Part III Category Theory

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Michaelmas 2016
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Contents

1 Definitions and Examples

Definition (Category)**.** *A category $\mathcal{C}$ consists of*

a. *a collection* $\operatorname{ob}\mathcal{C}$ *of* ***objects*** *A, B, C, ...*

b. *a collection* $\operatorname{mor}\mathcal{C}$ *of* ***morphisms*** *f, g, h, ...*

c. *two operations* dom, cod *from morphisms to objects. We write $f : A \to B$ or $A \xrightarrow{f} B$ to mean 'f is a morphism and* $\operatorname{dom} f = A$ *and* $\operatorname{cod} f = B$*'*

d. *an operation assigning to each object A a morphism $1_A : A \to A$*

e. *a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff$ $\operatorname{dom} g = \operatorname{cod} f$, and then $gf : \operatorname{dom} f \to \operatorname{cod} g$*

satisfying

f. *$f1_A = f$ and $1_B f = f$ $\forall f : A \to B$*

g. *$h(fg) = (hg)f$ whenever gf and hg are defined*

Definition (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor*** *$\mathcal{C} \to \mathcal{D}$ consists of*

a. *a mapping $A \to FA$ from* $\operatorname{ob}\mathcal{C}$ *to* $\operatorname{ob}\mathcal{D}$

b. *a mapping $f \to Ff$ from* $\operatorname{mor}\mathcal{C}$ *to* $\operatorname{mor}\mathcal{D}$

satisfying $\mathrm{dom}\, Ff = F\mathrm{dom}\, f$, $\mathrm{cod}\, Ff = F\mathrm{cod}\, f$ *for all* f, $F(1_A) = 1_{FA}$ *for all* A, *and* $F(gf) = (Fg)(Ff)$ *whenever* gf *is defined.*

Definition. *By a* ***contravariant functor*** $\mathcal{C} \to \mathcal{D}$ *we mean a functor* $\mathcal{C} \to \mathcal{D}^{op}$ *(or equivalently* $\mathcal{C}^{op} \to \mathcal{D}$*). A functor* $\mathcal{C} \to \mathcal{D}$ *is sometimes said to be* ***covariant****.*

Definition (Natural transformation)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories and* $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ *two functors. A* ***natural transformation*** $\alpha : F \to G$ *assigns to each* $A \in \mathrm{ob}\, \mathcal{C}$ *a morphism* $\alpha_A : FA \to GA$ *in* $\mathcal{D}$*, such that*

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given $\alpha : F \to G$ and $\beta : G \to H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A-component of a natural transformation $\beta\alpha : F \to H$.

Definition. *Given categories* $\mathcal{C}, \mathcal{D}$*, we write* $[\mathcal{C}, \mathcal{D}]$ *for the category of all functors* $\mathcal{C} \to \mathcal{D}$ *and natural transformations between them.*

Lemma 1. *Given* $F, G : \mathcal{C} \to \mathcal{D}$ *and* $\alpha : F \to G$*,* α *is an isomorphism in* $[\mathcal{C}, \mathcal{D}]$ $\iff$ *each* α_A *is an isomorphism in* $\mathcal{D}$*.*

Definition (Faithful and full)**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor.*

a. *We say that* F *is* ***faithful*** *if, given* $f, g \in \mathrm{mor}\, \mathcal{C}$*, the equations* $\mathrm{dom}\, f = \mathrm{dom}\, g$, $\mathrm{cod}\, f = \mathrm{cod}\, g$ *and* $Ff = Fg$ *imply* $f = g$*.*

b. F *is* ***full*** *if, given any* $g : FA \to FB$ *in* $\mathcal{D}$*, there exists* $f : A \to B$ *in* $\mathcal{C}$ *with* $Ff = g$*.*

c. *We say a subcategory* $\mathcal{C}'$ *of* $\mathcal{C}$ *is* ***full*** *if the inclusion* $\mathcal{C}' \hookrightarrow \mathcal{C}$ *is a full functor.*

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition (Equivalence of categories)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. An* ***equivalence*** *between* $\mathcal{C}$ *and* $\mathcal{D}$ *is a pair of functors* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ *together with natural isomorphisms* $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$*. We write* $\mathcal{C} \simeq \mathcal{D}$ *to mean that* $\mathcal{C}$ *and* $\mathcal{D}$ *are equivalent.*

We say a property P *of categories is* ***categorical*** *if whenever* $\mathcal{C}$ *has* P *and* $\mathcal{C} \simeq \mathcal{D}$ *then* $\mathcal{D}$ *has* P*.*

For example, being a groupoid is a categorical property, but being a group is not.

Definition (Slice category)**.** *Given an object B of a category $\mathcal{C}$, define the* ***slice category*** *$\mathcal{C}/B$ to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \to (A' \xrightarrow{f'} B)$ are morphisms $h : A \to A'$ making*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commute.

Lemma 2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and* ***essentially surjective****, i.e. for every $B \in \mathrm{ob}\,\mathcal{D}$, there exists $A \in \mathrm{ob}\,\mathcal{C}$ s.t. $FA \cong B$.*

Definition. *a. A* ***skeleton*** *of a category $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ containing exactly one object from each isomorphism class of objects of $\mathcal{C}$.*

b. We say $\mathcal{C}$ is ***skeletal*** *if it's a skeleton of itself. Equivalently, any isomorphism f in $\mathcal{C}$ satisfies $\mathrm{dom}\, f = \mathrm{cod}\, f$.*

For example, $\mathbf{Mat}_K$ is skeletal. The full subgategory of standard vector spaces K^n is a skeleton of $\mathbf{fd\ Mod}_K$.

Remark. *The following statements are each equivalent to the Axiom of Choice:*

1. *Every small category has a skeleton*

2. *Any small category is equivalent to each of its skeletons*

3. *Any two skeletons of a given small category are isomorphic*

Definition. *Let $f : A \to B$ be a morphism in a category $\mathcal{C}$.*

a. f is a ***monomorphism*** *if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightarrowtail B$ if f is monic.*

b. Dually, f is an ***epimorphism*** *if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.*

c. $\mathcal{C}$ is a ***balanced*** *category if every $f \in \mathrm{mor}\,\mathcal{C}$ which is both monic and epic is an isomorphism.*

2 The Yoneda Lemma

Definition. *A category $\mathcal{C}$ is* ***locally small*** *if, for any two objects A, B of $\mathcal{C}$, the morphism $A \to B$ are parametrised by a set $\mathcal{C}(A, B)$.*

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, \cdot) : \mathcal{C} \to \mathbf{Set}$: if $g : B \to B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \to \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \to B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \to \mathbf{Set}$.