University of Cape Town
Bayesian Computation
Lab 3

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Solutions:

(a) Non-linear regression of the form:

$$y_i = \alpha - \beta\gamma^{x_i} + e_i$$

for $i = 1, 2, ..., n$, and e_i is distributed normally with zero mean and $\sigma^2 = 1/\tau$.

Priors:

- α is normal
- β is normal
- τ is gamma (σ^2 is inverted gamma)
- γ is uniform on [0,1]

The means and variances of the normal priors are set to be non-informative (i.e. large variances).

The posterior conditionals on α, β and τ have the following form:

$$\pi(\alpha|\beta,\gamma,\tau,\boldsymbol{x}) \propto exp[-\frac{n\tau+\tau_\alpha}{2}(\alpha - \frac{\tau\Sigma(y_i+\beta\gamma^{x_i})+\tau_\alpha\mu_\alpha}{n\tau+\tau_\alpha})^2]$$

$$\pi(\beta|\alpha,\gamma,\tau,\boldsymbol{x}) \propto exp[-\frac{\tau\Sigma\gamma^{x_i}+\tau_\beta}{2}(\beta - \frac{\tau\Sigma(\alpha-y_i)\gamma^{x_i}+\tau_\beta\mu_\beta}{\tau\Sigma\gamma^{x_i}+\tau_\beta})^2]$$

$$\pi(\tau|\alpha,\beta,\gamma,\boldsymbol{x}) \propto \tau^{a+\frac{n}{2}-1}exp[-\tau(b+\frac{1}{2}\Sigma(y_i-\alpha+\beta\gamma^{x_i})^2)]$$

which means:

$$\pi(\alpha|\beta,\gamma,\tau,\boldsymbol{x}) = N(\frac{\tau\Sigma(y_i+\beta\gamma^{x_i})+\tau_\alpha\mu_\alpha}{n\tau+\tau_\alpha}, \frac{1}{n\tau+\tau_\alpha})$$

$$\pi(\beta|\alpha,\gamma,\tau,\boldsymbol{x}) = N(\frac{\tau\Sigma(\alpha-y_i)\gamma^{x_i}+\tau_\beta\mu_\beta}{\tau\Sigma\gamma^{x_i}+\tau_\beta}, \frac{1}{\tau\Sigma\gamma^{x_i}+\tau_\beta})$$

$$\pi(\tau|\alpha,\beta,\gamma,\boldsymbol{x}) = Gamma(a+\frac{n}{2}, b+\frac{1}{2}\Sigma(y_i-\alpha+\beta\gamma^{x_i})^2)$$

Hence, the implementation of a Metropolis-Hastings algorithm will also include aspected of multi Gibbs sampling when sampling these parameters

The conditional on γ has a non-standard form:

$$\pi(\gamma|\alpha,\beta,\tau,\boldsymbol{x}) \propto exp[-\frac{\tau}{2}\Sigma(y_i-\alpha+\beta\gamma^{x_i})^2]$$

and so the M-H algorithm will have to be used to sample γ.

The easiest choice of a proposal dist. for the M-H part is a normal distribution. But $0 \leq \gamma \leq 1$. Therefore, by transforming from γ to $g = log(\frac{\gamma}{1-\gamma})$, we can set $q = N(g_{old}, \sigma_q^2)$.

Algorithm:

(i) Initialise parameters at the arbitrary starting values $\alpha_0 = 1, \beta_0 = 1, \gamma_0 = 0.9, \tau_0 = 1$. Set $i = 1$

(ii) Sample $\alpha_i, \beta_i, \gamma_i$ directly from the conditional distributions, updating the values for each time before sampling the next value.

(iii) $g_{old} = log(\frac{\gamma_i}{1-\gamma_i})$ and sample g_{new} from $N(g_{old}, \sigma_q^2)$

(iv) Calculate acceptance value:

$$\alpha(g_{new}, g_{old}) = min(1, \frac{\pi(g_{new}|\alpha, \beta, \tau, \boldsymbol{x})}{\pi(g_{old}|\alpha, \beta, \tau, \boldsymbol{x})})$$

Note we do not need normalising constant for this posterior conditional of g, and it has the form:

$$\pi(g|\alpha, \beta, \tau, \boldsymbol{x}) \propto exp[-\frac{\tau}{2}\Sigma(y_i - \alpha + \beta(\frac{e^g}{1+e^g})^{x_i})^2]\frac{e^g}{(1+e^g)^2}$$

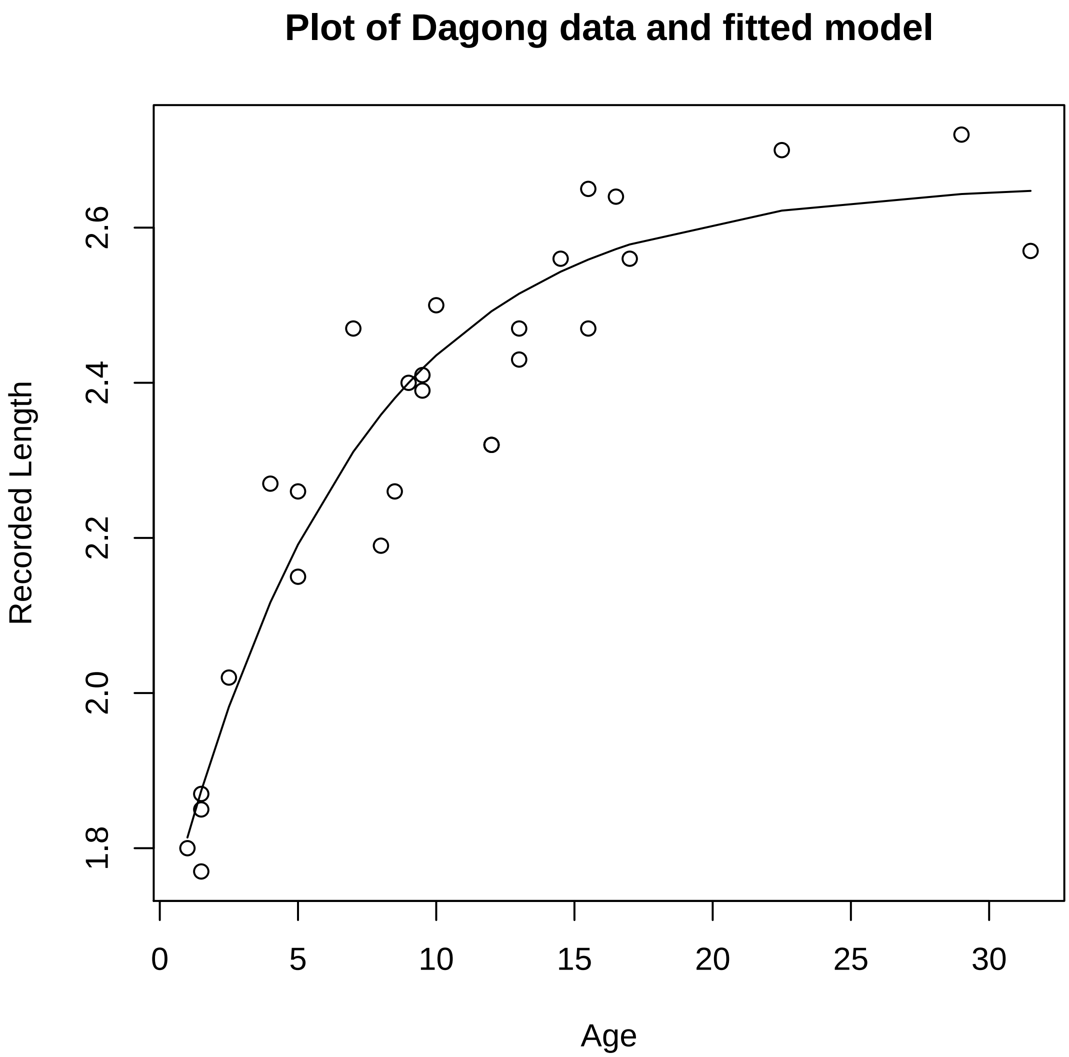
(v) Set $g_i = g_{new} if \alpha(g_{new}, g_{old}) < u$ where u is U[0,1]
Else $g_i = g_{old}$

(vi) i = i+1 and return to step (ii)

Repeat until i = 11000

(b)

	Estimate	Standard Deviation
α	2.656463	0.08377426
β	0.977808	0.07868861
σ^2	0.01008137	0.003259338
γ	0.861854	0.03719972



Plot of Dagong data and fitted model
Recorded Length
1.8
2.0
2.2
2.4
2.6
0
5
10
15
20
25
30
Age

(c)