Part III Algebraic Geometry

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Contents

1 Sheaves

Definition (Presheaf)**.** *Let X be a topological space. A* ***presheaf*** *$\mathcal{F}$ consists of a collection of abelian groups, $\mathcal{F}(U)$, where $U \subseteq X$ are the open subsets of X s.t. $\mathcal{F}(\emptyset) = 0$.*

$\exists$ a homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$, $s \mapsto s|_V$ for each inclusion $V \subseteq U$ of open sets. $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map. If $W \subseteq V \subseteq U$ are open sets then $\forall s \in \mathcal{F}(U)$, $(s|_V)|_W = s|_W$.

Definition (Sheaf)**.** *A* ***sheaf*** *$\mathcal{F}$ is a presheaf s.t. if $U = \bigcup U_i$, U, U_i open and if $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j$ then $\exists! s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$ $\forall i$.*

Definition (Stalk)**.** *Let X be a topological space, $\mathcal{F}$ a presheaf, $x \in X$. Define the* ***stalk*** *of $\mathcal{F}$ at x by $\mathcal{F}_x = \lim_{U \ni x} \mathcal{F}(U)$.*

More explicitly, each element of $\mathcal{F}_x$ is given by a pair (U, s) where $x \in U$ open, $s \in \mathcal{F}(U)$ subject to the condition

$$(U, s) = (V, t) \text{ if } \exists x \in W \subseteq U \cap V \text{ s.t. } s|_W = t|_W$$

Definition (Morphism)**.** *Let X be a topological space, $\mathcal{F}$, $\mathcal{G}$ presheaves. A* ***morphism*** *$\varphi : \mathcal{F} \to \mathcal{G}$ is given by a collection of homomorphisms $\mathcal{F}(U) \stackrel{\varphi(U)}{\to} \mathcal{G}(U)$ s.t. if $V \subseteq U$, the diagram*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

commutes. We say φ is an ***isomorphism*** *if it has an inverse.*

Remark. *$\forall x \in X$, any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, we get a homomorphism*

$$\begin{aligned} \varphi_x : \mathcal{F}_x &\to \mathcal{G}_x \\ (U, s) &\mapsto (U, \varphi_U(s)) \end{aligned}$$

Definition. *Let X be a topological space, $\mathcal{F}$ a presheaf. Then $\exists$ a sheaf $\mathcal{F}^+$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}^+$ s.t. if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism into a sheaf $\mathcal{G}$, then φ factors uniquely*

$$\begin{tikzcd} & \mathcal{F}^+ \arrow[d] \\ \mathcal{F} \arrow[ur, "\alpha"] \arrow[r, "\varphi"'] & \mathcal{G} \end{tikzcd}$$

for some morphism $\mathcal{F}^+ \to \mathcal{G}$. We call $\mathcal{F}^+$ the sheaf ***associated*** *to $\mathcal{F}$.*

$\mathcal{F}^+$ is constructed as follows:

$$\mathcal{F}^+(U) := \left\{ \text{functions } s : U \to \bigsqcup_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \forall x \in U,\ s(x) \in \mathcal{F}_x,\ \exists x \in W \subseteq V \text{ and} \\ t \in \mathcal{F}(W) \text{ s.t. } s(y) = (V, t) \in \mathcal{F}_y\ \forall y \in W \end{array} \right\}$$

Definition (Kernel and Image)**.** *Let X be a topological space, $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ a morphism of presheaves. The* ***kernel*** *of φ, denoted $\operatorname{Ker}\varphi$, is defined by*

$$(\operatorname{Ker}\varphi)(U) = \operatorname{Ker}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

The ***presheaf image*** *of φ, denoted $\operatorname{Im}(\varphi^{pre})$ is defined by*

$$(\operatorname{Im}\varphi^{pre})(U) = \operatorname{Im}(\varphi_U)$$

Now assume $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Define the kernel of $\varphi = \operatorname{Ker}\varphi$ as above, which is a sheaf. Define the image of φ by $\operatorname{Im}(\varphi^{pre})^+$, denoted $\operatorname{Im}\varphi$.

Theorem 1. *Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on a topological space X. Then*

i. φ is injective $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective $\forall x \in X$

ii. φ is surjective $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective $\forall x \in X$

iii. φ is an isomorphism $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism $\forall x \in X$

Definition. *Let X be a topological space. A* ***complex of sheaves*** *is a sequence*

$$\cdots \to \mathcal{F}_{-2} \overset{\varphi_{-2}}{\to} \mathcal{F}_{-1} \overset{\varphi_{-1}}{\to} \mathcal{F}_0 \overset{\varphi_0}{\to} \mathcal{F}_1 \overset{\varphi_1}{\to} \mathcal{F}_2 \overset{\varphi_2}{\to} \dots$$

of sheaves s.t. $\operatorname{Im}\varphi_i \subseteq \operatorname{Ker}\varphi_{i+1}\ \forall i$. *We say it is an* ***exact sequence*** *if* $\operatorname{Im}\varphi_i = \operatorname{Ker}\varphi_{i+1}\ \forall i$. *An exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is called a ***short exact sequence****.*

Definition (Constant sheaf)**.** *Let X be a topological space and A an abelian group. Define a presheaf $\mathcal{F}$ by $\mathcal{F}(U) = A\ \forall$ open $U \neq \emptyset$. We call $\mathcal{F}^+$ the* ***constant sheaf*** *associated to A.*

Definition (Direct image)**.** *Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. The* ***direct image*** *$f_*\mathcal{F}$ is defined by*

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$$

Definition (Skyscraper sheaf)**.** *Let X be a topological space, $x \in X$, A an abelian group. Define $\mathcal{F}$ by*

$$\mathcal{F} = \begin{cases} A \text{ if } x \in U \\ 0 \text{ if } x \notin U \end{cases}$$

We call $\mathcal{F}$ the ***skyscraper sheaf*** *associated to A at x.*