Quantum algorithms

Lecture 2: Quantum algorithms

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October 2, 2023

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Outline

Our first quantum algorithm: Deutsch's algorithm

Quadratic speedup: Grover's algorithm

- Implementation details

IBM Quantum Practicals

- Groups/Schedule on Moodle
- One week after the class, Send me (Julien Renard) a commented jupyter notebook showing/explaining your results
- Grade: Jupyter notebooks+Oral exam

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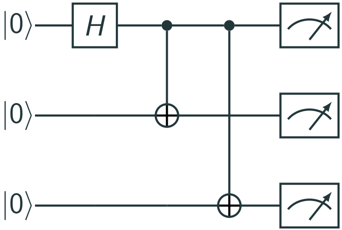
Implementation details

Reminder: Structure of a quantum circuit

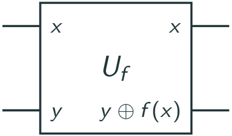
Quantum circuit: single qubit/two-qubit gates and measurements:



Algorithm: a quantum circuit to retrieve the solution of a problem in the measurement data with high probability.

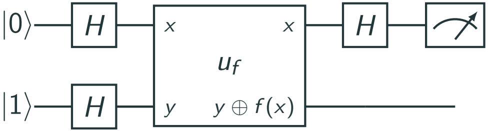
Deutsch's algorithm

- Problem: Given a single bit Boolean function $f(x)$, is f constant i.e $f(0) = f(1)$, or balanced, i.e $f(0) \neq f(1)$?
- We need to introduce an object called an *Oracle*, aka quantum black box.
- An oracle evaluates the classical function f on quantum states



- Complexity will refer here to the number of oracles evaluation.
- Note: a quantum algorithm will be of practical use if the oracle can be implemented easily (see Lecture 6 for a practical implementation of an oracle)

Deutsch's algorithm



One measurement gives me the solution, I would need two function evaluations in the classical case: quantum speedup

Deutsch's algorithm

After the first Hadamards

$$|\psi\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$$

After the oracle

$$|\psi\rangle' = \frac{1}{2}\left(|0, 0 \oplus f(0)\rangle - |0, 1 \oplus f(0)\rangle + |1, 0 \oplus f(1)\rangle - |1, 1 \oplus f(1)\rangle\right)$$

If $f(0) = f(1)$, let $0 \oplus f(0) = 0 \oplus f(1) = a$, $1 \oplus f(0) = 1 \oplus f(1) = b = 1 - a$

$$|\psi\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|a\rangle - |b\rangle)$$

After the last Hadamard,

$$|\psi\rangle \quad = \quad \frac{1}{\sqrt{2}}\,|0\rangle\left(|a\rangle - |b\rangle\right)$$

I measure $|0\rangle$ with probability 1

Deutsch's algorithm

If $f(0) \neq f(1)$, let $0 \oplus f(0) = 1 \oplus f(1) = a$, $1 \oplus f(0) = 0 \oplus f(1) = b$

$$|\psi\rangle = \frac{1}{2}(|0\rangle - |1\rangle)(|a\rangle - |b\rangle)$$

After the last Hadamard,

$$|\psi\rangle \quad = \quad \frac{1}{\sqrt{2}} |1\rangle \left(|a\rangle - |b\rangle\right)$$

I measure $|1\rangle$ with probability 1

Further reading

Some related algorithm using oracles:

- Deutsch Joza algorithm: generalization of Deutsch's algorithm to multiple qubits: oracle separation between **P** and **EQP** (exact quantum polynomial)
- Bernstein Vazirani and Simon's algorithm: Prove an oracle separation between **BPP** (bounded error classical complexity) and **BQP** (bounded-error quantum complexity).

Outline

Our first quantum algorithm: Deutsch's algorithm

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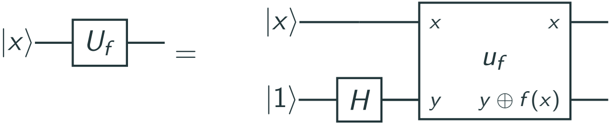
Grover's algorithm

- Unstructured search problem: Given a $n-$bit Boolean function $f(x)$, such that there exists a unique w such that $f(w) = 1$, find w.
- Application: Subroutine in various classical algorithms (example minimization problem, or machine learning)
- Input: A n-bit phase oracle



For any input x, we can mark to the solution

$$U_f |x\rangle = (-1)^{f(x)} |x\rangle$$

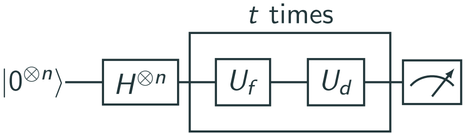
The ancilla qubit has been 'uncomputed'.

Grover's algorithm

- Classical algorithm: $O(2^n)$ evaluations (Just test in a loop...)
- Grover's quantum algorithm $O(\sqrt{2^n})$ oracle evaluations: quadratic speedup
- Possible applications: solving NP-complete problems that allow for oracle implementations (eg Lecture 6 on the 3-SAT problem), brute-force attacks on cryptographic keys ...

Grover's algorithm

So simple...



- with the diffuser $U_d = 2|\psi\rangle\langle\psi| - 1$, with $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_x |x\rangle$ the superposition on all $N = 2^n$ bitstrings $x = x_1, \ldots, x_n$.

Grover's algorithm

After the first Hadamards ($N = 2^n$), the state is

$$H^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{N}}(|0\rangle + |1\rangle)^{\otimes n} = \frac{1}{\sqrt{N}} \sum_x |x\rangle = |\psi\rangle$$

Introducing, $|\alpha\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq w} |x\rangle$, we can write

$$|\psi\rangle = \sin(\theta/2) |w\rangle + \cos(\theta/2) |\alpha\rangle ,$$

with $\sin(\theta/2) = 1/\sqrt{N}$.

Combined application of oracle and diffuser will lead to a rotation of the state $|\psi\rangle$ towards the solution.

$$U_f |\psi\rangle = -\sin(\theta/2) |w\rangle + \cos(\theta/2) |\alpha\rangle ,$$

Grover's algorithm

$$
\begin{aligned}
U_d \left|\alpha\right\rangle &= \cos(\theta)\left|\alpha\right\rangle + \sin(\theta)\left|w\right\rangle \\
U_d \left|w\right\rangle &= -\cos(\theta)\left|w\right\rangle + \sin(\theta)\left|\alpha\right\rangle
\end{aligned}
$$

After one iteration,

$$
\left|\psi_1\right\rangle = U_d U_f \left|\psi\right\rangle = \sin(3\theta/2)\left|w\right\rangle + \cos(3\theta/2)\left|\alpha\right\rangle
$$

After t iterations,

$$
\left|\psi_t\right\rangle = \sin((2t+1)\theta/2)\left|w\right\rangle + \cos((2t+1)\theta/2)\left|\alpha\right\rangle
$$

Grover's algorithm: time complexity

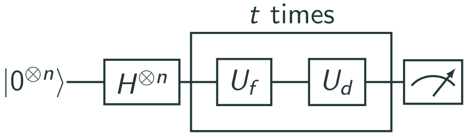
- Success probability

$$p_t = |\langle w|\psi_t\rangle|^2 = \sin((2t+1)\theta/2)^2,$$

which becomes of order one for $\theta t = \mathcal{O}(1)$.

- Remember that $\sin(\theta/2) = 1/\sqrt{N} = 1/\sqrt{2^n}$, thus $\theta \approx 2/\sqrt{2^n}$, we obtain t should be of the order of $\sqrt{2^n}$.

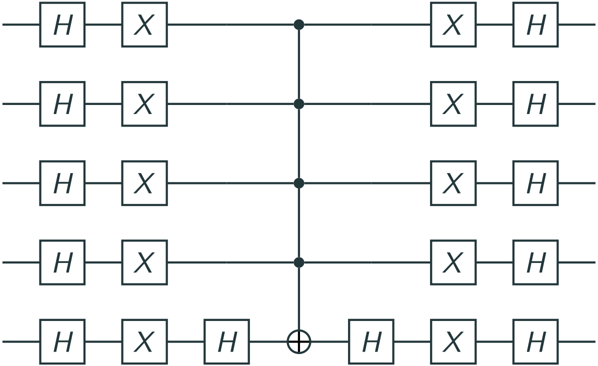
Implementation details



- Implentation of the oracle U_f depending on the function f: Careful Boolean logic to 'mark' solution without knowing the solution, eg test Boolean assertions using $CNOT$s and ancillas (Lecture 6).

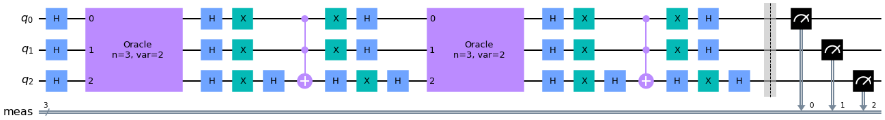
Implementation details

- Implementation of the diffuser $U_d = 2\left|\psi\right\rangle\left\langle\psi\right| - 1$: This can be done with a few gates, including a N-qubit Toffoli gate (see TD2)

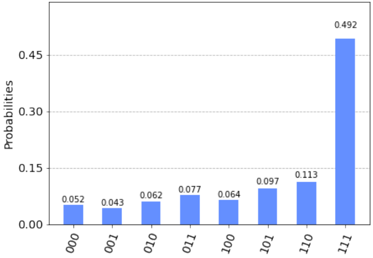


- In practice, the Toffoli gate must be decomposed in elementary CNOT gates, in an optimal way that is platform dependent

Illustration with an IBM quantum computer (c.f., Quantum Practical 2)



- The measurement gives you the solution (if errors are not too large)



- Take-Home Message: The required number of oracle evaluations $\sim \sqrt{N}$ is smaller than the number of entries N of the database!

Grover's algorithm: final remarks

- The quadratic speedup $\sqrt{N = 2^n}$ of Grover's algorithm is optimal for any quantum algorithm for unstructured search (see eg Preskill).
- This is sad news!!!: With an exponential speedup, some *NP*-complete problems could have been solved in polynomial time in the size n, thus *any NP* problem could have been solved in polynomial time. . . .

1. Consider a *NP*-complete problem of size n represented by a Boolean function f (eg 3-sat)
2. Implement the corresponding Grover oracle with n qubits (cf Lecture 6).
3. Run Grover's algorithm