Quantum Algorithms 2021/2022: Exercices 3

Benoît Vermersch (benoit.vermersch@lpmmc.cnrs.fr) -October 25, 2021

1 Measurement of ZZ stabilizers

The error syndromes in the three qubit bit flip code correspond to the measurement of Z_1Z_2, and Z_2Z_3.

1. An error syndrome associated with Z_1Z_2 consists in projecting the state following Born's rules. We define P_1, P_{-1} the projection operators associated with the eigenvalue $+1$ (-1) of the operator Z_1Z_2 $(P_1 + P_{-1} = 1)$. Then after the measurement, the state is transformed as

$$|\psi\rangle \to P_\epsilon |\psi\rangle \tag{1}$$

   with probability $\langle\psi| P_\epsilon |\psi\rangle$.

   For instance, after a bit flip error, our logical qubit state might be transformed as $|\psi\rangle = (|100\rangle + \langle 011|)/\sqrt{2}$. A measurement Z_1Z_2 will leave the state unchanged with probability 1, and with measurement result $\epsilon = -1$.

   The combined measurement of Z_1 and Z_2 does not correspond to the same physical process. I will project the state ψ via the product of the two projection operators $P_{\epsilon_1}P_{\epsilon_2}$ associated with Z_1 and Z_2.

   For instance, with $|\psi\rangle = |100\rangle + \langle 011|$, I will obtain $|100\rangle$ with probability 1/2, and $|011\rangle$ with probability 1/2, and therefore leave the code world.

2. C.f., lecture 3, we first entanglement the ancilla qubit with the two physical qubits via two CNOTs. We obtain

$$\begin{aligned}
|\psi\rangle &= (|0\rangle_1 \langle 0| + |1\rangle_1 \langle 1|)(|0\rangle_2 \langle 0| + |1\rangle_2 \langle 1|) |\psi\rangle \\
|\psi\rangle |0\rangle &\to (|00\rangle \langle 00| + |11\rangle \langle 11|) |\psi\rangle |0\rangle + (|01\rangle \langle 01| + |10\rangle \langle 10|) |\psi\rangle |1\rangle \\
&= |\psi'\rangle = P(1) |\psi\rangle |0\rangle + P(-1) |\psi\rangle |1\rangle
\end{aligned} \tag{2}$$

Therefore, a measurement of the ancilla qubit in the 0 state occurs with probability $\langle\psi'| |0\rangle \langle 0| |\psi'\rangle = \langle\psi| P(1) |\psi\rangle$, and projects the state in $P(1) |\psi\rangle |0\rangle$. Same thing for $P(-1)$. An ancilla qubit allows is therefore to realize the mesurement of Z_1Z_2 as described above.

2 The three qubit phase flip code

The three qubit phase flip code can correct against qubit phase errors Z_1, Z_2, Z_3 based on error syndromes associated with the measurement of X_1X_2, X_2X_3, X_1X_3.

1. We define the stabilizer group of commuting operators $S = \{X_1X_2, X_2X_3\}$. We have $n = 3$ physical qubits. The number of k logical qubits that can be implemented is that the number of elements in S is $n - k = 2$. Therefore, $k = 1$, the vector space V_S stabilized is of dimension 2, i.e I can encode one logical qubit. The code is a $[3, 1]$ stabilizer code.

2. We define the error set $E = \{I, Z_1, Z_2, Z_3\}$. For each $Z_i, Z_j \in E$, $Z_i^\dagger Z_j$ is in S or anticommutes with one element of S. Note that $I \in S$, because $X_1X_2X_1X_2 = I$.

3. In the spirit of the bit flip code, we define

$$\begin{aligned}
|0_L\rangle &= HHH |000\rangle = (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \\
|1_L\rangle &= HHH |111\rangle = (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)
\end{aligned} \tag{3}$$

   The two states are orthonal and stabilized by S, i.e $X_1X_2 |a_L\rangle = |a_L\rangle$ (using $XH |0\rangle = X(|0\rangle + |1\rangle) = (|0\rangle + |1\rangle) = H |0\rangle$, and $XH |1\rangle = -H |1\rangle$).

   Suppose a phase error Z_1 occurs on the first qubit

$$a |0_L\rangle + b |1_L\rangle \to a(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + b(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) \tag{4}$$

   Then the error syndromes give $\langle X_1X_2\rangle = -1$, $\langle X_2X_3\rangle = 1$. I detect this error, which I can fix by applying Z_1.

4. First two CNOTS targeted on the second and third qubit
$$(a|0\rangle + b|1\rangle)|0\rangle|0\rangle \to (a|00\rangle + b|11\rangle)|0\rangle \to a|000\rangle + b|111\rangle \tag{5}$$
Then three Hadamard
$$a|000\rangle + b|111\rangle \to aHHH|000\rangle + bHHH|111\rangle\,. \tag{6}$$

5. $\langle X_1X_2\rangle = \langle H_1H_2Z_1Z_2H_1H_2\rangle$. This means I can repeat the recipe of the previous exercise with application of two Hadamard gates before and after the CNOTs.
We obtain
$$\begin{aligned} H^{\otimes 3}|\psi\rangle &= (|0\rangle_1\langle 0| + |1\rangle_1\langle 1|)(|0\rangle_2\langle 0| + |1\rangle_2\langle 1|)H^{\otimes 3}|\psi\rangle \\ |\psi\rangle|0\rangle &\to H^{\otimes 3}(|00\rangle\langle 00| + |11\rangle\langle 11|)H^{\otimes 3}|\psi\rangle|0\rangle + H^{\otimes 3}(|01\rangle\langle 01| + |10\rangle\langle 10|)H^{\otimes 3}|\psi\rangle|1\rangle \\ &= |\psi'\rangle = P_X(1)|\psi\rangle|0\rangle + P_X(-1)|\psi\rangle|1\rangle \end{aligned} \tag{7}$$

3 Fault tolerance with the surface code

Adapted from https://arxiv.org/pdf/1208.0928.pdf.

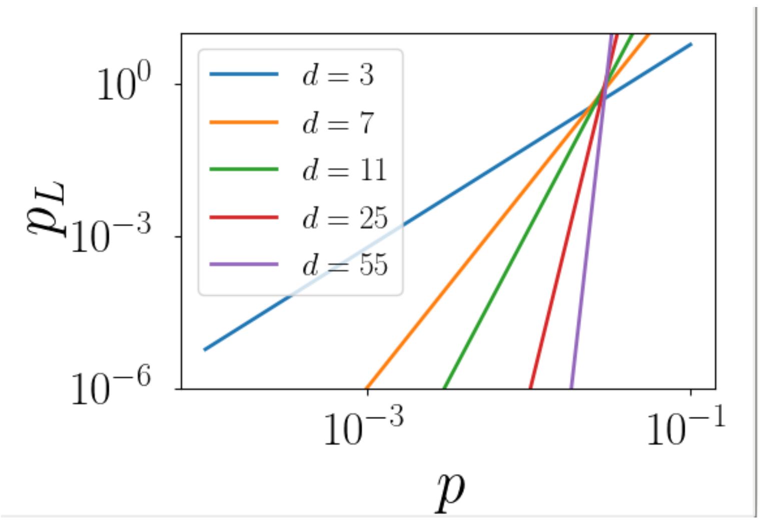
1. Consider a single row of the code of length $d = 5$ (number of white physical qubits).

By definition of a stabilizer code, any logical state $|\psi\rangle$, i.e the code world is stabilized by the stabilizers, i.e for any $i = 1, \ldots, 4$
$$Z_iZ_{i+1}|\psi\rangle = |\psi\rangle\,. \tag{8}$$
For an X_i error on a certain qubit, the state becomes $|\psi'\rangle = X_i|\psi\rangle$. This will be detected via the measurements of
$$\langle\psi'|Z_iZ_{i\pm 1}|\psi'\rangle = \langle\psi|X_iZ_iX_iZ_{i\pm 1}|\psi\rangle = -\langle\psi|Z_iZ_{i\pm 1}|\psi\rangle = -1. \tag{9}$$

2. Suppose the error syndrome step gives $-1, 1, -1, 1$. With two errors, the assignement is X_2X_3. The complementary error $X_1X_4X_5$ would give the same syndrome, giving rise to a logical X error.

3. A given pattern of such error occurs with probability $(8p)^{d_e}$. There are $C_{d,de} = d!/[(d-d_e)!d_e!]$ such patterns. This gives a logical error
$$p_L = C_{d,(d+1)/2}(8p)^{(d+1)/2} \tag{10}$$

4. In a $2D$ code, the logical error rate is multiplied by d
$$p_L = dC_{d,(d+1)/2}(8p)^{(d+1)/2} \tag{11}$$



For $p < p_c$, the logical error rate decreases as d increases, i.e we can reach arbitrary accuracy by adding physical qubits. This is the notion of fault tolerance.