Binary Relations

1 Basic definitions

Suppose we have a set A and suppose that for any two elements $x, y, \in A$ we can ask the question: "*Is x related to y*"? The exact notion of "*related*" can have many meanings, as we shall see. The pairs for which the answer is "*yes*" define the specific *relation* over A. Binary relations are used in many areas of computer science to simply define how elements within sets are *connected* or *related* to each other. We see applications of binary relations in areas of networking, scheduling, and often indirectly in data structures and algorithms.

Many binary relations can be found in our mathematical notation, although one may not have been aware that they were binary relations. For example, the symbol '$<$' is a binary relation on the real numbers, because for any two $x, y \in \mathbb{R}$ we can ask "*Is $x < y$*". All the pairs of real numbers for which $x < y$ belong to this relation. Thus the binary relation $'<'$ connects *all pairs* of real numbers for which $x < y$. This is an example of a binary relation from the set $\mathbb{R}$ to $\mathbb{R}$, which is an example of a relation where A and B are the *same* set.

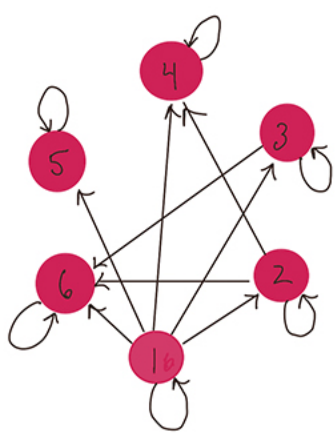
The formal definition is given below:

Definition. *A binary relation, R, on two sets A and B is a subset of the Cartesian product $A \times B$. We write $(a, b) \in R$ or aRb. The two sets A and B do not need to be distinct.*

Let's take a closer at how to visualize binary relations, using as an example the *divisibility* relation, "|" over the set $A = \{1, 2, 3, 4, 5, 6\}$. In other words, $(x, y) \in R$ iff x divides y. The binary relation is a subset of $A \times A$ and it contains all the following pairs:

$$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(5,5),(6,6)\}$$

We could draw this as a graph where each *node* of the graph is an element of A and an edge is drawn from a to b if a divides b:



The usual "=" sign is a binary relation over the real numbers. In this case, xRy whenever $x = y$. So the binary relation would include all pairs for which $x = x$. For example, it would include $(1, 1), (2.3, 2.3), (4.6, 4.6), \ldots$. There are an infinite number of real numbers!

Functions are also binary relations. Suppose $f : X \to Y$. Then $f \subseteq X \times Y$ where xRy whenever $f(x) = y$. This is a special type of binary relation because for *every* $x \in X$ we have exactly *one* pair $(x, y) \in R$. Let's take a look at a specific example and how we can describe this as a relation. Suppose $f(x) = x^2$ is a function from $A = \{1, 2, 3\}$ to $B = \{1, 2, \ldots, 10\}$. Then the binary relation that represents the function is simply the pairs (a, b) where $f(a) = b$. This would be exactly the following pairs:

$$(1, 1), (2, 4), (3, 9)$$

Notice that there is exactly one pair for each element in A, and no other pairs from $A \times B$ would be included.

Any binary relation is *not necessarily* a function, since it is possible that the relation does not satisfy the definition of a function. Suppose we use the same sets as above: $A = \{1, 2, 3\}$ and $B = \{1, 2, \ldots, 10\}$. Define a new binary relation on $A \times B$ using the following pairs:

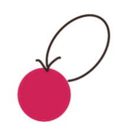
$$\{(1, 2), (1, 3), (2, 4)\}$$

The above set is a *binary relation* on $A \times B$ but **not** a *function*. It is impossible that $f(1) = 2$ and $f(1) = 3$ in a properly defined function.

We now look at certain properties of binary relations. We begin by giving their formal definitions, but in practice we can identify these properties fairly easily using their "*visual*" interpretation in a graph.

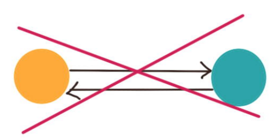
Definition. *Let R be a binary relation on the set $A \times A$ and $x, y, z \in A$. Then R is:*

Reflexive if "x is related to x". That is, $(x, x) \in R$ for all $x \in A$. This means that every node is related to itself, so there is a "loop" on each node:

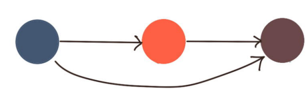


Symmetric if "x related to y implies y related to x". That is, if $(x, y) \in R$, then $(y, x) \in R$ for any $x, y \in A$. This means that each arrow goes in both directions.

AntiSymmetric if "x related to y implies that y is not related to x unless x=y". That is, for $x \neq y$, $(x, y) \in R$ and $(y, x) \in R$ are not both possible. This means that there are no arrows in both directions between two nodes.



Transitive if "x related to y, and y related to z, implies x is related to z". That is, for $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$. This means that every chain of two arrows must have a single arrow that goes from start to finish.



Examples:

- The binary relation "$\leq$" is reflexive (ex. $5 \leq 5$) and transitive (ex. $2 \leq 3$ and $3 \leq 5$ then $2 \leq 5$), but not symmetric. In fact it is *antisymmetric* since if $x \leq y$ and $y \leq x$ then this is only possible if $x = y$.

- The binary relation "$|$", divides, which is described above, is reflexive. Note that any x divides x. It is also transitive since if x divides y and y divides z, then x divides z. It is not symmetric since if x divides y, then y cannot divide x unless $x = y$. It is antisymmetric.

- For $a, b \in \mathbb{N}$ define the binary relation aRb if $a - b$ is *divisible by 3*. This relation is reflexive, since $a - a$ is divisible by 3. It is symmetric, since if $a - b$ is divisible by 3, then so is $b - a$. And it is transitive since if 3 divides both $a - b$ and $b - c$, then it divides $a - c$, (you should be able to check this on your own...) This is an example of an **equivalence relation**.

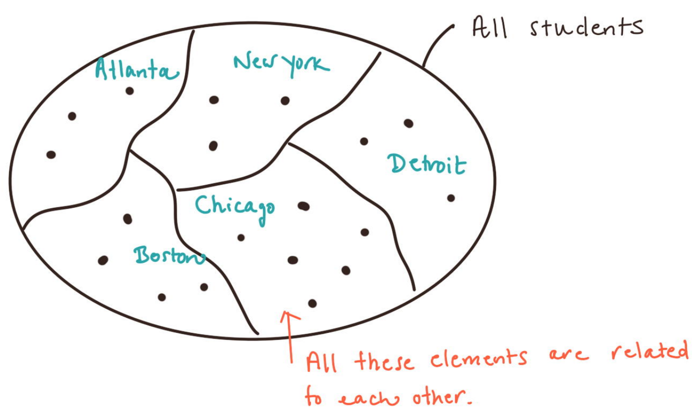
Definition. *A binary relation on the set A that is reflexive, symmetric and transitive is an equivalence relation. The notation "≡" or ∼ is sometimes used for an equivalence relation.*

Examples:

- Let R be the relation on the real numbers such that $(a, b) \in R$ if and only if $a - b$ is an integer. Let's check if this is an equivalence relation. First we check *reflexivity:*. $(a, a) \in R$. Note that $a - a = 0$ is an integer, and so this relation is reflexive. Next we check symmetry. Suppose that $(a, b) \in R$, which means that $a - b$ is an integer. Then clearly $b - a$ is also an integer, and this relation is symmetric. Finally, if $a - b$ is an integer and $b - c$ is an integer, then $a - c = (a - b) + (b - c)$ is also an integer. Thus the relation is also transitive. This is an example of an *equivalence relation*.
- Let R be a relation on the power set, $P(\mathbb{N})$. Recall that $P(\mathbb{N})$ is the set of all subsets of the natural numbers. So the relation R must relate two *subsets*. We define R as: $(x, y) \in R$ for two sets $x, y \in P(\mathbb{N})$ if and only if $x \subseteq y$. Let's check the properties of binary relations. First, we check if the relation is reflexive, i.e. if $(x, x) \in R$. Note that $x \subseteq x$, so this relation is reflexive. Is it symmetric? If $(x, y) \in R$ then $x \subseteq y$. But this does not mean that $y \subseteq x$ unless the sets are equal: $x = y$. Thus the relation is *antisymmetric*, NOT symmetric. The relation is transitive, since if $x \subseteq y$ and $y \subseteq z$, then $x \subseteq z$. This is *not* an equivalence relation.

2 Partitions and equivalence classes

One effect of an **equivalence relation** on $A \times A$ is that it *partitions* your set A into distinct groups. Before giving the definition, let's start off with a simple example. Suppose that our set A is the students at NYU and the relation R consists of all pairs (x, y) where x and y were born in the same city. Then if we are given any student x, we can form the set of *all* students equivalent to x with respect to the relation R. This set will include all the students at NYU who were born in the same city as x. The effect is that all of the students will be categorized into their birth cities, as shown in the picture below:



Each of the "*categories*" above is called an **equivalence class**.

Definition. *Let R be an equivalence relation on the set A. The set of all elements that are related to an element x is called the equivalence class of x, and it is denoted $[x]_R$, or simply $[x]$ if it is clear which relation we are referring to. That is,*

$$[x]_R = \{y \in A | (x, y) \in R\}$$

Using this specific notation is not so important in this course. The concept of an equivalence class is simply the set of all elements that are related to each other, and we use the notation $[x]$ to denote all the elements in the same category as x.

Example 1. *Let R be the relation on the natural numbers from 1 to 30 where aRb if and only if $a - b$ is divisible by 4. Determine the equivalence class of the elements* $1, 2, 3, 4$.

Solution: Let's start with the element 1. We need to consider all natural numbers $b \leq 30$ such that $1 - b$ is divisible by 4. Clearly, 1 is in this set, and 5 is in this set since $5 - 1$ is divisible by 4, and we notice that other elements in this set follow the pattern:

$$[1] = \{1, 5, 9, 13, 17, 21, 25, 29\}$$

Next, the element 2 has the equivalence class:

$$[2] = \{2, 6, 10, 14, 18, 22, 26, 30\}$$

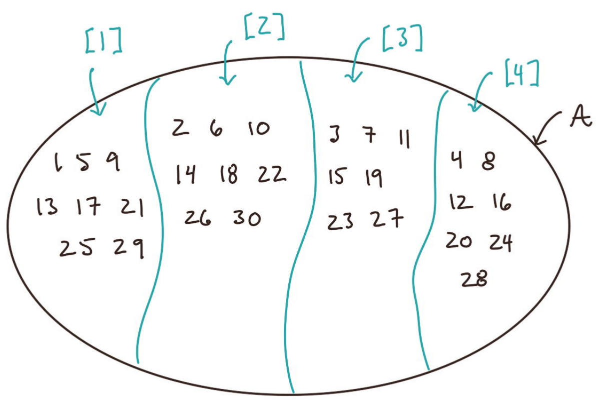
and

$$[3] = \{3, 7, 11, 15, 19, 23, 27\}$$

and lastly,

$$[4] = \{4, 8, 12, 16, 20, 24, 28\}$$

Suppose we continue to find the equivalence class of the element 5. We would simply get all elements where $5 - b$ is divisible by 4. Notice that this would be the set $\{1, 5, 9, 13, 17, 21, 25, 29\}$. In other words, we would get the same set as [1]. Thus we have found that $[1] = [5]$. Similary, $[9] = [1]$, etc. In other words, there are exactly four *distinct* equivalence classes over the set A defined by this relation. Any other equivalence class ends up being identical to one we already found. Thus the set A can be divided into these four classes:



Note that the numbers from 1 to 30 are partitioned into four classes, and each number is in exactly one class. This fact is stated formula in the theorem below: This demonstrates an important fact regarding equivalence classes: (we will not see the proof here).

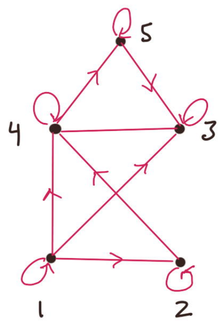
Theorem 1. *If R is an equivalence relation over A, then every element $a \in A$ belongs to exactly one equivalence class. If $b \in [a]$ then $[b] = [a]$. The relation forms a partition of the elements in A.*

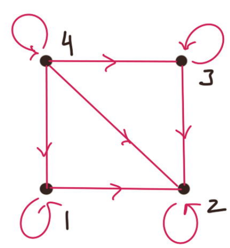
3 Partial Orders and Total Orders

Relations are often used over sets to order *some* or *all* of the elements. For example, we could define a relation over a set of jobs, where (x, y) means that job x must be completed before job y. Such a relation would order some of the jobs in our set, but not necessarily all of them. This is an example of a *partial order*. As a second example, note that the integers are ordered using the relation $\leq$. In this case, *all* the elements of the set are ordered, which is referred to as a *total order*.

Definition. *A relation R on a set A is a partial order if it is reflexive, antisymmetric, and transitive. The set A and the relation R is called a partially ordered set.*

Partially ordered sets can also be identified by their graphs. If the graph has no symmetric edges, and includes all the transitive edges and reflexive edges then it is a partially ordered set. Any two elements that do not have an edge between them are *not comparable*. In the graphs below, the figure on the left is *not* a partial order, although it contains all the reflexive edges and no symmetric edges. Since $(1, 4) \in R$ and $(4, 5) \in R$, but $(1, 5) \notin R$, the relation is not transitive. Thus it is not a partial order. The figure on the right however *is* a partial order, since it is reflexive, transitive, and antisymmetric.





From the examples we saw earlier, many of them were partial orders because we showed they were reflexive, transitive and antisymmetric:

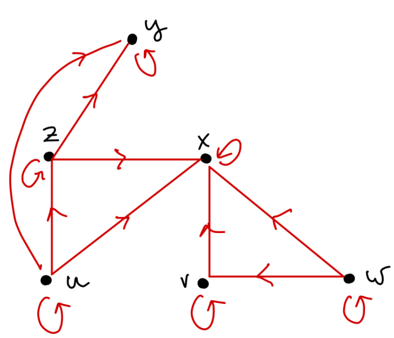
- The relation "$\leq$" is a partial order on the set of integers.
- The relation "*divides*" on the set of integers is reflexive, antisymmetric and transitive, so it is also a partial order.
- The relation "$\subseteq$" on the set $P(X)$ is a partial order, for any set X.

In a partially ordered set, it is possible that some pairs are *not comparable*: in other words neither xRy nor yRx. For example, in the partial order "$\subseteq$" on the set $P(S)$ for $S = \{1, 2, 3\}$, suppose $a = \{1\}$ and $b = \{2, 3\}$. Then $a \nsubseteq b$ and $b \nsubseteq a$. In this case we say that a and b are *not comparable*.

Partially ordered sets do not necessarily have a *maximum* or *minimum*. Instead, they may (or may not) have a **maximal** and/or a **minimal** element. The idea of a maximal element, m, is simply that there is no "*bigger*" element in the partial order, and the idea of a minimal element, s is that there is no "*smaller*" element, s in the partial order. The formal definition is given below.

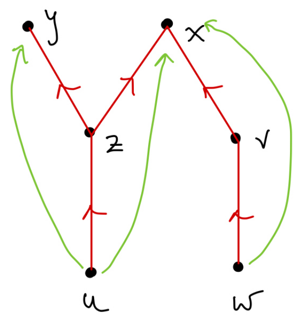
Definition. *An element m is maximal in a partially ordered set S if there is* ***no*** *$b \in S$ such that $(m, b) \in R$ (unless $b = m$). An element s is minimal if there is* ***no*** *$b \in S$ such that $(b, s) \in R$ (unless $b = s$).*

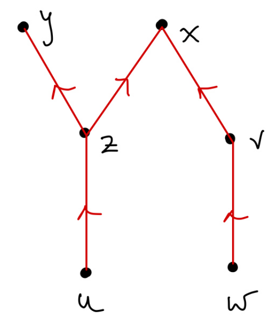
These elements are not necessarily unique. In the partial order below, the elements u and w are all minimal and the elements x and y are maximal.



3.1 Simplified drawings: Hasse Diagram

In the drawing of a partial order, many edges don't need to be shown (if we are part of a true partial order they can be assumed to be present). For example, we could redraw the above graph without the loops (reflexive edges) and we could "*order*" the nodes by height so that nodes on the same height are not comparable, and the minimal elements are at the bottom and the maximal elements at the top. In all partial orders, the relation is transitive, thus we can just assume that all the transitive edges (those in green on the left) are there, are not draw them. This simplified way of drawing a partial order is shown on the right, and is called a **Hasse Diagram**





3.2 Total Order

If we have a partial order where *every* pair of elements is *comparable*, i.e: if $a, b \in A$ then either aRb or bRa, then the order is called a **total order**. Examples such as "$\leq$" are total orders over the integers, since for any integers, such as $x = 2$, $y = 5$, then we can conclude either $x \leq y$ or $y \leq x$. In this case, $2 \leq 5$. Note that in total orders, if there is a maximal or minimal element, then it is *unique*.

4 Operations on Relations

Because relations from A to B (or over the same set A) are subsets of $A \times B$, two relations can be combined in the same way sets are combined. For example, we can define the *union, intersection* and *complement* as we saw previously in this course for sets.

The union of the relations R_1 and R_2 over $A \times B$ is simply the pairs $\{(a, b) \in R_1\} \cup \{(a, b) \in R_2\}$. Notice that we just used the usual definition of set union.

The intersection of the relations R_1 and R_2 is defined as the pairs $\{(a, b) \in R_1\} \cap \{(a, b) \in R_2\}$.

The complement of the relation R_1 is defined as the pairs $(a, b) \notin R_1$.

These definitions are made quite clear in the following example. **Example:** Suppose that our relation R_1 is over the set $S = \{1, 2, 3, 4, 5\}$, where $(a, b) \in R_1$ if and only if a divides b. The relation R_2 on the same set is defined as $(a, b) \in R_2$ if and only if $b - a$ is divisible by 2. Then the union of these two relations $R_1 \cup R_2$ is all the pairs where either the first number divides the second, OR the difference is divisible by 2. The pairs are therefore:

$$\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)\}$$

The intersection $R_1 \cap R_2$ is:

$$\{(1, 1), (2, 2), (3, 3,), (4, 4), (5, 5), (2, 4), (1, 3), (1, 5)\}$$

Lastly, we define the *composition* of binary relations. A composition is defined as it is for functions. In this course, we will focus on compositions of relations that are all over the same set, A. Certainly more general compositions of relations are defined.

Definition. *Let R_1 be a relation on $A \times A$ and R_2 be a relation on $A \times A$, then the composition, $R_2 \circ R_1$ is the relation consisting of the pairs (a, c) where $(a, b) \in R_1$ and $(b, c) \in R_2$.*

Example:

Suppose that R_1 is the relation on $A = \{1, 2, 3, 4, 5\}$ defined by $(a, b) \in R_1$ iff $|b - a| = 2$, and R_2 is the relation where $a < b$. We will consider the composition $R_2 \circ R_1$.

Let's list the pairs of R_1, which are all the pairs whose difference is 2:

$$R_1 = \{(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3)\}$$

The pairs in R_2 are all those for which $a < b$:

$$R_2 = \{((1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$$

To create the composition, start with a pair in R_1, say $(1, 3)$ and using the second element, select pairs of the form $(3, c) \in R_2$ (those would be $\{(3, 4), (3, 5)\}$). Then add to $R_2 \circ R_1$ all the new pairs of the form $(1, c)$: so we would add $\{(1, 4), (1, 5)\}$. The '3' is sort of an "*in between*" element. Using this technique, the pairs in the composition are:

$$\{(1, 4), (1, 5), (3, 2), (3, 3), (3, 4), (3, 5), (2, 5), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$$