# Control System Synthesis by Root Locus Method

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**Synopsis:** The root locus method determines all of the roots of the differential equation of a control system by a graphical plot which readily permits synthesis for desired transient response or frequency response. The base points for this plot on the complex plane are the zeros and poles of the open loop transfer function, which are readily available. The locus of roots is a plot of the values of $s$ which make this transfer function equal to $-1$ as loop gain is increased from zero to infinity. The plot can be established in approximate form by inspection and the significant parts of the locus calculated accurately and quickly by use of a simple device. For multiple loop systems, one solves the innermost loop first, which then permits the next loop to be solved by another root locus plot. The resultant plot gives a complete picture of the system, which is particularly valuable for unusual systems or those which have wide variations in parameters.

THE root locus method is the result of an effort to determine the roots of the differential equation of a control system by using the concepts now associated with frequency response methods.[1] The roots are desired, of course, because they describe the natural response of the system. The simplifying feature of the control system problem is that the open loop transfer function is known as a product of terms. Each term, such as $1/(1+Ts)$, can be easily treated in the same manner as an admittance such as $1/(R+jx)$. It is treated as a vector in the sense used by electrical engineers in solving a-c circuits. The phase shift and attenuation of a signal of the form $e^{st}$ being transmitted is represented by $1/(1+Ts)$ in which $s$ in general is a complex number. The key idea in the root locus method is that the values of $s$ which make transfer function around the loop equal to $-1$ are roots of the differential equation of the system.

The opening section in this paper, Background Theory, outlines the over-all pattern of analysis. The following section on Root Locus Plot points out the great usefulness of knowing factors of the open loop transfer function in finding the roots.

The graphical nature of the method requires that specific examples be used to demonstrate the method itself under the topics: Single Loop Example, Multiple Loop System, and Corrective Networks. The topic Correlation with Other Methods suggests methods by which experience in frequency methods can be extended to this method. The topic Other Applications includes the classic problem of solving an $n$th degree polynomial. Finally, the section on Graphical Calculations describes the key features of a plastic device called a "Spirule", which permits calculations to be made from direct measurement on the plot.

## Background Theory

The over-all pattern of analysis can be outlined before explaining the technique of sketching a root locus plot. Thus consider the general single loop system shown in Figure 1.

Note that each transfer function is of the form $KG(s)$ in which $K$ is a static gain constant and $G(s)$ is a function of the complex number. In general, $G(s)$ has both numerator and denominator known in factored form. The values of $s$ which make the function zero or infinite can therefore be seen by inspection and are called zeros and poles respectively. The closed loop transfer function can be expressed directly from Figure 1 as given in equation 1

$$\frac{\theta_0}{\theta_i}(s)=\frac{K_\mu G_\mu(s)}{1+K_\mu G_\mu(s)K_\beta G_\beta(s)} \tag{1}$$

The problem of finding the roots of the differential equation here appears in the form of finding values of $s$ which make the denominator zero. After these values are determined by the root locus method, the denominator can be expressed in factored form. The zeros of the function $\theta_0/\theta_i$ can be seen from equation 1 to be the zeros of $G_\mu(s)$ and the poles of $G_\beta(s)$. The function can now be expressed as shown in equation 2

$$\frac{\theta_0}{\theta_i}(s)=K_c s^\gamma \frac{(1-s/q_1)(1-s/q_2)\ \ldots}{(1-s/r_1)(1-s/r_2)\ \ldots.} \tag{2}$$

The constant $K_c$ and the exponent $\gamma$ depend upon the specific system but for control systems $\gamma$ is often zero and $K_c$ is often 1.

The full power of the Laplace Transform[2] or an equivalent method now can be used. The transient response of the output for a unit step input, for example, is given by equation 3

$$\theta_0(t)=1-\sum_{i=1}^{i=n} A_i e^{r_i t} \tag{3}$$

The amplitude $A_i$ is given by equation 4

$$A_i=\left[\frac{\theta_0}{\theta_i}(s)(1-s/r_i)\right]_{s=r_i} \tag{4}$$

The closed loop frequency response, on the other hand, can be obtained by substituting $s=j\omega$ into equation 2. Fortunately, the calculation in finding $A_i$ or $\theta_0/\theta_i(j\omega)$ involves the same problem of multiplying vectors that arises in making a root locus plot, and can be calculated quickly from the resultant root locus plot.

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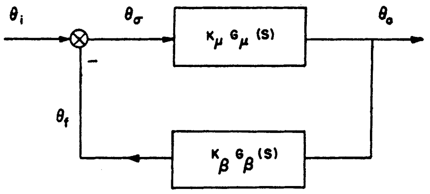
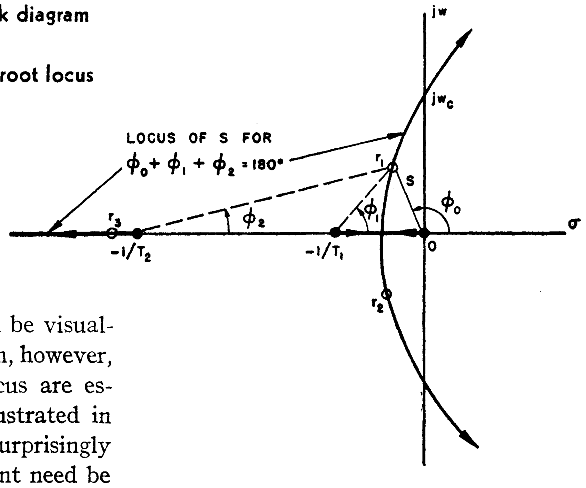



Figure 1 (left). General block diagram

Figure 3 (right). Single loop root locus




## Root Locus Plot

The open loop transfer function is typically of the form given in equation 5.

$$K_\mu G_\mu(\boldsymbol{s})K_\beta G_\beta(\boldsymbol{s}) = \frac{K(1+T_2\boldsymbol{s})[\sigma_3^2+\omega_3^2]}{\boldsymbol{s}(1+T_1\boldsymbol{s})[(\boldsymbol{s}+\sigma_3)^2+\omega_3^2]} \quad (5)$$

The parameters such as $T_1$ are constant for a given problem, whereas $\boldsymbol{s}$ assumes many values; therefore, it is convenient to convert equation 5 to the form of equation 6.

$$K_\mu G_\mu(\boldsymbol{s})K_\beta G_\beta(\boldsymbol{s}) = \frac{K(1/T_2+\boldsymbol{s})T_2[\sigma_3^2+\omega_3^2]}{\boldsymbol{s}(1/T_1+\boldsymbol{s})T_1[(\boldsymbol{s}+\sigma_3+j\omega_3)(\boldsymbol{s}+\sigma_3-j\omega_3)]} \quad (6)$$

The poles and zeros of the function are plotted and a general value of $\boldsymbol{s}$ is assumed as shown in Figure 2.

Note that poles are represented as dots, and zeros as crosses. All of the complex terms involved in equation 6 are represented by vectors with heads at the general point $\boldsymbol{s}$ and tails at the zeros or poles. The angle of each vector is measured with respect to a line parallel to the positive real axis. The magnitude of each vector is simply its length on the plot.

In seeking to find the values of $\boldsymbol{s}$ which make the open loop function equal to $-1$, the value $-1$ is considered as a vector whose angle is 180 degrees $\pm$ $n$ 360 degrees, where $n$ is an integer, and whose magnitude is unity. Then one can consider first the problem of finding the locus of values for which the angle condition alone is satisfied. In general, one pictures the exploratory $\boldsymbol{s}$ point at various positions on the plane, and imagines the lines from the poles and zeros to be constructed so that the angles in turn can be visualized. For any specific problem, however, many special parts of the locus are established by inspection as illustrated in examples in later sections. Surprisingly few trial positions of the $\boldsymbol{s}$ point need be assumed to permit the complete locus to be sketched.

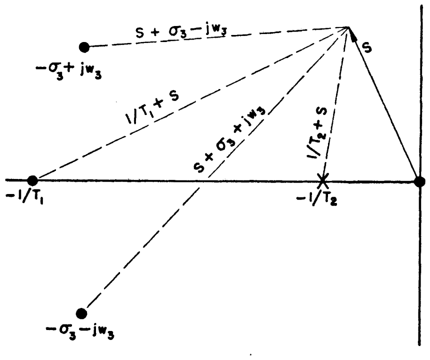



Figure 2. Root locus plot

After the locus has been determined, one considers the second condition for a root, that is, that the magnitude of $K_\mu G_\mu(s)K_\beta G_\beta(s)$ be unity. In general, one selects a particular value of $\boldsymbol{s}$ along the locus, estimates the lengths of the vectors, and calculates the static gain $K_\mu K_\beta = 1/G_\mu(s)G_\beta(s)$. After acquiring some experience, one usually can select the desired position of a dominant root to determine the allowable loop gain. The position of roots along other parts of the locus usually can be determined with less than two trials each.

An interesting fact to note from equation 6 is that for very low gain, the roots are very close to the poles in order that corresponding vectors be very small. For very high gain, the roots approach infinity or terminate on a zero.

## Single Loop Example

Consider a single loop system such as shown in Figure 1 in which the transfer functions are given in equation 7.

$$K_\mu G_\mu(\boldsymbol{s}) = \frac{K}{(1+T_1\boldsymbol{s})(1+T_2\boldsymbol{s})\boldsymbol{s}}; K_\beta G_\beta(\boldsymbol{s}) = 1 \quad (7)$$

The poles of the open loop function are at 0, $-1/T_1$ and $-1/T_2$ as represented by dots in Figure 3.

The locus along the real axis is determined by inspection because all of the angles are either 0 degrees or 180 degrees. An odd number of angles must therefore be 180 degrees as shown by the intervals between 0 and $-1/T_1$, and from $-1/T_2$ to $-\infty$. Along the $j\omega$ axis, $\phi_0$ is 90 degrees so that $\phi_2$ must be the complement of $\phi_1$, as estimated at $\boldsymbol{s}=j\omega_c$. For very large values of $\boldsymbol{s}$, all angles are essentially equal so the locus for the complex roots finally approaches radial lines at $\pm 60$ degrees.

The point where the locus breaks away from the real axis is found by considering a value of $\boldsymbol{s}$ just above the real axis. The decrease in $\phi_0$ from 180 degrees can be made equal to the sum of $\phi_1$ and $\phi_2$ if the reciprocal of the length from the trial point to the origin is equal to the sum of the reciprocals of lengths from the trial point to $-1/T_1$ and $-1/T_2$. If a damping ratio of 0.5 for the complex roots is desired, the roots $r_1$ and $r_2$ are fixed by the intersection with the locus of radial lines at $\pm 60$ degrees with respect to negative real axis.

In calculating $K$ for $\boldsymbol{s}=r_1$, it is convenient to consider a term $(1+T_1\boldsymbol{s})$ as a ratio of lengths from the pole $-1/T_1$ to the $\boldsymbol{s}$ point and from $\boldsymbol{s}$ to the origin respectively. After making gain $K=1/$-$[G(s)]_{\boldsymbol{s}=r_1}$ a good first trial for finding $r_3$ is to assume that it is near $-1/T_2$ and solve for $(1/T_2+\boldsymbol{s})$. After the roots are determined to the desired accuracy, the over-all transfer function can be expressed as given in equation 8.

$$\frac{\theta_0}{\theta_i} = \frac{1}{\left(1-\frac{s}{r_1}\right)\left(1-\frac{s}{r_2}\right)\left(1-\frac{s}{r_3}\right)} \quad (8)$$

The procedure in handling a multiple loop system now can be explained.

## Multiple Loop System

Consider a multiple loop system in which the single loop system just solved is the forward path of another loop, as shown in Figure 4.

$\theta_o/\theta_i$ is given in factored form by equation 8 so the roots of the inner loop now serve as base points for the new locus plot. For convenience, however, neglect the effect of the term $(1-\boldsymbol{s}/r_3)$ so that the locus for the outer loop is shown in Figure 5.

The locus for the outer loop would be a circle about the $-1/T$ point as a center if the effect of $\theta_o/\theta_i$ were completely neglected. Actually, the vectors from the points $r_1$ and $r_2$ introduce net angles so that the locus is modified as shown. The

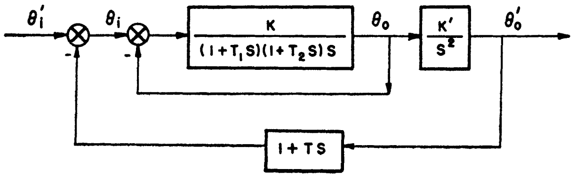
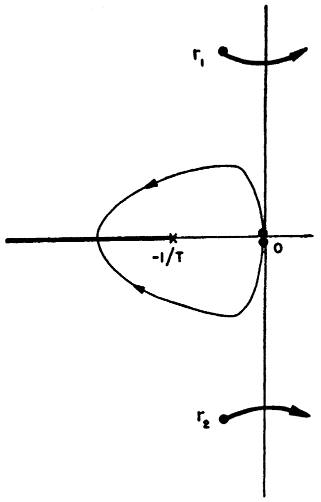



**Figure 4 (above). Multiple loop block diagram**

**Figure 5 (right). Multiple loop root locus**




angle at which the locus emerges from $r_1$ can be found by considering a value of $\boldsymbol{s}$ close to the point $r_1$, and solving for the angle of the vector $(\boldsymbol{s}-\boldsymbol{r}_1)$.

Assume that the static loop gain desired is higher than that allowed by the given system. The first modification suggested by the plot is to move the $r_1$ and $r_2$ points farther to the left by obtaining greater damping in the inner loop. If these points are moved far to the left, the loci from these points terminate in the negative real axis and the loci from the origin curve back and cross the $j\omega$ axis. Moving the $-1/T$ point closer to the origin would then be effective in permitting still higher loop gain. The next aspect of synthesis involves adding corrective networks.

## Corrective Networks

Consider a somewhat unusual system which arises in instrument servos whose open loop transfer function is identified by the poles $p_1$ and $p_2$ in Figure 6(A). As loop gain is increased from zero, the roots which start from $p_1$ and $p_2$ move directly toward the unstable half plane. These roots could be made to move away from the $j\omega$ axis if 180 degrees phase shift were added. A simple network to add is three lag networks in series, each having a time constant $T$ such that 60 degrees phase shift is introduced at $p_1$. The resultant locus plot is shown in Figure 6(B).

The gain now is limited only by the requirement that the new pair of roots do not cross the $j\omega$ axis. A value of gain is selected to obtain critical damping of these roots and the corresponding positions of all the roots are shown in Figures 6(A) and 6(B) as small circles.

Actually, greater damping could be achieved for roots which originate at $p_1$ and $p_2$ if a phase shifting bridge were used rather than the 3-lag networks. Its transfer function is $(3-Ts)/(1+Ts)$ and is of the "nonminimum phase" type of circuit.

Since these types of correction are somewhat unusual, it is perhaps well to point out that the analysis has been verified by actual test and application.

These examples serve to indicate the reasoning process in synthesizing a control system by root locus method. An engineer draws upon all of his experience, however, in seeking to improve a given system; therefore, it is well to indicate the correlation between this method and other methods.

## Correlation with Other Methods

The valuable concepts of frequency response methods[1] are in a sense merely extended by the root locus system. Thus a transfer function with $\boldsymbol{s}$ having a complex value rather than just a pure imaginary value corresponds to a damped sinusoid being transmitted rather than an undamped one. The frequency and gain for which the Nyquist plot passes through the $-1$ point are exactly the same values for which the root locus crosses the $j\omega$ axis. Many other correlations appear in solving a single problem by both methods.

The results of root locus analysis can be easily converted to frequency response data. Thus one merely assumes values of $\boldsymbol{s}$ along the $j\omega$ axis, estimates the phase angles and vector lengths to the zeros and poles, and calculates the sum of the angles for total phase shift and the product of lengths for attenuation. The inverse problem of determining zeros and poles from experimental data is the more difficult one. Many techniques are already available, however, such as drawing asymptotes to the logarithmic attenuation curve. For unusual cases, particularly those in which resonant peaks are involved, the conformal mapping technique originated by Dr. Profos of Switzerland is recommended.[3]

The transient response is described by the poles of the transfer function. The inverse problem in this case is to locate the poles from an experimental transient response. One might use dead time, maximum build-up rate, overshoot, natural frequency of oscillation, and the damping rate as effective clues in solving this problem.

## Other Applications

Many systems require a set of simultaneous equations to describe them and are said to be multicoupled. The corresponding block diagrams have several inputs to each loop so that the root locus method cannot be applied immediately. One should first lay out the diagram so that the main line of action of the signals forms the main loop with incidental coupling effects appearing as feedbacks and feed forwards. One then proceeds to isolate loops by replacing a signal which comes from within a loop by an equivalent signal at the output, replacing a signal entering a loop by an equivalent signal at the input. One can and should keep the physical picture of the equivalent system in mind as these manipulations are carried out.

The techniques of the root locus method can be used effectively in analyzing electric circuits. As a simple example, consider the lead-lag network of Figure 7(A).

It can be shown that the transfer function of this network is as given in equation 9

$$\frac{V_0}{V_i}=\frac{(1+R_1C_1\boldsymbol{s})(1+R_2C_2\boldsymbol{s})R_3}{(1+R_1C_1\boldsymbol{s})(1+R_2C_2\boldsymbol{s})R_3+R_1[1+(R_2+R_3)C_2\boldsymbol{s}]} \tag{9}$$

The denominator can be factored algebraically by multiplying out and finding the zeros of the resulting quadratic. As an alternative, it will be noted that the zeros of the denominator must satisfy equation 10

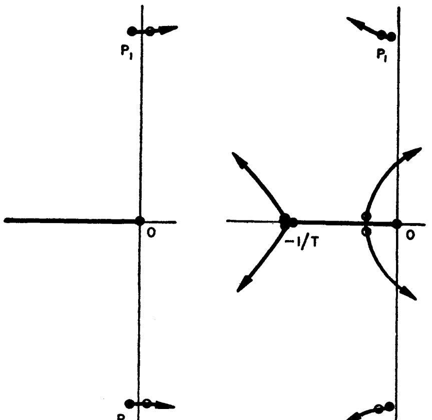



**Figure 6. (A) Basic system. (B) Corrected system**

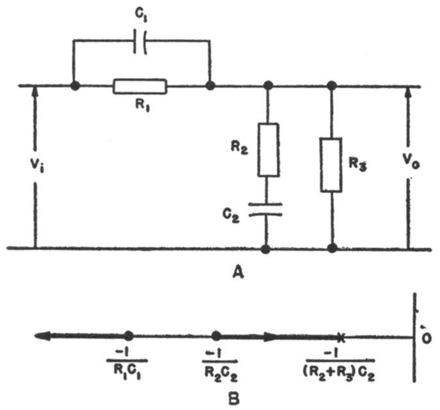
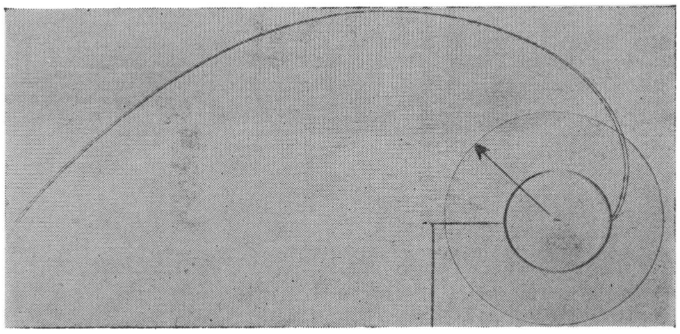



Figure 7 (left). (A) Circuit diagram. (B) Root locus

Figure 8 (right). Spirule



$$\frac{(1/R_1C_1+s)(1/R_2C_2+s)R_3(R_1C_1)(R_2C_2)}{[1/(R_2+R_3)C_2+s](R_2+R_3)C_2R_1} = -1 \quad (10)$$

The vectors in this expression are represented according to the root locus scheme in Figure 7(B). The two roots are thereby bounded as shown by the two dots and the cross. Their exact locations could be estimated or accurately determined by graphical methods.

The locus of roots now is simply intervals along the negative real axis between the open loop zeros and poles as shown in Figure 7 (B). The exact location of the roots along these intervals is determined in the usual way. Note that the constant in equation 10 is of the form $R'C_1$, in which $R'$ is the effective value of $R_2$ and $R_3$ in parallel.

In more complicated networks, the advantages of the root locus concept over algebraic methods becomes greater; its particular advantage is in retaining at all times a clear picture of the relationships between the over-all network parameters and the parameters of individual circuit elements.

In the classical problem of finding roots, the differential equation is given in the form of a sum of terms of successively higher order. This can be converted to the form shown in equation 10

$$s^n+as^{n-1}+bs^{n-2}+\ldots+m = [(s+a)s+b]s+\ldots+m \quad (11)$$

This corresponds to a block diagram with another loop closed for each higher order term. Solve for the roots of the first loop which corresponds to the quantities in brackets above and proceed as before for the multiple loop system. If the roots close to the origin are of most interest, substitute $s=1/x$ first and solve for root values of $x$. Other combinations are, of course, possible because a single root locus basically determines the factors of the sum of two terms.

The root locus method is thus an analytical tool which can be applied to other problems than control system synthesis for which it was developed. But in attacking a new problem one would probably do well to try first to develop a method of analysis which is natural for that problem rather than seek to apply any existing methods.

## Graphical Calculations

The root locus plot is first established in approximate form by inspection. Any significant point on the locus then can be checked by using the techniques indicated in this section. Note that only two calculations are involved, adding angles and multiplying lengths. Fortunately, all of these angles and lengths can be measured at the $s$ point. Thus angles previously pictured at the zeros and poles also appear at the $s$ point but between a horizontal line to the left and lines to the zeros and poles. A piece of transparent paper or plastic pivoted at the $s$ point can be rotated successively through each of these angles to obtain their sum.

The reader can duplicate the "spirule" with two pieces of transparent paper, one for the disk and the other for the arm. Several procedures are possible, but the over-all purpose is to successively rotate the arm with respect to the disk through each of the angles of interest. Thus for adding phase angles, the disk is held fixed while the arm is rotated from a pole to the horizontal, whereas the two move together in getting aligned on the next pole. For multiplying lengths, the disk is held fixed while the arm is rotated from the position where a pole is on the straight line to the position where the pole is on the logarithmic curve. Rotations are made in the opposite directions for zeros than they are for poles.

## Conclusions

The definite opinion of engineers using this method is that its prime advantage is the complete picture of a system which the root locus plot presents. Changing an open loop parameter merely shifts a point and modifies the locus. By means of the root locus method, all of the zeros and poles of the over-all function can be determined.

Any linear system is completely defined by this determination, and its response to any particular input function can be determined readily by standard mathematical or graphical methods.

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# No Discussion