

Part III Local Fields

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Definition (Absolute value). Let K be a field. An **absolute value** on K is a function $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ s.t.

- i. $|x| = 0 \iff x = 0$
- ii. $|xy| = |x| |y| \quad \forall x, y \in K$
- iii. $|x + y| \leq |x| + |y|$

Definition (Valued field). A **valued field** is a field with an absolute value.

Definition (Equivalence of absolute values). Let K be a field and let $|\cdot|, |\cdot|'$ be absolute values on K . We say that $|\cdot|$ and $|\cdot|'$ are **equivalent** if the associated metrics induce the same topology.

Definition (Non-archimedean absolute value). An absolute value $|\cdot|$ on a field K is called **non-archimedean** if $|x + y| \leq \max(|x|, |y|)$ (the **strong triangle inequality**).

Metrics s.t. $d(x, z) \leq \max(d(x, y), d(y, z))$ are called **ultrametrics**.

Assumption: unless otherwise mentioned, all absolute values will be non-archimedean. These metrics are weird!

Proposition 1. Let K be a valued field. Then $\mathcal{O} = \{x \mid |x| \leq 1\}$ is an open subring of K , called the **valuation ring** of K . $\forall r \in (0, 1]$, $\{x \mid |x| < r\}$ and $\{x \mid |x| \leq r\}$ are open ideals of \mathcal{O} .

Moreover, $\mathcal{O}^\times = \{x \mid |x| = 1\}$.

Proposition 2. Let K be a valued field.

- i. Let (x_n) be a sequence in K . If $x_n - x_{n+1} \rightarrow 0$ then (x_n) is Cauchy

Assume that K is complete

- ii. Let (x_n) be a sequence in K . If $x_n - x_{n+1} \rightarrow 0$ then (x_n) converges
- iii. Let $\sum_{n=0}^{\infty} y_n$ be a series in K . If $y_n \rightarrow 0$, then $\sum_{n=0}^{\infty} y_n$ converges