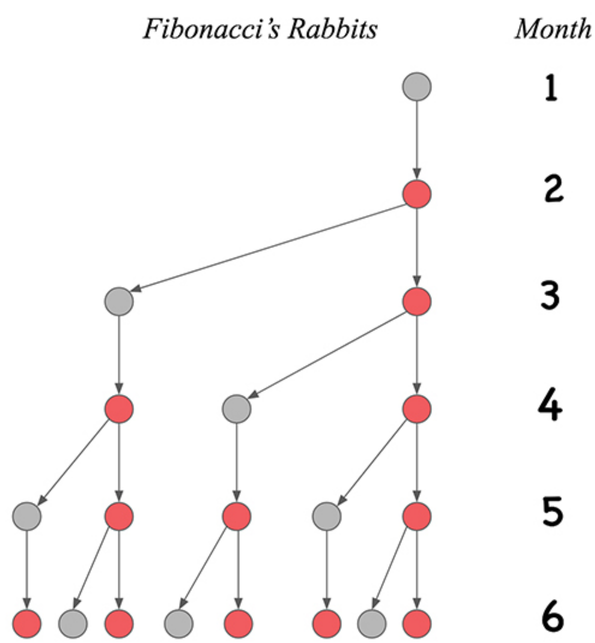
Recursive functions part 2

1 The Fibonacci Numbers

The Italian mathematician Leonardo Fibonacci asked the following question in the 13th century:

A farmer raises rabbits. Each rabbit gives birth to one rabbit when it turns 2 months old and then to one rabbit each month thereafter. Rabbits never die, and we ignore male rabbits. How many rabbits will the farmer have in the nth month if he starts with one newborn?



For small values one can figure this out fairly easily. At month one, there is one rabbit. At the second month, there is still only 1 rabbit since it is not old enough to give birth. At the third month, there are 2 rabbits (the original one and a newborn). At the fourth month there are 3 rabbits since the original rabbit has now given birth twice, but neither of his children are old enough to have given birth. The figure above shows the number of rabbits at each month. The newborns are grey, and when they are old enough to give birth, they are red. One can notice that the number of rabbits at each month is simply the number of rabbits from the *previous* month, plus the number of newborns. A newborn will be created for every rabbit that existed 2 months earlier. If we let $F(n)$ represent the number of rabbits at month n, then initially we have $F(1) = 1$ and $F(2) = 1$ and in general the number of rabbits at month n is simply the number of rabbits that existed the previous month (at generation $n-1$) *plus* the number of rabbits who have given birth (all those from generation $n-2$):

$$F(n) = F(n-1) + F(n-2)$$

Thus the rule for the number of rabbits at month n can be modelled with the recursive function:

$$F(1) = 1$$

$$F(2) = 1$$
$$F(n) = F(n-1) + F(n-2)$$

These numbers are called the Fibonacci numbers. A list of the first terms is given below:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597...$$

Interestingly, you might notice that if we sum up the first k Fibonacci numbers, then the total is one less than $F(k+2)$. For example, $1+1+2+3+5 = 12$ which is one less than $F(7) = 13$. We can prove this result in general:

Example 1. *Prove that*

$$F(1) + F(2) + F(3) + \ldots + F(n) = F(n+2) - 1$$

Solution: This can be proved quite easily by induction: For the base case, when $n = 1$, then certainly $F(1) = F(3) - 1 = 2 - 1$, and when $n = 2$, then $F(1) + F(2) = 2$ and $F(4) - 1 = 3 - 1 = 2$. Thus the statement is true for $n = 1$ and $n = 2$. Now assume the induction hypothesis that

$$F(1) + F(2) + \ldots + F(k) = F(k+2) - 1$$

We need to prove that this hypothesis implies

$$F(1) + F(2) + \ldots + F(k+1) = F(k+3) - 1$$

We use the induction hypothesis in the sum, and then the fact that $F(k+2) + F(k+1) = F(k+3)$:

$$\begin{aligned} F(1) + F(2) + \ldots + F(k+1) &= F(1) + F(2) + \ldots + F(k) + F(k+1) \\ &= (F(k+2) - 1) + F(k+1) \\ &= F(k+3) - 1 \end{aligned}$$

The above definition of the Fibonacci numbers is based on recursion - which means that the numbers are found iteratively, starting with $F(1), F(2), \ldots$. The next theorem provides us with a **closed-form** equation to the Fibonacci numbers, which means that given any n value, we can directly find the corresponding Fibonacci number.

Theorem 1. *If we define $F(0) = 0$ and $F(1) = 1$ then the nth Fibonacci number is given by the formula*

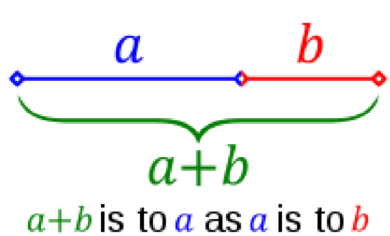
$$F(n) = \frac{1}{\sqrt{5}}(\phi^n - (-\frac{1}{\phi})^n)$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

The number

$$\phi = \frac{1+\sqrt{5}}{2}$$

is called the golden ratio and a little time on google and you will discover lots has been written about its occurrence in nature, art and architecture. In the image below, the ratio between the segments a and b is the golden ratio.



There are many more interesting identities involving the Fibonacci numbers, including some fascinating parallels between these numbers and objects that appear in nature, science, and biology. We will not dive into the Fibonacci numbers as a full topic in this lecture. Instead, our next topic will be how to solve for **closed-form** solutions to recurrences.

2 Solving Recurrence Relations:

2.1 Guessing a solution and proving it is correct

Often by simply examining the terms of a recursive sequence or function, it is possible to guess the closed-form equation. The next example is that of a recurrence for which we can **guess** and **verity** the closed-form equation.

Example 2. *Let* $a_n = 2a_{n-1} - a_{n-2}$ *for* $n \geq 2$ *and* $a_0 = 3$ *and* $a_1 = 4$. *Determine a closed-form equation for this recurrence.*

Solution: Using the recurrence relation, we find the first few terms of the sequence to be:

$$a_n = (3, 4, 5, 6, 7, 8, 9, \ldots)$$

By observation we could **guess a closed formula** for the sequence a_n is

$$a_n = n + 3$$

Once we have this guess, we cannot simply conclude naively that this is the proper equation. Our guess may in fact be incorrect. The next step is to **verity the closed-form equation by induction**:

- **Base case:**
  - $n = 0$. $a_0 = 3$ and $n + 3 = 0 + 3 = 3$ for $n = 0$. Thus the equation is valid when $n = 0$.
  - $n = 1$. $a_1 = 4$ and $n + 3 = 1 + 3 = 4$ for $n = 1$. Thus the equation is valid when $n = 1$.
- **Inductive step:** Since the recurrence relation involves two previous terms, we proceed by strong induction. Assume $a_k = k + 3$ for all $0 \leq k \leq n$. Show that this implies that $a_{n+1} = (n + 1) + 3$. Using the recurrence relation and applying the inductive hypothesis:

  $$a_{n+1} = 2a_n - a_{n-1} = 2(n + 3) - (n - 1 + 3) = n + 6 - 2 = n + 4 = (n + 1) + 3$$

  Therefore this sequence has closed formula $a_n = n + 3$ for all $n \geq 0$.

In the example above, the sequence was simple enough that we were able to *observe* what the closed formula was. If it is not immediately obvious what the equation is, sometimes we can guess the equation by iteratively plugging in values into the recurrence relation and looking for a pattern. This works best for very *simple* recurrences, where the formula for n only depends on one earlier term. The following example demonstrates this technique.

Example 3. *Solve the recurrence* $a_n = 3a_{n-1} + 2$ *subject to* $a_0 = 1$.

Solution: Using the recurrence, we find $a_1 = 3 + 2$. We leave this written as-is, (do not sum the terms), so that afterwards we can identify the a pattern that will help us find the closed-formula. Next, $a_2 = 3a_1 + 2 = 3(3 + 2) + 2 = 3^2 + 3 \cdot 2 + 2$. Again, do *not* determine the numerical value of this term. Instead, continue in this way, finding more elements of the sequence, until an equation becomes obvious:

$$a_3 = 3a_2 + 2 = 3(3^2 + 3 \cdot 2 + 2) + 2 = 3^3 + 3^2 \cdot 2 + 3 \cdot 2 + 2$$

$$a_4 = 3a_3 + 2 = 3(3^3 + 3^2 \cdot 2 + 3 \cdot 2 + 2) + 2 = 3^4 + 3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2$$

and so in general, it looks as though the pattern is:

$$a_n = 3^n + 3^{n-1} \cdot 2 + 3^{n-2} \cdot 2 + \ldots + 3 \cdot 2 + 2$$

If we common factor out the 2 to the front, we are left with a *geometric series* for the back terms ($r = 3$).

$$\begin{aligned} a_n &= 3^n + 2(3^{n-1} + 3^{n-2} + \ldots + 3 + 1) \\ &= 3^n + 2\left(\frac{3^n - 1}{3 - 1}\right) \\ &= 3^n + 3^n - 1 \\ &= 2 \cdot 3^n - 1 \end{aligned}$$

Thus we can finally "*guess*" that the closed-equation is $a_n = 2 \cdot 3^n - 1$. Again, we cannot simply end here. Our observation of the pattern is not necessarily correct. We finish things off by verifying our equation using nduction:

- **Base case:** For $n = 0$, $a_0 = 1$ and $2 \cdot 3^0 - 1 = 2 - 1 = 1$. Thus the equation is valid for $n = 0$.
- **Induction step:**. Assume the kth term of the sequence satisfies $a_k = 2 \cdot 3^k - 1$, for an arbitrary k. Then
$$a_{k+1} = 3a_k + 2 = 3(2 \cdot 3^k - 1) + 2 = 2 \cdot 3^{k+1} - 3 + 2 = 2 \cdot 3^{k+1} - 1$$
which shows the equation is satisfied for $k + 1$. Therefore, by induction, $a_n = 2 \cdot 3^n - 1$.

Once we have recurrence relations with several preceding terms, this technique will quickly become too complicated. We will now introduce a more sophisticated technique to handle several previous terms.

2.2 Linear Homogeneous Recurrence Relations

We begin with a very specific type of recurrence.

Definition. *A linear homogeneous recurrence of degree k, with constant coefficients $c_1, c_2, \ldots, c_k$, has the form:*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

It is called **homogeneous** because there are no constant terms included, and it is called **linear** because the right-hand side is simply a *sum* of the previous terms (multiplied by constants only). The recurrence $a_n = 1 + a_{n-1}^2$ is not linear. In general, this definition can include up to k preceding terms. We will focus on solving linear homogeneous recurrences that include only two previous terms, such as:

$$a_n = 2a_{n-1} + 3a_{n-2}$$

The solutions of this type of recurrence will be of the form $a_n = r^n$. The exact form of the solution depends on the following definition.

Definition. *Given a recurrence of the form $a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$, the corresponding **characteristic equation** is*

$$r^2 + c_1 r + c_2 = 0$$

From our example above where $a_n = 2a_{n-1} + 3a_{n-2}$, the characteristic equation is $r^2 - 2r - 3 = 0$. Suppose we find the roots of this equation. Since it is *quadratic*, there are 2 possible roots. In the case that the roots are distinct, we have the following theorem for finding a solution to our recurrence:

Theorem 2. *Suppose $r^2 + c_1 r + c_2 = 0$ has two distinct real roots, labelled r_1 and r_2. Then the recurrence $a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$ has solution:*

$$a_n = ar_1^n + br_2^n$$

for some constants $a, b \in \mathbb{R}$.

The specific constants a, b are determined by the initial conditions a_0 and a_1

Example 4. *Solve the recurrence $a_n = 2a_{n-1} + 3a_{n-2}$ where $a_0 = 1$ and $a_1 = 2$.*

Solution: The characteristic equation for the recurrence is $r^2 - 2r - 3 = 0$, which has roots $r_1 = 3$ and $r_2 = -1$. Since they are distinct, the general solution is

$$a_n = a3^n + b(-1)^n$$

It remains to solve for the constants a and b. In order to do this, we let $n = 0$ and set $a_0 = 1$ and repeat for $n = 1$. So

$$a_0 = a3^0 + b(-1)^0$$

gives us the equation $1 = a + b$. Similarly,

$$a_1 = a(3) + b(-1)$$

which gives us the equation $2 = 3a - b$. Solving for a, b we find that $b = 1/4$ and $a = 3/4$. Therefore the final solution is:

$$a_n = \frac{3}{4}3^n + \frac{1}{4}(-1)^n$$

In our next case, we consider the form of the solution if the roots are repeated.

Theorem 3. *Suppose that $r^2 + c_1 r + c_2 = 0$ has only one root, r_1. Then the recurrence $a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$ has solution:*

$$a_n = ar_1^n + bnr_1^n$$

Example 5. *Solve the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ where $a_0 = 1$ and $a_1 = 6$.*

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$ which has a single (repeated) root, $r_1 = 3$. The general form of the recurrence solution is

$$a_n = a3^n + bn3^n$$

Again we use the conditions for $n = 0$ and $n = 1$ to solve for a and b.

$$a_0 = a(3^0) + 0$$

and so $a = 1$. Next,

$$a_1 = a(3) + b(3)$$

and so $b = (6 - 3)/3 = 1$. The final solution is

$$a_n = 3^n + n3^n$$

The above two theorems can easily be generalized to characteristic equations of larger than degree 2. For example, if one is given a characteristic equation such as

$$r^3 - 6r^2 + 11r - 6 = 0$$

which has roots $r_1 = 1, r_2 = 2, r_3 = 3$, then the recurrence $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ has general solution

$$a_n = a(1)^n + b(2)^n + c(3)^n$$

In the practice problems and the chalkboard lecture we will look at cases of higher degree with repeated roots.

If the recurrence is **not homogeneous** then the solution technique is more involved. We will not cover that topic in this class.