Quantum algorithms: Exercices 1

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1 Universal reversible classical computing

Adapted from Jones/Jaksche, Oxford. The Toffoli gate is universal for resersible classical computing. We will illustrate this result by expressing common gates in terms of the Toffli gate.

We flip the last bit if and only the two control bits are activated.

1.

abc	$a' = a, b' = b, c' = (ab) \oplus c$
000	000
001	001
010	010
011	011
100	100
101	101
110	111
111	110

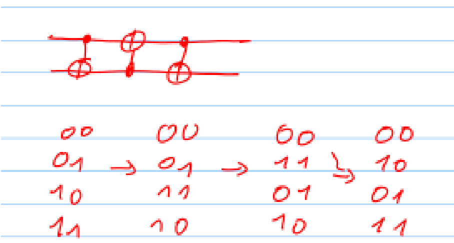
The toffoli gate is its own inverse. So it's reversible.

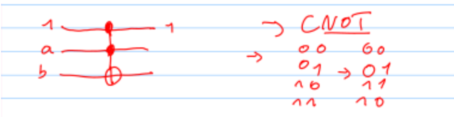
2.

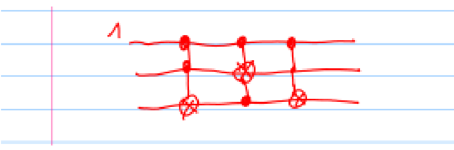
ab	$a' = b, b' = a$
00	00
01	01
10	10
11	11

3.

abc	$a' = a, b' = (1-a)b + ac, c' = (1-a)c + ab$
000	000
001	001
010	010
011	011
100	100
101	110
110	101
111	111

4. 

5. 

6. 

2 Universal quantum gates and measurement operations

The set of $(H, P, T, CNOT)$ form a set of universal quantum gates.

1. c.f Lecture 1

2.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1}$$

$$HH^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

With bra-ket notations, it is not so much simpler in this case

$$HH^\dagger = \frac{1}{2} \left(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1| \right)^2 = |0\rangle \langle 0| + |1\rangle \langle 1| = \mathbf{1} \tag{3}$$

$$H |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{4}$$

3. $Z = P^2$

4. $X = HZH = HP^2H$

$$\begin{aligned} CZ &= |0\rangle \langle 0| \otimes 1 + |1\rangle \langle 1| \otimes Z \\ &= |0\rangle \langle 0| \otimes 1 + |1\rangle \langle 1| \otimes HXH \\ &= |0\rangle \langle 0| \otimes HH + |1\rangle \langle 1| \otimes HXH \\ &= (1 \otimes H)(|0\rangle \langle 0| \otimes 1 + |1\rangle \langle 1| \otimes X)(1 \otimes H) \\ &= (1 \otimes H) CNOT (1 \otimes H) \end{aligned} \tag{5}$$

where we have used in the first line the mixed product property of the Kronecker product: $AC \otimes BD = (A \otimes B)(C \otimes D)$

3 Measurements

1.

$$\begin{aligned} \langle Z \rangle &= \langle \psi | Z | \psi \rangle \\ &= \langle \psi | (|0\rangle \langle 0| - |1\rangle \langle 1|) | \psi \rangle \\ &= | \langle \psi | |0\rangle |^2 - | \langle \psi | |1\rangle |^2 \end{aligned} \tag{6}$$

Therefore, we need to measure in the comput. basis and substract the two measured probabilities

2.

$$\begin{aligned} \langle X \rangle &= \langle \psi | X | \psi \rangle = \langle \psi | HZH | \psi \rangle \\ &= \langle \psi | H(|0\rangle \langle 0| - |1\rangle \langle 1|)H | \psi \rangle \\ &= | \langle \psi H | |0\rangle |^2 - | \langle \psi H | |1\rangle |^2 \end{aligned} \tag{7}$$

Therefore, we need to apply H, then measure in the comput. basis and substract the two measured probabilities.