Propositional Logic

The rules of logic are at the core of understanding mathematical reasoning. Logic is what enables us to establish what is mathematically *valid* and what is not. Using the tools of logic, we will explore what constitutes a valid argument, that is, a proof. Proofs play an important role in this course: as we progress through the different topics of this class, it is essential that we correctly justify our reasoning and claims. In computer science, we rely on proofs to verify the output of programs, to ascertain that algorithms do what they are supposed to and in the correct number of steps, and to develop artificial intelligence.

The ability to write a proper *proof* that supports your argument or idea is at the core of this course. Thus we begin with the precursor to proofs: the rules of propositional logic.

1 Propositions

We begin with the definition of a *statement* or *proposition*.

Definition. *A Proposition is a statement that is either true or false.*

This may seem like a rather simple definition, yet it is fundamental to propositional logic. For example, the following sentences are *propositions:*

1. $2 + 1 = 3$
2. $2 + 1 = 2$
3. A dog is not a cat.
4. 2 is a prime number.
5. Every number is a prime number.
6. Either x is a multiple of 7 or it is not.

For each of the above sentences, it is clear that they are either true (numbers 1,3,4,6) or false, and that no other *input* is necessary to make that decision.

The following are **not** propositions:

1. What day is it today?
2. x is a prime number
3. Give me my homework back

The above statements do **not** have either true or false values, and thus are not propositions. In particular, item number 2 above can only be decided if we had knowledge on the value of x.

It is important to note that it may not be easy to determine *if* a proposition is true or false. In fact, one famous example dates back to 1742 and is due to the german mathematician Christian Goldbach:

Conjecture 1. *(Goldbach) Every even integer greater than 2 is the sum of two primes*

The above statement is either true or false. Unfortunately, no *proof* is known of this conjecture and so to this day, it remains unsolved, and thus it's *actual* truth value is unknown. In such cases, we refer to these propositions as *conjectures*, because they have not yet been proven. Nevertheless, the sentence is a proposition, because it is either true or false.

In propositional logic, a capital letter is often used to denote a particular propsition. For example,

P: *All numbers are prime*

where the symbol P is the short form for the proposition.

2 Negation, conjunction, disjunction

In English, we can modify statements using words such as '*it is not true*", or *"and"*. New propositions are created in this way. The next few definitions give us the *logical operators* that are necessary to create these new compound propositions.

Definition. *Let P be a proposition. The negation of P is denoted $\neg P$, or $\bar{P}$ and is itself also a proposition with the opposite truth value of P.*

Below shows the **truth table** for $\neg P$:

P	$\neg P$
T	F
F	T

The first row shows that when P is true, the proposition $\neg P$ is false, and vice versa for the second line.

The negation of a statement in English can be worded in many ways. Take for example the proposition,

P: *All numbers are primes*

The negation of P can be stated as:

$\neg P$: *Not all numbers are primes*

$\neg P$: *It is not true that all numbers are primes*

The next operator combines two propositions.

Definition. *The conjunction of propositions P and Q is denoted by $P \wedge Q$ and is the new proposition "P and Q". The conjunction $P \wedge Q$ is true when both P and Q are true.*

In general, a **truth table** indicates the true/false values for all possible values of the variables. In this case, the truth table for $P \wedge Q$ has four rows:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition. *The disjunction of propositions P and Q is denoted $P \vee Q$ and is simply the new proposition "P or Q", it is false when both P and Q are false, otherwise it is true.*

The truth table for a disjunction is shown below:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

It is important to note that the word "*or*" in english is sometimes used as an *inclusive or* or an *exclusive or*. On the other hand, the disjunction, $\vee$, is an *inclusive* or. For example, consider the following statement:

"*If you are over 3 feet tall or weigh over 40 pounds, you do not need a car seat.*"

Here we mean that if you are over 3 feet, or weigh over 40 pounds, or in fact if you satisfy both options, you do not need a car seat. In the above sentence, the or is being used *inclusively* and corresponds to a disjunction. On the other hand, the english "or" can also be used *exclusively* , as in the following example:

"*You can have french fries or potatoes with your steak*"

Here it is clear that you cannot have both. This is not an example of a disjunction. The above meaning is an example of an *exclusive or*.

Definition. *Let P and Q be propositions. The exclusive or, denoted $P \oplus Q$, is the statment "P or Q but not both". It is true as long as P and Q have opposite truth values.*

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

The next operator is one of the most widely used in developing proofs and is consequential to many arguments in this course. It is often expressed in English following the word *if*, as in "*If if is raining, I carry an umbrella*".

Definition. *Let P and Q be propositions. The implication, denoted $P \Rightarrow Q$ is the statement "If P then Q". It is false when P is true and Q is false. Otherwise it is true.*

We first examine the truth table for the implication:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

From the last two lines in the above table we note that if P is false, it has no implication on Q, and the implied statement is considered true. In other words, the implication is automatically true when P is false because there is no consequence for the proposition Q. On the other hand, when P is true, the implication is only true when Q is also true.

The English expression of an implication is not always trivial, often because there are such a variety of ways of saying the same thing. Suppose that P is the proposition "*Gigi got a 92 on the exam*" and Q is the proposition "*Gigi gets an A*". Some examples of how we could express $P \Rightarrow Q$ include:

If Gigi gets 92, she gets an A
Gigi will get an A when she gets 92 on the exam
For Gigi to get an A, it is sufficient for her to get 92 on the exam

Note however, that if she does **not** get 92 on the exam, we have no expectation on her final letter grade. She may indeed still get an A or not. In other words, when P is false, we have no expectation on Q.

Examples of translating implications from English:.

- You can only make a withdrawal (W) if you have money in your account (A)

  Here the word "only" means that making the withdrawal implies that there is money in the account: $W \Rightarrow A$

- I cry (C) whenever I see the stars(S)

  The word "whenever" is used to state that the proposition S implies the proposition C.

  $S \Rightarrow C$.

- I only ride my bike (B) when it's sunny (S)

  Here the word "only" means that bike riding implies it was sunny.

  $B \Rightarrow S$

- I will cancel the trip ($\neg T$) unless Gigi comes (G)

  The word "unless' means that if Gigi is not coming, then the trip is cancelled.

  $\neg G \Rightarrow \neg T$

- It is sufficient to pay attention (A) in order to pass the course (P)

  The word "sufficient" means that proposition A implies P. $A \Rightarrow P$

- If you only Study (S) when you are under pressure (P), you will not learn ($\neg L$).

  $(S \Rightarrow P) \Rightarrow \neg L$.

- A function is integrable (I) provided the function is continuous (C).

  The word "provided" means that the second proposition C implies the first. $C \Rightarrow I$.

3 The Converse

The converse of an implication $P \Rightarrow Q$ is simply the opposite implication $Q \Rightarrow P$. Suppose that A is "*It is raining*" and B is "*I have an umbrella*". Then

$A \Rightarrow B$: *If it is raining, I have an umbrella*
$B \Rightarrow A$: *If I have an umbrella, it is raining*

The first sentence guarantees that whenever it is raining, you will definitely have your umbrella with you. However, it gives no information on the case when it is not raining - maybe you had your umbrella with you anyway. The second sentence states that if you are holding an umbrella that day, it was definitely raining. In other words, it is not possible that you bring your umbrella with you, even though it is not raining that day. Notice that these do **not** have the same truth tables, and thus they are **not equivalent** statements.

A	B	$A \Rightarrow B$	$B \Rightarrow A$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

4 The Contrapositive

The contrapositive of an implication $P \Rightarrow Q$ is the proposition $\neg Q \Rightarrow \neg P$. We use contrapositives often in English, as in:

"*If I don't study, I will not pass this course*"

Where P is passing the course, and Q is studying, the above is simply $\neg Q \Rightarrow \neg P$.

With some thought, you might notice that one might also express the same thought as:

"*I need to study to pass this course*"

which is $P \Rightarrow Q$.

The question is whether the two statements, which are **contrapositives**, are actually **equivalent** statements. We can verify this with a truth table:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The contrapositive is an extremely powerful proof technique. We shall see later that often proving $\neg Q \Rightarrow \neg P$ is much easier than proving $P \Rightarrow Q$. Since the two logical statements are **equivalent**, by proving the former, we have **also** proved the latter.

For example, suppose you need to prove:

if $x > 2$ **then** $x + 1 > 3$.

This implication can be written as $A \Rightarrow B$ where A: $x > 2$ and B: $x+1 > 3$. This statement is relatively easy to prove the way it is written, simply by adding 1 to each side. However, we could *equivalently* prove the contrapositive of this statement, $\neg B \Rightarrow \neg A$:

if $x + 1 <= 3$ **then** $x <= 2$,

and in doing so we would have also proved the original implication. Let's work out a proof using the contrapositive:

Example. *Prove that if n is an integer and n^2 is even (P) then n is even (Q).*

Proof: The contrapositive statement of the above is $\neg Q \Rightarrow \neg P$: "*if n is odd then n^2 is odd*", which can be shown quite easily. If n is indeed odd, then n^2 is simply the product of two odd numbers, so it is also odd. Therefore by using the contrapositive we have proved that if n^2 is even, then n is even.

5 If and only if

Equivalence of propositions P and Q is denoted $P \Leftrightarrow Q$, and is called "*P if and only if Q*" or "*P iff Q*". It refers to the proposition $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. The value of $P \Leftrightarrow Q$ is only true if both P and Q are true, or both P and Q are false.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

6 Logic Laws:

We saw above that the statements $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ were equivalent, by examining the truth tables of each proposition. However calculating a truth table with n variables requires 2^n rows and this becomes unfeasible quickly.

In this section we will develop and list some laws and axioms that establish *logical equivalence* between propositions. One can use these laws to simplify compound propositions and prove that they are equivalent to a perhaps much simpler expression. The symbol we will use for logical equivalence (i.e. identical truth tables) is $\equiv$. One could also use $\Leftrightarrow$.

Theorem 1. *De Morgan's Laws, named after the English mathematician Augustus De Morgan (19th century).*

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

The above laws can quickly be verified by truth tables. Here is another simple logical equivalence:

Theorem 2. *The statements $P \Rightarrow Q$, and $\neg P \vee Q$ are equivalent.*

Proof: The implication $P \Rightarrow Q$ is only false when P is true and Q is false. The same is true with $\neg P \vee Q$.

The table below shows a collection of some of the fundamental laws of logic, which can be verified by truth tables. It is important to read through them and get comfortable with their meaning.

Equivalence	Name
$A \wedge B \equiv B \wedge A$ $A \vee B \equiv B \vee A$	Commutative
$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ $(A \vee B) \vee C \equiv A \vee (B \vee C)$	Associative
$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	Distributive
$A \wedge A \equiv A$	Idempotence
$A \wedge \neg A \equiv \mathbf{F}$	Contradiction
$\neg(\neg A) \equiv A$	Double negative
$A \vee \neg A \equiv \mathbf{T}$ $A \wedge \neg A \equiv \mathbf{F}$	Validity
$A \vee (A \wedge B) \equiv A$ $A \wedge (A \vee B) \equiv A$	Absorption

Here is an example of using the above Logic laws in order to simplify a compound proposition:

Example. *Show that $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ is logically equivalent to $P \Rightarrow (Q \wedge R)$.*

Proof: We start with the result of Theorem 2, and then use the distributive laws:

$$\begin{aligned} P \Rightarrow (Q \wedge R) &\equiv \neg P \vee (Q \wedge R) \\ &\equiv (\neg P \vee Q) \wedge (\neg P \vee R) \\ &\equiv (P \Rightarrow Q) \wedge (P \Rightarrow R) \end{aligned}$$

7 Satisfiability and Tautology

The term *tautology* refers to a proposition that is $\mathbf{T}$, which is sometimes also referred to as a valid proposition. For example, it is clear that $P \vee \neg P$ is a valid proposition, because regardless of P, it evaluates to $\mathbf{T}$. Suppose we are asked to show that $(P \wedge Q) \Rightarrow (P \vee Q)$ is a tautology, we can use the above logic rules to simplify the expression:

$$\begin{aligned} (P \wedge Q) \Rightarrow (P \vee Q) &\equiv \neg(P \wedge Q) \vee (P \vee Q) \\ &\equiv (\neg P \vee \neg Q) \vee (P \vee Q) \\ &\equiv (\neg P \vee P) \vee (\neg Q \vee Q) \\ &\equiv \mathbf{T} \vee \mathbf{T} \\ &\equiv \mathbf{T} \end{aligned}$$

We could have also used a truth table to show that for all possible values of P and Q, the proposition is $\mathbf{T}$.

A set of propositions is called consistent or satisfiable if there *exists* an assignment of the truth values to its variables that results in all the propositions being true. Suppose we take the set of propositions,

$$P \vee Q$$

$$\neg P \Rightarrow R$$

$$Q \Rightarrow \neg R.$$

To determine if this is a *satisfiable* set of propositions, we need to determine if there is a way to set the variables such that each proposition is satisfied. Note that by setting $P = \mathbf{T}$, $Q = \mathbf{T}$ and $R = \mathbf{F}$, each proposition is satisfied.

8 Logic Puzzles

Logical reasoning can be used to solve puzzles, using a combination of the tools seen in the above section on propositional logic.

Example. *Suppose that in a boarding school there are 5 girls: Katie, Hilary, Raven, Val, and Amy. Now suppose that some of them are not in their rooms. We are told the following: Either Katie or Hilary is missing. Raven or Val is missing, but not both. If Amy is missing, Raven is missing because she always follows her. Val is missing if and only if Katie is missing because they do everything together. And If Hilary is missing, then Amy and Katie are both missing.*

Proof: Let K, H, R, V and A represent the statements that Katie, Hilary, Raven, Val, and Amy are missing, respectively. In other words: K : Kate is missing, etc. We have the following logical statements converted from the above english:

1. $K \vee H$
2. $R \oplus V$
3. $A \Rightarrow R$
4. $V \Leftrightarrow K$
5. $H \Rightarrow (A \wedge K)$

Case 1: Start with the particular proposition H, and assume for now that $H =$**T**. The idea is to see if this "works" in the sense that we can satisfy the above propositions. From the last statement above, since H=**T**, this implies that both A=**T** and K=**T**. Now since we know that A is true, then the 3rd statement above implies that $R =$**T**. And the second statement now implies that $V =$**F**. Next by the 4th statement, since $V =$**F**, it must be that $K =$**F**. But this contradicts the fact that we already determined that $K =$**T**. So the whole argument falls apart and it must therefore be impossible that $H =$ **T**.

Case 2: Suppose that we try to satisfy the statements by letting $H =$**F**. Then the first statement above implies that $K =$**T**. And the 4th statement now implis that $V =$**T**. The second statement forces us to set $R =$**F**. And finally the 3rd statement implies that $A =$**F**. One can check that with these assignments all statements above are satisfied.