Assignment 1 Solutions

Question 1

(a)

$A \vee B$ is equivalent to $\neg A \to B$ since both are only false in the case where $A = B = F$. $A \wedge B$ is equivalent to $\neg(A \to \neg B)$ since both are only true in the case where $A = B = T$.

$$((A \wedge (B \vee C)) \to \neg(A \wedge C)) \vee C$$
$$\neg((A \wedge (B \vee C)) \to \neg(A \wedge C)) \to C$$
$$\neg((A \wedge (B \vee C)) \to (\neg A \vee \neg C)) \to C$$
$$\neg((A \wedge (B \vee C)) \to (A \to \neg C)) \to C$$
$$\neg((A \wedge (\neg B \to C)) \to (A \to \neg C)) \to C$$
$$\neg(\neg(A \to \neg(\neg B \to C)) \to (A \to \neg C)) \to C$$

The statement is satisfiable. By setting $C = T$, then the above implication is true, regardless of the assignments of A, B.

(b)

- P= pass course, E = do all exercises. $P \to E$. The negation is $\neg(\neg P \vee E) = P \wedge \neg E$, which is "I passed the course but didn't do all the exercises".
- H= wear hat, S = sunny. $S \to H$. The negation is $\neg(\neg S \vee H) = S \wedge \neg H$, which is "It is sunny and i'm not wearing a hat"
- M = wear mask, S = getting sick $M \to \neg S$. The negation is $M \wedge S$, which is "I wear a mask and get sick".
- "doesn't imply" means that if you get sick, you were either wearing a mask or not. Both are possible. So in fact the statement can be written as $S \to (M \vee \neg M)$. The negation is $S \wedge (M \wedge \neg M)$ which is "I got sick but I was wearing and not wearing a mask", which is clearly not possible.

(c)

If a proposition is not satisfiable, then under all possible assignments of the variables, the proposition evaluates to F. If we take the negation, this means all possible assignments evaluate to T, so certainly it is satisfiable, and in fact it is also valid, since it is always true.

(d)

Define the following propositions:
I: vacation in italy
F: vacation in france
M: vacation in Mexico
V: go on vacation
C: buy new clothes
A: afford a vacation
U: go to a museum

$I \to C$: *If I go to Italy on vacation, then I need to buy new clothes*:
$\neg C \to A$: *If I don't buy new clothes, then I can afford to go on vacation*
$V \leftrightarrow I \vee F \vee M$: *A vacation takes place in italy, france or mexico*
$V \to A$: *Going on vacation implies that you had the money to do it*
$\neg F \to \neg U$: *If I don't go to France, I don't go to a museum*
$U \vee C$: *I buy new clothes or go to a museum*
$\neg(I \vee F) \vee \neg(I \vee M) \vee \neg(M \vee F)$: *can't go to more than one country*

This can be satisfied by setting $I = M = false$ and $F = V = true$. We also don't buy new clothes ($C = false$), we have money for vacation ($A = true$)and we go to a museum: ($U = true$).

Question 2:

(a)

- This statement is *true* since $S(x) \to P(x)$ for all students x. When $x = a, b$, it evaluates to $T \to T$ and when $x = c$ it evaluates to $F \to F$ and when $x = d$ it evaluates to $F \to T$.
- This statement is *true* since $\forall x S(x)$ is false (not all students are studying) and therefore the implication evaluates to true.
- This statement is equivalent to $\exists x(P(x) \wedge \neg S(x))$, which is *true* since student d is passing and not studying.

(b) Define the predicates:
$C(x, y)$: student x is in class y where y comes from the set of classes.
$G(x)$: student x is graduating
$P(x)$: student x is patient

- $\neg\exists y \forall x C(x,y)$
- $\forall y \exists x \exists z C(x,y) \wedge C(z,y) \wedge x \neq z$
- $\exists x \forall y C(x,y)$
- $\forall x (G(x) \rightarrow \exists y_1 \exists y_2 C(x,y_1) \wedge C(x,y_2) \wedge y_1 \neq y_2)$
- $\exists x \forall y (C(x,y) \wedge \neg G(x))$
- $\forall x (G(x) \rightarrow P(x))$

(c) Original statement:
$$\forall z \in \mathbb{Z}, \exists n \in \mathbb{N}(z = 2n \vee z = 2n+1 \vee z = 0)$$

Take the negation:

$$\begin{aligned}
\neg(\forall z \in \mathbb{Z}, \exists n \in \mathbb{N}(z = 2n \vee z = 2n+1 \vee z = 0)) &= \exists z \in \mathbb{Z}, \neg(\exists n \in \mathbb{N}(z = 2n \vee z = 2n+1 \vee z = 0)) \\
&= \exists z \in \mathbb{Z}, \forall n \in \mathbb{N} \neg(z = 2n \vee z = 2n+1 \vee z = 0) \\
&= \exists z \in \mathbb{Z}, \forall n \in \mathbb{N}(z \neq 2n \wedge z \neq 2n+1 \wedge z \neq 0)
\end{aligned}$$

(d)
1) Everyone who takes a positive test, gets sick.
2) If everyone has a negative test, then there exists a person who is not sick.
The statements are not equivalent. For example, consider the case where person a took a positive test, but didn't get sick. Then the first statement above is false. However the second statement is true, because $\forall x \neg T(x)$ is false.

(e) Let $A = \{1\}$ and $B = \{\{1\}, 2, 3, 4, 5, 6\}$ and we can see that $A \in B$. Then define C so that $B \in C$ and $B \subseteq C$:

$$C = \{\{1\}, 2, 3, 4, 5, 6, \{\{1\}, 2, 3, 4, 5, 6\}\}$$

(f)

- False. Suppose that $B = \emptyset$ and $A = \{1\}$. Then $A \cap B = \emptyset \neq A$, but $A - B = A$.
- False. $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ and $P(\{1,2\}) \cup P(\{2,3\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \cup \{\emptyset, \{2\}, \{3\}, \{2,3\}\} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$

Question 3:
(a)

- Not injective, since $f(2) = f(-2) = 4$ and not surjective since there is no $n \in \mathbb{Z}$ such that $f(n) = 5$.
- Not injective, since $f(2) = f(-2) = 2$. The function is surjective, since any $m \in \mathbb{N}$ has a pre-image: if m is positive, then $n = m$, and if m is negative, the pre-image is just the absolute value of m.
- Injective, since if $a \neq b$, then $\sqrt{a} \neq \sqrt{b}$. Not surjective since there is not $n \in \mathbb{R}^+$ such that $f(n) = \sqrt{-5}$.
- Not injective since $f(\{1,2\}) = 2$ and $f(\{2,3\}) = 2$. Not surjective, since no B exists such that $f(B) = 7$.

(b) No. Suppose the elements in B were $b_1, b_2, \ldots b_n$. If f were injective, then the function would map these elements to values $a_1, a_2, \ldots, a_n$ in the set A, and they would all be unique. This would mean that A had at least n elements in it, which is not possible If $|B| > |A|$.

(c) $f^{-1}(A) = \{1, 2, 3, 4, 5. \ldots, 25\}$. These are the only natural numbers whose square root falls in the range $5 \leq x \leq 5$.

(d)

- The set is finite, it has size 19 elements in it. It is countable. It does not have the same cardinality as A because A is infinite.
- The set is infinite. It is also uncountable, since it consists of a subset of the real numbers. It has the same cardinality as A, since $f(x) = 5x - 50$ is a bijective function that maps A to this set.
- The set is infinite, but it is countable since it is all multiples of 7 which can be counted as $f(n) = 7n$ using a bijection from the natural numbers. This set does not have the same cardinality as A because it is countable and A is not.

Question 4:

(a)

- $f(n) = n^{7/4} + \log n$ is $O(n^{7/4})$

TRUE

Show $f(n) \le C \cdot n^{7/4}$:

$$f(n) = n^{7/4} + \log n \le n^{7/4} + n^{7/4} = 2n^{7/4}$$

$C = 2$

$\log n < n^{7/4}$ for $n \ge K$.

- $f(n) = n^{7/4} \log n$ is $O(n^2)$

TRUE

Show $f(n) \le C \cdot n^2$:

$$f(n) = n^{7/4} \cdot \log n \le n^{7/4} \cdot n^{1/4} = n^2$$

$C = 1$

$\log n < n^{1/4}$ for $n \ge K$.

(b)

- $f(n) = \sqrt{n}$ is $\Omega(g(n))$, $g(n) = \log n + (\log n)^2$

Goal: Show $f \ge C \cdot g$

$$f(n) = \sqrt{n}$$
$$= \frac{1}{2}\sqrt{n} + \frac{1}{2}\sqrt{n} \ge \frac{1}{2}\log n + \frac{1}{2}(\log n)^2 = \frac{1}{2} g(n)$$

$C = 1/2$

$\sqrt{n} \ge \log n$ for $n \ge K$

- $f(n) = 3^n \cdot n^2 + n^2 \cdot \log n + 6^n \cdot \log n + 2^n \cdot \log^2 n$ $\quad g(n) = 6^n$.

f is $\Omega(g)$: Show $f \ge C \cdot g$:

$$f(n) = 3^n \cdot n^2 + n^2 \cdot \log n + 6^n \cdot \log n + 2^n \cdot \log^2 n$$
$$\ge 6^n \cdot \log n$$
$$\ge 6^n$$

$C = 1$

- $f(n) = \sum_{1}^{n} (K+2) = \frac{n(n+1)}{2} + 2n = \frac{n^2}{2} + \frac{n}{2} + 2n$ $\quad g(n) = 4n^2$.

f is $O(g)$:

Show $f \le C \cdot g$

$$f = \frac{n^2}{2} + \frac{n}{2} + 2n$$
$$\le \frac{n^2}{2} + \frac{n^2}{2} + 2n^2$$
$$= 3n^2 \le 4n^2 = g$$

$C = 1$

f is $\Omega(g)$

Show $f \ge C \cdot g$

$$f = \frac{n^2}{2} + \frac{n}{2} + 2n$$
$$\ge \frac{n^2}{2} = \frac{1}{8}(4n^2)$$

$C = 1/8$

- $f = \log n^{.2} = .2 \log n$ $\quad g = n^{.2} + (\log n)^{.2}$

f is $O(g)$:

Show $f \le C \cdot g$

$$f = .2 \log n$$
$$\le .2\, n^{.2}$$
$$\le n^{.2} + (\log n)^{.2} = g$$

$\log n \le n^{.2}$ for $n \ge K$

$C = 1$

(c)

Assume $n + 3n^2 \cdot \log n$ is $O(n^2)$

Then

$$n + 3n^2 \log n \le C \cdot n^2$$

$$\Rightarrow \quad \frac{n}{n^2} + \frac{3n^2 \log n}{n^2} \le C$$

$$\Rightarrow \quad 1/n + 3 \cdot \underbrace{\log n}_{\text{gets huge! } \log n \to \infty} \le \underbrace{C}_{\text{CONSTANT}}$$

} Impossible! $\therefore$ f is NOT $O(n^2)$.

Question 5

(a)

$$\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \frac{27}{16} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n \cdot \frac{1}{3}$$, geometric sum

$\frac{3}{2} > 1$ $\therefore$ sum to infinity doesn't exist.

(b)

$$\sum_{n=1}^{\infty} (-2)^{\lceil n/2 \rceil} = (-2)^1 + (-2)^1 + (-2)^2 + (-2)^2 + (-2)^3 + (-2)^3 + \cdots$$

Partial sums.

$S_1 = -2$

$S_2 = -2 - 2 = -4$

$S_3 = -2 - 2 + 4 = 0$

$S_4 = -2 - 2 + 4 + 4 = 4$

$S_5 = -2 - 2 + 4 + 4 - 8 = -4$

$S_6 = -2 - 2 + 4 + 4 - 8 - 8 = -12$

$\downarrow$ NOT CONVERGING

$\therefore$ Sum doesn't exist.

(c)

$$\sum_{i=-2}^{\infty} \frac{2^{i-1}}{3^i} = \sum_{i=-2}^{\infty} \left(\frac{2}{3}\right)^i \cdot \frac{1}{2} = \frac{1}{2}\left[\frac{1}{1 - 2/3}\right] + \underbrace{\frac{2^{-2}}{3^{-1}}}_{i=-1} + \underbrace{\frac{2^{-3}}{3^{-2}}}_{i=-2}$$

$$= \frac{3}{2} + \frac{3}{4} + \frac{9}{8}$$

(d)

At day n, I deposit $(100) \cdot \frac{1}{2^{n-1}}$ into the account, starting at $n = 1$.

told in question

Total after 50 days: $\sum_{n=1}^{50} 100 \left(\frac{1}{2^n}\right) \cdot 2$

$$= 200 \sum_{n=1}^{50} \left(\frac{1}{2}\right)^n = 200 \left[\frac{(1/2)^{51} - 1}{1/2 - 1} - 1\right]$$

$$= 200 \left(2 \cdot \left(1 - \frac{1}{2^{51}}\right) - 1\right)$$

$$= 200 \left(1 - \frac{1}{2^{50}}\right)$$

You will never achieve \$200 because you can't sum to infinity! $\sum_{n=1}^{\infty} 100 \left(\frac{1}{2^n}\right) \cdot 2 = 200 \left[\frac{1}{1 - 1/2} - 1\right] = 200$