Texas A&M University
Department of Mechanical Engineering

MEEN 689: Convex Optimization Methods for Control System Design
EXAM - II

December 1, 2019

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1. Markov-Lucaks Theorem states that

   (a) a polynomial $P(z) \geq 0$ for all $z \geq 0$ if and only if there exist polynomials $f(z), g(z)$ so that $P(z)$ can be expressed as:

   $$P(z) = f^2(z) + zg^2(z).$$

   (b) an even polynomial $P(z)$ that is non-negative on $[0,1]$ can be represented as

   $$P(z) = f^2(z) + z(1-z)g^2(z),$$

   for some polynomials $f(z)$ and $g(z)$,

   (c) an odd polynomial $P(z)$ that is non-negative on $[0,1]$ can be expressed as

   $$P(z) = zf^2(z) + (1-z)g^2(z),$$

   for some polynomials, $f(z)$ and $g(z)$.

   Prove Markov-Lucaks theorem using the following steps:

   - Show that $P(z)$ cannot have real positive roots of odd multiplicity. (Hint: Use Taylor's series expansion about such a root).
   - Show that $P(z)$ can be factored as

   $$P(z) = A\underbrace{\left[\Pi_{i=1}^{l}(z+\alpha_i)\right]}_{P_1(z)}\underbrace{\left[\Pi_{j=1}^{k}(z-r_j)^2\right]}_{P_2(z)}\underbrace{\left[\Pi_{k=1}^{m}((z-\sigma_k)^2+\omega_k^2)\right]}_{P_3(z)}.$$

- If $P(z) = az + b$ is non-negative for $z \geq 0$, find polynomials f and g such that $P(z) = f^2(z) + zg^2(z)$.
- Suppose $P(z) = az^2 + bz + c$ is non-negative for $z \geq 0$, find polynomials f and g such that $P(z) = f^2(z) + zg^2(z)$.
- Prove Markov-lucaks theorem by induction on the degree of $P(z)$ by first showing (a) and then proving (b) and (c) through the use of (a). [Hint: You may find the following identity useful for the proof by induction:

  $$(f_1^2 + zf_2^2)(g_1^2 + zg_2^2) = (f_1g_1 + zf_2g_2)^2 + z(f_1g_2 - f_2g_1)^2.$$

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NAME:

2. This problem concerns with the construction of a largest volume ellipsoid in a convex set:

   - Show that the largest volume ellipsoid contained in a polyhedron described by inequalities $< a_i, x > \leq b_i,\ i = 1, 2, \ldots, m$ can be expressed by the convex program:

   $$J^* = \max \quad \log(det(P)), \quad \textit{subject to}$$
   $$\|Pa_i\| \leq b_i - < a_i, x_c >, \ \ i = 1, 2, \ldots, n, \qquad P(= P^T) \succ 0.$$

   (Note that $det(P)$ represents the volume of the ellipsoid, $\log(det(P))$ for $P \succ 0$ is concave, and an ellipsoid can be expressed as the set, $\{x = x_c + Pu, \ \ \|u\| \leq 1\}$.) Alternatively, show that the constraint may also be expressed as

   $$\begin{bmatrix} b_i - < a_i, x_c > & a_i^T P \\ Pa_i & (b_i - < a_i, x_c >)I \end{bmatrix} \succeq 0, \quad P \succ 0.$$

   by showing the equivalency of the feasible sets.

   - Show also that the largest volume ellipsoid contained in the intersection of spheres $\|x - c_i\| \leq r_i$, where c_i is the center and r_i is of radius the i^{th} sphere:

   $$\begin{aligned} J &= \max \ \log(det(P)), \quad \textit{subject to} \\ \|x_c + Pu - c_i\| &\leq r_i, \ i = 1, \ldots, m, \quad \|u\| \leq 1. \end{aligned}$$

   Show that the above set of constraints can be recast as:

   $$\begin{bmatrix} r_i - \lambda & (x_c - b_i)^T & 0 \\ x_c - b_i & r_i I & P \\ 0 & P & \lambda I \end{bmatrix} \succeq 0, \quad P \succeq 0, \quad \lambda \geq 0.$$

NAME:

3. Consider a LTI system:

$$\dot{x} = Ax + Bw, \quad y = Cx, \quad x(0) = 0.$$

Note that $w(t) \in \mathcal{L}_2$ implies $\int_0^\infty w^*(\tau)w(\tau)d\tau < \infty$. Suppose $w(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Show the following:

- For all $t \geq 0$, $\|x(t)\|^2 \leq \lambda_{max}(P)\|w\|_2^2 \iff$ there exists a

$$P \succeq 0, \quad AP + PA^T + DD^* = 0.$$

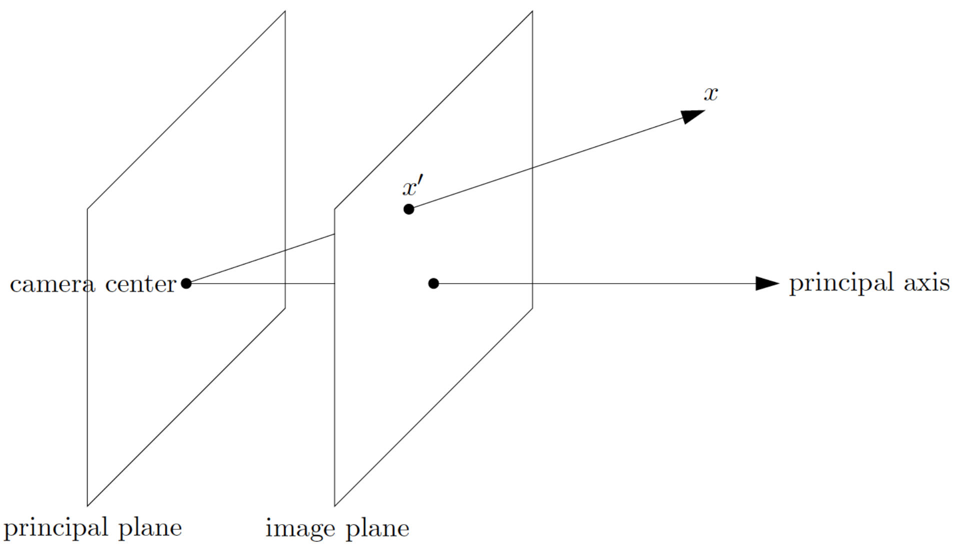
- Show that $\|y(t)\|^2 \leq \gamma\|w\|_2^2$ for all t, where $\gamma = trace(CPC^*)$ subject to $P \succ 0,\ AP + PA^T + DD^* = 0$.
- Using Julia/JuMP, calculate the gain γ from $\|w\|_2$ to $\|y\|_\infty$ for the following system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \tau\dot{x}_3 &= -k_p x_1 - (k_v + hk_p)x_2 - x_3 + w, \\ y &= k_p x_1 + k_v x_2 + k_a x_3. \end{aligned}$$

You may choose $\tau = 0.5\ s,\ k_p = 1,\ k_v = 2,\ k_a = 0.5,\ h = 0.8.$

NAME:

4. This problem deals with triangulation from images obtained by multiple cameras will be employed for determining the location of the object denoted by $x \in \Re^3$.



A projective camera can be described by a linear-fractional function $f : \Re^3 \to \Re^2$ as:

$$f(x) = \frac{1}{c^T x + d}(Ax + b), \quad \mathbf{dom}\ f = \{x : c^T x + d > 0\},$$

with $A \in \Re^{2\times 3}$, $b \in \Re^2$ and

$$\mathbf{rank} \begin{bmatrix} A \\ c^T \end{bmatrix} = 3.$$

Essentially, the domain of f consists of points in front of the camera. The 3×4matrix

$$P = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix},$$

is called the camera matrix and has rank 3. Since f is invariant with respect to scaling, i.e., it does not change if we were to replace A, b, c, d with kA, kb, kc, kd for any constant $k \neq 0$, we can normalize the parameters and assume that $\|c\|_2 = 1$. The number $c^T x + d$ is then the distance of the point x from the plane $\{z : c^T z + d = 0\}$. This plane is called the principal plane. The point

$$x_c = -\begin{bmatrix} A \\ c^T \end{bmatrix}^{-1} \begin{bmatrix} b \\ d \end{bmatrix}$$

lies in the principal plane and is called the camera center. The ray $\{x_c + \theta c : \theta \geq 0\}$ is called the principal axis and is perpendicular to the principal plane and passes through the camera center. The image plane is parallel to the principal plane, at a unit distance from the principal plane.

The point x' in the figure is the intersection of the image plane and the line connecting camera center and x, and is given by:

$$x' = x_c + \frac{1}{c^T(x - x_c)}(x - x_c).$$

(Show why this is true).

Using the definition of x_c, we can write $f(x)$ as:

$$f(x) = \frac{1}{c^T(x - x_c)}A(x - x_c) = A(x' - x_c) = Ax' + b.$$

This shows that the mapping $f(x)$ can be interpreted as a projection of x onto the image plane to get x', followed by an affien transformation of x'. We can interpret $f(x)$ as the coordinates of a point x' in a two-dimensional frame attached to the image plane.

In this exercise, consider the problem of determining the position $x \in \Re^3$ from its image in N cameras. Each camera is characterized by a known linear-fractional transformation f_k and a camera matrix P_k:

$$f_k(x) = \frac{1}{c_k^T x + d_k}(A_k x + b_k), \quad P_k = \begin{bmatrix} A_k & b_k \\ c_k^T & d_k \end{bmatrix}.$$

The image of the point x in k^{th} camera will be denoted by $u^{(k)} \in \Re^2$. Due to camera calibration errors, we do not expect the equations $f_k(x) = y^{(k)},\ k = 1, 2, \ldots, N$ to be consistent. To estimate the point x, we can therefore minimize the maximum error in the equations by solving

$$\mathit{minimize} \;\; g(x) = \max_{k=1,2,\ldots,N} \|f_k(x) - y^{(k)}\|_2.$$

Formulate the problem and demonstrate how you would use convex optimization techniques to solve the problem for the following instance:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 10 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1 & 1 & 1 & -10 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 10 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$y^{(1)} = \begin{bmatrix} 0.98 \\ 0.93 \end{bmatrix}, \quad y^{(2)} = \begin{bmatrix} 1.01 \\ 1.01 \end{bmatrix}, \quad y^{(3)} = \begin{bmatrix} 0.95 \\ 1.05 \end{bmatrix}, \quad y^{(4)} = \begin{bmatrix} 2.04 \\ 0.0 \end{bmatrix}.$$

NAME:

5. A small object is located at an unknown position $x \in \Re^3$, and is viewed by a set of m cameras. The goal is to find a box $\mathcal{B} \in \Re^3$:

$$\mathcal{B} = \{z \in \Re^3 : l \preceq z \preceq u\},$$

for which there is a guarantee that $x \in \mathcal{B}$. Clearly, the smallest bounding box is desirable; we can use volume to judge the smallest among possible solutions.

The object at a location $x \in \Re^3$ creates an image in the image plane of the i^{th} camera at a location

$$v_i = \frac{1}{c_i^T x + d_i}(A_i x + b_i) \in \Re@,$$

where the matrices $A_i \in \Re^{2\times 3}$, vectors $b_i \in \Re^2$, $c_i \in \Re^3$ and $d_i \in \Re$ are dependent on the camera positions and orientations and may be treated as parameters specific to the camera. One may assume that $c_i^T x + d_i > 0$ and define a 3×4 matrix

$$P_i := \begin{bmatrix} A_i & b_i \\ c_i^T & d_i \end{bmatrix},$$

as the camera "calibration" matrix (for camera i). It is often (but not always) the case that the first three columns of P_i form an orthogonal matrix, in which case the camera is called orthographic.

We do not have direct access to the image point v_i; we only know the (square) pixel it lies on. In other words, the camera gives us a measurement $\hat{v}_i$ (the center of the pixel that the image point lies in); we are guaranteed that

$$\|v_i - \hat{v}_i\| \leq \frac{\rho_i}{2},$$

where ρ_i is the pixel width (and height) of the i^{th} camera. We know nothing else about v_i; it could be any point in the pixel.

Given the data $A_i, b_i, c_i, d_i, \hat{v}_i, \rho_i$, formulate a mathematical program to determine the smallest box $\mathcal{B}$, (i.e., determine vectors l and u) that is guaranteed to contain x. In other words, formulate a mathematical program to determine the smallest box in $\Re^3$ that contains all points consistent with the observations from the camera.

Explain how you would solve the problem using convex optimization methods and be sure to include your julia file as camera_data_yourname.jl that outputs the desired bounding box.