QR Decomposition with Gram-Schmidt

The QR decomposition also called QR factorization is a decomposition of matrix into an orthogonal matrix and a upper triangular matrix.

A, QR decomposition of a real square matrix / rectangular matrix A is a decomposition of A as

$$A = QR$$

where Q is an orthogonal matrix

ie $Q^T Q = I$

and R is an upper triangular matrix. If A is non singular then this factorization is unique

Gram-Schmidt process

Consider the Gram Schmidt procedure, with the vectors to be considered in the process as columns of the matrix A. That is

$$A = \left[a_1 \mid a_2 \mid \cdots \mid a_n \right] \text{ or } A \left[v_1 \mid v_2 \mid \cdots \mid v_n \right]$$

Then $u_1 = v_1 \qquad e_1 = \dfrac{u_1}{\|u_1\|}$

$$u_2 = v_2 - \text{proj}_{u_1} v_2 = v_2 - \frac{(u_1 \cdot v_2)}{\|u_1\|^2} u_1 \quad ; \quad e_2 = \frac{u_2}{\|u_2\|}$$

or $u_2 = v_2 - (v_2 \cdot e_1) e_1$

$$u_3 = v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3 = v_3 - \frac{(u_1 \cdot v_3)}{\|u_1\|^2} u_1 - \frac{(u_2 \cdot v_3)}{\|u_2\|^2} u_2$$

or $u_3 = v_3 - (v_3 \cdot e_1) e_1 - (v_3 \cdot e_2) e_2$

Illy $u_4 = v_4 - (v_4 \cdot e_1) e_1 - (v_4 \cdot e_2) e_2 - (v_4 \cdot e_3) e_3.$

The resulting QR factorization is

$$A = \left[v_1 \,|\, v_2 \,|\, v_3 \right] = \left[e_1 \,|\, e_2 \,|\, e_3 \right] \begin{bmatrix} v_1 \cdot e_1 & v_2 \cdot e_1 & v_3 \cdot e_1 \\ 0 & v_2 \cdot e_2 & v_3 e_2 \\ 0 & 0 & v_3 e_3 \end{bmatrix}$$

Note that once we find $e_1, e_2, \ldots e_n$ it is not hard to write the QR factorization

Example: problem

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ decompose it into QR

factorization

let $v_1 = (1\ 1\ 0)^T$, $v_2 = (1\ 0\ 1)^T$ $v_3 = (0\ 1\ 1)^T$

performing the Gram. Schmidt procedure, we obtain

$$u_1 = v_1 = (1, 1, 0)$$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$u_2 = v_2 - (v_2 \cdot e_1) e_1 = (1, 0, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (1, 0, 1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$u_2 = \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$e_2 = \frac{u_2}{\|u\|_2} = \frac{1}{\sqrt{3/2}}\left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$u_3 = v_3 - (v_3 \cdot e_1) e_1 - (v_3 \cdot e_2) e_2$$

$$= (0, 1, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$= (0, 1, 1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) - \left(\frac{1}{6}, -\frac{1}{6}, \frac{2}{6}\right)$$

$$= \left(0 - \frac{1}{2} - \frac{1}{6},\ 1 - \frac{1}{2} + \frac{1}{6},\ 1 - 0 - \frac{2}{6}\right)$$

$$= \left(\frac{-3-1}{6},\ \frac{6-3+1}{6},\ \frac{6-2}{6}\right) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$u_3 = \left(-\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$$

$$\|u_3\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}}$$

$$= \sqrt{\frac{12}{9}}$$

$$= \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{\left(-\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right)}{\frac{2}{\sqrt{3}}} = \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

Thus $Q = \left[\,e_1 \mid e_2 \mid e_3\,\right] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$$R = \begin{bmatrix} v_1\cdot e_1 & v_2\cdot e_1 & v_3\cdot e_1 \\ 0 & v_2\cdot e_2 & v_3\cdot e_2 \\ 0 & 0 & v_3\cdot e_3 \end{bmatrix}$$

$$= \begin{bmatrix} (1,1,0)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix} & (1,0,1)\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix} & (0\,1\,1)\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix} \\ 0 & (1\,0\,1)\begin{pmatrix}\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{pmatrix} & (0\,1\,1)\begin{pmatrix}\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{pmatrix} \\ 0 & 0 & (0\,1\,1)\cdot\begin{pmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$$

2) Consider the matrix $A = \begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

find its QR Decomposition

The Gram Schmidt process on the matrix A proceeds as follows

$$u_1 = v_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad e_1 = \frac{u_1}{\|u_1\|} = \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}{\sqrt{4+4+1}} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$u_2 = v_2 - (v_2 \cdot e_1)e_1 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - \left[\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \cdot \left(\tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{1}{3}\right)\right] \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - \left(-4/3 + 2/3 + 2/3\right)\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1/3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - 0 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{4+1+4}} = \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$u_3 = v_3 - (v_3 \cdot e_1)e_1 - (v_3 \cdot e_2)e_2$$

$$= \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} - \left(\begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} \cdot \left(\tfrac{2}{3}\ \tfrac{2}{3}\ 1\right)\right)\begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} - \left(\begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} \cdot \left(-\tfrac{2}{3}\ \tfrac{1}{3}\ \tfrac{2}{3}\right)\right)\begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} - 12\begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} + 12\begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ 8 \\ 12 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}}{\sqrt{4+16+16}} = \frac{1}{6}\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

Thus the orthogonalized matrix resulting from Gram-Schmidt process is

$$\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$$

The Component R of the QR decomposition Can also be found from the Calculations made in Gram schmidt process as defined

$$R = \begin{bmatrix} v_1 \cdot e_1 & v_2 \cdot e_1 & v_3 \, e_1 \\ 0 & v_2 \, e_2 & v_2 \cdot e_2 \\ 0 & 0 & v_3 \, e_3 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\left(2/3 \; 2/3 \; 1/3\right) & \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\left(2/3 \; 2/3 \; 1/3\right) & \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix}\left(2/3 \; 2/3 \; 1/3\right) \\ 0 & \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\left(-2/3 \; 1/3 \; 2/3\right) & \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix}\left(-2/3 \; 1/3 \; 2/3\right) \\ 0 & 0 & \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix}\left(1/3 \; -2/3 \; 2/3\right) \end{bmatrix}$$

$$R = \begin{pmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\therefore \; Q \cdot R = A.$$

3 Find the QR decomposition (Gram Schmidt process)

$$A = \begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix}$$

$$Q = [u_1, u_2]$$

$$u_1 = v_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \qquad e_1 = \frac{u_1}{\|u_1\|} = \frac{\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}}{\sqrt{3^2+4^2+0^2}} = \frac{\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}}{5}$$

$$= \frac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$$

$$u_2 = v_2 - (v_2 \cdot e_1)\, e_1$$

$$= \begin{pmatrix} -6 \\ -8 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} -6 \\ -8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3/5 & 4/5 & 0 \end{pmatrix}\right) \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ -8 \\ 1 \end{pmatrix} - \left(\left(\frac{-18}{5} - \frac{32}{5}\right)\right) \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ -8 \\ 1 \end{pmatrix} + 10 \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ -8 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6+6 \\ -8+8 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$R = \begin{bmatrix} v_1 \cdot e_1 & v_2 \cdot e_1 \\ 0 & v_2 \cdot e_2 \end{bmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3/5 & 4/5 & 0 \end{pmatrix} & \begin{pmatrix} -6 \\ -8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3/5 & 4/5 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} -6 \\ -8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{5} & \frac{-18-32}{5} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 0 & 1 \end{bmatrix}$$

$$A = Q \cdot R$$

4. Find the QR decomposition (Gram Schmidt process)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

Sol

$A = [V_1, V_2, V_3]$ $\quad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $V_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ $V_3 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

$$u_1 = V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_1 = \frac{u_1}{\|u_1\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_2 = V_2 - (V_2 \cdot e_1) e_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \left[\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \cdot (1, 0, 0)\right] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \left(2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}}{3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_3 = V_3 - (V_3 \cdot e_1) e_1 - (V_3 \cdot e_2) e_2$$

$$= \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \left(\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot (1\ 0\ 0)\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot (0\ 0\ 1)\right] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \left(4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) - 6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 4-4 \\ 5-0 \\ 6-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow \|u_3\| = \sqrt{0+25+0} = \sqrt{25} = 5$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{1}{5} \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$R = \begin{pmatrix} v_1 \cdot e_1 & v_2 \cdot e_1 & v_3 \cdot e_1 \\ 0 & v_2\, e_2 & v_3 \cdot e_2 \\ 0 & 0 & v_3\, e_3 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot (1\,0\,0) & \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} (1\,0\,0) & \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \cdot (0\,1\,0) \\ 0 & \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} (0\,0\,1) & \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} (0\,0\,1) \\ 0 & 0 & \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} (0\,1\,0) \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A = Q \cdot R$$