Source Seeking for Underactuated Vehicles

Bo Wang

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Assistant Professor

Department of Mechanical Engineering
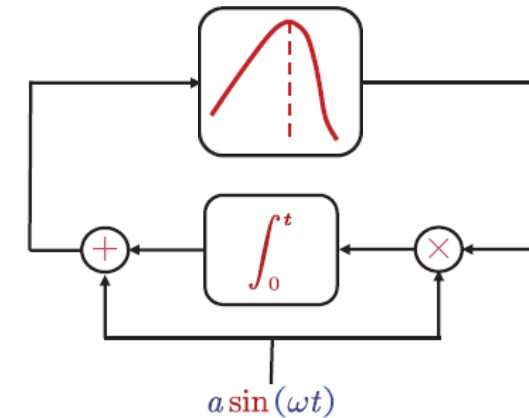
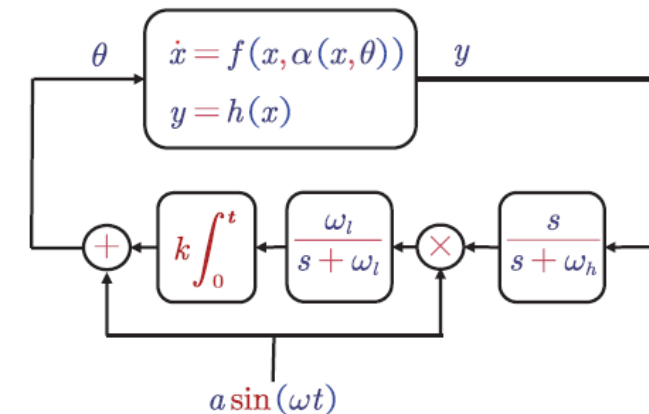The City College of New York

- The college has graduated 12 Nobel Prize winners, 1 Fields Medalist, 1 Turing Award winner.
- Henry Kissinger, Colin Luther Powell, Michio Kaku (加來道雄), …
- Albert Einstein gave the first of his series of United States lectures at CCNY in 1921.

Extremum Seeking Control (ESC)

- Extremum seeking (ES)
  - Online black-box reward-seeking
  - Real-time, model-free optimization algorithm

- ES history
  - Sinusoid-based ES was popular in the 1950s
  - Stability proved in [Krstic & Wang '00]
  - Torrent of advances in theory & applications (12,000+ papers since 2000)
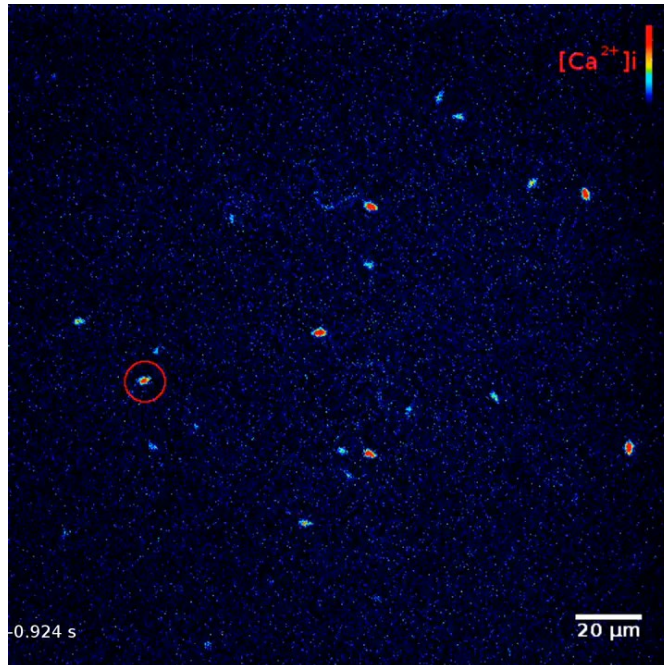

(a) ES scheme for a static map

$a \sin(\omega t)$

θ $\quad \dot{x} = f(x, \alpha(x, \theta)) \quad y$
$y = h(x)$

$k \int_0^t \quad \dfrac{\omega_l}{s + \omega_l} \quad \times \quad \dfrac{s}{s + \omega_h}$

$a \sin(\omega t)$

(b) ES for a dynamical system

Krstic & Wang (2000)

Extremum Seeking in Nature

Sperm seeking an egg

Sea urchin sperm exploit extremum seeking
control to find the egg

Mahmoud Abdelgalil[1,*], Yasser Aboelkassem[2,3], Haithem Taha[1]

[1] Mechanical and Aerospace Engineering Department, University of California at Irvine, Irvine, CA 92697
[2] College of Innovation and Technology, University of Michigan at Flint, Flint MI 48502
[3] Michigan Institute for Data Science, University of Michigan, Ann Arbor MI 48109

Phototropism

- **Motivation**: Locate the Source of a Signal (w/o GPS or INS)

- Electro-magnetic, optical, chemical, etc.

- GPS and INS signals are not available

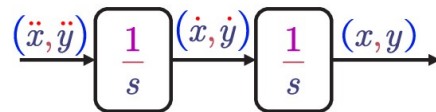- Locate and repair the leakage, avalanche victim search, etc.

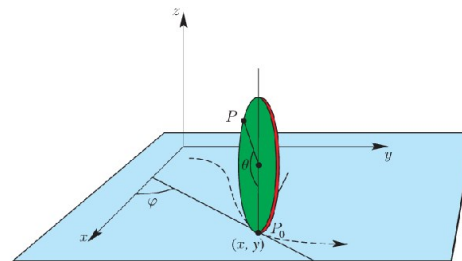Without GPS or INS, Source Seeking algorithms are very useful!

On Vehicle Models

- Single/double-integrators: [Zhang'07]
- Unicycle: [Zhang'07; Ghods; Durr'13,'17]
- Fully-actuated systems: [Suttner'21,'22]
- Underactuated systems
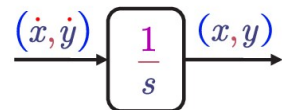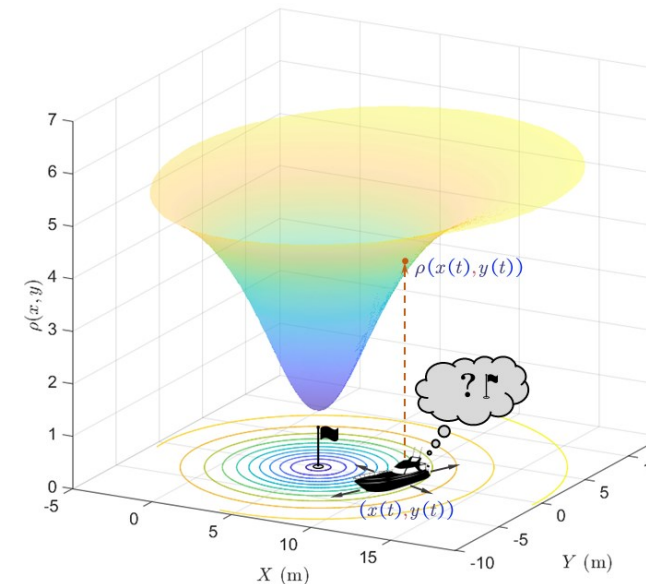

(a) Single integrator

(c) Unicycle

(b) Double integrator

(d) Underactuated boat

Control/Optimization Objective

- The position-dependent function $\rho: \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ has a global minimum at : $(x^\star, y^\star)$
- Both $(x^\star, y^\star)$ and $\nabla \rho$ are <u>unknown</u>.
- Vehicle can **only** measure the value of $\rho(x(t), y(t))$ in <u>real time</u>.
- Without position & velocity measurements

$$\lim_{t \to \infty} |(x(t), y(t)) - (x^\star, y^\star)| \leq \varepsilon.$$
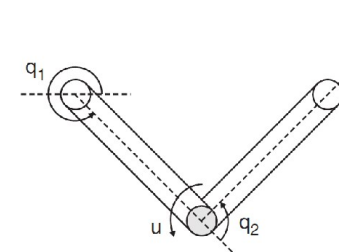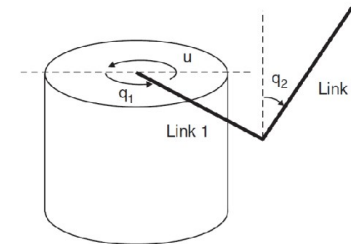
Keyword: <u>Underactuated</u> Mechanical Systems

- Mechanical systems

$$\underbrace{M(q)\ddot{q}+C(q,\dot{q})}_{\text{geometric acceleration}}+\underbrace{D(\dot{q})}_{\text{damping}}+\underbrace{g(q)}_{\text{gravity}}=\underbrace{G(q)u}_{\text{external forces}}$$
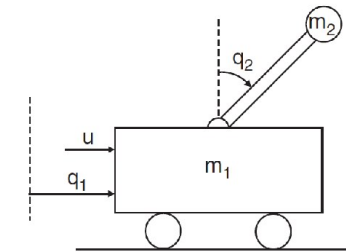
- $q \in \mathbb{R}^n$: configuration variable
- $u \in \mathbb{R}^m$: input variable, $m < n$
- $G(q) \in \mathbb{R}^{n \times m}$: input matrix

- Fully-actuated, if $\text{rank}\{G(q)\} = n$
- Underactuated, if $\text{rank}\{G(q)\} < n$

- Simple, power saving, make better robots…
- Almost all real vehicles are underactuated!

Acrobot

cart-pole

rotating pendulum

surface vessel

Fig: Underactuated systems.

1. **Classical averaging-based (2000-2013)**

Consider the system

$$\dot{x} = \varepsilon f(t, x, \varepsilon) \qquad (10.23)$$

where f and its partial derivatives with respect to (x, ε) up to the second order are continuous and bounded for $(t, x, \varepsilon) \in [0, \infty) \times D_0 \times [0, \varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset R^n$ is a domain. Moreover, $f(t, x, \varepsilon)$ is T-periodic in t for some $T > 0$ and ε is positive. We associate with (10.23) an autonomous average system

$$\dot{x} = \varepsilon f_{av}(x) \qquad (10.24)$$

where

$$f_{av}(x) = \frac{1}{T} \int_0^T f(\tau, x, 0) \, d\tau \qquad (10.25)$$

PERGAMON

Automatica 36 (2000) 595–601

www.elsevier.com/locate/automatica

Brief Paper

Stability of extremum seeking feedback for general nonlinear dynamic systems

Miroslav Krstić[a],*, Hsin-Hsiung Wang[b,1]

[a] Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA
[b] Department of Electrical Engineering, Oriental Institute of Technology, 58, Sec. 2, Szu-Chuan Road, Panchiao, Taipei County, 220 Taiwan

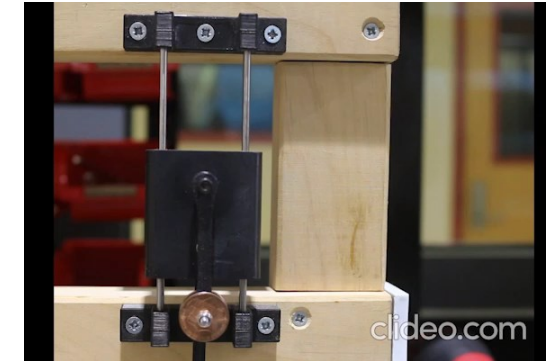Received 4 March 1997; revised 16 February 1999; received in final form 16 August 1999

2. **Lie-bracket approximation-based (2013-)**

Automatica 49 (2013) 1538–1552

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Lie bracket approximation of extremum seeking systems

Hans-Bernd Dürr[a,1], Miloš S. Stanković[b], Christian Ebenbauer[a], Karl Henrik Johansson[c]

[a] Institute for Systems Theory and Automatic Control, University of Stuttgart, Germany
[b] Innovation Center, School of Electrical Engineering, University of Belgrade, Serbia
[c] ACCESS Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden

obtain an input-affine system of the form

$$\dot{x} = b_1(x)\sqrt{\omega}\, u_1(\omega t) + b_2(x)\sqrt{\omega}\, u_2(\omega t) \qquad (2)$$

with $b_1(x) = \alpha$ and $b_2(x) = f(x)$. Interestingly, if one computes the so called Lie bracket system involving $[b_1, b_2]$, i.e.

$$\dot{z} = \frac{1}{2}[b_1, b_2](z) = \frac{\alpha}{2}\nabla_z f(z), \qquad (3)$$

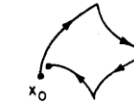PROCEEDINGS OF THE IEEE, JANUARY 1976

Fig. 1. Motivating the definition of ubiquitous bracket of Lie.

is the so-called *Lie bracket* (also sometimes called the Jacobi bracket, also sometimes defined with the opposite sign) of the vector fields f_1 and f_2. This calculation, which "everyone should do once in his life" is most significant. Everything else depends on it. If $[f_1, f_2]$ is not a linear combination of $f_1, f_2, \cdots, f_m$ then $[f_1, f_2]$ represents a "new" direction in which the solution can move and the original problem of finding a manifold such that $f_1, f_2, \cdots, f_m$ span the tangent space will not be solvable.

3. **Symmetric-product approximation based (2021-)**

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AVERAGING AND VIBRATIONAL CONTROL OF MECHANICAL SYSTEMS*

FRANCESCO BULLO[†]

Abstract. This paper investigates averaging theory and oscillatory control for a large class of mechanical systems. A link between averaging and controllability theory is presented by relating the key concepts of averaged potential and symmetric product. Both analysis and synthesis results are presented within a coordinate-free framework based on the theory of affine connections. The analysis focuses on characterizing the behavior of mechanical systems forced by high amplitude high frequency inputs. The averaged system is shown to be an affine connection system subject to an appropriate forcing term. If the input codistribution is integrable, the subclass of systems with Hamiltonian equal to "kinetic plus potential energy" is closed under the operation of averaging. This result precisely characterizes when the notion of averaged potential arises and how it is related to the symmetric product of control vector fields. Finally, a notion of vibrational stabilization for mechanical systems is introduced, and sufficient conditions are provided in the form of linear matrix equality and inequality tests.

Key words. mechanical system, averaging, vibrational stabilization, nonlinear controllability

- Linear **homogeneous ODE**

$$y^{(n)}(x) + \sum_{i=0}^{n-1} a_i(x) y^{(i)}(x) = 0$$

Solution: $y(x) = \sum_{i=1}^{n} c_i y_i(x)$

- Linear **non-homogeneous ODE**

$$y^{(n)}(x) + \sum_{i=0}^{n-1} a_i(x) y^{(i)}(x) = f(x)$$

Solution: $y(x) = \sum_{i=1}^{n} c_i(x) y_i(x)$

Theorem (Variation of constants) [Bullo '02]

Consider the dynamical system

$$\dot{x} = f(t, x) + g(t, x), \quad x(0) = x_0.$$

If $z(t)$ is the solution of the (pull-back) system

$$\dot{z}(t) = \left(\left(\Phi_{0,t}^g \right)^* f \right)(t, z), \quad z(0) = x_0,$$

then the solution $x(t)$ of

$$\dot{x} = g(t, x), \quad x(0) = z(t)$$

is the solution of the original system.

Symmetric Product Approximations

- EL vehicle model

$$\dot{q} = J(q)v,$$
$$M\dot{v} + C(v)v + Dv = Gu$$

- **High-amplitude high-frequency** input

$$u = b_0 + \frac{1}{\varepsilon}\sum_{i=1}^{m} b_i(q)w_i\left(\frac{t}{\varepsilon}\right),$$

Theorem (Variation of constants) [Bullo '02]

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then the solution $x(t)$ of

$$\dot{x} = g(t, x), \quad \boxed{x(0) = z(t)}$$

is the solution of the original system.

---

1. The closed-loop system **in time scale** $\tau = t/\varepsilon$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} q \\ v \end{bmatrix} = \varepsilon \underbrace{\begin{bmatrix} J(q)v \\ -M^{-1}[C(v)v + Dv - B_0] \end{bmatrix}}_{f(q,v): \text{ time-invariant}} + \underbrace{\begin{bmatrix} 0 \\ \sum_{i=1}^{m} B_i(q)w_i(\tau) \end{bmatrix}}_{g(\tau,q)}$$

2. "Pull-back": $\dot{z}(t) = \left(\left(\Phi_{0,t}^g\right)^* f\right)(t, z) =: F(t, z)$

$$F = f + \sum_{i=1}^{m}\left[\begin{pmatrix}\frac{\partial f_2}{\partial v}\end{pmatrix}B_i - \begin{pmatrix}\frac{\partial B_i}{\partial q}\end{pmatrix}J(q)v\right]\int_0^\tau w_i(s_1)\mathrm{d}s_1 - \sum_{i,j=1}^{m}\begin{bmatrix} 0 \\ \langle B_i : B_j\rangle \end{bmatrix}\int_0^\tau\int_0^{s_1} w_i(s_1)w_j(s_2)\mathrm{d}s_2\mathrm{d}s_1$$

3. "Homogeneous": $\dot{x} = g(t, x), \quad x(0) = z(t)$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \sum_{i=1}^{m} B_i(q)w_i(\tau) \end{bmatrix}, \quad \begin{bmatrix} q(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} \hat{q}(\tau) \\ \hat{v}(\tau) \end{bmatrix}\right\} \implies [q(\tau) \equiv q(0) = \hat{q}(\tau)]$$

4. "Averaged pull-back": Symmetric Product System

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \bar{q} \\ \bar{v} \end{bmatrix} = \frac{\varepsilon}{T}\int_0^T F(\tau, \bar{q}, \bar{v})\mathrm{d}\tau\right\} \implies \left\{\begin{array}{c} \dot{\bar{q}} = J(\bar{q})\bar{v}, \\ M\dot{\bar{v}} + C(\bar{v})\bar{v} + D\bar{v} = B_0 - M\sum_{i,j=1}^{m}\Lambda_{ij}\langle B_i : B_j\rangle(\bar{q}) \end{array}\right\}$$

$$\boxed{q(\tau) \equiv \hat{q}(\tau) \approx \bar{q}(\tau)} \text{ i.e., (Partial) Converging Trajectories Property}$$
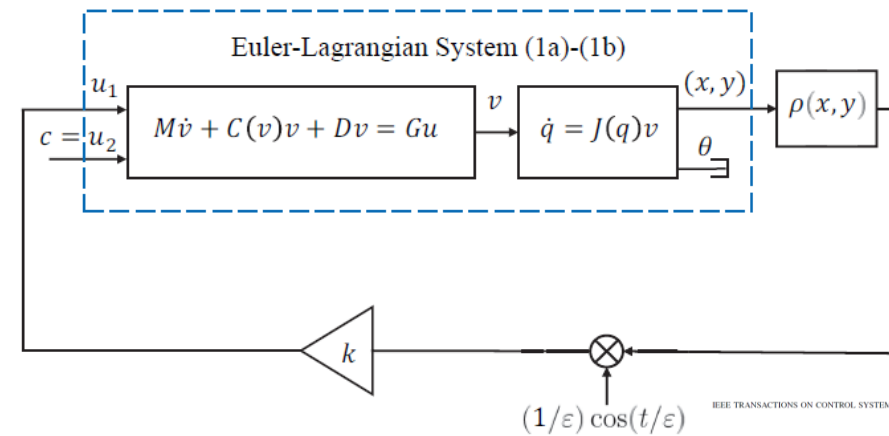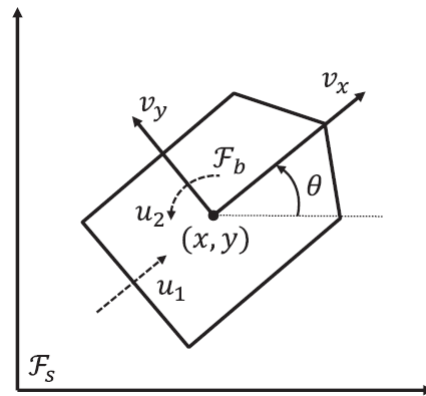
Theorem (Source seeking: surge force tuning) [Wang et al., TCST, CDC'22]

Consider the underactuated vehicle system with inputs

$$u_1 = \frac{k}{\varepsilon} \cos\left(\frac{t}{\varepsilon}\right) \rho(x, y),$$

$$u_2 = c.$$

Then, $\exists \, \hat{c} > 0$ and for any $c \in (0, \hat{c})$, $\exists \, \hat{\varepsilon} > 0$ such that for the given c and any $\varepsilon \in (0, \hat{\varepsilon})$ and $k > 0$, the closed-loop system is SPAS w.r.t. $(x - x^\star, y - y^\star, v_x, v_y)$ uniformly in $(\theta(0), \omega(0))$.

Euler-Lagrangian System (1a)-(1b)

$M\dot{v} + C(v)v + Dv = Gu$ $\dot{q} = J(q)v$ $\rho(x, y)$

$(1/\varepsilon)\cos(t/\varepsilon)$

IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY

Underactuated Source Seeking by Surge Force
Tuning: Theory and Boat Experiments

Bo Wang, *Graduate Student Member, IEEE*, Sergey Nersesov, *Member, IEEE*,
Hashem Ashrafiuon, *Senior Member, IEEE*, Peiman Naseradinmousavi, and Miroslav Krstić, *Fellow, IEEE*

Key idea:

Gradient Decent?

$F(x_-(t))$ $F(x_+(t))$

$F(x^*)$

$x_-(t)$ $x_+(t)$

Symmetric product system (34a)-(34b)

(Unicycle model)

$$\dot{\bar{x}} = \bar{v}_x\cos(\bar{\theta}) \qquad -\bar{v}_y\sin(\bar{\theta})$$

$$\dot{\bar{y}} = \bar{v}_x\sin(\bar{\theta}) \qquad +\bar{v}_y\cos(\bar{\theta})$$

$\bar{v}_y$

$$\dot{\bar{\theta}} = \bar{\omega}$$

PD feedback

$$\dot{\bar{v}}_x = -\frac{d_{11}}{m_{11}}\bar{v}_x - \alpha\rho\left(\rho'_x\cos(\bar{\theta}) + \rho'_y\sin(\bar{\theta})\right) + C_1(\bar{v}_y,\bar{\omega})$$

$$\dot{\bar{\omega}} = -\frac{d_{33}}{m_{33}}\bar{\omega} + \frac{c}{m_{33}} + C_3(\bar{v}_x,\bar{v}_y)$$

(Underactuated equation) $(\bar{v}_x,\bar{\omega})$

$$\dot{\bar{v}}_y = -\frac{d_{22}}{m_{22}}\bar{v}_y + C_3(\bar{v}_x,\bar{\omega})$$

$\rho(x(t),y(t))$

$(x(t),y(t))$

$\rho(x,y)$

X (m)

Y (m)
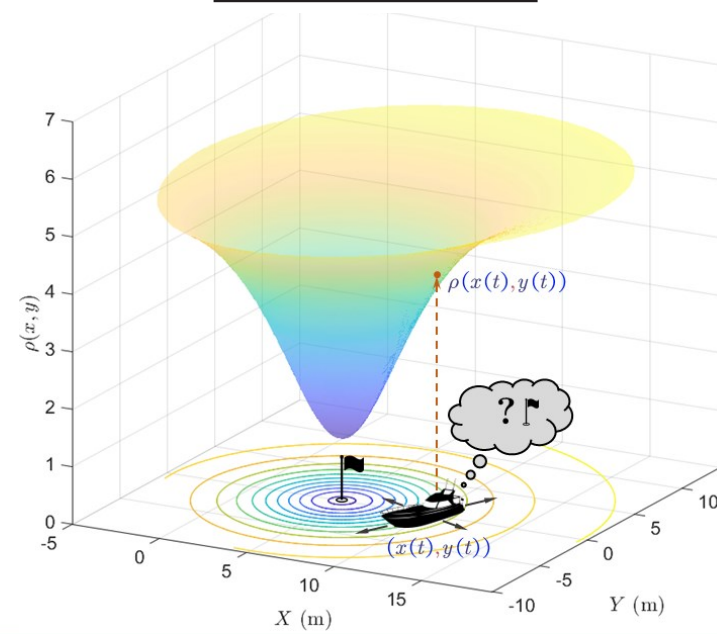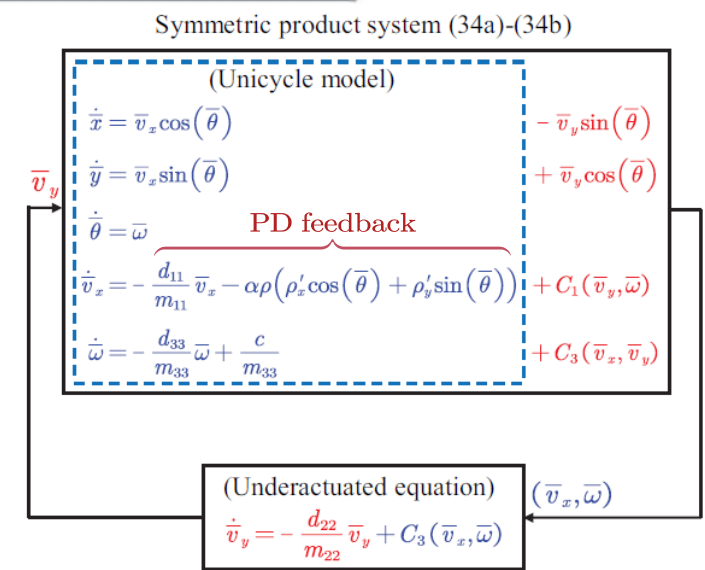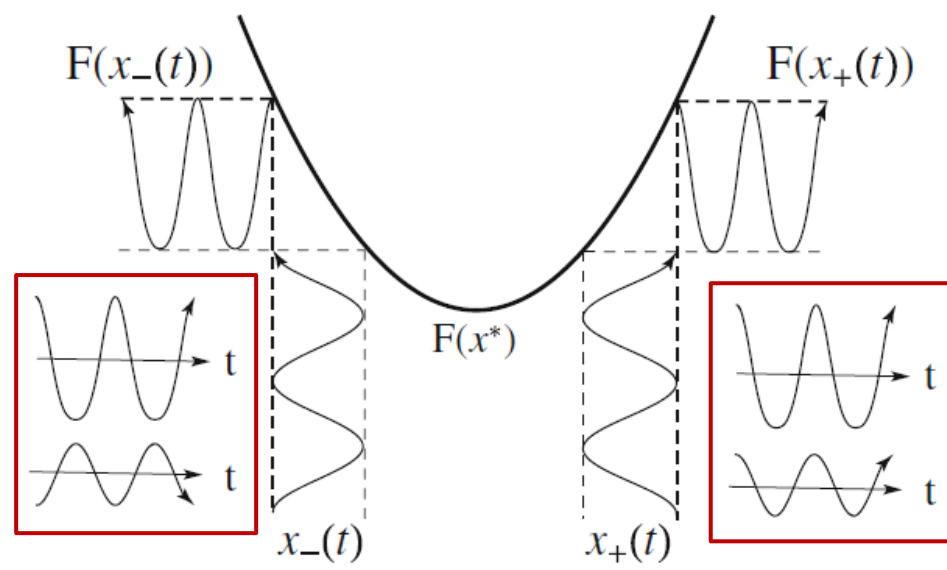
- Simulation Results: (Boat)
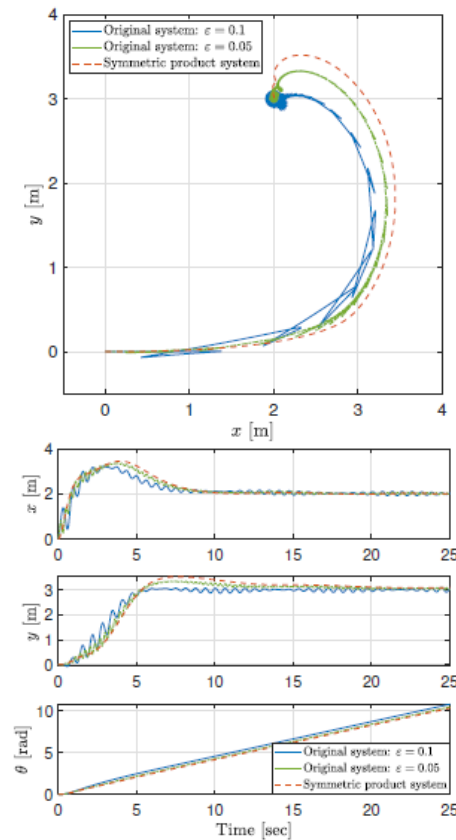
$(c = 1, k = 1)$
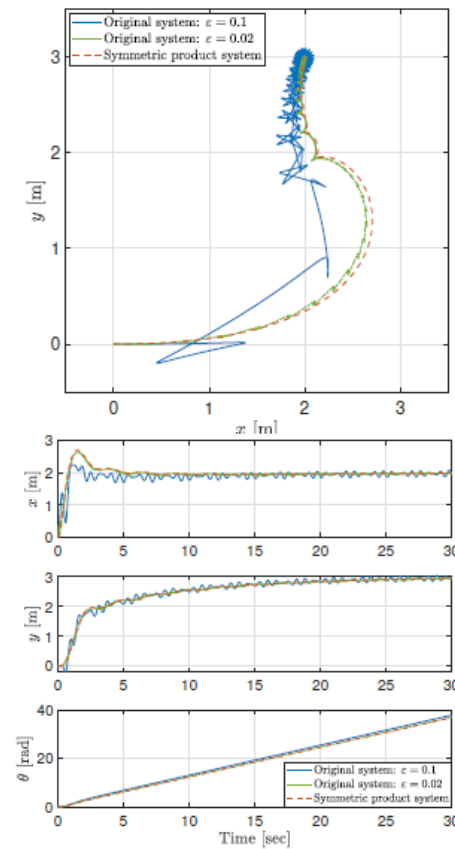
$(c = 3, k = 1)$
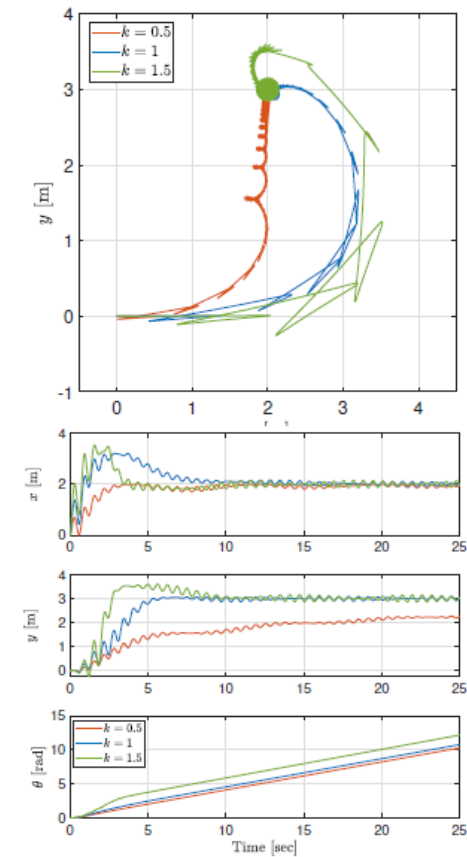
$(c = 1, \varepsilon = 0.1)$

- Experimental Results
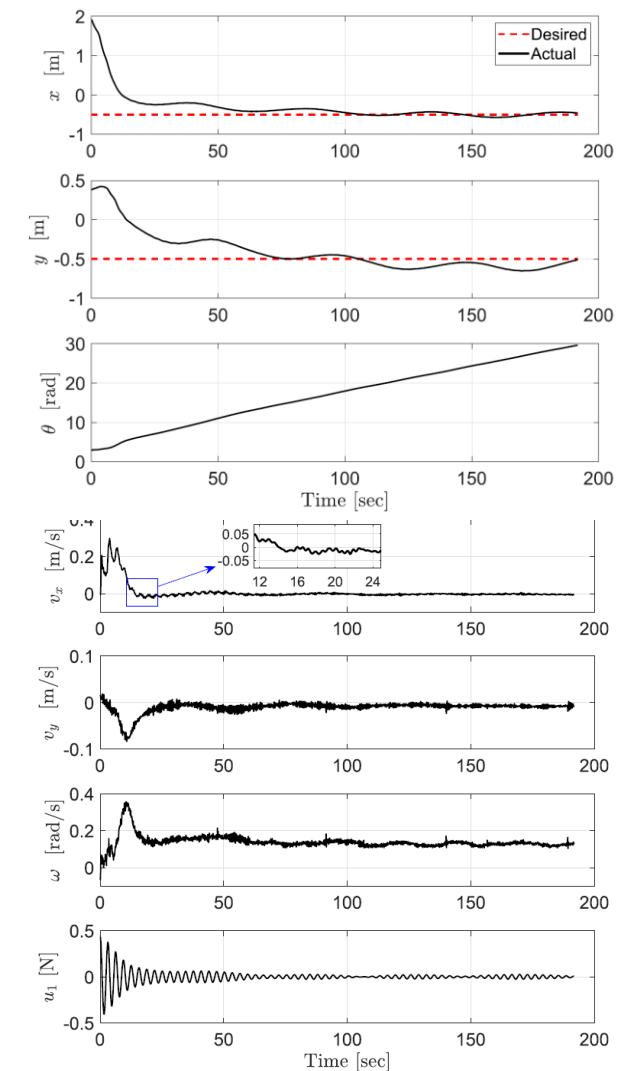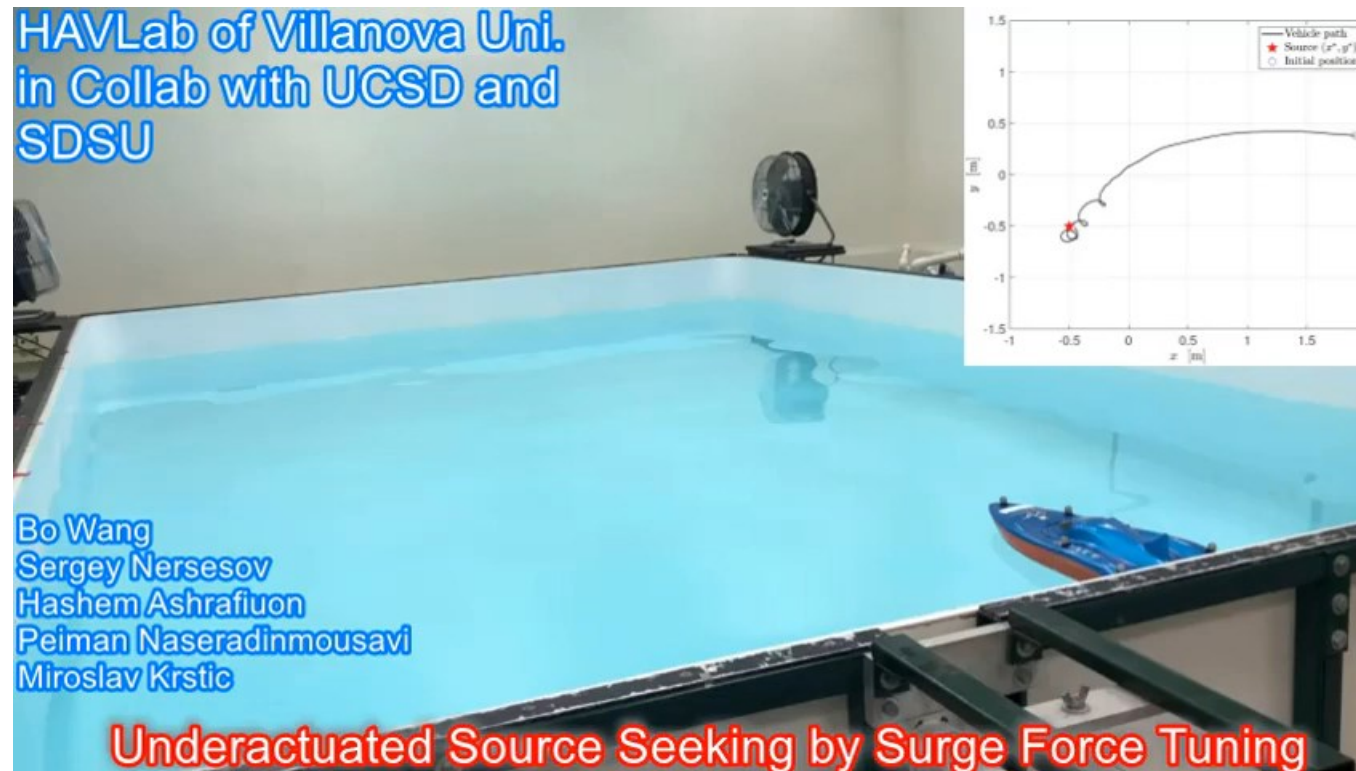
Fig: State trajectories and control input
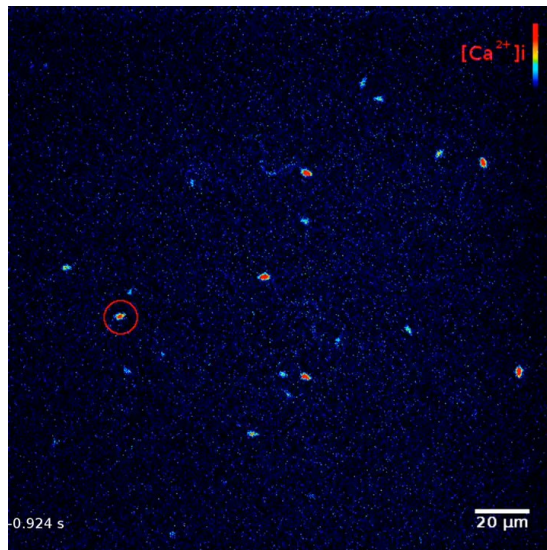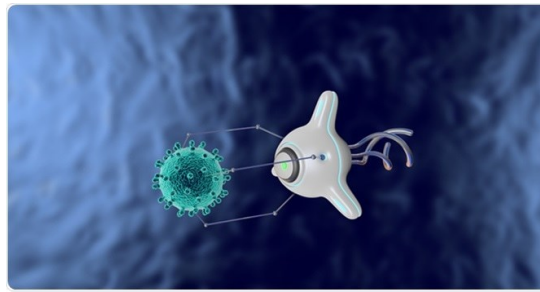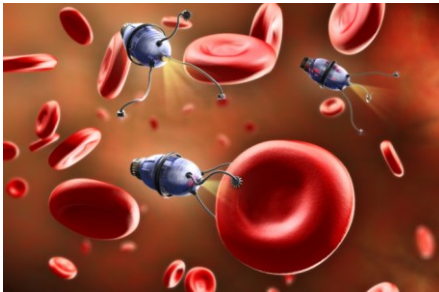
- From Force Tuning to Torque Tuning
  - [Wang, Krstic, (Semi-Global Nonholonomic Source Seeking by Torque Tuning, in prepare)]
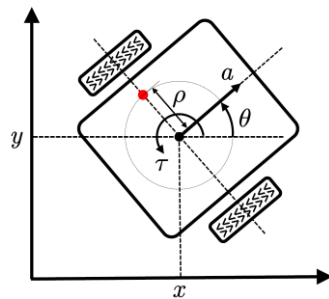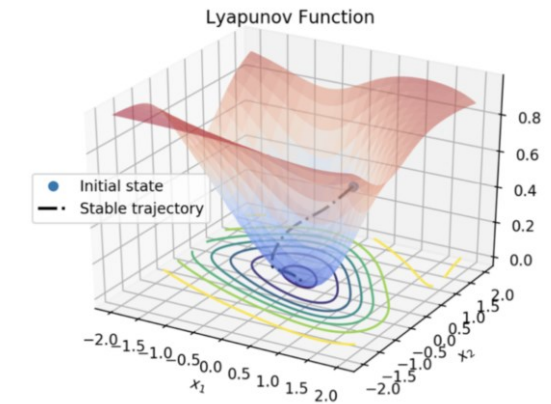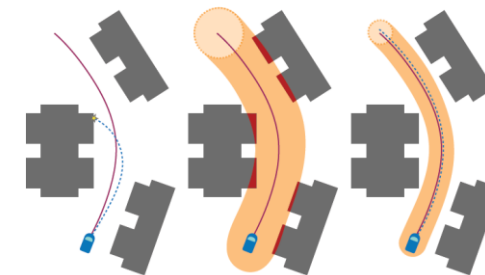
Constant Surge Force +

"High-freq. high-amp." Torque
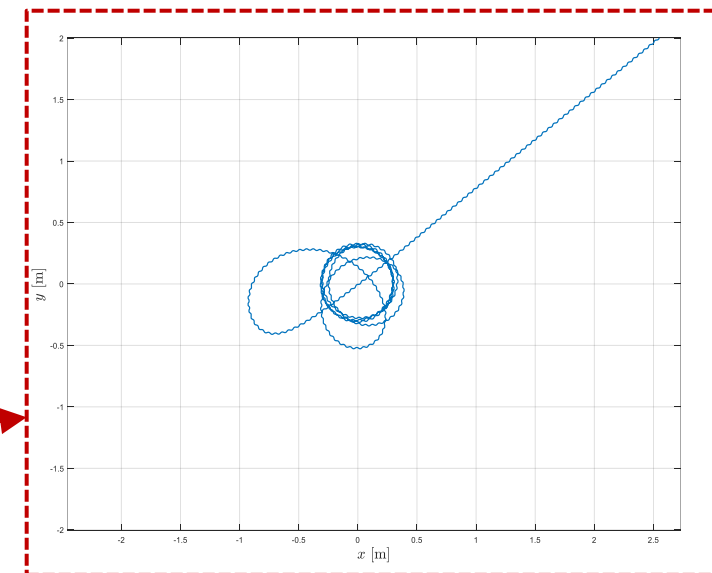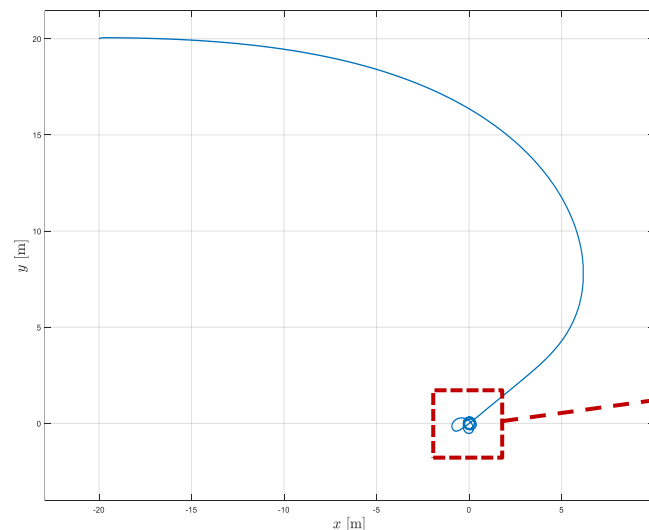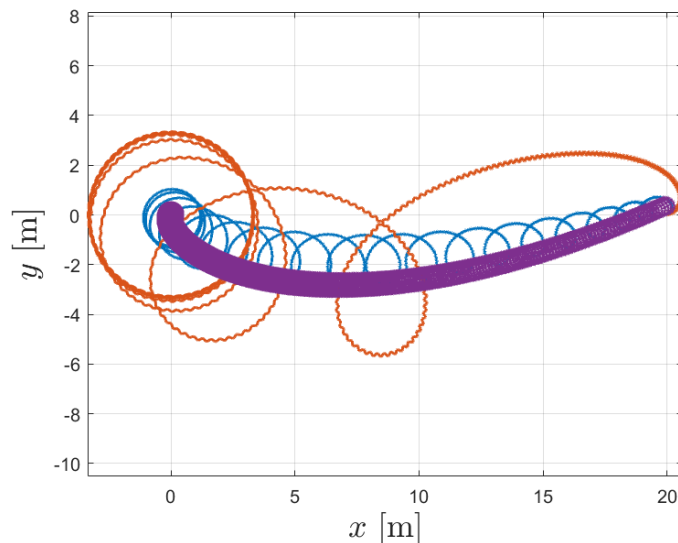
- **Medical Robotics**
  - **Micro/Nano Robotics** with <u>Source Seeking</u> ability
  (Tumor, cancel cells seeking and treatment)

- **Learning-Based Control**
  - **Data-driven, seeking the minimum of a "Lyapunov"**

- **Safety Critical Control**
  - Guarantee safety in complex environment (CBF)
  - Fixed/prescribed-time control

- Looking for outstanding **Ph.D. students** interested in Control Theory and Underactuated Systems

- **Collaboration is welcome!**

  - Email: bwang1@ccny.cuny.edu   /   bwang.ccny@gmail.com

  - Website: https://bwang-ccny.github.io

Thank you for the attention!