Problem: Obtain PCNF of the of the following formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$ without using truth table.

Solution: given formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$$\equiv (\neg(\neg P) \vee R) \wedge (Q \rightarrow P) \wedge (P \rightarrow Q) \quad \text{since } P \rightarrow Q \equiv \neg P \vee Q$$

$$\equiv (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$$

$$\equiv [(P \vee R) \vee F] \wedge [(\neg Q \vee P) \vee F] \wedge [(\neg P \vee Q) \vee F] \text{ since } P \vee F \equiv P$$

$$\equiv [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [(\neg Q \vee P) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$$

Since for any variable P, $P \wedge \neg P \equiv F$

$$\equiv (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

$$\equiv (P \vee R \vee Q \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \text{ since } P \wedge P \equiv P$$

Problem: Obtain PDNF of the of the following formula $(P \wedge \neg Q \wedge \neg R) \vee (Q \wedge R)$ without using truth table

Solution: given formula is $(P \wedge \neg Q \wedge \neg R) \vee (Q \wedge R)$

$$\equiv (P \wedge \neg Q \wedge \neg R) \vee [(Q \wedge R) \wedge T] \quad \text{since } P \wedge T \equiv P$$

$$\equiv (P \wedge \neg Q \wedge \neg R) \vee [(Q \wedge R) \wedge (P \vee \neg P)] \quad \text{since } P \vee \neg P \equiv T$$

$$\equiv (P \wedge \neg Q \wedge \neg R) \vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P) \text{ by distributive law}$$

Problem: Obtain PDNF of the of the following formula $(\neg P \vee Q)$ for two variables P,Q, without using truth table

Solution: given formula $(\neg P \vee Q)$

$$\equiv (\neg P \wedge T) \vee (Q \wedge T)$$

$$\equiv (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$

$$\equiv(\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

$$\equiv(\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

The following table contains some basic equivalent formulas which will be found useful.

S. No	Formulas		
1	$(P \vee P) \Leftrightarrow P$	$(P \wedge P) \Leftrightarrow P$	Idempotent law
2	$(P \vee Q) \Leftrightarrow (Q \vee P)$	$(P \wedge Q) \Leftrightarrow (Q \wedge P)$	Commutative law
3	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	Associative law
4	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	Distributive law
5	$P \vee F \Leftrightarrow P$	$P \wedge T \Leftrightarrow P$	
6	$P \vee T \Leftrightarrow T$	$P \wedge F \Leftrightarrow F$	
7	$P \vee \neg P \Leftrightarrow T$	$P \wedge \neg P \Leftrightarrow F$	
8	$P \vee (P \wedge Q) \Leftrightarrow P$	$P \wedge (P \vee Q) \Leftrightarrow P$	Absorption Law
9	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	De Morgan's Law

<u>Example 1.8.2</u>: If PDNF of A is $(P \wedge Q) \vee (\sim P \wedge Q)$, find it's PCNF.

Given that PDNF of A is $(P \wedge Q) \vee (\sim P \wedge Q)$, then PDNF of $\sim A$ is the disjunction of the remaining minterms which do not appear in PDNF of A

i.e. $\sim A \Leftrightarrow (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

Applying negation on both sides, we get,

$$\sim\sim A \Leftrightarrow \sim((P \wedge \sim Q) \vee (\sim P \wedge \sim Q))$$

Therefore $\mathbf{A} \Leftrightarrow \sim(P \wedge \sim Q) \wedge \sim(\sim P \wedge \sim Q)$

$$\Leftrightarrow (\sim P \vee Q) \wedge (P \vee Q)$$

Hence PCNF of $A \Leftrightarrow (\sim P \vee Q) \wedge (P \vee Q)$

Remarks:

(1) If a formula is tautology(contradiction), then all the minterms (maxterms) appear in it's PDNF(PCNF)

(2) Every formula which is not a contradiction (Tautology) has an equivalent PDNF (PCNF).

(3) The PDNF (PCNF) of a formula (if exists) is unique except for the rearrangement of minterms in the disjunction.

Automatic Theorem Proving

Before going to study the Automatic theorem proving , we should have idea about the terminology such as String, sequent, Axiom, Theorem etc.

String: i) if P, Q, R, S are Primary variables then α is called string of primary variables.

ii) if A, B, C, D are statement formulas then α is called string of statement formulas.

Examples : α : P, Q, R where P,Q, R are primary variables

α : A, B, C where A, B, C are statement formulas.

Sequent: if α, β are strings of formulas then $\alpha \xrightarrow{s} \beta$ is called sequent. We read this $\alpha \xrightarrow{s} \beta$ as "α string implies to β" in which α is called antecedent and β is called consequent.

i) If $\alpha \xrightarrow{s} \beta$ is true iff at least one formula of α is F (Contradiction) OR atleast one formula of β is T (Tautology).

ii) If $\alpha \xrightarrow{s} \beta$ is true, then we write this as $\alpha \overset{s}{\Rightarrow} \beta$, read it as "α string tautological implies to β".

iii) let A be a statement formula

a) A is Tautology iff $\xrightarrow{s} A$ is True iff $\overset{S}{\Rightarrow} A$

b) A is a Contradiction iff $A \xrightarrow{s}$ is True iff $A \overset{s}{\Rightarrow}$

Axiom: let α, β are strings of formulas then $\alpha \xrightarrow{s} \beta$ is axiom iff α, β have at least one variable in common.

Example: $P, Q, R \xrightarrow{s} P, S$ is axiom.

Note: if $\alpha \xrightarrow{s} \beta$ is an axiom, then "α string tautological implies to β".

Theorem:

a) Every axiom is a theorem.

b) if a sequent α is a theorem and a sequent β results from α through the use of one of the 10 rules of the system which are given below, then β is a theorem.

Rules: The following rules are used to combine formulas within strings by introducing connectives. Corresponding to each of the connectives there are two rules, one for the introduction of the connective in the antecedent and the other for its introduction in the consequent. In the description of these rules, $\alpha, \beta, \gamma, \ldots.$ are strings of formulas while X and Y are formulas to which the connectives are applied.

Antecedent rules:

$\neg\Rightarrow$: if $\overset{s}{\Rightarrow} X$ then $\neg X \overset{s}{\Rightarrow}$

$\wedge\Rightarrow$: if $X, Y \overset{s}{\Rightarrow}$ then $X \wedge Y \overset{s}{\Rightarrow}$

$\vee\Rightarrow$: if $X \overset{s}{\Rightarrow}$ and $Y \overset{s}{\Rightarrow}$ then $X \vee Y \overset{s}{\Rightarrow}$

$\rightarrow\Rightarrow$: if $\overset{s}{\Rightarrow} X$ and $Y \overset{s}{\Rightarrow}$ then $X \rightarrow Y \overset{s}{\Rightarrow}$

$\leftrightarrow\Rightarrow$: if $X, Y \overset{s}{\Rightarrow}$ and $\overset{s}{\Rightarrow} X, Y$ then $X \leftrightarrow Y \overset{s}{\Rightarrow}$

Consequent rules:

$\Rightarrow\neg$: if $X \overset{s}{\Rightarrow}$ then $\overset{s}{\Rightarrow} \neg X$

$\Rightarrow\wedge$: if $\overset{s}{\Rightarrow} X$ and $\overset{s}{\Rightarrow} Y$ then $\overset{s}{\Rightarrow} X \wedge Y$

$\Rightarrow\vee$: if $\overset{s}{\Rightarrow} X, Y$ then $\overset{s}{\Rightarrow} X \vee Y$

$\Rightarrow\rightarrow$: if $X \overset{s}{\Rightarrow} Y$ then $\overset{s}{\Rightarrow} X \rightarrow Y$

$\Rightarrow\leftrightarrow$: if $X \overset{s}{\Rightarrow} Y$ and $Y \overset{s}{\Rightarrow} X$ then $\overset{s}{\Rightarrow} X \leftrightarrow Y$

Problem: Show that $P \vee Q$ follows from P.

Solution: We need to show that

$$\text{(1)} \qquad \overset{s}{\Rightarrow} P \to (P \vee Q)$$

$$\text{(1) If (2)} \quad P \overset{s}{\Rightarrow} P \vee Q \quad (\Rightarrow\to)$$

$$\text{(2) If (3)} \quad P \overset{s}{\Rightarrow} P, Q \quad (\Rightarrow \vee) \quad \text{Axiom.}$$

Hence $P \vee Q$ follows from P.

Problem: Does P follows from $P \vee Q$?

Solution:

$$\text{(1)} \qquad \overset{s}{\Rightarrow} (P \vee Q) \to P$$

$$\text{(1) If (2)} \quad P \vee Q \overset{s}{\Rightarrow} P \qquad (\Rightarrow\to)$$

$$\text{(2) If (3)} \quad P \overset{s}{\Rightarrow} P \ \text{ and (4)} \ Q \overset{s}{\Rightarrow} P \qquad (\vee \Rightarrow)$$

Note that (3) is an axiom, but (4) is not.

Hence P does not follow from $P \vee Q$.

Problem: Prove that $P \to \neg P$ is not a Tautology.

Solution: let us assume that $P \to \neg P$ is Tautology.

$$\overset{s}{\Rightarrow} P \to \neg P \qquad (1)$$

$$(1) \ \ if \ \ (2) \ \ P \overset{s}{\Rightarrow} \neg P \qquad (2) \ (\because By \Rightarrow\to)$$

Since (2) is not an axiom.

Therefore (1) is not true.

Hence $P \to \neg P$ is not a Tautology.