Is Brouwers intuitionism formalisable?

A closer look on Brouwers synthetic a-priori experience.

Kevin De Keyser
ETH Zurich

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To begin with I will try to state how I understand Brouwers intuitionism. Secondly the origin of abstraction in numbers in ancient iraq will be analysed, in order to question whether our intuition of natural numbers is a-priori or a-posteriori.
Furthermore this essay then compares Brouwers intuitionism against other alternative mathematics, specifically *Random mathematics* as described by van Bendegem, in order to show its distinction to formalism.
Finally the question of whether Brouwers intuitionism is truly an a-priori experience will be tackled.

In intuitionism a mathematical statement is true, as soon as one constructs and experiences a proof of said statement.
These constructions are built upon an ur-phenomena (first act of intuitionism) [1]: "[..] by the perception of the movement of time and the falling apart of a life moment into two distinct things: what was, 1, and what is together with what was, 2, and from there to 3, 4,..". What he means is that numbers are experienced by the subject who constructs mathematics. Although Brouwer meant our sense of time, another way of understanding the experience behind numbers is by listening to your heartbeats, from which after the first heartbeat follows the second and so on. Important here is that every human subject doing math can experience this two-ness of time **a-priori**, which then leads to a shared intuitive understanding of time[1].

A concrete example of such a mathematical statement can be found in Heyting [2]: l is defined as the largest number for which $l - 2$ is prime or

[1]Contrary to time, a heartbeat is not something every human can experience 'a-priori', since there are people with artificial hearts. Brouwer feels strongly that the intuition is not experienced due to the outside (a-posteriori), but from the inside (a-priori), which helps to give his theory more credibility.

if this does not exist $l = 1$. An existential proof that $l \neq 1$[2] would not matter in intuitionism, since one would need a way of constructing l. This construction additionally has to be experienced "[..] by a direct, mental construction [..]" of a subject (as pointed out in [1] p.1774).

If I tell you the problem $4+? = 6$ has the solution 2, there can be an exact moment in which one realises this statement is true. At the moment of realisation this mathematical fact is added to the memory of all previously experienced mathematical facts. Brouwer: "This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory."

Being able to abstract from this procedure a general formula for $x+? = y$, namely by trying $? = y - x$, is what Brouwer calls *ur-intuitionism*. This general (abstract) result can then always be tested with concrete cases if need-be. Brouwer: "[...] (the) full development of the mechanism of mathematical activity [..] (is) mathematical abstraction, by means of which one deprives twoness of its substantial content and retains only the empty form, the common substrate of all twonesses."

I claim that both this ur-phenomena and ur-intuition are not a-priori, but instead a-posteriori. Some evidence for this can be found in ancient iraq, where the first abstractions from quantities to numbers were recorded.

In excavations of Tell Sabi Abyad in 6000 BCE tokens were found, which likely were used for accounting of goods, such as wheat. Since wheat had to be stored dry, multiple farmers probably put their wheat into the same hut and in order to keep track how much wheat belongs to whom, tokens were utilised (To name a sample use-case for tokens).
This first abstraction from concrete objects such as wheat to tokens and later to empty sealed envelopes came from a necessity of sustaining a larger population[3], which was a very practical, a-posteriori motivation for the development of this numbers.

Since multiple goods were used (loaf of bread, jugs of bear) not only quantity, but also the type of the goods had to be stored. Originally the size of the tokens and type of seals might have helped the people doing the transaction to remember the type of the transaction, however "[..] these systematisations were local at best" [3].

[2] We already know that either twin, cousin or sexy primes are infinite, but not which one. This is another example of a non-intuitionistic result.

[3] It has to be noted here that tokens have been excavated from times before the agriculture revolution, potentially for similar trading purposes.

These numeric abstractions had nothing to do with time, although the association with time and numbers is Brouwers main justification for his ur-intuitionism.

Only under centralisation efforts in Ur, with the two driving forces behind it being a strict government and religion, did people first start to use calendars[4].

As part of these centralisation efforts a switch from commodity-specific metrologies (term used by Robson) to a unified sexagesimal system lead the Babylonians to develop a more sophisticated time measuring system, not only being able to measure days but seconds, a system on which our time tracking system is based upon to this day. This is the first time consciously quantifiable moments were measured.

The abstraction and development of numbers was far from finished however, for example also in Babylonian times did humans start to abstract numbers not only from quantity, but also from length and area.

The development of abstract numbers as a concept seemingly took hundreds of years to mature and was not purely the result of a spontaneous ur-phenomena of a mathematically gifted individual human.

According to a religious hymn of the people in Sumer, even one of their deities (Nibasa) was responsible for teaching / inventing this early mathematics: "Nisaba leads your fingers on the clay. [..] A yardstick, and a writing board which gives you wisdom; Nisaba generously bestows them on you." [3]

This would put Brouwers ur-phenomenon even outside of the human experience!

Although a-posteriori objects were clearly an influence in the invention of the natural numbers and the first recorded history of time tracking came hundreds of years after the use of tokens as numbers, this is not sufficient evidence to disprove that Brouwers intuitionism comes from an a-priori experience.

Brouwer could have said, that every person learning about numbers experienced this two-ness and they all had some notion of 2 coming after 1 and we simply do not have a record of this.
Brouwer could also not have considered this 'mathematics', since no proofs were involved in these constructions, although clearly algorithms for computing numeric operations were available, which seemed convincing enough for the people to believe.

[4] see Umma calendar

To tackle the first opposition I ask whether it is possible to do mathematics without experiencing this two-ness of time, without consciousness or even without time?

Clearly if there would be mathematics without consciousness or without time, Brouwer's temporal ur-intuition would not be a necessary framework for mathematics. Since human mathematicians both experience time and are conscious[5], it might be easier to first discuss whether being human is a necessairy condition for doing mathematics.

Early mathematics could easily be done by an automatized scribe or a computer algebra system, which clearly would work without ur-intuition, but even there Brouwer might argue that the machine feels such a two-ness of time, with this thought train leading to the chinese room problem.

In *random mathematics* as proposed by van Bendegem [4] arithmetic statements are of the form $a + b = c$. At the beginning one is at some configuration, for e.g. $a = 2, b = 3, c = 5$. Then either a or b is increased or decreased randomly and c is modified accordingly, which can be thought of as a movement on a grid.

I would argue this is mathematics, which requires no thought and therefore is a mathematics not relying on any intuition.

This brings me to the second opposition: Brouwer might not call this mathematics, since for him mathematics has to be done by a *subject*.

However, when this act of randomisation is performed by a mathematical subject[6], this actor, with a similar logic to Brouwer, might experience a two-ness of space, which is why such random movements on a grid make intuitionistic sense. This grid-intuitionistic mathematics allows induction, since one can move from one tile to the next, but it would also exclude the law of the excluded middle, since it is not known whether one ever reaches A or $\neg A$ (if this statement even exists on the board[7]).

One might say this spatial two-ness is only temporal two-ness in disguise. It is the act of moving from one tile to the other, which gives temporal ordering and the two-ness arises from realising the new tile and remembering it.

However if this were the case, any formal system with rules of inference

[5] With maybe extreme cases, where people keep forgetting every few seconds that they were conscious. See the documentary: 'The man with the seven second memory'.

[6] One might be inclined to call this act of randomization "choice", although I refrain from this, since it would lead to a discussion on free-will.

[7] In classical mathematics the truth value of the continuum hypothesis does also depend on the formalisation you use.

would have this temporal two-ness. You start at a statement (axiom) and move to the next statement by any possible rule of inference. This act of going from one statement to the next could also be referred as a two-ness of time, which in return would make any formal system intuitionistic, also classical logic for example.

The point is that Brouwer would not call an 'inference rule' a mathematical object, because he never experienced the ur-phenomenon of this object as a human being. Likewise he would not call random mathematics intuitionistic, since he never was trapped in a grid, where the only thing you could do is walk from tile to tile. In other words the domain of his intuitionistic logic is $\mathbb{N}$ precisely because he as a human being experiences time, which I would argue is a-posteriori.

Conversely, according to [5] there are indeed formalisations of his intuitionism where the subject is included. There the subject experiencing these truths is denoted $\Box$. If a prove of A has been realised at time n it is written as: $\Box_n A$, if not as $\neg\Box_n A$. The axioms are respectively:

$$\Box_n A \vee \neg\Box_n A \qquad \Box_n A \rightarrow \Box_{n+m} A \qquad \exists n \Box_n A \leftrightarrow A$$

From there several interpretations of his work, such as BHK (Brouwer Heyting Kolmogoroff) interpretation start to build up this intuitionist mathematic.

However these formalisations are very much up to interpretation and as Brouwer said in his own words: "Syntactic rules of formalism cannot be finite since with human experience; formal systems, because they are static, are in principle unable to capture the dynamic and open-ended realm of creative mathematical activity." [1]

The problem is, that Brouwer cannot formalise his experience of time in a satisfactory manner, as he himself said that language (and by extension other formalisations) are a most imperfect tool of expressing his thoughts.

With this I want to highlight, that Brouwer's intuitionism does not stem from an a-priori experience, but from a synthetic a-posteriori experience, since both time and the way we interpret this two-ness, is dependent on our outer-world (call it the law of general relativity).

Even more startling to the nature of Brouwers intuitionism it might come that not all people connect this sensation of time with the construction of natural numbers. Beside the historic indication for this claim given beforehand, I might add that many people still prefer doing classical mathematics, which might be another case that Brouwers intuitionism is simply a product of Brouwers interpretation of the world.

Ultimately if you are a materialist and have not experienced his ur-intuition, it is still possible to figure out what precisely Brouwers axioms and inference rules are, by running a physical simulation of the world sufficient enough to sustain Brouwer and his experience, in order to then ask him what he thinks about including a new axiom or to ask him to clarify the semantic of his system, making his entire intuitionism an incredibly complicated formal system and his previously termed a-posteriori foundation potentially an a-priori analytic product.

References

[0] The main source of this essay are discussions from the course 851-0125-65L "A Sampler of Histories and Philosophies of Mathematics FS2017" as tought by Dr. Roy Wagner, of which this essay was the final assignment.

[1] L.E.J. Brouwer, pp. 1170-1178 (sections 1-2), 1180-1183

[2] Arend Heyting, "Disputation", pp. 66-76

[3] Eleanor Robson, "Mathematics in Ancient Iraq: A Social History", pp. 28, 33[top]-44, 75-78, 81-83, 86-90, 117-122

[4] Jean Paul van Bendegem, "Can there be an alternative mathematics, really?", pp. 349-359

[5] Iemhoff, Rosalie, "Intuitionism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Winter 2016 Edition), Edward N. Zalta (ed.)