Counting: part 2

1 The Multinomial coefficient

In the previous lecture, we saw that the the number of possible arrangements of n distinct items is exactly $n!$. What if we would like to count the possible arrangements of n items where certain items are considered **identical to others**? For example, how many different ways are there of scrambling the letters in the word BOOKKEEPER? Notice that some letters are **repeated** - switching the position of two identical letters does not count as a new scrambled word. For example, switching the position of the first two O's does not count as a new arrangement. Thus we cannot simply say that number of possible arrangements for this word is 10!. Instead, we can imagine selecting *final positions* for each of the 6 types of letters. There are 10 possible final locations. The first letter, B, has only one instance. There are therefore 10 possible locations for letter B to be placed. The next letter, O appears twice, thus we can simply select 2 positions out of the remaining 9 spots and place the O's in those locations. That is a total of $\binom{9}{2}$ options for placing the O's. Continuing in this way, he have $\binom{7}{2}$ possible positions for the K's, and then $\binom{5}{3}$ possible positions for the E's, and then then $\binom{2}{1}$ possibilities for the P's and finally 1 option left for the last letter, the R.

Putting this all together using the product rule gives us a total of:

$$10\binom{9}{2}\binom{7}{2}\binom{5}{3}\binom{2}{1}(1)$$

and it can be simplified to:

$$\frac{10!}{2!2!3!}$$

You may notice that the expression above relates nicely to the numbers given in the original problem. There were 10 letters, and O was repeated twice, K was repeated twice and E was repeated 3 times. The expression above is called the multinomial coefficient, and we can use this formula directly whenever we are counting the number of permutations where some items are repeated.

Definition. *Given a set A of n elements where element i is repeated k_i times, then the number of permutations of A is the* ***multinomial coefficient:***

$$\frac{n!}{k_1!k_2!\ldots k_m!}$$

Example 1. *Suppose we have 3 pennies, 4 nickels, 2 dimes and 3 quarters. How many different ways are there to arrange these coins in a row?*

Solution: By the multinomial coefficient, the number of possible arrangements is:

$$\frac{12!}{3!4!2!3!}$$

Note that this solution assumes that the nickels are *identical* to each other. For example, if we had to simply line up 12 coins (each of which was distinct) then the number of ways would be 12!.

1.1 Distinct balls into Distinct Bins

The multinomial coefficient is also used to count the number of ways of distributing n **distinct objects** into k **distinct bins** - each with a certain capacity. For example, suppose we have 16 presents and we wanted to distribute them into 4 different boxes: the first box holds 5, the second box holds 4, the third box holds 6, the last box holds 1. The multinomial coefficient can count the number of ways that this can be done.

Theorem 1. *The number of ways to distribute n objects (each one is unique) into m distinguishable boxes so that k_i objects are placed into box i is given by:*

$$\frac{n!}{k_1!k_2!\dots k_m!}$$

Thus there are

$$\frac{16!}{5!4!6!1!}$$

different ways of distributing the 16 presents into the boxes of size $5, 4, 6$ and 1.

Example 2. *How many ways are there to distribute hands of 5 cards to each of four players (standard 52 card deck)*

Solution: Imagine each player as one of the boxes, and the remaining cards will go into a 5th imaginary box which will be the "*left-over*" cards. We distribute 52 cards using the multinomial coefficient:

$$\frac{52!}{5!5!5!5!32!}$$

Example 3. *How many ways can* 12 *people be assigned to three different tables? Table one can seat* 3 *people, table two can seat* 4 *people and table three can seat* 5 *people. (Ignore the actual seating positions at each table).*

Solution:. Imagine each table as a box. Then the 12 people are each distinct objects. The number of ways of assigning them to tables is:

$$\frac{12!}{5!4!3!}$$

2 Generalized Combinations: repetitions and multisets

In the previous section, we use the multinomial to count the number of ways of distributing n **distinct objects** into **distinct bins**. Suppose however that we have objects that are **indistinguishable**, and we are distributing them into distinct bins. In this section, we will focus on two scenarios: one where we allow for the fact that a bin may be empty, and one where we assume each bin gets *at least one* object.

2.1 Indistinguishable objects into boxes: Empty boxes allowed

Suppose we have 15 pennies (all the same) and we want to distribute them to 5 children (where some child might actually get no pennies). How many ways are there of doing this? The only feature that we are interested in is the *number* of pennies that each child gets - the *particular* penny does not matter, since the objects in this case are *indistinguishable*. In order to decide on how to distribute the pennies, we illustrate a technique involving counting dots.

- Imagine laying out 19 "*dots*" in a row:

- Next, select exactly 4 *"dividing dots"* out of the 19 (we mark them red in the figure below).

- Notice that by selecting these red dots, we have 15 black dots left, divided into 5 groups. We can then count the number of pennies between each of the dividing dots, and distribute exactly that many pennies to each child.

For example, the first child gets 1 penny, since there is exactly 1 penny to the left of the first dot. The second child gets 4 pennies, the third child gets no pennies, the fourth child gets 4 pennies, and the last child gets 6 pennies.

This technique provides a method of distributing the pennies to the children. In fact, the number of ways of doing that depends only on the number of ways we had of selecting the red dots: $\binom{19}{4}$. This can be generalized in the following way: assume we have n **identical objects** (the pennies in the above example) and we distribute them into k distinct bins (the children in the above example), allowing for a bin to possibly be empty. The general formula to determining the number of ways this can be done is given below:

Theorem 2. *The number of ways to distribute n identical objects to k distinct bins is*

$$\binom{n+k-1}{k-1}$$

This includes the possibility that any bin may be empty.

Example 4. *Suppose we have a till containing bills of type: \$1,\$5,\$10,\$20,\$50. We want to select 8 bills, and we can take any number of any type of bill (including none of a specific bill type). How many ways are there of selecting these 8 bills?*

Solution: It may at first seem that this is not at all related to Theorem 2, since we are not distributing any objects. However, we can form a direct correspondence between the bill-selection problem and the balls in bins of Theorem 2. Imagine we have five boxes - each one "representing" a bill type. If we take 8 imaginary balls and distribute them into the 5 types of boxes, then the number of balls that appear in each box is exactly how many bills of that type we take. Of course, this is simply an *illustration* of how to count the number of ways of selecting 8 bills. Nevertheless, since it corresponds perfectly to Theorem 2, then we conclude that the number of bill choices is exactly:

$$\binom{n+k-1}{k-1} = \binom{8+5-1}{5-1} = \binom{12}{4} = 495$$

Example 5. *Suppose there are 4 different types of jelly beans in a shop. How many different ways are there of selecting 6 jelly beans?*

Solution: Again, we can assume that we have 6 possible jelly bean *choices* and we want to "distribute" them among our four possible types of jelly beans. Thus the number of ways of selecting the beans is

$$\binom{6+4-1}{4-1} = \binom{9}{3} = \frac{9!}{3!6!} = 84$$

Example 6. *How many non-negative integer solutions does the equation $w + x + y + z = 20$ have?*

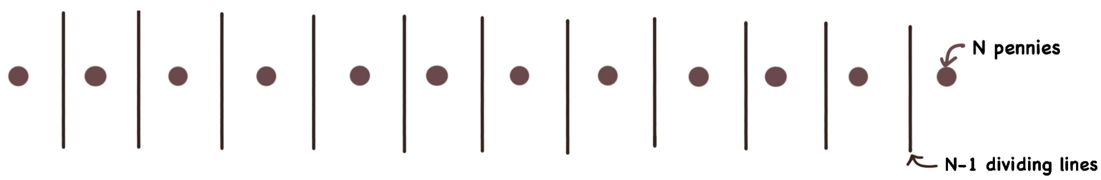
Solution: Imagine we have 20 balls and we distribute them into boxes labelled w,x,y, and z. The number of balls that fall into a specific box determine the value of that particular variable. In such a way, the total of all the variables will be exactly 20. Thus the number of possible solutions is:

$$\binom{20+4-1}{4-1} = 1771$$

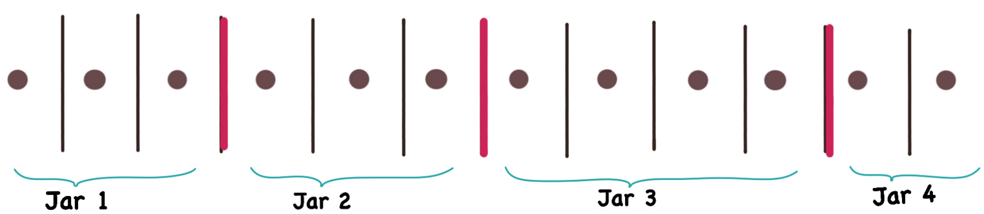
2.2 Identical objects distributed into distinct non-empty bins

Suppose as in the example above, that we have n pennies (they are all identical) and we want to place them into k labelled jars, $J_1, J_2, \ldots, J_k$ with the requirement that *each jar contains at least one penny.*. In this scenario, we develop a slightly different illustration of how to carry out the selection. Suppose that $n = 12$ and there are 4 labelled jars.

- We line up our 12 pennies and mark 11 possible "*division positions*" between the pennies.



- We select exactly 3 of the *dividing* positions, out of the 11 possible dividing positions.



  The result is that we have divided the pennies into the 4 jars - 3 in the first, 3 in the second, 4 in the third, and 2 in the fourth. Notice that by using the dividing lines, no jar is empty. Thus the total number of ways of distributing the pennies is exactly the number of ways we had of selecting the dividing positions: $\binom{11}{3}$.

Theorem 3. *The number of ways to distribute n identical objects to k distinct bins such that no bin is empty is:*

$$\binom{n-1}{k-1}$$

Example 7. *How many ways are there to select 8 books from a library containing books in the subjects of math, music and history, if you must select at least one book from each subject?*

Solution: The bins in this case are the 3 subjects, and we are "distributing" our choice of 8 books over the 3 subjects, such that no subject is "empty". Thus by Theorem 3, the number of ways is $\binom{8-1}{3-1} = \binom{7}{2}$.

Example 8. *In a certain country there are three main political parties. A committee must be formed of* 12 *representatives, and there must be at least one representative from each party. How many different ways are there of forming such a committee? Consider only the number of representatives from each party, and not the individual person.*

Solution: Consider $n = 12$ as the number of choices, and $k = 3$ the number of parties to chose from. Then we are distributing our 12 choices over 3 parties, none of which can be empty. Therefore the number of ways of doing this is: $\binom{12-1}{3-1} = \binom{11}{2}$.