Branch and Bound

→ This technique is mostly used to solve optimization problems.

→ In branch and bound, a state space tree is constructed in such a way that all children of an E-node are generated before any other live node becomes an E-node.

→ Generated nodes that cannot possibly lead to a feasible solution are discarded. The remaining nodes are added to the list of live nodes, and then one node from this list is selected to become the next E-node. This expansion process continues until either the answer node is found or the list of live nodes becomes empty.

→ The next E-node can be selected in one of three ways:

1. FIFO (or) Breadth First Search: This scheme extracts nodes from the list of live nodes in the same order as they are placed in it. The live nodes list behaves as a queue.
2. LIFO (or) D Search: The live nodes list behaves as a stack.
3. LC Search (Least Cost Search) (or) Best First Search: The nodes are assigned ranks based on certain criteria and they are extracted in the order of Best-Rank-First.

→ Brach and Bound involves two iterative steps:

1. Branching:
   Splitting the problem into a number of subproblems.
   (or)
   Generating all the children of an E-node in the state space tree.

2. Bounding:
   Finding an ***optimistic estimate*** of the best solution to the subproblem.

Optimistic estimate: Upper bound for maximization problems.
Lower bound for minimization problems.

→ In case of LC Search Brach and Bound (LCBB) method, after generating the children of an E-node, the node with the best bound value (i.e., smallest lower bound in case of minimization problems or largest upper bound in case of maximization problems) is chosen from the list of all live nodes and is made the next E-node.

→ We terminate the search process at the current node of an LCBB algorithm because of any one of the following reasons:

1. The node represents an infeasible solution because constraints are not satisfied.

2. The bound value of the node is not better than the value of the best solution seen so far.

0/1 Knapsack problem:

Given *n* items with profits *(p1, p2, ..., pn)* and weights *(w1, w2, ..., wn)* and knapsack capacity *M*.

maximize

Subject to the constraints

$\leq M$

and

$\{0,1\}$, $1 \leq i \leq n$.

Solution to the 0/1 Knapsack problem using LCBB:

→ 0/1 knapsack problem is maximization problem.

How to find the optimistic estimate (i.e., upper bound)?

We relax the integral constraint, i.e., $\{0,1\}$, $1 \leq i \leq n$. That means, fractions of the items are allowed.

→ Arrange the items in the decreasing order of profit densities (*pi/wi* values).

→ In the state space tree, at every node we record three values, viz.,
W: Sum of the weights of the objects considered till now
P: Sum of the profits of the objects considered till now
UB: Upper bound on the optimal profit

→ The upper bound is computed as follows:
Upper Bound = Sum of the profits of the items provided the total weight is less than or equal to knapsack capacity considering the fractions of items.

Example:

$n=4$; $(w1, w2, w3, w4) = (4,7,5,3)$; $(p1, p2, p3, p4) = (40,42,25,12)$ and $M=10$.

Solution:

i	*1*	*2*	*3*	*4*
Pi	*40*	*42*	*25*	*12*
wi	*4*	*7*	*5*	*3*

pi/wi	*10*	*6*	*5*	*4*

The optimal solution is: $(x1, x2, x3, x4) = (1,0,1,0)$

Exercise:

$n=3$; $(w1, w2, w3) = (2,1,3)$; $(p1, p2, p3,p4) = (10,4,6)$ and $M=5$.

Traveling Salesman Problem using LCBB:

→ Let G = (V, E) be a directed graph defining an instance of the traveling salesman problem.

→ We assume that the tour starts at node *1* and ends at node *1*. The graph is represented by its cost adjacency matrix C, where *C[i, j]* denotes the cost of the edge <*i, j*>. If <*i, j*> *E* or if *i=j*, then *C[i, j]*= ∞.

→ In the state space tree, a lower bound on the optimal tour length is maintained at each node.

→ The *lower bound* can be obtained by using the *reduced cost matrix*.

Reduced cost matrix:

A matrix is said to be reduced iff all its rows and columns are reduced.

A row is said to be reduced iff it contains *at least one zero* and all remaining elements are non-negative. Similarly, A column is said to be reduced iff it contains *at least one zero* and all remaining elements are non-negative.

→ We associate a reduced cost matrix with every node in the state space tree using which we compute the lower bound at that node.

Obtaining the reduced cost matrix and lower bound at root node:

1. In each row of the given cost adjacency matrix, subtract the minimum value of that row from all the entries of that row.
2. In each column, subtract the minimum value of that column from all the entries of that column.
3. Lower Bound at root node = Total amount subtracted from all rows and columns.

Obtaining the reduced cost matrix and lower bound at other nodes in the spate space tree:

→ Let *A* be the reduced cost matrix for node *R* in the state space tree. Let *S* be a child of *R* such that the tree edge <*R, S*> corresponds to including edge <*i, j*> in the tour.

→ If *S* is not a leaf node, then the reduced cost matrix for *S* can be obtained as follows:

1. Change all entries in row *i* and column *j* of *A* to ∞ and set *A[j, 1]* to ∞.
2. Reduce all rows and columns in the matrix obtained in step-1 except for those rows and columns which contain only ∞.
3. If *r* is the total amount subtracted in step-2, then

LB(S)=LB(R)+ A[i, j]+r.

Example: Solve the following TSP instance using LCBB.

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}$$

Solution:

→ 1

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix} \begin{array}{c} \text{reduced by} \\ 10 \\ 2 \\ 2 \\ 3 \\ 4 \end{array} \xrightarrow[\text{reduction}]{\text{After row}} \begin{bmatrix} \infty & 10 & 20 & 0 & 1 \\ 13 & \infty & 14 & 2 & 0 \\ 1 & 3 & \infty & 0 & 2 \\ 16 & 3 & 15 & \infty & 0 \\ 12 & 0 & 3 & 12 & \infty \end{bmatrix}$$

reduce by → 1 — 3

↓ After column reduction

$r = 25$

$lb(1)$

So, $lb(1) = 10+2+2+3+4+1+3$

$= \underline{25}$

$$M_1 = \begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

Considering the path $1 \to 2$ at node 2 in state-space tree :

In the M_1 matrix, all entries in row 1 and column 2 and the entry at $M_1[2,1]$ should be made ∞.

$$M_1 = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ \infty & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{bmatrix} \Rightarrow M_2 .$$

is already reduced matrix. So total reduction $r = 0$

$$lb(2) = lb(1) + M_1[1,2] + r$$
$$= 25 + 10 + 0 = 35 .$$

Considering the path $1 \to 3$ at node 3 in state-space tree:

In the M_1 matrix, all entries in row 1 & column 3 and the entry at $M_1(3,1)$ should be made ∞

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 15 & 3 & \infty & \infty & 0 \\ 11 & 0 & \infty & 12 & \infty \end{bmatrix} \xrightarrow{\text{After column reduction}} \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 4 & 3 & \infty & \infty & 0 \\ 0 & 0 & \infty & 12 & \infty \end{bmatrix} = M_3 .$$

$$\begin{matrix} 11 & - & - & - & - \end{matrix}$$

$r = 11$.

$$lb[3] = lb[1] + M_1[1,3] + r$$
$$= 25 + 17 + 11 = 53$$

considering the path $1 \to 4$, at node 4 in state-space tree:

In M_1 matrix, all entries in row 1 & column 4 and the entry at $M_1(4,1)$ should be made ∞

$$M_1 = \begin{bmatrix} \infty & 10 & 13 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

$$= \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix} \to \underbrace{\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}}_{M_4}$$

$$r = 0$$

$$lb(4) = lb(1) + M_1[1,4] + r$$
$$= 25 + 0 + 0 = 25$$

considering the path $1 \to 5$ at node 5 in state space tree

In M_1 matrix, all entries in row 1 & column 5 and the entry at $M_1(5,1)$ should be made ∞

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & 2 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 15 & 3 & 12 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix} \begin{matrix} \\ 2 \\ \\ 3 \\ \\ \end{matrix} \to \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix} = M_5$$

$r = 2 + 3 = 5$

$$lb(5) = lb(1) + M_1[1,5] + r$$
$$= 25 + 5 + 1$$
$$= 31$$

$lb(2) = 35$, $lb(3) = 53$, $lb(4) = 25$ (min)

$lb(5) = 31$

so node 4 will be next E-node & its children nodes 6, 7, 8 will be generated.

Considering the path $1 \to 4 \to 2$ at node 6

In the matrix M_4, all entries in row 4 and col 2 & entry at $M_4(2,1)$ should be made ∞

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix} = M_6$$

so $lb(6) = lb(4) + M_4[4,2] + r$

$= 25 + 3 + 0 = 28$.

Considering the path $1 \to 4 \to 3$ at node 7.

In the matrix M_4, all entries in row 4 & column 3 & entry at $M_4(3,1)$ should be made ∞.

$$M_4 = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$

row 4 & 3 col

$$= \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & \infty & 0 \\ \infty & 3 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & 0 & \infty & \infty & \infty \end{bmatrix} \begin{array}{c} - \\ - \\ 2 \\ - \\ - \end{array} = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & \infty & \infty & \infty \end{bmatrix}$$

$r = 13$ $\qquad\qquad = M_7$

so $lb[7] = lb(4) + M_4[4,3] + r$

$= 25 + 12 + 13$

$= 50$.

→ considering the path 1→4→5 at node 1 .

In the matrix M_4, all entries in Row 4 & column 5 & entry $M(5,1)$ should be made ∞ .

$$= \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix} \begin{array}{c} - \\ \to 11 \\ - \\ - \\ - \end{array} \to \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix}$$

$r = 11$ $\qquad\qquad M_8$.

$lb(8) = lb(4) + M_4(4,5) + r$.

$= 25 + 0 + 11 = 36$

considering the path from $1 \to 4 \to 2 \to 3$, it made 9.

In the matrix M_6, all entries in row 2 & column 3 & entry at $M(3,1)$, should be made ∞.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & \infty & \infty & \infty \end{bmatrix} \begin{matrix} - \\ - \\ 2 \\ - \\ - \end{matrix}$$

$$\begin{matrix} 11 & - & - & - & - \end{matrix}$$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \end{bmatrix} = M_9.$$

$$r = 13$$

$$lb(9) = lb(6) + M_6(2,3) + r$$

$$= 28 + 11 + 13$$

$$= 52$$

→ considering the path from $1 \to 4 \to 2 \to 5$ it made 10.

In the matrix M_6, all entries in row 2 & col 5 & entry at $M_6(5,1)$ should be made ∞.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix} \to M_{10} \qquad r = 0$$

$$lb(10) = lb(6) + M(2,5) + r$$

$$= 28 + 0 + 0 = 28$$

→ Considering the path from $1 \to 4 \to 2 \to 5 \to 3$ at node 11.

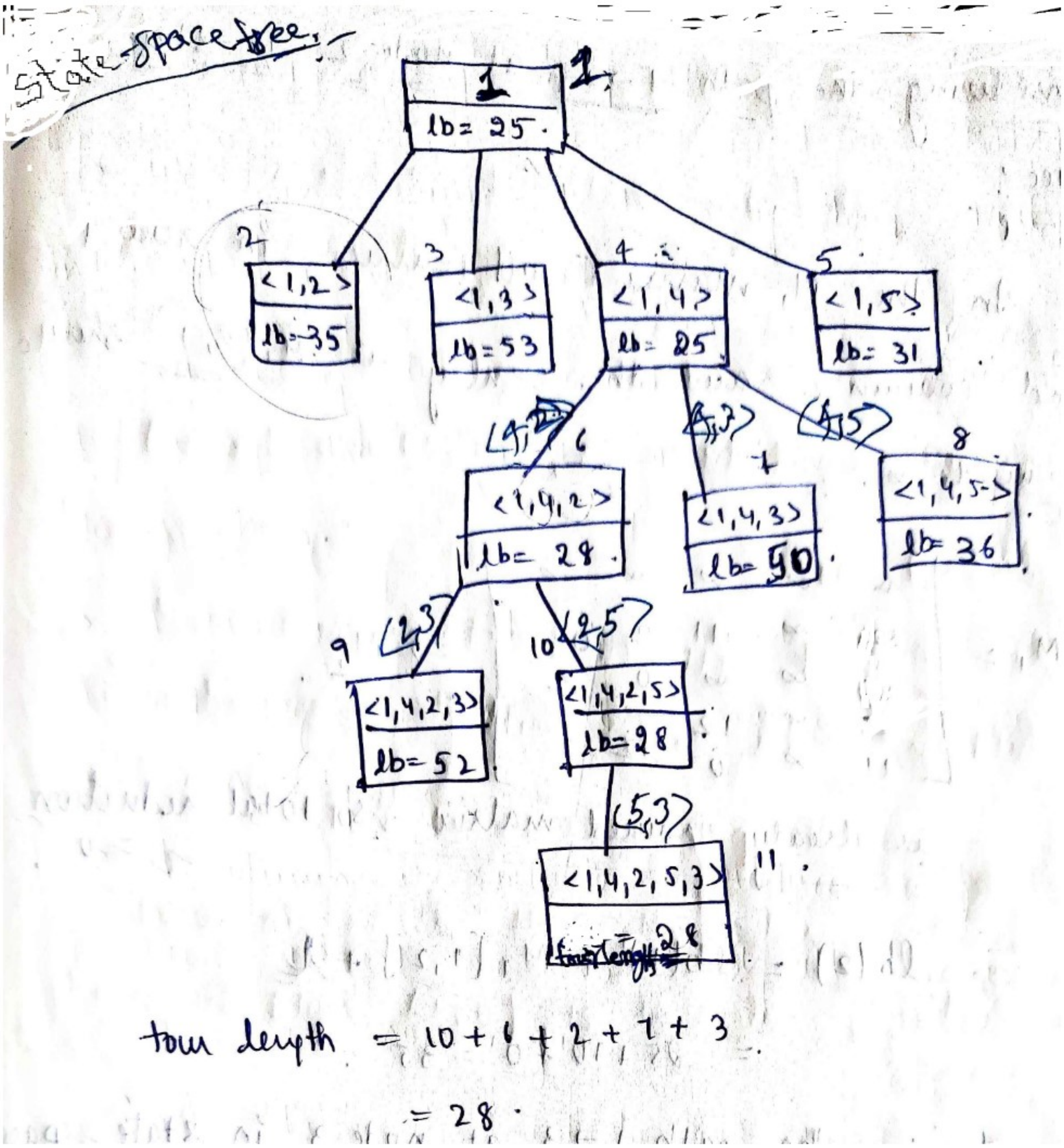
In the matrix M_{10}, all elements in row 5 & col 3 & at $M(3,1)$ is made ∞

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix} = M_{11}$$

$$r = 0$$

$$lb(11) = lb(10) + M(5,3) + r$$

$$= 28 + 0 + 0 = 28$$



The optimal tour is: 1 — 4 — 2 — 5 — 3 — 1

H|W.

$$\begin{bmatrix} \infty & 7 & 3 & 12 & 8 \\ 3 & \infty & 6 & 4 & 9 \\ 5 & 8 & \infty & 6 & 18 \\ 9 & 3 & 5 & \infty & 11 \\ 18 & 4 & 9 & 8 & \infty \end{bmatrix}$$