Brandon Vo

1. Suppose that f(n) = Θ(g(n)). Assume that both are increasing functions.
(a) Must it be true that log f(n) = Θ(log g(n))? Prove or disprove.

True: Use overestimation

A function always outgrows a constant.

$$c * g(n) \leq f(n) \leq d * g(n), \textit{where } c \leq d$$

$$\log\big(c * g(n)\big) \leq \log\big(f(n)\big) \leq \log\big(d * g(n)\big)$$

$$\textit{Because } f(n) = \Theta\big(\mathrm{g(n)}\big), \textit{then} \log\big(g(n)\big) \leq \log\big(f(n)\big) \leq \log\big(g(n)\big)$$

$$\log(c) + \log\big(g(n)\big) \leq \log\big(f(n)\big) \leq \log(d) + \log\big(g(n)\big)$$

$$\log(c) + log\big(g(n)\big) \leq \log(d) + \log\big(g(n)\big) \ \textit{where } c \leq d$$

$$\textit{Because c and d are constants, exaggerate} \log(c) \textit{ and} \log(d) \ \textit{ to be} \log\big(g(n)\big)$$

$$\textit{where} \log(c) + \log\big(g(n)\big) \leq \log\big(g(n)\big) + \log\big(g(n)\big)$$

$$\log\big(g(n)\big) + \log\big(g(n)\big) \leq \log\big(f(n)\big) \leq \log\big(g(n)\big) + \log\big(g(n)\big)$$

$$2\log\big(g(n)\big) \leq \log\big(f(n)\big) \leq 2\log\big(g(n)\big)$$

$$\textit{This is remains to be } \Theta(\log g(n)), \textit{so the statement holds true}$$

(b) Must it be true that $2^{f(n)} = \Theta(2^{g(n)})$? Prove or disprove.

$$2^{\mathrm{f(n)}} \ = \ \Theta\big(2^{\mathrm{g(n)}}\big)$$

$$2^{c_1 * g(n)} \leq 2^{f(n)} \leq 2^{c_2 * g(n)}$$

$$O(n) \textit{ is not equal to } O(2^n), \textit{these yield different hierarchies of time complexities}$$

Checking for O(g(n))

$$2^{2\,\mathrm{lo}} \quad \leq c_2 * 2^{\log n}, \textit{flipping the n to yield a constant exponent}$$

$$n^{2*\log 2} \leq c_2 * n^{\log} \ \ , \textit{where } 2\log 2 > \log 2$$

$$f(n) \textit{ would be } O(n^2) \textit{ while g is only } O(n)$$

$$\textit{There is no value of } c_2 \textit{ where g would be the upper bound of } f(n)$$

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Same issue can be applied for Omega(g(n))

$$c_1 * g(n) \leq f(n)$$

$$f(n) = n, g(n) = 4n$$

$$c_1 * 4n \leq n, c_1 \text{ can be any value} \leq \frac{1}{4} \text{ to match with } n$$

$$\text{With } 2^{f(n)}:\ c_1 * 2^{4n} \leq 2^n$$

$$c_1 * 16^n \leq 2^n, \text{no values of } c_1 \text{ can cause } g(n) \leq f(n) \text{ due to } g(n) \text{ being exponentially higher}$$

$$\text{The statement does not hold true}$$

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2. a) Suppose you have a function of two variables, n and k. What would it mean, mathematically, to say that this function is O(n + k)?

$$Both\ variables\ n\ and\ k\ are\ necessary\ to\ determine\ the\ asymptotic\ upper\ bound\ of\ g(n,k)$$

$$While\ f(n) \leq c * g(n)\ translates\ to\ f(n) \leq c * n\ when\ describing\ O(n)$$

$$With\ O(n+k), f(n,k) \leq c * g(n+k)\ for\ all\ n\ and\ k$$

$$This\ translates\ to\ f(n,k) \leq c * (n+k)$$

$$There\ exists\ constants\ where\ n \geq n_0\ and\ k \geq k_0\ and\ c > 0\ such\ that:$$

$$f(n,k) = O\big(g(n+k)\big)\ or\ f(n,k) \leq c(n+k)\ and\ g(n+k)\ serves\ as\ the\ upper\ bound\ for\ f(n,k)$$

(b) Let f(n) = O(n) and g(n) = O(n). Let c be a positive constant. Prove or disprove that f(n) + c · g(k) = O(n + k)

$$f(n) \leq c_1 * n$$

$$Assuming\ that\ g(n) \leq c_2 * g(k)\ so\ that\ g(k)\ is\ O(k)$$

$$If\ g(k) = O(k)$$

$$then\ g(k) \leq c_2 * k$$

$$f(n) = O(n), g(n) = O(n)\ where\ f(n) + g(n) = O(n)$$

$$If\ f(n) + g(k) = O(n+k)$$

$$then\ f(n) + g(k) \leq d_2(n+k)$$

$$Substituting\ g(n)\ with\ c_2 * k$$

$$f(n) + c_2 * k \leq d_2(n+k)$$

$$Substituting\ f(n)\ with\ c_1 * n\ since\ f(n)\ is\ O(n)$$

$$c_1 * n + c_2 * k \leq d_2(n+k)$$

$$c_1 * n + c_2 * k \leq d_2(n+k)$$

$$If\ c_1 \leq d_2\ and\ c_2 \leq d_2, then\ we\ can\ exaggerate\ and\ simplify\ c_1\ and\ c_2$$

$$c_1 * n + c_2 * k \leq d_2(n+k)\ where\ (c_1 + c_2) \leq d_1 \leq d_2$$

$$c_1 * n + c_2 * k = d_1(n+k)\ where\ (c_1 + c_2) \leq d_1\ and\ substitute\ (c_1 + c_2)\ with\ d_1$$

$$d_1(n+k) \leq d_2(n+k)\ where\ d_1 \leq d_2$$

$$f(n) + cg(k) = d_1(n+k) \leq d_2(n+k)\ where\ c \leq d_1 \leq d_2$$

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$$Implies\ that\ f(n) + c * g(k) \leq O(n + k)$$

$$Further\ implying\ that\ f(n) + c * g(k) = O(n + k)\ where\ c \leq d$$

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3. Let f(n) = $\sum_{y=1}^{n} n^6 * y^{23}$
   Find a simple g(n) such that f(n) = Θ(g(n)), by proving that f(n) = O(g(n)), and that f(n) = Ω(g(n)).
   Don't use induction / substitution, or calculus, or any fancy formulas. Just exaggerate and simplify for big-O, then underestimate and simplify for **Ω**.

Exaggerate and simplify:

This creates a geometric series

$$\sum_{y=1}^{n} n^6 * y^{23} = n^6 * 1^{23} + n^6 * 2^{23} + \cdots n^6 * n^{23}$$

$$n^6(1^{23} + 2^{23} + 3^{23} + \cdots + n^{23})$$

$$Exaggerate\ all\ terms\ to\ n\ since\ y \leq n$$

$$n^6(n^{23} + n^{23} + \cdots + n^{23})$$

$$n^6 * (n^{23} * n)$$

$$f(n) = O(n^{30})$$

For **Ω**, underestimate and simplify

$$\sum_{y=1}^{n} n^6 * y^{23} = n^6 * 1^{23} + n^6 * 2^{23} + \cdots n^6 * n^{23}$$

Taking half of the summation works for underestimating

$$\sum_{y=\lceil \frac{n}{2} \rceil}^{n} n^6 * y^{23} = n^6 * 1^{23} + n^6 * 2^{23} + \cdots n^6 * n^{23}$$

$$\sum_{y=\lceil \frac{n}{2} \rceil}^{n} n^6 * y^{23} = n^6 * 0 + n^6 * 0 + \cdots + n^6 * \left(\frac{n}{2}\right)^{23} + \cdots n^6 * \left(\frac{n}{2}\right)^{23}$$

$$n^6\left(\left(\frac{n}{2}\right)^{23} + \cdots + n^{23}\right) = n^6 * \left(\frac{n}{2} * \left(\frac{n}{2}\right)^{23}\right)$$

Underestimate the geometric series

$$n^6 * \left(\frac{n}{2}\right)^{24}$$

$$f(n) \leq c * \frac{1}{2^{24}} * n^{30}$$

$$\frac{1}{2^{24}} * n^{30} \leq f(n) \leq n^{30}$$

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$$g(n) = \Theta(\mathrm{n}^{30})$$