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DMS-Assignment-4

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CSE-4

(1) Solve the recurrence relation $a_n = a_{n-1} + n^2$ where $a_0 = 7$ by substitution method.

Ans Sol:- Given RR: $a_n = a_{n-1} + n^2$ and $a_0 = 7$

for $n=1$; $a_1 = a_0 + (1)^2 = 7+1 = 8$

for $n=2$; $a_2 = a_1 + (2)^2 = 8+4 = 12 = 7+(1+4)$

for $n=3$; $a_3 = a_2 + (3)^2 = 12+9 = 21 = 7+(1+4+9)$

⋮

$a_n = a_0 + (1^2+2^2+3^2+\cdots+n^2) = a_0 + \frac{(n)(n+1)(2n+1)}{6}$.

∴ The general solution of given recurrence is

$$a_n = 7 + \frac{n(n+1)(2n+1)}{6}.$$

(2) Find generating function for a_r = the number of non-negative integral solutions of $e_1+e_2+e_3+e_4+e_5 = r$ where $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd and $1 \le e_5 \le 9$.

Sol:- Given that,

$0 \le e_1 \le 3$, then $e_1 = 0,1,2,3$, $A_1(x) = 1+x+x^2+x^3$.

$0 \le e_2 \le 3$, then $e_2 = 0,1,2,3$, $A_2(x) = 1+x+x^2+x^3$.

$2 \le e_3 \le 6$, then $e_3 = 2,3,4,5,6$, $A_3(x) = x^2+x^3+x^4+x^5+x^6$.

$2 \le e_4 \le 6$, then $e_4 = 2,3,4,5,6$, $A_4(x) = x^2+x^3+x^4+x^5+x^6$.

e_5 is odd & $1 \le e_5 \le 9$, then $e_5 = 1,3,5,7,9$, $A_5(x) = x+x^3+x^5+x^7+x^9$.

Thus the generating function we required is.

$A(x) = A_1(x) \cdot A_2(x) \cdot A_3(x) \cdot A_4(x) \cdot A_5(x)$

$A(x) = (1+x+x^2+x^3)^2 (x^2+x^3+x^4+x^5+x^6)^2 (x+x^3+x^5+x^7+x^9)$.

(3) Find the coefficient of x^5 in $\frac{1}{x^2-5x+6}$.

Sol:- First we decompose the function into partial function.

$$x^2-5x+6 = x^2-3x-2x+6 = (x-3)(x-2)$$

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)} = \frac{A}{(x-3)} + \frac{B}{(x-2)}.$$

$$1 = A(x-2)+B(x-3) \longrightarrow (1)$$

$$A=1,\ B=-1$$

$$\frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}.$$

Now we want to find x^5 coefficient in $\frac{1}{(x-3)} - \frac{1}{(x-2)}$.

$$= \frac{1}{(-3)\left(1-\frac{x}{3}\right)} - \frac{1}{(-2)\left(1-\frac{x}{2}\right)}$$

$$= \frac{1}{2\left(1-\frac{x}{2}\right)} - \frac{1}{3\left(1-\frac{x}{3}\right)}$$

$$\frac{1}{2\left(1-\frac{x}{2}\right)} - \frac{1}{3\left(1-\frac{x}{3}\right)} = \frac{1}{2}\sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^r - \frac{1}{3}\sum_{r=0}^{\infty}\left(\frac{x}{3}\right)^r \text{ (Take } r=5)$$

$$\therefore x^5 \text{ coefficient} = \frac{1}{2}\left(\frac{1}{2^5}\right) - \frac{1}{3}\left(\frac{1}{3^5}\right). = \frac{1}{2^6} - \frac{1}{3^6}.$$

(4) Let $G = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : a,b,c,d \in R\right\}$ and defined as $A*B = A\cdot B$, $\forall A,B \in G$ then $(G,*)$ is a monoid.

Sol:- In order to prove $\langle G,*\rangle$ is a monoid it is enough to show '$*$' is a binary, associative and existence of identity.

Binary:- For this let us assume $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1\end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2\end{bmatrix} \in G$.

Now we have to show '$*$' is binary operation.

i.e, we have to show '$*$' is a mapping from $G \times G \to G$.

Consider,

$$A*B = A\cdot B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1\end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2\end{bmatrix} = \begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ a_2c_1+d_1c_2 & c_1b_2+d_1d_2\end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix} \in G.$$

Hence '$*$' is a binary operation.

Associative:- Let $A=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, $C=\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \in G$.

Now consider

$$A*(B*C)=A*BC=A*\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2a_3+b_2c_3 & a_2b_3+b_2d_3 \\ c_2a_3+d_2c_3 & c_2b_3+d_2d_3 \end{bmatrix}$$

$$=\begin{bmatrix} a_1a_2a_3+a_1b_2c_3+b_1c_2a_3+b_1d_2c_3 & a_1a_2b_3+a_1b_2d_3+b_1c_2b_3+b_1d_2d_3 \\ c_1a_2a_3+c_1b_2c_3+d_1c_2a_3+d_1d_2c_3 & c_1a_2b_3+c_1b_2d_3+d_1c_2b_3+d_1d_2d_3 \end{bmatrix}$$

and $(A*B)*C=AB*C=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}*C=\begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ a_2c_1+d_1c_2 & c_1b_2+d_1d_2 \end{bmatrix}*\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$

$$=\begin{bmatrix} a_1a_2a_3+a_1b_2c_3+b_1c_2a_3+b_1d_2c_3 & a_1a_2b_3+a_1b_2d_3+b_1c_2b_3+b_1d_2d_3 \\ c_1a_2a_3+c_1b_2c_3+d_1c_2a_3+d_1d_2c_3 & c_1a_2b_3+c_1b_2d_3+d_1c_2b_3+d_1d_2d_3 \end{bmatrix}$$

$\therefore (A*B)*C = A*(B*C)$. Hence "$*$" is associative.

Existence of identity:- Let $A=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in G$. Let $e=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$A*e=A.e=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}=A.$$

Now, $e*A=e.A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}=\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}=A.$

$\therefore A*e=e*A=A.$

$\therefore e=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity.

Hence, $\langle G, * \rangle$ is a monoid.

~~(7). Define and give example~~

(6) Compute the inverse of each element in Z_7 using Fermat's theorem.

Sol:- Given that $Z_7 = \{0,1,2,3,4,5,6\}$.

"0" has no inverse, since inverse exists for non-zero elements.

Inverse of remaining:-

$1^{7-1} \equiv 1 \pmod 7 \Rightarrow 1^6 \equiv 1 \pmod 7 \Rightarrow 1\cdot 1^5 \equiv 1 \pmod 7 \Rightarrow 1^5 = 1$ is the inverse of 1.

$2^{7-1} \equiv 1 \pmod 7 \Rightarrow 2^6 \equiv 1 \pmod 7 \Rightarrow 2\cdot 2^5 \equiv 1 \pmod 7 \Rightarrow 2^5 = 2\cdot 2^3 \cdot 2^2 \equiv 1 \pmod 7$

$\Rightarrow 1\cdot 2^2 = 4$ is inverse of 2.

$3^{7-1} \equiv 1 \pmod 7 \Rightarrow 3^6 \equiv 1 \pmod 6 \Rightarrow 3\cdot 3^5 \equiv 1 \pmod 7 \Rightarrow 3^5 = 3\cdot 3\cdot 3^2 \cdot 3^2 \equiv 1 \pmod 7$

$\Rightarrow 3\cdot 2\cdot 2 = 12 = 5$ is inverse of 3.

$4^{7-1} \equiv 1 \pmod 7 \Rightarrow 4^6 \equiv 1 \pmod 7 \Rightarrow 4\cdot 4^5 \equiv 1 \pmod 7 \Rightarrow 4^5 = 4\cdot 4\cdot 4^2\cdot 4^2 \equiv 1 \pmod 7$

$\Rightarrow 4\cdot 2\cdot 2 = 16 = 2$ is inverse of 4.

$5^{7-1} \equiv 1 \pmod 7 \Rightarrow 5^6 \equiv 1 \pmod 7 \Rightarrow 5\cdot 5^5 \equiv 1 \pmod 7 \Rightarrow 5^5 = 5\cdot 5\cdot 5^2\cdot 5^2 \equiv 1 \pmod 7$

$\Rightarrow 5\cdot 4\cdot 4 = 80 = 3$ is inverse of 5.

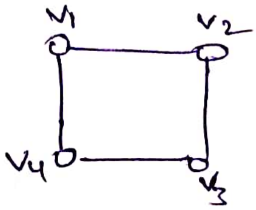
$6^{7-1} \equiv 1 \pmod 7 \Rightarrow 6^6 \equiv 1 \pmod 7 \Rightarrow 6\cdot 6^5 \equiv 1 \pmod 7 \Rightarrow 6^5 = 6\cdot 6\cdot 6^2\cdot 6^2 \equiv 1 \pmod 7$

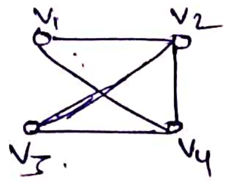
$\Rightarrow 6\cdot 1\cdot 1 = 6$ is inverse of 1.

(7) Define and give examples of (i) Bipartite Graph.
(ii) K-regular graph (iii) Complete Bipartite Graph.

Sol:- (i) Bipartite Graph:- A non-directed graph $G = \langle V, E \rangle$ is said to be a bipartite graph if 'V' can be partitioned into two sets 'V_1' and 'V_2' in such a way that every edge of 'G' joins a vertex in V_1 to a vertex in V_2.

Ex:-



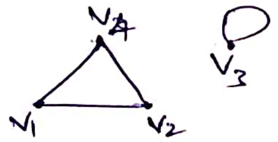


A non-directed graph $G = \langle V, E \rangle$ is a bi-partite graph if 'V' can be partitioned into two vertices V_1 and V_2 (2 sets).

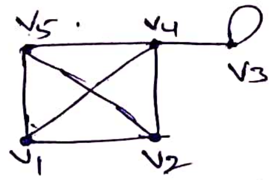
$V = \{v_1, v_2, v_3, v_4\}$, $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$.

(ii) K-regular Graph:- If every vertex of a graph 'G' has same degree (say) K, then we say that G is a K-regular graph.

Ex:-



2-regular graph.

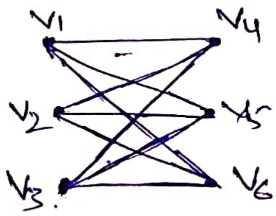


Not 3-regular graph.

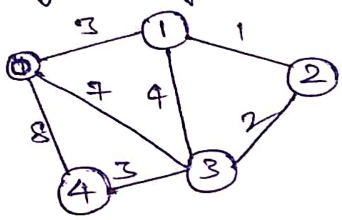
(iii) Complete bipartite graph:- A bipartite graph $G \langle V_1 \cup V_2, E \rangle$ is said to be complete iff every vertex of V_1 is adjacent to every vertex of V_2.

Ex:- $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{v_4, v_5, v_6\}$.

The graph is partitioned into two sets V_1 & V_2.
Every vertex of V_1 is adjacent to every vertex of V_2.



(8) Using Kruskal's algorithm find a minimum spanning tree for the weighted graph shown below.



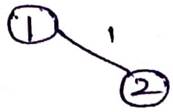
The graph contains 5 vertices. So, the minimum spanning tree formed will be having (5-1) = 4 edges.

Sorted list of edges:-

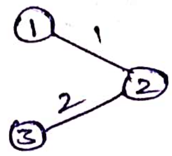
Weight	Src	Dest
1	1	2
2	2	3
3	0	1
3	3	4
4	1	3
7	0	3
8	0	4.

Pick all edges one by one from sorted list of edges.

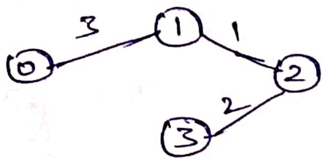
(i) Edge 1-2 :- No cycle is formed, so include it



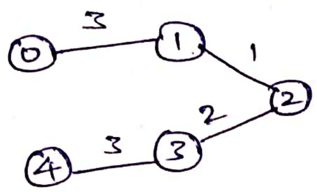
(ii) Edge 2-3 :- No cycle is formed, so include it



(iii) Edge 0-1 :- No cycle is formed, so include it

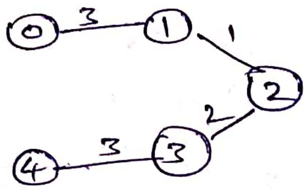


(iv) Edge 3-4 :- No cycle is formed, so include it.

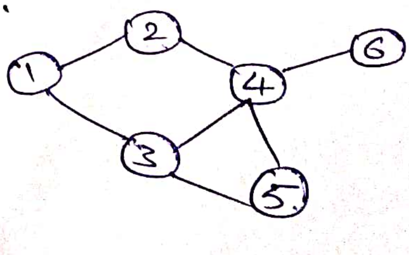


Since, the no. of edges included equals (5-1) = 4, the algorithm stops here.

∴ The obtained minimum spanning tree is



(9) Use BFS (Breadth First Search) Technique and find a spanning tree for the following graph.

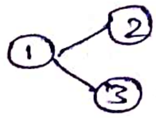


Sol:- Consider the ordering of the vertices "1, 2, 3, 4, 5, 6"

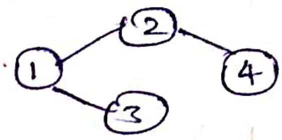
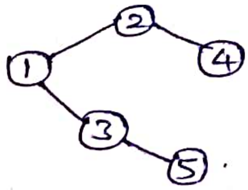
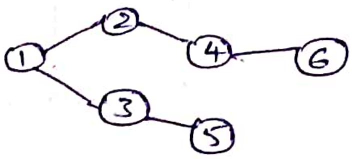
Select '1' as the first vertex for the spanning tree T and designate it as the root of 'T'.

Set $V = \{1\}$ T: 

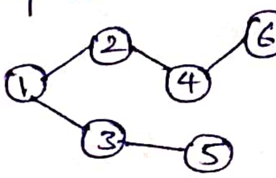
Select all those edges $\{1, x\}$ where 'x' runs from $2, \ldots, 6$. in that order that do not form a cycle in T'.

$\{1,2\}, \{1,3\}$ are added to T   T: 

Set $V = \{2, 3\}$.

First examine, edges incident at 2.
At '2', include $\{2,4\}$ as a tree edge   T: 
At '3', $\{3,4\}$ is not included since it forms a cycle.
But include $\{3,5\}$ T: 
Now. set $V = \{4, 5\}$.
At 4, include the edge $\{4, 6\}$.   T: 
At 5, we have no edges.
Set $V = \{6\}$.
But no edges can be added from 6. So, algorithm stops here.

$\therefore$ The obtained spanning tree for the given graph is



(10) Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$, $a_0 = 10$, $a_1 = 41$, using generating function method.

Sol:- Given recurrence relation, $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$, $a_1 = 41$, $a_0 = 10$.

Multiplying the above equation with x^n both sides and ta-

-king summation from $n=2$ to ∞, we get

$$\sum_{n=2}^{\infty}(a_n x^n - 7a_{n-1}x^n + 10a_{n-2}x^n)=0.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1}x^n + 10\sum_{n=2}^{\infty} a_{n-2}x^n = 0.$$

$$\Rightarrow (a_2x^2+a_3x^3+a_4x^4+\cdots) - 7(a_1x^2+a_2x^3+a_3x^4\cdots) + 10(a_0x^2+a_1x^3+a_2x^4+\cdots)=0.$$

$$\Rightarrow (A(x)-a_0-a_1x) - 7(x(A(x)-a_0)) + 10x^2A(x)=0.$$

$$\Rightarrow (A(x)-10-41x) - 7(x(A(x)-10)) + 10x^2(A(x))=0.$$

$$\Rightarrow A(x)(1-7x+10x^2)-10+29x=0.$$

$$A(x)=\frac{10-29x}{(1-7x+10x^2)} = \frac{10-29x}{(5x-1)(2x-1)} = \frac{A}{(5x-1)} + \frac{B}{(2x-1)}.$$

$$A(x) = \frac{-7}{(5x-1)} + \frac{(-3)}{(2x-1)}$$

$$A(x)=\frac{7}{(1-5x)}+\frac{3}{(1-2x)}. \qquad \left(\because \left(\frac{1}{1-ax}\right)=\sum_{r=0}^{\infty}a^r x^r\right)$$

$$\sum_{r=0}^{\infty} a_r x^r = 7.\sum_{r=0}^{\infty} 5^r.x^r + \sum_{r=0}^{\infty} 2^r.x^r.$$

Equating x^n coefficient on both sides.

$$a_n = 7 5^n + 3.2^n.$$

(11) Find the coefficient of x^{32} in $(1+x^5+x^9)^{10}$.

Sol: x^{32} coefficient $\rightarrow (1+x^5+x^9)^{10}$. & $n_1+n_2+n_3=10$.

$\downarrow \; \downarrow \; \downarrow$

$x_1 \quad x_2 \quad x_3$.

Consider $(x^9)^3.(x^5)^1$ then, $n_2=1, n_3=3$.

then $n_1=10-1-3=6$.

$\therefore (1)^6$.

$\therefore (1)^6.(x^5)^1.(x^9)^3$ coefficient in $(1+x^5+x^9)^{10}$ is $\dfrac{n!}{n_1!.n_2!n_3!} = \dfrac{10!}{1!3!6!}$

(12) Solve the recurrence relation $a_n = 4a_{n-1} + 5a_{n-2}$, $n \geq 2$, $a_0 = 2$, $a_1 = 6$ using characteristic roots method.

Sol: Given RR: $a_n - 4a_{n-1} - 5a_{n-2} = 0$ for $n \geq 2 \rightarrow (1)$

Substitute $a_n = ck^n$ in eq (1).

$$ck^n - 4ck^{n-1} - 5ck^{n-2} = 0.$$

$$ck^{n-2}(k^2 - 4k - 5) = 0.$$

$k^2 - 4k - 5 = 0 \rightarrow (2)$ is characteristic equation of (1).

On solving (2), we get $k = -1, 5$ (real & distinct).

General solution of (1) is $a_n = c_1(-1)^n + c_2(5)^n \rightarrow (3)$.

Given, $a_0 = 2$.	$a_1 = 6$
$c_1(-1)^0 + c_2(5)^0 = 2$	$c_1(-1) + c_2(5) = 6$
$c_1 + c_2 = 2$.	$5c_2 - c_1 = 6$.

$$c_1 = \frac{4}{3}, \quad c_2 = \frac{2}{3}.$$

$\therefore a_n = (-1)^n\left(\frac{4}{3}\right) + (5)^n\left(\frac{2}{3}\right)$ is the particular solution.

(13) Find the coefficient of x^{20} in $(x^3 + x^4 + x^5 + \cdots)^5$.

Sol: Given that $(x^3 + x^4 + x^5 + \cdots)^5$.

$$= (x^3(1 + x + x^2 + \cdots))^5.$$

$$= x^{15}(1 + x + x^2 + \cdots)^5.$$

$$= x^{15}((1-x)^{-1})^5 = x^{15}(1-x)^{-5}.$$

$$= x^{15}\left(\frac{1}{(1-x)^5}\right) = x^{15}\sum_{r=0}^{\infty} {}^{4+r}C_r \, x^r.$$

for $r = 5$, we get x^{20} coefficient

$$= {}^9C_5 = C(9,5).$$

$\therefore x^{20}$ coefficient in $(x^3 + x^4 + x^5 + \cdots)^{15} = {}^9C_5$.