Part III Category Theory

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Contents

1 Definitions and Examples

Definition 1.1 (Category)**.** A category $\mathcal{C}$ consists of

    a. a collection $\operatorname{ob}\mathcal{C}$ of **objects** A, B, C, ...

    b. a collection $\operatorname{mor}\mathcal{C}$ of **morphisms** f, g, h, ...

    c. two operations dom, cod from morphisms to objects. We write $f : A \to B$ or $A \xrightarrow{f} B$ to mean 'f is a morphism and $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$'

    d. an operation assigning to each object A a morphism $1_A : A \to A$

    e. a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff$ $\operatorname{dom} g = \operatorname{cod} f$, and then $gf : \operatorname{dom} f \to \operatorname{cod} g$

satisfying

    f. $f1_A = f$ and $1_B f = f \ \forall f : A \to B$

g. $h(fg) = (hg)f$ whenever gf and hg are defined

Definition 1.2 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $\mathcal{C} \to \mathcal{D}$ consists of

a. a mapping $A \to FA$ from ob $\mathcal{C}$ to ob $\mathcal{D}$

b. a mapping $f \to Ff$ from mor $\mathcal{C}$ to mor $\mathcal{D}$

satisfying dom $Ff = F$dom f, cod $Ff = F$cod f for all f, $F(1_A) = 1_{FA}$ for all A, and $F(gf) = (Fg)(Ff)$ whenever gf is defined.

Definition 1.3. By a **contravariant functor** $\mathcal{C} \to \mathcal{D}$ we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or equivalently $\mathcal{C}^{op} \to \mathcal{D}$). A functor $\mathcal{C} \to \mathcal{D}$ is sometimes said to be **covariant**.

Definition 1.4 (Natural transformation)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two functors. A **natural transformation** $\alpha : F \to G$ assigns to each $A \in \text{ob}\,\mathcal{C}$ a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$, such that

$$
\begin{array}{ccc}
FA & \xrightarrow{\ Ff\ } & FB \\
\downarrow{\alpha_A} & & \downarrow{\alpha_B} \\
GA & \xrightarrow{\ Gf\ } & GB
\end{array}
$$

commutes.

We can compose natural transformations: given $\alpha : F \to G$ and $\beta : G \to H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A-component of a natural transformation $\beta\alpha : F \to H$.

Definition 1.5. Given categories $\mathcal{C}, \mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category of all functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

Lemma 1.6. *Given $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \to G$, α is an isomorphism in $[\mathcal{C}, \mathcal{D}] \iff$ each α_A is an isomorphism in $\mathcal{D}$.*

Definition 1.7 (Faithful and full)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

a. We say that F is **faithful** if, given $f, g \in \text{mor}\,\mathcal{C}$, the equations dom $f =$ dom g, cod $f = $ cod g and $Ff = Fg$ imply $f = g$.

b. F is **full** if, given any $g : FA \to FB$ in $\mathcal{D}$, there exists $f : A \to B$ in $\mathcal{C}$ with $Ff = g$.

c. We say a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is **full** if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition 1.8 (Equivalence of categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to mean that $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

We say a property P of categories is **categorical** if whenever $\mathcal{C}$ has P and $\mathcal{C} \simeq \mathcal{D}$ then $\mathcal{D}$ has P.

For example, being a groupoid is a categorical property, but being a group is not.

Definition 1.9 (Slice category)**.** Given an object B of a category $\mathcal{C}$, define the **slice category** $\mathcal{C}/B$ to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \to (A' \xrightarrow{f'} B)$ are morphisms $h : A \to A'$ making

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;h\;\;} & A' \\
& \searrow{\scriptstyle f} \quad \swarrow{\scriptstyle f'} & \\
& B &
\end{array}
$$

commute.

Lemma 1.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and **essentially surjective**, i.e. for every $B \in \mathrm{ob}\,\mathcal{D}$, there exists $A \in \mathrm{ob}\,\mathcal{C}$ s.t. $FA \cong B$.*

Definition 1.11.   a. A **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ containing exactly one object from each isomorphism class of objects of $\mathcal{C}$.

   b. We say $\mathcal{C}$ is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism f in $\mathcal{C}$ satisfies $\mathrm{dom}\,f = \mathrm{cod}\,f$.

For example, $\mathbf{Mat}_K$ is skeletal. The full subgategory of standard vector spaces K^n is a skeleton of $\mathbf{fd\ Mod}_K$.

Remark 1.12. The following statements are each equivalent to the Axiom of Choice:

   1. Every small category has a skeleton

   2. Any small category is equivalent to each of its skeletons

   3. Any two skeletons of a given small category are isomorphic

Definition 1.13. Let $f : A \to B$ be a morphism in a category $\mathcal{C}$.

   a. f is a **monomorphism** if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightarrowtail B$ if f is monic.

b. Dually, f is an **epimorphism** if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.

c. $\mathcal{C}$ is a **balanced** category if every $f \in \mathrm{mor}\,\mathcal{C}$ which is both monic and epic is an isomorphism.

2 The Yoneda Lemma

Definition 2.1. A category $\mathcal{C}$ is **locally small** if, for any two objects A, B of $\mathcal{C}$, the morphism $A \to B$ are parametrised by a set $\mathcal{C}(A, B)$.

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$: if $g : B \to B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \to \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \to B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \to \mathbf{Set}$.

Lemma 2.2 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, $A \in \mathrm{ob}\,\mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Set}$. Then*

  i. There is a bijection between natural transformations $\mathcal{C}(A, -) \to F$ and elements of FA.

  ii. Moreover, this bijection is natural in both A and F.

Proof. Bijection: given $\alpha : \mathcal{C}(A, -) \to F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.

Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \to F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$ is natural: given $g : B \to C$, we have

$$
\begin{aligned}
\Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\
&= F(gf)(x) \\
&= (Fg)(Ff)(x) \\
&= (Fg)\Psi(x)_B(f)
\end{aligned}
$$

$\Phi\Psi(x) = x$ since $F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$ since, for any $f : A \to B$,

$$
\begin{aligned}
\Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\
&= Ff(\alpha_A(1_A)) \\
&= \alpha_B(\mathcal{C}(A, f)(1_A)) \\
&= \alpha_B(f)
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Corollary 2.3. *The mapping $A \to \mathcal{C}(A, -)$ is a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C}, \textbf{Set}]$.*

Proof. Given two objects A, B, 2.2(i) gives us a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ (by taking $F : C \mapsto \mathcal{C}(B, C)$). We need to show this is functorial, but given $f \in \mathcal{C}(B, A)$, $\Psi(F)_A$ sends 1_A to $\mathcal{C}(B, f)(1_A) = f$, so it's the natural transformation $g \mapsto gf$.

Hence, given $e : C \to B$, $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$ $\qquad\square$

We call this functor the **Yoneda embedding**. Hence any locally small category $\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{C}^{op}, \textbf{Set}]$.

Definition 2.4. A functor $\mathcal{C} \to \textbf{Set}$ is **representable** if it's isomorphic to $\mathcal{C}(A, -)$ for some A.

A **representation** of $F : \mathcal{C} \to \textbf{Set}$ is a pair (A, x) where $A \in \mathrm{ob}\,\mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, -) \to F$ is an isomorphism. We also call x a **universal element** of F.

Corollary 2.5 ('Representations are unique up to unique isomorphism')**.** *If (A, x) and (B, y) are both representations of $F : \mathcal{C} \to \textbf{Set}$, then there's a unique isomorphism $f : A \to B$ s.t $Ff(x) = y$.*

Definition 2.6 (Product and coproduct)**.** Given two objects A, B of a locally small category $\mathcal{C}$, we define their **product** to be a representation of the functor

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{op} \to \textbf{Set}$$

i.e. an object $A \times B$ equipped with morphisms $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ s.t. given any pair $(f : C \to A, g : C \to B)$, there exists a unique $h : C \to A \times B$ s.t. $\pi_1 h = f$ and $\pi_2 h = g$.

More generally, we can define the product $\prod_{i \in I} A_i$ of a family $\{A_i \mid i \in I\}$ of objects, or the product of the empty family, i.e. a **terminal object** 1 s.t. for every A there's a unique $A \to 1$.

Dualizing, we get the notion of **coproduct** or **sum**.

Definition 2.7 (Equaliser and coequaliser)**.** Given a parallel pair $f, g : A \rightrightarrows B$ in a locally small category $\mathcal{C}$, the assignment $C \mapsto FC = \{h : C \to A \mid fh = gh\}$ is a subfunctor F of $\mathcal{C}(-, A)$. A representation of F is called an **equaliser** of (f, g).

In elementary terms, it's an object E equipped with $e : E \to A$ s.t. $fe = ge$, s.t. any h with $fh = gh$ factors uniquely as $h = ek$

Dually, we have the notion of **coequaliser**, i.e. a morphism $q : B \to Q$ satisfying $qf = qg$, and universal among such.

Definition 2.8.   a. We say a monomorphism is **regular** if it occurs as an equaliser (dually, regular epimorphism).

b. We say $f : A \to B$ is a **split monomorphism** if there exists $g : B \to A$ with $gf = 1_A$.

Every split monomorphism is regular: if $gf = 1_A$, f is an equaliser of $(1_B, fg)$ [see sheet 1, q2].

Definition 2.9. Let $\mathcal{C}$ be a (locally small) category, $\mathcal{G}$ a collection of objects of $\mathcal{C}$.

a. Say $\mathcal{G}$ is a **separating family** if the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ are jointly faithful, i.e. if given $f, g : A \rightrightarrows B$ with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \to A$ with $fh \neq gh$.

b. Say $\mathcal{G}$ is a **detecting family** if the $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ jointly reflect isomorphisms, i.e. if given $f : A \to B$ s.t. every $g : G \to B$ with $G \in \mathcal{G}$ factors uniquely through f, f is an isomorphism.

Lemma 2.10.   *i. If $\mathcal{C}$ is balanced, then any separating family is detecting*

ii. If $\mathcal{C}$ has equalisers, then every detecting family is separating

Definition 2.11. An object P is **projective** if $\mathcal{C}(P, -)$ preserves epimorphisms, i.e. if given

$$
\begin{array}{c}
P \\
\downarrow f \\
A \xrightarrow{\ e\ } B
\end{array}
$$

there exists $g : P \to A$ with $eg = f$.

Dually, P is **injective** in $\mathcal{C}$ if it's projective in $\mathcal{C}^{op}$.

If P satisfies this property $\forall e$ in some class $\mathcal{E}$ of epimorphisms, we call it $\mathcal{E}$-projective.

Corollary 2.12. *Representable functors are (pointwise) projective in $[\mathcal{C}, \boldsymbol{Set}]$*

Proof. Given

$$
\begin{array}{c}
\mathcal{C}(A, -) \\
\downarrow \beta \\
F \xrightarrow{\ \alpha\ } G
\end{array}
$$

β corresponds to some $y \in GA$. α_A is surjective, so $\exists x \in FA$ with $\alpha_A(x) = y$. x corresponds to $\gamma : \mathcal{C}(A, -) \to F$ with $\alpha\gamma = \beta$. $\square$

3 Adjunctions

Definition 3.1 (D.M. Khan, 1958)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be two functors. An **adjunction** between F and G is a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and morphisms $A \to GB$ in $\mathcal{C}$, which is natural in A and B.

(If $\mathcal{C}$ and $\mathcal{D}$ are locally small, this says that $(A, B) \to \mathcal{D}(FA, B)$ and $(A, B) \to \mathcal{C}(A, GB)$ are naturally isomorphic functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$).

We say F is **left adjoint** to G, or G is **right adjoint** to F, and write $F \dashv G$.

Theorem 3.2. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Given $A \in \operatorname{ob} \mathcal{C}$, let $(A \downarrow G)$ be the category whose objects are pairs (B, f) with $B \in \operatorname{ob} \mathcal{D}$, $f : A \to GB$ and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ in $\mathcal{D}$ such that*

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & GB \\
 & {\scriptstyle f'} \searrow & \big\downarrow {\scriptstyle Gg} \\
 & & GB'
\end{array}
$$

commutes. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A.

Proof. First suppose G has a left adjoint F. Let $\eta_A : A \to GFA$ be the morphism corresponding to $1_{FA} : FA \to FA$. The pair (FA, η_A) is an object of $(A \downarrow G)$. We'll show it's initial.

Given $g : FA \to B$, the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$ must correspond to $FA \xrightarrow{1} FA \xrightarrow{g} B$ under the adjunction.

So, for any object (B, f) of $(A \downarrow G)$, the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$ is the morphism $FA \to B$ corresponding to f.

Conversely, suppose we're given an initial object (FA, η_A) of $(A \downarrow G)$ for each G. Given $f : A \to A'$, the composite $A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA'$ is an object of $(A \downarrow G)$, so there's a unique morphism $Ff : FA \to FA'$ making

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & GFA \\
{\scriptstyle f}\big\downarrow & & \big\downarrow {\scriptstyle GFf} \\
A' & \xrightarrow{\eta_{a'}} & GFA'
\end{array}
$$

commute.

$f \mapsto Ff$ is functorial: given $f' : A' \to A''$, then $(Ff')(Ff)$ and $F(f'f)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''} f'f)$ in $(A \downarrow G)$, so they're equal.

Finally, given $f : A \to GB$, the morphism $g : FA \to B$ corresponding to it is the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$.

The naturality of this bijection is given by naturality of η, and naturality in B is immediate. $\qquad\square$

Corollary 3.3. *If F, F' are both left-adjoint to G, then there's a canonical natural isomorphism $F \to F'$.*

Proof. For each A, (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \to (F'A, \eta'_A)$.

α is natural: given $f : A \to A'$, $\alpha_{A'}f$ and $(Ff)\alpha_A$ are both morphisms $(FA, \eta_A) \to (F'A', \eta'_{A'}f)$ in $(A \downarrow G)$. So they're equal. $\square$

Lemma 3.4. *Given $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$, if $F \dashv G$ and $H \dashv K$ then $HF \dashv GK$.*

Proof. We have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

which are natural in A and C. $\square$

Corollary 3.5. *Given a commutative square* $\begin{array}{ccc} \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle H} \\ \mathcal{E} & \overset{K}{\longrightarrow} & \mathcal{F} \end{array}$ *of categories and functors, suppose all the functors in the diagram have left adjoints. Then the diagram* $\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{C} \end{array}$ *of left adjoints commutes up to natural isomorphism.*

Given $F \dashv G$, we have a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ defined as in 3.2. We call η the **unit** of the adjunction.

Dually, we have $\epsilon : FG \to 1_{\mathcal{D}}$, the **counit**. $\epsilon_B : FGB \to B$ corresponds to $1_{GB} : GB \to GB$.

Theorem 3.6. *Suppose we're given $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\epsilon : FG \to 1_{\mathcal{D}}$ such that*

$$\begin{array}{ccc} F & \overset{F\eta}{\longrightarrow} & FGF \\ & {\scriptstyle 1_F}\searrow & \downarrow{\scriptstyle \epsilon_F} \\ & & F \end{array} \qquad and \qquad \begin{array}{ccc} G & \overset{\eta_G}{\longrightarrow} & GFG \\ & {\scriptstyle 1_G}\searrow & \downarrow{\scriptstyle G\epsilon} \\ & & G \end{array}$$

*commute. (We say η and ϵ satisfy the **triangular identities**).*

Proof. Given $F \dashv G$, we define η and ϵ as already described. Since $\epsilon_{FA} : FGFA \to FA$ corresponds to 1_{GFA}, the composite $\epsilon_{FA}(F\eta_A)$ corresponds to $A \overset{\eta_A}{\longrightarrow} GFA \overset{1_{GFA}}{\longrightarrow} GFA$, so it must be 1_{FA}.

Similarly for the other identity.

Conversely, given η and ϵ satisfying the $\triangle^r$ identities, we map $f : A \to GB$ to the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$ and $g : FA \to B$ to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$.

We have

$$\Phi(\ A \xrightarrow{f} GB\ ) = \ FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$$

$$\Psi(\ FA \xrightarrow{g} B\ ) = \ A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$$

So

$$\Psi\Phi(f) = \ A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB$$

$$= \ A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB$$

$$= f$$

And dually $\Phi\Psi(g) = g$.

Naturality of Φ in A is immediate from its definition, and naturality in B follows from that of ϵ. $\square$

Lemma 3.7. *Suppose given* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *and natural isomorphisms* $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. *Then there exist natural isomorphisms* α', β' *which additionally satisfy the triangular identities. In particular* $(F \dashv G)$.

Proof. We define $\alpha' = \alpha$ and take β' to be the composite

$$FG \xrightarrow{\beta_{FG}^{-1}} FGFG \xrightarrow{F\alpha_G^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that, since $\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ {\scriptstyle \beta_{FG}}\downarrow & & \downarrow{\scriptstyle \beta} \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$ commutes and β is monic, we have $FG\beta = \beta_F G$.

Similarly, $GF\alpha = \alpha_{GF} : GF \to GFGF$.

Now

$$\beta_F' \circ F_{\alpha'} = \ F \xrightarrow{F\alpha} FGF \xrightarrow{\beta_{FGF}^{-1}} FGFGF \xrightarrow{F\alpha_{GF}^{-1}} FGF \xrightarrow{\beta_F} F$$

$$= \ F \xrightarrow{\beta_F^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{FGF\alpha^{-1}} FGF \xrightarrow{\beta_F} F$$

$$= 1_F$$

and

$$G\beta' \circ \alpha_G' = \ G \xrightarrow{\alpha_G} GFG \xrightarrow{GFG\beta^{-1}} GFGFG \xrightarrow{GF\alpha_G^{-1}} GFG \xrightarrow{G\beta} G$$

$$= \ G \xrightarrow{G\beta^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha_{GFG}^{-1}} GFG \xrightarrow{\beta_F} G$$

$$= 1_G$$

$\square$

Lemma 3.8. *Suppose* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *,* $(F \dashv G)$ *is an adjunction with counit* ϵ. *Then*

    i. ϵ *is (pointwise) epic* $\iff$ G *is faithful*

    ii. ϵ *is an isomorphism* $\iff$ G *is full and faithful*

Proof.   i. Given $g : B \to B'$, the morphism $Gg : GB \to GB'$ corresponds to

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$$

    So, for fixed B, composition with ϵ_B is injective on morphisms $B \to B'$ $\iff$ $(g \mapsto Gg)$ is injective on morphisms $B \to B'$.

    Hence G is faithful $\iff$ ϵ_B is epic $\forall B$.

  ii. Similarly, ϵ_B is 0 $\forall B$ $\implies$ G is bijective on morphisms with given domain and codomain, i.e. G is full and faithful.

    Conversely, if G is full and faithful, 1_{FGB} factors uniquely as
$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} FGB$, so ϵ_B is split monic. But it's epic by (i), hence an isomorphism.

$\square$

Definition 3.9.   i. A **reflection** is an adjunction satisfying the conditions of 3.8(ii).

  ii. A **reflective** subcategory of $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ for which the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint.

   Dually, **coreflection** and **coreflective** subcategory.

4   Limits

Definition 4.1.   a. Let J be a category (almost always small, often finite). A **diagram of shape J** in a category $\mathcal{C}$ is a functor $D : J \to \mathcal{C}$.

    E.g. if J is the finite category , a diagram of shape J is a

    commutative square. If J is the category , a diagram of shape

    J is a not-necessarily-commutative square.

The objects $D(j)$, $j \in \mathrm{ob}\, J$ are called **vertices** of D, and the morphisms $D(\alpha)$, $\alpha \in \mathrm{mor}\, J$ are called **edges** of D.

b. Let $D : J \to \mathcal{C}$ be a diagram in $\mathcal{C}$. A **cone over D** is a pair $(A, (\lambda_j \mid j \in \mathrm{ob}\, J))$ where $\lambda_j : A \to D(j)\ \forall j$, and

$$
\begin{array}{ccc}
 & A & \\
 \lambda_j \swarrow & & \searrow \lambda_{j'} \\
 D(j) & \xrightarrow{\ D(\alpha)\ } & D(j')
\end{array}
$$

commutes for each $\alpha : j \to j'$ in J.

A is called the **apex** of the cone, and the λ_j are its **legs**.

Equivalently, λ is a natural transformation $\triangle A \to D$, where $\triangle A$ is the **constant diagram** with all vertices A and all edges 1_A.

A **morphism** $f : (A, (\lambda_j)) \to (B, (\mu_j))$ of cones over D is a morphism $f : A \to B$ s.t.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
 \lambda_j \searrow & & \swarrow \mu_j \\
 & D(j) &
\end{array}
$$

commutes for each j. We have a category **Cone**(D) of cones over D.

Note that $A \mapsto \triangle A$ is a functor $\mathcal{C} \to [J, \mathcal{C}]$ and **Cone**(D) is in fact the category $(\triangle \downarrow D)$.

A cocone over $D : J \to \mathcal{C}$ is a cone over $D : J^{op} \to \mathcal{C}^{op}$. We write **Cocone**$(D)$ for the category of cocones over D.

Definition 4.2.   i. A **limit** (resp. **colimit**) for a diagram $D : J \to \mathcal{C}$ is a terminal object of **Cone**(D) (respectively an initial object of **Cocone**(D)).

ii. We say $\mathcal{C}$ has limits (resp. colimits) of shape J if $\triangle : \mathcal{C} \to [J, \mathcal{C}]$ has a right (resp. left) adjoint.

(This is equivalent to making a choice of limit (resp. colimit) for every diagram of shape J).

Definition 4.3 (Pullback)**.** Let J be
$$
\begin{array}{ccc}
 & & \cdot \\
 & & \downarrow \\
 \cdot & \longrightarrow & \cdot
\end{array}
$$
. A diagram of shape J looks

like
$$
\begin{array}{ccc}
A & & \\
\downarrow f & & \\
B & \xrightarrow{\ g\ } & C
\end{array}
$$
. A cone over it consists of
$$
\begin{array}{ccc}
D & \xrightarrow{\ h\ } & A \\
\downarrow k & \searrow l & \\
C & & B
\end{array}
$$
satisfying $fh = l = gk$. Equivalently, it's a pair
$$
\begin{array}{ccc}
D & \xrightarrow{\ h\ } & A \\
\downarrow k & & \\
C & &
\end{array}
$$
completing the diagram to a commutative square.

A universal such pair is called a **pullback** (or **fibre product**); in **Set** it can be defined as $\{(a, b) \in A \times B \mid f(a) = g(b)\}$. A colimit of shape J^{op} is called a **pushout**.

Theorem 4.4. *Let $\mathcal{C}$ be a category.*

    i. If $\mathcal{C}$ has equalisers and all finite (resp. all small) products, then $\mathcal{C}$ has all finite (resp. all small) limits.

    ii. If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.

Proof.   i. Given $D : J \to \mathcal{C}$, first form the products

$$P = \prod_{j \in \mathrm{ob}\, J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \mathrm{mor}\, J} D(\mathrm{cod}\, \alpha)$$

Define $\;P \overset{f}{\underset{g}{\rightrightarrows}} Q\;$ by $\pi_\alpha f = \pi_{\mathrm{cod}\,\alpha} : P \to D(\mathrm{cod}\, \alpha)$ and $\pi_\alpha g = D(\alpha) \circ \pi_{\mathrm{dom}\,\alpha} : P \to D(\mathrm{dom}\, \alpha) \to D(\mathrm{cod}\, \alpha)$, and let $e : E \to P$ be the equaliser of (f, g).

Claim $(E, (\pi_j e \mid j \in \mathrm{ob}\, J))$ is a limit cone for D. It is a cone since, for any $\alpha : j \to j'$, $D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e$.

Given any cone $(C, (\lambda_j \mid j \in \mathrm{ob}\, J))$, the λ_j define a unique $\lambda : C \to P$, and $f\lambda = g\lambda$ since $\pi_\alpha f \lambda = \pi_\alpha g \lambda$ $\forall \alpha$. So λ factors uniquely through e.

ii. Let 1 be a terminal object of $\mathcal{C}$. For any pair of objects (A, B) the pullback

of $\begin{matrix} A \\ \downarrow \\ B \longrightarrow 1 \end{matrix}$ has the universal property of a product $A \times B$, so $\mathcal{C}$

has binary products. Then we can define any finite product $\prod_{i=1}^n A_i$ as $(((A_1 \times A_2) \times A_3) \times \dots) \times A_n$.

So we need to show $\mathcal{C}$ has equalisers. Given $\;A \overset{f}{\underset{g}{\rightrightarrows}} B\;$, consider the

pullback of $\begin{matrix} & & B \\ & & \downarrow {\scriptstyle (1_A, f)} \\ A & \overset{(1_A, g)}{\longrightarrow} & A \times B \end{matrix}$.

It consists of $\begin{matrix} P & \overset{h}{\longrightarrow} & B \\ \downarrow {\scriptstyle k} & & \\ A & & \end{matrix}$ satisfying $1_A h = 1_A k$ and $fh = gk$, and uni-

versal among such.

But this forces $h = k$, and h has the universal property of an equaliser for (f, g). So by (i), $\mathcal{C}$ has all finite limits.

$\square$

Definition 4.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

a. We say F **preserves** limits of shape J if, given $D : J \to \mathcal{C}$ and a limit cone $(L, (\lambda_j : j \in \mathrm{ob}\, J))$ for D, the cone $(FL, (F\lambda_j : j \in \mathrm{ob}\, J))$ is a limit for $FD : J \to \mathcal{D}$.

b. We say F **reflects** limits of shape J if, given $D : J \to \mathcal{C}$ and a cone $(L, (\lambda_j))$ such that $(FL, (F\lambda_j))$ is a limit for FD, then $(L, (\lambda_j))$ is a limit for D.

c. We say F **creates** limits of shape J if, given $D : J \to \mathcal{C}$ and a limit $(M, (\mu_j))$ for FD, there exists a cone (L, λ_j) over D whose image is isomorphic to $(M, (\mu_j))$, and any such cone is a limit for D.

Lemma 4.6. *Suppose $\mathcal{D}$ has limits of shape J. Then $[\mathcal{C}, \mathcal{D}]$ has limits of shape J, and they're constructed pointwise (i.e. the forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\mathrm{ob}\, \mathcal{C}}$ creates them).*

Proof. Consider a functor $D : J \times \mathcal{C} \to \mathcal{D}$. For each $A \in \mathrm{ob}\, \mathcal{C}$, let $(LA, (\lambda_{j,A} : LA \to D(j, A) \mid j \in \mathrm{ob}\, J))$ be a limit for the diagram $D(-, A) : J \to \mathcal{D}$.

Given any $f : A \to B$ in $\mathcal{C}$, the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, so they induce a unique $Lf : LA \to LB$ such that

$$
\begin{array}{ccc}
LA & \xrightarrow{Lf} & LB \\
\downarrow{\scriptstyle \lambda_{j,A}} & & \downarrow{\scriptstyle \lambda_{j,B}} \\
D(j, A) & \xrightarrow{D(j,f)} & D(j, B)
\end{array}
$$

commutes for all j. Uniqueness assures $L(gf) = L(g)L(f)$, so L is a functor $\mathcal{C} \to \mathcal{D}$, and the $\lambda_{j,-}$ are natural transformations $L \to D(j, -)$.

Suppose we're given any cone over D in $[\mathcal{C}, \mathcal{D}]$ with apex M and legs $\mu_j : M \to D(j, -)$. Then $(MA, (\mu_{j,A} : MA \to D(j, A) \mid j \in \mathrm{ob}\, J))$ is a cone over $D(-, A)$ in $\mathcal{D}$, so we get a unique $\nu_A : MA \to LA$ s.t. $\lambda_{j,A}\nu_A = \mu_{j,A}$ for all j.

Uniqueness tells us that

$$
\begin{array}{ccc}
MA & \xrightarrow{Mf} & MB \\
\downarrow{\scriptstyle \nu_A} & & \downarrow{\scriptstyle \nu_B} \\
LA & \xrightarrow{Lf} & LB
\end{array}
$$

commutes for all $f \in \mathrm{mor}\, \mathcal{C}$, so $\nu : M \to L$ in $[\mathcal{C}, \mathcal{D}]$, so it's the unique factorisation of the $\mu_{j,-}$ through the $\lambda_{j,-}$. $\square$

Lemma 4.7. *A morphism $f : A \to B$ is monic $\iff$*

$$
\begin{array}{ccc}
A & \xrightarrow{\ 1_A\ } & A \\
{\scriptstyle 1_A}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

is a pullback.

Proof. f is monic $\iff$ any cone (g,h) over (f, f) has $g = h$ $\iff$ (g, h) factors uniquely through $(1_A, 1_A)$. $\qquad\square$

Hence, provided $\mathcal{D}$ has pullbacks, a morphism $\alpha : F \to G$ in $[\mathcal{C}, \mathcal{D}]$ is monic $\iff$ $\alpha_A : FA \to GA$ is monic for each A.

Theorem 4.8. *If $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, then G preserves all limits which exist in $\mathcal{D}$.*

Proof. Suppose $\mathcal{C}$ and $\mathcal{D}$ both have limits of shape J and let $(F \dashv G)$. The diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{D} \\
{\scriptstyle \triangle}\big\downarrow & & \big\downarrow{\scriptstyle \triangle} \\
[J, \mathcal{C}] & \xrightarrow{\ [J, F]\ } & [J, \mathcal{D}]
\end{array}
$$

commutes and $[J, F]$ has a right adjoint $[J, G]$. So by 3.5 the diagram of right adjoints

$$
\begin{array}{ccc}
[J, \mathcal{D}] & \xrightarrow{\ [J, G]\ } & [J, \mathcal{C}] \\
{\scriptstyle \lim_J}\big\downarrow & & \big\downarrow{\scriptstyle \lim_J} \\
\mathcal{D} & \xrightarrow{\ G\ } & \mathcal{C}
\end{array}
$$

commutes up to isomorphism, i.e. G preserves limits of shape J. $\qquad\square$

Proof. Let $D : J \to \mathcal{D}$ be a diagram with limit $(L, (\lambda_j \mid j \in \text{ob } J))$. Given a cone $(A, (\mu_j : A \to GD(j) \mid j \in \text{ob } J))$ in $\mathcal{C}$, we get a cone $(FA, (\bar{\mu}_j : FA \to D(j) \mid j \in \text{ob } J))$ in $\mathcal{D}$, and hence a unique $\bar{\nu} : FA \to L$ such that $\lambda_j \bar{\nu} = \bar{\mu}_j$ for all j.

Then $\nu : A \to GL$ is the unique morphism such that $(G\lambda_j)\nu = \mu_j \forall j$. $\qquad\square$

The 'primeval' Adjoint Functor Theorem says that if $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits, then G has a left adjoint.

This depends on two ideas:

Lemma 4.9. *$\mathcal{C}$ has an initial object $\iff$ $1_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ has a limit.*

Proof. Suppose $\mathcal{C}$ has an initial object 0. The morphisms $(0 \to A \mid A \in \text{ob } \mathcal{C})$ form a cone over $1_\mathcal{C}$. If we had another, say $(L, (\lambda_A \mid A \in \text{ob } \mathcal{C}))$, then $\lambda_0 : L \to 0$ would make

$$L \xrightarrow{\lambda_0} 0$$
$$\searrow{}_{\lambda_A} \swarrow$$
$$A$$

commute for all A, and it's the only morphism which does.

Conversely, suppose $(I, (\lambda_A : I \to A \,|\, A \in \mathrm{ob}\,\mathcal{C}))$ is a limit for $1_\mathcal{C}$.

If $f : I \to A$, then

$$I \xrightarrow{\lambda_I} I$$
$$\searrow{}_{\lambda_A} \swarrow{}_f$$
$$A$$

commutes. In particular, $\lambda_A \lambda_I = \lambda_A$ for all A, so $\lambda_I = 1_I$ since both are factorisations of the limit cone through itself. So $f = \lambda_A$, and hence I is initial. $\qquad\square$

Lemma 4.10. *Suppose $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then, for each $A \in \mathrm{ob}\,\mathcal{C}$, $(A \downarrow G)$ has limits of shape J and the forgetful functor $(A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Suppose given $D : J \to (A \downarrow G)$. Write $D(j)$ as $(UD(j), f_j : A \to GUD(j))$ for each j. Let $(L, (\lambda_j \,|\, j \in \mathrm{ob}\,J))$ be a limit for UD, then $(GL, (G\lambda_j \,|\, j \in \mathrm{ob}\,J))$ is a limit for GUD. But the f_j form a cone over GUD with apex A, so there's a unique $h : A \to GL$ such that

$$A \xrightarrow{h} GL$$
$$\searrow{}_{f_j} \qquad \downarrow{}_{G\lambda_j}$$
$$GUD(j)$$

commutes for all j. So there's a unique lifting of the cone over D in $(A \downarrow G)$.

Suppose we're given a cone $((B, g), (\mu_j \,|\, j \in \mathrm{ob}\,J))$ over D. Then

$$A \xrightarrow{g} GB$$
$$\searrow{}_{h} \qquad \downarrow{}_{G_k}$$
$$GL$$

commutes since both ways round are factorisations of $(f_j \,|\, j \in \mathrm{ob}\,J)$ through the limit GL. $\qquad\square$

Combining 4.10 and 4.9 with 3.2, we've proved the primeval Adjoint Functor Theorem. However, this requries $\mathcal{D}$ to have limits for diagrams 'as big as $\mathcal{D}$ itself', and the only such categories are preorders (c.f. Q6, sheet 2).

In practice, the most we can hope for is that $\mathcal{D}$ has all small limits. We call such a $\mathcal{D}$ **complete**.

Theorem 4.11 (General Adjoint Functor Theorem). *Suppose that $\mathcal{D}$ is complete and locally small. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if it preserves all small limits and satisfies the 'solution set condition': for any $A \in \mathrm{ob}\,\mathcal{C}$, there is a set $\{f_i : A \to GB_i \,|\, i \in I\}$ of objects of $(A \downarrow G)$ such that any $h : A \to GC$ factors as*

$$A \xrightarrow{\ f_i\ } GB_i \xrightarrow{\ Gg\ } Gc$$

for some $i \in I$ and $g : B_i \to C$.

Proof. If G has a left adjoint, then it preserves small limits by 4.8, and $\{\eta_A : A \to GFA\}$ is a singleton solution set at A.

Conversely, each $(A \downarrow G)$ is complete by 4.10, and locally small since it admits a faithful functor to $\mathcal{D}$. So we need to show: if $\mathcal{A}$ is complete and locally small, and has a weakly initial set of objects $\{S_i \,|\, i \in I\}$, then $\mathcal{A}$ has an initial object.

First form $P = \prod_{i \in I} S_i$: then P is weakly initial.

Now form the limit $I \xrightarrow{\ a\ } P$ of the diagram $P \rightrightarrows P$ whose edges are all morphism $P \to P$ in $\mathcal{A}$.

Claim I is initial: it's weakly initial since it admits a morphism to P.

Suppose we had $I \underset{g}{\overset{f}{\rightrightarrows}} A$. Let $b : E \to I$ be an equaliser for (f, g): then there exists $c : P \to E$.

Now $P \xrightarrow{\ c\ } E \xrightarrow{\ b\ } I \xrightarrow{\ a\ } P$ is an edge of the diagram whose limit is I, but so is 1_P; so $abca = 1_P a = a$. But a is monic, so $bca = 1_I$. So b is (split) epic, and $f = g$. So all the $(A \downarrow G)$ have initial objects, hence by 3.2 G has a left adjoint. $\qquad\qquad\square$

The Special Adjoint Functor Theorem imposes additional conditions on $\mathcal{C}$ and $\mathcal{D}$ which ensure that every functor $\mathcal{D} \to \mathcal{C}$ preserving small limits has a left adjoint.

Definition 4.12.   a. A **subobject** of an object A is a monomorphism $A' \rightarrowtail A$. We write $\mathbf{Sub}_{\mathcal{C}}(A)$ for the full subcategory of $\mathcal{C}/A$ whose objects are subobjects of A: note that this category is a preorder.

   b. We say $\mathcal{C}$ is **well-powered** if each $\mathbf{Sub}_{\mathcal{C}}(A)$ is equivalent to a small category, i.e. up to isomorphism each object has only a set of subobjects.

       Dually, $\mathcal{C}$ is **well-copowered** if $\mathcal{C}^{op}$ is well-powered.

Lemma 4.13. *Suppose given a pullback*

$$P \xrightarrow{\;k\;} A$$
$$\downarrow{\scriptstyle h} \qquad \downarrow{\scriptstyle f}$$
$$B \xrightarrow{\;g\;} C$$

with f monic. Then h is monic.

Proof. Suppose $D \underset{y}{\overset{x}{\rightrightarrows}} P$ satisfy $hx = hy$. Then $fkx = fky = ghx = ghy$ and f is monic so $kx = ky$.

Now $x = y$ since both are factorisations of the same cone through the pullback. $\qquad\square$

Theorem 4.14 (Special Adjoint Functor Theorem)**.** *Suppose both $\mathcal{C}$ and $\mathcal{D}$ are locally small, and $\mathcal{D}$ is complete, well-powered and has a separating set. Then $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint $\iff$ G preserves all small limits.*
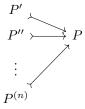
Proof. The forward implication is 4.8 again.

Conversely, we first show that $(A \downarrow G)$ has the properties we've assumed for $\mathcal{D}$: it's complete by 4.10, and locally small as in 4.11. It's well-powered since subobjects of (B, f) in $(A \downarrow G)$ are in bijection with subobjects $B' \rightarrowtail B$ such that f factors through $GB' \rightarrowtail GB$.

It has a coseparating set: if $\{S_i \mid i \in I\}$ is a coseparating set for $\mathcal{D}$, then $\{(S_i, f) \mid i \in I, f : A \to GS_i\}$ is a coseperating set for $(A \downarrow G)$, since if $(B, f) \underset{g'}{\overset{g}{\rightrightarrows}} (B', f')$ satisfies $g \neq g'$, there exists $h : B' \to S_i$ for some i with $hg \neq hg'$, and then h is a morphism $(B', f') \to (S_i, (Gh)f')$ in $(A \downarrow G)$.

Now we show that if $\mathcal{A}$ is complete, locally small and well-powered and has a coseperating set, then it has an initial object.
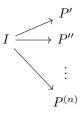
First form $P = \prod_{i \in I} S_i$, where $\{S_i \mid i \in I\}$ is a coseparating set.

Consider the diagram

$$P'$$
$$P'' \rightarrowtail P$$
$$\vdots$$
$$P^{(n)}$$

whose edges are a representative set of subobjects of P.

Form its limit

$$
\begin{array}{c}
\nearrow \; P' \\
I \xrightarrow{\quad} P'' \\
\searrow \vdots \\
P^{(n)}
\end{array}
$$

by the argument of 4.13 the legs $I \to P^{(-)}$ are monic, so $I \rightarrowtail P$ is monic and it's the least subobject of P.

Hence in particular I has no proper subobjects, so any two maps $I \overset{f}{\underset{g}{\rightrightarrows}} A$ must be equal, since their equaliser is an isomorphism.

Now given $A \in \mathcal{A}$, form the product $Q = \prod_{i, f : A \to S_i} S_i$. The canonical morphism $h : A \to Q$ defined by $\pi_{i,f} h = f$ is monic since the S_i form a coseparating set.

We also have $k : P \to Q$ defined by $\pi_{i,f} k = \pi_i$, and we can form the pullback

$$
\begin{array}{ccccc}
I & \longrightarrow & B & \xrightarrow{\;m\;} & A \\
& \searrow & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle h} \\
& & P & \xrightarrow{\;k\;} & Q
\end{array}
$$

By 4.13 l is monic and hence isomorphic to an edge of the diagram defining I, so $I \rightarrowtail P$ factors through it. So there exists a morphism $I \to A$, hence I is initial. $\qquad\qquad\square$

5   Monads

Suppose given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $F \dashv G$. How much of this can we describe purely in terms of $\mathcal{C}$?

We have the composite $T = GF : \mathcal{C} \to \mathcal{C}$, and the unit $\eta : 1_{\mathcal{C}} \to T$. We also have $G\epsilon_F : GFGF \to GF$, which we'll denote $\mu : TT \to T$.

These satisfy the commutative diagrams

$$
\begin{array}{ccccc}
T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta_T} & T \\
& {\scriptstyle 1_T}\searrow\; ① & \downarrow{\scriptstyle \mu} & ②\;\nearrow{\scriptstyle 1_T} & \\
& & T & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
TTT & \xrightarrow{T\mu} & TT \\
\downarrow{\scriptstyle \mu_T} & ③ & \downarrow{\scriptstyle \mu} \\
TT & \xrightarrow{\mu} & T
\end{array}
$$

from the $\triangle^r$ identities and naturality of ϵ.

Definition 5.1. A **monad** $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \to T$, $\mu : TT \to T$ satisfying the commutative diagrams ①, ② and ③.

Definition 5.2. Let $\mathbb{T}$ be a monad on $\mathcal{C}$. A $\mathbb{T}$**-algebra** is a pair (A, α) where $A \in \mathrm{ob}\,\mathcal{C}$, and $\alpha : TA \to A$ satisfies

$$
\begin{array}{ccc}
A \xrightarrow{\eta_A} TA & \quad\text{and}\quad & TTA \xrightarrow{T\alpha} TA \\
\textcircled{4}\;\searrow{\scriptstyle 1_A}\;\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mu_A} \quad \textcircled{5} \quad \downarrow{\scriptstyle \alpha} \\
A & & TA \xrightarrow{\alpha} A
\end{array}
$$

A **homomorphism** $f : (A, \alpha) \to (B, \beta)$ of $\mathbb{T}$-algebras is a morphism $f : A \to B$ such that

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow{\scriptstyle \alpha} & \textcircled{6} & \downarrow{\scriptstyle \beta} \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes. We write $\mathcal{C}^{\mathbb{T}}$ for the category of $\mathbb{T}$-algebras.

Lemma 5.3. *The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ has a left adjoint F, and the adjunction $(F \dashv G)$ induces the monad $\mathbb{T}$.*

Proof. We define $FA = (TA, \mu_A)$ (which is an algebra by ② and ③), and $F(A \xrightarrow{f} B) = Tf$ (which is a homomorphism by naturality of μ).

Clearly $GF = T$ and $\eta : 1_{\mathcal{C}} \to GF$.

We define $\epsilon : FG \to 1_{\mathcal{C}^{\mathbb{T}}}$ by $\epsilon_{(A, \alpha)} = \alpha : (TA, \mu_A) \to (A, \alpha)$ (which is a homomorphism by ⑤).

The triangular identities for η and ϵ follow from ④ and ①, so $(F \dashv G)$.

Finally, $G_{\epsilon_{FA}} = \mu_A$ by the definitions of FA and ϵ, so the adjunction incudes $\mathbb{T}$. $\qquad\qquad\square$

Note that if $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ induces $\mathbb{T}$, then so does $\mathcal{C} \underset{G/\mathcal{D}'}{\overset{F}{\rightleftarrows}} \mathcal{D}'$ where $\mathcal{D}'$ is the full subcategory of objects of the form FA. So in seeking to construct $\mathcal{D}$, we may require F to be bijective on objects. But then morphisms $FA \to FB$ in $\mathcal{D}$ correspond bijectively to morphisms $A \to GFB = TB$ in $\mathcal{C}$.

Definition 5.4. Given a monad $\mathbb{T}$ on $\mathcal{C}$, the **Kleisi category** $\mathcal{C}_{\mathbb{T}}$ is defined by: $\mathrm{ob}\,\mathcal{C}_{\mathbb{T}} = \mathrm{ob}\,\mathcal{C}$, morphisms $A{\to}B$ in $\mathcal{C}_{\mathbb{T}}$ are morphisms $A \to TB$ in $\mathcal{C}$, the identity $A{\to}A$ is $A \xrightarrow{\eta_A} TA$, and the composite of $A\xrightarrow{f}B\xrightarrow{g}C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} C$.

We check

$$
\begin{aligned}
A\xrightarrow{1_A}A\xrightarrow{f}B = A &\xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} B \\
&= A \xrightarrow{f} TA \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} B \\
&= f \text{ by } \textcircled{2}
\end{aligned}
$$

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$$A \xrightarrow{f} B \xrightarrow{1_B} B = A \xrightarrow{f} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} B$$

$$= f \text{ by } \textcircled{1}$$

Given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

$$(hg)f = A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD$$

$$= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD \text{ by } \textcircled{3}$$

$$= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{T_h} TTD \xrightarrow{\mu_D} TD$$

$$= h(gf)$$

Lemma 5.5. *There exists an adjunction* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_{\mathbb{T}}$ *inducing* $\mathbb{T}$.

Proof. We define $FA = A$ and $F(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. This clearly preserves identities, and

$$(Fg)(Ff) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC$$

$$= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \text{ by } \textcircled{1} \text{ and naturality of } \eta$$

$$= F(gf)$$

We define $GA = TA$ and $G(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. G preserves identities by $\textcircled{1}$ and

$$G(A \xrightarrow{f} B \xrightarrow{g} C) = TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC$$

$$= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC \text{ by } \textcircled{3}$$

$$= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \text{ by naturality of } \mu$$

$$= (Gg)(Gf)$$

Clearly $GFA = TA$ and

$$GF(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} TB$$

$$= Tf \text{ by } \textcircled{1}$$

so $GF = T$ and $\eta : 1_{\mathcal{C}} \to GF$.

We define $FGA \xrightarrow{\epsilon_A} A$ to be $TA \xrightarrow{\eta_{TA}} TA$. To verify naturality of ϵ, consider

$$\begin{array}{ccc} FGA & \xrightarrow{FGf} & FGB \\ \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_B} \\ A & \xrightarrow{f} & B \end{array}$$

The top and right edges yield

$$TA \overset{Tf}{\to} TTB \overset{\mu_B}{\to} TB \overset{\eta_{TB}}{\to} TTB \overset{1_{TTB}}{\to} TTB \overset{\mu_B}{\to} TB$$

and the left and bottom yield

$$TA \overset{1_{TA}}{\to} TA \overset{Tf}{\to} TTB \overset{\mu_B}{\to} TB$$

For the $\triangle^r$ identities,

$$GA \overset{\eta_{GA}}{\to} GFGA \overset{G\epsilon_A}{\to} GA = TA \overset{\eta_{TA}}{\to} TTA \overset{1_{TTA}}{\to} TTA \overset{\mu_A}{\to} TA = 1_{TA}$$

and

$$FA \overset{F\eta_A}{\to} FGFA \overset{\epsilon_{FA}}{\to} FA = A \overset{\eta_A}{\to} TA \overset{\eta_{TA}}{\to} TTA \overset{1_{TTA}}{\to} TTA \overset{\mu_A}{\to} TA$$

$$= A \overset{\eta_A}{\to} TA \ (= FA \overset{1_{FA}}{\to} FA)$$

Finally, $G\epsilon_{FA} = TTA \overset{1_{TTA}}{\to} TTA \overset{\mu_A}{\to} TA = \mu_A$, so the adjunction induces the monad $\mathbb{T}$. □

Theorem 5.6. *Given a monad $\mathbb{T}$ on $\mathcal{C}$, let **Adj**$(\mathbb{T})$ be the category whose objects are adjunctions $\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{D}$ inducing $\mathbb{T}$, and whose morphisms $(\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{D}) \to (\mathcal{C} \overset{F'}{\underset{G'}{\rightleftarrows}} \mathcal{D}')$ are functors $K : \mathcal{D} \to \mathcal{D}'$ satisfying $KF = F'$ and $G'K = G$.*

*Then the Kleisi category $\mathcal{C}_{\mathbb{T}}$ is initial in **Adj**$(\mathbb{T})$, and the Eilenberg-Moore category $\mathcal{C}^{\mathbb{T}}$ is terminal.*

Proof. Given $(\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{D})$ in **Adj**$(\mathbb{T})$, we define the **Eilenberg-Moore comparison functor** $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\epsilon_B)$ (note that $G\epsilon_B$ is an algebra structure on GB: the unit condition ④ follows from a $\triangle^r$ identity, and ⑤ follows from the naturality of ϵ).

$K(B \overset{g}{\to} B') = Gg : (GB, G\epsilon_B) \to (GB', G\epsilon_{B'})$ (a homomorphism since ϵ is natural).

It's clear that K is a functor, that $G^{\mathbb{T}}K = G$ and that $KFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$ and $KF(A \overset{f}{\to} B) = Tf = F^{\mathbb{T}}f$.

Uniqueness: suppose $\bar{K}$ also satisfies $G^{\mathbb{T}}\bar{K} = G$ and $\bar{K}F = F^{\mathbb{T}}$. Then $\bar{K}B$ is of the form (GB, β_B) for some algebra structure β_B, and that $\beta_{FA} = \mu_A = G\epsilon_{FA}$ for all A.

Given any B, consider the diagram

$$
\begin{array}{ccc}
GFGFGB & \overset{GFG\epsilon_B}{\longrightarrow} & GFGB \\
\downarrow{\scriptstyle \mu_{GB}} & & \downarrow{\scriptstyle \beta_B} \\
GFGB & \overset{G\epsilon_B}{\longrightarrow} & GB
\end{array}
$$

which must commute, since $G\epsilon_B$ is an algebra homomorphism. But it would also commute with $G\epsilon_B$ in place of β_B, and $GFG\epsilon_B$ is (split) epic, so $\beta_B = G\epsilon_B$.

For the **Kleisi comparison functor** $K : \mathcal{C}_\mathbb{T} \to \mathcal{D}$, we define $KA = FA$,
$K(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB$.

To verify this is functorial, consider

$$K(A \xrightarrow{f} B \xrightarrow{g} C) = FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{FG\epsilon_{FC}} FGFC \xrightarrow{\epsilon_{FC}} FC$$

$$= FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{\epsilon_{FGFC}} FGFC \xrightarrow{\epsilon_{FC}} FC$$

$$= FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB \xrightarrow{Fg} FGFC \xrightarrow{\epsilon_{FC}} FC$$

$$= (Kg)(Kf)$$

$GKA = GFA = TA = G_\mathbb{T}A$
$GK(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB = G_\mathbb{T}(f)$
And $KF_\mathbb{T}A = FA$,
$$KF_\mathbb{T}(A \xrightarrow{f} B) = \begin{array}{ccc} & FA \xrightarrow{\;Ff\;} FB \xrightarrow{\;F\eta_B\;} FGFB & \\ & \searrow{\scriptstyle 1_{FB}} \quad \downarrow{\scriptstyle \epsilon_{FB}} & \\ & FB & \end{array}$$

So K is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Uniqueness: suppose $\bar{K}$ is any other morphism $\mathcal{C}_\mathbb{T} \to \mathcal{D}$ in $\mathbf{Adj}(\mathbb{T})$. Then $\bar{K}A = FA = KA$ for all A; since $\bar{K}$ commutes with both the Fs and the Gs, we have $\bar{K}(\epsilon_A) = \epsilon_{FA}$.

We can write $A \xrightarrow{f} B$ as $A \xrightarrow{F_\mathbb{T}f} F_\mathbb{T}G_\mathbb{T} \xrightarrow{\epsilon_B} B$, so $\bar{K}(f) = \bar{K}(\epsilon_B)Ff = K(f)$. $\qquad\square$

The Kleisi category $\mathcal{C}_\mathbb{T}$ inherits coproducts from $\mathcal{C}$ if $\mathcal{C}$ has them, but it has few other limits or colimits in general.

Theorem 5.7.   *i. The forgetful functor $G : \mathcal{C}^\mathbb{T} \to \mathcal{C}$ creates all limits which exist in $\mathcal{C}$.*

   ii. If $T : \mathcal{C} \to \mathcal{C}$ preserves colimits of shape J, then $G : \mathcal{C}^\mathbb{T} \to \mathcal{C}$ creates them.

Proof.   i. Let $D : J \to \mathcal{C}^\mathbb{T}$ be a diagram, write $D(j) = (GD(j), \delta_j)$.

Let $(L, (\lambda_j : L \to GD(j)))$ be a limit for GD. The composites $TL \xrightarrow{T\lambda_j} TGD(j) \xrightarrow{\delta_j} GD(j)$ form a cone over GD, since the edges of GD are algebra homomorphisms.

So they induce a unique $l : TL \to L$ such taht

$$\begin{array}{ccc} TL & \xrightarrow{\;T\lambda_j\;} & TGD(j) \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \delta_j} \\ L & \xrightarrow{\;\lambda_j\;} & GD(j) \end{array}$$

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commutes for each j.

l is an algebra structure: $l\eta_L = 1_L$ since both are factorisations of (λ_j) through itself, and $lTl = l\mu_L$ since they're factorisations of the same cone through L.

So $((L, l), (\lambda_j))$ is the unique lifting of $(L, (\lambda_j))$ to a cone over D in $\mathcal{C}^{\mathbb{T}}$.

Any cone over D in $\mathcal{C}^{\mathbb{T}}$ factors uniquely through L, and the factorisation is an algebra homomorphism.

ii. Similarly, given $D : J \to \mathcal{C}^{\mathbb{T}}$ as before and a colimit $(L, (\lambda_j : GD(j) \to L))$ for GD, we get a unique $l : TL \to L$ making

$$
\begin{array}{ccc}
TGD(j) & \xrightarrow{T\lambda_j} & TL \\
\downarrow{\scriptstyle \delta_j} & & \downarrow{\scriptstyle l} \\
GD(j) & \xrightarrow{\lambda_j} & L
\end{array}
$$

commute, since $(TL, (T\lambda_j))$ is a colimit. The rest of the proof is similar to (i). $\qquad\square$

Definition 5.8. An adjunction $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$, $(F \dashv G)$, is **monadic** if the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is part of an equivalence, where $\mathbb{T}$ is the monad induced by $(F \dashv G)$. We also say $G : \mathcal{D} \to \mathcal{C}$ is monadic if it has a left adjoint and the adjunction is monadic.

Note that K preserves all limits which exist in $\mathcal{D}$, since G preserves them and $G^{\mathbb{T}}$ creates them.

Lemma 5.9. *Suppose given* $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$, $(F \dashv G)$ *inducing a monad* $\mathbb{T}$ *on* $\mathcal{C}$. *Suppose, for each* $\mathbb{T}$-*algebra* (A, α), *the pair* $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ *has a coequaliser in* $\mathcal{D}$. *Then* $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ *has a left adjoint* L.

Proof. We define $L(A, \alpha) = \operatorname{coeq}(FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA)$.

Given $(A, \alpha) \xrightarrow{f} (B, \beta)$, we get

$$
\begin{array}{ccc}
FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA & \longrightarrow & L(A, \alpha) \\
\downarrow{\scriptstyle FGFf} \qquad \downarrow{\scriptstyle Ff} & & \vdots\, {\scriptstyle Lf} \\
FGFB \underset{\epsilon_{FB}}{\overset{F\beta}{\rightrightarrows}} FB & \longrightarrow & L(B, \beta)
\end{array}
$$

So $\exists! Lf$ making the right hand square commute. Uniqueness ensures L is functorial.

Morphisms $L(A, \alpha) \to B$ in $\mathcal{D}$ correspond bijectively to morphisms $f : FA \to B$ such that $f(F\alpha) = f(\epsilon_{FA})$ and hence to morphisms $\bar{f} : A \to GB$ such that $f\bar{\alpha} = Gf|_{GFA} = G\epsilon_B \circ GF\bar{f}$, i.e. to algebra homomorphisms $(A, \alpha) \to (GB, G\epsilon_B) = KB$.

So $(L \dashv K)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 5.10.   a. A parallel pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is **reflexive** if $\exists r : B \to A$ such that $fr = gr = 1_B$.

Note that $FGFA \overset{F\alpha}{\underset{\epsilon_{FA}}{\rightrightarrows}} FA$ is reflexive, with common splitting $FA \overset{F\eta_A}{\to} FGFA$.

A **reflexive coequaliser** is the coequaliser of a reflexive pair.

b. A **split coequaliser diagram** is a diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} \overset{}{\underset{t}{\rightsquigarrow}} B \overset{h}{\underset{s}{\rightarrow}} C$$

satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$.

If these equations hold, h **is** a coequaliser of f and g: given $k : B \to D$ with $kf = kg$, we have $k = kgt = kft = ksh$, so k factors through h, uniquely since h is split epic.

Note that **any** functor preserves split equalisers.

c. Given $G : \mathcal{D} \to \mathcal{C}$, a pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{D}$ is **G-split** if there exists a split coequaliser

$$GA \overset{Gf}{\underset{Gg}{\rightrightarrows}} \overset{}{\underset{t}{\rightsquigarrow}} GB \overset{}{\underset{s}{\rightarrow}} C$$

in $\mathcal{C}$.

The pair $FGFA \overset{F\alpha}{\underset{\epsilon_{FA}}{\rightrightarrows}} FA$ of 5.9 is G-split:

$$GFGFA \overset{Gf\alpha}{\underset{G\epsilon_{FA}}{\rightrightarrows}} \overset{}{\underset{\eta_{GFA}}{\rightsquigarrow}} GFA \overset{\alpha}{\underset{\eta_A}{\rightarrow}} A$$

is a split coequaliser diagram.

Theorem 5.11 (Precise Monadicity Theorem)**.** $G : \mathcal{D} \to \mathcal{C}$ *is monadic* $\iff$ G *has a left adjoint, and c reates coequalisers of G-split pairs.*

Theorem 5.12 (Crude Monadicity Theorem)**.** *Suppose $G : \mathcal{C} \to \mathcal{C}$ has a left adjoint, that $\mathcal{D}$ has and G preserves reflexive coequalisers, and G reflects isomorphisms. Then G is monadic.*

Proof. For the forward implication in 5.11, it's enough to show that $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates coequalisers of $G^{\mathbb{T}}$-split pairs. This follows from 5.7, given that T and TT both preserve split coequalisers.

Conversely in either case, $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ has a left adjoint L by 5.9. Now $LKB = \mathrm{coeq}(FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB)$ and the counit $LKB \to B$ is the factorisation of $FGB \underset{\epsilon_B}{\overset{\epsilon_{FGB}}{\rightrightarrows}} B$ through this coequaliser.

But $\quad GFGFGB \rightrightarrows GFGB \longrightarrow GB\quad$ is a split coequaliser diagram.

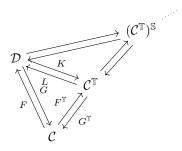So either set of hypotheses ensures that $LKB \to B$ is an isomorphism.

$KL(A,\alpha) = K(\mathrm{coeq}(FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA))$. Either hypothesis implies that $G = G^{\mathbb{T}}K$ preserves this coequaliser, but

$$GFGFGA \underset{G\epsilon_{FA}}{\overset{GF\alpha}{\rightrightarrows}} GFA \overset{\alpha}{\longrightarrow} A$$

is a split coequaliser, so $GL(A,\alpha) \cong A$.

The unit $(A,\alpha) \to KL(A,\alpha)$ is mapped to this isomorphism by $G^{\mathbb{T}}$, so it's an isomorphism in $\mathcal{C}^{\mathbb{T}}$. $\qquad\square$

Remark 5.13. Let $\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{D}$ be an adjunction, and suppose $\mathcal{D}$ has reflexive coequalisers. The **monadic tower** of $(F \dashv G)$ is the diagram

where $\mathbb{T}$ is the monad induced by $(F \dashv T)$, K is the Eilenberg-Moore comparison functor, $(L \dashv K)$ (5.9), $\mathbb{S}$ is the monad induced by $(L \dashv K)$, etc.

We say $(F \dashv G)$ has **monadic length** n if we reach an equivalence after n steps. **Top** $\to$ **Set** has monadic length ∞.

6 Regular Categories

Definition 6.1. The **image** of a morphism $A \xrightarrow{f}$ is the smallest subobject of B through which f factors, if this exists.

We say $\mathcal{C}$ **has images** if every $f \in \operatorname{mor} \mathcal{C}$ has an image.

$A \xrightarrow{f} B$ is a **cover** if its image is 1_B, i.e. it doesn't factor through any proper subobject of B. We write $A \xrightarrow{f} B$ to indicate that f is a cover.

Lemma 6.2. *If $\mathcal{C}$ has finite limits, then covers in $\mathcal{C}$ coincide with strong epimorphisms.*

Proof. Recall that f is strong epic if and only if given $(*)$ $\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & {\scriptstyle l}\nearrow & \uparrow{\scriptstyle k} \\ B & \xrightarrow{h} & D \end{array}$ with k monic, $\exists l : B \to C$ with $kl = h$ and $lf = g$.
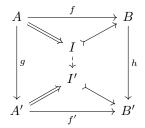
Being a cover is the special case of this condition with $h = 1_B$, so strong epimorphisms are covers.

Conversely, if f is a cover then it's epic, since if $gf = hf$ then f factors through the equaliser of g and h, so this must be an isomorphism. Given $(*)$, we can form the pullback $\begin{array}{ccc} P & \xrightarrow{n} & C \\ {\scriptstyle m}\downarrow & & \uparrow{\scriptstyle k} \\ B & \xrightarrow{h} & D \end{array}$, and m is monic by 4.13.

f factors through m, so m is an isomorphism and $B \xrightarrow{nm^{-1}} C$ is the diagonal fill in for $(*)$. $\qquad\square$

It follows that image factorisation is functorial: given

we get a unique $I \to I'$ making everything commute. So image factorisation defines a functor $[\mathbf{2}, \mathcal{C}] \to [\mathbf{3}, \mathcal{C}]$.

Definition 6.3. A **regular category** is a category which preserves finite limits and images, in which strong epimorphisms are stable under pullback.

A **regular functor** is one which preserves finite limits and strong epimorphisms.

Theorem 6.4. *In a regular category, the strong epimorphisms coincide with regular epimorphisms.*

Proof. Regular $\implies$ strong is true in general (see sheet 1). Suppose $A \xrightarrow{f} B$ is strong epic. Let $R \overset{a}{\underset{b}{\rightrightarrows}}$ be the **kernel-pair** of f, i.e. the pullback of f against itself. Certainly $fa = fb$.

Suppose $A \xrightarrow{g} C$ with $ga = gb$. Form the image

$$A \Longrightarrow I \overset{(k,l)}{\rightarrowtail} B \times C$$

of (f, g).

Claim k is an isomorphism: given this, the composite $B \xrightarrow{k^{-1}} I \xrightarrow{l} C$ satisfies $lk^{-1}f = lk^{-1}kh = lh = g$, and it's unique since f is epic.

We know k is strong epic, since $kh = f$ is strong epic. So we need to show it's monic.

Suppose $D \overset{x}{\underset{y}{\rightrightarrows}}$ satisfy $kx = ky$. Form the pullback

$$
\begin{array}{ccc}
F & \xrightarrow{\quad\quad m \quad\quad} & D \\
\downarrow{\scriptstyle (n,p)} & & \downarrow{\scriptstyle (x,y)} \\
A \times A \overset{1 \times h}{\Longrightarrow} & A \times I \overset{h \times 1}{\Longrightarrow} & I \times I
\end{array}
$$

then $h \times 1$ and $1 \times h$ are strong epimorphisms, and hence so is m.

Now $fn = khn = kxm = kym = khp = fp$, so (n, p) factors through (a, b), say by $E \xrightarrow{q} R$. Since $kha = khb$ and $lha = lhb$, and (k, l) is monic, we have $ha = hb$. So $xm = hn = haq = hbq = hp = ym$. But m is epic, so $x = y$. $\qquad\square$

Remark 6.5. In many textbooks, regular categories are defined as categories with (some) finite limits, in which every morphism factors as a regular epimorphism and a monomorphism, and regular epimorphims are stable under pullback.

Note that if $A \xrightarrow{f} B$ has kernel pair $R \overset{a}{\underset{b}{\rightrightarrows}} A$ and image factorisation $A \xrightarrow{g} I \rightarrowtail B$, then (a, b) is also the kernel-pair of g since h is monic. So g may be obtained as the coequaliser of the kernel-pair of f, and f as the factorisation of f through this.