The averages of some arithmetic functions. An exercise in complex variables

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1 Introduction

"There is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds". G.H. Hardy, A Mathematician's Apology.

These are intended to be a short set of notes (maybe even to be denigrated to 'comments'). Their purpose is to explain how the tools of complex variable theory, when applied to various relations on the Riemann zeta function, provides asymptotic estimates for the averages of various sums over the divisors of an integer. For example, if

$$\tau(n) = \sum_{d|n} 1, \tag{1}$$

then, noting that $\tau(n)$ behaves erratically with n it is natural to ask about its behaviour in the 'mean'. This is the 'Dirichlet divisor problem'. This question gains weight when one notes that a general Dirichlet series,

$$A(s) = \sum \frac{a_n}{n^s}, \tag{2}$$

acts as a generating function for such sums. The situation is analogous to how power series act as generating functions for standard convolutions,

$$c(n) = \sum_{0 \le r \le n} a(r) b(n-r), \tag{3}$$

with division taken as a partial ordering on the set of integers. That is, if

$$b(n) = \sum_{d|n} a_d, \tag{4}$$

then

$$B(s) = \zeta(s)A(s). \tag{5}$$

And so,

$$\sum \frac{\tau(n)}{n^s} = \zeta(s)^2. \tag{6}$$

The fascinating analytic properties of $\zeta(s)$ (which I don't wish to review, or prove here) ought to provide some insights into this question. The answer seems to be that they do, but the matter appears far from closed. A full answer would require complete knowledge of the Riemann Zeta function in the critical strip, $0 < \Re s < 1$, either in the form of the average of certain integrals ('Mean value theorems'); bounds on its growth as $\Im s \to \infty$ (eg 'Lindelhof Hypothesis'); or, for some arithmetic functions, the locations of poles of $1/\zeta(s)$ ('Riemann Hypothesis'). The moral might be that as analytical objects Dirichlet series are much more difficult beasts than power series. There are no new insights here, my sources are Apostol (1976), Edwards (2001) and Ivic (2003).

On an autobiographical aside, the above question is very natural and it has interested me since I came across it as an over-ambitious A-level student. Since that is now over 20 years ago, writing these notes might be a worthwhile exercise in psychological closure, on the other hand it might be best to let sleeping dogs lie!

2 The number of divisors of an integer

Given an integer n let $\tau(n)$ denote the number of divisors of n, that is,

$$\tau(n) = \sum_{d|n} 1. \tag{7}$$

This function is multiplicative. Not only that, but equation 7 defines a convolution of Dirichlet type, and so we can encode the convolution sum in terms of a generating function, or Dirichlet series.

$$\sum \frac{\tau(n)}{n^s} = \zeta(s)^2. \tag{8}$$

I will define,

$$T(x) = \sum_{0<n\leq x} \tau(n). \tag{9}$$

and,

$$\sum_n T(n)\left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right] = \zeta(s)^2. \tag{10}$$

$$s\sum_n T(n)\int_n^{n+1} x^{-s-1}dx = \zeta(s)^2. \tag{11}$$

$$s\int_0^\infty T(x)x^{-s-1}dx = \zeta(s)^2. \tag{12}$$

It follows from the inversion theorem for Mellin transforms that,

$$T(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)^2 \frac{x^s}{s} ds. \tag{13}$$

It is widely appreciated that when considered as a function of the complex variable, s, $\zeta(s)$ has some remarkable properties. For example,

1. $\zeta(s)$ is analytic in $\Re s > 0$ (it is obviously analytic in $\Re s > 1$) with the exception of a single pole at $s = 1$.

2. It has the Laurent series,

$$\zeta(s) = \frac{1}{s-1} + \sum (-1)^n \frac{\gamma_n}{n!} s^n \tag{14}$$

3. It has a functional equation,

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi \frac{s}{2})\Gamma(1-s)\zeta(1-s) \tag{15}$$

   which provides analytic continuation through the whole complex plane. Moreover, it continues the trivial region, $\Re s > 1$, into $\Re s < 0$. The 'critical strip' $0 \le \Re s \le 1$ is mapped onto itself.

When taken in combination with the residue theorem these will provide asymptotic estimates for the growth of $T(x)$. For example, moving the contour of integration from $\Re s > 1$ into the critical strip $0 < \Re s < 1$, we will pick up the contribution from the pole of $\zeta(s)$ at $s = 1$. Moreover, near $s = 1$,

$$\zeta(s)^2 \frac{x^s}{s} = \left[\frac{1}{(s-1)^2} + \frac{2\gamma}{(s-1)} + ...\right][x(1 + \log x(s-1) + ...)][1 - (s-1) + ...]. \tag{16}$$

So,

$$T(x) = x(\log x + (2\gamma - 1)) + \Delta(x), \tag{17}$$

where

$$\Delta(x) = \frac{1}{2\pi i} \lim_{t\to\infty}\left[\int_{\sigma+it}^{c+it} + \int_{\sigma-it}^{\sigma+it} + \int_{c-it}^{\sigma-it}\right]\zeta(s)^2 \frac{x^s}{s} ds. (\sigma \ge 1/2) \tag{18}$$

If it is true that,

$$\zeta(s) = O(t^{\delta}) \text{ for } \frac{1}{2} < \Re s < 1 \tag{19}$$

then,

$$\left|\left(\int_{\sigma+it}^{c+it} + \int_{c-it}^{\sigma-it}\right)\zeta(s)^2 \frac{x^s}{s} ds\right| < t^{(2\delta-1)} x^{\sigma}, \tag{20}$$

and

$$\left|\int_{\sigma-it}^{\sigma+it} \zeta(s)^2 \frac{x^s}{s} ds\right| < Cx^{\sigma} + x^{\sigma} \int_{t_0}^{t} t^{2\delta-1} ds, \tag{21}$$

and,

$$\Delta(x) = O(x^{\frac{1}{2}+2\delta}) \tag{22}$$

This is not a tight bound, but if it were it might suggest that the divisor function could be modelled as a random process, where

$$\tau(n) = \log(n) + 2\gamma + r_n \tag{23}$$

and r_n is a random process with zero mean.

We can apply this technique to many other arithmetic functions. Here are some more examples,

3 The sum of powers of divisors of an integer

One generalisation of 7 is to consider the sum of the ath powers of the divisors of n. Lets denote this $\sigma_a(n)$ so,

$$\sigma_a(n) = \sum_{d|n} d^a. \tag{24}$$

This function is also multiplicative. But now,

$$\sum \frac{\sigma_a(n)}{n^s} = \zeta(s)\zeta(s-a). \tag{25}$$

I will define,

$$S_a(x) = \sum_{0<n\leq x} \sigma_a(n) \tag{26}$$

and turning the handle of the machine outlined in section 1,

$$S_a(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s)\zeta(s-a) \frac{x^s}{s} ds. \tag{27}$$

If $a > 0$ then moving the contour of integration from $\Re s > 1$ into the critical strip $0 < \Re s < 1$, we will pick up the contribution from the pole of $\zeta(s)$ at $s = 1$ and a second from the pole of $\zeta(s-a)$ at $s = a+1$. Moreover, near $s = 1$,

$$\zeta(s)\zeta(s-a)\frac{x^s}{s} = \left[\frac{1}{(s-1)} + \gamma + ...\right][x + ...][1 + ...]\zeta(1-a). \tag{28}$$

whilst, near $s = a+1$,

$$\zeta(s)\zeta(s-a)\frac{x^s}{s} = \left[\frac{1}{(s-a-1)} + \gamma + ...\right]\left[x^{a+1} + ...\right]\left[\frac{1}{a+1} + ...\right]\zeta(a+1). \tag{29}$$

,

$$S_a(x) = \zeta(a+1)\frac{x^{a+1}}{a+1} + x\zeta(1-a) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s)\zeta(s-a)\frac{x^s}{s}ds.(c \geq 1/2) \tag{30}$$

4 The variance in the number of divisors of an integer

Clearly, the number of divisors of an integer when considered over the interval $[1, x]$ of the natural numbers behaves erratically. However, section 1 says that the mean of this function approaches a logarithm. What about its variance? I want to consider the sum,

$$T_2(x) = \sum_{0<n\leq x}\tau^2(n). \tag{31}$$

Since $\tau(n)$ is multiplicative so is $\tau^2(n)$. Moreover,

$$\sum\frac{\tau(n)^2}{n^s} = \prod_p\sum_i\frac{(i+1)^2}{p^{is}} \tag{32}$$

$$= \prod_p(1 + \frac{1}{p^s})(1 - \frac{1}{p^s})^{-3}, \tag{33}$$

$$= \frac{\zeta(s)^4}{\zeta(2s)}. \tag{34}$$

So,

$$T_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s)^4}{\zeta(2s)}\frac{x^s}{s}ds. \tag{35}$$

I just need the residue of the integrand at $s = 1$,

$$\zeta(s)^4 = \frac{1}{(s-1)^4} + \frac{4\gamma}{(s-1)^3} + \frac{6\gamma^2 + 4\gamma_1}{(s-1)^2} + \frac{4\gamma^3 + 6\gamma\gamma_1 + 4\gamma_2}{(s-1)} + \dots \tag{36}$$

$$x^s = x + (s-1)x\log(x) + (s-1)^2 x\log(x)^2 + (s-1)^3 x\log(x)^3 + \dots \tag{37}$$

$$\frac{1}{s} = 1 - (s-1) + (s-1)^2 + (s-1)^3 + \dots \tag{38}$$

$$\frac{1}{\zeta(2s)} = \frac{6}{\pi^2} + c_1(s-1) + c_2(s-1)^2 + c_3(s-1)^3 + \dots \tag{39}$$

The residue provides the asymptotic behaviour,

$$\frac{6}{\pi^2}x\log(x)^3 + \left(\frac{24\gamma}{\pi^2} + c_1 - \frac{6}{\pi^2}\right)x\log(x)^2 + O(x\log(x)) \tag{40}$$

The fact that this leading order term exceeds that of 17 helps explain the 'scratchiness' seen in figure 1.

5 Further remarks and speculations

5.1 Numerical evidence

The divisor sum can be easily evaluated numerically, namely

$$T(x) = \sum_{ad \le n} 1, \tag{41}$$

$$= 2\sum_{d=1}^{\sqrt{x}} \left[\frac{x}{d}\right] + [\sqrt{x}]^2. \tag{42}$$

Both Apostol (1976) and Hardy and Wright (2008) show how this provides the $O(x^{\frac{1}{2}})$ bound without complex analysis. We can use this to compute $\Delta(x)$ and in figure 1 we compare its growth to several known bounds. Of course, such an investigation ultimately proves nothing, and conveys at most an impression, not least because it is meaningless without first assuming some scaling for a 'big-O' term.

Similarly,

$$S_1(x) = \frac{1}{2}\sum_{d=1}^{x} \left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right] + 1\right) \tag{43}$$

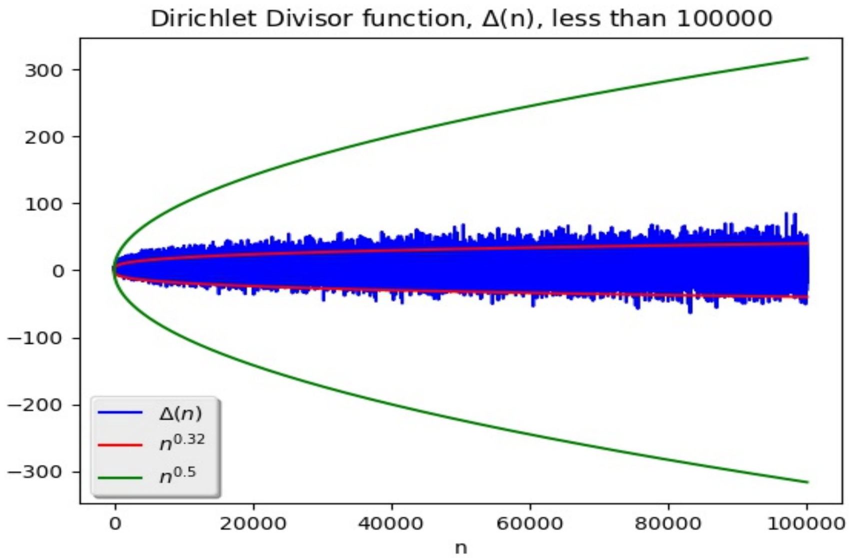

Figure 1: Growth of $\Delta(n)$ (blue) compared to $n^{0.5}$ (green) and $n^{0.32}$ (red) for $1 \leq n \leq 1,000,000$

Equation 30 is missing a 'big-O' estimate. It really needs one, Apostol (1976) and Hardy and Wright (2008) give,

$$S_1(x) = \frac{\pi^2 x}{12} + O(x \log x). \tag{44}$$

In figure 2, we plot the value of,

$$E_1(x) = S_1(x) - \frac{\pi^2}{12} x^2. \tag{45}$$

against $x \log(x)$.

5.2 Perron's formula

My clumsy derivation of equation 13 can actually be used to prove a more general result,

$$\sum_{n \leq x} \frac{a_n}{n^\alpha} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A(s+\alpha) \frac{x^s}{s} ds. \tag{46}$$

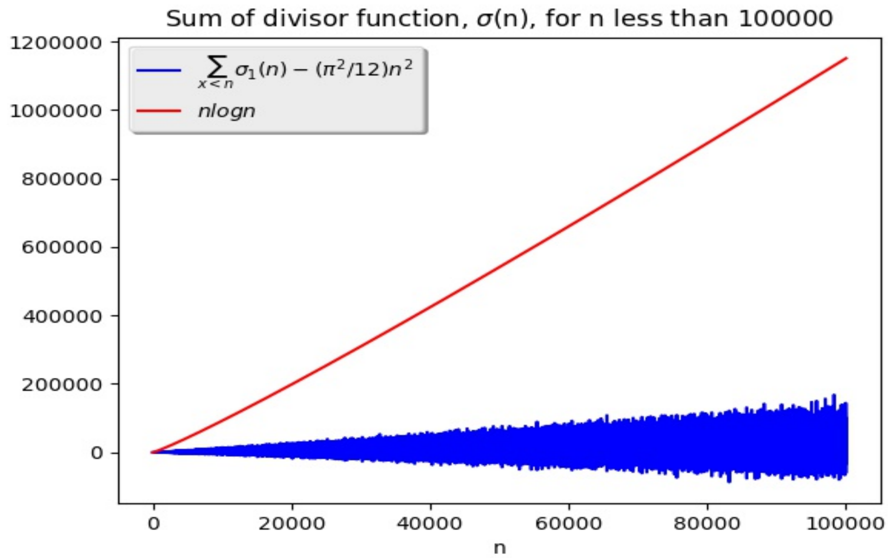

Figure 2: Growth of $E_1(n)$ for $1 \leq n \leq 100,000$

This is Perron's formula, see Ivic (2003) or Apostol (1976). It has deeper consequences such as establishing approximate functional equations for Dirichlet series. One consequence that is closer to my purposes is that,

$$\frac{1}{x}\sum_{n\leq x}\frac{\sigma_n}{n} \sim \frac{\pi^2}{6}. \tag{47}$$

Since,

$$\frac{\pi^2}{6} = 1.645... < 2, \tag{48}$$

the average number must be less than perfect, but not hugely so. This is surely a blow to numerologists.

5.3 Voronoi summation formula

The bound,

$$\Delta(x) = O(x^{\frac{1}{2}}), \tag{49}$$

was known to Dirichlet and is not tight. This came as something of a disappointment to me as I hoped that simply applying the residue theorem and bounding the integral in the strip would provide a tight bound. Ivic (2003) describes a very neat method of Voronoi that works over similar lines but did manage to achieve a tighter bound. Starting from,

$$T(x) = x(\log x + (2\gamma - 1)) + \frac{1}{4} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)^2 \frac{x^s}{s} ds \;\; (c < 0). \tag{50}$$

Now use the functional equation and since $\Re(1-s) > 1$, we can integrate termwise, to get,

$$\int_{c-i\infty}^{c+i\infty} \zeta(s)^2 \frac{x^s}{s} ds = \int_{c-i\infty}^{c+i\infty} 2^s \pi^{s-1} \sin^2(\pi\frac{s}{2})\Gamma^2(1-s)\zeta^2(1-s)\frac{x^s}{s} \tag{51}$$

$$= \sum \tau(n) \int_{c-i\infty}^{c+i\infty} 2^s \pi^{s-1} \sin^2(\pi\frac{s}{2})\Gamma^2(1-s)n^{s-1}\frac{x^s ds}{s} \tag{52}$$

The Mellin transform can be evaluated in terms of Bessel functions this together with well-known formulae for the Bessel function to gives the amazing formula,

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le T} \tau(n) n^{-3/4} \cos(4\pi\sqrt{nx} - 4) - \frac{\pi}{4}) + O(x^{\epsilon}) + O(x^{1/2+\epsilon}T^{-1/2}). \tag{53}$$

It would be great if I could find a similar simplification for,

$$\int_{c-i\infty}^{c+i\infty} \zeta(s)\zeta(s-1)\frac{x^s}{s} ds = -\sum \sigma(n) \int_{c-i\infty}^{c+i\infty} (2\pi)^{s-1} \sin(\pi s)\Gamma^2(1-s)\frac{1-s}{s} n^{s-1} x^s ds, \tag{54}$$

Possibly there is one, either a Bessel function or Confluent Hypergeometric function, compare with Whittaker and Watson (2000)?

References

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