Sets and Predicates

1 Basic Sets

In this section, we discuss **sets**, which are the fundamental discrete structure used in order to group objects together. Often the objects in the set are *similar*. For example a set could consist of the names of the students in the class, or your favourite types of fruit, or all the integers larger than 7.

Definition. *A set is an unordered collection of objects.*

In order to visualize this "unordered collection", we can think of the set as a "bag" of objects, and the objects are all sitting in there with no particular order. This is quite different from a list of objects, for example, where there is a first, second, third item in the list.

The items in the set are called the *elements* or *members*

In order to denote that an element, x is **in the set A**, we write

$$x \in A$$

whereas if x is **not in the set A**, we write

$$x \notin A$$

Example. *Let S be the set of positive integers. Then $2 \in S$, and $-1 \notin S$.*

In order to *describe* a set, we can use set notation in several different ways. One option is to simply write out the elements of the set, using *set notation*. For example,

$$V = \{a, e, i, o, u\}$$

where V is the set of vowels. Recall from the definition that the set has *no order*, and is defined only by the elements it contains. Thus the set V could equivalently be written as $V = \{e, a, o, u, i\}$.

Some sets have their own special symbols:

Definition. *The set of integers is denoted $\mathbb{Z}$.*

The elements of the set $\mathbb{Z}$ are all the integers, positive and negative, including 0.

Set notation can include a restriction which adds a condition to type of elements that belong to the set. For example,

$$W = \{x \in \mathbb{Z} | x > 5\}$$

is the set of integers with the added condition that they must be greater than 5. In the above notation, x is simply a placeholder for describing an element in W. It appears on the right-hand side to explain what condition is required on any element (temporarily called x) in order for it to be in W. Describing a set in this way is much easier than listing all the elements. The next three sets of commonly used in mathematics and have their own symbols:

Definition. *The set of positive integers is:*

$$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$$

The set of rational numbers:

$$\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$$

The set of real numbers consists of all rational and irrational numbers is denoted $\mathbb{R}$.

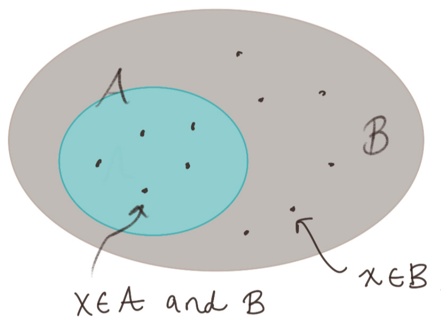
The set notation in the above definition for $\mathbb{Q}$ explain that the rationals are the set of all numbers that can be written as one integer over another.

Two sets A and B are equal only if they have the exact same set of elements, and we can write $A = B$. In other words, every $x \in A$ is also in B. And every $y \in B$ is also in A.

For example, the sets $\{1, 5, 7\}$ and $\{5, 7, 1\}$ are equal.

Definition. *The set which is empty is denoted* $\emptyset$.

Definition. *A set A is a subset of B if every $x \in A$ is also in B. The notation is $A \subseteq B$.*



On the other hand if A is *not* a subset of B, then we would write $A \not\subseteq B$. By the definition of subsets, if A is not a subset of B, this would mean that *not all* the elements of A are in B. In other words, *someone* in A is *not* in B.

Example. *The set $A = \{2, 4, 6\}$ is not a subset of $B = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$ because $4 \notin B$.*

The definition above allows for two imporant consequences. Firstly, that

$$\emptyset \subseteq A$$

for any set A. And secondly that

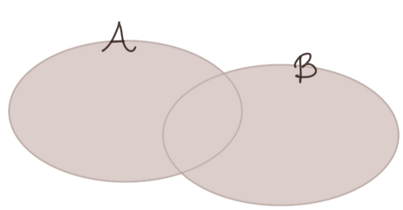
$$A \subseteq A$$

by noticing that certainly a set A is a subset of itself since all of its elements are inside A.

We can combine sets to create new and more complicated sets. Below are the definitions of the operators that combine sets.

Definition. *The union of sets A and B is denoted $A \cup B$ and is defined as the elements x that are in either A or B or both.*

$$x \in A \cup B \quad \Leftrightarrow \quad x \in A \text{ or } x \in B$$

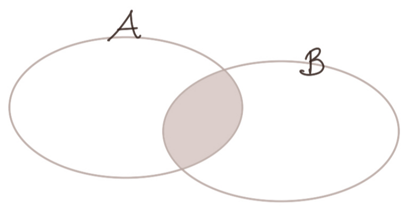


One can also define the union of a *collection* of sets. In that case, the notation is

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

Definition. *The intersection of sets A and B is denoted $A \cap B$ and is defined as the elements x that are in both A and B.*

$$x \in A \cap B \qquad \Leftrightarrow \qquad x \in A \;\; \textit{and} \;\; x \in B$$



The intersection of sets A and B includes all the elements that are in common in both sets. If however, there are *no such elements* then we say that the sets are disjoint, and write

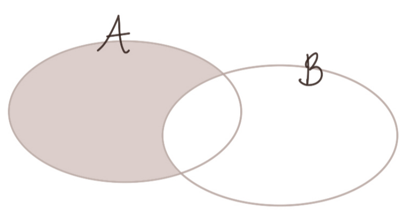
$$A \cap B = \emptyset$$

One can also define the intersection of a *collection* of sets. In that case, the notation is

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Definition. *The set difference of sets A and B is denoted $A - B$ and is defined as the elements x that are in A but not in B.*

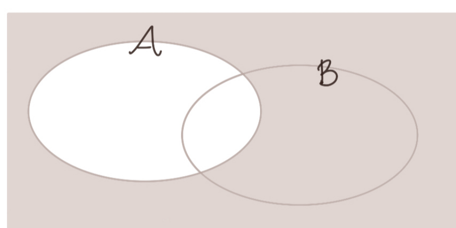
$$x \in A - B \qquad \Leftrightarrow \qquad x \in A \;\; \textit{and} \;\; x \notin B$$



Definition. *The complement of set A is denoted $\bar{A}$ or A^c and is defined as the elements x that are not in A.*

$$x \in \bar{A} \qquad \Leftrightarrow \qquad x \notin A$$

In set notation, $\bar{A} = \{x | x \notin A\}$.



To illustrate the above examples, suppose that our elements all come from the set of integers, $\mathbb{Z}$, and we define A to be the set of positive integers, $A = \{x \in \mathbb{Z} | x > 0\}$, and B the set of even integers, $B = \{x \in \mathbb{Z}|$ x is even $\}$. Then

$$A \cap B = \{2, 4, 6, \ldots\}$$
$$A \cup B = \{\ldots, -6, -4, -2, 0, 1, 2, 3, 4, \ldots\}$$
$$A - B = \{1, 3, 5, 7, \ldots\}$$
$$\bar{A} = \{\ldots, -4, -3, -2, -1, 0\} = \{x \in \mathbb{Z} | x \leq 0\}$$
$$\bar{B} = \{x \in \mathbb{Z} | \text{x is odd}\}$$

1.1 Cardinality and Power Sets

Sets may have any number of elements in them, they may be finite and contain a fixed number of elements, or they may be infinite.

Definition. *For a finite set S, the number of elements in the set is denoted $|S|$ and is referred to as the cardinality of the set.*

For example, the set of integers is infinite. So is the set of positive integers. But the set of positive integers less than 5 is finite, and has cardinality 4.

We saw previously what it means for a set S to have a subset. Clearly, there are many possible subsets of a set S. For example, if $S = \{1, 2, 3\}$, then one possible subset is $\{2, 3\}$, among many others. If we consider *all* the possible subsets of S, this is called the *power set* of S.

Definition. *The Power set of S is the set of all possible subsets of S, denoted $P(S)$.*

Note that $\emptyset$ is always a subset of S and S is also a subset of itself. So the power set by default will include those two sets. For example, if $S = \{1, 2, 3\}$ the the power set is

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

If set S has $|S| = n$, then the cardinality of the power set is $|P(S)| = 2^n$.

Power sets can be built from infinite sets as well. For example, the set $P(\mathbb{Z})$ is the set of all possible *subsets* of integers. One such subset is the *even* integers. So for example,

$$\{x \in \mathbb{Z} | x \text{ is even}\} \in P(\mathbb{Z})$$

1.2 Set identities

Previously we discussed the concept of two sets being *equal*. We defined equality of sets A and B to mean that *each* element x in A is *also* in B **and** *each* element y in B is also in A. Thus equality of sets is defined formally as:

$$(x \in A \Rightarrow x \in B) \wedge (y \in B \Rightarrow y \in A)$$

Let's start off with an example showing how to prove a set identity.

Example. *Prove directly that $A - B = \bar{B} \cap A$.*

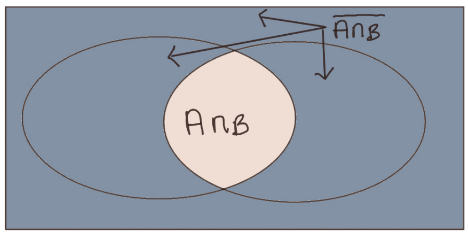
Proof: In order to prove an equality, we need to carefully test the definition of *set equality.* Start by assuming $x \in A - B$, and then show it *also* belongs to $\bar{B} \cap A$. If $x \in A - B$, then by definition $x \in A$ but $x \notin B$. Thus $x \in \bar{B} \cap A$. Next we need to assume that if $x \in \bar{B} \cap A$, then this means that $x \in A - B$. Since $x \in \bar{B} \cap A$, then by definition $x \in \bar{B}$, and $x \in A$. Since $x \in \bar{B}$, then $x \notin B$. Thus $x \in A - B$.

There are several important **set identities** that establish equality between certain sets, and in fact there is a correspondance between these inequalities and those we saw in Propositional Logic. The two **de Morgan's laws** from propositional logic are echoed here:

Theorem 1. *De Morgan's laws for sets state that:*

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$
$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$

The proof of this theorem is done in the exact same way as the proof for the corresponding propositional logic version. A visualization of the set $\overline{(A \cap B)}$ is shown below:



In the list of set identities below, the set **U** refers to the *univeral* set, in other words, the set of *all* the elements possible, from which all sets and subsets are created. As you read through these rules, most of them should make intuitive sense to you (except perhaps the distributive laws which are less important in this class).

Identity	**Name**
$A \cup B = B \cup A$	Commutative
$A \cup \emptyset = A$	Identity
$A \cap \emptyset = \emptyset$	
$A \cup A = A$ $A \cap A = A$	Idempotent
$A \cup (\bar{A}) = U$ $A \cap \bar{A} = \emptyset$, $\overline{(\bar{A})} = A$	Complement laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive
$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$ $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$	De Morgan's
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption

1.3 Cartesian products

In sets, the *order* of the elements is not important. One structure that *does* allow for order is the *ordered n-tuple*, $(a_1, a_2, \ldots, a_n)$. This collection specifically refers to a_1 as its first element, a_2 as the second, etc. For two tuples to be *equal*, each of their elements must be equal and in the same order.

If the $n-$tuple has only 2 elements in it, it is called an *ordered pair*. For example, (a, b) is an ordered pair, where a is the first element and b is the second.

We can create ordered pairs by selecting elements first from set A and next by set B. This concept is called the *Cartesian product*.

Definition. *If A and B are sets, the Cartesian product is denoted $A \times B$ and it consists of all ordered pairs, where the first element is from A and the second element is from B.*

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Since the order of the elements matters in the pair above, the cartesian product $A \times B \neq B \times A$, unless of course $A = B$.

2 Predicates and Quantifiers

2.1 Predicates

A Predicate differs from a proposition in that it's truth value *depends* on one or more variables. For example, "x is even" depends on the value of x. We also use letter names to denote predicates, and function-like notation is used for the variable. Ex,

$$P(x) = \text{x is even}$$

Given a particular x value, we can determine the truth value. In this case, $P(3)$ is false and $P(4)$ is true.

Predicates with two variables are written as $Q(x, y)$, as in "$x = 2y$", where now the truth value depends on 2 variables.

2.2 Quantifiers

Some examples of quantifiers in english are words such as *many, none, some, all* etc, which suggest the extent to which the predicate is true. We will focus on two quantifiers: the *universal* quantifier and the *existential* quantifier.

Definition. *The universal quantifier is denoted $\forall$, and is used as $\forall x P(x)$ to mean exactly*

"P(x) is true for all values of x".

The quantified statement itself may be true or false. For example, if $P(x)$ is defined as "$x < x + 1$", then the statement $\forall x P(x)$ is true, for x a real number. On the other hand, if $Q(x)$ is defined as "x is prime". Then the statement $\forall x Q(x)$ is clearly false: not all real numbers are primes. Specifically, the case $Q(4)$ is a *counterexample* since 4 is not a prime number.

The family from which the variable is selected from is called the *domain*. In the example above, we assumed x was a real number, and then wrote $\forall x P(x)$. In this case the domain is the reals, and the statement is true. Chosing a specific domain can change the truth value of the statement. In order to be specific, one writes: $\forall x \in \mathbb{R}, P(x)$.

The domain of x could be restricted even further. If $x \in \mathbb{R}$, then we could specify new domains as in :

$$\forall x < 0, (x^3 < 0)$$

$$\forall (x \neq 0), (x^2 > 0)$$

The first statement means that for all real numbers that are negative, their cubed value is also negative. The second statement means that for all real numbers other than 0, their squared value is positive.

Definition. *The existential quantifier is denoted $\exists$, and is used as $\exists x P(x)$ to mean exactly*

"There exists an element x in the domain for which $P(x)$ is true"

If we assume $P(x)$ is defined as above, as "x is prime", then certainly if the domain is real numbers, then $\exists x P(x)$ is true.

2.3 Negation of quantifiers

Negation of a statement containing quantifiers is an important part of predicate logic. Let's take the sentence "*Not everyone likes this class*". If $P(x)$ is "*x likes this class*" then the sentence can be translated as:

$$\neg(\forall x, P(x))$$

You may notice that the above english statement is equivalent to saying "*Someone doesn't like this class*" which is translated as:

$$\exists x(\neg P(x))$$

These sentences have the same meaning, and we have the logical equivalence:

$$\neg(\forall x, P(x)) \Leftrightarrow \exists x(\neg P(x))$$

Suppose we look at another negative sentence, "*It is not true that someone in this class speaks French*". Letting $F(x)$ be the predicate that "x speaks French", then we can translate this sentence as

$$\neg(\exists x, F(x))$$

Saying that there is "*not someone*" is like sayin *"no one"*. So the above sentence can be also expressed as "*No one in this class speaks French*", or equivalently "*Everyone doesn't speak French*", thus translating this into logic results in:

$$\forall x(\neg F(x))$$

This gives us the second equivalence for quantifiers:

$$\neg(\exists x, P(x)) \Leftrightarrow \forall x(\neg P(x))$$

The above two rules are de Morgan's rules for quantifiers, which we have already seen in Propositional Logic and Set theory.

We can use de Morgan's rules to simplify negated statements.

Example. *What is the negation of $\forall x(x > x - 1)$.*

The negation is $\neg\forall x(x > x - 1)$, which is simplified as $\exists x \neg(x > x - 1)$, or even further, $\exists x(x \leq x - 1)$.

Example. *What is the negation of "no one is perfect"*

Written in logic, the expression is $\forall x(\neg P(x))$ where $P(x)$ is "x is perfect". Simplifying this first (before taking the negative), we get $\neg\exists x, P(x)$. This can be read as "*it is not the case that a perfect person exists*". Now that it is simplified, we can simply remove the negative in front in order to take the negative, and the result is $\exists x, P(x)$. Finally, we can conclude that the negative of the above sentence is "Somone is perfect".

2.4 Examples of translating from English:

In this section, we will look at translating sentences from English into logical expressions, and taking their negative.

- "*Someone in your class speaks French*"

  This is expressed as $\exists x F(x)$ where $F(x)$ is the predicate that x speaks French. The negation is $\neg\exists x F(x) \Leftrightarrow \forall x(\neg F(x))$, which is "*No one in your class speaks French*"

- "*At least someone in the class will not pass*'"

  This is expressed as $\exists x \neg P(x)$ where $P(x)$ is "x passes the course". Simplifying this using de Morgan's laws is $\neg\forall x P(x)$, which is the same as saying "not everyone will pass". Now we take the negative and we get $\forall x P(x)$, which means "everyone will pass'.

- "*No ducks can swim*"

  This is expressed as $\forall x \neg S(x)$, where $S(x)$ is "x can swim". We could simplify this using de Morgan's laws before we negate, as we did in the previous example, but for variety, we will negate this statement as is: $\neg\forall x \neg S(x)$. Using de Morgan's laws this is equivalent to $\exists x \neg\neg S(x)$, which is $\exists x S(x)$. Thus the negation is "*There exists a duck that can swim*".

2.5 Nesting quantifiers

We can nest the quantifiers one after the other. There are 4 cases using 2 quantifiers:

- $\forall x \forall y P(x, y)$. This means that P(x,y) is true for every pair x, y.
- $\forall x \exists y P(x, y)$. This means that for every x, there *is* a y such that $P(x, y)$ is true.
- $\exists x \forall y P(x, y)$. This *exists* an x such that for every y, $P(x, y)$ is true.
- $\exists x \exists y P(x, y)$. This *exists* some pair x, y for which $P(x, y)$ is true.

Examples:

- $\forall x \forall y(x + y = 0)$, which means that for all x and y, $x + y = 0$. This statement is **not** true, since certainly $2 + 3 \neq 0$.
- $\forall x \exists y(x + y = 0)$

  This means that for *every* x, you can find a y where x + y =0. To check if this is true, assume that you are given any real x, then how you would find the magic y where $x+y = 0$? Simply using $y = -x$ shows that this statement is true.

- Let $L(x, y)$ be the predicate "*x loves y*". Then
  - *Everybody loves somebody* can be written as $\forall x \exists y L(x, y)$
  - *There is somebody whom everybody loves* is $\exists y \forall x L(x, y)$
  - *Everyone loves himself* is $\forall x L(x, x)$
  - *Nobody loves everybody* is $\forall x \neg\forall y L(x, y)$
- Apply de Morgan's rule to $\neg\exists x \exists y P(x, y)$.

  $$\neg\exists x \exists y P(x, y) \Leftrightarrow \forall x \neg\exists y P(x, y) \Leftrightarrow \forall x \forall y \neg P(x, y)$$

  The statement above says "*It is not true that there exists a pair x, y for which P(x,y) is true*". After applying de Morgan's rules it can be read as *"For all pairs x, y, P(x,y) is not true"*. The two statements are equivalent.

- Apply de Morgan's rule to $\neg\forall x \exists y P(x, y)$.

$$\neg\forall x \exists y P(x, y) \Leftrightarrow \exists x \neg\exists y P(x, y) \Leftrightarrow \exists x \forall y \neg P(x, y)$$

  The statement above says "*it is not true that every x has a y for which P(x,y) is true*". After applying de Morgan's rules, the statement can be read as "*there exists an x for which P(x,y) is false for all y*". The two statements are equivalent.