Quantum algorithms

Lecture 4: Quantum Error Correction

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Outline

What is an error in quantum computing?

The bit flip code

The Steane code

The surface code and fault-tolerance

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An error in a quantum computer?

- Example: Spontaneous emission with an atomic qubit $|\psi\rangle = |1\rangle$

$$|1\rangle \rightarrow \sqrt{1-p}\,|1\rangle\,|0\rangle_{\mathrm{photon}} + \sqrt{p}\,|0\rangle\,|1\rangle_{\mathrm{photon}} \tag{1}$$

- Spontaneous emission process corresponds to a ‘bitflip error’ $|\psi\rangle \rightarrow X\,|\psi\rangle$

$$|\psi\rangle \rightarrow |\psi\rangle\,|E\rangle_I + X\,|\psi\rangle\,|E\rangle_X \tag{2}$$

An error in a quantum computer?

- For a general qubit state $|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle)$, a decoherence process can always be interpretated as a sum of 'Pauli Errors':

$$|\psi\rangle \rightarrow |\psi\rangle |E\rangle_I + X|\psi\rangle |E\rangle_X + Y|\psi\rangle |E\rangle_Y + Z|\psi\rangle |E\rangle_Z \tag{3}$$

- Quantum error correction: How to detect an error without destroying the quantum superposition?

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The bit flip code

- Our first code: The bit flip code

$$|\psi\rangle = \alpha |0\rangle_L + \beta |1\rangle_L \tag{4}$$

with a logical qubit that is made of three physical qubits

$$|0\rangle_L = |000\rangle \quad |1\rangle_L = |111\rangle \tag{5}$$

- The code aims at tracking and correcting X errors occurring on one of the three physical qubits

$$|\psi\rangle \rightarrow |\psi\rangle |E\rangle_I + \sum_{i=1,2,3} X_i |\psi\rangle |E\rangle_{X_i} \rightarrow_{\text{QEC}} \quad |\psi\rangle \tag{6}$$

The bit flip code

- There are two mesurements to be made $\langle Z_1 Z_2 \rangle$, $\langle Z_2 Z_3 \rangle$, giving rise to unique error syndromes, independently of the qubit superposition state.

Error	State	$\langle Z_1 Z_2 \rangle$, $\langle Z_2 Z_3 \rangle$
none	$\alpha \|000\rangle + \beta \|111\rangle$	1,1
X_1	$\alpha \|100\rangle + \beta \|011\rangle$	-1,1
X_2	$\alpha \|010\rangle + \beta \|101\rangle$	-1,-1
X_3	$\alpha \|001\rangle + \beta \|110\rangle$	1,-1

- Code distance: Number of errors that map one logical state to the other. Here it's $d = 3$. For a general d, we can correct t errors if $d \geq 2t + 1$.
- How to measure and correct errors?

The bit flip code: Collective measurements

- We require a collective measurement of $\langle Z_1 Z_2 \rangle$ with two measurement outcomes (eigenvalues) $\epsilon = \pm 1$:

$$Z_1 Z_2 = \underbrace{|00\rangle\langle 00| + |11\rangle\langle 11|}_{P_1} - \underbrace{(|01\rangle\langle 01| + |10\rangle\langle 10|)}_{P_{-1}}$$

- A measurement on $|\psi'\rangle$ gives a mesurement outcome ϵ and a projection

$$|\psi'\rangle \rightarrow P_\epsilon |\psi'\rangle \text{ with probability } \langle\psi|P_\epsilon|\psi\rangle$$

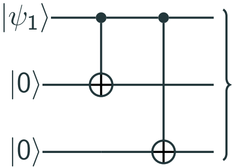
- If $|\psi'\rangle$ is proportional to $|\psi\rangle, X_1|\psi\rangle, X_2|\psi\rangle$, we obtain a deterministic measurement $\epsilon = 1$, or $\epsilon = -1$, and the state is unchanged.
- For a quantum superposition of errors, the outcome is probabilitic, but the post-measured state is compatible with such outcome.

The bit flip code: Collective measurements

- Example with a bitflip process on qubit 3, $|\psi'\rangle = X_3 |\psi\rangle |E\rangle_X$
- Measurement of $Z_1 Z_2$: Projection onto the same state with $|\psi'\rangle$ with outcome 1.
- Measurement of $Z_2 Z_3$: I measure outcome -1. I can perform error recovery by applying X_3 and obtain $X_3^2 |\psi\rangle = |\psi\rangle$
- Note: With a ‘naive’ non-collective measurement sequence of Z_1, Z_2, Z_3, I would always project the state $|\psi'\rangle$ in a classical state, such as $|000\rangle$ and destroy quantum superposition $\alpha |000\rangle + \beta |111\rangle$.

The bit flip code: Implementation aspects

- Step 1: Encoding from a physical qubit state $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$:

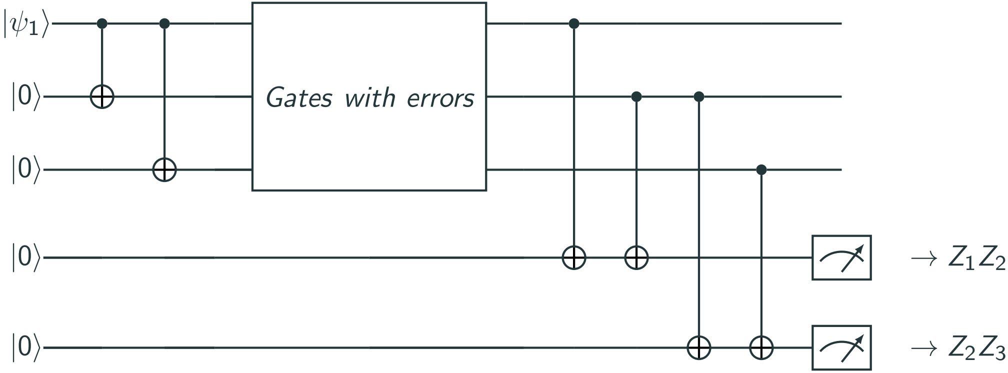


$$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$$

- Side remark: This is very different from quantum cloning $|\psi\rangle \to |\psi\rangle^{\otimes 3}$, which can be proven to be strictly impossible.

The bit flip code: Implementation aspects

- Step 2: Error syndromes and recoveries: One requires ancilla qubits (see also Exercices 4)



- The logical gates $X_L = |0\rangle_L \langle 1| + h.c = X_1X_2X_3$, $Z_L = |0\rangle_L \langle 0| - |1\rangle_L \langle 1| = Z_1$,

The bit flip code: Limitations

- The bit flip code fails for two and three qubit bit flip errors with probability

$$p_L = 3p^2(1-p) + p^3 \tag{7}$$

  with p the single qubit error

- Notion of threshold: Quantum error correction is only useful when the logical qubit lifetime is larger than the physical qubit lifetime, i.e when $p_L \leq p$, this means when $p \leq 1/2$.
- What about combined presence of X, Y, Z errors?

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Steane code

- One logical qubit made of seven physical qubits.
- The error syndromes are defined as the set $S = \{Z_4Z_5Z_6Z_7, Z_2Z_3Z_6Z_7, Z_1Z_3Z_5Z_7, X_4X_5X_6X_7, X_2X_3X_6X_7, X_1X_3X_5X_7\}$.
- These operators commute, i.e errors can be measured successively
- The 'code world' (distance $d = 3$)

$$\begin{aligned} |0\rangle_L &= 1/\sqrt{8}\,(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle) \\ |1\rangle_L &= X_1X_2X_3\,|0\rangle_L \end{aligned} \tag{8}$$

- The code is 'stabilized' by S: For any $|\psi\rangle = \alpha\,|0\rangle_L + \beta\,|1\rangle_L$, for any $g \in S$, $g\,|\psi\rangle = |\psi\rangle$.
- The logical gates are $X_L = \prod_i X_i$, $Z_L = \prod_i Z_i$

Steane code

- The Steane code is an example of *stabilizer codes*, whose error syndromes are elements of a commuting Pauli subgroup.
- For the purpose of this lecture, we will simply check that the syndromes do the job.
- General rules:
  - If Z_i is present in an error syndrome g, it will detect X_i errors (because $X_i Z_i X_i = -Z_i$, and operators acting on different sites i, j commute.)
  - Similarly, Z_i errors are detected by X_i operators .
  - $Y = iXZ$, therefore a Y error is a Z error followed by an X error.

Steane code

Error	$Z_4 Z_5 Z_6 Z_7$	$Z_2 Z_3 Z_6 Z_7$	$Z_1 Z_3 Z_5 Z_7$	$X_4 X_5 X_6 X_7$	$X_2 X_3 X_6 X_7$	$X_1 X_3 X_5 X_7$
none	1	1	1	1	1	1
X_1	1	1	-1	1	1	1
X_2	1	-1	1	1	1	1
X_3	1	-1	-1	1	1	1
X_4	-1	1	-1	1	1	1
X_5	-1	1	-1	1	1	1
X_6	-1	-1	1	1	1	1
X_7	-1	-1	-1	1	1	1
Z_1	1	1	1	1	1	-1
$\vdots$						
Y_1	1	1	-1	1	1	-1
$\vdots$						

Steane code: Conclusion

- The Steane corrects any single qubit errors.
- As the bitflip code, it does not corrected double errors (ex: X_1X_2).
- A first option to achieve Fault tolerance (reaching arbitrary precision in presence of a finite error probability): Concatenated Steane Codes.
- Another approach: Surface codes.

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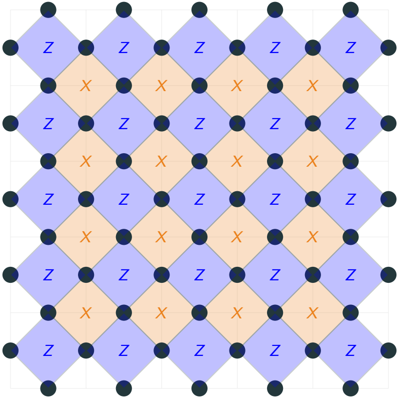
Surface code

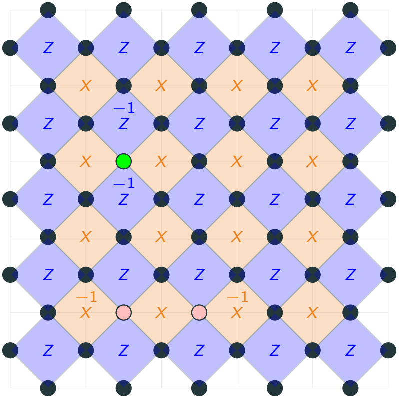


- Kitaev, Bravyi (1997), following works on 'Toric codes'.
- The physical qubits sit on a 2D lattice.
- The stabilizer operators, i.e the measurements to be made for error detection, are

$$Z_{i_1} Z_{i_2} Z_{i_3} Z_{i_4} \text{ on plaquettes}$$
$$X_{j_1} X_{j_2} X_{j_3} X_{j_4} \text{ on vertices}$$

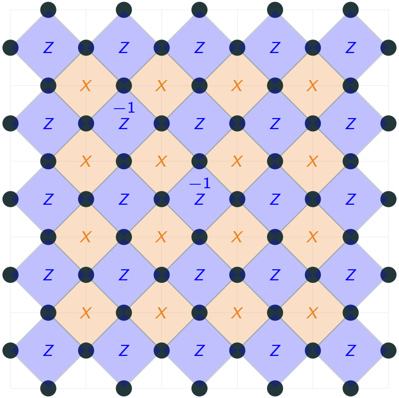
- Code world is 'stabilized' by all such operators $g \left|\psi\right\rangle = \left|\psi\right\rangle$

Surface code: Error detection



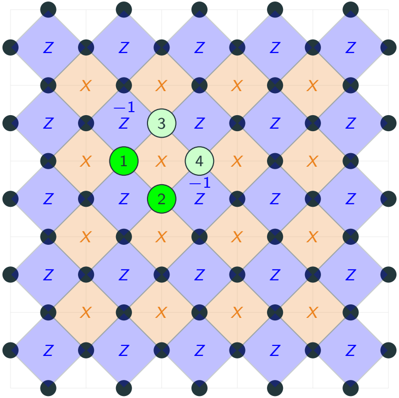
- A green qubit indicates an X error.
- A pink qubit indicates an Z error.
- This is detected by the neighboring plaquettes.
- Any error can be detected provided the lattice is sufficiently large.
- One can then apply recovery operations (or adapt in the software the definition of the code with $g \to -g$)

Surface code: Error detection



- Example inspired from Google's demonstration of the toric code [Science 374, 1237-1241 (2021)]
- I observe stabilizer expectations $g = -1$ as indicated here. What is the error recovery operation?

Surface code: Error detection

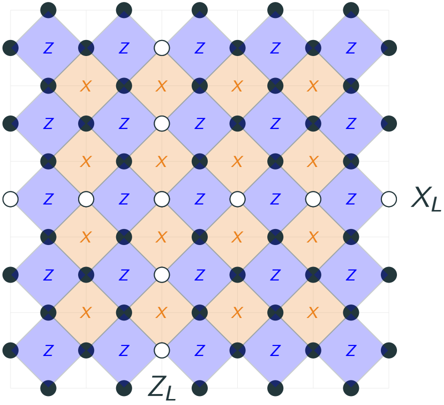


- Apparent ambiguity: I have two 'explanations' (X_1X_2 or X_3X_4)
- However these are fixed by the same recovery operation
- Assume eg the error that happened is $|\psi'\rangle = X_3X_4|\psi\rangle$, but you apply the 'other' recovery operation X_1X_2, you still obtain the right state

$$X_1X_2X_3X_4|\psi\rangle = |\psi\rangle \qquad (9)$$

- All these properties can be checked using the stabilizer formalism (eg Preskill notes).

Surface code: Single qubit initialization and quantum logic

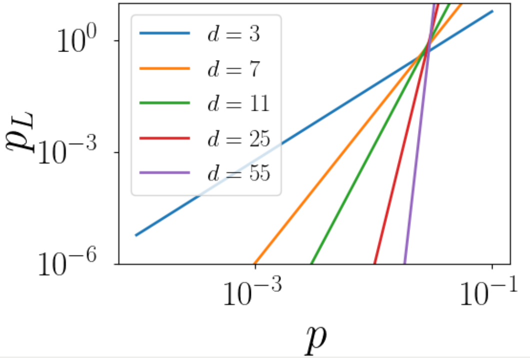


- Probabilistic initialization: Take a random state. Measure stabilizers. Perform error recovery. I obtain a random state in the code world $|\psi\rangle$.
- Let $Z_L = \prod_j Z_{i=L/2,j}$. We can prepare our logical $|0_L\rangle$ by measuring Z_L, such that $Z_L \left|0\right\rangle_L = \left|0\right\rangle_L$ (Note that Z_L commutes with all stabilizer operators).
- Let $X_L = \prod_i X_{i,j=L/2}$, and we define our logical $\left|1\right\rangle_L = X_L \left|0\right\rangle_L$. Exercice: prove that $\left|1\right\rangle_L$ also belongs to the code world and that $\langle 0_L | 1_L \rangle = 0$.

Notion of fault tolerance

- ‘Macroscopic errors’ cannot be detected with a surface code (eg an error of the type X_L, see also Exercices 4)
- Possible fix: increase the code size. But, if I add more and more noisy qubits, I also increase the probabilities of individual errors ...
- Quantum threshold theorem [Knill,Laflamme,Zurek, Aharonov,Ben-or,Kitaev], we can achieve arbitrary precision small ϵ on arbitrary quantum circuits with quantum error correction, provided the physical qubit error probability is below a threshold $p < p_{th}$.
- If $p < p_{th}$, adding more qubits help in achievieng quantum error correction.

Fault tolerance in the surface code



- Exercises 4: We will roughly estimate the logical error probability p_L versus p, for different system sizes d.
- Optimistic scenario for practical applications: More than 1000 physical required for implemented a single logical qubit. We are not there yet...
- Recent highlights
  - Google's toric code
  - Fault tolerant two qubit gates (Postler et al, Nature 2022)