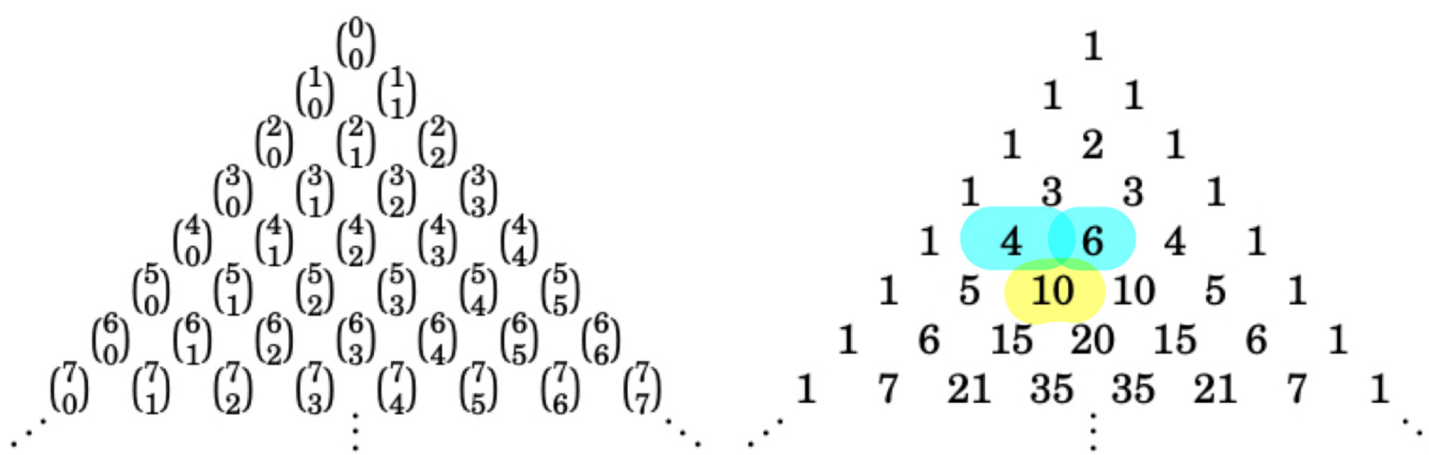
Counting: part 3

1 Binomial Theorem and Pascals Triangle

Until now we have defined the binomial coefficient, $\binom{n}{k}$, as the number of ways of selecting a subset of size k out of a group of n objects. There are some significant and beautiful patterns among the numbers $\binom{n}{k}$. To demonstrate these patterns, we start by arranging the binomial coefficients into a triangular scheme: At the top vertex, we put $\binom{0}{0}$. In the row just below, we put $\binom{1}{0}$ and $\binom{1}{1}$. In the row below that we put $\binom{2}{0}, \binom{2}{1}, \binom{2}{2}$. We shift the rows so that the items form a triangle. In general, then nth row contains the numbers

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

The resulting triangle is known as Pascal's triangle after the French mathematician Blaise Pascal (1623-1662).



On the right hand side, the binomial coefficients are replaced with their numerical values. If we look at a particular row of the triangle (ex: $1, 5, 10, 10, 5, 1$), we notice that the triangle is symmetric with respect to the vertical line through its tip. For example, $\binom{6}{4} = \binom{6}{2} = 15$. This is an important fact regarding the binomial coefficient, that in general:

$$\binom{n}{k} = \binom{n}{n-k}$$

There are many identities satisfied by these coefficient, one of the most important being Pascal's identity. In the figure above, you may notice that the sum of the numbers highlighted in blue is equal to the number highlighted in yellow. In general, every number in Pascal's triangle is the sum of the two numbers immediately above it (except for the 1's on the boundary).

Pascal's Identity:

Theorem 1. *Let n and k be positive integers with $n \geq k$. Then*

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

For example, the above theorem tells us that $\binom{5}{2} = \binom{4}{2} + \binom{4}{1} = 6 + 4$. The proof of this identity can be given algebraically (by manipulating the formula $\binom{n}{k}$) or combinatorially. We will introduce combinatorial proofs in the next section and provide a proof for Theorem 1.

We now turn to one of the most important properties of Pascal's triangle. One might notice that the nth row of Pascal's triangle lists the coefficients of $(x+y)^n$. For example using the row $n = 3$ from the triangle, we have the coefficients $(1, 3, 3, 1)$:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

and from the 4th row, $(1, 4, 6, 4, 1)$:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

This fact is true for every n, and is known as the **binomial theorem**:

Theorem 2. *The Binomial Theorem states that if n is any non-negative integer, then*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Using the binomial theorem, we can quickly expand expressions such as $(x+y)^7$ and determine their coefficients. For example,

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

We now turn to some combinatorial facts regarding the binomial coefficients.

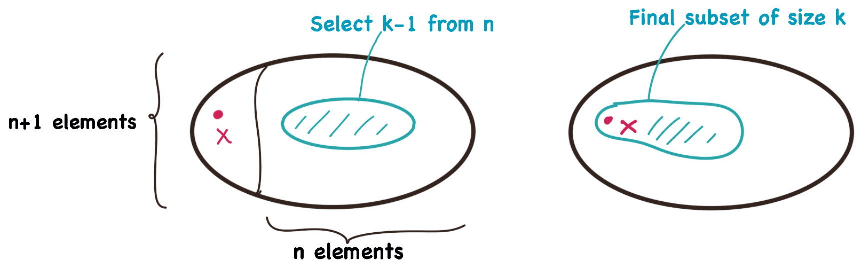
2 Combinatorial proofs

There are many interesting identities involving the binomial numbers. Some of them can be proved by algebraic manipulation, or by what is referred to as a **combinatorial proof**. In a combinatorial proof, we argue that we can count the same thing in *two* different ways: one way which represents the expression on the the left side of the equation, and one way which represents the expression on the right. If we can argue that both methods **count the same thing**, then the left and right sides of the expression **must be equal**. In order to demonstrate this technique, we provide a combinatorial proof of Theorem 1, which states that for $n \geq k$:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

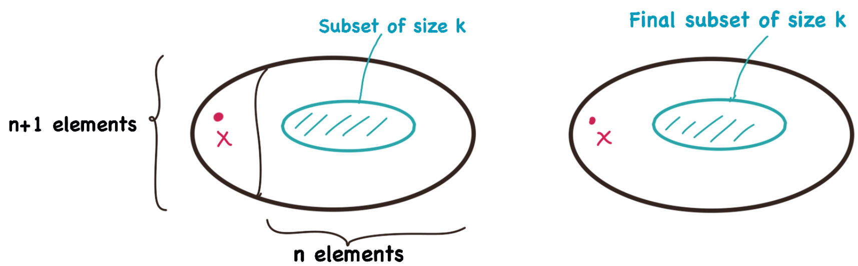
Notice that the left-hand side above is the number of ways of selecting a subset of size k out of $n+1$ elements. We now show that the right-hand side **also counts** the number of ways of selecting a subset of size k out of a set of $n+1$ elements. We do this by isolating a specific element and considering two cases for the subset. Imagine that we label some element in the set S as x. Any subset of size k will either *include* x (**Case 1**) or *not include* x (**Case 2**).

- **Case 1:** If the subset includes x, then in order to select a total of k elements, we would need to select $k-1$ *more* elements from the n that remain. The subset would then be the element x combined with the $k-1$ other elements.



The number of ways of selecting a subset in this way is exactly $\binom{n}{k-1}$.

- **Case 2:** On the other hand, suppose the subset does not include x. Then we would need to select k elements from the n that remain. This is a total of $\binom{n}{k}$ possibilities.



We can *sum* these two cases together, because they represent disjoint cases. Thus the total of number of ways of selecting a subset of size k from $n+1$ elements (using two cases) is exactly:

$$\binom{n}{k-1} + \binom{n}{k}$$

This is a combinatorial proof that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

Example 1. *Show that*

$$1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Solution: The right hand side of the above expression is simply the number of possible subsets (of *any* size) from a set of n elements. We can count the number of subsets in a **different way** and obtain the expression on the left. The result is a combinatorial proof of the above equation.

The idea is to split the number of subsets of an n-element set into disjoint cases, based on the number of possible elements in any subset.

- Case 1: the subset has size 0. There is only one possibility: the empty set.
- Case 2: the subset has size 1. There are $\binom{n}{1}$ possible ways of choosing a subset of size 1.
- Case 3: the subset has size 2. There are $\binom{n}{2}$ ways of choosing a subset of size 2.
- ...
- Case n: the subset has size n-1. There are $\binom{n}{n-1}$ ways of choosing a subset of size $n-1$.
- Case n+1: the subset has size n. There are $\binom{n}{n}$ ways of choosing a subset of size n.

Since these cases are all disjoint and they represent *all* possible subset sizes over n elements, then their *total sum* is the number of subsets of *any* size:

$$1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$$

which must be equivalent to the right hand side, 2^n.

Example 2. *Use a combinatorial proof to show that*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Solution: The right-hand side is the number of ways of selecting n items from a set S of $2n$ elements. We can count these options in another way and show that it represents the total on the left-hand side. Let's divide the set S in half. The first half we will call set A and the second half, set B. Note that they will each have size n. Now, any subset of size n that we select from the original set S will contain a certain number of elements from set A and a certain number of elements from set B. If we consider *all* possible cases for the different ways of selecting the n elements across A and B, then we will have considered *all* subsets of size n from the original set S. Thus the different possibilities for selecting a set of size n from S is :

- Case 1: 0 items from A, and n items from B: $\binom{n}{0}\binom{n}{n}$
- Case 2: 1 item from A, and $n-1$ items from B: $\binom{n}{1}\binom{n}{n-1}$
- Case 3: 2 items from A, and $n-2$ items from B: $\binom{n}{2}\binom{n}{n-2}$
- Case 4: 3 items from A, and $n-3$ items from B : $\binom{n}{3}\binom{n}{n-3}$
- ...
- Case n+1: n items from A, and 0 items from B. $\binom{n}{n}\binom{n}{0}$

Each of these cases is *disjoint*. Thus we can sum them up and they will cover all possible ways of selecting n items from S:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$$

Using the fact that $\binom{n}{k} = \binom{n}{n-k}$ the above equation is equivalent to $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Example 3. *Show that*

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

Solution:. The left-hand side is the number of ways of selecting 2 items from a set S of size $2n$ elements. As above, let's divide the set S into two sets A and B each of size n. Then if we select 2 items there are 3 possibilities:

- Case 1: Both elements came from set A: $\binom{n}{2}$
- Case 2: Both elements came from set B: $\binom{n}{2}$
- Case 3: One item from A and one item from B: n^2

Summing these cases together gives us the result.