Sequences

1 Sequences and Summations

Unlike sets, sequences are *ordered* lists of elements. We can think of sequences as *lists* of elements from a set S. The list may be infinite or finite. For example, $(1, 1/2, 1/3, 1/4, \ldots)$ is an infinite sequence of real numbers, whereas $(1, 2, 3, 4)$ is a finite sequence. We can use variables to represent the terms in the sequence, and they are labelled with their position, as in:

$$a_0, a_1, a_2, a_3, \ldots$$

where the element in the nth position is referred to as a_n. Note that repetitions are allowed, for example $(1, 1, 1, \ldots)$ is also a sequence.

When referring to the entire sequence, we often write a_n, or you may also see (a_n). Note that the notation for sequences uses *round* brackets, and that for sets uses *curly* brackets, to distinguish between the former (which has a order) and the latter (where order of the elements doesn't matter).

The formal definition of a sequence is given here:

Definition. *A sequence is a countable collection of objects, where order matters and repetitions are allowed. We denote the sequence as a_n, where the nth item is the value a_n.*

Some sequences start at a_1 and some start at a_0. The exact starting variable of the sequence is not important. Often the sequence has a **closed formula**, which is simply an equation for determining each value of the sequence. For example, the sequence above, $(1, 1/2, 1/3, 1/4, \ldots)$ could be represented by the formula $a_n = 1/n$. Using this closed formula we can determine any element of the sequence. For example, the $100th$ item in the sequence is $\frac{1}{100}$. The sequence $(1, 1, 1, \ldots)$ is defined by the formula $a_n = 1$, which says that any nth term of the sequence is 1.

Functions can also be defined **inductively (or recursively)**. Such a definition involves a formula that *relates* the terms in the sequence to the previous terms. For example:

$$a_0 = 1, a_1 = 2, a_n = 2a_{n-1} - a_{n-2}$$

The equation above allows us to "*reconstruct*" the sequence element by element:

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 2\\
a_2 &= 2a_1 - a_0 = 2 \cdot 2 - 1 = 3\\
a_3 &= 2a_2 - a_1 = 2 \cdot 3 - 2 = 4\\
a_4 &= 2a_3 - a_2 = 2 \cdot 4 - 3 = 5
\end{aligned}$$

etc. Finding formulas for sequences is not always trivial. In some cases, the sequence is of a *known type*, and can be related to a general formula for a specific type of sequence. We discuss two of those below:

Definition. *A* *geometric sequence* *has the form* $a_n = ar^n$ *for constants* a *and* r*.*

A geometric sequence can be recognized by the fact that the *ratio* between successive terms is constant, and is the value r

Definition. *A* *arithmetic sequence* *has the form* $a_n = a + nd$ *for constants* a *and* d*.*

An arithmetic sequence can be recognized by the fact that the *differentce* between successive terms is constant, and is the value d.

Example 1. *What type of sequence is* $(1, 1/2, 1/4, 1/8, 1/16, \ldots)$*?*

Solution:. Notice that the terms of this sequence seem to be powers of $1/2$. Thus it represents a geometric sequence, with $r = 1/2$. The formula for the sequence is then $a_n = (\frac{1}{2})^n$. If we let the sequence start at $n = 0$, then the first term is a 1, the second is $1/2$, etc. Thus the sequence is represented by the closed formula of a geometric sequence:

$$a_n = \left(\frac{1}{2}\right)^n, n \geq 0$$

Example 2. *What type of sequence is* $(3, 7, 11, 15, \ldots)$*?*

Solution:. The difference between each term of the sequence is consistently 4. Thus it is an *arithmetic* sequence with $d = 4$. Since the first term is 3, then we can set $a = 3$ and represent the sequence as an arithmetic sequence starting at at $n = 0$:

$$a_n = 3 + 4n$$

2 Summations

In this section we describe summation notation, which allows us to efficiently represent the summation of the terms of a sequence. This summation may be infinite or finite. Suppose that we wish to sum the first n terms of a sequence. Then this sum is represented as:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

The index below the summation term indicates the *first* item in the sum, and the index at the top of the summation represent the *last* index in the sum. The particular *variable* in the sum, either an n or an i is not important, and any variable name can be used. Below is an example of a finite sum of 5 terms:

$$\sum_{n=1}^{5} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

2.1 Geometric sums

If a geometric *sequence* is used to construct a summation, it is called a geometric sum. The sum may be finite or infinite. We look first at the case where the sum is finite, and show the *exact expression* for the sum of the first n terms

Theorem 1. *If a and r are real numbers and $r \neq 0, r \neq 1$ then*

$$\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r - 1}$$

In the case where $r = 1$ the summation is simply $a + a + a + ... + a = a(n + 1)$.

In the example below, $a = 5$ and $r = 1/2$:

$$\sum_{i=0}^{5} \frac{5}{2^i} = \frac{5(1/2^6) - 5}{(1/2) - 1}$$

Note that the summation does not *have* to start at $i = 0$. In this case we simply use the formula above and then subtract however many terms are necessary:

$$\sum_{i=2}^{4} ar^i = \frac{ar^5 - a}{r - 1} - a - ar$$

where the $i = 0$ term is simply a and the $i = 1$ term is ar. Both of these terms were subtracted off because they were not included in the sum.

In other cases, some manipulation is required before applying the geometric series formula:

$$\begin{aligned}\sum_{i=0}^{10} 2^{i+1} - 3^{i-1} &= \sum_{i=0}^{10} 2(2^i) - \sum_{i=0}^{10} 3^{-1}(3^i) \\ &= \frac{2(2^{11}) - 2}{1} - \frac{\frac{1}{3}(3^{11} - 1)}{2}\end{aligned}$$

The geometric sum may also be *infinite*, in which case we often refer to it is an *infinite geometric series.* . There is an exact value for this sum also, as long as $|r| < 1$.

Theorem 2. *If a and r are real numbers and $|r| < 1$ then*

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r}$$

Example 3. *Find the exact value of the infinite sum $3 + 3/2 + 3/4 + 3/8 + 3/16 + \ldots$.*

Solution: This is the geometric sum

$$\sum_{i=0}^{\infty} 3\left(\frac{1}{2}\right)^i = \frac{3}{1 - \frac{1}{2}} = 6$$

2.2 Other Summation Formulae

There are other formulae for many types of summations. The two formulae below are very common in computer science, and they will be applied often during the course. They represent cases where one sums up the first n integers, or the *square* of the first n integers:

Sum	Formula
$\sum_{k=1}^{n} a$	an
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$

We work through some examples of the above formulas here, (there are more examples in the chalkboard lecture). The examples below are from the Rosen text.

- Find a formula for $\sum_{k=1}^{n}(2k-1)$
  Solution:
  $$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n}(k) - \sum_{k=1}^{n}(1) = 2n(n+1)/2 - n$$
- Find a formula for $\sum_{k=1}^{n}(1+(-1)^k)$
  Solution: Notice that when k is odd, the term in the sum is 0. Thus we only need to consider the cases where k is even. If n is an even number, then
  $$\sum_{k=1}^{n}(1+(-1)^k) = 0+2+0+2+\ldots+2 = 2\cdot(n/2) = n$$
  If n is odd, then
  $$\sum_{k=1}^{n}(1+(-1)^k) = 0+2+0+2+\ldots+0 = 2\cdot(n-1)/2 = n-1$$

2.3 Infinite Series

As we saw with geometric sums, summations can be **infinite**, in which case we refer to them as **infinite series**. The notation for an infinite series is:
$$\sum_{i=1}^{\infty} a_n$$
The definition of what it means to sum an *infinite* number of terms is rather delicate. There is technically no such concept as *adding terms forever*. Thus $\sum_{i=1}^{\infty} a_n \neq a_1 + a_2 + a_3 + a_4 \ldots$, because the addition of an infinite number of terms is not defined.

The infinite series is instead defined using the concept of *partial sums*, which refers to the value of the sum if we stopped at a certain index. The definition is given below, but it will be made clearer in the next examples.

Definition. *If (a_n) is a sequence of elements in $\mathbb{R}$, then the infinite series, written $\sum_{n=1}^{\infty} a_n$,is the new sequence:*

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \end{aligned}$$

etc. The numbers a_n are the ***terms*** *of the series and the numbers s_n are the* ***partial sums****.*

The value of $\sum_{n=0}^{\infty} a_n$ corresponds to the the behaviour of these partial sums, s_n. If the partial sums get closer and closer to a final result, then this is the value of the infinite sums. If they don't, then the infinite sum does not exist.

In this course, proving when these sums exist is *way* beyond the scope of the material. It is important to simply know how an infinite sum is defined and when it exists, and to understand how to apply certain well-known formulas for certain infinite sums, such as the geometric sum. We look at an example of an infinite sum that does not exist.

Example 4. *Show that the infinite series $\sum_{i=0}^{\infty}(-1)^i = 1-1+1-1+1-1+1\ldots$ does not exist.*

Solution: It is very common to look at this infinite sum and assume that the answer is 0. It may seem like overall the 1's are cancelling with the 0's and we are left with nothing. However it is important to determine the existence of the sum based on the definition of what it means to have an infinite sum: and it does *not* mean to average the result of adding together the numbers forever.

The partial sums for this sum are as follows:

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1-1=0 \\
s_3 &= 1-1+1=1 \\
s_4 &= 1-1+1-1=0 \\
s_5 &= 1-1+1-1+1=1
\end{aligned}$$

Notice that the partial sums are alternating between 1 and 0. Thus it is impossible that the partial sums are getting *closer and closer* to a final number. In this case then, there is *no final sum* and we say that the infinite sum *does not exist.*

We finish this section by giving some examples of how to use the geometric series formula from Theorem 2. Recall that this formula applied for any $|r| < 1$, and we had that

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

Example 5. $\sum_{k=0}^{\infty} \frac{4}{3^{k+1}}$

Solution: In some cases, the formula may at first not seem as though it matches up with the formula in Theorem 2. With some simply manipulation, we can reorganize the inside of the summation so that we have one term that is raised to the proper exponent.

$$\sum_{k=0}^{\infty} \frac{4}{3^{k+1}} = \sum_{k=0}^{\infty} \frac{4}{3}(1/3)^k = \frac{4}{3}\left(\frac{1}{1-(1/3)}\right) = \frac{4}{3}\cdot\frac{3}{2} = 2$$

Example 6. *Evaluate $\sum_{k=1}^{\infty} x^{2k}$ for $|x| < 1$.*

Solution: In this solution, we simply need to rewrite the exponent, so that we have a term that is raised to k. Secondly, since the sum starts at 1 and not 0, we need to use the formula and then subtract the first term.

$$\begin{aligned}\sum_{k=1}^{\infty} x^{2k} &= \sum_{k=1}^{\infty} (x^2)^k \\ &= \frac{1}{1-x^2} - (x^2)^0 \\ &= \frac{1}{1-x^2} - 1\end{aligned}$$

2.4 Notation for double summation

Double summations occur when two summations signs are nested. Let's look at a double summation where the inner summation depends on the value i of the outer summation. In this example, the inner summation is executed for $j = 0$ to $j = i$. The red term on the right corresponds to $i = 0$, the next term (orange) corresponds to $i = 1$ (and we sum $j = 0, j = 1$), the next blue term corresponds to $i = 2$ (and we sum from $j = 0, 1, 2$), etc.

$$\sum_{i=0}^{3} \sum_{j=0}^{i} (i+j) = [0] + [(1+0) + (1+1)] + [(2+0) + (2+1) + (2+2)] + [(3+0) + (3+1) + (3+2) + (3+3)]$$