frac{6}{-2} = -3$$

$$Q:- \oint_C \frac{z^2+3z+2}{(z-1)^2(z-3)} dz \quad C: |z|=2$$

Singularities : 1, 3

Only $z=1$ lies inside C .

$$\oint_C \frac{z^2+3z+2}{(z-1)^2(z-3)} dz = 2\pi i f'(1)$$

$$= -8\pi i$$

$$f'(z) = \frac{(z-3)(2z+3) - (z^2+3z+2)}{(z-3)^2}$$

$$f'(1) = -\frac{2(5)}{4} = -\frac{16}{4} = -4$$

$$Q:- \oint_C (z-a)^n dz \quad C: |z|>1 \text{ & } a>1$$

Solⁿ :- By Cauchy I.T :

$$\oint_C (z-a)^n dz = 0 \because \text{there are no sing}$$

$$Q:- \oint_C \frac{1}{z-a} dz \quad C: |z|=1 \quad a<1 \text{ (a is inside)}$$

By C.I.F :

$$\oint_C \frac{1}{z-a} dz = 2\pi i f(a) = 2\pi i$$

$$f(z) = 1, \quad f(a) = 1$$

$$Q:- \oint_C \frac{1}{(z-a)^n} dz ; \quad C: |z|=1 ; \quad a \text{ is inside} \\ C \quad \text{& } n > 1$$

$$\oint_C \frac{1}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a) = 0$$

$$f(z) = 1, \quad f(a) = 1$$

$$f'(z) = 0, \quad f^{(n-1)}(z) = 0$$

Isolated singularity -

There is no other singularity within some mbd of singularity "a", then "a" is isolated singularity.

Laurent's Theorem :-

If "a" is isolated singularity of $f(z)$ then $f(z)$ can be expressed as L.S (Laurent's series) abt $z=a$ (L.S with centre at $z=a$)

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Analytic Part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{\text{Principle Part}}$$

Def - If the principle part of L.S representation of $f(z)$ about singularity "a" contains

- (i) No terms \rightarrow "a" is called removable singularity.
- (ii) Infinite terms \rightarrow "a" is called essential singularity.
- (iii) finite terms (say m) \rightarrow "a" is called pole of order m.

Residue of $f(z)$ at isolated singularity "a"

Residue = b_1 = coefficient of $\frac{1}{z-a}$
 = coefficient of 1st - ve power of $(z-a)$ in L.S. representation of $f(z)$ abt $z=a$.

Result -

$$\text{Let } f(z) = \frac{g(z)}{(z-a)^m} \text{ then}$$

$f(z)$ has pole of order m at $z=a$
 if $g(z)$ is analytic at a &
 $g(a) \neq 0$

$$Q:- f(z) = \frac{z^3 + 3z + 4}{(z-2)^2 (z-3)^5}$$

Singularities : 2, 3

$z=2$: pole of order 2

Residue of $f(z)$ at "a" where "a" is pole of order m,

$$\text{Res } f(z) = \lim_{z \rightarrow a} \left[\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \right]$$

pole of order 2 \rightarrow double pole.

pole of order 1 \rightarrow simple pole.

Residue at simple pole $z=a$

$$\text{Res } f(z) = \lim_{z \rightarrow a} [(z-a) f(z)]$$

Residue at double pole $z=a$

$$\text{Res } f(z) = \lim_{z \rightarrow a} \left[\frac{d}{dz} \left[(z-a)^2 f(z) \right] \right]$$

Q:- Find residue of $f(z)$ at singularities

$$(1) f(z) = \frac{z^3 + 5z^2 + 6z + 5}{(z-3)^3}$$

$z=3$ is pole of order 3.

$$\text{Res } f(z) = \lim_{z \rightarrow 3} \left[\frac{1}{(3-1)!} \frac{d^2}{dz^2} \left[(z-3)^3 \frac{z^3 + 5z^2 + 6z + 5}{(z-3)^3} \right] \right]$$

$$\Rightarrow \lim_{z \rightarrow 3} \left[\frac{1}{2} \frac{d}{dz} [3z^2 + 10z + 6] \right]$$

$$\Rightarrow \lim_{z \rightarrow 3} \left[\frac{1}{2} [6z + 10] \right] = \frac{28}{2} = 14$$

Q:- Find residues of $f(z)$ at singularities

$$f(z) = \frac{z^2 + 6z + 5}{(z-1)^2 (z-2)}$$

$z=1$ is pole of order 2 (double pole).

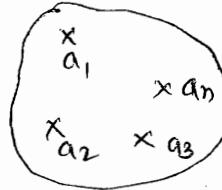
$z=2$ is simple pole.

$$\begin{aligned} R_1 &= \text{Res } f(z) = \lim_{z \rightarrow 1} \left[\frac{d}{dz} \left[(z-1)^2 \frac{z^2 + 6z + 5}{(z-1)^2 (z-2)} \right] \right] \\ &= \lim_{z \rightarrow 1} \frac{(z-2)(2z+6) - (z^2 + 6z + 5) \cdot 1}{(z-2)^2} \\ &= -8 - 12 = -20 \end{aligned}$$

Let $f(z)$ be analytic inside & on C except for finite no. of isolated singularities a_1, a_2, \dots, a_n inside C .

Let R_1, R_2, \dots, R_n be residues at a_1, a_2, \dots, a_n respectively.

$$\oint_C f(z) dz = 2\pi i [R_1 + R_2 + \dots + R_n]$$



$$Q:- \oint_C \frac{z^2 + 6z + 5}{(z-1)^2(z-2)} dz ; C: |z|=3$$

Singularities $\rightarrow 1, 2$ both lies inside C .

$z=1$ is double pole;
 $z=2$ is simple pole.

$$R_1 = -20 ; R_2 = 21$$

\therefore By Residue Theorem:

$$\begin{aligned} \oint_C \frac{z^2 + 6z + 5}{(z-1)^2(z-2)} dz &= 2\pi i [R_1 + R_2] \\ &= 2\pi i [21 - 20] \\ &= 2\pi i \end{aligned}$$

$$Q:- \oint_C \tan z dz \quad C: |z|=2$$

Singularities: $\cos z = 0 \Leftrightarrow z = (2n+1)\frac{\pi}{2}$

$$n = 0, \pm 1, \pm 2, \dots$$

$$\Rightarrow \dots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \dots$$

$z = -\frac{\pi}{2}$ & $\frac{\pi}{2}$ lies inside C

$z = \frac{\pi}{2}$ is a simple pole.

$z = -\frac{\pi}{2}$ is a simple pole.

$$R_{\frac{\pi}{2}} = \lim_{z \rightarrow \frac{\pi}{2}} (z - \frac{\pi}{2}) \cdot \tan z$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} \frac{(z - \frac{\pi}{2}) \sin z}{\cos z}$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} \frac{(z - \frac{\pi}{2}) \cos z + \sin z}{-\sin z} = \frac{1}{-1} = -1$$

$$R_{-\frac{\pi}{2}} = \lim_{z \rightarrow -\frac{\pi}{2}} \frac{(z + \frac{\pi}{2}) \sin z}{\cos z} = \lim_{z \rightarrow -\frac{\pi}{2}} \frac{(z + \frac{\pi}{2}) \cos z + \sin z}{-\sin z} = -1$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} \frac{(z + \frac{\pi}{2}) \cos z + \sin z}{-\sin z} = -1$$

$$\oint_C \tan z dz = 2\pi i [-1 - 1] = -4\pi i$$

$$Q:- \oint_C \frac{e^{2z}}{z^2 + 1} dz \quad C: |z|=3$$

$$\text{Singularities: } z^2 + 1 = 0 ; z = \pm i$$

$z = \pm i$ lies inside C .

$z = i$ simple pole, $z = -i$ simple pole.

$$R_i = \lim_{z \rightarrow i} \frac{(z-i)e^{2z}}{(z-i)(z+i)} = \frac{e^{2i}}{2i}$$

$$R_{-i} = \lim_{z \rightarrow -i} \frac{(z+i)e^{2z}}{(z+i)(z-i)} = \frac{e^{-2i}}{-2i}$$

$$\begin{aligned} \oint_C \frac{e^{2z}}{z^2 + 1} dz &= 2\pi i [R_i + R_{-i}] \\ &= 2\pi i \left[\frac{e^{2i} - e^{-2i}}{2i} \right] \\ &= 2\pi i (\sin 2) \end{aligned}$$

$$Q:- f(z) = \frac{\sin z}{z^5} \text{ has } \underline{\quad} \text{ at } z=0$$

a) pole of order 3 b) 4 c) 5 d) 6

$$\begin{aligned} f(z) &= \frac{1}{z^5} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right] \\ &= \frac{1}{(z^4)} - \frac{1}{z^2 3!} + \frac{1}{5!} - \frac{z^2}{7!} + \dots \end{aligned}$$

$\frac{\sin z}{z^5}$ has pole of order 4 at $z=0$ as highest -ve power is $\cancel{z^4}$.

Residue of $\frac{\sin z}{z^5}$ at $z=0$ is 0

\therefore coeff. of $\frac{1}{z}$ is 0.

$$\frac{1}{z^4} + \frac{1}{z^3} \cancel{0} - \frac{1}{z^2 3!} + \frac{1}{5!} \cancel{z^0} + \frac{1}{5!}$$

- a) Removable Singularity
- b) Essential Sing.
- c) Simple pole.
- d) Double pole.

Soln:

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{z^3 \cdot 3!} + \frac{1}{z^5 \cdot 5!} - \dots$$

Residue is 1 \because coeff. of $\frac{1}{z}$ is 1.
As max^m power is ∞ .

VECTOR CALCULUS

Gradient \rightarrow Scalar f's do vector f's.

Divergence \rightarrow Vector f's to scalar f's.

Curl \rightarrow Vector f's to vector f's.

Del or Nabla -

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

Gradient -

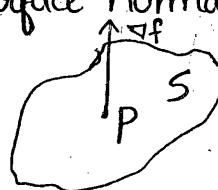
$$\text{grad } f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = \nabla f$$

1) Unit Normal :-

Let $f(x, y, z) = c$ denote surface.

∇f at P denotes surface normal vector at P.

$$\hat{N} = \frac{\nabla f}{|\nabla f|}$$



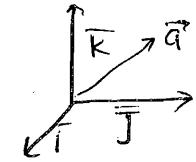
2) Angle b/w two surfaces $f(x, y, z) = c_1$
 $\& g(x, y, z) = c_2$ at P

$$\cos \theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|} \text{ at P}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

direcⁿ of -

1. \vec{i} (x-axis) is $\frac{\partial f}{\partial x}$
2. \vec{j} (y-axis) is $\frac{\partial f}{\partial y}$
3. \vec{k} (z-axis) is $\frac{\partial f}{\partial z}$



(4) D.D. of f at P in direction \vec{a}

$$D \cdot D_{\vec{a}} f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|} \text{ at P.}$$

(5) D.D. of f will be max in direcⁿ

$$\nabla f \cdot \vec{a} = |\vec{a}| |\nabla f| \cos \theta \\ \text{max at } \theta = 0^\circ.$$

(6) Max. value of D.D. = $|\nabla f|$

$$D \cdot D_{\nabla f} f = \nabla f \cdot \frac{\nabla f}{|\nabla f|} \\ = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$$

Q:- Find unit normal to the surface

$$f = 3x^2y - y^3z^2 \text{ at } (1, -2, -1)$$

$$\text{Soln: } \hat{N} = \frac{\nabla f}{|\nabla f|}$$

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \\ &= (6xy - 0) \vec{i} + (3x^2 - 3y^2z^2) \vec{j} + (0 - 2y^3z) \vec{k} \end{aligned}$$

$$\nabla f(1, -2, -1) = -12\vec{i} - 9\vec{j} - 16\vec{k}$$

$$|\nabla f| = \sqrt{(-12)^2 + (-9)^2 + (-16)^2} = \sqrt{481}$$

$$\hat{N} = \frac{-12\vec{i} - 9\vec{j} - 16\vec{k}}{\sqrt{481}}$$

Q:- Find unit normal to the sphere

$$x^2 + y^2 + z^2 = 4$$

$$f: x^2 + y^2 + z^2 = 4$$

$$\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\begin{aligned} & \nabla f = \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2} [x\hat{i} + y\hat{j} + z\hat{k}] \\ &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \quad \text{on surface of sphere } x^2 + y^2 + z^2 = 4 \\ &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{2} \end{aligned}$$

Q:- Find D.D. of $f = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direcⁿ of $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$

Solⁿ :- $D_D_{\vec{a}} f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$ at P.

$$\nabla f = [2xyz + 4z^2]\hat{i} + [x^2z + 0]\hat{j} + [x^2y + 8xz]\hat{k}$$

$$(\nabla f)_{(1, -2, -1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

$$\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\begin{aligned} D_D_{\vec{a}} f &= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3} \\ &= \frac{16+1+20}{3} = \frac{37}{3} \end{aligned}$$

~~Q:- greatest value of D.D. of f, $f = x^2yz + 4xz^2$ at $(1, -2, -1)$~~

Solⁿ :- $|\nabla f| = \sqrt{64+1+100} = \sqrt{165}$ is the greatest value or max^m value of the f^n .

Position Vector of point (x, y, z) :-

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{length } \vec{r} = |\vec{r}|$$

Results -

$$\begin{aligned} (1) \frac{\partial r}{\partial x} &= \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} \\ &= \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{r} \end{aligned}$$

$$(2) \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$$

$$(3) \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

$$\begin{aligned} (4) \nabla r &= \frac{\partial r}{\partial x} \hat{i} + \frac{\partial r}{\partial y} \hat{j} + \frac{\partial r}{\partial z} \hat{k} \\ &= \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j} + \frac{z}{r} \hat{k} \\ &\boxed{\nabla r = \frac{\vec{r}}{r}} \end{aligned}$$

$$(5) \nabla f(r) = \frac{\partial}{\partial x} f(r) \hat{i} + \frac{\partial}{\partial y} f(r) \hat{j} + \frac{\partial}{\partial z} f(r) \hat{k}$$

$$\frac{\partial}{\partial x} f(r) = f'(r) \frac{\partial r}{\partial x} = f'(r) \frac{x}{r}$$

$$\begin{aligned} \nabla f(r) &= f'(r) \frac{x}{r} \hat{i} + f'(r) \frac{y}{r} \hat{j} + \\ &f'(r) \frac{z}{r} \hat{k} \end{aligned}$$

$$\boxed{\nabla f(r) = \frac{f'(r)}{r} [\vec{r}]}$$

$$(6) \nabla \log r = \frac{1}{r} \nabla r = \frac{1}{r} \frac{\vec{r}}{r} = \boxed{\frac{\vec{r}}{r^2}}$$

$$(7) \nabla \left(\frac{1}{r} \right) = -\frac{1}{r^2} \nabla r = \boxed{\frac{-\vec{r}}{r^3}}$$

$$\begin{aligned} (8) \nabla(r^n) &= n r^{n-1} \nabla r \\ &= n r^{n-1} \frac{\vec{r}}{r} \end{aligned}$$

$$\boxed{\nabla(r^n) = n r^{n-2} \vec{r}}$$

$$\text{Div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{Curl } - \text{Curl } F = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Def :— $\nabla \cdot \vec{F} = 0$ then \vec{F} is solenoidal vector.

$\nabla \times \vec{F} = \vec{0}$ then \vec{F} is irrotational

Ex :— (1) Find P, $\vec{F} = (x+3y)\hat{i} + (y+2z)\hat{j} + (x+yz)\hat{k}$ is solenoidal.

(2) $\vec{F} = r^n \vec{r}$ is solenoidal then $n = \underline{\quad}$

$$\text{Sol}^n : \text{ (1)} \nabla \cdot \vec{F} = 0 ; \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y+2z) + \frac{\partial}{\partial z}(x+yz) = 0$$

$$1+1+p=0 \Rightarrow p=-2$$

$$\text{(2)} \vec{F} = r^n [x\hat{i} + y\hat{j} + z\hat{k}] = r^n x \hat{i} + r^n y \hat{j} + r^n z \hat{k}$$

$$\nabla \cdot \vec{F} = 0$$

$$\frac{\partial}{\partial x}(r^n x) = r^n + x n r^{n-1} \frac{\partial r}{\partial x} = r^n + n x r^{n-1} \frac{x}{r}$$

$$\frac{\partial}{\partial x}(r^n x) = r^n + n x^2 r^{n-2}$$

$$\text{Similarly, } \frac{\partial}{\partial y}(r^n y) = r^n + n y^2 r^{n-2} ; \frac{\partial}{\partial z}(r^n z) = r^n + n z^2 r^{n-2}$$

$$\frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) = 3r^n + n^2 r^{n-2} [x^2 + y^2 + z^2]$$

$$\nabla \cdot \vec{F} = 3r^n + n r^{n-2} r^2 = 0 \Rightarrow 3r^n + n r^n = 0$$

$$(3+n) r^n = 0 \Rightarrow n = -3$$

Q :— Find Curl of $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$\text{Sol}^n : \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \hat{i}[x-x] - \hat{j}[y-y] + \hat{k}[z-z] = \vec{0}$$

\vec{F} is irrotational.

Def :— \vec{F} is conservative field if there exist a scalar $f^n \phi$ such that

$$\boxed{\vec{F} = \nabla \phi}$$

\vec{F} is irrotational.

$$\vec{F} = \nabla\phi \text{ iff } \nabla \times \vec{F} = 0$$

$\phi \rightarrow$ [scalar potential]

Q:- Find scalar potential of

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$1) \nabla \times \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is conservative field.

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Partial Integration:

$$\frac{\partial\phi}{\partial x} = yz \Rightarrow \phi = xyz + C_1$$

$$\frac{\partial\phi}{\partial y} = zx \Rightarrow \phi = xyz + C_2$$

$$\frac{\partial\phi}{\partial z} = xy \Rightarrow \phi = xyz + C_3$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow \boxed{\phi = xyz + C}$$

Line Integral -

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$\underline{\text{Ex:}} - 1) \vec{F} = 3xy\hat{i} - y^2\hat{j}$$

Solⁿ: - $\int_{C_1} \vec{F} \cdot d\vec{r}$ where C_1 is arc of parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} 3xy dx - y^2 dy$$

Parametric Form :-

$$x = t \quad \& \quad y = 2t^2$$

$$dx = dt \quad \& \quad dy = 4t dt$$

x varies from 0 to 1
 $x=t$ varies from 0 to 1

$$\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= \int_{t=0}^1 3t(2t^2) dt - (4t^4) \cdot 4t dt \\ &= \int_{t=0}^1 [6t^3 - 16t^5] dt \\ &= \left[\frac{6t^4}{4} - \frac{16t^6}{6} \right]_0^1 \\ &= \frac{6}{4} - \frac{16}{6} = -\frac{7}{6} \quad \underline{\text{Ans.}} \end{aligned}$$

② $\int_{C_2} \vec{F} \cdot d\vec{r}$ where C_2 is line joining pts from $(0,0)$ to $(1,2)$

Remember -

Parametric form of line joining pts (x_1, y_1, z_1) to (x_2, y_2, z_2)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$$

Solⁿ: - Parametric form :

$$\frac{x-0}{1-0} = \frac{y-0}{2-0} = t \Rightarrow x=t, y=2t$$

$$dx = dt \quad \& \quad dy = 2dt$$

$$(0,0) \text{ to } (1,2)$$

x varies from 0 to 1

$t=x$ varies from 0 to 1

$$\begin{aligned} \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_{t=0}^1 3xy dx - y^2 dy \\ &= \int_{t=0}^1 3t(2t) dt - 4t^2 \cdot 2dt \\ &= -2 \left[\frac{t^3}{3} \right]_0^1 \\ &= -\frac{2}{3} \quad \underline{\text{Ans.}} \end{aligned}$$

1) In general,

$$\int_{P_1}^{P_2} \bar{F} \cdot d\bar{r} \neq \int_{P_1}^{P_2} \bar{F} \cdot d\bar{r}$$

along C₁ along C₂

2) If \bar{F} is conservative field ($\bar{F} = \nabla \phi$)

$$\text{then } \int_{P_1}^{P_2} \bar{F} \cdot d\bar{r} = \int_{P_1}^{P_2} d\phi = [\phi]_{P_1}^{P_2}$$

along only C

$$\text{Ex: } \bar{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

Find $\int_C \bar{F} \cdot d\bar{r}$ where C is the line joining point (1,1,1) to (1,2,3)

$$1) \nabla \times \bar{F} = 0 \quad 2) \bar{F} = \nabla \phi$$

$$3) \phi = xyz + c$$

$$\begin{aligned} \int_C \bar{F} \cdot d\bar{r} &= \int_{(1,1,1)}^{(1,2,3)} d\phi = [\phi]_{(1,1,1)}^{(1,2,3)} \\ &= xyz + c \Big|_{(1,1,1)}^{(1,2,3)} \end{aligned}$$

$$= 6 + c - 1 - c = 5 \text{ Ans.}$$

Multiple Integrals

$$\text{Ex: } ① \int_0^2 \int_0^1 xy dx dy = \int_0^2 \frac{x^2}{2} \Big|_0^1 dy$$

$$= \int_0^2 \frac{y}{2} dy = \frac{y^2}{4} \Big|_0^2 = 1$$

$$\text{② } \int_{x=0}^1 \int_{y=0}^2 xy dy dx = \int_0^1 x \left[\frac{y^2}{2} \right]_0^2 dx$$

$$\int_0^1 2x dx = x^2 \Big|_0^1 = 1$$

$$\text{③ } \int_0^1 x dx \int_0^2 y dy = \frac{x^2}{2} \Big|_0^1 \frac{y^2}{2} \Big|_0^2 = \frac{1}{2} \cdot 2 = 1$$

We cannot separate in terms of x & y, so ③ is not applicable here.

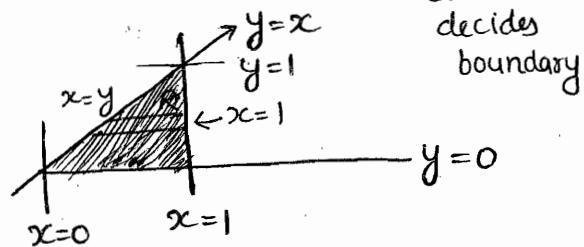
$$\begin{aligned} \int_0^2 \frac{x^2}{2} + yx \Big|_0^1 dy &= \int_0^2 \left(\frac{1}{2} + y \right) dy \\ &= \frac{y}{2} + \frac{y^2}{2} \Big|_0^2 = \cancel{y^2} = 2+1=3 \\ \text{Ex: } \int \int x^2 y dy dx & \\ x=0 & \end{aligned}$$

→ Double integration is carried out over a region.

$$\begin{aligned} \int_0^1 x^2 y^2 \Big|_0^x dx &= \int_0^1 \frac{x^4}{2} dx \\ &= \frac{x^5}{10} \Big|_0^1 = \frac{1}{10} \text{ Ans.} \end{aligned}$$

Change of order → variable limits decides shape

$$y = \overbrace{0 \text{ to } x}^{\text{variable limit decides shape}}, x = \underbrace{0 \text{ to } 1}_{\substack{\text{const limit} \\ \text{decides boundary}}}$$



limits becomes: $x = 0 \text{ to } 1$ &

$$\begin{aligned} \int_{x=0}^1 \int_{y=0}^x x^2 y dy dx &= \int_{y=0}^1 \int_{x=y}^1 x^2 y dx dy \\ &= \int_0^1 \left[\frac{x^3}{3} \right]_y^1 y dy = \int_0^1 \left[\frac{1}{3} - \frac{y^3}{3} \right] y dy \\ &= \frac{1}{3} \left[\frac{1}{2} - \frac{1}{6} \right] = \frac{1}{3} \left[\frac{5}{10} \right] \\ &= \frac{1}{10} \text{ Ans.} \end{aligned}$$

over a region.

- Triple integrals are carried out over volumes.

Formula -

Area of Region :

$$A = \iint_R dx dy$$

Note :— $\iint_R f(x,y) dx dy$ is not area of region.

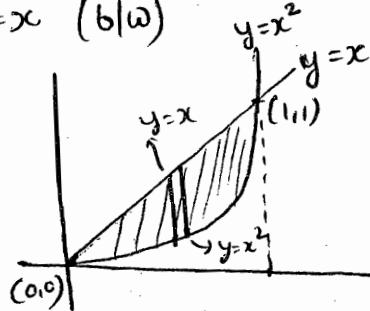
Volume enclosed by closed surface :

$$V = \iiint_V dx dy dz$$



Q:- Find the area of region bounded by $y=x^2$ & $y=x$ (b/w)

$$A = \iint_R dx dy$$



Limits :

$$y=x \quad y=x^2 \Rightarrow x^2=x \\ x(x-1)=0 \\ x=0, 1$$

Limits of Region :

$$y=x^2 \text{ to } x \\ x=0 \text{ to } 1$$

$$A = \int_0^x \int_x^{x^2} dy dx = \int_0^x [x-x^2] dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \underline{\text{Ans}}$$

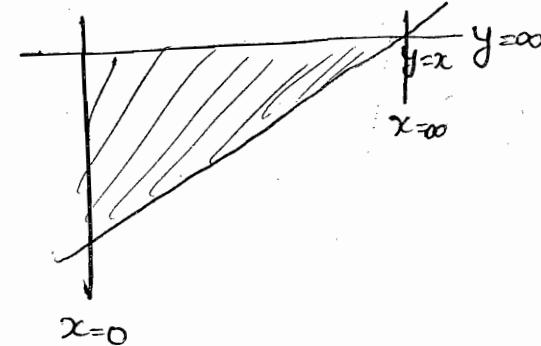
$$x=y \rightarrow x=\sqrt{y}$$

$$y=0 \text{ to } 1$$

$$A = \int_0^1 \int_{\sqrt{y}}^y dx dy = \int_0^1 [\sqrt{y}-y] dy \\ = \frac{1}{6} \underline{\text{Ans}}$$

$$\text{Ex:— } \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dx dy$$

$$y=x \text{ to } \infty ; x=0 \text{ to } \infty$$



Change the order :-

$$\text{Limits: } x=0 \text{ to } y \text{ & } y=0 \text{ to } \infty$$

$$\int_0^\infty \int_y^\infty \frac{e^{-y}}{y} dy dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [xy]_0^y dy = \int_0^\infty e^{-2y} dy$$

$$= 0+1 = \underline{\text{Ans}}$$

Jacobian of the transformation :-

1) Let $U = f_1(x,y) + V = f_2(x,y)$

$$J = \frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}$$

2) Let $U = f_1(x,y,z)$,

$$V = f_2(x,y,z)$$

$$W = f_3(x,y,z)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} vx & vy & vz \\ wx & wy & wz \end{vmatrix}$$

$$\theta = 0 \text{ to } 2\pi$$

Formulas for change of variables :-

Double Integral :

$$R \iint f(u,v) dudv = \iint f(f_1, f_2) |J| dx dy.$$

Triple Integral :

$$\sqrt{V} \iiint f(u,v,w) dudvdw =$$

$$\sqrt{V} \iiint f(f_1, f_2, f_3) |J| dx dy dz.$$

Polar coordinates -

$$x = r \cos \theta, y = r \sin \theta$$

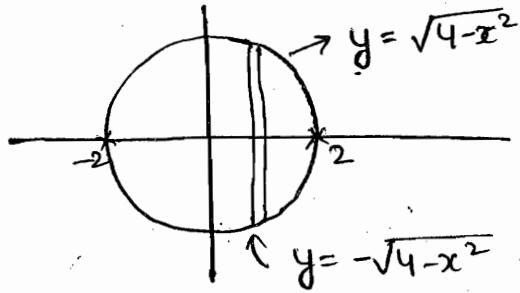
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} xr & x\theta \\ yr & y\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r [\cos^2 \theta + \sin^2 \theta] = r$$

Ex :-

$$R \iint f(x,y) dx dy = \iint f(r \cos \theta, r \sin \theta) r dr d\theta$$

Ex :- Circle $x^2 + y^2 = 4$



$$R: x^2 + y^2 = 4$$

$$\text{Limits : } y = -\sqrt{4-x^2} \text{ to } +\sqrt{4-x^2} \\ x = -2 \text{ to } 2$$

Changing into polar coordinates :

$$x = r \cos \theta, y = r \sin \theta$$

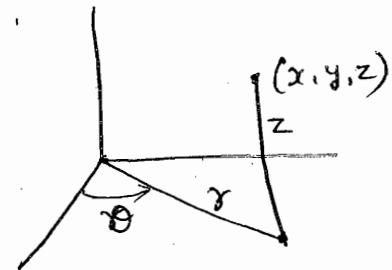
$$dx dy = r dr d\theta$$

Cylindrical Polar Coordinates -

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$



$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} xr & x\theta & xz \\ yr & y\theta & yz \\ zr & z\theta & zz \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

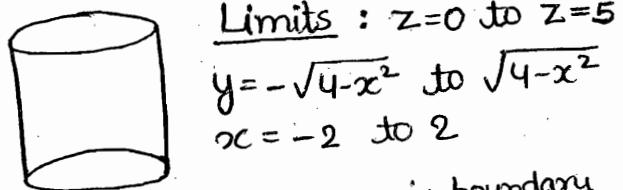
Result :- Cartesian to cylindrical polar coordinates.

$$\iiint f(x,y,z) dx dy dz =$$

$$\sqrt{V} \iiint f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

Ex :- Cylinder :

$$x^2 + y^2 = 4, z = 0 \text{ to } z = 5$$



$$\text{Limits : } z = 0 \text{ to } z = 5 \\ y = -\sqrt{4-x^2} \text{ to } \sqrt{4-x^2} \\ x = -2 \text{ to } 2$$

x & y takes care of circle, z take care of shape in the space. \rightarrow ie. boundary.

Using cylindrical polar :-

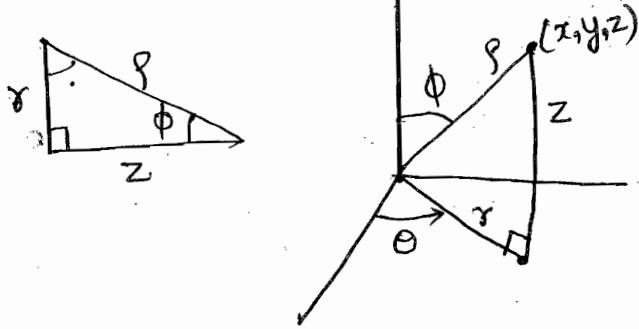
$$x = r \cos \theta, y = r \sin \theta, z = z$$

$$dx dy dz = r dr d\theta dz$$

$$\text{Limits : } z = 0 \text{ to } z = 5$$

$$r = 0 \text{ to } 2$$

$$\theta = 0 \text{ to } 2\pi$$



$$\sin \phi = \frac{y}{\rho} \quad \cos \phi = \frac{z}{\rho}$$

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z =$$

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

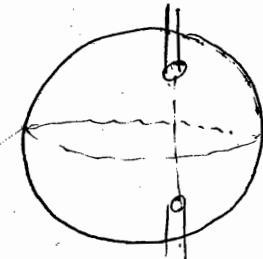
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi$$

Cartesian to spherical polar coordinates

$$\iiint f(x, y, z) dx dy dz =$$

$$\iiint f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

$$\text{Ex: } - \text{Sphere } x^2 + y^2 + z^2 = 4$$



$$z = \sqrt[3]{4 - x^2 - y^2}$$

$$z = -\sqrt[3]{4 - x^2 - y^2} \text{ to } +\sqrt[3]{4 - x^2 - y^2}$$

$z \rightarrow$ takes cone shape in the space.

$$z=0, y = -\sqrt{4-x^2} \text{ to } +\sqrt{4-x^2}$$

$$z, y=0, x = -2 \text{ to } +2$$

Limits of shell :-

Using spherical polar coordinates

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$dxdydz = \rho^2 \sin \phi d\rho d\phi d\theta$$

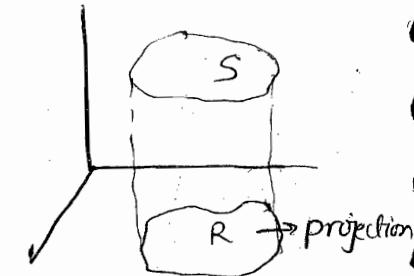
Limits : $\rho = 0$ to $\rho = 2$

$$\phi = 0 \text{ to } \pi$$

$$\theta = 0 \text{ to } 2\pi$$

Surface Integrals

$$S \iint \bar{F} \cdot \hat{N} ds$$



Formulas

1) Let R_1 be the projection of S on xy plane.

$$S \iint \bar{F} \cdot \hat{N} ds = \iint_{R_1} \bar{F} \cdot \hat{N} \frac{dx dy}{|\hat{N} \cdot \hat{r}|}, \quad \hat{N} \cdot \hat{r} \neq 0$$

2) Let R_2 be projection S on yz plane.

$$S \iint \bar{F} \cdot \hat{N} ds = \iint_{R_2} \bar{F} \cdot \hat{N} \frac{dy dz}{|\hat{N} \cdot \hat{i}|}, \quad \hat{N} \cdot \hat{i} \neq 0$$

3) Let R_3 be projection of S on xz plane.

$$S \iint \bar{F} \cdot \hat{N} ds = \iint_{R_3} \bar{F} \cdot \hat{N} \frac{dx dz}{|\hat{N} \cdot \hat{j}|}, \quad \hat{N} \cdot \hat{j} \neq 0$$

Q:- Find the surface integral of

$$S \iint \bar{F} \cdot \hat{N} ds. \quad \bar{F} = z\bar{i} + x\bar{j} - 3y^2\bar{k}$$

S: Surface of cylinder, $x^2 + y^2 = 16$

included in 1st octant b/w

$$z=0 \text{ & } z=5$$

$$f: x^2 + y^2 = 16$$

$$\hat{N} = \frac{\nabla f}{|\nabla f|},$$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

$$\hat{N} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$$

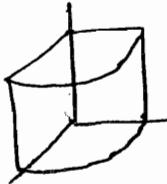
$$\text{on surface } x^2 + y^2 = 16$$

Let R be the projection of S on yz plane.

$$S \iint \bar{F} \cdot \hat{N} dS = R \iint \bar{F} \cdot \hat{N} \frac{dy dz}{|N \cdot \hat{i}|}$$

$$\begin{aligned}\bar{F} \cdot \hat{N} &= (z\hat{i} + x\hat{j} - 3yz^2\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j}}{4} \right) \\ &= \frac{xz + xy}{4} = \frac{x}{4}(y+z)\end{aligned}$$

$$|N \cdot \hat{i}| = \frac{x}{4}$$



Limits of R :-

$$\begin{aligned}x^2 + y^2 &= 16 \\ \text{on } yz \text{ plane}\end{aligned}$$



$$x=0, y^2=16, y=\pm 4$$

$$y=0 \text{ to } 4 \text{ and } z=0 \text{ to } 5.$$

$$S \iint = \iint_0^4 \left(\frac{x}{4}(y+z) \cdot \frac{dy dz}{x^2 y} \right)$$

$$= \int_{z=0}^5 \int_{y=0}^4 (y+z) dy dz$$

$$= 90$$

$$1) \nabla r = \vec{r}/r$$

$$2) \nabla \cdot \vec{r} = 3$$

$$3) \nabla \times \vec{r} = \vec{0}$$

$$4) \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$5) \nabla \cdot (\nabla \phi) = \nabla^2 \phi$$

Div. (Grad ϕ)

$$6) \nabla \times (\nabla \phi) = \vec{0}$$

Curl (Grad ϕ)

$$7) \nabla \cdot (\nabla \times \bar{F}) = 0$$

Div. (Curl \bar{F})

$$8) \nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2(\bar{F})$$

$$(2) \nabla r = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1=3$$

$$(3) \nabla \times \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

Integral Theorems :-

Closed Curve $\begin{cases} 2D \rightarrow \text{Green's Theorem} \\ 3D \rightarrow \text{Stokes Theorem} \end{cases}$

Surfaces $\begin{cases} \text{open} \\ \text{closed} \rightarrow \text{Divergence Theorem} \end{cases}$

Green's Theorem in a plane -

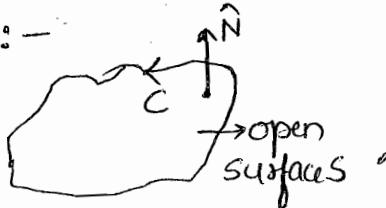
Let P, Q be differentiable fns defined over a region R bounded closed curve C



$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y} \right) dx dy$$

boundary of open surface S.

Stoke's Theorem:

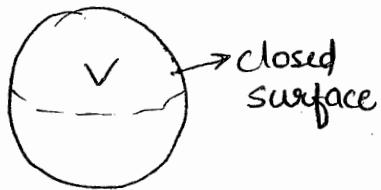


Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ be differentiable vector function defined over the open surface S bounded by closed curve C

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds$$

Gauss Divergence Theorem:

Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ be differentiable vector function defined over the closed surface S enclosing volume V.



$$\iint_S \vec{F} \cdot \hat{N} ds = \iiint_V (\nabla \cdot \vec{F}) dv$$

$$Q: \textcircled{1} \iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds = \dots$$

Solⁿ:— Divergence :

$$\begin{aligned} \iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds &= \iiint_V \nabla \cdot (\nabla \times \vec{F}) dv \\ &= 0. \end{aligned}$$

② Let \vec{F} be conservative field.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds$$

$$\vec{F} = \nabla \phi = \iiint_S (\nabla \times \nabla \phi) \cdot \hat{N} ds = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{N} ds = 0$$

④ $\iint_S \vec{F} \cdot \hat{N} ds = \dots$ where S is closed surface enclosing volume V.

- a) $2V$ b) $3V$ c) V d) 0

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{N} ds &= \iiint_V (\nabla \cdot \vec{F}) dv \\ &= 3 \iiint_V dv \\ &= 3V. \end{aligned}$$

Q:— Find $\iint_S \vec{F} \cdot \hat{N} ds$ where S is surface of the sphere $x^2 + y^2 + z^2 = 4$

$$\vec{F} = (x+z) \hat{i} + (y+z) \hat{j} + (x+y) \hat{k}$$

Solⁿ:— By Divergence Theorem :—

$$\iint_S \vec{F} \cdot \hat{N} ds = \iiint_V (\nabla \cdot \vec{F}) dv$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y) \\ &= 1+1+0 = 2. \end{aligned}$$

$$= 2 \iiint_V dv = 2V$$

Changing into spherical polar co-ordinates :—

$$x = r \sin \phi \cos \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \phi$$

$$dx dy dz = r^2 \sin \phi dr d\theta d\phi$$

Limits:— $r = 0$ to 2

$$\phi = 0 \text{ to } \pi$$

$$\theta = 0 \text{ to } 2\pi$$

$$= -2 \left[\frac{e^3}{3} \right]_0^2 \left[-\cos \phi \right]_0^\pi \left[0 \right]_0^{2\pi} = \frac{64}{3} \pi \text{ Ans.}$$

$$\cos n\pi = (-1)^n$$

Method 2 :-

$$2 \iiint_V dx dy dz - ① \Rightarrow 2 \text{ Volume of sphere} = 2 \cdot \frac{4}{3} \pi r^3 = \frac{8}{3} \pi (2^3) = \frac{64\pi}{3}$$

Q:- $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$. Find $\iint_S \vec{F} \cdot \hat{N} dS$ where S is Surface of cylinder $x^2 + y^2 = 4$, $z=0$ & $z=3$

Soln:- By Gauss Divergence Theorem -

$$\iint_S \vec{F} \cdot \hat{N} dS = \iiint_V (\nabla \cdot \vec{F}) dx dy dz, \quad \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) \\ = 4 - 4y + 2z.$$

$$\Rightarrow \iiint_V (4 - 4y + 2z) dx dy dz - ①$$

Changing cylindrical polar, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dx dy dz = r dr d\theta dz$

Limits :- $r = 0$ to 2, $\theta = 0$ to 2π , $z = 0$ to 3

$$\begin{aligned} ① &= \int_0^{2\pi} \int_0^2 \int_0^3 [4 - 4r\sin\theta + 2z] r dz dr d\theta = \int_0^{2\pi} \int_0^2 [(4 - 4r\sin\theta)[z]_0^3 + [z^2]_0^3] r dr d\theta \\ &= \int_0^{2\pi} \int_0^2 [3(4 - 4r\sin\theta) + 9] r dr d\theta = \int_0^{2\pi} \int_0^2 [21r - 12r^2 \sin\theta] dr d\theta \\ &= \int_0^{2\pi} \left[21 \left[\frac{r^2}{2} \right]_0^2 - 12 \left[\frac{r^3}{3} \right]_0^2 \sin\theta \right] d\theta = \int_0^{2\pi} [42 - 32\sin\theta] d\theta \\ &= \int_0^{2\pi} 42 d\theta - 32 \int_0^{2\pi} \frac{\sin\theta d\theta}{\sin(2\pi - \theta)} = -32 \int_0^{2\pi} \frac{\sin\theta d\theta}{-\sin\theta} = 32 \int_0^{2\pi} 1 d\theta = 42 [0]_0^{2\pi} = 84\pi \text{ Ans.} \end{aligned}$$

Q:- $\oint_C [(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}] \cdot d\vec{r}$ where C is the upper half of sphere $x^2 + y^2 + z^2 = 1$ \therefore it is open surface

Stoke's theorem :- $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{N} dS$

Soln:-



$$\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{k}; \quad \iint_S \vec{E} \cdot \hat{N} dS; \quad \hat{N} = \frac{\nabla f}{|\nabla f|}, \quad \text{Surface } f: x^2 + y^2 + z^2 = 1$$

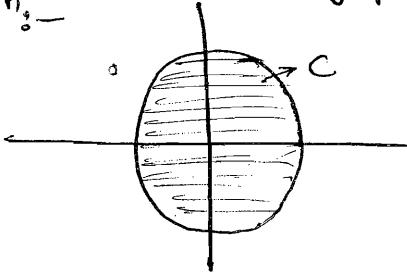
$$\hat{N} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$\iint \bar{K} \cdot \hat{N} dS = \iint \bar{K} \cdot \hat{N} dx dy / |\bar{N} \cdot \bar{K}| = \iint dx dy$$

\Rightarrow Area of Circle $= \pi r^2 = \pi 1^2 = \pi$ Ans.

Q:- Find $\int (x^2 + y^2) dx + 3xy^2 dy$ where
C: $x^2 + y^2 = 4$ on xy-plane.

Soln:-



By Green's theorem:

$$P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

HINT Directly: $x = 2\cos\theta, y = 2\sin\theta$
 $\theta = 0 \text{ to } 2\pi$

$$dx = -2\sin\theta d\theta, dy = 2\cos\theta d\theta$$

$$\iint \frac{x^2 + y^2}{P} dx + \frac{3xy^2}{Q} dy$$

$$= \iint \left[\frac{\partial}{\partial x} (3xy^2) - \frac{\partial}{\partial y} (x^2 + y^2) \right] dx dy$$

$$= \iint [3y^2 - 2y] dx dy$$

Using polar coordinates;

$$x = r\cos\theta, y = r\sin\theta, dx dy = r dr d\theta$$

limit: $r = 0 \text{ to } 2; \theta = 0 \text{ to } 2\pi$

$$= \iint_0^{2\pi} \left[3r^2 \sin^2\theta - 2r\sin\theta \right] r dr d\theta$$

$$= \iint_0^{2\pi} \left[3r^3 \sin^2\theta - 2r^2 \sin\theta \right] dr d\theta$$

$$= \int_0^{2\pi} \left[3 \left[\frac{r^4}{4} \right]_0^2 \sin^2\theta - 2 \left[\frac{r^3}{3} \right]_0^2 \sin\theta \right] d\theta$$

$$= \int_0^{2\pi} 12\sin^2\theta d\theta - \frac{16}{3} \int_0^{2\pi} \sin\theta d\theta$$

$$= 12\pi \text{ Ans.}$$

planes $x+y+z=1, x=0, y=0 \text{ & } z=0$

$$V = \iiint dxdydz$$

$$= \iint_0^1 \int_0^{1-x-y} dz dy dx$$

$$= \iint_0^1 \int_0^{1-x-y} dy dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2} dx$$

$$= \frac{1}{6} \text{ Ans.}$$



$$z=0, z=1-x-y$$

$$y=0 \text{ to } y=1-x$$

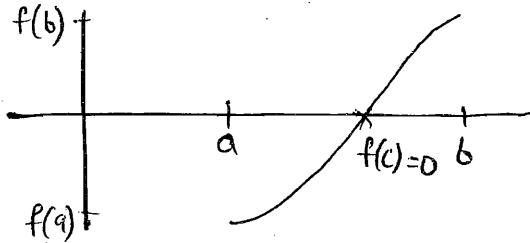
$$x=0 \text{ to } x=1$$

NUMERICAL METHODS

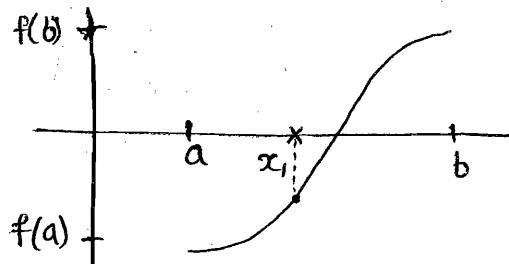
Numerical soln to find roots of $f(x)=0$

Result - If $f(a) \neq f(b)$ are of opposite sign then there exist a root of the eqn

$$f(x)=0 \text{ in } [a, b]$$



Bisection Method -



Procedure :- To find $f(x)=0$ in $[a, b]$

Let $f(a) < 0 \wedge f(b) > 0$

First approx: $x_1 = \frac{a+b}{2}$

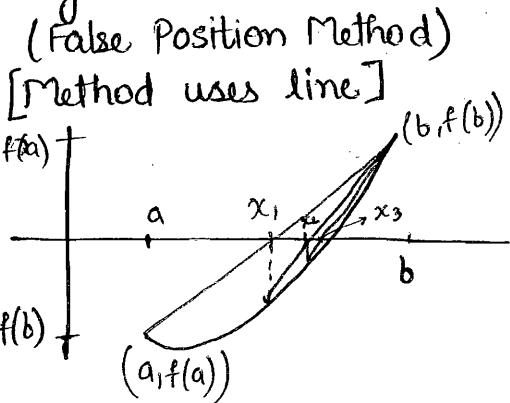
If $f(x_1) = 0 \Rightarrow x_1$ is root

If $f(x_1) < 0 \Rightarrow$ Put $a = x_1$

If $f(x_1) > 0 \Rightarrow$ Put $b = x_1$

Second approx: $x_2 = \frac{a+b}{2}$

Rule 6



Procedure :— To find $f(x) = 0$ in $[a, b]$
Let $f(a) < 0$ & $f(b) > 0$

First approx: $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

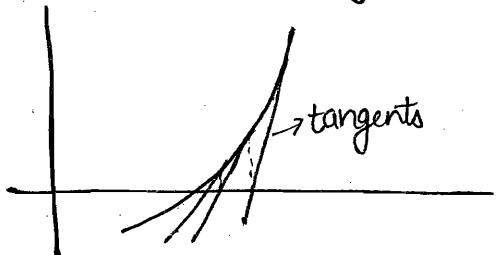
If $f(x_1) = 0 \Rightarrow x_1$ is root

If $f(x_1) < 0 \Rightarrow$ Put $a = x_1$

If $f(x_1) > 0 \Rightarrow$ Put $b = x_1$

Second approx: $x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

Newton-Raphson Method:
[Method uses tangents]



Procedure : To find $f(x) = 0$.

Let x_0 be initial guess.

First approx: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Second approx: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

General Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Bisection	Slow	always conv. (1)
Regular Falsi	Slow	-u-
N-R Method	Fast	May diverge (2)

Order
of convergence

Result :—

→ N-R Method does not work: $f'(x) = 0$

2) The min^m no. 'n' of iterations reqd. to find root of $f(x) = 0$ in $[a, b]$ using bisection method so that error is ϵ is given by

$$\frac{|b-a|}{2^n} < \epsilon$$

Q:- Find root of the eqⁿ $f(x) = x^3 - x - 1$ in $[1, 2]$ using Bisection method in four stages.

Solⁿ:— $f(1) = -1$, $f(2) = 5$

	a	b	$f(a)$	$f(b)$	$x_i = \frac{a+b}{2}$	$f(x_i)$	
①	1	2	-1	5	$\frac{1+2}{2} = 1.5$	0.875	
②	1	1.5	-1	0.875	1.25	-0.296	
③	1.25	1.5	-0.296	0.875	1.375	0.2246	
④	1.25	1.375	-0.296	0.2246	1.3125	Ans.	

Q:- Min^m no. of iterations reqd. for bisection method in solving the eqⁿ $x^3 + 3x - 7 = 0$ in $[1, 2]$ so that $|\text{error}| \leq 10^{-2}$

Solⁿ:— $\epsilon = 10^{-3}$, $a = 1$, $b = 2$

$$\frac{|b-a|}{2^n} < \epsilon ; \frac{2-1}{2^n} < 10^{-3}$$

$$2^n > 1000 \Rightarrow n > \log_2 1000$$

$$n > 9.965 \Rightarrow n = 10. \text{ Ans.}$$

Q:- Find roots of the eqⁿ $f(x) = x^3 - 4x - 9$ in $[2, 3]$ using Regular Falsi in 2 stages.

$$f(2) = -9 \quad f(3) = 6$$

$$y_{n+1} = y_n + k$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$k_1 = h f(x_n, y_n), k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h f\left(x_n + h, y_n + k_3\right)$$

Q:- $\frac{dy}{dx} - y = 0$ $y(0) = 0$ using Euler's

Method with step size $0.1 = h$

Find $y(0.3)$

Solⁿ:- $\frac{dy}{dx} = x + y \Rightarrow f(x, y) = x + y$

$$\underline{x}$$

$$x_0 = 0$$

$$x_1 = 0.1$$

$$x_2 = 0.2$$

$$x_3 = 0.3$$

$$\underline{y}$$

$$y_0 = 0$$

$$y_1 = y(0.1) = 0$$

$$y_2 = y(0.2) = 0.01$$

$$y_3 = y(0.3) = 0.03$$

$$y_1 = y_0 + h f(x_0, y_0) = 0 + 0.1 \times 0 = 0$$

$$y_2 = y_1 + h f(x_1, y_1) = 0 + 0.1 f(0.1, 0) = 0.01$$

$$y_3 = y_2 + h f(x_2, y_2) = 0.03$$

Q:- $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$ using R-K

method of 4th order with step size

0.2 , find $y(0.2)$

Solⁿ:- $f(x, y) = 1 + y^2$, $h = 0.2$

$$\underline{x}$$

$$x_0 = 0$$

$$x_1 = 0.2$$

$$\underline{y}$$

$$y_0 = y(0) = 0$$

$$y_1 = y(0.2) =$$

$$y_1 = y_0 + k$$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$k_1 = h f(x_0, y_0) = 0.2 f(0, 0) = 0.2 \times 1 = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f(0.1, 0.1)$$

$$= 0.2 [1 + 0.1^2] = 0.202$$

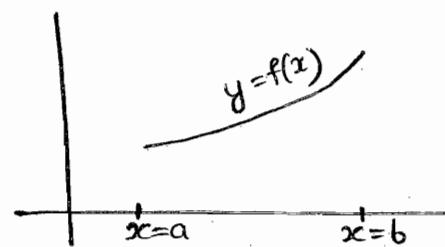
$$= 0.20204$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.20816$$

$$k_4 = \cancel{0.20816} 0.2027$$

$$y_1 = y_0 + k = 0.2027$$

Length of arc :-



Length of arc : $y = f(x)$ b/w $x=a$ to $x=t$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

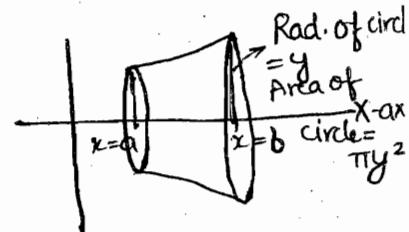
Length of arc : $x = g(y)$ b/w $y=c$ to $y=d$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Volume of solid of revolution -

1) The volume of solid of revolution by revolving $y = f(x)$ b/w $x=a$ to $x=b$ around x-axis

$$V = \int_a^b \pi y^2 dx$$



2) The volume of solid of revolution by revolving $x = g(y)$ b/w $y=c$ to $y=d$ around y-axis

$$V = \int_c^d \pi x^2 dy$$

Find length of arc ;

$$\text{Soln:- } \frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{1/2} = x^{1/2}$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1+x} dx$$

$$= \frac{2}{3} [2\sqrt{2} - 1] \text{ Ans.}$$

Q:- Parabolic arc: $y = \sqrt{x}$ $1 \leq x \leq 2$ is revolved around x-axis. Volume of solid of revolution.

$$V = \int_1^2 \pi y^2 dx = \int_1^2 \pi x dx$$
$$= \frac{3\pi}{2} \text{ Ans.}$$

Gamma function :-

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n+1) = \begin{cases} n! & n \text{ is integer} \\ n\Gamma(n) & o/w. \end{cases}$$

$$\Gamma(3) = 2! = 2$$

$$\Gamma(5) = 4! = 24$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx ; x^2=t \quad 2xdx=dt$$

$x=0 ; t=0$
 $x \rightarrow \infty ; t \rightarrow \infty$

$$= 2 \int_0^\infty e^{-t} \frac{dt}{2(t)^{1/2}} = \int_0^\infty e^{-t} t^{1/2-1} dt$$
$$= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$Q:- \int_0^\infty e^{-y^3} y^{1/2} dy$$

$$y^3 = t \quad \begin{matrix} \text{Limits} \\ y=0 \Rightarrow t=0 \\ y \rightarrow \infty \Rightarrow t \rightarrow \infty \end{matrix}$$
$$3y^2 dy = dt$$

$$dy = \frac{dt}{3(y)^2} = \frac{dt}{3t^{2/3}}$$

$$\int_0^\infty e^{-t} t^{1/6} \frac{dt}{3t^{2/3}} = \frac{1}{3} \int_0^\infty e^{-t} t^{\frac{1}{6}-\frac{2}{3}} dt$$
$$= \frac{1}{3} \int_0^\infty e^{-t} t^{1/2-1} dt = \frac{1}{3} \Gamma\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{\pi}}{3}$$

LAPLACE TRANSFORM

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$1) L\{1\} = 1/s$$

$$2) L\{e^{at}\} = \frac{1}{s-a}$$

$$3) L\{\sin ht\} = \frac{a}{s^2-a^2}$$

$$4) L\{\cos ht\} = \frac{s}{s^2-a^2}$$

$$L\{ \cos at + i \sin at \} = \frac{s+ia}{s^2+a^2}$$

$$5) L\{ \cos at \} = \frac{s}{s^2+a^2}$$

$$6) L\{ \sin at \} = \frac{a}{s^2+a^2}$$

$$7) L\{ t^n \} = \int_0^\infty e^{-st} t^n dt$$

Put $st = x$ Limit

$$sdt = dx \quad t=0 \quad x=0 \\ t \rightarrow \infty \quad x \rightarrow \infty$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)} dx$$

$$L\{ t^n \} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\text{Q:- } L\{ t \} = \frac{1}{s^2}$$

$$L\{ t^2 \} = \frac{2}{s^3}$$

$$L\{ t^6 \} = \frac{\Gamma(7)}{s^7} = \frac{6!}{s^7}$$

$$L\{ t^{1/2} \} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

$$L\{ t^{3/2} \} = \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{s^{5/2}}$$

$$L\{ t^{5/2} \} = \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{s^{7/2}}$$

$$L\{ t^{9/2} \} = \frac{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{s^{11/2}}$$

Unit Step f^n :-

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

$$= \frac{e^{-as}}{s}$$

Unit Impulse F^n :-

$$s(t-a) = \begin{cases} \frac{1}{\epsilon} & a < t < a+\epsilon \\ 0 & \text{elsewhere} \end{cases}$$

Dirac Delta
 F^n

$\epsilon \rightarrow 0$

$$L\{ s(t-a) \} = e^{-as}$$

L.T. of periodic f^n s : - of period T

$$L\{ f(t) \} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

L.T. of $f(t)$ of period 2π

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi \leq t \leq 2\pi \end{cases}$$

$$L\{ f(t) \} = \frac{1}{1-e^{-2\pi s}} \left[\int_0^\pi e^{-st} \sin t dt \right] + \int_\pi^{2\pi} e^{-st} 0 dt$$

Remember -

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-s\pi}}{s^2+1} [-s \cdot 0 - (-1)] - \frac{e^0}{s^2+1} [s \cdot 0] \right]$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)} [e^{-s\pi} + 1]$$

$$= \frac{e^{\pi s}}{(e^{\pi s}-1)(s^2+1)}$$

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$1) \mathcal{L}\{e^{at} t^n\} = \frac{F'(n+1)}{(s-a)^{n+1}}$$

$$2) \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$3) \mathcal{L}\{e^{at} \cos bt\} = \frac{(s-a)}{(s-a)^2 + b^2}$$

$$\underline{\mathcal{L}\{f(t)\} = F(s)}$$

$$(1) \mathcal{L}\{1\} = 1/s$$

$$(2) \mathcal{L}\{t\} = 1/s^2$$

$$(3) \mathcal{L}\{e^{at}\} = 1/s-a$$

$$(4) \mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$$

$$(5) \mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$$

$$(6) \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$(7) \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$(8) \mathcal{L}\{t^n\} = \frac{F'(n+1)}{s^{n+1}}$$

$$\begin{aligned} &= \mathcal{L}^{-1}\left\{\frac{(s-2)+2}{(s-2)^3}\right\} \\ &= e^{2t} \mathcal{L}^{-1}\left\{\frac{(s+2)^2}{(s-2)^3}\right\} = e^{2t} \mathcal{L}^{-1}\left\{\frac{s^2+4s+4}{s^3}\right\} \\ &= e^{2t} \left[\mathcal{L}^{-1}\frac{1}{s} + 4 \mathcal{L}^{-1}\frac{1}{s^2} + 4 \mathcal{L}^{-1}\frac{1}{s^3} \right] \\ &= e^{2t} \left[1 + 4t + \frac{4t^2}{2} \right] \text{ Ans.} \end{aligned}$$

$$\underline{\mathcal{L}^{-1}\{F(s)\} = f(t)}$$

$$\mathcal{L}^{-1}\{1/s\} = 1$$

$$\mathcal{L}^{-1}\{1/s^2\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a} \sinhat$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \sin at$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma(n)}$$

First Translation of ILT :-

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at} \mathcal{L}^{-1}\{F(s)\}}$$

$$\text{Q:- } \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4s+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4s+4+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+5}\right\}$$

$$= e^{-2t} \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} = e^{-2t} \cos \sqrt{5}t$$

(Derivative of transform) -

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$\underline{\text{Ex:}} - L\{t \cos t\}$$

$$L\{\cos t\} = \frac{s}{s^2 + 1}$$

$$L\{t \cos t\} = -\frac{d}{ds} \left[\frac{s}{s^2 + 1} \right] \\ = \left[\frac{s^2 - 1}{(s^2 + 1)^2} \right]$$

$$\underline{\text{Ex:}} - L\{t^2 \sin at\}$$

$$L\{e^{iat} t^2\}$$

$$L\{t^2\} = \frac{2}{s^3}$$

$$L\{e^{iat} \cdot t^2\} = \frac{2}{(s-ia)^3}$$

$$L\{t^2 \sin at\} = \text{Im} \frac{2(s+ia)^3}{(s-ia)^3 (s+ia)^3}$$

$$L\{t^2 \sin at\} = \frac{6s^2 a - 2a^3}{(s^2 + a^2)^3} \quad \underline{\text{Ans.}}$$

Division by t [Integral L.T.]

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$(1) \quad L\left\{\frac{\sin t}{t}\right\}; \quad L\{\sin t\} = \frac{1}{s^2 + 1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2 + 1} ds = \tan^{-1}s \Big|_s^\infty \\ = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\underline{\text{Q:}} - \int_0^\infty \frac{\sin t}{t} dt$$

$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}s$$

$$\int_0^\infty e^{-st} \frac{\sin t}{t} dt = \cot^{-1}s$$

$$\int_0^\infty \frac{\sin t}{t} dt = \cot^{-1} 0 = \pi/2$$

$$\underline{\text{Ex:}} - L\left\{\frac{e^{at}}{t}\right\}$$

$$L\{e^{at}\} = \frac{1}{s-a}; \quad L\left\{\frac{e^{at}}{t}\right\} = \int_s^\infty \frac{1}{s-a} ds \\ = \log(s-a) \Big|_s^\infty \quad \text{It does not exist.}$$

$$\underline{\text{Ex:}} - L\left\{\frac{e^{at} - e^{bt}}{t}\right\}$$

$$L\{e^{at} - e^{bt}\} = \frac{1}{s-a} - \frac{1}{s-b}$$

$$L\left\{\frac{e^{at} - e^{bt}}{t}\right\} = \int_s^\infty \left[\frac{1}{s-a} - \frac{1}{s-b} \right] ds$$

$$= [\log(s-a) - \log(s-b)] \Big|_s^\infty = \log \frac{s-a}{s-b} \Big|_s^\infty$$

$$\lim_{s \rightarrow \infty} \log \left(\frac{s-a}{s-b} \right) - \log \left(\frac{s-a}{s-b} \right)$$

$$\lim_{s \rightarrow \infty} \log \left(\frac{(s-a)s}{(s-b)s} \right) - \log \left(\frac{s-a}{s-b} \right) = 0 - \log \left(\frac{s-a}{s-b} \right) \\ = \log \left(\frac{s-b}{s-a} \right)$$

L.T. of derivative (Multiplication by s):-

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$\underline{\text{Ex:}} - y'' + 4y = 12t; \quad y(0) = 0; \quad y'(0) = 9$$

$$\underline{\text{Sol:}} - L\{y\} = Y(s)$$

$$L\{y'' + 4y\} = 12 L\{t\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{12}{s^2}$$

$$[s^2 + 4] Y(s) - 9 = \frac{12}{s^2}$$

$$Y(s) = \frac{12 + 9s^2}{s^2(s^2 + 4)} = \frac{9s^2 + 36 - 36 + 12}{s^2(s^2 + 4)}$$

$$= \frac{9(s^2 + 4)}{s^2(s^2 + 4)} - \frac{24}{s^2(s^2 + 4)} \xrightarrow{24} \left[\frac{1}{s^2} - \frac{1}{s^2 + 4} \right]$$

$$= \frac{9}{s^2} - \frac{6}{s^2} + \frac{6}{s^2 + 4} = \frac{3}{s^2} + \frac{6}{s^2 + 4}$$

$$y(t) = 3t + 3\sin 2t.$$

$$F(s) = \log(s-3) - \log(s-2)$$

$$F'(s) = \frac{1}{s-3} - \frac{1}{s-2}$$

$$\mathcal{L}\{F'(s)\} = e^{3t} - e^{2t}$$

$$-tf(t) = e^{3t} - e^{2t}$$

$$f(t) = \frac{e^{2t} - e^{3t}}{t} \quad \underline{\text{Ans}}$$

The end of math