Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ — ① be a 3×3 matrix

$AX = \lambda X \rightarrow X$ is a Column vector

Characteristic Equation is $|A - I\lambda| = 0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

This gives a Cubic Equation in "λ" whose roots are Eigen values.

Corresponding to each eigen value we have a non Zero Solution $X = [x_1, x_2, x_3]$ which is Called an eigen vector.

Such an equation Can ordinarily be Solved easily.

· Rayleigh power method:

(Largest eigen value and Corresponding Eigen vector)

We start with a Column vector X which is near the solution as possible and evaluate AX which is written as $\lambda^{1} X^{(1)}$ after normalisation.

This gives the first approximation $\lambda^{(1)}$ to the eigen value and $X^{(1)}$ to eigen vector.

Similarly we evaluate $AX^{(1)} = \lambda^{(2)} X^{(2)}$ which gives the Second approximation

We repeat this process till $[X^{(r)} - X^{(r-1)}]$ becomes negligible

Then $\lambda^{(r)}$ will be the largest eigen value of (1)

and $X^{(r)}$ the Corresponding eigen vector.

This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.

1. Determine the largest eigen value and the Corresponding eigen vector of the matrix using power method

(i) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Let the initial approximation to the eigen vector Corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then $AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^{(1)} X^{0}$

So the first approximation to the eigen value is $\lambda^{(1)} = 5$ and the Corresponding eigen vector is $X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$

Now $AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^{(2)} X^{(2)}$

Thus the second approximation to the eigen value is $\lambda^{(2)} = 5.8$, and corresponding eigen vector is $X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$ repeating the above process, we get

$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^{(3)} X^{(3)}$

$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.250 \end{bmatrix}$
$\lambda^{(4)} \quad X^{(4)}$

$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 5.99\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)} X^{(5)}$

$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(6)} X^{(6)}$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ upto 3 decimal places

Hence the largest eigen value is 6 and the Corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.25 \end{pmatrix}$

Determine the largest eigen value and the corresponding vector of matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let the initial approximation to the required eigen vector $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Then $AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix}$

$= \lambda' X'$

So the first approximation to the eigen value $\lambda^{(1)} = 2$ and the corresponding eigen vector

$x^{(1)} = (1, -0.5, 0)$

Hence $Ax^{(1)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -2 \\ 0.5 \end{pmatrix}$

$= 2.5\begin{pmatrix} 1 \\ -0.8 \\ 0.2 \end{pmatrix}$.

$= \lambda^{(2)} x^{(2)}$

Repeating the above process, we get

$Ax^{(2)} = 2.8\begin{pmatrix} 1 \\ -1 \\ 0.43 \end{pmatrix}$

$= \lambda^{(3)} . x^{(3)}$

$Ax^{(3)} = 3.43\begin{pmatrix} 0.87 \\ -1 \\ 0.54 \end{pmatrix}$

$\lambda^{(4)} \quad x^{(4)}$

$Ax^{(4)} = 3.41\begin{pmatrix} 0.86 \\ -1 \\ 0.61 \end{pmatrix}$

$\lambda^{(5)} \quad x^{(5)}$

$A(X^5) = 3.41\begin{pmatrix} 0.76 \\ -1 \\ 0.65 \end{pmatrix} = \lambda^{(6)} x^{(6)}$

$$Ax^{(6)} = 3.41 \begin{pmatrix} 0.74 \\ -1 \\ 0.67 \end{pmatrix}$$

$$= \lambda^{(7)} x^{7}$$

clearly $\lambda^{(6)} = \lambda^{(7)}$ and $x^{(6)} = x^{(7)}$

approximately

Hence the largest eigen value is 3.41.

and the Corresponding eigen vector is $(0.74, -1.067)^T$

3. Find the power method for finding dominant Eigen value and Eigen vector

given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ with initial approximation

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ax_0 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_1 = \frac{1}{4} \begin{bmatrix} 1 \\ -0.75 \\ 0.25 \end{bmatrix}$$

2nd Iteration

$$Ax_1 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 13 \\ -12.25 \\ 5.75 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_2 = \frac{1}{13} \begin{bmatrix} 13 \\ -12.25 \\ 5.75 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.94 \\ 0.44 \end{bmatrix}$$

3rd iteration

$$Ax_2 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.94 \\ 0.44 \end{bmatrix} = \begin{bmatrix} 14.54 \\ -14.37 \\ 7.1 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_3 = \frac{1}{14.54}\begin{bmatrix}14.54\\-14.37\\7.1\end{bmatrix} = \begin{bmatrix}1\\-0.99\\0.49\end{bmatrix}$$

4 th iteration

$$Ax_3 = \begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}\begin{bmatrix}1\\-0.99\\0.49\end{bmatrix} = \begin{bmatrix}14.9\\-14.87\\7.42\end{bmatrix}$$

and by scaling we obtain the approximation

$$x_4 = \frac{1}{14.9}\begin{bmatrix}14.9\\-14.87\\7.42\end{bmatrix} = \begin{bmatrix}1\\-1\\0.5\end{bmatrix}$$

5 th iteration

$$Ax_4 = \begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}\begin{bmatrix}1\\-1\\0.5\end{bmatrix} = \begin{bmatrix}14.98\\-14.97\\7.48\end{bmatrix}$$

and by scaling we obtain the approximation

$$x_5 = \frac{1}{14.98}\begin{bmatrix}14.98\\-14.97\\7.48\end{bmatrix} = \begin{bmatrix}1\\-1\\0.5\end{bmatrix}$$

6 th iteration

$$Ax_5 = \begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}\begin{bmatrix}1\\-1\\0.5\end{bmatrix} = \begin{bmatrix}15\\-14.99\\7.5\end{bmatrix}$$

and by scaling we obtain the approximation

$$x_6 = \frac{1}{15}\begin{bmatrix}15\\-14.99\\7.5\end{bmatrix} = \begin{bmatrix}1\\-1\\0.5\end{bmatrix}$$

7 th iteration

$$Ax_6 = \begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}\begin{bmatrix}1\\-1\\0.5\end{bmatrix} = \begin{pmatrix}15\\-15\\7.5\end{pmatrix}$$

and by scaling we obtain the approximation

$$x_7 = \frac{1}{15}\begin{pmatrix}15\\-15\\7.5\end{pmatrix} = \begin{pmatrix}1\\-1\\0.5\end{pmatrix}$$

The dominant Eigen Value $\lambda = 15$
and the dominant Eigen Vector is

$$= \begin{bmatrix} 1 \\ -1 \\ 0.5 \end{bmatrix}$$

Find the largest eigen value and the Corresponding eigen vector of the matrices

(a) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$

Ans: $\lambda = 7$

Eigen vector $\begin{bmatrix} 2.099/7 \\ 0.467/7 \\ 1 \end{bmatrix} = x$

(b) $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ $\lambda = 25.0182$

$x = [1 \quad 0.045 \quad 0.068]^T$

(c) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial approximation $(1, 1, 0)^T$