Mathematical Induction

Often we come across statements in mathematics that assert something is true *for all positive integers*, or for some *set* of the positive integers, such as *for all numbers* $n \geq 2$. In the lecture that is presented here, we describe a powerful technique, that of **mathematical induction**. This is an extremely powerful tool in discrete mathematics and computer science, and is used to prove a tremendous variety of results.

1 Introduction

Let's start by looking at a typical statement that asks us to prove something is true **for all natural numbers**. We will take a look at the difference between proving these types of statements, and those that we saw in the last lecture.

Example 1. *The sum of the first n numbers is $\frac{n(n+1)}{2}$*

This statement as asking us to show that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ *for any value of n* . We could certainly *test* if the statement is true for certain values of n:

	Sum	Formula
$n = 1$	1	1
$n = 2$	$1 + 2 = 3$	$2(3)/2 = 3$
$n = 3$	$1 + 2 + 3 = 6$	$4(3)/2 = 6$
$n = 4$	$1 + 2 + 3 + 4 = 10$	$5(4)/2 = 10$
$n = 5$	$1 + 2 + 3 + 4 + 5 = 15$	$6(5)/2 = 15$

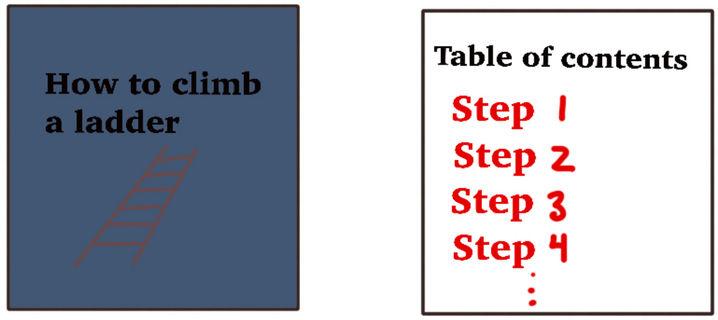
and it seems that indeed, yes, the statement is true so far. Yet this is **not a *proof*** ... we cannot possibly verify that the formula for the sum is true for all possible values of n. The problem is to determine how to show that this is *always* true, for any n.

Mathematical induction is the tool that can be used tackle these problems. The idea behind the concept is that you don't need to show *each* statement is true for *every* possible n: instead, you provide a logical argument that establishes why the statement is true for all n. The result is that one is able to prove the statement is true for all n, *without manually checking them all*.

We start things off with an example of how this can be done for a simple task, like climbing a ladder.

1.1 Induction Analogy

Suppose we are given an **infinite** ladder (one that goes up forever, with an infinite number of steps). Imagine we need to provide an instruction book that teaches a climber how to get to step n on the ladder, for *any* n. In other words, we need to *prove* that this climber can get as far up as he wants on thie infinite ladder, by *explaining* exactly how it is done. If $n = 1$, we would need to explain how to climb to step 1. If $n = 2$, we would need to explain how to climb to step 2, which involves taking steps 1,2. If $n = 3$, we would need to explain how to climb to step 3, which involves taking steps 1,2,3. If $n = 4$, we would need to explain how to climb to step 4, which involves taking steps 1,2,3,4. Does it seem ridiculous to teach the climber how to climb to step n starting from very bottom for every value of n? Do we need to provide a manual where we explain how to reach *each and every step*? That would mean an instruction book with an infinite number of pages! Certainly not.



Almost everyone would agree that we only need the following instructions for our climber:

Therefore we only need the following:

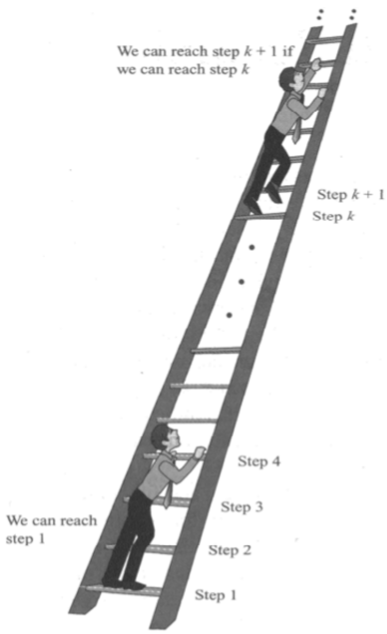
- Go from the ground to the very first step by lifting your feet and getting on the first step.
- Go from any step to the *next* step by stepping from one rung to the next.

Using the above two instructions, the climber can use the first instruction to get *on* the ladder, and then can *repeatedly* use the second instruction to go from one step to the next, as long as he wants. In doing so, can can reach the step n, no matter how how up n is.

With that idea in mind, we now look at mathematical induction more formally.

2 Mathematical Induction

We assume the statements we are trying to prove are of the form: $P(n)$, where n is *any* positive integer. From the example above, $P(n)$ would be the statement that the sum of the first n integers is $n(n+1)/2$. The goal is to prove that $P(n)$ is true for *all* $n \in \mathbb{N}$. This is identical to the climber on the ladder, where our goal was to prove we could climb to *all* steps $n \in \mathbb{N}$.



There are **two parts** to a proof by mathematical induction. The first step is analogous to "*getting on the ladder*" and the second step is like the instruction on how to "*go from one step to the next*"

- A **Base Step**: We simply verify that $P(1)$ is true. In other words, set n equal to the smallest value given, and verify that the statement is true.
- An **Inductive Step:** We show that *if* $P(k)$ is true, then that *implies* that $P(k+1)$ is also true.

The second step above works as follows: if $P(k)$ is true for an arbitrary integer k, (this is called the inductive hypothesis), then we show that *under this assumption* $P(k+1)$ is true. This is analogous to the argument that if you are already on the kth step, then you can get to the $(k+1)$th step on the ladder.

Once the steps of mathematical induction are complete, we have shown that the following is true:

$$\forall n \in \mathbb{N}, P(n)$$

We can now complete the proof of Example 1 using mathematical induction.

2.1 Proof of Example 1

We define the statement $P(n)$ to be the proposition that the sum of the first n numbers is $\frac{n(n+1)}{2}$. The goal is to prove $P(n)$ is true for all n.

Base Step:. $P(1)$: We need to verify the statement for $n = 1$. In other words, does the sum of the first 1 numbers satisfy the formula $n(n+1)/2$ when $n = 1$. Indeed, setting $n = 1$ into the formula yields $1(2)/2 = 1$ Thus the statement is true for $n = 1$.

Induction Step:. We assume the induction hypothesis is true (this is analogous to assuming we have already arrived at the kth step). Thus we **assume** that

$$1 + 2 + 3 + \ldots k = \frac{k(k+1)}{2}$$

Now we need to show that we can **use the above fact** to show that $P(k+1)$ is true. In other words we set $n = k+1$ in the statement and the goal is to prove that the sum of the first $(k+1)$ terms satisfies:

$$1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

We start with the left hand side and notice that we can collect the first terms so that we can apply the induction hypothesis:

$$\begin{aligned} 1 + 2 + \ldots + (k+1) &= 1 + 2 + \ldots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

and thus we have shown that $P(k+1)$ is true.

Conclusion: By mathematical induction, we conclude that the statement

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

is true for all n.

3 Examples

The following examples illustrate how to prove theorems using induction. There are a wide variety of possibilities for induction proofs: summations, inequalities, divisibility results, counting facts, etc.

Example 2. *The sum of the first n positive odd integers is n^2:*

$$1+3+5+\ldots+(2n-1)=n^2$$

Solution: Let $P(n)$ be the proposition that the sum of the first n odd positive integers is n^2.

Base step: $P(1)$: The sum of the first odd integer is 1. And certainly $1=1^2$.

Inductive step: We assume the inductive hypothesis, $P(k)$ is true:

$$1+3+\ldots+(2k-1)=k^2$$

Now we need to show that $P(k+1)$ is true, using the above equation in orange. Note that $P(k+1)$ is simply $P(n)$ where $n=k+1$:

$$1+3+\ldots+(2k-1)+(2k+1)$$

and we need to show that this is *equal* to $(k+1)^2$. We start with $P(k+1)$ and organize the terms so that we can use the assumption $P(k)$. Below we color the terms of $P(k)$ in orange to demonstrate how we can use the fact that $P(k)$ is true.

$$\begin{aligned}1+3+5\ldots+(2k-1)+(2k+1)&=(1+3+5\ldots+(2k-1))+2k+1\\&=k^2+2k+1\\&=(k+1)^2\end{aligned}$$

in the second line above, we replaced $1+3+5\ldots+(2k-1)$ with k^2 since that was our inductive hypothesis.

We have completed the inductive step, and thus shown that $P(n)$ is true for all natural numbers.

Example 3. *Prove the formula from our section on sums:*

$$\sum_{i=0}^{n} ar^i=\frac{ar^{n+1}-a}{r-1}$$

where $r\neq 1$ and a is a real number.

Solution: The statement $P(n)$ represents the formula above. Notice that in this case, the sum starts at $i=0$, so in fact the *smallest n value* that we could use in this formula is $n=0$. Our base step of induction will therefore use $n=0$:

Base step: $P(0)$ is the formula

$$\sum_{i=0}^{0} ar^i=\frac{ar^{0+1}-a}{r-1}$$

The sum on the left has only 1 term, when $i = 0$: $ar^0 = a$. The equation on the right simplifies to $\frac{ar-a}{r-1} = a$. Thus $P(0)$ is true.

Inductive Step: We assume that $P(k)$ is true for an arbitrary $k \geq 0$. The statement $P(k)$ is:

$$a + ar + ar^2 + \ldots + ar^k = \frac{ar^{k+1} - a}{r - 1}$$

Our goal is to prove $P(k+1)$, (the original formula with $n = k+1$):

$$a + ar + ar^2 + \ldots + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}$$

In order to prove the above, we begin with the left-hand side and organize the terms in order to use $P(k)$:

$$\begin{aligned} a + ar + ar^2 + \ldots + ar^k + ar^{k+1} &= a + r + ar^2 + \ldots + ar^k + ar^{k+1} \\ &= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} \\ &= \frac{ar^{k+1} - a + ar^{k+1}(r-1)}{r - 1} \\ &= \frac{ar^{k+2} - a}{r - 1} \end{aligned}$$

And therefore we have been able to prove that $P(k+1)$ is true, under the assumption that $P(k)$ is true. The conclusion is that the the statements $P(n)$ is true for all n.

Example 4. *Show by induction that $n^3 - n$ is divisible by 3 for any positive integer n.*

Solution: The statement $P(n)$ will be that $n^3 - n$ is divisible by 3. Note in this case the question specifies for all $n \geq 1$. Thus our base case is $n = 1$.

Base step: For $n = 1$, we have $1^3 - 1 = 0$, which is divisible by 3, and thus $P(1)$ is true.

Inductive step: We assume $P(k)$ is true:

$$k^3 - k$$

is divisible by 3. Now the job remains to prove $P(k+1)$, in other words that $(k+1)^3 - (k+1)$ is divisible by 3. We can expand this equation and again collect terms that allow us to exploit $P(k)$:

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 - k + 3(k^2 + k) \end{aligned}$$

On the right hand side we have $k^3 - k$ which is divisible by 3 by $P(k)$, and $3(k^2 + k)$, which is also divisible by 3. Thus the entire expression is divisible by 3. Therefore $P(k+1)$ is true. By induction then, we can conclude that $n^3 - n$ is divisible by 3 for all natural numbers n.

Example 5. *Suppose that a doctor administers 1mg of medicine to a person on the morning day 1. During the day, exactly* ***half*** *of the medicine in the person's blood stream is eliminated. However, each morning, the patient receives another 1mg dose. Prove that the amount of medicine in the patient's blood stream on any day is always less than 2 mg.*

Solution:

We can create a variable for the amount of medicine in the blood stream: let $M(n)$ be the amount of medicine in the patient on day n. Thus the goal is to show that:

$$M(n) < 2$$

for all values of n.

Base step:. When $n = 1$, the patient has received only a single dose, and thus the amount of medicine is exactly $1mg$, so $M(1) = 1$. Therefore $M(1) < 2$ and the statement is true for $n = 1$

Induction step: We assume that $P(k)$ is true: that $M(k) < 2$. We now need to show that the amount of medicine after $k + 1$ days is also less than 2, in other words that $M(k + 1) < 2$. Since the amount of medicine on day $k + 1$ is simply half of what was there the previous day, plus 1 extra milligram, then

$$M(k+1) = \frac{M(k)}{2} + 1$$

We apply the assumption that $M(k) < 2$ and conclude:

$$M(k+1) < 2/2 + 1 = 2$$

and thus $M(k + 1) < 2$. Therefore the statement $P(k + 1)$ is true.

By induction, the amount of medicine in the blood stream is less than 2 on any day, n.

Example 6. *Prove that for all $n \in \mathbb{N}$, that $6^n - 1$ is a multiple of 5*

Solution:

Base step: When $n = 1$, $P(1)$ is the statement that $6^1 - 1 = 5$ is a multiple of 5. Thus $P(1)$ is true.

Inductive step: If $P(k)$ is true, then $6^k - 1$ is a multiple of 5. This means that $6^k - 1 = 5m$ for some integer m. The job is then to show that $6^{k+1} - 1$ is a multiple of 5. Notice that:

$$6^{k+1} - 1 = 6^{k+1} - 6 + 6 - 1 = 6(6^k - 1) + 6 - 1 = 6(5m) + 5$$

and the result is a multiple of 5. Therefore $P(k + 1)$ is true. By induction, we conclude that $6^n - 1$ is a multiple of 5, for all $n \geq 1$.

Example 7. *Prove that for $n \geq 1$, if $A_1, A_2, \ldots, A_n$, B are sets, then*

$$(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \cup B)$$

Solution:

Base step: The statement $P(1)$ is that

$$A_1 \cup B = (A_1 \cup B)$$

and certainly this is true.

Inductive step: The statement $P(k)$ is that

$$(A_1 \cap A_2 \cap \ldots \cap A_k) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B)$$

If $P(k)$ is true, we need to show that this implies $P(k + 1)$. We begin with the left side of the expression with $k + 1$ sets, and as usual manipulate the expression in order to use $P(k)$:

$$\begin{aligned}
(A_1 \cap A_2 \cap \ldots \cap A_{k+1}) \cup B &= ((A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}) \cup B \\
&= ((A_1 \cap A_2 \cap \ldots \cap A_k) \cup B) \cap (A_{k+1} \cup B) \\
&= ((A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B)) \cap (A_{k+1} \cup B)
\end{aligned}$$

and thus we have shown that $P(k + 1)$ is true. By induction, we conclude that the statement $P(n)$ is true for all $n \geq 1$.

Example 8. *Use mathematical induction to show that if S is a set with $n \geq 2$ elements, then the number of subsets of size 2 is $n(n-1)/2$.*

Solution:

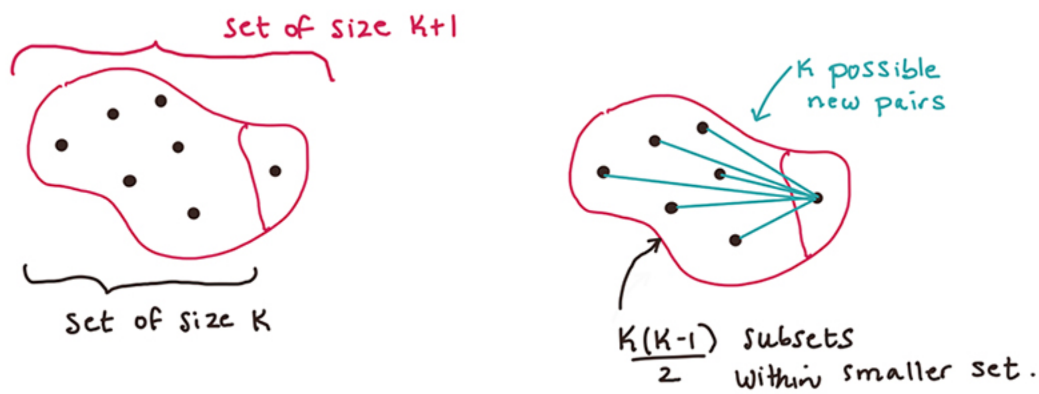
Base Step: The base step here will be when $n = 2$. Thus we need to determine the number of subsets of size on a set with only 2 elements. Certainly the number of possible subsets of size 2 in that case is 1. And $2(2-1)/2 = 1$. Thus $P(2)$ is true.

Inductive Step: We will now assume that $P(k)$ is true: that the number of subsets of size 2 from a set S of size k is $k(k-1)/2$. The goal is to show that this implies that the number of subsets of size 2 from a set of size $k+1$ is $(k+1)k/2$.

Suppose then that we have a set S_{k+1} of size $k+1$. In order to use the inductive hypothesis, we need to somehow relate this larger set to a set of size k. If we isolate one element, (label him x), from S_{k+1} then the remaining elements form a set of size k (see figure below). The number of subsets of size 2 in the smaller set is $k(k-1)/2$ by the inductive hypothesis. However, that does *not* count the possible subsets of size 2 that *include* x. Therefore, we need to also manually count the number of subsets of size 2 which have x as a member. The element x could be paired with any of the k elements in the smaller subset: that is a total of k possible *more* subsets. Therefore the new total of subsets of size 2 in the set S_{k+1} is:

$$k(k-1)/2 + k = (k(k-1) + 2k)/2 = (k+1)k/2$$

Thus the total number of subsets of size 2 is $(k+1)k/2$, which is exactly $n(n-1)/2$ when $n = k+1$. Therefore $P(k+1)$ is true.



4 Strong Induction

There are cases when it is quite difficult to prove $P(k+1)$ follows from $P(k)$ alone. Depending on the algebra of the particular problem, or the nature of the subproblems, we need "*more information*" in order to conclude that $P(k+1)$ is true. **Strong Induction** can be useful in these cases. The base step of strong induction remains the same. However, the inductive step is slightly different. Instead of proving that *if* $P(k)$ is true, *then* $P(k+1)$ is true, we show that **if all** $P(1), P(2), \ldots, P(k)$ are true, *then* it follows that $P(k+1)$ is true.

The same visualization with the infinite ladder can be used to understand strong induction. The new "*steps*" that would tell you how to climb the ladder would be:

- Get on the first step
- If you have already climbed to all the first k rungs, then you can climb to the $(k+1)$st rung.

Example 9. *Prove that any natural number greater than 1 is either prime or can be written as the product of primes.*

Solution: The statement $P(n)$ is "*n is either prime or the product of primes*", and we are asked to show it is true for all $n \geq 2$.

Base case: For $n = 2$, $P(2)$ is the statement that "*2 is prime or the product of primes*". Since 2 is a prime number, the statement is valid.

Inductive step: For strong induction, we assume that $P(2), P(3), \ldots, P(k)$ are true. Now we need to show that $P(k+1)$ is true. If $k+1$ is a prime number, then certainly $P(k+1)$ is true, and we are done. If on the other hand, $k+1$ is not a prime number, then by definition of not being prime, it has some divisor other than 1. Thus we can write

$$k + 1 = m_1 \cdot m_2$$

where m_1 and m_2 are the divisors of $k + 1$. Since both the m_1 and m_2 are *less than* $k + 1$, then by the inductive hypothesis, they must each be prime or the product of primes. In either case, we have expressed $k + 1$ as the product of primes. Thus $P(k + 1)$ is true.

Example 10. *Show that you can make any postage of at least 18 cents using* 7*-cent and* 4*-cent stamps.* *Also presented in chalkboard lecture*

Solution: The statement $P(n)$ is that we can make a postage of n cents using 7-cent and 4-cent stamps. This is equivalent to asking if $n = 7t + 4s$ for $s, t \in \mathbb{N}$ when $n \geq 18$. In this example, we will show that sometimes we need to verify more than one base case..

Base case: If $n = 18$, then we need to determine if we can make exactly 18 cents of postage. We could do this by using 2 of the 7-cent stamps, and 1 of the 4-cent stamps: $18 = 7(2) + 4(1)$.

Induction Step: Suppose that we can make as postage of $18, 19, 20 \ldots, k$. We would like to use this assumption to make a postage of $k + 1$ cents. Suppose we use a 4-cent stamp on our postage $k + 1$. This would mean that we have a remaining $k - 3$ cents to create with out stamps. By the induction hypothesis, we could assume that this is possible, since we assumed that all postages from $18, 19, 20, \ldots, k$ were possible, and certainly $k - 3$ should be in this range. The only problem may be if k is so small that $k - 3$ is not in this range (for example, if $k = 19$, then $k - 3 < 18$. Thus we can only apply our induction hypothesis if $k - 3$ is at least 18. If this were the case, then we could use the induction hypothesis to create a postage for the remaining $k - 3$ cents. If we set $k - 3 \geq 18$, then $k \geq 21$. This means that our induction hypothesis must **start at a k value of at least** 21. The important consequence of this is that we must **manually test** $n = 18, 19, 20, 21$ and include this values of n in the base case.

Revised base case:. We show that $P(n)$ is true for $n = 18, 19, 20, 21$:

n	**Postage**
$n = 18$	$7(2) + 4(1)$
$n = 19$	$7(1) + 4(3)$
$n = 20$	$7(0) + 4(5)$
$n = 21$	$7(3) + 4(0)$

We can now continue with the induction step. Given a postage of $k + 1$ cents, we can simply "*use*" a 4-cent stamp, and the remaining postage of $k - 3$ cents exists due to the induction hypothesis. Thus we have shown by strong induction that a postage exists for all n values of 18 or more.