Quantum Algorithms 2021/2022: Exercices 5

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1 Density matrix and quantum state tomography

The density matrix ρ summarizes all the physical properties of a quantum S. For a system S embedded in an environment E, it is defined as

$$\rho = \mathrm{Tr}_E \left|\psi_{SE}\right\rangle \left\langle\psi_{SE}\right|, \tag{1}$$

where Tr_E is the trace over the environment, defined as $\mathrm{Tr}_E(.) = \sum_{i_E} \langle i_E| \,.\, |i_E\rangle$, and where $|\psi_{SE}\rangle$ is the combined state of the system and environnement.

1. Calculate ρ when the system is decoupled from the environnement, i.e., $|\psi_{SE}\rangle = |\psi_S\rangle \otimes |\psi_E\rangle$. Describe the physical meaning of this situation when S is a quantum computer.

   Solution:

$$\rho = \sum_{i_E} \langle i_E| \left(|\psi_S\rangle \otimes |\psi_E\rangle\right)\left(\langle\psi_S| \otimes \langle\psi_E|\right) |i_E\rangle = |\psi_S\rangle\langle\psi_S| \sum_{i_E} \langle i_E| \,|\psi_E\rangle\langle\psi_E|\, |i_E\rangle = |\psi_S\rangle\langle\psi_S| \tag{2}$$

   In this case, the system is isolated from its environment. This is the ideal scenario for a quantum computer: Quantum algorithm create a state $|\psi_S\rangle$, before the influence of the enviroment, i.e. errors, starts playing a role.

2. Let us define an observable O acting on the system, i.e $O = O_S \otimes 1$. Write the expression of the expectation value $\langle O\rangle$ as a function of ρ.

   Solution:

$$\langle O\rangle = \langle\psi_{SE}| O |\psi_{SE}\rangle = \mathrm{Tr}_{SE}(O |\psi_{SE}\rangle\langle\psi_{SE}|) \tag{3}$$

   as we can always perform the trace in a basis involving $|\psi_{SE}\rangle$

$$\langle O\rangle = \langle\psi_{SE}| O |\psi_{SE}\rangle = \mathrm{Tr}_S(\mathrm{Tr}_E((O_S \otimes 1) |\psi_{SE}\rangle\langle\psi_{SE}|)) = \mathrm{Tr}_S(O_S\rho) \tag{4}$$

   where we have used

$$\mathrm{Tr}_E((A \otimes 1)C) = \sum_{i_E} \langle i_E| (A \otimes 1)C |i_E\rangle = \sum_{i_E} A \langle i_E| C |i_E\rangle = A\mathrm{Tr}_E(C) \tag{5}$$

3. Write the evolution of a density matrix via a unitary operation, i.e gate, U?

   Solution:

$$|\psi_{SE}\rangle' = (U \otimes 1) |\psi_{SE}\rangle \tag{6}$$

$$\rho' = \mathrm{Tr}_E[(U \otimes 1) |\psi_{SE}\rangle\langle\psi_{SE}| (U^\dagger \otimes 1)] = U\rho U^\dagger \tag{7}$$

4. Quantum state tomography describes a protocol to measure the matrix ρ in a quantum computer. It is based on decomposing ρ is a basis of Pauli strings.

$$\rho = \sum_\sigma c_\sigma \sigma \tag{8}$$

   with $\sigma = \bigotimes_i \sigma_i$, $\sigma_i = 1_i, X_i, Y_i, Z_i$. Write the expression of c_σ as a function of ρ and σ.

   Solution:

$$\mathrm{Tr}(\rho\sigma) = \sum_{\sigma'} \mathrm{Tr}(c'_\sigma \sigma\sigma') = c_\sigma 2^N \tag{9}$$

5. Write a quantum circuit to measure c_σ. We recall the identities $X = HZH$, $Y = SXS^\dagger = SHZHS^\dagger$. **Solution:** Based on the identities above, we have

$$\mathrm{Tr}(\rho U^\dagger \sigma_Z U) = \mathrm{Tr}(U\rho U^\dagger \sigma_Z) \tag{10}$$

   We then have to measure the multi-qubit operators σ_Z, which involve only Z or 1 operators, i.e which is diagonal in the computational basis. This measurement is performed after application of the gate U which transform the X and Y of the Pauli string σ into Z.