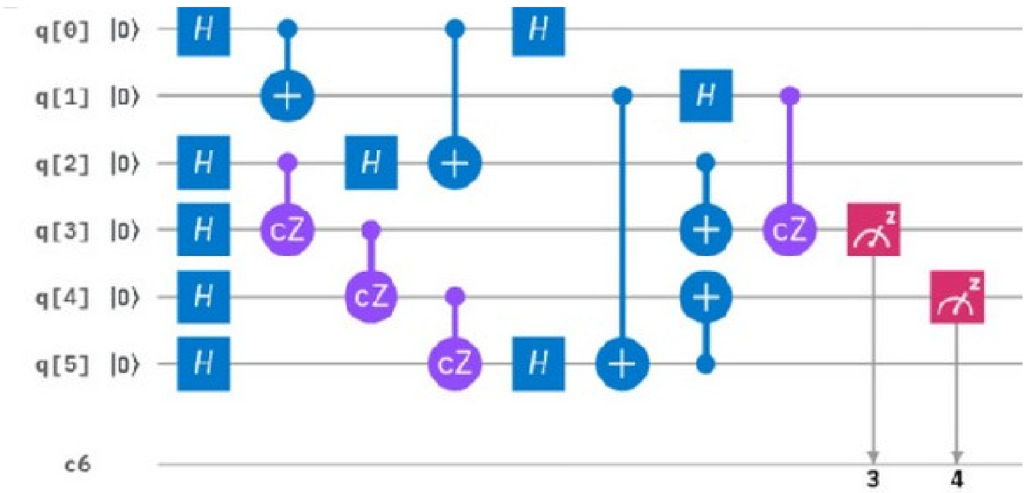
Lecture 2

Quantum algorithms in the quantum circuit model



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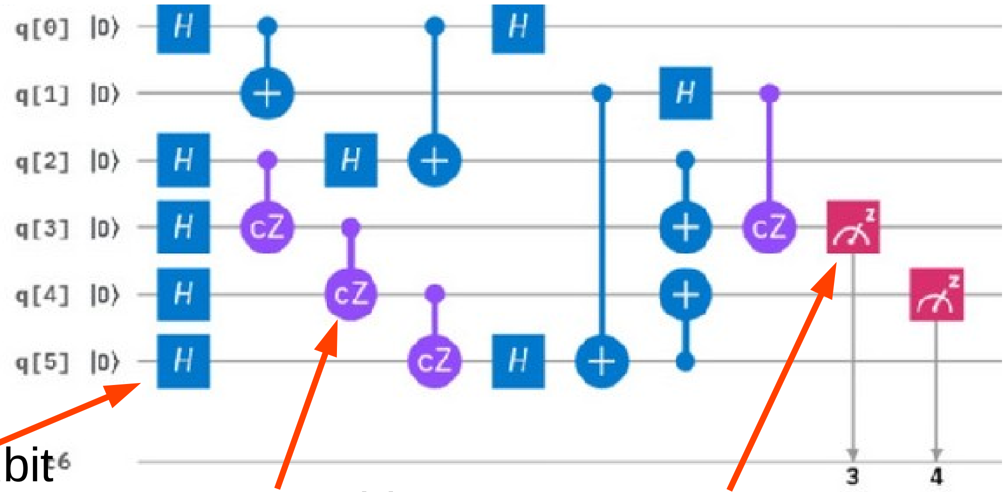
Reminder : A quantum circuit

Qubits



Single qubit Gate

Two-qubit gate

Measurement

Goal 1: Having algorithms that are faster (less operations) than classical algorithms

Goal 2: Having algorithms that are protected against errors (Lecture 3)

Warm-up : Deutsch's algorithm

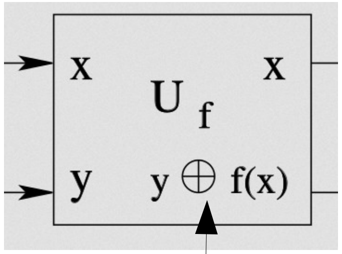
Problem: Given binary function f : [0,1]→ [0,1]. Is f(0)=f(1)?

Classical solution:

Two iterations needed (Iteration 1, I measure f(0). Iteration 2, I measure f(1))

Quantum solution: we will test the two input states simultaneously

Function f implemented via a two-qubit 'quantum oracle'

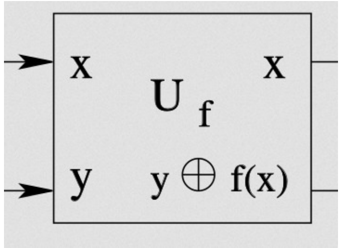


ie y is flipped iif f(x) is 1

XOR (sum mod 2)

Warm-up : Deutsch’s algorithm

Remark: How to implement a quantum oracle?

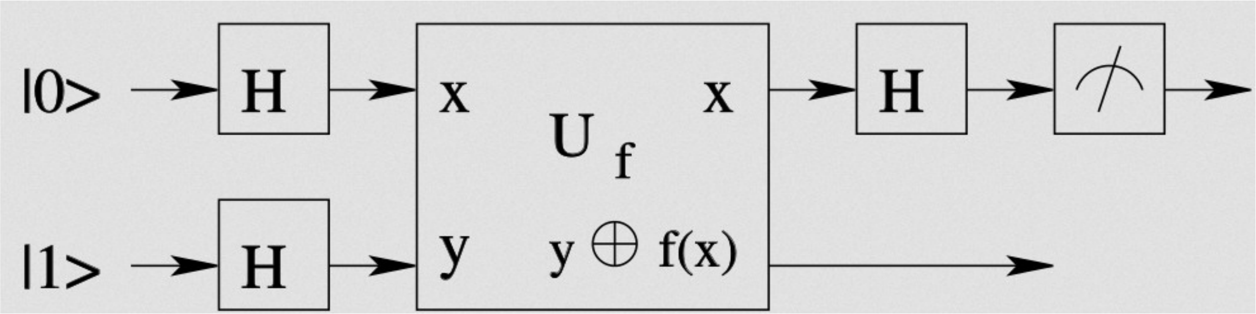


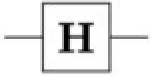
- In **quantum query complexity**, one assumes the oracle given and counts the number of oracle queries to define the complexity

- In practice, if the function can be computed classically via a reversible circuit, we can map the circuit to a quantum circuit, using the technique of `uncomputation’.

- In the rest of this lecture, we won’t bother anymore about oracles. However, this does not mean this questions is not important.

Warm-up : Deutsch's algorithm

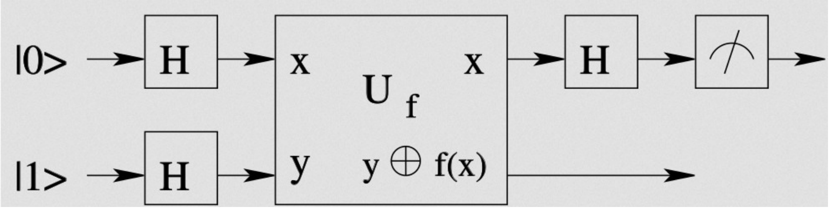


Hadamard (H) 

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

What do I measure for f(0)=f(1), for f(0) != f(1) ? (using a single measurement!)

Warm-up : Deutsch's algorithm



$$|\psi\rangle = (|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$$

$$|\psi\rangle' = |0, 0 \oplus f(0)\rangle - |0, 1 \oplus f(0)\rangle + |1, 0 \oplus f(1)\rangle - |1, 1 \oplus f(1)\rangle$$

If $f(0) = f(1)$, let $0 \oplus f(0) = 0 \oplus f(1) = a$, $1 \oplus f(0) = 1 \oplus f(1) = b$

$$|\psi\rangle' = (|0\rangle + |1\rangle)(|a\rangle - |b\rangle)$$

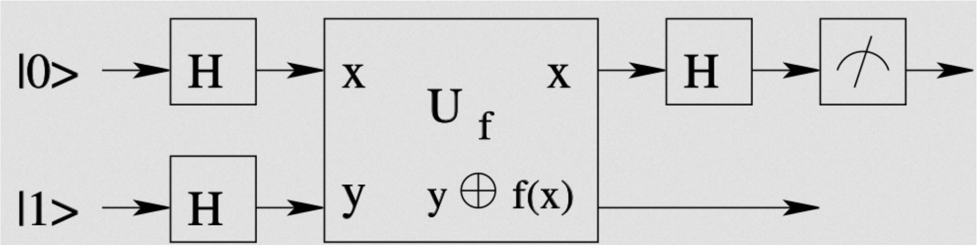
Else, if $f(0) \neq f(1)$, let $0 \oplus f(0) = 1 \oplus f(1) = a$, $1 \oplus f(0) = 0 \oplus f(1) = b$

$$|\psi\rangle' = (|0\rangle - |1\rangle)(|a\rangle - |b\rangle)$$

After the last Hadamard,

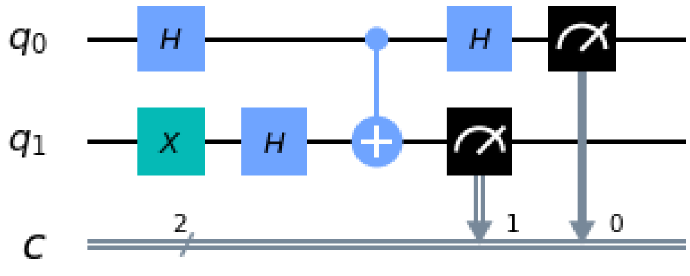
$$\begin{aligned} |\psi\rangle' &= |0\rangle\,(|a\rangle - |b\rangle) \quad , f(0) = f(1) \\ |\psi\rangle' &= |1\rangle\,(|a\rangle - |b\rangle) \quad , f(0) \neq f(1) \end{aligned}$$

Warm-up : Deutsch's algorithm

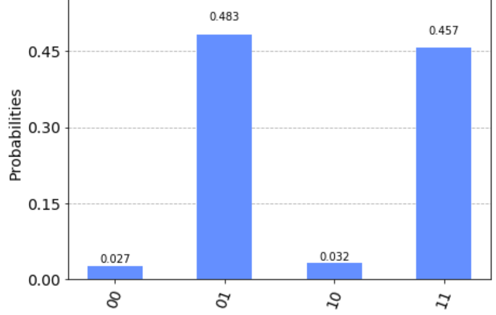


Implementation with IBM Qiskit

Suppose f(x)=x. Then the oracle becomes a CNOT gate.

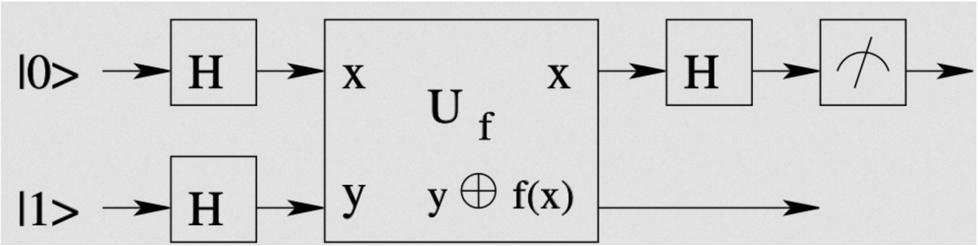


Demo with IBMQ



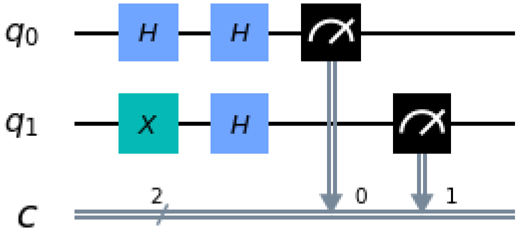
Up to errors, the first qubit ends up in |1> !

Warm-up : Deutsch's algorithm

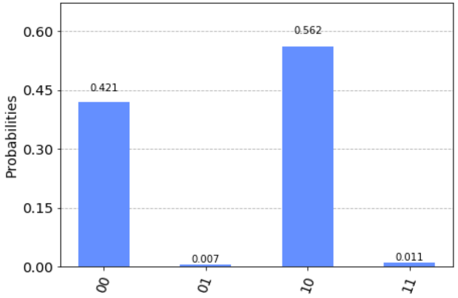


Implementation with IBM Qiskit

Suppose f(x)=0. Then the oracle becomes the identity



Demo with IBMQ



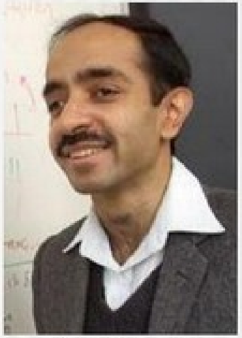
Up to errors, the first qubit ends up in |0> !

Warm-up : Deutsch's algorithm

Conclusion : First algorithm that outperfoms classical algorithms using quantum parallelism.

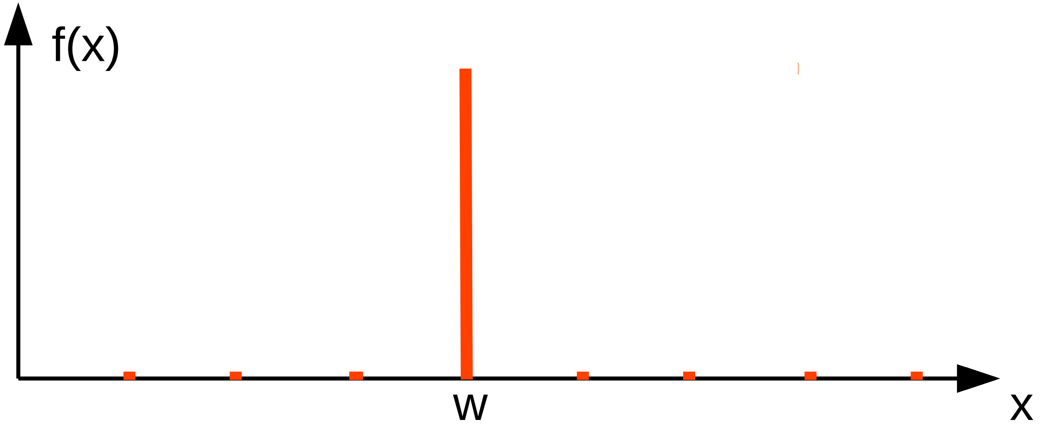
Generalizes to n qubits : Deutsch-Josza algorithm

Grover’s algorithm (1996)



Lov Grover

Problem (Data search): Given binay function with f(w)=1 for a single n-bit string w
($N=2^n$ is the number of configurations), find w



Application: Database search (applications: SAT problems (circuit design, automatic theorem proving, etc..)

Grover’s algorithm (1996)

Classical solution : $O(N=2^n)$ function evaluation

Quantum Grover’s algorithm : Simultaneous testing via quantum parallelism

Grover’s algorithm (1996)

Grover’s oracle : $$U_f = I - 2\left|w\right\rangle\left\langle w\right|$$

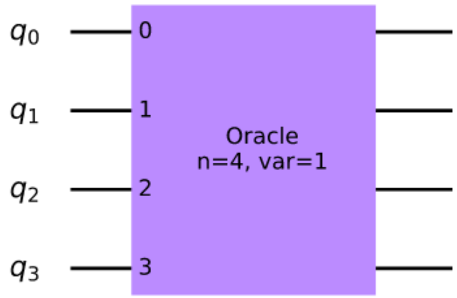
The oracle ‘marks’ the solution:

$$U_f\left|x \neq w\right\rangle = \left|x\right\rangle$$

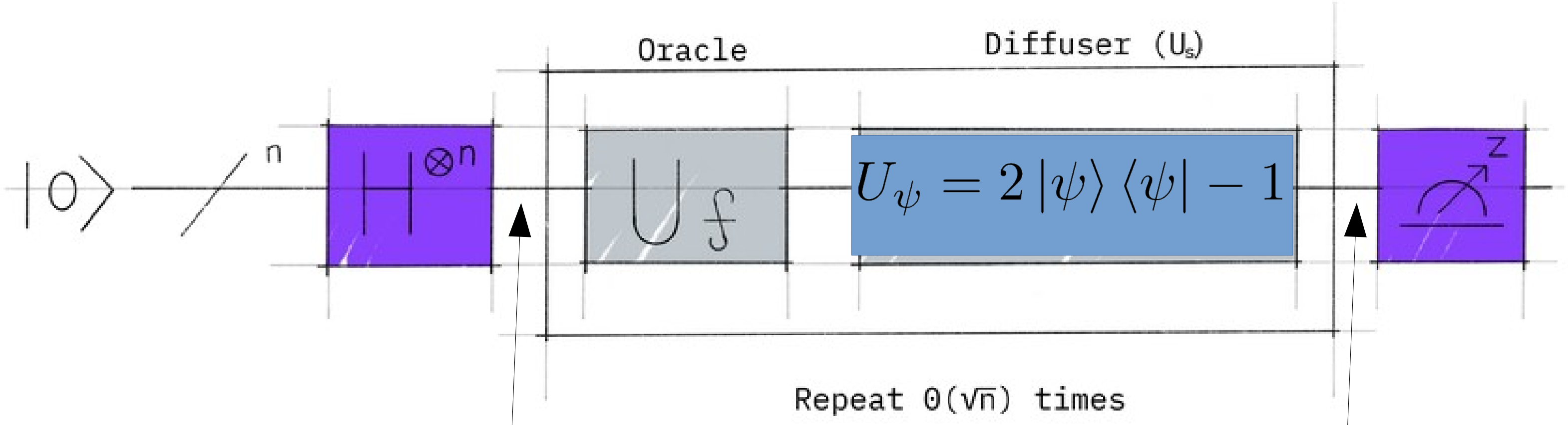
$$U_f\left|w\right\rangle = -\left|w\right\rangle$$

Ex: Qiskit’s implementation
(the details are not our concern for an oracle..)

```
n = 4
oracle = grover_problem_oracle(n, variant=1)
```



Grover's algorithm (1996)

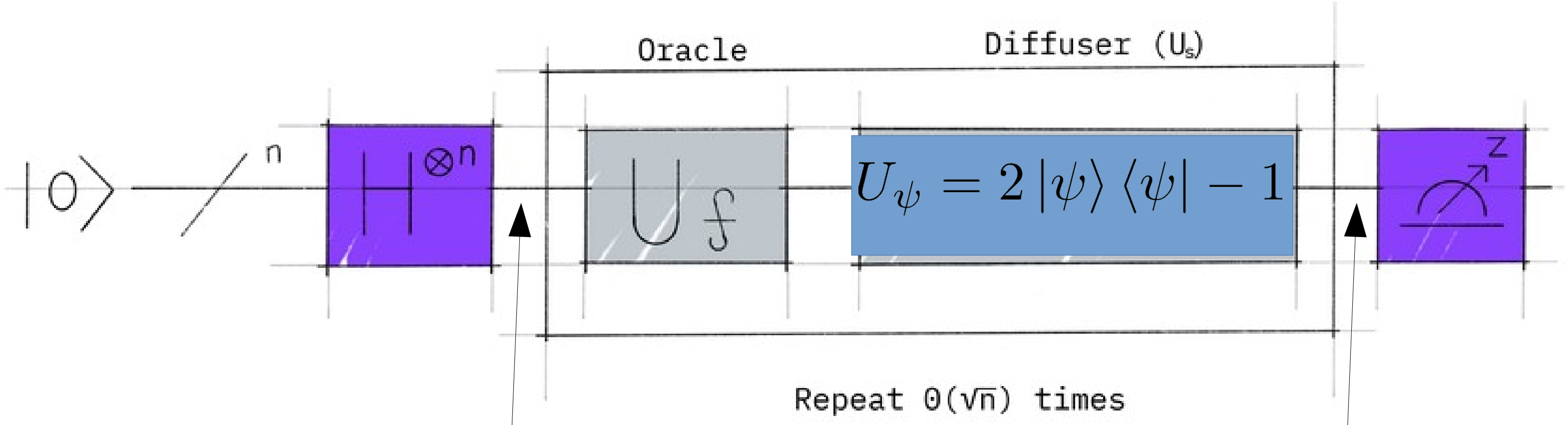


$$U_\psi = 2\left|\psi\right\rangle\left\langle\psi\right| - 1$$

$$\left|\psi\right\rangle = \frac{1}{\sqrt{N}}\sum_x \left|x\right\rangle$$

$$\left|\psi_t\right\rangle = (U_\psi U_f)^t \left|\psi\right\rangle$$

Grover's algorithm (1996)



$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle$$

$$|\psi_t\rangle = (U_\psi U_f)^t |\psi\rangle$$

Grover's algorithm (1996)

After the first Hadamards:

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_x |x\rangle = \underbrace{\sqrt{\frac{N-1}{N}}}_{\cos(\theta/2)}|\alpha\rangle + \underbrace{\sqrt{\frac{1}{N}}}_{\sin(\theta/2)}|w\rangle$$

$$|\alpha\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq\omega}|x\rangle$$

Oracle

$$U_f = I - 2|w\rangle\langle w|$$

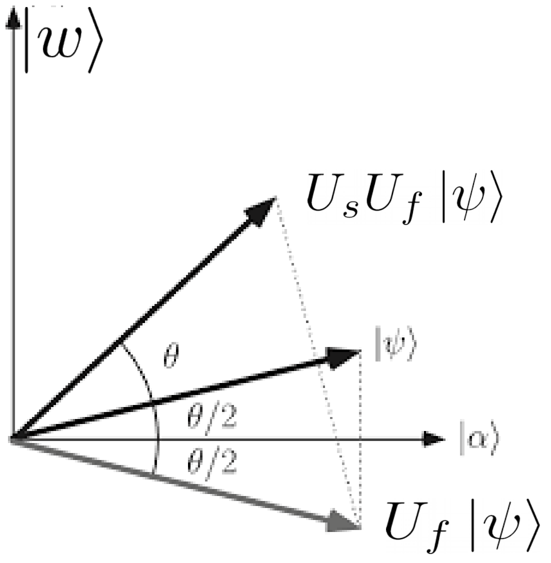
$$U_f|\psi\rangle = \cos(\theta/2)|\alpha\rangle - \sin(\theta/2)|w\rangle$$

Reflection versus $|\alpha\rangle$

Diffuser

$$U_\psi = 2|\psi\rangle\langle\psi| - 1$$

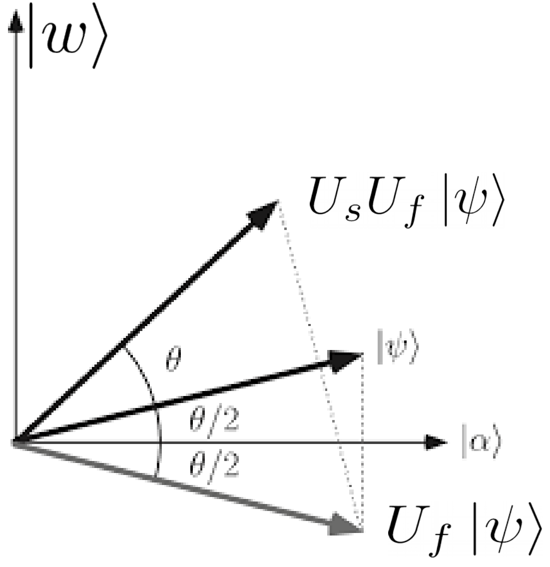
Reflection versus $|\psi\rangle$



Grover's algorithm (1996)

After one Grover iteration (c.f TD)

$$|\psi_1\rangle = U_s U_f |\psi\rangle = \cos(3\theta/2)|\alpha\rangle + \sin(3\theta/2)|w\rangle$$

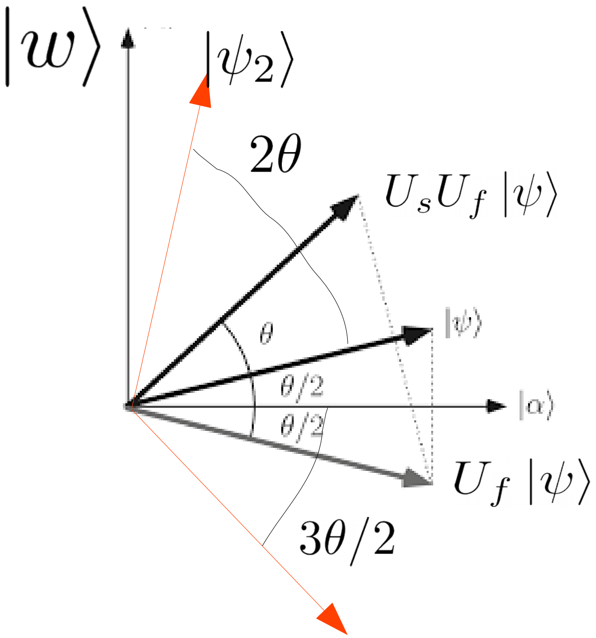


The algorithm brings the quantum state towards the solution w

Grover's algorithm (1996)

Performance

After t iterations $$|\psi_t\rangle = (U_\psi U_f)^t\,|\psi\rangle = \cos[(2t+1)\theta/2]\,|\alpha\rangle + \sin[(2t+1)\theta/2]\,|w\rangle$$



Solution obtained for:

$$\theta t \approx \pi/2 \longrightarrow t \approx (\pi/4)\sqrt{N}$$

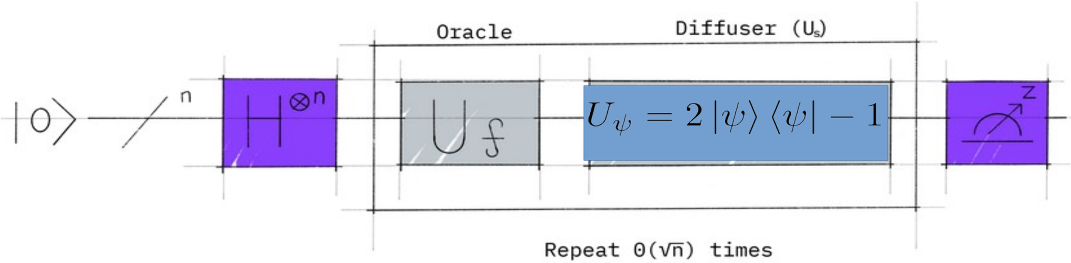
Quadratic speedup !

Ex : 128-bit key 2^{64} iterations instead of 2^{128}

Note: for multiple targets → $t \approx (\pi/4)\sqrt{N/k}$

Grover's algorithm (1996)

Implementation

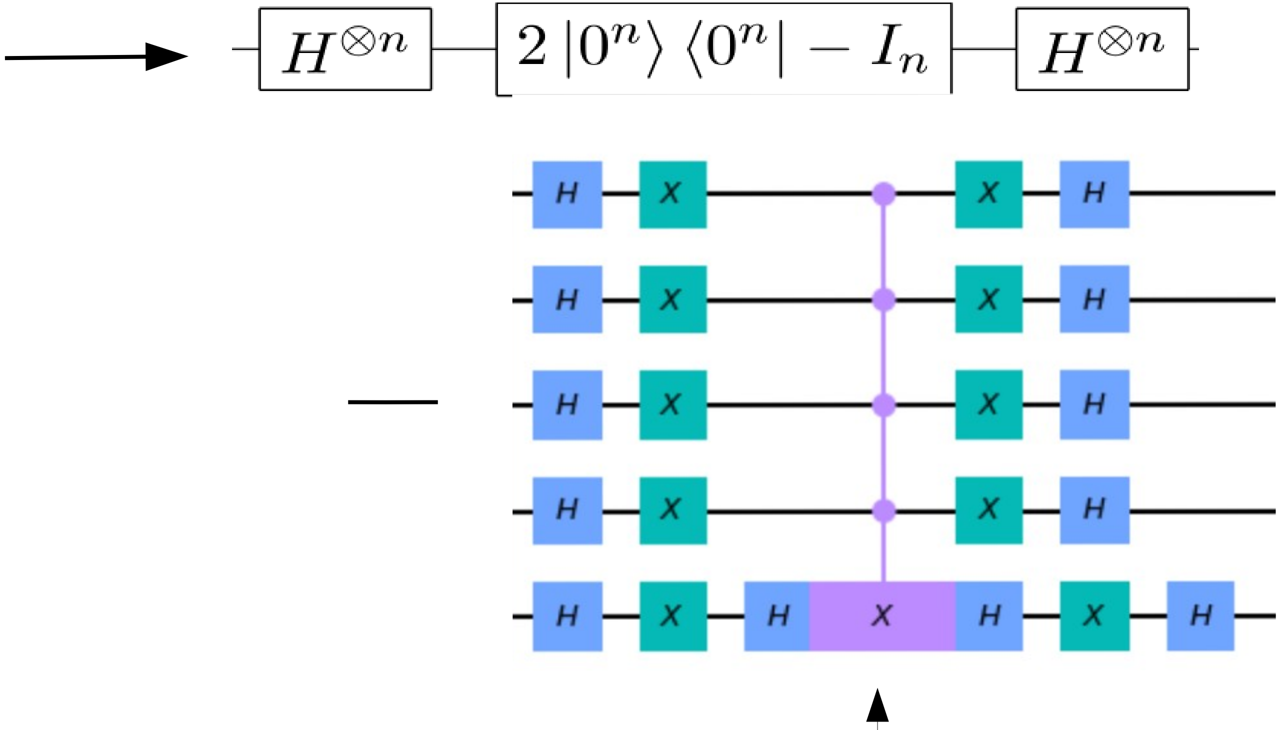


Efficient algorithm for Grover's diffuser (cf TD)

$$U_\psi = 2|\psi\rangle\langle\psi| - 1 \qquad |\psi\rangle = H^{\otimes n}|000\ldots0\rangle$$

$$\longrightarrow \quad \boxed{H^{\otimes n}} - \boxed{2|0^n\rangle\langle 0^n| - I_n} - \boxed{H^{\otimes n}}$$

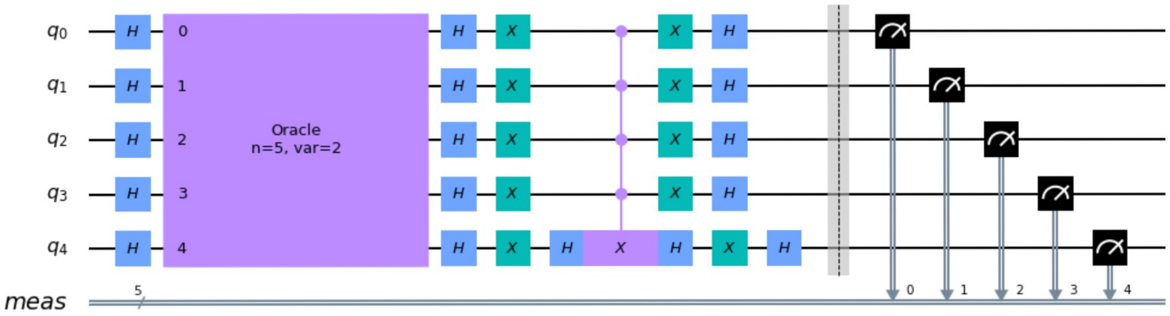
Grover’s algorithm (1996)

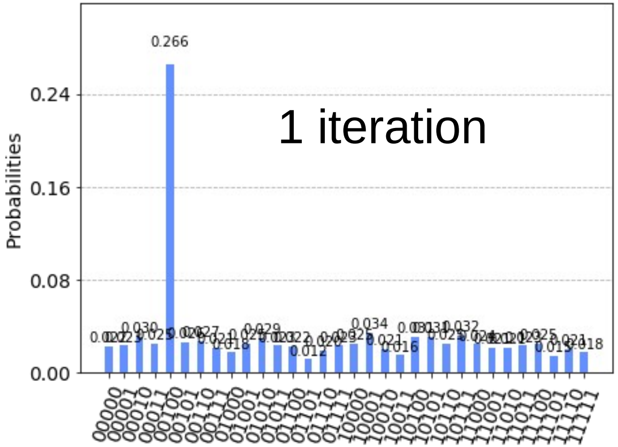


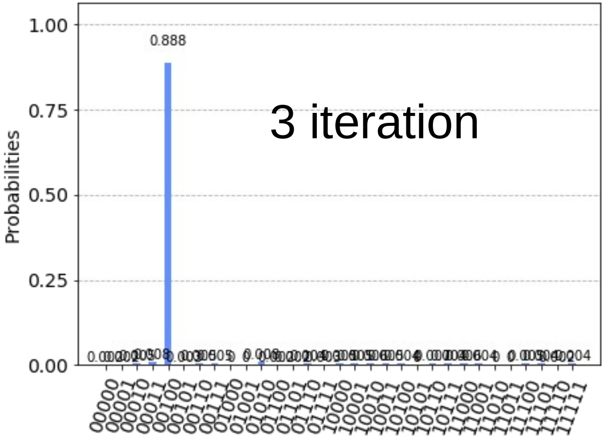
N qubit Toffoli gate

Grover’s algorithm (1996)

Illustration with Qiskit’s Aer simulator







Grover’s algorithm (1996)

Remark: Is the complexity of Grover’s algorithm a good news for quantum computers?

If an `improved Grover’s’ algorithm would provide **an exponential speedup** (I.e scaling polynomially with the number of qubits),

Then I could solve any NP problem in polynomial time on a quantum machine!!

- Take a NP problems with 2^n possible solutions
- Each solution can be tested in polynomial time (NP property) → I can define an oracle function f
- Use the oracle in Grover’s algorithm → I could find the solution in polynomial time

Unfortunately, the current Grover’s algorithm with only quadratic speedup has been shown to be optimal. $\sqrt{N = 2^n}$

Is there an algorithm that can solve a specific NP problem in polynomial time?

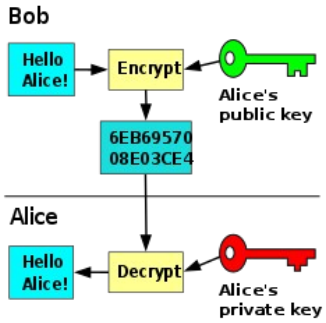
Shor's algorithm (1995)

Factorization Problem of a number N

Classical algorithm: (sub-)exponential in n (number of bits to represent n)

Quantum algorithm: polynomial in n: **Exponential speedup**

→ A potential threat to RSA cryptography...



Shor's algorithm (1995)

Prerequisites from arithmetic:

If N, product of two coprimes, divides b^2-1 ,
Then gcd(N,b-1) and gcd(N,b+1) are non-trivial factors of N

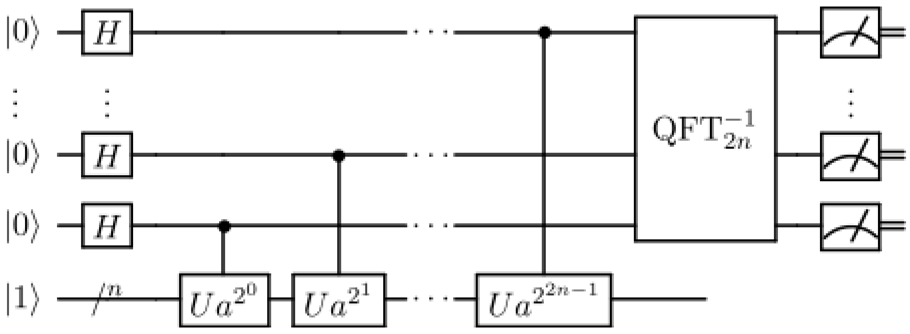
Proof: see e.g., Nielsen and Chuang

Example**:** N=91. For b=64. N divides b^2-1=4095.
Therefore, gcd(91,63)=7 and gcd(91,65)=13 divide 91

Algorithm

- Take a random in [1,N]
- Find r such that **a^r=1 mod (N)** by finding the period of f(x) = a^x mod(N)
  Then N divides a^r-1
- If r is even, b=$a^{r/2}$, and, N divides b^2-1

Shor’s algorithm (1995)



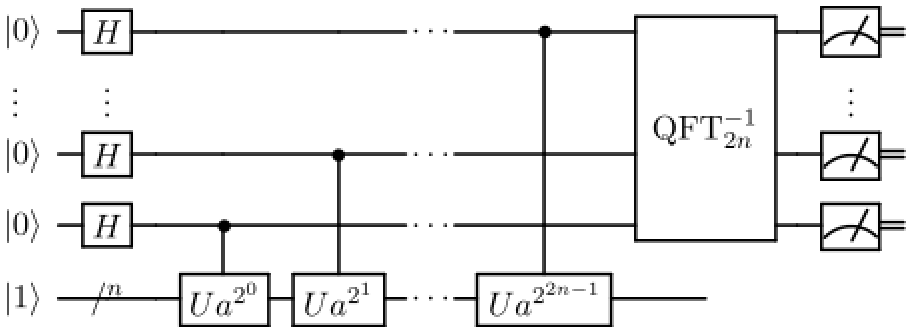
Quantum subroutine : find the period r of f(x) = a^x mod(N)

- Choose q so that $Q=2^q > N^2$ and consider a 2q qubit quantum computer (to provide sufficient spectral resolution in finding r)
- Prepare the first q qubits in a superposition state
- Apply **modular exponentiation**
- Apply **the inverse quantum Fourier transform** on the first q qubits

$$|x\rangle\,|1\rangle^{\otimes N} \rightarrow |x\rangle \otimes |a^x \mathrm{mod}(N)\rangle$$

$$|\psi\rangle = \frac{1}{Q}\sum_x \Big(\sum_y e^{2i\pi xy/Q}\,|y\rangle\Big) \otimes |f(x)\rangle$$

Shor's algorithm (1995)



Measurement:

$$|\psi\rangle = \frac{1}{Q}\sum_x\Big(\sum_y e^{2i\pi xy/Q}|y\rangle\Big)\otimes|f(x)\rangle \longrightarrow |\psi\rangle = \frac{1}{Q}\sum_y\left(|y\rangle\otimes\sum_x e^{2i\pi xy/Q}|f(x)\rangle\right) \qquad [f(x) = a^x \bmod(N)]$$

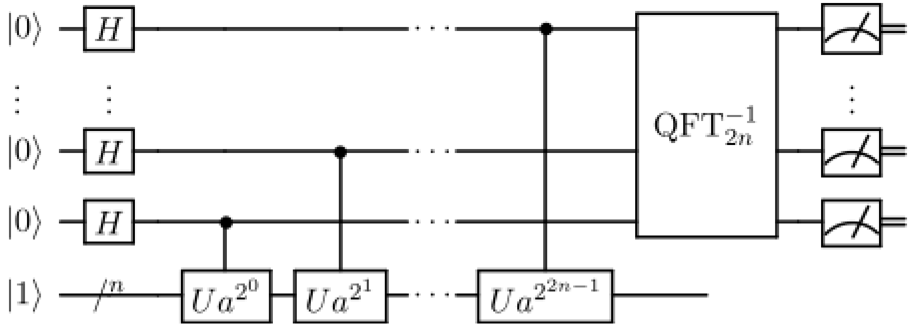
$$P(y) = \frac{1}{Q^2}\sum_{x,x'} e^{2i\pi(x'-x)y/Q}\langle f(x)\,|\,f(x')\rangle \qquad P(y) \approx \sum_{n,x-x'=nr} e^{2i\pi nry/Q}$$

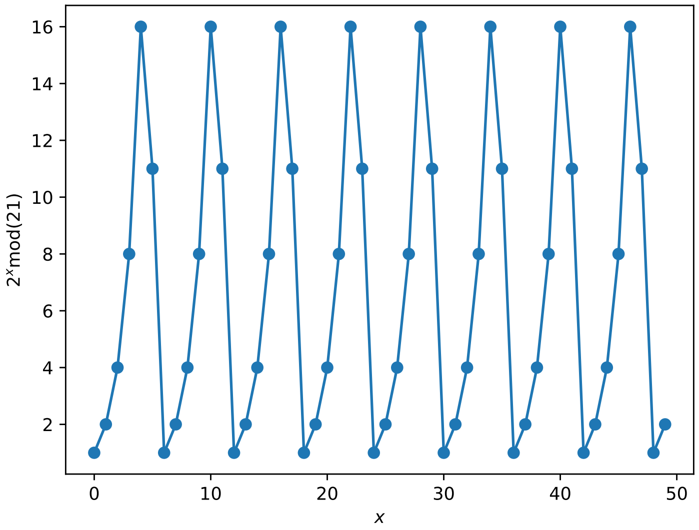
Maximum for yr/Q integer (as a constructive interference in optics)

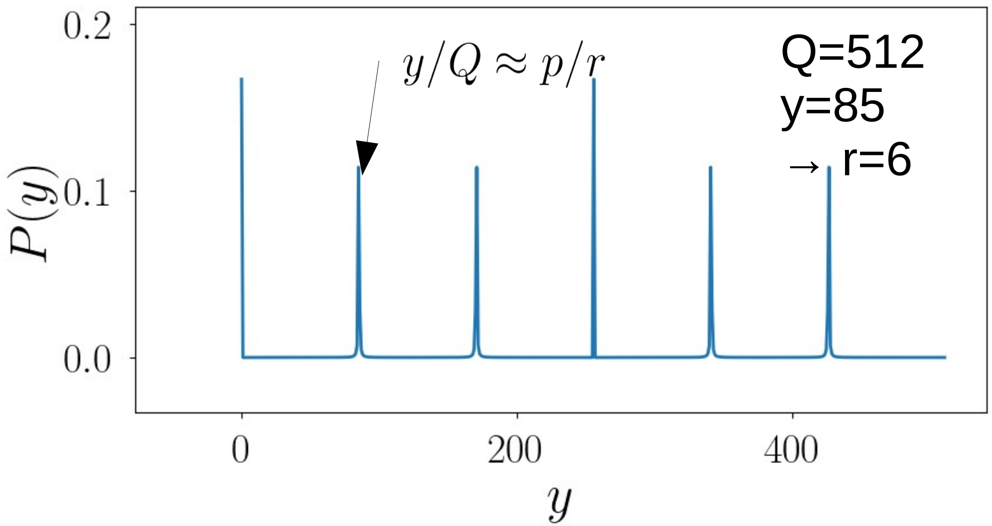
→ r can be extracted (via continued fraction algorithms, see Nielsen)

Shor’s algorithm (1995)

Example: Factorizing 21 with a=2 (TD2)

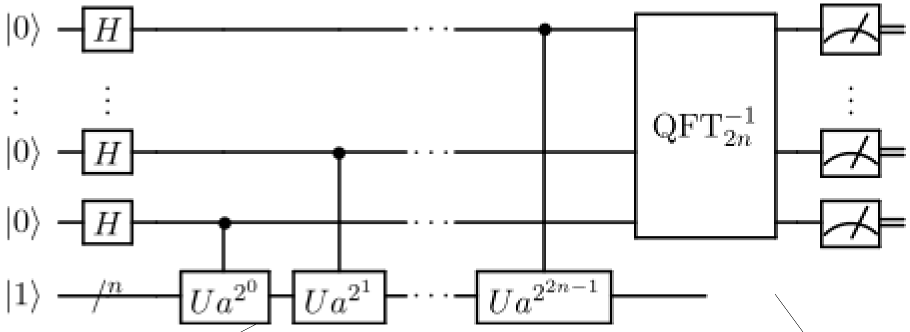






Shor's algorithm (1995)

Implementation aspects



Modular exponentiation
(multiplication in the ancilla space)

Cost $O(n^3)$

Quantum Fourier Transform

Cost $O(n^2)$ (see TD)

The practical implementation of Shor's algorithm is difficult: Many qubits and many gates..