Detailed understanding of differentials

by Sven Nilsen

When learning the differential of functions, it is common to assume that the function must be differentiable at a point, or you use a limit notation that tells which side you approach it. In numerical analysis this can be ambiguous because the meaning of which side you approach something is derived from multiple expressions. This article explains the connection between limits and the use of a small number to find the differential for an arbitrary function.

Assume that we analyze a function that is continuous, but not differential. It has a well defined value at the point and very close to it. What we find is that approaching the point from one side give a different answer than approaching it from the other side:

$$\lim_{\varepsilon \rightarrow +0}\left(\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right) \neq \lim_{\varepsilon \rightarrow -0}\left(\frac{f(x)-f(x-\varepsilon)}{\varepsilon}\right)$$

Formula 1.1

One can think of the function as two sub functions glued together. At the non-differential point i has two differentials, one for each direction:

$$f'_{+0}(x) \neq f'_{-0}(x)$$

Formula 1.2

So, a continuous function does not have to be strictly non-differential, but can have multiple ones. How we deal with this depends on the tools we have to approach a point from different sides.

The surprising thing about this is that a function can have more than one differential at a point irrespective of the choice of how close you approach it. By definition, of course. One could define that the differential smoothes the transition for less values than a certain limit. This would give the function a single differential.

When deriving the formula of double differential for a function, it is more rigorous to include the notation of which side we are working with:

$$f'_{+0}(x) = \lim_{\varepsilon \rightarrow +0}\left(\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right)$$

Formula 1.3

We introduce a new infinitesimal, labeling the first 1 and the new 2:

$$f'_{+0}(x+\varepsilon_2) = \lim_{\varepsilon_1 \rightarrow +0}\left(\frac{f(x+\varepsilon_1+\varepsilon_2)-f(x+\varepsilon_2)}{\varepsilon_1}\right)$$

Formula 1.4

This is done by substitution:

$$x \rightarrow x+\varepsilon_2$$
$$\varepsilon \rightarrow \varepsilon_1$$

Then we use Formula 1.3 again, substituting twice:

$$f''_{+0}(x)=\lim_{\varepsilon_2\to+0}\left(\frac{f'(x+\varepsilon_2)-f'(x)}{\varepsilon_2}\right)$$

Formula 1.5

$$f\to f'$$
$$\varepsilon\to\varepsilon_2$$

To insert Formula 1.4 in 1.5, we have to make the following assumption:

$$\lim_{\varepsilon_2\to+0}(f'(x+\varepsilon_2)=f'_{+0}(x+\varepsilon_2))$$

Formula 1.6

This is true because we are approaching it from the same side as we are approaching the double differential.

This is a problem for:

$$\lim_{\varepsilon_2\to+0}(f'(x)=f'_{+0}(x))$$

Formula 1.7

Because it does not bind to the limit variable.

However, let us pretend this work for a moment and insert labels in Formula 1.3:

$$f'_{+0}(x)=\lim_{\varepsilon_1\to+0}\left(\frac{f(x+\varepsilon_1)-f(x)}{\varepsilon_1}\right)$$

Formula 1.8

We can expand the right side in Formula 1.7 with Formula 1.8:

$$\lim_{\varepsilon_2\to+0}\left(f'(x)=\lim_{\varepsilon_1\to+0}\left(\frac{f(x+\varepsilon_1)-f(x)}{\varepsilon_1}\right)\right)$$

Now we see this work because the two limits are approaching the point from the same side.

Let us use what we have and insert into Formula 1.5:

$$f''_{+0}(x)=\lim_{\varepsilon_2\to+0}\left(\frac{\lim_{\varepsilon_1\to+0}\left(\frac{f(x+\varepsilon_1+\varepsilon_2)-f(x+\varepsilon_2)}{\varepsilon_1}\right)-\lim_{\varepsilon_1\to+0}\left(\frac{f(x+\varepsilon_1)-f(x)}{\varepsilon_1}\right)}{\varepsilon_2}\right)$$

Formula 1.9

Now we set the two infinitesimals equal to each other and simplify by removing the inner limits:

$$f''_{+0}(x)=\lim_{\varepsilon\to+0}\left(\frac{f(x+2\varepsilon)-2f(x+\varepsilon)+f(x)}{\varepsilon^2}\right)$$

Formula 1.10

This formula can be used to find the differential at one side without putting the restriction on the point to have only one differential.

When there is only one differential, we can “slide” the formula to take a value on both sides:

$$f''_{\pm0}(x)=\lim_{\varepsilon\to\pm0}\left(\frac{f(x+\varepsilon)-2f(x)+f(x-\varepsilon)}{\varepsilon^2}\right)$$

Formula 1.11

This is done by substitution:

$$x\to x-\varepsilon$$

When applying Formula 1.11 in numerical analysis, it will “smooth” out the transition of the differential.

When applying Formula 1.10, it will give different answers depending on which side you approach the point.

These are assumptions we make when doing numerical simulations.

THE END