Counting!

For most of us, the notion of counting refers to the process of pointing to each object and calling off "*one,two,three,...*". In general, counting, or enumerating objects with certain properties can be quite complex, and this primitive approach becomes inadequate. The study of counting is an area of *Combinatorics* that has been studied for hundreds of years, and today has many applications in computer science ranging from algorithm analysis to game theory.

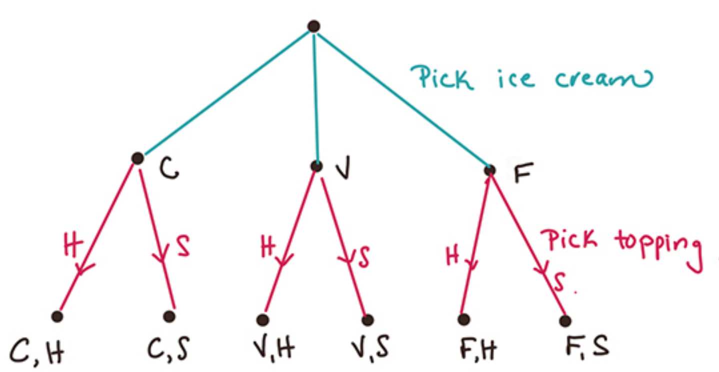
1 Basic Counting

We have already introduced some notions of *counting* when we studied the cardinality of sets. We noticed that we could *count* the rational numbers, by finding an enumeration $1, 2, \ldots$ of each element. Essentially we were creating a bijection from the natural numbers to the rational numbers. In this type of example, we are able to count the elements of a set directly, by *listing* them in order of the natural numbers.

In more complex situations, we can use certain techniques to reduce the need to directly count (one by one) the objects in a set.

1.1 Product Rule

When we are counting the number of possibilities of completing two tasks, where the first task has n_1 possibilities and the second task has n_2 possibilities, then the **total** number of ways of completing **both** tasks is $n_1 n_2$. For example, if we have three possible ice cream choices (Chocolate, Vanilla, Fudge) and two possible toppings (Hot sauce or Sprinkles) then we can imagine visualizing our possible ice cream options as shown below:



By the product rule, $3 \cdot 2 = 6$, and thus as we can see above, there are 6 possible final choices of ice cream orders.

Example 1. *The number of possible subsets of a set S on n elements is 2^n.*

Solution: We presented a proof to this fact in our lecture on induction. We can now present a new argument, using the product rule. If we order the elements of S as $a_1, a_2, \ldots, a_n$, then the task of creating a subset can be broken down into the task of deciding for each element a_i if it will be *in* or *out* of the subset. Thus for the first element a_1, there are *two* options (in or out), and for the second element a_2 there are two options (in or out), etc. Once we have gone through all the elements (of which there are n), then the total number of ways of creating a subset is $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$.

Example 2. *How many different license plates are there consisting of 3 letters followed by 3 digits (1-9)?*

Solution: The first character has 26 possible choices, as does the second and third. By the product rule, this makes 26^3 possible choices. Next, we have 9 choices for the 4th, 5th and 6th characters. By the product rule, the total number of possibilities is $26^3 9^3$.

1.2 Sum Rule

If a task can be done in **either** n_1 ways **or** n_2 ways where the options n_1 do not overlap with any of the options n_2, then the **total** number of ways to carry out the task is $n_1 + n_2$.

In set theory notation, this is equivalent to writing

$$|A \cup B| = |A| + |B|$$

as long as the sets A, B are disjoint.

Suppose you have to make a 3-character long password, where the password must consist of three alphabetic letters, OR of three digits. The number of ways of creating a password out of three alphabetic letters is 26^3 by the product rule. Similarly, the number of ways of creating a password out of three digits is 10^3. Since these represent different, non-overlapping tasks, then the total number of ways of creating such a password is:

$$26^3 + 10^3$$

As long as the sets we are counting are *disjoint* then we can use the sum rule.

We can also *rearrange* the sum rule as follows:

$$|A \cup B| - |B| = |A|$$

for disjoint sets A and B. This allows to use the sum rule to actually *subtract off* the cases that we might *not want to count*, as shown in the example below.

Example 3. *How many ways can you create a* ***non-empty*** *subset from set S, where $|S| = n$.*

Solution: The total number of subsets of a set of size 10 is 2^{10}, as we saw in Example 1. However, this includes *all possible* subsets. Since we want to exclude the number of subsets that are *empty*, then we can subtract those cases off as follows:

$$\text{Total number of non-empty subsets} = (\text{\# of subsets of } S) \text{ - } (\text{\# of empty subsets of } S)$$

Therefore the total number of non-empty subsets of S is exactly $2^{10} - 1$.

Example 4. *A certain school offers 9 courses every day: 5 in arts and 4 in science. On any day of the week, a student can chose to participate in either one of five arts classes, or one of four science classes. How many different choices does the student have over the whole week? What if the student must take at least one arts class?*

Solution: On any given day, the student has 9 courses from which to chose. By the product rule, this results in 9^7 different ways of selecting courses over the 7 days. However, if the student must take at least one arts class, then we must subtract the cases that represent *no* arts classes:

Possible selections with at least one arts class = 9^7 - (# of course selections having no arts classes)

Having *no* arts classes means that student would have selected only science courses over the week, for which there are 4^7 possibilities. Therefore, the number of possible course selections over the week which include at least one arts class is simply $9^7 - 4^7$.

1.3 Inclusion-Exclusion

In some cases, the sets that define each task are **not** disjoint. In other words, the tasks in the first list can *coincide* with some of the tasks in the second list. We cannot simply add possibilities from the first list to the second list, because then we would have *double counted* some possibilities.

For example, suppose we create a password of length 4 out of the characters $\{A, B, C\}$. What are the number of ways of having a password with three A's in a row? In other words, we want to consider the ways of creating passwords of the form $AAA\_$ or $\_AAA$.

- The number of ways of having $AAA\_$ can be counted as follows: for each character we count the number of options available, 1 option for the first three characters, and then 3 options for the last character. Using the product rule that gives a total of $1 \cdot 1 \cdot 1 \cdot 3 = 3$.
- Similarly, the number of ways of having $\_AAA$ is 3.
- If we **sum** the above two results, we will have **double counted** the cases where the password starts with AAA and also ends with AAA: which is possible if the password were $AAAA$.
- The principle of **inclusion-exclusion** is simply to subtract what was double-counted: thus the number of passwords is $3 + 3 - 1 = 5$.

This concept of subtracting off the cases that were double-counted, is the principle of inclusion-exclusion on two sets:

The principle of Inclusion-Exclusion for sets A and B states that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 5. *How many ways are there of throwing 2 dice such that at least one three appears?*

Solution: Let A be the cases where the first dice is a three, and B be the cases where the second dice is a three. Then $A = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$, and so $|A| = 6$. Similarly, $|B| = 6$. However, if we sum $6+6$, we have double-counted the cases where a three appears on both die. Thus the total number of ways of throwing a die such that at least one three appears is:

$$|A| + |B| - |A \cap B| = 6 + 6 - 1 = 11$$

The principle of inclusion-exclusion can be extended to more than 2 sets. For example, for sets A, B, C which are not necessarily disjoint we have:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

2 Pigeonhole

The pigeonhole principle is a fundamental counting technique which is applied to situations where we have *more objects* than *bins*. The concept is illustrated nicely by imagining a set of at least $k + 1$ pigeons flying over a set of k pigeonholes. If each pigeon needs to find a hole, and since there are *more pigeons* than holes, someone will have to share. This is the basic principle of the pigeonhole method.

> **The Pigeonhole Principle** If k is a positive integer and we have at least than $k+1$ objects placed into k boxes, then there is at least one box containing more than one object.

For example, if we have 10 pigeons and only 9 holes, that *at least one hole* will have two pigeons in it. Notice that it *is* possible that there are empty holes (which happens if several pigeons decide to share). For example, we could have 4 pigeons in the first hole, 4 in the second, and two pigeons in the third hole - leaving 6 holes empty. The pigeonhole method does *not* tell us about *empty* holes, it only **guarantees that some holes will consist of more than one object.**

This fact can be generalized to even more objects, in which case we have a box that is shared by more than two objects.

> **The generalized Pigeonhole Principle** states that if N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects.

Example 6. *If 10 pigeons fly into 4 holes, show that there must be some hole with at least 3 pigeons in it.*

Solution: By the pigeonhole principle, there must be some hole with at least $\lceil 10/4 \rceil = 3$ pigeons in it.

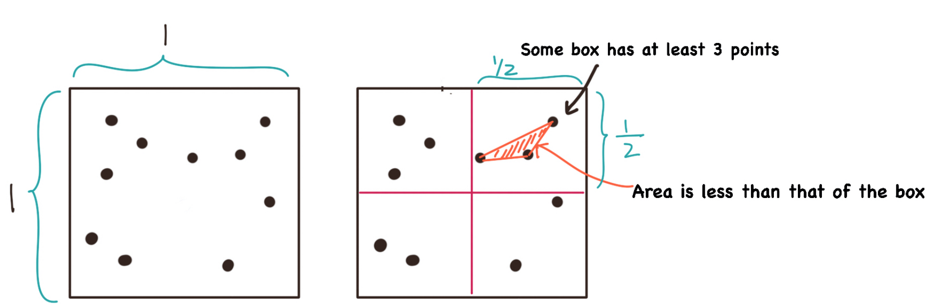
Example 7. *In a group of 367 people, there are at least 2 people who have the same birthday*

Solution: There are 365 possible birthdays (your pigeonholes) and 367 people. Thus at least 2 people are born on the same day.

Example 8. *Suppose a gumball machine is full of red, green, and blue gumballs. The store has a deal that if you buy 5 of the same color, then you win \$5. What is the fewest number of gumballs you need to buy to be certain that you will win? (assuming that the gumballs come out of the machine with some random color).*

Solution: In this example, our "*holes*" are the color categories. We would like a color category to contain at least 5 gumballs. By the pigeonhole principle, if we buy n gumballs, there is some color category containing $\lceil n/3 \rceil$ balls. If we want this number to be 5, then we need to buy 13 gumballs ($\lceil 13/3 \rceil = 5$).

Example 9. *Suppose we have a square of unit area. Inside the square we drop 10 points randomly. Show that at least 3 of these points form a triangle whose area is $\frac{1}{4}$ or less.*



Solution: We can slice the square up into 4 smaller squares, each of area $\frac{1}{4}$. If these are our compartments, then of the 10 points that are placed into the larger square, at least $\lceil 10/4 \rceil = 3$ of them must fall into the same smaller square. Three points can always be connected to form a triangle. The area of the triangle must be smaller than the area of the square, since the triangle is inside the square. Thus there exists a triangle whose area is less than $1/4$.

Example 10. *Show that if we take $n+1$ numbers from the set $\{1, 2, 3, \ldots, 2n\}$, then some pair of numbers will have no common factors*

Solution: As an example, suppose $n = 10$. Then in the list of numbers from 1 to 20, there is certainly at least one pair with no common factors: $(5, 7)$ is one such pair. If we use the pigeonhole principle to solve this problem, we need to determine what out "*pigeonholes*" will be. Suppose we pair numbers up in the following way:

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2n-1, 2n\}$$

Notice that there will be exactly n of these pairs. They will represent our pigeonholes. Now we select $n+1$ numbers in the range $1 \ldots 2n$, (these are our "*pigeons*"), and they will each land in one of the pairs above. However there are $n+1$ numbers and only n pairs, so at least two numbers will fall into the same pair. Therefore at least two numbers will be *consecutive*. And consecutive numbers have no common divisors. Therefore at least two numbers will have no common divisors.

3 Permutations and Combinations

3.1 Permutations

Many counting problems deal with the number of ways to *arrange* distinct elements, where the *order* of the elements matters. For example, how many ways can we seat people at a table, how many ways can we line up 3 students out of 5 for a picture?

A permutation of a set of objects is an **ordered** arrangement of the objects. For example, if we are given the set $S = \{a, b, c\}$ then a particular permutation is the arrangement c, a, b. We are interested in determining the **number of possible arrangements** in a set S of size n. There are two scenarios that we consider here: the first where we arrange *all* the elements of the set, the second where we arrange only some subset of the elements.

Theorem 1. *The number of possible arrangements of n objects is:*

$$n(n-1)(n-2)\ldots 1 = n!$$

Theorem 2. *The number of possible arrangements of k elements out of a set of size n is :*

$$n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Before looking at some examples, we introduce the second related concept.

3.2 Combinations

Given a set S of n distinct elements, another question we might ask is how many ways we can **select** a subset of a certain size out of the set S. Contrary to a permutation, we are not concerned with ordering the elements, we simply want to know how many different options exist if we select a smaller subset from the larger one. For example, how many ways can we select 3 people out of 5? This leads us to the notion of a *combination*.

A k-combination of elements of a set is an **unordered** selection of k elements from a set of size n. We use the notation

$$\binom{n}{k}$$

to denote the number of ways of selecting k elements out of n. Again, this is an **unordered** selection.

Theorem 3. *The number of ways of selecting k elements out of a set of size n (where n is a nonnegative integer and $0 \leq k \leq n$) equals:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

3.3 Examples:

Example 11. *How many different ways are there of creating a hand of 5 cards (out of a 52 card deck) which consists of 2 clubs and 3 hearts?*

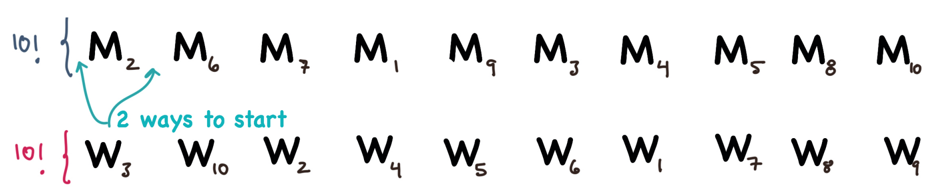
Solution: First we can consider the number of ways of selecting the clubs. There are 13 clubs and we want to select 2. The number of ways is thus $\binom{13}{2}$. We want to also select 3 hearts out of 13, and the number of ways of doing that is $\binom{13}{3}$. These two totals should be *multiplied* together because of the product rule discussed above. Thus the total number of ways of creating such a hand is

$$\binom{13}{2}\binom{13}{3} = \frac{13!13!}{(2!11!)(3!10!)}$$

Example 12. *A group consists of 10 women and 10 men. How many ways are there to arrange these people in a row if the men and women must alternate?*

Solution: Since they must alternate, it is best to start with the men, arrange them, and then consider the number of ways of placing the women in between them. There are 10 men and thus 10! ways or arranging the men. Now we consider how to arrange the women around the men. There are also 10! arrangements for the women, and now we consider how to line them up in between the men. Since the men and women must alternate, we don't have a lot of choice. We will simply take the women's arrangement and place them one at a time in between the men. However we do have one choice left: either we start with a man or we start with a women. There are 2 options. Thus the total number of possibilities is:

$$2(10!)(10!)$$



Example 13. *How many different ways are there to set 10 people at a circular table?*

Solution: Normally when we arrange n elements, we consider them arranged in a line, because the person who is standing in position 1 is clearly different from the person who is standing in the last position. When people are seated at a *circular* table, there is no first and last, the arrangement is simply based on who is seated to the left and the right of you. Therefore the *first* person who sits down may sit anywhere. Afterwards, the 9 remaining people need to be arranged with respect to his position. Therefore the number of possible seatings is 9! (not 10!)

Example 14. *There are seven women and nine men in a school department. How many ways can you select a committee of 5 people which includes a woman president? What if you just need one woman on the committee and no president?*

Solution: There are 7 possible ways of selecting a woman president to put on the committee. After a woman is chosen, there are $6 + 9 = 15$ remaining people from which we can chose the remaining members. Out of this 15 we only need 4 more members to complete the committee. Thus the total number of possibilities is:

$$7\binom{15}{4}$$

Now suppose we just want at least one woman on the committee. There are $\binom{16}{5}$ different ways of making a committee of 5 people. If we made a committee with *no* woman, there would be $\binom{9}{5}$. The the difference:

$$\binom{16}{5} - \binom{9}{5}$$

gives us the total number of ways of having at least one woman on the committee.

Example 15. *Suppose we have 4 math books, 3 chemistry books and 7 history books. We want to count the number of ways of arranging them on the shelf so that in the final arrangement, the books are organized by subject (i.e. all the math books are together, etc).*

Solution: This can be solved in two steps:

- First we decide the arrangement of the *subjects*. In other words, we decide if we order them as math, chemistry, history, or history, math, chemistry, etc. There are three subjects, thus the number of possible ways of organizing the subjects is $3! = 6$.
- Next, we imagine that we have decided how the subjects will be arranged, and as we place the books on the shelf, we can also arrange the books *within* each subject. There are 4! ways of arranging the math books, 3! ways of arranging the chemistry books and 7! ways of arranging the history books.

The total number of final arrangements of the books is:

$$6(4!)(3!)(7!)$$