Singular Value Decomposition

let A be given matrix

$$[A]_{m\times n} = U_{(m\times r)} \Sigma_{(r\times r)} (V_{(n\times r)})^T$$

A : Input data matrix

where U : is left singular vectr ($m\times r$ matrix)

Σ : Singular values

$r\times r$ diagonal matrix

Singular values are in decreasing order

$$\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \quad T_1 > T_2$$

V : Right Singular vectors where r is 'rank'

$n\times r$ matrix

SVD. properties:

It is always possible to decompose a real matrix A into $A = U \Sigma V^T$ where

U, Σ, V : unique

U, V : Column orthonormal vectors (matrices)

$UU^T = VV^T = I$

Columns are orthogonal unit vectrs

Σ : diagonal : $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$

Entries of principal diagonal positions (singular values) are positive and they are stored in decreasing order ($\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$).

Find singular value decomposition (SVD) of a matrix

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \qquad A = U \Sigma V^T \qquad \boxed{v_i = \frac{1}{\sigma_i} A^T u_i}$$

$$(AA^T - I\lambda_1)x = 0$$

$$A^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \qquad A \cdot A^T = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$$

characteristic equation $|A \cdot A^T - I\lambda| = 0$

$$\begin{vmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{vmatrix} = 0 \qquad (16-\lambda)(34-\lambda) - 144 = 0$$

$$\lambda^2 - 50\lambda + 544 - 144 = 0$$

$$\lambda^2 - 50\lambda + 400 = 0$$

$$\lambda^2 - 40\lambda - 10\lambda + 400 = 0$$

$$\lambda(\lambda - 40) - 10(\lambda - 40) = 0$$

$$(\lambda - 10)(\lambda - 40)$$

The Eigen values of AA^T are 10 and 40

The Eigen vector for $\lambda = 40$ is

$$(AA^T - I\lambda)x = 0 \qquad \begin{pmatrix} 16-40 & 12 \\ 12 & 34-40 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -24 & 12 \\ 12 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_2 \to 2R_2 + 1R_1$$

$$= \begin{pmatrix} -24 & 12 \\ -24+24 & -12+12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -24 & 12 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= -24x_1 + 12x_2 = 0$$

$$= -12(2x_1 - x_2) = 0$$

$$2x_1 - x_2 = 0 \qquad \text{let } x_1 = K \text{ then}$$

$$x_2 = 2K$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ 2K \end{pmatrix} = K \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The eigen vector for $\lambda = 10$ is

$(AA^T - I\lambda)X = 0$

$$\begin{pmatrix} 16-10 & 12 \\ 12 & 34-10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 12 \\ 12 & 24 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\begin{pmatrix} 6 & 12 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$6x_1 + 12x_2 = 0 \quad 6(x_1 + 2x_2) = 0$

let $x_2 = k$ then

$x_1 = -2x_2 = -2k$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2k \\ k \end{pmatrix} = +k\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\therefore$ ~~For~~ eigen vector for $\lambda = 40$, is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \|u_1\| = \sqrt{1+4} = \sqrt{5}$

$$e_1 = \left(\frac{u_1}{\|u_1\|}\right) = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

for Eigen vector for $\lambda = 10$ is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$u_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \|u_2\| = \sqrt{(-2)^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}{\sqrt{5}} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

V is found using formula $v_i = \frac{1}{\sigma_i} A^T u_i$

$$v_1 = \frac{1}{\sigma_1} A^T e_1$$

$$= \frac{1}{\sqrt{40}} \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{40}} \begin{bmatrix} \frac{10}{\sqrt{5}} \\ \frac{-10}{\sqrt{5}} \end{bmatrix} = \begin{pmatrix} \frac{10}{\sqrt{200}} \\ \frac{-10}{\sqrt{200}} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$V_2 = \frac{1}{\sigma_2} A^T \cdot e_2$$

$$= \frac{1}{\sqrt{10}} \begin{pmatrix} 4 & 3 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} \frac{-8+3}{\sqrt{5}} \\ \frac{-5}{\sqrt{5}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{10}} \begin{pmatrix} -5/\sqrt{5} \\ -5/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore V = (v_1, v_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

verification

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{pmatrix} \sqrt{8} & -2\sqrt{8} \\ 2\sqrt{8} & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{4}+\sqrt{4} & 0 \\ \sqrt{16}-1 & -\sqrt{16}-1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix}$$

Singular Value Decomposition

let $\{v_1, v_2, v_3 \cdots v_n\}$ be the eigen vectors of $(A^TA)_{n\times n}$ and $\{u_1, u_2, \cdots u_m\}$ be the eigen vectors corresponding to $(AA^T)_{m\times m}$

let $\lambda_i = \sigma_i^2$ be the eigen values of A^TA

Then σ_i's are called as the singular values of A. and $\{v_1, v_2, \cdots v_n\}$ and $\{u_1, u_2 \cdots u_m\}$ are the singular vectors of A

$$A = U\Sigma V^T = (u_1\ u_2, \cdots u_m)_{m\times m} \begin{bmatrix} \sigma_1 & 0 & 0 \cdot & 0 & 0 \cdot 0 \\ 0 & \sigma_2 & 0 \cdots & 0 & \cdots \\ \vdots & & & & \\ 0 & \cdots & \cdots & \sigma_m & 0 \quad 0 \end{bmatrix}$$

$$[v_1, v_2, \cdots v_n]^T_{n\times n}$$

Find Singular value decomposition

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 5 & 13 \end{bmatrix}$$

Eigen values of $AA^T = |AA^T - \lambda I| = 0$

$$\begin{vmatrix} 13-\lambda & 5 \\ 5 & 13-\lambda \end{vmatrix} = 0$$

$$(13-\lambda)^2 - 25 = 0$$

$$169 + \lambda^2 - 26\lambda - 25 = 0$$

$$\lambda^2 - 26\lambda + 144 = 0$$

$$\lambda^2 - 18\lambda - 8\lambda + 144 = 0$$

$$(\lambda - 8)(\lambda - 18) = 0$$

$$\lambda = 8, \ \lambda = 18.$$

Now consider $A \cdot A^T$ for finding Eigen vector for $\boxed{\lambda_1 = 18}$ $|A \cdot A^T - I\lambda| = 0$

$$\left[\begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} - \begin{pmatrix} 18 & 0 \\ 0 & 18 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 + R_1 \sim \begin{pmatrix} -5 & 5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-5x_1 + 5x_2 = 0 \Rightarrow 5(-x_1 + x_2) = 0 \Rightarrow -x_1 + x_2 = 0$

let $x_2 = K$, then $x_1 = K$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ K \end{pmatrix} = K\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $K \in \mathbb{R} - \{0\}$

$\therefore e_1 = \dfrac{u_1}{\|u_1\|} = \dfrac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{1+1}} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

The Eigen vector for the Eigen value $\boxed{\lambda_2 = 8}$

$|A \cdot A^T - I\lambda| = 0$

$$\left[\begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1 \sim \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$5(x_1 + x_2) = 0$ $x_1 + x_2 = 0$

One Equation with 2 unknowns we get infinite numbers of solution let $x_2 = K$ then $\boxed{x_1 = -K}$

$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -K \\ K \end{pmatrix} = K\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $K \in \mathbb{R} - \{0\}$

$u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore$ $e_2 = \dfrac{u_2}{\|u_2\|} =$ $\|u_2\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$e_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

v is found using formula

$$v_i = \frac{1}{\sigma_i} A^T u_i$$

$$\boxed{\sigma_1 > \sigma_2}$$

$$v_1 = \frac{1}{\sigma_1} A^T e_1$$

$$= \frac{1}{\sqrt{18}} \begin{pmatrix} 3 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{18}} \begin{pmatrix} \frac{6}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{36}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_2 = \frac{1}{\sigma_2} A^T e_2$$

$$= \frac{1}{\sqrt{8}} \begin{pmatrix} 3 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{8}} \begin{pmatrix} 0 \\ -\frac{4}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-4}{\sqrt{16}} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\therefore \quad U = \begin{array}{c} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ \begin{matrix} e_1 & \quad e_2 \end{matrix} \end{array} \qquad \Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{8} \end{bmatrix}$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$V^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>Verification:</u> $U \Sigma V^T$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{18}}{\sqrt{2}} & -\frac{\sqrt{8}}{\sqrt{2}} \\ \frac{\sqrt{18}}{\sqrt{2}} & \frac{\sqrt{8}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{9}\sqrt{2}}{\sqrt{2}} & -\frac{\sqrt{4}\sqrt{2}}{\sqrt{2}} \\ \frac{\sqrt{9}\sqrt{2}}{\sqrt{2}} & \frac{\sqrt{4}\sqrt{2}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$$

Find Singular value decomposition (SVD) of a matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2}$$

$$A \cdot A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

- $A = U \Sigma V^T$

$\Sigma \rightarrow$ Singular values (Eigen values)

$\lambda_1 > \lambda_2$

U = Eigen vectors of AA^T. (left Singular values)

let them be (e_1, e_2)

V = (Right Singular values)

$$V_i = \frac{1}{\sigma_i} A^T \cdot e_1$$

$$\boxed{\begin{aligned} V_1 &= \frac{1}{\sigma_1} A^T e_1 \\ V_2 &= \frac{1}{\sigma_2} A^T e_2 \end{aligned}}$$

Step1: Singular values: Characteristic Equation is

$$|AA^T - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 3; \ \lambda_2 = 2$$

$$\therefore \ \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$\boxed{\begin{aligned} \sigma_1 &= \sqrt{3} \\ \sigma_2 &= \sqrt{2} \end{aligned}}$$

Step 2: Computation of left Singular Values

$|A \cdot A^T - I\lambda| = 0$ Eigen vector for $\lambda = 3$

$$\begin{pmatrix} 2-3 & 0 \\ 0 & 3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x_1 = 0 \Rightarrow \boxed{x_1 = 0}$ Here x_2 is missing variable $\therefore x_2 = K$

$\therefore u_1 = \begin{pmatrix} 0 \\ K \end{pmatrix} = K\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $K \in \mathbb{R} - \{0\}$

$e_1 = \frac{u_1}{\|u_1\|} = \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\sqrt{1}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eigen vector $\lambda = 2$

$$\begin{pmatrix} 2-2 & 0 \\ 0 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_2 = 0$ x_1 is missing variable $\therefore$ let $x_1 = K$

$\therefore u_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix} = K\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \|u_2\| = \sqrt{1+0} = \sqrt{1} = 1$

$\therefore U = (e_1, e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Step 3: $V_i = \frac{1}{\sigma_i} A^T e_i$

$$V_1 = \frac{1}{\sigma_1} A^T e_1 = \frac{1}{\sqrt{3}}\underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}}\underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$V_2 = \frac{1}{\sigma_2} A^T . e_2$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}_{3\times 2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2\times 1}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$$\therefore \; V = (v_1, v_2) = \begin{pmatrix} \frac{1}{\sqrt{3}} & 1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -1/\sqrt{2} \end{pmatrix}$$

$$\therefore \; A = U \; \Sigma \; V^T$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$