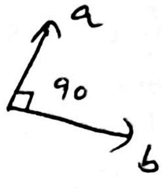
Orthogonal vectors:

The vectors $v_1, v_2, v_3 \ldots v_n$ are said to be orthogonal if the inner product of any two different vectors equals to zero.



ie $\langle v_i, v_j \rangle = 0 \quad \forall i \neq j$

For example: $\vec{a} = (5,4)$. and $\vec{b} = (8,-10)$

$a \cdot b^T = (5,4) \cdot \begin{pmatrix} 8 \\ -10 \end{pmatrix} = 40-40=0$

The vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

Ex: (i) Check whether the vectors $a=(2,3,1)$ and $b=(3,1,-9)$ are orthogonal or not

Consider the dot product $a \cdot b^T = (2,3,1) \begin{pmatrix} 3 \\ 1 \\ -9 \end{pmatrix}$

$= 6+3-9=0$

As the dot product is zero. hence these 2 vectors in three dimensional plane are orthogonal in nature.

(ii) Check whether the 2 vectors $a=(2,4,1)$ and $b=(2,1,-8)$ are orthogonal

We will calculate dot product of $a \cdot b^T$

$(2,4,1) \cdot \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix} = 4+4-8=0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are orthogonal in a three dimensional plane.

Vector Norm:

The length of a vector is referred to as the vector norm' or vector's magnitude.

The length of vector is always a positive number except for a vector of all zero values.

Example: find the norm of the vector $\vec{u} = (2, -2, 3)$

Since $u \in \mathbb{R}^3$, we use the formula $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

$$= \sqrt{(2)^2 + (-2)^2 + (3)^2} = \sqrt{4+4+9} = \sqrt{17}.$$

Orthonormal vectors The vectors $u_1, u_2, \ldots u_n$ are said to be orthonormal if

(i) They are orthogonal vectors

ii) They are unit vectors

Mathematically: $\langle u_i, u_j \rangle = 0 \quad \forall i \neq j$

$\|u_i\| = 1 \quad \forall i = 1, 2, 3, \ldots n$

Basis: The set of vectors $S = \{v_1, v_2, \ldots v_n\} \subseteq V$ in a vector space V is called a basis for V.

if S spans V (ie $\text{Span}(S) = V$)

S is linearly independent

$\therefore$ S is called a basis for V

The standard basis

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

let $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Vector Space: Basis

Example: Consider the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

is this set of vectors a "basis" for $\mathbb{R}^3$

Since $\mathbb{R}^3$, need 3 linearly independent vectors

we check the 3 vectors are linearly independent or not

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 1 & 4 & 1 \end{pmatrix} \quad R_3 \to R_3 - R_1 \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 6 & 1 \end{pmatrix}$$

$$R_3 \to R_3 - 6R_2 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2 \to R_2 - 3R_3 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_3 \to R_3 + 2R_1 \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3 pivots
3 linearly independent vectors exists

other way to check linearly independent vectors:

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 1(1-12) + 2(0-3)$$

$$= -11 + (-6) = -17 \neq 0 \Rightarrow \text{linearly independe-}$$

nt columns exists

yes. Set forms a basis for "$\mathbb{R}^3$"

Example of orthonormal vectors

Find the orthonormal vectors of $\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$

Sol: The set of vectors $v_1=\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $v_2=\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}$, $v_3=\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$ are mutually orthogonal.

$$(1,0,-1)\cdot(1,\sqrt{2},1)^T=(1,0,-1)\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}=1-1=0$$

$$(1\ \sqrt{2},\ 1)\cdot(1,-\sqrt{2},1)^T=(1,\sqrt{2},1)\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}=1-2+1=0$$

$$(1\ -\sqrt{2},1)\cdot(1,0,-1)^T=(1,-\sqrt{2},1)\begin{pmatrix}1\\0\\-1\end{pmatrix}=1-1=0$$

Let $\vec{u_1}=\dfrac{v_1}{\|v_1\|} \Rightarrow \|v_1\|=\sqrt{1^2+0+(-1)^2}=\sqrt{2}$

$$\therefore \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$$

$u_2=\dfrac{v_2}{\|v_2\|} \Rightarrow \|v_2\|=\sqrt{1+2+1}=\sqrt{4}=2$

$$\therefore \frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}$$

$u_3=\dfrac{v_3}{\|v_3\|} \Rightarrow \|v_3\|=\sqrt{1^2+(-\sqrt{2})^2+1^2}=\sqrt{1+2+1}=\sqrt{4}=2$

$$u_3=\frac{v_3}{|v_3|}=\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}=\begin{pmatrix}1/2\\-\sqrt{2}/2\\1/2\end{pmatrix}$$

The set of vectors $\{\vec{u_1},\vec{u_2}\ u_3\}$ is orthogonal.

An orthogonal set of non zero. vectors is linearly independent.

Gram Schmidt process

Given a set of linearly independent vectors.
it is often useful to convert them into an orthonormal set of vectors.

We first define the projection operator:

Def: Let $\vec{u}$ and $\vec{v}$ be two vectors. The projection of vector $\vec{v}$ on $\vec{u}$ is defined as follows:

$$\text{proj}_{\vec{u}} v = \frac{(\vec{u}\cdot\vec{v})}{\|u\|^2} u$$

Ex: Consider the two vectors $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

These two vectors are linearly independent.

The vectors are not orthogonal to each other.

We create an orthogonal vector in the following manner

$$\vec{v_1} = \vec{v} - (\text{proj}_{\vec{u}} v)$$

$$\text{proj}_u v = \frac{(1,1)\cdot(1,0)}{\sqrt{1^2+0^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{1}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\therefore$ v_1, constructed is orthogonal to $\vec{u}$

The Gram-Schmidt Algorithm

Let $v_1, v_2, \dots v_n$ be a set of n linearly independent vectors in $\mathbb{R}^n$. Then we can construct an orthonormal set of vectors as follows

Step (1) let $\vec{u_1} = \vec{v_1}$; $\vec{e_1} = \frac{\vec{u_1}}{\|\vec{u_1}\|}$

Step (2) let $\vec{u_2} = \vec{v_2} - \text{proj}_{u_1} \vec{v_2}$; $\vec{e_2} = \frac{\vec{u_2}}{\|\vec{u_2}\|}$

Step 3. let $\vec{u_3} = \vec{v_3} - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3$; $\vec{e_3} = \frac{\vec{u_3}}{\|\vec{u_3}\|}$

Step 4: let $\vec{u_4} = \vec{v_4} - \text{proj}_{u_1} v_4 - \text{proj}_{u_2} \vec{v_4} - \text{proj}_{u_3} v_4$; $\vec{e_4} = \frac{\vec{u_4}}{\|\vec{u_4}\|}$

$\vdots$

Ex: Apply the Gram-Schmidt algorithm to orthonormalize the set of vectors

$$V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

To apply the Gram-Schmidt, we first need to check that the set of vectors are linearly independent

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(0-1) - 1(-2-1) + 1(-1)$$
$$= -1 + 3 - 1 = 1 \neq 0$$

Therefore the vectors are linearly independent

Gram-Schmidt algorithm

Step 1: let $u_1 = v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\qquad \boxed{u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$

$\vec{e_1} = \frac{\vec{u_1}}{\|u_1\|} \Rightarrow \|u_1\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$$e_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Step 2: $\vec{u_2} = v_2 - \text{proj}_{u_1} \vec{v_2}$

$$\text{proj}_{\vec{u_1}} u_2 = \frac{(1,0,1)\cdot(1,-1,1)}{(1^2+(-1)^2+1)} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{1+0+1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$u_2 = v_2 - \text{proj}_{\vec{u_1}} v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{u_2} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}}$$

$$e_2 = \frac{u_2}{\|u_2\|} \Rightarrow \|u_2\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$$

$$\therefore \vec{e_2} = \frac{3}{\sqrt{6}} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

Step 3: $\vec{u_3} = v_3 - \text{proj}_{\vec{u_1}} v_3 - \text{proj}_{\vec{u_2}} v_3$

$$\text{proj}_{u_1} v_3 = \frac{(1,1,2)\cdot(1,-1,1)}{(1^2+(-1)^2+1^2)} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{proj}_{u_2} v_3 = \frac{(1,1,2)\cdot(1/3, 2/3, 1/3)}{\left[(1/3)^2 + (2/3)^2 + (1/3)^2\right]} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{\left(\frac{1}{3} + \frac{2}{3} + \frac{2}{3}\right)}{\left(\frac{1+4+1}{9}\right)} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{5}{3} \cdot \frac{9}{6} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$= \frac{5}{2} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{5}{2}\begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix} - \begin{pmatrix} 5/6 \\ 5/3 \\ 5/6 \end{pmatrix}$$

$$\boxed{u_3 = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}}$$

$$e_3 = \frac{\vec{u_3}}{\|u_3\|} \Rightarrow \|u_3\| = \sqrt{\frac{1}{4} + 0\,\frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}}$$

$$\Rightarrow e_3 = \sqrt{2}\begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

2. Apply Gram Schmidt algorithm to orthonormalize the set of vectors $(1,1,-1)$ $(-1,1,0)$, $(1,0,1)$

Given that $v_1 = (1,1,-1)$, $v_2 = (-1,1,0)$, $v_3 = (1,0,1)$

first we set $u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \boxed{u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}$

Step 1

$\vec{e_1} = \dfrac{\vec{u_1}}{\|\vec{u_1}\|}$ $\quad \|\vec{u_1}\| = \sqrt{1+1+1} = \sqrt{3}$

$\vec{e_1} = \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$

Step 2: $\vec{u_2} = v_2 - \text{proj}_{u_1} \vec{v_2}$

$\text{proj}_{u_1} v_2 = \dfrac{(v_2 \cdot u_1)}{\|u_1\|^2} u_1$

$\vec{u_2} = (-1,1,0) - \dfrac{((-1,1,0)\cdot(1,1,-1))}{(\sqrt{3})^2} (1,1,-1)$

$= (-1,1,0) - \dfrac{(-1+1+0)}{3} (1,1,-1)$

$= (-1,1,0) - \dfrac{0}{3} (1,1,-1)$

$= (-1,1,0) - 0 = (-1,1,0)$

$e_2 = \dfrac{u_2}{\|u_2\|} \Rightarrow \|u_2\| = \sqrt{(-1)^2+(1)^2+0} = \sqrt{1+1} = \sqrt{2}$

$\therefore e_2 = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

Step 3: $\vec{u_3} = v_3 - \text{proj}_{u_1} \vec{v_3} - \text{proj}_{u_2} v_3$

$= (1,0,1) - \dfrac{(v_3 \cdot u_1)}{\|u_1\|^2} \cdot \vec{u_1} - \dfrac{(v_3 \cdot \vec{u_2})}{\|u_2\|^2} \vec{u_2}$

$$= (1,0,1) - \left(\frac{(1,0,1)\cdot(1,1,-1)}{\|(1,1,-1)\|^2}\right)(1,1,-1) - \frac{\langle(1,0,1)\cdot(-1,1,0)\rangle}{\|(-1,1,0)\|^2}(-1,1,0)$$

$$= (1,0,1) - \frac{(1+0-1)}{\sqrt{3}}(1,1,-1) - \frac{(-1+0+0)}{(\sqrt{1+1})^2}(-1,1,0)$$

$$= (1,0,1) - 0 + \frac{+1}{2}(-1,1,0)$$

$$= (1,0,1) + \frac{1}{2}(-1,1,0)$$

$$= \left(1-\frac{1}{2},\ 0+\frac{1}{2},\ 1+0\right) = \left(\frac{1}{2}, +\frac{1}{2}, 1\right)$$

$$v_3 = \left(\frac{1}{2}, \frac{1}{2}, 1\right) = \frac{1}{2}(1,1,2)$$

let us take $v_3 = (1,1,2)$ $\quad \|v_3\| = \sqrt{1+1+4} = \sqrt{6}$

$$\therefore e_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}}(1,1,2) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$\therefore$ The orthonormal basis is $\left\{\frac{(1,1,-1)}{\sqrt{3}}, \frac{(-1,1,0)}{\sqrt{2}}, \frac{(1,1,2)}{\sqrt{6}}\right\}$

3. Apply Gram Schmidt algorithm to orthonormalize the set of vectors $\beta_1 = (1,2,1)$, $\beta_2 = (1,4,3)$, $\beta_3 = (3,1,1)$

Sol: Given that $v_1 = (1,2,1)$, $v_2 = (1,4,3)$, $v_3 = (3,1,1)$

First we set $u_1 = v_1 = (1,2,1)$

$$q = \frac{u_1}{\|u_1\|} \qquad \|u_1\| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$q = \frac{1}{\sqrt{6}}(1,2,1)$$

Step 2: $\vec{u_2} = v_2 - \text{proj}_{u_1} \vec{v_2}$

$$\text{proj}_{u_1} v_2 = \frac{(v_2 \cdot u_1)}{\|u_1\|^2} u_1$$

$$\therefore u_2 = (1,4,3) - \left\langle \frac{(1,4,3)\cdot(1,2,1)}{(\sqrt{6})^2} \right\rangle (1,2,1)$$

$$= (1,4,3) - \frac{12}{6}(1,2,1)$$

$$= (1,4,3) - 2(1,2,1) = (1,4,3) - (2,4,2)$$

$$= (-1,0,1)$$

$$e_2 = \frac{\vec{u_2}}{\|u_2\|} = \frac{(-1,0,1)}{\sqrt{1+1}} = \frac{(-1,0,1)}{\sqrt{2}}$$

Step 3: $\vec{u_3} = v_3 - \text{proj}_{u_1} \vec{v_3} - \text{proj}_{u_2} \vec{v_3}$

$$= v_3 - \frac{(v_3 \cdot u_1)}{\|u_1\|^2} \cdot u_1 - \frac{(v_3 \cdot u_2)}{\|u_2\|^2} u_2$$

$$= (3\ 1\ 1) - \left\langle \frac{(3\ 1\ 1)\cdot(1,2,1)}{\|(1\ 2\ 1)\|^2} \right\rangle (1\ 2\ 1) - \left\langle \frac{(3\ 1\ 1)\ (-1,0,1)}{\|(-1\ 0\ 1)\|^2} \right\rangle (-1\ 0\ 1)$$

$$= (3\ 1\ 1) - \frac{(3+2+1)}{6}(1\ 2\ 1) - \frac{(-3+0+1)}{2}(-1\ 0\ 1)$$

$$= (3\ 1\ 1) - 1(1,2,1) - (-1)(-1\ 0\ 1)$$

$$= (3\ 1\ 1) - (1,2,1) + (-1,0,1)$$

$$u_3 = (1,-1,1)$$

$$e_3 = \frac{u_3}{\|u_3\|} \Rightarrow \|u_3\| = \sqrt{1+1+1} = \sqrt{3}$$

$$e_3 = \frac{(1,-1,1)}{\sqrt{3}}$$

$\therefore$ the following vectors form an orthonormal basis for the given vectors $\left\{ \frac{(1,2,1)}{\sqrt{6}}, \frac{(-1,0,1)}{\sqrt{2}}, \frac{(1,-1,1)}{\sqrt{3}} \right\}$