Discrete Mathematical Structures

Assignment - 3

on

Unit - 3 : Recurrence Relations

Unit - 4 : Algebraic Structures

Unit - 5 : Graph Theory

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CSE-4

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DMS Assignment - 3

1. Solve the recurrence relation $a_n = a_{n-1} + n^2$ where $a_0 = 7$ by substitution method.

Sol. Given $a_n = a_{n-1} + n^2$

So, $a_1 = a_0 + 1^2$

$a_2 = a_1 + 2^2 = (a_0 + 1^2) + 2^2$

$a_3 = a_2 + 3^2 = (a_0 + 1^2 + 2^2) + 3^2$

$\vdots \quad \vdots \quad \vdots$

$$a_n = a_0 + (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= a_0 + \Sigma n^2$$

$$= a_0 + \frac{n(n+1)(2n+1)}{6}$$

$$= 7 + \frac{n(n+1)(2n+1)}{6}$$

2. Find a generating function for a_r = the number of non-negative integral solutions of $e_1 + e_2 + e_3 + e_4 + e_5 = r$ where $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd and $1 \le e_5 \le 9$.

Sol. $e_1 = 0, 1, 2, 3$ $\quad A_1(x) = (1 + x + x^2 + x^3)$.

$e_2 = 0, 1, 2, 3$ $\quad A_2(x) = (1 + x + x^2 + x^3)$

$e_3 = 2, 3, 4, 5, 6$ $\quad A_3(x) = (x^2 + x^3 + x^4 + x^5 + x^6)$

$e_4 = 2, 3, 4, 5, 6$ $\quad A_4(x) = (x^2 + x^3 + x^4 + x^5 + x^6)$

$e_5 = 1, 3, 5, 7, 9$ $\quad A_5(x) = (x + x^3 + x^5 + x^7 + x^9)$

Thus, the generating function required is

$$A_1(x) \cdot A_2(x) \cdot A_3(x) \cdot A_4(x) \cdot A_5(x) = [(1 + x + x^2 + x^3)^2][(x^2 + x^3 + x^4 + x^5 + x^6)^2]$$
$$[(x + x^3 + x^5 + x^7 + x^9)]$$

3. Find the coefficient of x^5 in $\frac{1}{x^2-5x+6}$

Sol. $x^2 - 5x + 6 = (x-3)(x-2)$

Now, $\frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)} \equiv \frac{A}{x-3} + \frac{B}{x-2}$

$\Rightarrow A(x-2) + B(x-3) = 1$

$A = 1, \ B = -1$

$\therefore \frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}$

$= \frac{1}{(-3)} \cdot \frac{1}{\left(1-\frac{x}{3}\right)} - \frac{1}{(-2)} \frac{1}{\left(1-\frac{x}{2}\right)}$

$= \frac{1}{2} \frac{1}{\left(1-\frac{x}{2}\right)} - \frac{1}{3} \frac{1}{\left(1-\frac{x}{3}\right)}$

$= \frac{1}{2} \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r - \frac{1}{3} \sum_{r=0}^{\infty} \left(\frac{x}{3}\right)^r$

x^5 coefficient is $\frac{1}{2^6} - \frac{1}{3^6}$

4. Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \right\}$ and "$*$" defined as $A * B = A \cdot B, \ \forall A, B \in G$ then $(G, *)$ is a monoid is to be proved.

Sol. i) Binary:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in G$

Now $A * B = A \cdot B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \in G$

$\therefore A * B \in G$

Hence, $*$ is a binary operation.

ii) Associative :

W.K.T., matrix multiplications are always associative i.e.,

Let $A, B, C \in G$

$A * (B * C) = A * (B.C)$

$= A.B.C$

$(A * B) * C = (A.B) * C$

$= A.B.C$

$\therefore *$ is associative

iii) Identity :

Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A * I = A$

$\Rightarrow A.I = A$

$\Rightarrow A.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$

G has identity

Hence, $(G, *)$ is a Monoid

5 Define Ring and give example.

Sol. An algebraic system $\langle S, +, . \rangle$ is called a ring if the binary operations + and . on S satisfy the following three properties.

i) $\langle S, + \rangle$ is an abelian group

ii) $\langle S, . \rangle$ is a semigroup

iii) The operation . is distributive over + i.e., for any $a, b, c \in S$,

$a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$

Eg. $\langle Z, +, . \rangle$, $\langle R, +, . \rangle$ are rings.

6. Compute the inverse of each element in Z_7 using Fermat's theorem

Sol. From Fermats theorem, $a^{p-1} \equiv 1 \pmod p$

$\Rightarrow a \cdot a^{p-2} \equiv 1 \pmod p$ [p → prime, (a, p) = 1]

$\therefore$ Inverse of a is a^{p-2}

Here $p = 7$

$\therefore$ inverse of a is a^5 — ①

Given, $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

W.K.T., 1, 2, 3, 4, 5, 6 has inverse and zero has no inverse.

By ①,

inverse of 1 : $1^5 \pmod 7 \equiv 1 \pmod 7$

$\Rightarrow$ 1 is inverse of 1

inverse of 2 : $2^5 \pmod 7 \equiv 32 \pmod 7 \equiv 4 \pmod 7$

$\Rightarrow$ 4 is the inverse of 2

inverse of 3 : $3^5 \pmod 7 \equiv 9 \cdot 9 \cdot 3 \pmod 7 \equiv 12 \pmod 7 \equiv 5 \pmod 7$

$\Rightarrow$ 5 is the inverse of 3

inverse of 4 : 2 is the inverse of 4

inverse of 5 : 3 is the inverse of 5

inverse of 6 : $6^5 \pmod 7 \equiv 36 \cdot 36 \cdot 6 \pmod 7 \equiv 6 \pmod 7$

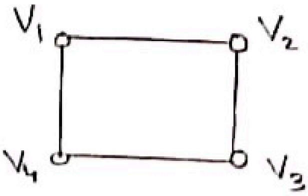
$\Rightarrow$ 6 is the inverse of 6

7. Define and give examples of

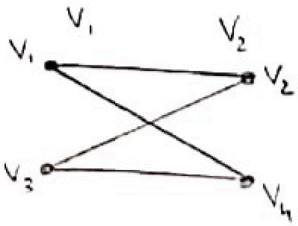
i) Bipartite Graph :-

Sol. A non-directed graph $G = \langle V, E \rangle$ is said to be a bipartite graph if 'V' can be partitioned into two sets 'V_1' and 'V_2' in such a way that every edge of 'G' joins a vertex in 'V_1' to a vertex in 'V_2'.

Eg- Consider the graph -



$G = \langle V, E \rangle$ where $V = \{v_1, v_2, v_3, v_4\}$



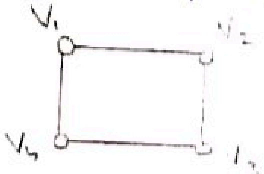
$V_1 = \{v_1, v_3\}$

$V_2 = \{v_2, v_4\}$

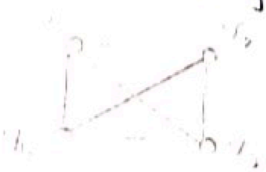
ii) K- Regular Graph :-

A graph is called regular graph if degree of each vertex is equal. A graph is called k regular if degree of each vertex in the graph is k.

Eg -



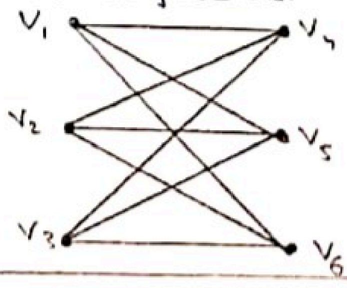
2 Regular Graph



3 Regular Graph

iii) Complete Bipartite graph :

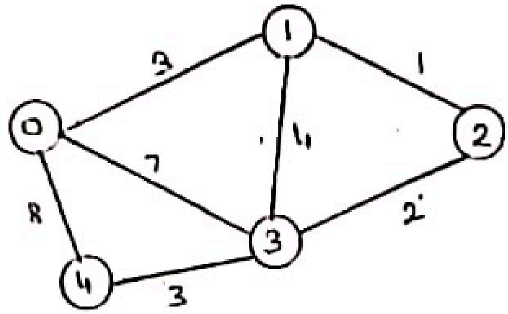
A bipartite graph $G \langle V_1 \cup V_2, E \rangle$ is said to be complete iff every vertex of V_1 is adjacent to every vertex of V_2



$V_1 = \{v_1, v_2, v_3\}$

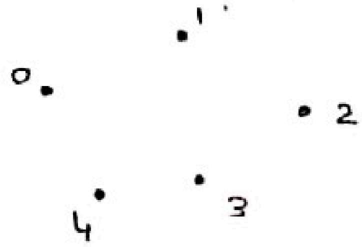
$V_2 = \{v_4, v_5, v_6\}$

8. Using Kruskal's algorithm find a minimal spanning tree for the weighted graph, shown below.



sol. Step-1:

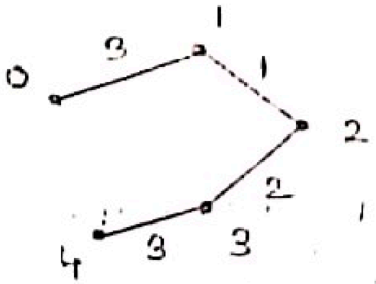
Consider the null graph formed by G.



~~Select the edges~~

Edge :	{1 2}	{2 3}	{3 4}	{0 1}	{1 3}	{0 3}	{0 4}
wts :	1	2	3	3	4	7	8

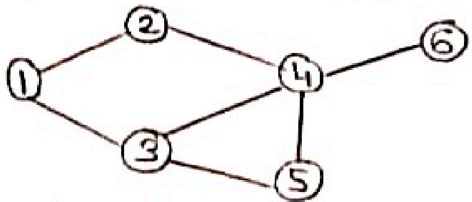
Step-2 : Select and add the edges in the above order such that addition of an edge doesn't produce a cycle. Select {1 2}, {2 3}, {3 4}, {0 1}.



This is the minimal spanning tree and the minimal cost for construction of this tree is

$3 + 1 + 2 + 3 = 9$

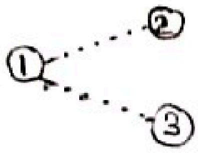
9. Use BFS technique and find a spanning tree for the following graph.



Sol. Let the order of vertices be '1, 2, 3, 4, 5, 6'.

Step-1:

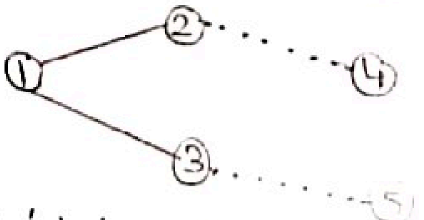
Select '1' as first vertex and connect all edges connected to '1' such that it don't form a cycle.



'2','3' are level-1 vertices from the root '1'.

Step-2:

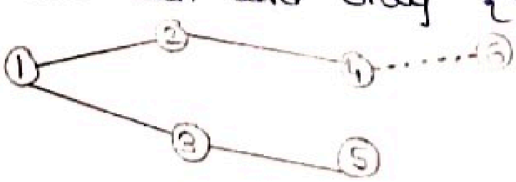
Add all edges connected to '2','3' that doesn't form a cycle (until all vertices are visited).



'4','5' will be level-2 vertices from roots '2','3'.

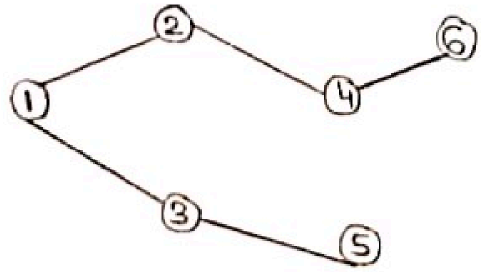
Step-3:

Add edges connected to '4','5' that doesn't form a cycle. Here {4 5} can't be added as it forms a cycle. So, we can add only {4 6}



Here, we can observe that all the vertices are visited.

∴ The spanning tree is given by



10. Solve RR $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$, $a_0 = 10$, $a_1 = 41$ using generating function method.

Sol. Given recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$

for $n \geq 2$, $a_0 = 10$, $a_1 = 41$ ——— (1)

Multiply (1) with x^n both sides and taking summation from $n = 2$ to ∞

$$\sum_{n=2}^{\infty} (a_n x^n - 7a_{n-1} x^n + 10a_{n-2} x^n) = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow (a_2 x^2 + a_3 x^3 + \ldots) - 7(a_1 x^2 + a_2 x^3 + \ldots) + 10(a_0 x^2 + a_1 x^3 + \ldots) = 0$$

$$\Rightarrow [A(x) - a_0 - a_1 x] - 7[x(A(x) - a_0)] + 10x^2 (A(x)) = 0$$

$$\Rightarrow A(x)[1 - 7x + 10x^2] - 10 + 29x = 0$$

$$\Rightarrow A(x) = \frac{10 - 29x}{(1 - 7x + 10x^2)} = \frac{A}{5x - 1} + \frac{B}{2x - 1}$$ (Here, $A = -7$, $B = -3$)

$$= \frac{(-7)}{5x - 1} + \frac{(-3)}{(2x - 1)}$$

$$\therefore a_n = 7(5)^n + 3(2)^n$$

11. Find x^{32} coefficient in $(1 + x^5 + x^9)^{10}$

Sol. Given expression, $(1 + x^5 + x^9)^{10} = (x^0 + x^5 + x^9)^{10}$

$$n_1 + n_2 + n_3 = 10$$

$$n_1 + 1 + 3 = 10 \Rightarrow n_1 = 6$$

$\therefore x^{32}$ coefficient $(1)^6 (x^5)^1 (x^9)^3$ in $(1 + x^5 + x^9)^{10}$

is $\dfrac{10!}{6!\,1!\,3!} = 840$

12. Solve the recurrence relation $a_n = 4a_{n-1} + 5a_{n-2}, n \geq 2$ where $a_0 = 2, a_1 = 6$, using characteristic root method

Sol. Given RR is $a_n - 4a_{n-1} - 5a_{n-2} = 0$ for $n \geq 2$ —①

Substitute $a_n = ck^n$

$\therefore ck^{n-2}[k^2 - 4k - 5] = 0$

↳ characteristic equation —②

Solving ②, we get $k = 5, -1$

So, the solution is $a_n = c_1(k_1)^n + c_2(k_2)^n$

$= c_1(5)^n + c_2(-1)^n$

At $n=0 \Rightarrow a_0 = c_1 + c_2$

$\Rightarrow c_1 + c_2 = 2$ —③

At $n=1 \Rightarrow a_1 = 5c_1 - c_2$

$\Rightarrow 5c_1 - c_2 = 6$ —④

Solving ③ & ④,

$c_1 = \frac{4}{3}, \quad c_2 = \frac{2}{3}$

$\therefore$ The solution is $a_n = \frac{4}{3}(5)^n + \frac{2}{3}(-1)^n$

13. Find the coefficient of x^{20} in $(x^3 + x^4 + x^5 + \ldots)^5$

Sol. Given $(x^3 + x^4 + x^5 + \ldots)^5$

$= [x^3(1 + x + x^2 + \ldots)]^5$

$= x^{15}(1 + x + x^2 + \ldots)^5$

$= x^{15}[(1-x)^{-1}]^5$

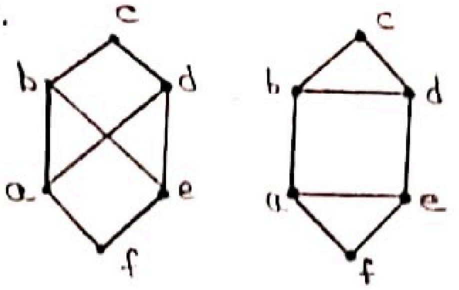
$= x^{15}(1-x)^{-5}$

$= \frac{x^{15}}{(1-x)^5}$

$= x^{15} \sum_{r=0}^{\infty} {}^{4+r}C_r \, x^r$

$\therefore$ coefficient of x^{20} is ${}^{9}C_5 = C(9,5)$

14. Are the following pair of graphs isomorphic? Justify your answer.

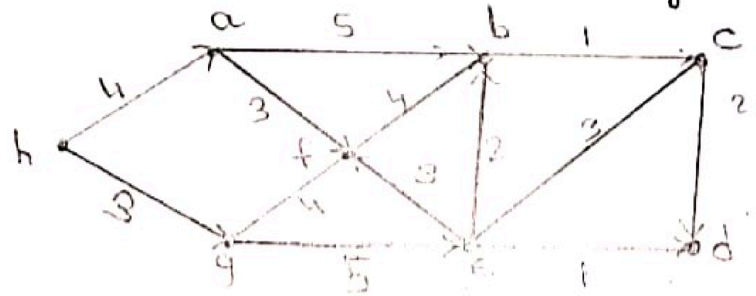


Sol. Vertices of two graphs is same i.e., 6
Edges of two graphs is same i.e., 8
Degree sequence is also same

But, first graph has 4-cycle and doesn't have any 3 cycle. Second graph has 3-cycle.

Hence, the given pair of graph are not isomorphic.

15. Find minimal spanning tree for the connected graph given below using Kruskal's algorithm.

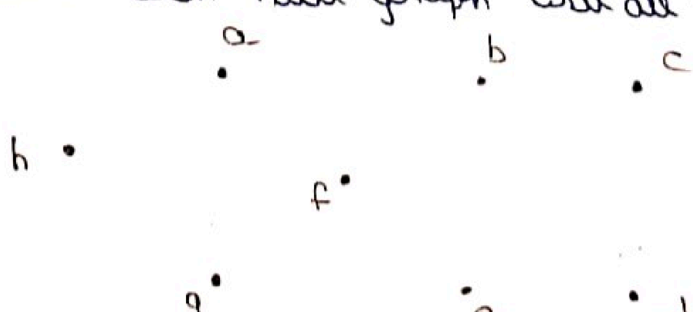


Sol. We've no. of vertices, n = 8

The edges whose weights in ascending order are given by

Edge:	{bc}	{ed}	{cd}	{be}	{af}	{ef}	{gh}	{ce}	{bf}	{ah}	{fg}	{ab}	{eg}
wts:	1	1	2	2	3	3	3	3	4	4	4	5	5

Step-1: Start with null graph with all vertices.



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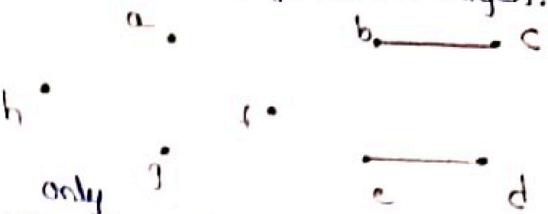
Step-2 :

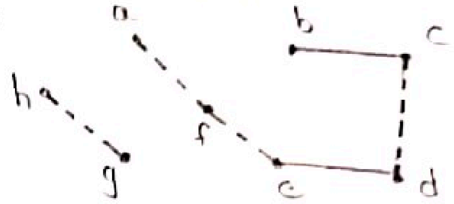
Add the edges in the above order such that they don't form any cycle & until we get (n-1) edges.

Select {bc}, {ed}

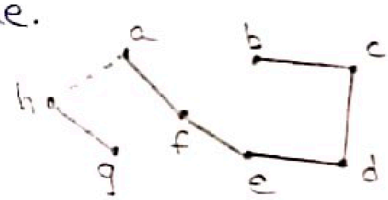


Now, we've option to choose only {b e} or {c d} to avoid the ~~loop~~ cycle. Select {c d}.

Next select {af}, {ef}, {gh}. Avoid {ce} inorder to avoid the cycle.

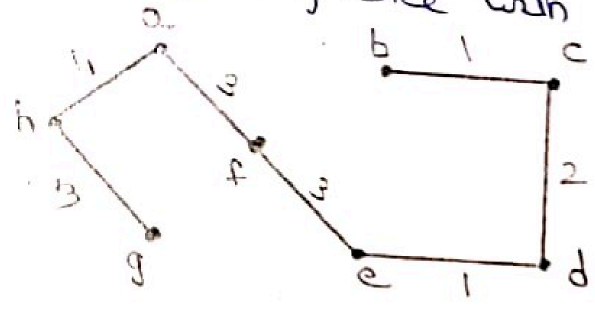


Then, diselect {b f} to avoid cycle and select {ah}. After selecting this, the total no. of edges becomes (8-1) = 7. So, we stop here.



Finally, we got the minimal spanning tree with 7 edges. The minimal cost for the construction is

1+1+2+3+3+3+4 = 17.



16. Prove that a tree with 'n' vertices has 'n-1' edges.

Sol. Let $G = (V, E)$ be a tree with 'n' vertices

To prove this, we use mathematical induction on the numbers of vertices (n) of 'G'.

If $n = 1$, then there are no edges in 'G'. Hence, the result is trivial for $n = 1$.

Assume that, for $K \geq 1$ all trees with 'k' vertices have exactly $(k-1)$ edges.

Let 'G' be a tree with $(K+1)$ vertices.

By a known theorem we've, there is a vertex (say) 'v' in G with $\deg(v) = 1$ i.e., there is only one edge (say) 'e' incident at 'v'

Let $G' = G - v$ (i.e., G' is the graph obtained by removing 'v' and edge 'e' from 'G')

then G' is also ~~known as~~ a tree with k vertices.

By induction hypothesis, 'G' has exactly $(K-1)$ edges.

Now, by adding the vertex 'v' and the edge 'e' to G' we get 'G' and then G has exactly $(K-1) + 1 = K$ edges.

$\therefore$ The theorem is true for any tree with 'n' vertices where $n \in Z^+$

17. Using Chinese Remainder Theorem, find a solution of linear congruence $17x \equiv 9 \pmod{276}$.

Sol. Given linear congruence is $17x \equiv 9 \pmod{276}$

Since, $276 = 3.4.23$

The given linear congruence is equivalent to the system of congruences.

$17x \equiv 9 \pmod 3$	$17x \equiv 9 \pmod 4$	$17x \equiv 9 \pmod{23}$
$2x \equiv 0 \pmod 3$	$x \equiv 1 \pmod 4$	$17x \equiv 9 \pmod{23}$
		$170x \equiv 90 \pmod{23}$
		$9x \equiv 21 \pmod{23}$
		$3x \equiv 7 \pmod{23}$
		$24x \equiv 56 \pmod{23}$
$x \equiv 0 \pmod 3$	$x \equiv 1 \pmod 4$	$x \equiv 10 \pmod{23}$

By comparing with $x \equiv a_1 \pmod{n_1}$, $x \equiv a_2 \pmod{n_2}$, $x \equiv a_3 \pmod{n_3}$

$a_1 = 0 \quad a_2 = 1 \quad a_3 = 10$

$n_1 = 3 \quad n_2 = 4 \quad n_3 = 23$

So, $N = n_1 . n_2 . n_3 = 276$

$N_1 = \frac{N}{n_1} = \frac{276}{3} = 92 \qquad N_2 = \frac{276}{4} = 69 \qquad N_3 = \frac{276}{23} = 12$

$N_1 x_1 \equiv 1 \pmod{n_1} \quad N_2 x_2 \equiv 1 \pmod{n_2} \quad N_3 x_3 \equiv 1 \pmod{n_3}$

$92 x_1 \equiv 1 \pmod 3 \quad 69 x_2 \equiv 1 \pmod 4 \quad 12 x_3 \equiv 1 \pmod{23}$

$x_1 = 2 \qquad x_2 = 1 \qquad x_3 = 2$

$\therefore \ x \equiv [a_1 N_1 x_1 + a_2 N_2 x_2 + \ldots + a_a N_a x_a] \pmod N$

$= [0.92.2 + 1.69.1 + 10.12.2] \pmod{276}$

$= [0 + 69 + 240] \pmod{276}$

$= 309 \pmod{276}$

$x \equiv 33 \pmod{276}$ is the solution of given system.

18. Compute the inverse of each element in Z_{12}, if excits, using Euler's formula.

Sol: By Euler's theorem,

$$a^{\varphi(m)} \equiv 1 \pmod{m} \quad — ①$$

Here, $m = 12$; $\varphi(m) = \varphi(12) = 4$ $(\{1, 5, 7, 11\})$

By ①, $a^4 \equiv 1 \pmod{12}$

$a.a^3 \equiv 1 \pmod{12}$

$\therefore$ Inverse of a is a^3

$\therefore$ Inverse of 1 : $1^3 \pmod{12} = 1 \pmod{12}$

$\Rightarrow$ 1 is the inverse of 1

Inverse of 5 : $5^3 \pmod{12} = 25.5 \pmod{12} = 5 \pmod{12}$

$\Rightarrow$ 5 is the inverse of 5

Inverse of 7 : $7^3 \pmod{12} = 49.7 \pmod{12} = 7 \pmod{12}$

$\Rightarrow$ 7 is the inverse of 7

Inverse of 11 : $11^3 \pmod{12} = 121.11 \pmod{12} = 11 \pmod{12}$

$\Rightarrow$ 11 is the inverse of 11