Relative Langlands duality

Bryan Wang Peng Jun

(Supervisor: Prof Gan Wee Teck)

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Executive summary

David Ben-Zvi, Yiannis Sakellaridis, Akshay Venkatesh: a proposed **duality** of Hamiltonian varieties whose quantizations lead to instances of the relative Langlands program in duality with each other.

Central question: Find more examples of this duality to understand it better!

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Central question: Find more examples of this duality to understand it better!

But before that: what does the above mean..?

Lightning introduction to the Langlands program

G: reductive group ($G = GL_n, Sp_{2n}, SO_n, \dots$)
$G^\vee$: Langlands dual group ($GL_n \leftrightarrow GL_n, Sp_{2n} \leftrightarrow SO_{2n+1}, SO_{2n} \leftrightarrow SO_{2n}$)

F: 'global' number field ($F = \mathbb{Q}$)
and corresponding 'local' fields F_v ($F_v = \mathbb{Q}_p, \mathbb{R}$)
Assemble all the local fields together to form the 'adele ring' $\mathbb{A}_F$

$$\left\{ \begin{matrix} \text{'Galois representations'} \\ \text{“}Gal(\bar{F}/F) \to G^\vee(\mathbb{C})\text{”} \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \text{'Automorphic forms'} \\ \text{Functions on } G(\mathbb{A}_F) \\ \text{satisfying certain properties...} \\ \text{(e.g. smoothness...)} \end{matrix} \right\}$$

A story of refinements...

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$\downarrow$ local-global principle

$$\left\{\begin{matrix}\text{'(Local) Langlands parameters'} \\ \text{“}Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to G^\vee(\mathbb{C})\text{”}\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{'Representations of } G(\mathbb{Q}_p)\text{'} \\ \text{satisfying certain properties} \\ \text{(e.g. smoothness...)}\end{matrix}\right\}$$

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$$\left\{\begin{matrix}\text{`Langlands parameters'} \\ \rho\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{`Automorphic representations' of } G(\mathbb{A}_F) \\ \pi = \otimes_v \pi_v\end{matrix}\right\}$$

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compatibly with L-functions:
(classical 'simplest' example: Riemann zeta function $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$):

$$\rho \quad \xleftrightarrow{\sim} \quad \pi$$

$$L(\rho, s) = L(\pi, s) \text{``}=\text{''} \prod_v L(\pi_v, s)$$

("Artin L-functions") ("Automorphic/Langlands L-functions")

L-functions and integrals

The Riemann zeta function $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ itself has nothing much to do with today's talk, but it does tell us a few important things:

- Studying L-functions is hard, and their expressions as *Euler products* $\prod_p$ are slightly mysterious (does the expression even make sense..?)
- Riemann was able to show non-trivial properties of ζ by expressing it as a certain integral.
- John Tate, in his landmark 1950 thesis, recast this in the language of adeles and automorphic forms.

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To cut a long story short:

Slogan: L-functions are best studied by relating them to certain integrals of automorphic forms.

Periods of automorphic forms

The prototypical example of such an integral is the H-**period** integral: for a subgroup H of G, take an automorphic form ϕ in an automorphic representation π, and integrate it against H:

$$\pi \to \mathbb{C}$$
$$\phi \mapsto \int_H \phi$$

This period integral should be related to certain (values of) L-functions.

But before we can talk about its values, we should first ask the simplest question: when is it nonzero?

Periods and representation theory

We can think of the period integral as a H-equivariant functional on π, i.e. an element of

$$\mathrm{Hom}_{H(\mathbb{A}_F)}(\pi, \mathbb{C}) \cong \otimes_v \mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C})$$

In order for this period functional to be non-zero, we must first have $\mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C})$ non-zero!

Such a local representation π_v (of $G(F_v)$) is called H-**distinguished**.

Relative Langlands program (Sakellaridis-Venkatesh)

The central problem in the *relative* ("relative to a subgroup H of G") Langlands program is therefore:

- (Local) When is π_v H-distinguished?
- (Global) When is the H-period of π nonzero? (We can also call such π H-distinguished.)
  If it is nonzero, what is its value - in terms of certain (values of) L-functions?

The answers to the above problems should be given in terms of Langlands parameters.

In order to understand how the answers can possibly be given in terms of Langlands parameters, we need to return to our lightning introduction to discuss the most important part of the Langlands program that we have earlier left out: **functoriality**.

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For two reductive groups G, G' and a (L-)homomorphism $\tau : G^\vee \to G'^\vee$ between their Langlands dual groups,

$$\begin{array}{ccc}
\left\{ \begin{matrix} \text{`Langlands parameters'} \\ \rho : Gal_F \to G^\vee \end{matrix} \right\} & \longleftrightarrow & \{ \text{ Automorphic representations of } G \ \} \\
\Big\downarrow {\scriptstyle \tau : G^\vee \to G'^\vee} & & \Big\downarrow \text{“Functorial lift”} \\
\left\{ \begin{matrix} \text{`Langlands parameters'} \\ \tau \circ \rho : Gal_F \to G'^\vee \end{matrix} \right\} & \longleftrightarrow & \{ \text{ Automorphic representations of } G' \ \}
\end{array}$$

A relative Langlands problem and its answer

An answer to the relative Langlands program (*Sakellaridis-Venkatesh conjecture*): given $X = (G, H)$, the H-distinguished representations are precisely those Langlands functorial lifted via a homomorphism $X^\vee \to G^\vee$, for some dual group $X^\vee$.

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Question: where is the duality in the relative Langlands program?

A relative Langlands problem and its answer

Ben-Zvi-Sakellaridis-Venkatesh's answer: they propose a **duality**

$$\left\{\begin{matrix}\text{Hamiltonian } G\text{-variety} \\ M\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{Hamiltonian } G^\vee\text{-variety} \\ M^\vee\end{matrix}\right\}$$

such that:

$$\left\{\begin{matrix}\text{representations appearing} \\ \text{in a quantization of } M\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{Langlands parameters} \\ \text{with fixed points on } M^\vee\end{matrix}\right\}$$

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This is the **relative Langlands duality**, and our aim is to study more examples of this duality to understand it better.

Hamiltonian varieties and quantization

Unanswered question: what are Hamiltonian G-varieties and their quantizations?

The 'physics background':

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Hamiltonian mechanics:}\\ \text{Classical system}\\ \text{(with } G\text{-symmetries)}\end{array}\right\} \arrow[r, leftrightarrow, "\text{modelled by}"] \arrow[d, "\text{Quantization}"] & \left\{\begin{array}{c}\text{Symplectic manifold } M\\ \text{(with Hamiltonian } G\text{-action)}\\ \text{(comes with moment map}\\ \mu : M \to \mathfrak{g}^*)\end{array}\right\} \arrow[d, "\text{Quantization}"] \\
\{\text{Quantum system}\} \arrow[r, leftrightarrow, "\text{modelled by}"] & \left\{\begin{array}{c}\text{Hilbert space } V\\ \text{with } G\text{-action}\end{array}\right\}
\end{tikzcd}$$

General principle: quantization takes a Hamiltonian G-variety M and produces a unitary G-representation on a Hilbert space V.

Hamiltonian varieties and quantization

How does this relate to the relative Langlands program?

The prototypical example of Hamiltonian G-varieties are cotangent bundles.

Given (G, H)

$\rightsquigarrow$ (*'spherical'*) variety $H\backslash G$

$\rightsquigarrow$ cotangent bundle $M = T^*(H\backslash G)$

$\rightsquigarrow$ quantizes to the G-representation $L^2(H\backslash G)$

$\rightsquigarrow$ decomposes into precisely the H-distinguished representations

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Aim: study more examples of this duality to understand it better!

Work in progress. . . . . .

Thank you!

Hamiltonian varieties and quantization

Now we have answered the 'what is'. How about the 'why'?

- Ben-Zvi-Sakellaridis-Venkatesh take the physics approach:
  - Kapustin-Witten (2006): relate Langlands duality to electric-magnetic duality in topological quantum field theories (TQFTs)
  - Hamiltonian G-varieties $\leftrightsquigarrow$ 'boundary conditions' of these TQFTs.

Hamiltonian varieties and quantization

- There is a more relevant perspective which also relates to a key tool used to address some of the earlier questions in the Langlands program.
  - Recall the H-distinguished representations should be those functorial lifted from one group G' to another group G''.

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  - One answer: **theta correspondence**.

Hamiltonian varieties and quantization

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  - Recall the H-distinguished representations should be those functorial lifted from one group G' to another group G''.
  - Question: How can we realise such a lifting explicitly?
  - One answer: **theta correspondence**.
  - Theta correspondence transfers representations of G' to G'' in a concrete and purely representation-theoretic way: by viewing them as part of a larger representation of $G' \times G''$, called the Weil representation.
  - The Weil representation arises as quantization of a *symplectic vector space* (with symplectic group acting on it in the obvious way)!

Relative Langlands duality

$$\left\{\begin{matrix}\text{Hamiltonian } G\text{-variety} \\ {\color{red}(\text{e.g. } M = T^*(H\backslash G))} \\ {\color{red}(\text{e.g. } M = \text{std} \curvearrowleft Sp_{2n})}\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{Hamiltonian } G^\vee\text{-variety} \\ M^\vee\end{matrix}\right\}$$

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Question: if theta correspondence both (a) fits into this framework and (b) is a tool for establishing instances of this framework, can we use this relationship to produce and study more instances of this duality?

Answer: work in progress. . . . . .