Quantum algorithms 2022/2023: Final exam

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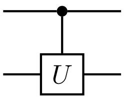
- Documents allowed: Slides of the lectures, documents of the exercices, hand-written notes
- You can only use your laptop to look at the documents from Moodle.
- You can also use printed versions of these documents.
- The use of smartphones or tablets is not allowed.

1 Warm-up on controlled-U operations

1. We consider a unitary matrix U that acts on m qubits. The controlled-U operation



acts on $1+m$ qubits, and is defined by the transformation

$$C[U] = |0\rangle\langle 0| \otimes \mathbf{1}_m + |1\rangle\langle 1| \otimes U. \tag{1}$$

Give an example of a controlled-U operation that was studied during the lecture for $m=1$, and write the corresponding output states $|\psi'\rangle$ for all the possible initial states $|\psi\rangle = |00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

Solution: The $CNOT$ is a controlled X operation acting $1+m=2$ qubits. The respective output states are $|00\rangle$, $|01\rangle$, $|11\rangle$, $|10\rangle$.

2. Let $|\psi\rangle = |x_1, y_1, \ldots, y_m\rangle$ an input state for the controlled-U operation. Show that

$$C[U]\,|\psi\rangle = |x_1\rangle \otimes U^{x_1}\,|y_1, \ldots, y_m\rangle\,. \tag{2}$$

Solution: If $x_1 = 0$, then

$$C[U]\,|\psi\rangle = |x_1, y_1, \ldots, y_m\rangle = |x_1\rangle \otimes U^0\,|y_1, \ldots, y_m\rangle\,. \tag{3}$$

else for $x_1 = 1$

$$C[U]\,|\psi\rangle = |x_1\rangle \otimes |y_1, \ldots, y_m\rangle = |x_1\rangle \otimes U^1\,|y_1, \ldots, y_m\rangle\,. \tag{4}$$

3. Write the state of the system $C[U]\,|\psi\rangle$ for input state of the form

$$|\psi\rangle = \sum_{x_1, y_1, \ldots, y_m} c_{x_1, y_1, \ldots, y_m}\,|x_1, y_1, \ldots, y_m\rangle \tag{5}$$

Solution: By linearity

$$C[U]\,|\psi\rangle = \sum_{x_1, y_1, \ldots, y_m} c_{x_1, y_1, \ldots, y_m} C[U]\,|x_1, y_1, \ldots, y_m\rangle = \sum_{x_1, y_1, \ldots, y_m} c_{x_1, y_1, \ldots, y_m}\,|x_1\rangle \otimes U^{x_1}\,|y_1, \ldots, y_m\rangle\,. \tag{6}$$

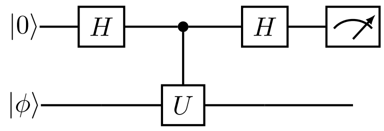
2 Quantum Phase Estimation with one qubit measurements

1. We consider a unitary matrix U acting on m qubits, and an eigenstate $|\phi\rangle$ of U. Show that we can define a real number $\delta \in [0, 1[$ such that

$$U\,|\phi\rangle = e^{2i\pi\delta}\,|\phi\rangle \tag{7}$$

Solution: $|\phi\rangle$ is an eigenstate $U\,|\phi\rangle = \epsilon\,|\phi\rangle$. $U^\dagger U = 1$ because U is unitary. Therefore, $\langle\phi|U^\dagger U|\phi\rangle = |\epsilon|^2 = 1$. This implies that, without loss of generality, I can define δ as above.

2. The goal of the quantum phase estimation algorithm (QPE) is to estimate the phase δ. We first consider a simplified version of the QPE using $1+m$ qubits. The first qubit is initialized in $|0\rangle$, the last m qubits are prepared in the eigenstate $|\phi\rangle$. Then, we apply the following circuit.



Write the wavefunction of the system after the first Hadamard gate

Solution:

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |\phi\rangle \tag{8}$$

3. Write the wavefunction of the system as a function of δ after the controlled-U operation.

4. Write the wavefunction of the system after the last Hadamard gate.

   Solution: After the C_U

   $$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle e^{2i\pi\delta}\right) |\phi\rangle \tag{9}$$

   After the last Hadamard

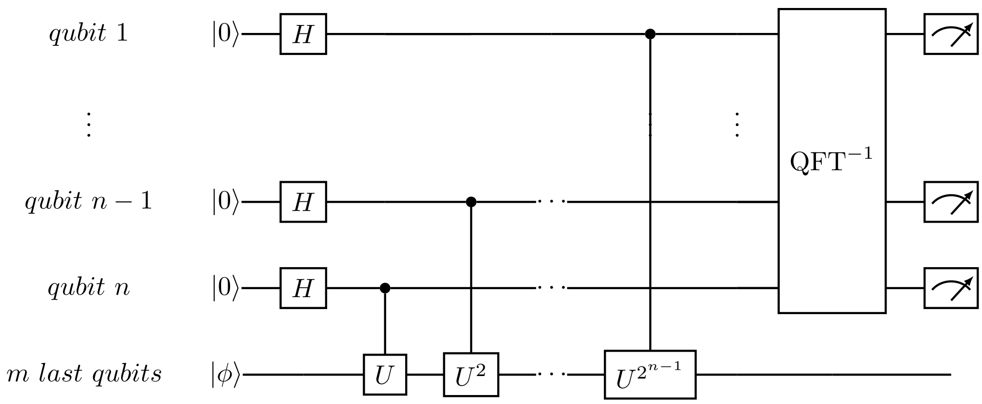
   $$|\psi\rangle = \frac{1}{2} \left(|0\rangle + |1\rangle + e^{2i\pi\delta}[|0\rangle - |1\rangle]\right) |\phi\rangle \tag{10}$$

5. We assume here that the phase can be either $\delta = 0$ or $\delta = 1/2$. Show that a single measurement of the first qubit allows us to extract δ with unit probability.

   Solution: We obtain the measurement probability $p(0) = 1$ if $\delta = 0$, and $p(1) = 1$ if $\delta = 1/2$. This means a single measurement is sufficient to reveal the phase.

3 Quantum phase estimation with n-qubit measurements

We consider the general QPE algorithm that uses a circuit of $n + m$ qubits. The choice of $n \geq 1$ controls the 'resolution' in determining δ, while the unitary U still acts on m qubits.



The first n qubits are first subject to a Hadamard gate, then each qubit $j = 1, \ldots, n$ controls a $U^{2^{n-j}}$ operation. At the end, the inverse quantum Fourier transform is applied before measurement. Again, note that the last m qubits are initialized in the eigenstate $|\phi\rangle$ of U.

1. Write the state of the system after the Hadamard gates. As in the lectures, we will use the notation $|x\rangle = |x_1, \ldots, x_n\rangle$, $x = \sum_{j=1}^{n} x_j 2^{n-j}$ to denote the 2^n basis states of the n-qubit system.

   Solution:

   $$|\psi\rangle = \frac{1}{\sqrt{2^n}} (|0\rangle + |1\rangle)^{\otimes n} |\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x_1,\ldots,x_n=0,1} |x_1, \ldots, x_n\rangle |\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |\phi\rangle \tag{11}$$

2. Consider now one of the controlled operations $C_j[U^{2^{n-j}}]$ that uses the qubit j as controlled qubit

   $$C_j[U^{2^{n-j}}] = |0\rangle_j \langle 0| \otimes \mathbf{1}_m + |1\rangle_j \langle 1| \otimes U^{2^{n-j}}. \tag{12}$$

   Show that the action on the state $|x\rangle |\phi\rangle$ can be written as

   $$C_j[U^{2^{n-j}}] |x\rangle |\phi\rangle = |x\rangle \otimes (U^{(2^{n-j})x_j} |\phi\rangle) \tag{13}$$

Solution: By definition

$$C_j[U^{2^{n-j}}] = |0\rangle_j \langle 0| \mathbf{1} + |1\rangle_j \langle 1| U^{2^{n-j}} \tag{14}$$

We obtain if $x_j = 0$

$$C_j[U^{2^{n-j}}] |x\rangle |\phi\rangle = |x\rangle |\phi\rangle \tag{15}$$

and if $x_j = 1$

$$C_j[U^{2^{n-j}}] |x\rangle |\phi\rangle = |x\rangle \otimes (U^{2^{n-j}} |\phi\rangle) \tag{16}$$

3. Write the state of the system after the last controlled operation $C_1[U^{2^{n-1}}]$, as a function of δ and x

   Solution: After the first controlled operation

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_x |x\rangle U^{x_n} |\phi\rangle \tag{17}$$

   After the last one

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2^n}} \sum_x |x\rangle \otimes (U^{x_1 2^{n-1} + \dots + x_n}) |\phi\rangle) = \frac{1}{\sqrt{2^n}} \sum_x e^{2i\pi\delta(x_1 2^{n-1} + \dots + x_n)} |x\rangle |\phi\rangle \\ &= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{2i\pi x\delta} |x\rangle |\phi\rangle \end{aligned} \tag{18}$$

4. We recall the expression of the inverse quantum Fourier transform operation

$$QFT^{-1} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2i\pi xy/2^n} |y\rangle \tag{19}$$

   Write the expression of the final state before measurement. You can define the amplitude c_y such that $|\psi\rangle = \sum_{y=0}^{2^n-1} c_y |y\rangle |\phi\rangle$.

   Solution: We obtain the final state before measurement

$$|\psi\rangle = \frac{1}{2^n} \sum_{x,y=0}^{2^n-1} e^{2i\pi x(\delta - y/2^n)} |y\rangle |\phi\rangle = \sum_{y=0}^{2^n-1} c_y |y\rangle |\phi\rangle \tag{20}$$

   with $c_y = 1/(2^n) \sum_{x=0}^{2^n-1} \alpha_y^x$, $\alpha_y = e^{2i\pi(\delta - y/2^n)}$.

5. We assume that the phase δ can be written as $\delta = s/2^n$, with $0 \leq s \leq 2^{n-1}$ an integer. What is the probability to reveal the correct phase from a single measurement?

   Solution: Consider the state $y = s$, we have $\alpha_s = 1$, $c_s = 1$ (which implies $c_{y \neq s} = 0$ as the state is normalized). Thus the probability to measure the right outcome $y = s$ is $P(y = s) = 1$ ($= 0$ for $y \neq s$). This means a single measurement gives us the correct result s.

6. We consider now the situation when $\delta 2^n$ is not an integer. Show that the probability to observe y can be written as

$$P(y) = \frac{1}{4^n} \frac{\sin^2(\pi 2^n(\delta - \tilde{y}))}{\sin^2(\pi(\delta - \tilde{y}))} \tag{21}$$

   with $\tilde{y} = y/2^n$.

   Solution: Using the expression of α_y, we have

$$\begin{aligned} c_y = &= \frac{1}{2^n} \frac{1 - \alpha_y^{2^n}}{1 - \alpha_y} \\ &= \frac{1}{2^n} \frac{\sin(\pi 2^n(\delta - \tilde{y}))}{\sin(\pi(\delta - \tilde{y}))} \end{aligned} \tag{22}$$

   and therefore

$$P(y) = |c_y|^2 = \dots \tag{23}$$

7. Describe qualitatively the shape of $P(y)$, and explain why n controls the accuracy of the estimation of the phase δ.

   Solution: The function P is maximal when $\tilde{y} \approx \delta$. As n increases, the width of the peak narrows, i.e we can approximate δ better and better (The statement can be made more quantitative, for instance by Taylor expanding $P(y)$ around δ).