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CSCI 115
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Lab 2

1.

a. isBrotherOf
   i. reflexive - no, a person cannot be his/her own brother
   ii. symmetric - no, Bob isBrotherOf of Anne holds true, but Anne isBrotherOf Bob does not hold
   iii. antisymmetric - no, if John isBrotherOf Bob holds true, John != Bob, but Bob isBrotherOf John holds true, which contradicts antisymmetry
   iv. transitive - yes, if 2 males are brothers, and they have another brother or sister, in either case a->b and b->c, then a->c

b. isFatherOf
   i. reflexive - no, a person cannot be his/her own father
   ii. symmetric - no, Bob isFatherOf John holds true, John isFatherOf Bob cannot hold true
   iii. antisymmetric - yes, if Bob isFatherOf John holds true, then Bob != John and John isFatherOf Bob doesn't hold true, then antisymmetric holds
   iv. transitive - no, if a isFatherOf b holds true and b isFatherOf c holds true, then a isFatherOf c would not hold true (grandfather)

c. {(x,y) | x^2 + y^2 = 1}
   i. reflexive - no, the only example of this working is the square root of ½. All other cases fail.
   ii. symmetric - yes, for any case where this statement holds, the opposite statement will be true due to the commutative property of addition
   iii. antisymmetric - no, x^2 + y^2 = 1 and y^2 + x^2 = 1 where x != y, so this property cannot be true
   iv. transitive - no, if a^2 + b^2 = 1, and b^2 + c^2 = 1, then a^2 and c^2 must be the same value. The only case this holds true is when a, b and c are all sqrt ½.

d. {(x,y) | x^2 = y^2}
   i. reflexive - yes, in any case where x = y, x^2 = y^2 will hold
   ii. symmetric - yes, there are no cases where x^2 = y^2 but y^2 != x^2
   iii. antisymmetric - no, if x = -1 and y = 1, x^2 = y^2 and y^2 = x^2, but x != y, so antisymmetry does not hold
   iv. transitive - yes, if x^2 = y^2 and y^2 = z^2, then x^2 = z^2 will always be true

e. {(x,y)| x mod y = 0}
   i. reflexive - yes, any number mod itself will be 0
   ii. symmetric - no, 4 mod 2 = 0, but 2 mod 4 = 2
   iii. antisymmetric - yes, because in any case where a mod b = 0 and b mod a = 0, a == b

iv. transitive - yes, if a mod b = 0, then a is a multiple of b. If b mod c = 0, then b is a multiple of c. If a is a multiple of b, and b is a multiple of c, then a is a multiple of c, so this holds true.

f. Empty relation on integers
g. Empty relation on the empty set

2.
a. equivalence relation of a and b where a + b is even
   i. reflexive - yes, any number added to itself is even
   ii. symmetric - yes, if a + b is even, then b + a is still even by its commutative property of addition
   iii. transitive - yes, if a + b is even, then a and b are both even or both odd. then, if b + c is even, then a, b and c are all even or odd. Therefore, a and b are both even or odd, so added together they will be even
b. equivalence relation of a and b where a + b is odd
   i. reflexive - no, the same number added to itself will not be odd, so the property does not hold, meaning that the equivalence relation does not hold
c. equivalence relation of a and b where a x b > 0 (a and b are real, nonzero numbers)
   i. reflexive - yes, any nonzero numbers multiplied together (squared) are positive
   ii. symmetric - yes, if two nonzero numbers are multiplied together and are positive, then the reverse multiplication will have the same result
   iii. transitive - yes, if a x b > 0 and b x c > 0, then a,b and c are all either positive or negative, so a x c will always be greater than 0
d. equivalence relation of a and b where a/b is an integer
   i. reflexive - yes, any number divided by itself will always be 1, which is an integer
   ii. symmetric - no, a/b can be an integer, but b/a can be a fraction
e. equivalence relation of a and b where a - b is an integer
   i. reflexive - yes, any number subtracted from itself will be an integer
   ii. symmetric - yes, for any two numbers where the distance between them is an integer, the magnitude of the distance between the two will always be the same
   iii. transitive - yes, if a - b results in a whole number, and a - c results in a whole number, then a - c will be a whole number.
      1. let a - b = x, which is an integer
      2. let b - c = y, which is an integer
      3. a - b = a - (c +y) = x
      4. so a - c = x + y
      5. since x and y are both integers, there sum is also an integer
f. equivalence relation of a and b where |a - b| <= 2
   i. reflexive - yes, any number subtracted from itself will be 0
   ii. symmetric - yes, as stated in the previous question, the magnitude of the difference between 2 numbers will be the same in the case of a - b or b - a. If |a-b| <= 2, the |b-a| <= 2
   iii. transitive -
      1. let |a - b| = x which is less than 2 -2 <= a -b <= 2

2. let |b - c| = which is less than 2 -2 <= b -c <= 2
3. so, c -2 <= b <= c+2
4. -2 <= a - (c +- 2) <= 2
5. -4 <= a - c <= 4, which does not work in this case
6. ex: a = 6, b = 4, c = 2
    a. |a - b| = |6 - 4| = 2 <= 2
    b. |b - c| = |4 - 2| = 2 <= 2
    c. |a - c| = |6 - 2| = 4 >= 2

3.

a. no, the isFatherOf relation is not transitive, so it cannot be a partial order
b. yes, it is antisymmetric, just like isFatherof, but it is also transitive. Father only goes up one level in the ancestry line, but an ancestor of an ancestor is still an ancestor
c. yes, you can treat isOlderThan as a greater than relation with age, and the greater than operation is a partial order
d. no, this fails antisymmetry, the sister relationship goes both ways if both subjects are female
e. no, <a,b> <b,a> fails antisymmetry
f. this passes antisymmetry because there are no cases where aRb ^ bRa exists. It also passes tranisitivity because there are no cases where aRb ^ bRc where aRc is missing

4. for n elements, the number of possible total orders is n!. This is because we can introduce any relationship between the elements in the set and *could* hold. For example, if we were to use the < operator for n=3, then we could have the total order {1,2,3}. In the same set, if we were to use the > operator, we could hvae the total order {3,2,1}. If we were to continue to place different operators on the set, we would end up with every permutation of the set n=3, which gives n! possibilities

29. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for $i \geq 1$

Proof by induction

Base case: $i = 1$

$$i^3 = \frac{n^2(n+1)^2}{4}$$

$$1^3 = \frac{1^2(1+1)^2}{4}$$

$$1 = \frac{(2)^2}{4} = 1 \checkmark$$

Inductive step: $i = n+1$

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + \sum_{n}^{n+1} i^3$$

By the IH, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

$$= \frac{n^2(n+1)^2}{4} + \frac{(n+1)^3 \cdot 4}{4}$$

$$= \frac{(n+1)^2(n^2+4(n+1))}{4}$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \frac{(n+1)^2((n+1)+1)^2}{4} \checkmark$$

32.

$T(n) = T(n-1) + n \quad T(1) = 1$

Closed form: $T(n) = n(n+1)/2$

IH: $T(n-1) = \frac{(n-1)(n)}{2}$

Base case: $T(1) = \frac{1(1+1)}{2} = 1 \checkmark$

Inductive case: $T(n) = T(n-1) + n$

$$= \frac{(n-1)(n)}{2} + \frac{n \cdot 2}{2}$$

$$= \frac{n^2 - n + 2n}{2}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2} \checkmark$$

33. $T(n) = 2T(n-1) + 1; \quad n > 0 \quad T(0) = 0$

$= 2(2T(n-2)+1)+1$

$= 2^2 T(n-2) + 2 + 1$

$= 2^2 T(n-2) + 3$

$= 2(2^2 T(n-3) + 3) + 1$

$= 2^3 T(n-3) + 6 + 1$

$= 2^3 T(n-3) + 7$

$= 2^n T(n-n) + 2^n - 1$

$= 2^n \cancel{T(0)} + 2^n - 1 = 2n - 1$

Closed form: $T(n) = 2^n - 1$, IH: $T(n-1) = 2^{n-1} - 1$

Base case: $T(1) = 2^1 - 1 = 1$ ✓

Inductive case: $T(n) = 2T(n-1) + 1$

$= 2(2^{n-1} - 1) + 1$

$= 2^n - 2 + 1 = 2^n - 1$ ✓

1 line
2 reg

a. $T(n) = T(n-1) + n \qquad T(0) = 1$

With each new line, it must intersect each existing line because no lines are parallel. At each intersection, the region the line went through will have been split into 2. For n previous lines, n+1 new sections will be created.

+2

2 line
4 reg

b. $T(n) = T(n-1) + n$

$= (T(n-2) + n-1) + n$

$= (T(n-3) + n-2 + n-1 + n)$

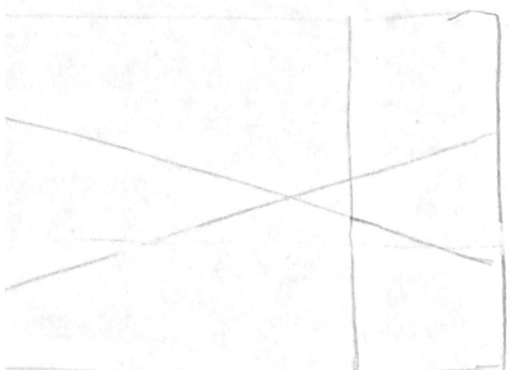
$\vdots$

$T(n-n) + (n-(n-1)) + (n-(n-2)) - 1$

$= T(0) + 1 + 2 + 3$

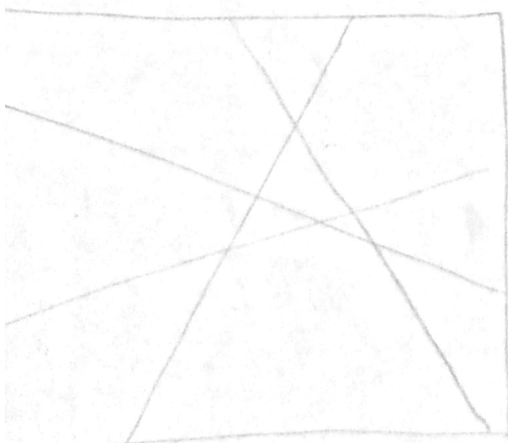
$= 1 + \sum_{i=1}^{n} i$

+3

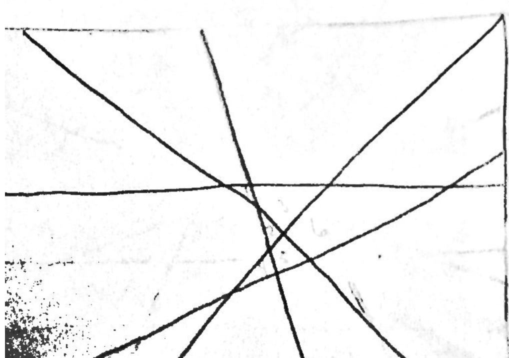


3 lines
7 reg

c. $\frac{n(n+1)}{2} + 1$



4 lines
11 reg



5 lines
16 reg

34. $T(n) = T(n-1) + 3n + 1$

$= \left(T(n-2) + 3(n-1) + 1\right) + 3n + 1$

$= T(n-2) + 3(n-1) + 3n + 1 + 1$

$= T(n-2) + 3(2n-1) + 2$

$= \left(T(n-3) + 3(n-2) + 1\right) + 3(2n-1) + 2$

$= T(n-3) + 3(n-2) + 3(2n-1) + 3$

$= T(n-3) + 3(3n-3) + 3$

$= \left(T(n-4) + 3(n-3) + 1\right) + 3(3n-3) + 3$

$= T(n-4) + 3(4n-6) + 4$

$= \left(T(n-5) + 3(n-4) + 1\right) + 3(4n-6) + 4$

$= T(n-5) + 3(5n-10) + 5$

$\vdots$

$T(n-n) + 3\left(n \cdot n - \frac{n(n+1)}{2}\right) + n$

$= 1 + 3\left(\frac{2n^2}{2} - \frac{n^2+n}{2}\right) + n$

$= 1 + 3\left(\frac{n(n+1)}{2}\right) + n$

$= \frac{3n(n+1)}{2} + \frac{2(n+1)}{2}$

$= \frac{(3n+2)(n+1)}{2}$ closed form

$(n-1)+2$

$\sum_{i=1}^{n} i = 1+2+3+4 \cdots n = \frac{n(n+1)}{2}$

Proof: $T(n) = T(n-1) + 3n + 1$; $T(0) = 1$

IH: $T(n) = \frac{(3n+2)(n+1)}{2}$

Base case: $T(0) = \frac{(0+2)(0+1)}{2} = 1$ ✓

Inductive case:

$T(n+1) = T(n) + 3(n+1) + 1$

$= \frac{(3n+2)(n+1)}{2} + \frac{(3n+4) \cdot 2}{2}$

$= \frac{3n^2 + 5n + 2 + 6n + 8}{2}$

$= \frac{3n^2 + 11n + 10}{2} = \frac{(3n+5)(n+2)}{2}$

$= \frac{(3(n+1)+2)((n+1)+1)}{2}$ ✓

$\frac{(3(n+1)+2)((n+1)+1)}{2}$

$(3n+5)(n+2)$