Hand Calculations for 2-2-2 Neural Network

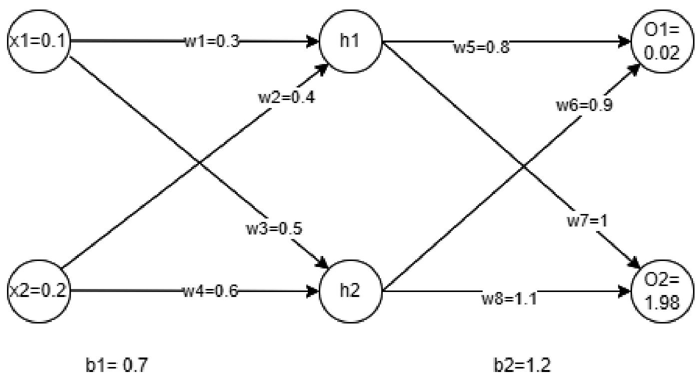


Figure 1: 2-2-2 ann

Given:

- Inputs: $x_1 = 0.1$, $x_2 = 0.2$
- Outputs: $o_1 = 0.02$, $o_2 = 1.98$
- Weights:
  - $w_1 = 0.3$
  - $w_2 = 0.4$
  - $w_3 = 0.5$
  - $w_4 = 0.6$
  - $w_5 = 0.8$
  - $w_6 = 0.9$
  - $w_7 = 1$
  - $w_8 = 1.1$
- Biases:
  - $b_1 = 0.7$ (for hidden layer)
  - $b_2 = 1.2$ (for output layer)
- Activation function: Sigmoid, $f(z) = \frac{1}{1+e^{-z}}$

Step 1: Calculate Hidden Layer Activations

Calculating h_1^{in} and h_2^{in}:

$$h_1^{\text{in}} = w_1 x_1 + w_2 x_2 + b_1$$

Substitute values:

$$h_1^{\text{in}} = (0.3)(0.1) + (0.4)(0.2) + 0.7 = 0.03 + 0.08 + 0.7 = 0.81$$

$$h_2^{\text{in}} = w_3 x_1 + w_4 x_2 + b_1$$

Substitute values:

$$h_2^{\text{in}} = (0.5)(0.1) + (0.6)(0.2) + 0.7 = 0.05 + 0.12 + 0.7 = 0.87$$

Apply Sigmoid Function to Get h_1^{out} and h_2^{out}:

$$f(z) = \frac{1}{1 + e^{-z}}$$

$$h_1^{\text{out}} = f(h_1^{\text{in}}) = \frac{1}{1 + e^{-0.81}} \quad \text{and} \quad h_2^{\text{out}} = f(h_2^{\text{in}}) = \frac{1}{1 + e^{-0.87}}$$

Compute Values:

$$h_1^{\text{out}} \approx 0.692 \quad \text{and} \quad h_2^{\text{out}} \approx 0.705$$

Step 2: Calculate Output Layer Activations

Calculating o_1^{in} and o_2^{in}:

$$o_1^{\text{in}} = w_5 h_1^{\text{out}} + w_6 h_2^{\text{out}} + b_2$$

Substitute values:

$$o_1^{\text{in}} = (0.8)(0.692) + (0.9)(0.705) + 1.2 = 0.5536 + 0.6345 + 1.2 = 2.3881$$

$$o_2^{\text{in}} = w_7 h_1^{\text{out}} + w_8 h_2^{\text{out}} + b_2$$

Substitute values:

$$o_2^{\text{in}} = (1)(0.692) + (1.1)(0.705) + 1.2 = 0.692 + 0.7755 + 1.2 = 2.6675$$

Apply Sigmoid Function to Get o_1^{out} and o_2^{out}:

$$o_1^{\text{out}} = f(o_1^{\text{in}}) = \frac{1}{1 + e^{-2.3881}} \quad \text{and} \quad o_2^{\text{out}} = f(o_2^{\text{in}}) = \frac{1}{1 + e^{-2.6675}}$$

Compute Values:

$$o_1^{\text{out}} \approx 0.916 \quad \text{and} \quad o_2^{\text{out}} \approx 0.935$$

Step 3: Calculate Total Error Using Mean Squared Error (MSE)

The formula for MSE:

$$E = \frac{1}{n}\sum_{i=1}^{n}(o_i^{\text{out}} - o_i)^2$$

Substitute values:

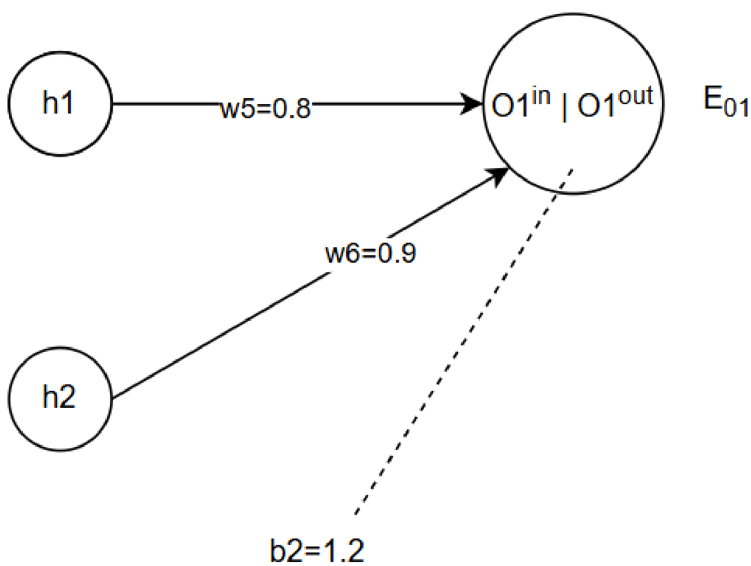
$$E = \frac{1}{2}\left[(0.916 - 0.02)^2 + (0.935 - 1.98)^2\right]$$

Compute:

$$E = \frac{1}{2}\left[(0.896)^2 + (-1.045)^2\right] = \frac{1}{2}\left[0.802816 + 1.092025\right] = \frac{1}{2} \cdot 1.894841 = 0.947$$

Step 4: Backpropagation to Update Weights

Gradients for w_5 and w_6



Gradient Formula for w_5:

$$\frac{\partial E}{\partial w_5} = \frac{\partial E}{\partial o_1^{\text{out}}} \cdot \frac{\partial o_1^{\text{out}}}{\partial o_1^{\text{in}}} \cdot \frac{\partial o_1^{\text{in}}}{\partial w_5}$$

1. Compute each term:

$$\frac{\partial E}{\partial o_1^{\text{out}}} = o_1^{\text{out}} - o_1 = 0.916 - 0.02 = 0.896$$

$$\frac{\partial o_1^{\text{out}}}{\partial o_1^{\text{in}}} = o_1^{\text{out}}(1 - o_1^{\text{out}}) = 0.916 \cdot 0.084 = 0.0769$$

$$\frac{\partial o_1^{\text{in}}}{\partial w_5} = h_1^{\text{out}} = 0.692$$

2. Combine terms:

$$\frac{\partial E}{\partial w_5} = 0.896 \cdot 0.0769 \cdot 0.692 \approx 0.0477$$

Gradient Formula for w_6:

$$\frac{\partial E}{\partial w_6} = \frac{\partial E}{\partial o_1^{\text{out}}} \cdot \frac{\partial o_1^{\text{out}}}{\partial o_1^{\text{in}}} \cdot \frac{\partial o_1^{\text{in}}}{\partial w_6}$$

$$\frac{\partial o_1^{\text{in}}}{\partial w_6} = h_2^{\text{out}} = 0.705$$

$$\frac{\partial E}{\partial w_6} = 0.896 \cdot 0.0769 \cdot 0.705 \approx 0.0485$$

Gradients for w_7 and w_8

Gradient Formula for w_7:

$$\frac{\partial E}{\partial w_7} = \frac{\partial E}{\partial o_2^{\text{out}}} \cdot \frac{\partial o_2^{\text{out}}}{\partial o_2^{\text{in}}} \cdot \frac{\partial o_2^{\text{in}}}{\partial w_7}$$

$$\frac{\partial E}{\partial w_7} = \frac{\partial E}{\partial o_2^{\text{out}}} \cdot \frac{\partial o_2^{\text{out}}}{\partial o_2^{\text{in}}} \cdot \frac{\partial o_2^{\text{in}}}{\partial w_7}$$

1. Compute each term:

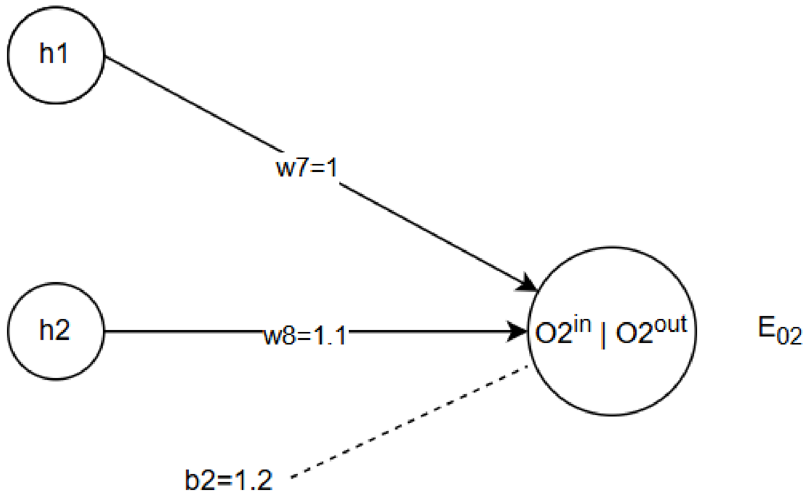
$$\frac{\partial E}{\partial o_2^{\text{out}}} = o_2^{\text{out}} - o_2 = 0.935 - 1.98 = -1.045$$

$$\frac{\partial o_2^{\text{out}}}{\partial o_2^{\text{in}}} = o_2^{\text{out}}(1 - o_2^{\text{out}}) = 0.935 \cdot 0.065 = 0.0608$$

$$\frac{\partial o_2^{\text{in}}}{\partial w_7} = h_1^{\text{out}} = 0.692$$

2. Combine terms:

$$\frac{\partial E}{\partial w_7} = -1.045 \cdot 0.0608 \cdot 0.692 \approx -0.044$$



Gradient Formula for w_8:

$$\frac{\partial o_2^{\text{in}}}{\partial w_8} = h_2^{\text{out}} = 0.705$$

$$\frac{\partial E}{\partial w_8} = -1.045 \cdot 0.0608 \cdot 0.705 \approx -0.0448$$

Updated Weights

Using the weight update rule:

$$w_{\text{new}} = w_{\text{old}} - \eta \cdot \frac{\partial E}{\partial w}$$

$$w_5^{\text{new}} = 0.8 - 0.1 \cdot 0.0477 \approx 0.795$$
$$w_6^{\text{new}} = 0.9 - 0.1 \cdot 0.0485 \approx 0.895$$
$$w_7^{\text{new}} = 1 - 0.1 \cdot (-0.044) \approx 1.004$$
$$w_8^{\text{new}} = 1.1 - 0.1 \cdot (-0.0448) \approx 1.105$$

Step 4: Calculate Gradients for w_1, w_2, w_3, w_4 (linked to hidden layer)

Gradients for weights in the hidden layer depend on the total error contributions from both outputs (o_1^{out} and o_2^{out}).

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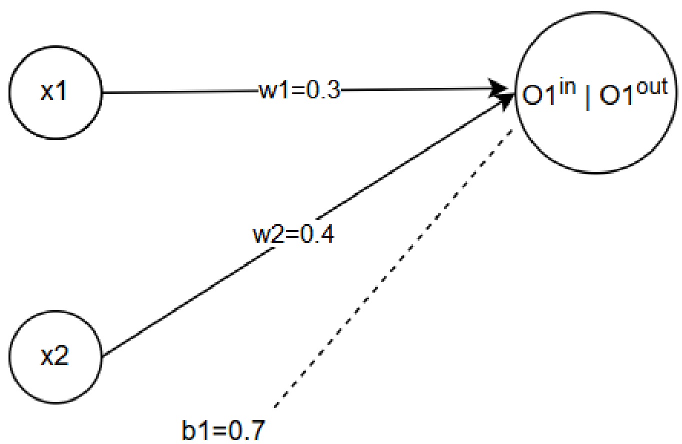
Gradient for w_1 and w_2:



Gradient Formula for w_1:

$$\frac{\partial E}{\partial w_1} = \frac{\partial E}{\partial h_1^{\text{out}}} \cdot \frac{\partial h_1^{\text{out}}}{\partial h_1^{\text{in}}} \cdot \frac{\partial h_1^{\text{in}}}{\partial w_1}$$

Step-by-Step: 1. $\frac{\partial E}{\partial h_1^{\text{out}}}$ is the sum of contributions from both outputs:

$$\frac{\partial E}{\partial h_1^{\text{out}}} = \frac{\partial E_1}{\partial h_1^{\text{out}}} + \frac{\partial E_2}{\partial h_1^{\text{out}}}$$

First Output Contribution (o_1^{out}):

$$\frac{\partial E_1}{\partial h_1^{\text{out}}} = \frac{\partial E_1}{\partial o_1^{\text{out}}} \cdot \frac{\partial o_1^{\text{out}}}{\partial o_1^{\text{in}}} \cdot \frac{\partial o_1^{\text{in}}}{\partial h_1^{\text{out}}}$$

- $\frac{\partial E_1}{\partial o_1^{\text{out}}} = o_1^{\text{out}} - o_1 = 0.916 - 0.02 = 0.896$ - $\frac{\partial o_1^{\text{out}}}{\partial o_1^{\text{in}}} = o_1^{\text{out}}(1 - o_1^{\text{out}}) = 0.916(1 - 0.916) = 0.0769$ - $\frac{\partial o_1^{\text{in}}}{\partial h_1^{\text{out}}} = w_5 = 0.8$

$$\frac{\partial E_1}{\partial h_1^{\text{out}}} = 0.896 \cdot 0.0769 \cdot 0.8 \approx 0.0552$$

Second Output Contribution (o_2^{out}):

$$\frac{\partial E_2}{\partial h_1^{\text{out}}} = \frac{\partial E_2}{\partial o_2^{\text{out}}} \cdot \frac{\partial o_2^{\text{out}}}{\partial o_2^{\text{in}}} \cdot \frac{\partial o_2^{\text{in}}}{\partial h_1^{\text{out}}}$$

- $\frac{\partial E_2}{\partial o_2^{\text{out}}} = o_2^{\text{out}} - o_2 = 0.935 - 1.98 = -1.045$ - $\frac{\partial o_2^{\text{out}}}{\partial o_2^{\text{in}}} = o_2^{\text{out}}(1 - o_2^{\text{out}}) = 0.935(1 - 0.935) = 0.0608$ - $\frac{\partial o_2^{\text{in}}}{\partial h_1^{\text{out}}} = w_7 = 1.0$

$$\frac{\partial E_2}{\partial h_1^{\text{out}}} = -1.045 \cdot 0.0608 \cdot 1.0 \approx -0.0636$$

Total Gradient for h_1^{out}:

$$\frac{\partial E}{\partial h_1^{\text{out}}} = 0.0552 - 0.0636 = -0.0084$$

2.
$$\frac{\partial h_1^{\text{out}}}{\partial h_1^{\text{in}}} = h_1^{\text{out}}(1 - h_1^{\text{out}}) = 0.692(1 - 0.692) = 0.692 \cdot 0.308 = 0.213$$

3.
$$\frac{\partial h_1^{\text{in}}}{\partial w_1} = x_1 = 0.1$$

4. Multiply terms:

$$\frac{\partial E}{\partial w_1} = -0.0084 \cdot 0.213 \cdot 0.1 \approx -0.00018$$

Update w_1:

$$w_1^{\text{new}} = w_1 - \eta \cdot \frac{\partial E}{\partial w_1}$$
$$w_1^{\text{new}} = 0.3 - 0.1 \cdot (-0.00018) \approx 0.300018$$

$$w_1^{\text{new}} = w_1 - \eta \cdot \frac{\partial E}{\partial w_1}$$
$$w_1^{\text{new}} = 0.3 - 0.1 \cdot (-0.00018) \approx 0.300018$$

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Gradient Formula for w_2:

$$\frac{\partial E}{\partial w_2} = \frac{\partial E}{\partial h_1^{\text{out}}} \cdot \frac{\partial h_1^{\text{out}}}{\partial h_1^{\text{in}}} \cdot \frac{\partial h_1^{\text{in}}}{\partial w_2}$$

Steps are similar to w_1, except:

$$\frac{\partial h_1^{\text{in}}}{\partial w_2} = x_2 = 0.2$$

$$\frac{\partial E}{\partial w_2} = -0.0084 \cdot 0.213 \cdot 0.2 \approx -0.00036$$

Update w_2:

$$w_2^{\text{new}} = 0.4 - 0.1 \cdot (-0.00036) \approx 0.400036$$

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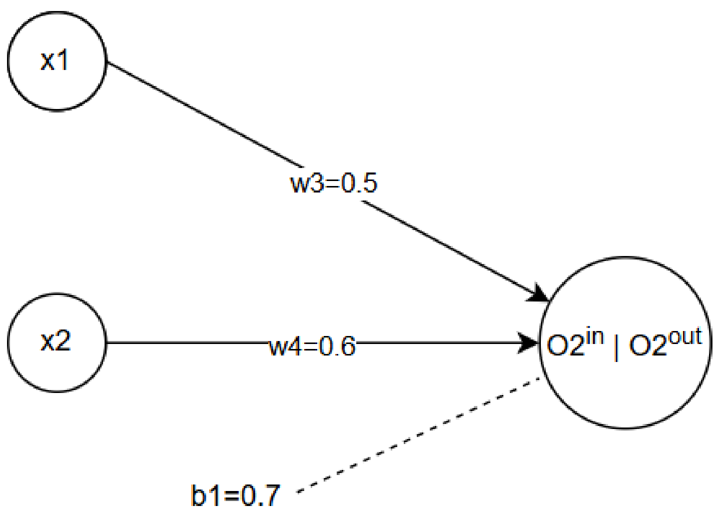
Gradient for w_3 and w_4:



For the second hidden unit, the steps mirror w_1 and w_2 with changes in gradients related to h_2^{out}.

Formula for w_3:

$$\frac{\partial E}{\partial w_3} = \frac{\partial E}{\partial h_2^{\text{out}}} \cdot \frac{\partial h_2^{\text{out}}}{\partial h_2^{\text{in}}} \cdot \frac{\partial h_2^{\text{in}}}{\partial w_3}$$

Steps: 1. $\frac{\partial E}{\partial h_2^{\text{out}}} = 0.0552 - 0.0636 = -0.0084$ 2. $\frac{\partial h_2^{\text{out}}}{\partial h_2^{\text{in}}} = 0.705 \cdot (1 - 0.705) = 0.207$ 3. $\frac{\partial h_2^{\text{in}}}{\partial w_3} = x_1 = 0.1$

$$\frac{\partial E}{\partial w_3} = -0.0084 \cdot 0.207 \cdot 0.1 \approx -0.000173$$

Update w_3:

$$w_3^{\text{new}} = 0.5 - 0.1 \cdot (-0.000173) \approx 0.5000173$$

Formula for w_4:

$$\frac{\partial h_2^{\text{in}}}{\partial w_4} = x_2 = 0.2$$

$$\frac{\partial E}{\partial w_4} = -0.0084 \cdot 0.207 \cdot 0.2 \approx -0.000346$$

Update w_4:

$$w_4^{\text{new}} = 0.6 - 0.1 \cdot (-0.000346) \approx 0.6000346$$

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Updated Weights Summary:

$$w_1^{\text{new}} \approx 0.300018, \quad w_2^{\text{new}} \approx 0.400036$$
$$w_3^{\text{new}} \approx 0.5000173, \quad w_4^{\text{new}} \approx 0.6000346$$