Notes on SIR modelling

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1 Basic SIR

A population can be divided up into three class Susceptible (to infection), S, Infected, I, and Recovered, R. Each contact between a Susceptible and Infective results in a rate of increase of infectives moving out of the S class into I. Similarly as the disease progresses infectives begin to recover moving from I to R, that is

$$\frac{\partial S}{\partial t} = -\beta IS \tag{1}$$

$$\frac{\partial I}{\partial t} = \beta IS - \gamma I \tag{2}$$

$$\frac{\partial R}{\partial t} = \gamma I \tag{3}$$

As the system is closed (no births or deaths), we have the additional constraint,

$$S + I + R = N. \tag{4}$$

We wish to analyze the dynamics of this coupled system of equations. Firstly, we non-dimensionalize the system,

$$\frac{\partial s}{\partial t} = -R_0 is \tag{5}$$

$$\frac{\partial i}{\partial t} = (R_0 s - 1)i \tag{6}$$

$$\frac{\partial r}{\partial t} = i \tag{7}$$

r depends only on s, which is evidenced by,

$$\frac{\partial r}{\partial t} = \frac{\partial r}{\partial s}\frac{\partial s}{\partial t} \tag{8}$$

$$\frac{\partial r}{\partial s} = \frac{-1}{R_0 s} \tag{9}$$

Thus,

$$s = \exp(-R_0 r), \tag{10}$$

using the fact that r=0 when s=1. Hence, the steady state is given by,

$$1 - r^* = \exp(-R_0 r^*), \tag{11}$$

with $i^* = 0$. Since $0 \leq r \leq 1$, this equation only has the solution $r^* = 0$, unless,

$$R_0 > 1. \tag{12}$$

As such, equation 12 provides the 'threshold condition' for an SIR epidemic to occur. Moreover, 10 gives us the complete ODE satisfied by r,

$$\frac{\partial r}{\partial t} + \exp(-R_0 r) + r - 1 = 0. \tag{13}$$

For small r, this reduces to the Logistic differential equation,

$$\frac{\partial r}{\partial t} = (R_0 - 1) r \left(1 - \frac{r}{r_1^*}\right), \tag{14}$$

with

$$r_1^* = \frac{2(R_0 - 1)}{R_0^2}. \tag{15}$$

r_1^* being the approximate solution to 11 under the same approximation for $\exp(-R_0 r^*)$ (obv.). It represents the carrying capacity for the disease.

2 SEIR

One extension of the SIR model occurs when the host first moves into an exposed (but critically not infectious) stage immediately after infection. Now the class system (or compartment model) is given by,

$$\frac{\partial S}{\partial t} = -\beta I S \tag{16}$$

$$\frac{\partial E}{\partial t} = \beta I S - \delta E \tag{17}$$

$$\frac{\partial I}{\partial t} = \delta E - \gamma I \tag{18}$$

$$\frac{\partial R}{\partial t} = \gamma I \tag{19}$$

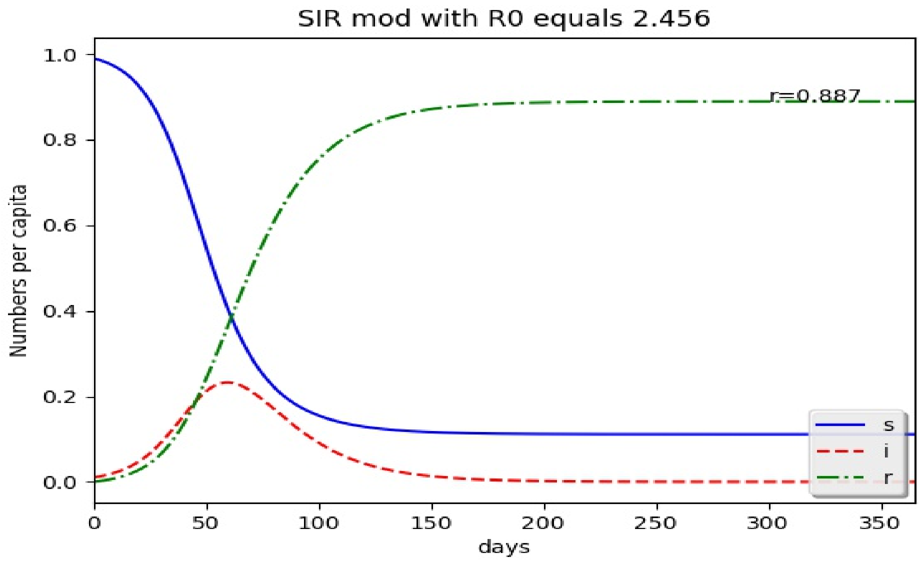

Figure 1: Simple SIR illustrating thresholding given by equation 11 with $R0 = 2.456$

Arguing as for the SIR case, at threshold,

$$I = 0, \tag{20}$$

and the dependence of R (or r) on S (or s) remains the same, so 10 still applies. It follows that,

$$s + i^{'} + r = 1, \tag{21}$$

with

$$i^{'} = \frac{E + I}{N}. \tag{22}$$

At threshold, $i = 0$, leading to the same threshold conditions as for SIR (obv.). The full time course is a bit more subtle,

$$\frac{\partial s}{\partial t} = -R_0 i s \tag{23}$$

$$\frac{\partial i^{'}}{\partial t} = i(R_0 s - 1) \tag{24}$$

$$i^{'} = a\frac{\partial i}{\partial t} + (1 + a)i \tag{25}$$

$$\frac{\partial r}{\partial t} = i \tag{26}$$

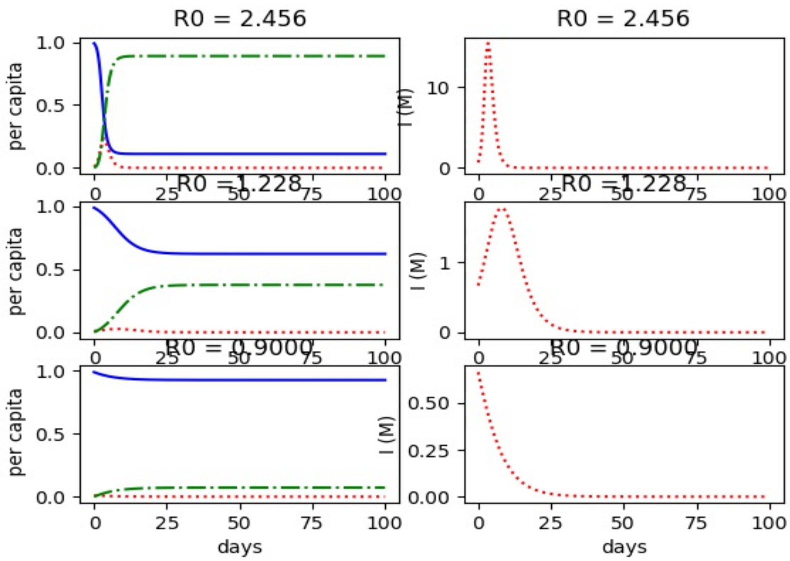

Figure 2: Simple SIR with varying R0

with $a = \frac{\gamma}{\delta}$. So, the complete ODE satisfied by r, is

$$a\frac{\partial^2 r}{\partial t^2} + (1 + a)\frac{\partial r}{\partial t} + \exp\left(-R_0 r\right) + r - 1 = 0. \tag{27}$$

Let us assume that infectives move out of the exposed class much faster than they recover, then

$$a << 1. \tag{28}$$

We can now look for an asymptotic solution of the form,

$$r = r_0 + ar_1. \tag{29}$$

It follows that for leading order, r_0 satisfies equation 13, whilst at second order

$$\frac{\partial r_1}{\partial t} + \left(1 - R_0 \exp\left(-R_0 r_0\right)\right) r_1 = -R_0 \exp\left(-R_0 r_0\right)\left(1 - r_0 - \exp\left(-R_0 r_0\right)\right). \tag{30}$$

One can apply the integrating factor method here. However, if we allow ourselves the earlier assumption that r_0 is small, then

$$\frac{\partial r_1}{\partial t} - (R_0 - 1)\left(1 - \frac{2r_0}{r_1^*}\right) r_1 = -R_0(R_0 - 1) r_0 (1 - R_0 r_0)\left(1 - \frac{r_0}{r_1^*}\right). \tag{31}$$

I think we get the picture.

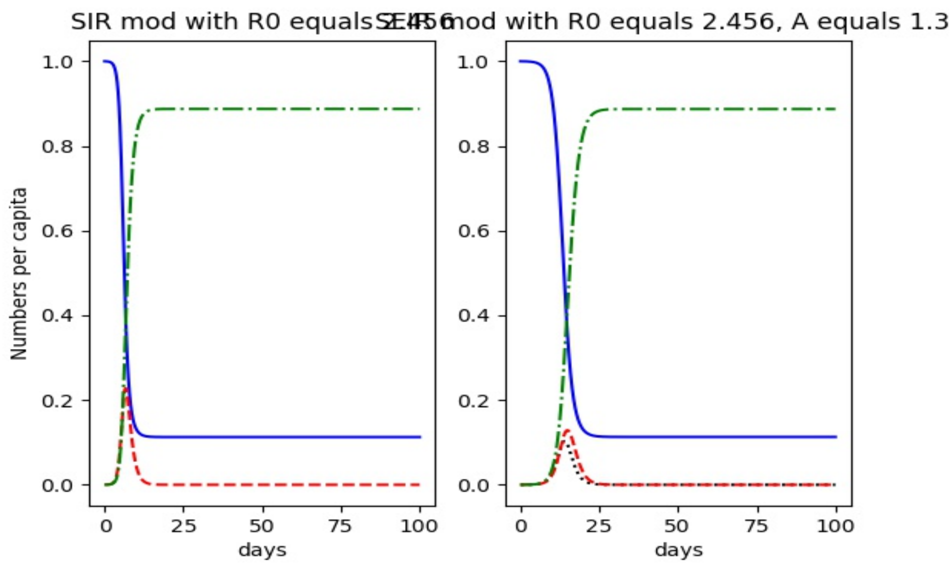

Figure 3: Comparison between SIR and SEIR illustrating the same threshold given by equation 11 with $R0 = 2.456$

3 N stage SIR models

To allow ourselves a little more generality let us suppose that once infected, each infective moves through the disease in N stages each with different rates of transmission (both to susceptibles and to the subsequent stage) and to recovery. Now the system takes the form,

$$\frac{\partial S}{\partial t} = -(\sum \beta_n I_n)S \tag{32}$$

$$\frac{\partial I_0}{\partial t} = (\sum \beta_n I_n)S - (\gamma_0 + \delta_0)I_0 \tag{33}$$

$$\frac{\partial I_n}{\partial t} = \delta_{n-1}I_{n-1} - (\gamma_n + \delta_n)I_n \quad \text{for } 1 \leq n < N \tag{34}$$

$$\frac{\partial I_N}{\partial t} = \delta_{N-1}I_{N-1} - \gamma_N I_N \tag{35}$$

$$\frac{\partial R}{\partial t} = (\sum \gamma_n I_n). \tag{36}$$

A minor simplification of these equations occurs if we assume that the infected move through the stages of the virus at a constant state, for instance if n represents each day of infection ($\delta_n = 1$), for this case,

$$\frac{\partial S}{\partial t} = -(\sum \beta_n I_n) S \tag{37}$$

$$\frac{\partial I_0}{\partial t} = (\sum \beta_n I_n) S - (1 + \gamma_0) I_0 \tag{38}$$

$$\frac{\partial I_n}{\partial t} = I_{n-1} - (1 + \gamma_n) I_n \quad \text{for } 1 \leq n < N \tag{39}$$

$$\frac{\partial I_N}{\partial t} = I_{N-1} - \gamma_N I_N \tag{40}$$

$$\frac{\partial R}{\partial t} = (\sum \gamma_n I_n). \tag{41}$$

More generally, we can pose any system of equations,

$$\frac{\partial S}{\partial t} = -(\sum \beta_n I_n) S \tag{42}$$

$$\frac{\partial I_0}{\partial t} + D_0 I_0 = (\sum \beta_n I_n) S - \gamma_0 I_0 \tag{43}$$

$$\frac{\partial I_n}{\partial t} + D_n I_n = -\gamma_n I_n \quad \text{for } n \geq 1 \tag{44}$$

$$\frac{\partial R}{\partial t} = (\sum \gamma_n I_n). \tag{45}$$

where D_n is a difference operator subject to the condition,

$$\sum_n D_n I_n = 0 \tag{46}$$

Let us assume that $I_n(t)$ is separable, so that,

$$I_n(t) = p_n \bar{I}(t). \tag{47}$$

It follows that,

$$\frac{\partial S}{\partial t} = -(\sum \beta_n p_n) \bar{I} S \tag{48}$$

$$\bar{I}\frac{\partial p_0}{\partial t} + p_0 \frac{\partial \bar{I}}{\partial t} = \left[(\sum \beta_n p_n) S - (\gamma_0 + \delta_0) p_0\right] \bar{I} \tag{49}$$

$$\bar{I}\frac{\partial p_n}{\partial t} + p_n \frac{\partial \bar{I}}{\partial t} = \left[\delta_{n-1} p_{n-1} - (\gamma_n + \delta_n) p_n\right] \bar{I} \quad \text{for } 1 \leq n < N \tag{50}$$

$$\bar{I}\frac{\partial p_N}{\partial t} + p_N \frac{\partial \bar{I}}{\partial t} = \left[\delta_{N-1} p_{n-1} - \gamma_N p_N\right] \bar{I} \tag{51}$$

$$\frac{\partial R}{\partial t} = (\sum \gamma_n p_n) \bar{I}. \tag{52}$$

Summing equations 49, 50 and 51 gives,

$$\bar{I}\sum\frac{\partial p_n}{\partial t}+\sum p_n\frac{\partial \bar{I}}{\partial t}=\left[\left(\sum\beta_n p_n\right)S-\left(\sum\gamma_n p_n\right)\right]\bar{I}. \tag{53}$$

Assuming that,

$$\sum p_n = 1, \tag{54}$$

for all t, we have the standard SIR equation,

$$\frac{\partial S}{\partial t} = -\beta_p\bar{I}S \tag{55}$$

$$\frac{\partial \bar{I}}{\partial t} = \left[\beta_p S-\gamma_p\right]\bar{I} \tag{56}$$

$$\frac{\partial R}{\partial t} = \gamma_p\bar{I}. \tag{57}$$

where, the p_n's are coupled to $\bar{I}$ through the matrix equation,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}\partial_t\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_4 \\ \vdots \\ p_{N-1} \\ p_N \end{bmatrix} = -\begin{bmatrix} 0 & 0 & & & \cdots & & \\ c_0 & d_1 & & & & & \\ & c_1 & d_2 & & & & \\ & & c_2 & d_3 & & & \\ & & & \ddots & \ddots & & \\ & & & & & d_{N-1} & \\ & & & & & c_{N-1} & d_N \end{bmatrix}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_4 \\ \vdots \\ p_{N-1} \\ p_N \end{bmatrix}. \tag{58}$$

where,

$$c_n = -\delta_n\bar{I}, \tag{59}$$

$$d_n = \begin{cases} \bar{I}_t + (\gamma_n+\delta_n)\bar{I}, & \text{for } 1 \le n < N, \\ \bar{I}_t + \gamma_n\bar{I}, & \text{for } n = N. \end{cases} \tag{60}$$

Or equivalently,

$$c_n = -\delta_n, \tag{61}$$

$$d_n = \begin{cases} \beta_p S + \gamma_n - \gamma_p + \delta_n, & \text{for } 1 \le n < N, \\ \beta_p S + \gamma_n - \gamma_p, & \text{for } n = N. \end{cases} \tag{62}$$

$$
\partial_t \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_4 \\ \vdots \\ p_{N-1} \\ p_N \end{bmatrix} = \begin{bmatrix} c_0, & c_1 + d_1, & c_2 + d_2, & c_3 + d_3, & \cdots & c_{N-1} + d_{N-1}, & d_N \\ -c_0 & -d_1 & & & & & \\ & -c_1 & -d_2 & & & & \\ & & -c_2 & -d_3 & & & \\ & & & \ddots & \ddots & & \\ & & & & & -d_{N-1} & \\ & & & & & -c_{N-1} & -d_N \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_4 \\ \vdots \\ p_{N-1} \\ p_N \end{bmatrix}. \tag{63}
$$

Alternating the update step for S, $\bar{I}$, R with solving the matrix equations for p allows another (more efficient, in python) method for updating the system.

4 SIRS models

These model the case where some of the recovered community relapse to the susceptible class, ie immunity is not permanent, and they satisfy the cycle equations,

$$\frac{\partial S}{\partial t} = -\beta IS + \delta R \tag{64}$$

$$\frac{\partial I}{\partial t} = \beta IS - \gamma I \tag{65}$$

$$\frac{\partial R}{\partial t} = \gamma I - \delta R \tag{66}$$

It follows that

$$\frac{\partial s}{\partial t} = -R_0 is + \alpha r \tag{67}$$

$$\frac{\partial i}{\partial t} = R_0 is - i \tag{68}$$

$$\frac{\partial r}{\partial t} = i - \alpha r \tag{69}$$

For the steady state, either

$$(s, i, r) = (1, 0, 0), \tag{70}$$

or,

$$(s, i, r) = \left(\frac{1}{R_0}, \frac{\alpha(R_0 - 1)}{R_0(1 + \alpha)}, \frac{R_0 - 1}{R_0(1 + \alpha)} \right), \tag{71}$$

Near this state,

$$\frac{\partial}{\partial t}\begin{bmatrix} \delta s \\ \delta i \\ \delta r \end{bmatrix} = \begin{bmatrix} -R_0 i & -R_0 s & \alpha \\ R_0 i & R_0 s - 1 & 0 \\ 0 & 1 & -\alpha \end{bmatrix}\begin{bmatrix} \delta s \\ \delta i \\ \delta r \end{bmatrix} \tag{72}$$

Adding rows 2 and 3 to row 1, gives,

$$\frac{\partial}{\partial t}\begin{bmatrix} \delta n \\ \delta i \\ \delta r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ R_0 i & R_0(s-i) - 1 & -R_0 i \\ 0 & 1 & -\alpha \end{bmatrix}\begin{bmatrix} \delta n \\ \delta i \\ \delta r \end{bmatrix}, \tag{73}$$

where

$$\delta n = \delta s + \delta i + \delta r. \tag{74}$$

The top column gives us that $\delta n = \text{const}$, which we already knew, since

$$\delta n = 0. \tag{75}$$

We consider the first steady state condition when $s = 1$, here

$$\frac{\partial}{\partial t}\begin{bmatrix} \delta i \\ \delta r \end{bmatrix} = \begin{bmatrix} R_0 - 1 & -R_0 i \\ 1 & -\alpha \end{bmatrix}\begin{bmatrix} \delta i \\ \delta r \end{bmatrix}, \tag{76}$$

The system matrix has eigenvalues $-\alpha$ and $R_0 - 1$, so the final solution is

$$\begin{bmatrix} \delta n \\ \delta i \\ \delta r \end{bmatrix} = C_1 e^{(R_0-1)t}\begin{bmatrix} -(R_0+\alpha) \\ (R_0 - 1 + \alpha) \\ 0 \end{bmatrix} + C_2 e^{-\alpha t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{77}$$

So, again, only if $R_0 > 1$ can the epidemic break out.

Now, consider the steady state where $R_0 s = 1$.

$$\frac{\partial}{\partial t}\begin{bmatrix} \delta i \\ \delta r \end{bmatrix} = \begin{bmatrix} -R_0 i^* & -R_0 i^* \\ 1 & -\alpha \end{bmatrix}\begin{bmatrix} \delta i \\ \delta r \end{bmatrix}. \tag{78}$$

Here the eigenvalues, λ, satisfy,

$$\lambda^2 + (\alpha + R_0 i^*)\lambda + (\alpha + 1)R_0 i^* = 0. \tag{79}$$

ie,

$$\lambda = -\frac{\alpha + R_0 i^*}{2} \pm \frac{1}{2}\sqrt{(\alpha - R_0 i^*)^2 - 4R_0 i^*}. \tag{80}$$

It is obvious, that,

$$(\alpha - R_0 i^*)^2 - 4R_0 i^* < (\alpha - R_0 i^*)^2 < (\alpha + R_0 i^*)^2 \tag{81}$$

So, either both roots are negative, or complex conjugate. The conditions for which α and R_0 conspire to the latter is a bit more tricky. Before jumping into that, we note that assuming both are complex conjugate, say,

$$\lambda = -a \pm ip, \tag{82}$$

then, the eigenvectors are given by,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -a \pm ip \end{bmatrix}. \tag{83}$$

And for this case,

$$\begin{bmatrix} \delta i \\ \delta r \end{bmatrix} = C_1 e^{-at} \begin{bmatrix} \cos pt \\ -a\cos pt + p\sin pt \end{bmatrix} + C_2 e^{-at} \begin{bmatrix} \sin pt \\ -a\sin pt + p\cos pt \end{bmatrix} \tag{84}$$

So damped oscillations are possible, when might this occur? The discriminant is given by,

$$\Delta = \alpha^2 \left[1 - \frac{(R_0 - 1)}{1 + \alpha}\right]^2 - \frac{4(R_0 - 1)\alpha}{\alpha + 1}, \tag{85}$$

$$\Delta = \frac{\alpha^2}{(\alpha + 1)^2} \left[(1 + \alpha) - (R_0 - 1)\right]^2 - 4(R_0 - 1)\alpha(\alpha + 1), \tag{86}$$

So the question is, given $R_0 > 1$, for what $u > 1$ is,

$$(u - 1)\left[u - (R_0 - 1)\right]^2 \le 4(R_0 - 1)u \tag{87}$$

or

$$u^3 - 2R_0 u^2 + (R_0 - 1)(R_0 - 3)u + (R_0 - 1)^2 \le 0. \tag{88}$$

Following the work of the previous section we can also investigate the case where immunity is gradually lost as a function of time since initial recovery (say after 6 months).

$$\frac{\partial S}{\partial t} = -(\sum \beta_n I_n)S + (\sum \kappa_n R_n) \tag{89}$$
$$\frac{\partial I_0}{\partial t} = (\sum \beta_n I_n)S - (\gamma_0 + \delta_0)I_0 \tag{90}$$
$$\frac{\partial I_n}{\partial t} = \delta_{n-1}I_{n-1} - (\gamma_n + \delta_n)I_n \quad \text{for } 1 \le n < N \tag{91}$$
$$\frac{\partial I_N}{\partial t} = \delta_{N-1}I_{N-1} - \gamma_N I_N \tag{92}$$
$$\frac{\partial R_0}{\partial t} = (\sum \gamma_n I_n) - (\kappa_0 + \epsilon_0)R_0 \tag{93}$$
$$\frac{\partial R_n}{\partial t} = \epsilon_{n-1}R_{n-1} - (\kappa_n + \epsilon_n)R_n \quad \text{for } 1 \le n < M \tag{94}$$
$$\frac{\partial R_M}{\partial t} = \epsilon_{M-1}R_{M-1} - \kappa_M R_M. \tag{95}$$

Following 3, we can express

$$R_n(t) = q_n \bar{R}(t) \tag{96}$$

to derive the standard SIRS equation,

$$\frac{\partial S}{\partial t} = -\beta_p \bar{I} S + \kappa_q \bar{R} \tag{97}$$
$$\frac{\partial \bar{I}}{\partial t} = \left[\beta_p S - \gamma_p\right] \bar{I} \tag{98}$$
$$\frac{\partial \bar{R}}{\partial t} = \gamma_p \bar{I} - \kappa_q \bar{R}. \tag{99}$$

where,

$$\partial_t \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_4 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix} = \begin{bmatrix} e_0, & e_1 + f_1, & e_2 + f_2, & e_3 + f_3, & \cdots & e_{N-1} + f_{N-1}, & f_N \\ -e_0 & -f_1 & & & & & \\ & -e_1 & -f_2 & & & & \\ & & -e_2 & -f_3 & & & \\ & & & \ddots & \ddots & & \\ & & & & & -f_{N-1} & \\ & & & & & -e_{N-1} & -f_N \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_4 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix}, \tag{100}$$

with

$$e_n = -\epsilon_n \bar{R}, \tag{101}$$

$$f_n = \begin{cases} \bar{R}_t + (\kappa_n + \epsilon_n)\bar{R}, & \text{for } 1 \le n < M, \\ \bar{R}_t + \epsilon_n \bar{R}, & \text{for } n = M. \end{cases} \tag{102}$$

5 Including deaths

We shall assume that natural deaths ('not from disease') occur at an equal rate for each of the S, I, R classes. Infants are born into the S class (neither immunity or infection is conferred from the mothers), we also assume that the rate of natural deaths increase with N. This allows a threshold for population growth without incidence of disease. So,

$$\frac{\partial S}{\partial t} = bN - \beta IS - dNS \tag{103}$$

$$\frac{\partial I}{\partial t} = \beta IS - \gamma I - cI - dNI \tag{104}$$

$$\frac{\partial R}{\partial t} = \gamma I - dNR \tag{105}$$

Summing these we have the population growth,

$$\frac{\partial N}{\partial t} = bN - cI - dN^2 \tag{106}$$

On the over simplification that $I = pN$, then

$$\frac{\partial N}{\partial t} = (b - cp)N\left(1 - \frac{d}{b - cp}N\right) \tag{107}$$

So I plays the role of lowering the exponential growth due to births, and the carrying capacity. At the steady state (with disease),

$$\beta IS + dNS = bN \tag{108}$$

$$\gamma + c + dN = \beta S \tag{109}$$

$$\gamma I = dNR \tag{110}$$

We use,

$$S = \frac{\gamma + c + dN}{\beta} \tag{111}$$

$$R = \frac{\gamma}{dN} I \tag{112}$$

$$\tag{113}$$

and plug this into the population equation,

$$N = \frac{dN + \gamma}{dN} I + \frac{dN + c + \gamma}{\beta} \tag{114}$$

so,

$$I = \frac{dN}{\beta}\left[\frac{\beta N - c}{dN + \gamma} - 1\right]. \tag{115}$$

$$\frac{\partial N}{\partial t} = \frac{b\beta + cd}{\beta} N - dN^2 - cN\left[\frac{1 - \frac{c}{\beta N}}{1 + \frac{\gamma}{dN}}\right] \tag{116}$$

Now, assuming $\gamma/(dN) << 1$,

$$\beta d \frac{\partial N}{\partial t} = c(cd + \gamma\beta) + d((b - c)\beta + cd)N - \beta d^2 N^2 \tag{117}$$

$$\frac{1}{d}\frac{\partial N'}{\partial t} = x^2 - N'^2 \tag{118}$$

$$\frac{x\partial N}{x^2 - N'^2} = xd\partial t \tag{119}$$

$$N' = x\frac{Ce^{2xdt} - 1}{Ce^{2xdt} + 1} \tag{120}$$

Otherwise,

$$\beta\frac{\partial N}{\partial t} = N\left((cd + b\beta) - \beta dN - cd\left[\frac{\beta N - c}{dN + \gamma}\right]\right) \tag{121}$$

$$\beta(dN + \gamma)\frac{\partial N}{\partial t} = N((c^2 d + b\gamma\beta + cd\gamma) + (c^2 d + bd\beta - cd\beta - d\beta\gamma)N - d^2\beta N^2) \tag{122}$$

$$\beta\frac{dN}{N}\frac{\gamma + dN}{y^2 + xN - \beta^2 d^2 N^2} = dt \tag{123}$$

where,

$$x = c^2 d + bd\beta - cd\beta - d\beta\gamma \tag{124}$$

$$y^2 = c^2 d + b\gamma\beta + cd\gamma \tag{125}$$

$$\beta\frac{dN}{N}\frac{\gamma + dN}{r^2 - \beta^2 d^2 (N + s)^2} = dt \tag{126}$$

where,

$$s = -\frac{x}{2\beta^2 d^2} \tag{127}$$

$$r^2 = y^2 + \frac{x^2}{4\beta^2 d^2} \tag{128}$$

partial fractions(1),

$$\beta \frac{dN}{N}\left\{\frac{\gamma - ds}{2r}\left[\frac{1}{r - \beta d(N+s)} + \frac{1}{r + \beta d(N+s)}\right]\right. \tag{129}$$

$$\left. + \frac{d}{2}\left[\frac{1}{r - \beta d(N+s)} - \frac{1}{r + \beta d(N+s)}\right]\right\} = dt \tag{130}$$

$$\frac{\beta}{2r}\frac{dN}{N}\left\{\frac{\gamma - d(s-r)}{r - \beta d(N+s)} + \frac{\gamma - d(s+r)}{r + \beta d(N+s)}\right\} = dt \tag{131}$$

partial fractions(2),

$$\beta ddN\left\{\frac{\gamma + d(r-s)}{r - ds}\left[\frac{1}{\beta dN} + \frac{1}{r - \beta d(N+s)}\right] + \frac{\gamma - d(r+s)}{r + ds}\left[\frac{1}{\beta dN} - \frac{1}{r + \beta d(N+s)}\right]\right\} = 2rdt \tag{132}$$

$$\frac{\gamma + d(r-s)}{r - ds}\left[\log \beta dN - \log(r - \log \beta d(N+s))\right] \tag{133}$$

$$+ \frac{\gamma - d(r+s)}{r + ds}\left[\log \beta dN - \log(r + \beta d(N+s))\right] = 2rt + C \tag{134}$$

$$\frac{(\gamma - ds + dr)(r + ds)}{r^2 - d^2 s^2}\log\left(\frac{\beta dN}{r - \beta d(N+s)}\right) + \tag{135}$$

$$\frac{(\gamma - ds - dr)(r - ds)}{r^2 - d^2 s^2}\log\left(\frac{\beta dN}{r + \beta d(N+s)}\right) = 2rt + C \tag{136}$$

$$((\gamma - ds)r + d^2 rs)\log\left(\frac{N^2}{r'^2 - (N+s)^2}\right) + \tag{137}$$

$$(dr^2 + ds(\gamma - ds))\log\left(\frac{r' + (N+s)}{r' - (N+s)}\right) = 2r(r^2 - d^2 s^2)t + C \tag{138}$$

where

$$r' = \frac{r}{\beta d}. \tag{139}$$

Giving a threshold equation

$$\left(\frac{N^2}{r'^2 - (N+s)^2}\right)\left(\frac{r' + (N+s)}{r' - (N+s)}\right)^{\mu} = ce^{\nu t} \tag{140}$$

with,

$$\mu = \frac{dr^2 + ds(\gamma - ds)}{(\gamma - ds)r + d^2 rs} \tag{141}$$

$$\nu = \frac{2r(r^2 - d^2 s^2)}{((\gamma - ds)r + d^2 rs)} \tag{142}$$

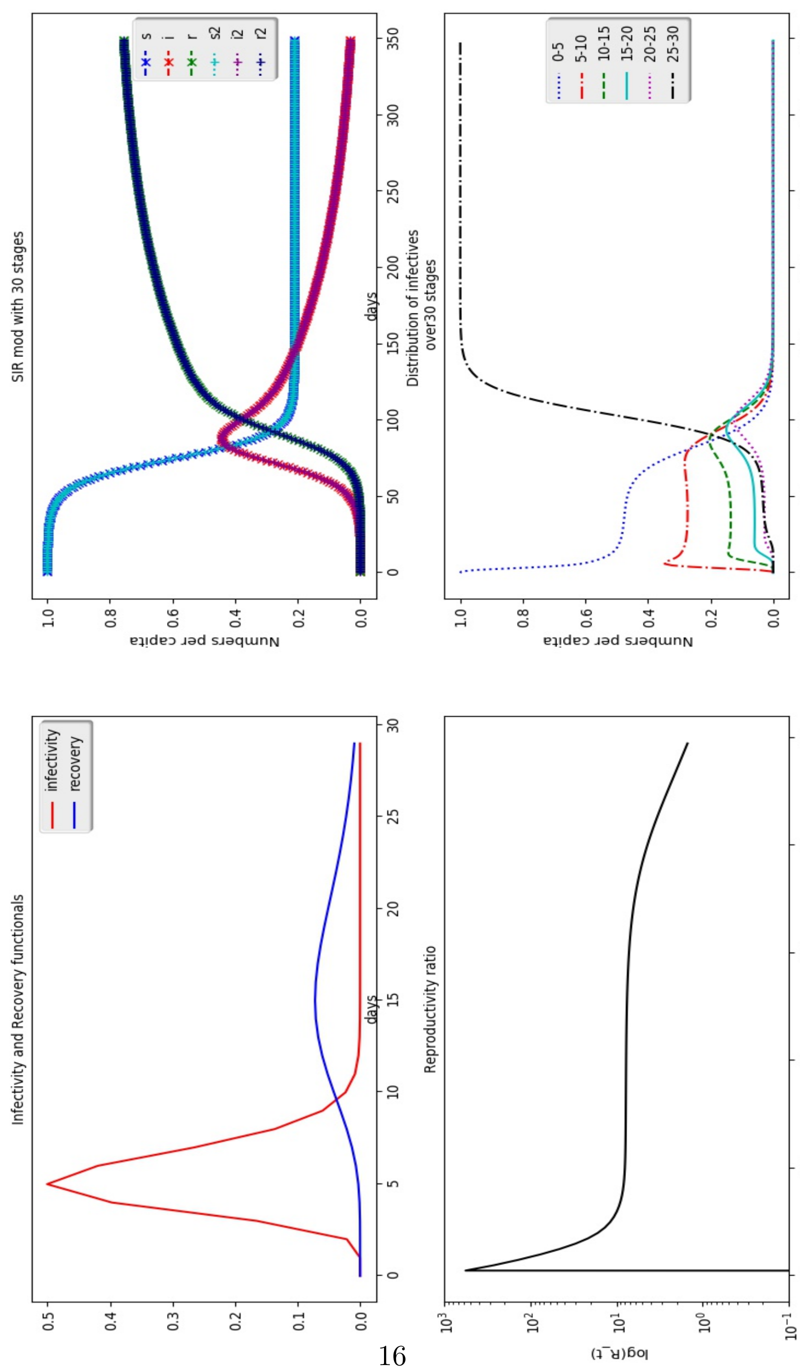

Figure 4: Illustration of SIR in nstages, gamma distribution is assumed for infectivity and recovery functions

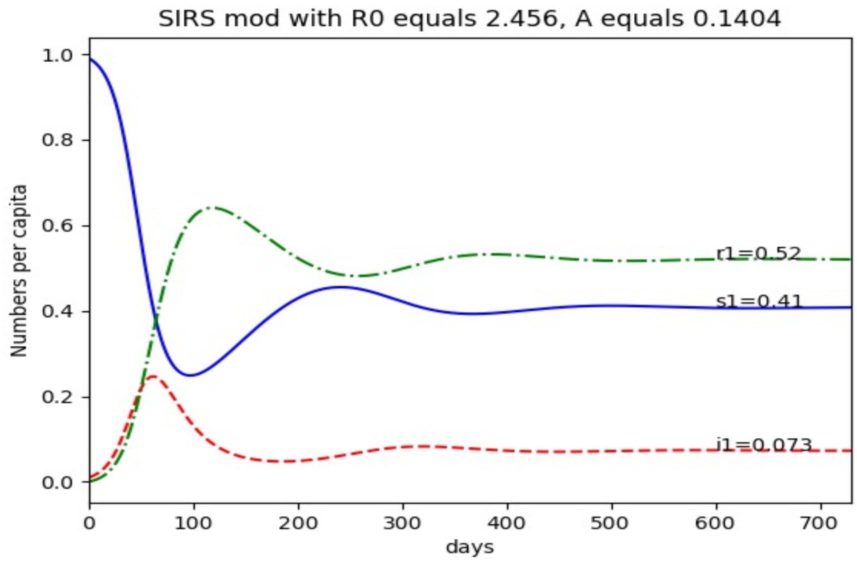

Figure 5: SIRS illustrating the thresholds given by equation 71 with $R0$ = 2.456

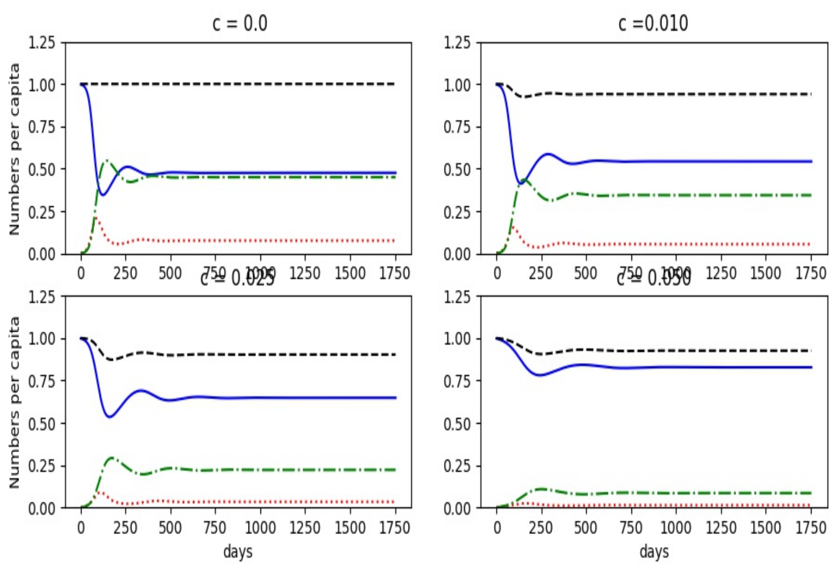

Figure 6: SIRN for varying disease death rate