Part III Category Theory

Based on lectures by Prof P.T. Johnstone

Michaelmas 2016
University of Cambridge

Contents

1 Definitions and Examples

Definition 1.1 (Category)**.** *A category $\mathcal{C}$ consists of*

a. *a collection* $\mathrm{ob}\,\mathcal{C}$ *of* ***objects*** *A, B, C, ...*

b. *a collection* $\mathrm{mor}\,\mathcal{C}$ *of* ***morphisms*** *f, g, h, ...*

c. *two operations* dom, cod *from morphisms to objects. We write $f : A \to B$ or $A \xrightarrow{f} B$ to mean 'f is a morphism and* $\mathrm{dom}\, f = A$ *and* $\mathrm{cod}\, f = B$*'*

d. *an operation assigning to each object A a morphism $1_A : A \to A$*

e. *a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff$* $\mathrm{dom}\, g = \mathrm{cod}\, f$*, and then* $gf : \mathrm{dom}\, f \to \mathrm{cod}\, g$

satisfying

f. *$f1_A = f$ and $1_B f = f$ $\forall f : A \to B$*

g. *$h(fg) = (hg)f$ whenever gf and hg are defined*

Definition 1.2 (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor*** *$\mathcal{C} \to \mathcal{D}$ consists of*

a. *a mapping* $A \to FA$ *from* $\mathrm{ob}\,\mathcal{C}$ *to* $\mathrm{ob}\,\mathcal{D}$

b. *a mapping* $f \to Ff$ *from* $\mathrm{mor}\,\mathcal{C}$ *to* $\mathrm{mor}\,\mathcal{D}$

satisfying $\mathrm{dom}\,Ff = F\mathrm{dom}\,f$, $\mathrm{cod}\,Ff = F\mathrm{cod}\,f$ *for all* f, $F(1_A) = 1_{FA}$ *for all* A, *and* $F(gf) = (Fg)(Ff)$ *whenever* gf *is defined.*

Definition 1.3. *By a* ***contravariant functor*** $\mathcal{C} \to \mathcal{D}$ *we mean a functor* $\mathcal{C} \to \mathcal{D}^{op}$ *(or equivalently* $\mathcal{C}^{op} \to \mathcal{D}$*). A functor* $\mathcal{C} \to \mathcal{D}$ *is sometimes said to be* ***covariant****.*

Definition 1.4 (Natural transformation)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories and* $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ *two functors. A* ***natural transformation*** $\alpha : F \to G$ *assigns to each* $A \in \mathrm{ob}\,\mathcal{C}$ *a morphism* $\alpha_A : FA \to GA$ *in* $\mathcal{D}$*, such that*

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

commutes.

We can compose natural transformations: given $\alpha : F \to G$ and $\beta : G \to H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A-component of a natural transformation $\beta\alpha : F \to H$.

Definition 1.5. *Given categories* $\mathcal{C}, \mathcal{D}$*, we write* $[\mathcal{C}, \mathcal{D}]$ *for the category of all functors* $\mathcal{C} \to \mathcal{D}$ *and natural transformations between them.*

Lemma 1.6. *Given* $F, G : \mathcal{C} \to \mathcal{D}$ *and* $\alpha : F \to G$*,* α *is an isomorphism in* $[\mathcal{C}, \mathcal{D}] \iff$ *each* α_A *is an isomorphism in* $\mathcal{D}$*.*

Definition 1.7 (Faithful and full)**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor.*

a. *We say that* F *is* ***faithful*** *if, given* $f, g \in \mathrm{mor}\,\mathcal{C}$*, the equations* $\mathrm{dom}\,f = \mathrm{dom}\,g$*,* $\mathrm{cod}\,f = \mathrm{cod}\,g$ *and* $Ff = Fg$ *imply* $f = g$*.*

b. F *is* ***full*** *if, given any* $g : FA \to FB$ *in* $\mathcal{D}$*, there exists* $f : A \to B$ *in* $\mathcal{C}$ *with* $Ff = g$*.*

c. *We say a subcategory* $\mathcal{C}'$ *of* $\mathcal{C}$ *is* ***full*** *if the inclusion* $\mathcal{C}' \hookrightarrow \mathcal{C}$ *is a full functor.*

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition 1.8 (Equivalence of categories)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to mean that $\mathcal{C}$ and $\mathcal{D}$ are equivalent.*

*We say a property P of categories is **categorical** if whenever $\mathcal{C}$ has P and $\mathcal{C} \simeq \mathcal{D}$ then $\mathcal{D}$ has P.*

For example, being a groupoid is a categorical property, but being a group is not.

Definition 1.9 (Slice category)**.** *Given an object B of a category $\mathcal{C}$, define the **slice category** $\mathcal{C}/B$ to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \to (A' \xrightarrow{f'} B)$ are morphisms $h : A \to A'$ making*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commute.

Lemma 1.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and **essentially surjective**, i.e. for every $B \in \operatorname{ob}\mathcal{D}$, there exists $A \in \operatorname{ob}\mathcal{C}$ s.t. $FA \cong B$.*

Definition 1.11.

a. *A **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ containing exactly one object from each isomorphism class of objects of $\mathcal{C}$.*

b. *We say $\mathcal{C}$ is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism f in $\mathcal{C}$ satisfies $\operatorname{dom} f = \operatorname{cod} f$.*

For example, $\mathbf{Mat}_K$ is skeletal. The full subgategory of standard vector spaces K^n is a skeleton of $\mathbf{fd\ Mod}_K$.

Remark 1.12. *The following statements are each equivalent to the Axiom of Choice:*

1. *Every small category has a skeleton*
2. *Any small category is equivalent to each of its skeletons*
3. *Any two skeletons of a given small category are isomorphic*

Definition 1.13. *Let $f : A \to B$ be a morphism in a category $\mathcal{C}$.*

a. *f is a **monomorphism** if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightarrowtail B$ if f is monic.*

b. *Dually, f is an* ***epimorphism*** *if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.*

c. *$\mathcal{C}$ is a* ***balanced*** *category if every $f \in \operatorname{mor} \mathcal{C}$ which is both monic and epic is an isomorphism.*

2 The Yoneda Lemma

Definition 2.1. *A category $\mathcal{C}$ is* ***locally small*** *if, for any two objects A, B of $\mathcal{C}$, the morphism $A \to B$ are parametrised by a set $\mathcal{C}(A, B)$.*

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$: if $g : B \to B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \to \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \to B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \to \mathbf{Set}$.

Lemma 2.2 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob} \mathcal{C}$ and $F : \mathcal{C} \to$* ***Set****. Then*

i. *There is a bijection between natural transformations $\mathcal{C}(A, -) \to F$ and elements of FA.*

ii. *Moreover, this bijection is natural in both A and F.*

Proof. Bijection: given $\alpha : \mathcal{C}(A, -) \to F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.

Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \to F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$ is natural: given $g : B \to C$, we have

$$\begin{aligned}
\Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\
&= F(gf)(x) \\
&= (Fg)(Ff)(x) \\
&= (Fg)\Psi(x)_B(f)
\end{aligned}$$

$\Phi\Psi(x) = x$ since $F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$ since, for any $f : A \to B$,

$$\begin{aligned}
\Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\
&= Ff(\alpha_A(1_A)) \\
&= \alpha_B(\mathcal{C}(A, f)(1_A)) \\
&= \alpha_B(f)
\end{aligned}$$

□

Corollary 2.3. *The mapping $A \to \mathcal{C}(A, -)$ is a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C}, \boldsymbol{Set}]$.*

Proof. Given two objects A, B, 2.2(i) gives us a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ (by taking $F : C \mapsto \mathcal{C}(B, C)$). We need to show this is functorial, but given $f \in \mathcal{C}(B, A)$, $\Psi(F)_A$ sends 1_A to $\mathcal{C}(B, f)(1_A) = f$, so it's the natural transformation $g \mapsto gf$.

Hence, given $e : C \to B$, $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$ □

We call this functor the **Yoneda embedding**. Hence any locally small category $\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$.

Definition 2.4. *A functor $\mathcal{C} \to \boldsymbol{Set}$ is **representable** if it's isomorphic to $\mathcal{C}(A, -)$ for some A.*

*A **representation** of $F : \mathcal{C} \to \boldsymbol{Set}$ is a pair (A, x) where $A \in \operatorname{ob}\mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, -) \to F$ is an isomorphism. We also call x a **universal element** of F.*

Corollary 2.5 ('Representations are unique up to unique isomorphism')**.** *If (A, x) and (B, y) are both representations of $F : \mathcal{C} \to \boldsymbol{Set}$, then there's a unique isomorphism $f : A \to B$ s.t $Ff(x) = y$.*

Definition 2.6 (Product and coproduct)**.** *Given two objects A, B of a locally small category $\mathcal{C}$, we define their **product** to be a representation of the functor*

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{op} \to \boldsymbol{Set}$$

i.e. an object $A \times B$ equipped with morphisms $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ s.t. given any pair $(f : C \to A, g : C \to B)$, there exists a unique $h : C \to A \times B$ s.t. $\pi_1 h = f$ and $\pi_2 h = g$.

*More generally, we can define the product $\prod_{i \in I} A_i$ of a family $\{A_i \mid i \in I\}$ of objects, or the product of the empty family, i.e. a **terminal object** 1 s.t. for every A there's a unique $A \to 1$.*

*Dualizing, we get the notion of **coproduct** or **sum**.*

Definition 2.7 (Equaliser and coequaliser)**.** *Given a parallel pair $f, g : A \rightrightarrows B$ in a locally small category $\mathcal{C}$, the assignment $C \mapsto FC = \{h : C \to A \mid fh = gh\}$ is a subfunctor F of $\mathcal{C}(-, A)$. A representation of F is called an **equaliser** of (f, g).*

In elementary terms, it's an object E equipped with $e : E \to A$ s.t. $fe = ge$, s.t. any h with $fh = gh$ factors uniquely as $h = ek$

*Dually, we have the notion of **coequaliser**, i.e. a morphism $q : B \to Q$ satisfying $qf = qg$, and universal among such.*

Definition 2.8. *a. We say a monomorphism is* ***regular*** *if it occurs as an equaliser (dually, regular epimorphism).*

b. We say $f : A \to B$ is a ***split monomorphism*** *if there exists $g : B \to A$ with $gf = 1_A$.*

Every split monomorphism is regular: if $gf = 1_A$, f is an equaliser of $(1_B, fg)$ [see sheet 1, q2].

Definition 2.9. *Let $\mathcal{C}$ be a (locally small) category, $\mathcal{G}$ a collection of objects of $\mathcal{C}$.*

a. Say $\mathcal{G}$ is a ***separating family*** *if the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ are jointly faithful, i.e. if given $f, g : A \rightrightarrows B$ with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \to A$ with $fh \neq gh$.*

b. Say $\mathcal{G}$ is a ***detecting family*** *if the $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ jointly reflect isomorphisms, i.e. if given $f : A \to B$ s.t. every $g : G \to B$ with $G \in \mathcal{G}$ factors uniquely through f, f is an isomorphism.*

Lemma 2.10. *i. If $\mathcal{C}$ is balanced, then any separating family is detecting*

ii. If $\mathcal{C}$ has equalisers, then every detecting family is separating

Definition 2.11. *An object P is* ***projective*** *if $\mathcal{C}(P, -)$ preserves epimorphisms, i.e. if given*

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

there exists $g : P \to A$ with $eg = f$.

Dually, P is ***injective*** *in $\mathcal{C}$ if it's projective in $\mathcal{C}^{op}$.*

If P satisfies this property $\forall e$ in some class $\mathcal{E}$ of epimorphisms, we call it $\mathcal{E}$-projective.

Corollary 2.12. *Representable functors are (pointwise) projective in $[\mathcal{C}, \textbf{Set}]$*

Proof. Given

$$\begin{tikzcd} & \mathcal{C}(A, -) \arrow[d, "\beta"] \\ F \arrow[r, two heads, "\alpha"] & G \end{tikzcd}$$

β corresponds to some $y \in GA$. α_A is surjective, so $\exists x \in FA$ with $\alpha_A(x) = y$. x corresponds to $\gamma : \mathcal{C}(A, -) \to F$ with $\alpha\gamma = \beta$. $\square$

3 Adjunctions

Definition 3.1 (D.M. Khan, 1958)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be two functors. An **adjunction** between F and G is a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and morphisms $A \to GB$ in $\mathcal{C}$, which is natural in A and B.*

*(If $\mathcal{C}$ and $\mathcal{D}$ are locally small, this says that $(A, B) \to \mathcal{D}(FA, B)$ and $(A, B) \to \mathcal{C}(A, GB)$ are naturally isomorphic functors $\mathcal{C}^{op} \times \mathcal{D} \to$ **Set**).*

*We say F is **left adjoint** to G, or G is **right adjoint** to F, and write $F \dashv G$.*

Theorem 3.2. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Given $A \in \operatorname{ob}\mathcal{C}$, let $(A \downarrow G)$ be the category whose objects are pairs (B, f) with $B \in \operatorname{ob}\mathcal{D}$, $f : A \to GB$ and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ in $\mathcal{D}$ such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"] & GB \arrow[d, "Gg"] \\ & GB' \end{tikzcd}$$

commutes. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A.

Proof. First suppose G has a left adjoint F. Let $\eta_A : A \to GFA$ be the morphism corresponding to $1_{FA} : FA \to FA$. The pair (FA, η_A) is an object of $(A \downarrow G)$. We'll show it's initial.

Given $g : FA \to B$, the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$ must correspond to $FA \xrightarrow{1} FA \xrightarrow{g} B$ under the adjunction.

So, for any object (B, f) of $(A \downarrow G)$, the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$ is the morphism $FA \to B$ corresponding to f.

Conversely, suppose we're given an initial object (FA, η_A) of $(A \downarrow G)$ for each G. Given $f : A \to A'$, the composite $A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA'$ is an object of $(A \downarrow G)$, so there's a unique morphism $Ff : FA \to FA'$ making

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{a'}"] & GFA' \end{tikzcd}$$

commute.

$f \mapsto Ff$ is functorial: given $f' : A' \to A''$, then $(Ff')(Ff)$ and $F(f'f)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''} f' f)$ in $(A \downarrow G)$, so they're equal.

Finally, given $f : A \to GB$, the morphism $g : FA \to B$ corresponding to it is the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$.

The naturality of this bijection is given by naturality of η, and naturality in B is immediate. $\square$

Corollary 3.3. *If F, F' are both left-adjoint to G, then there's a canonical natural isomorphism $F \to F'$.*

Proof. For each A, (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \to (F'A, \eta'_A)$.

α is natural: given $f : A \to A'$, $\alpha_{A'} f$ and $(Ff)\alpha_A$ are both morphisms $(FA, \eta_A) \to (F'A', \eta'_{A'} f)$ in $(A \downarrow G)$. So they're equal. $\square$

Lemma 3.4. *Given $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$, if $F \dashv G$ and $H \dashv K$ then $HF \dashv GK$.*

Proof. We have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

which are natural in A and C. $\square$

Corollary 3.5. *Given a commutative square* $\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{E} \arrow[r, "K"] & \mathcal{F} \end{tikzcd}$ *of categories and functors, suppose all the functors in the diagram have left adjoints. Then the diagram* $\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{C} \end{tikzcd}$ *of left adjoints commutes up to natural isomorphism.*

Given $F \dashv G$, we have a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ defined as in 3.2. We call η the **unit** of the adjunction.

Dually, we have $\epsilon : FG \to 1_{\mathcal{D}}$, the **counit**. $\epsilon_B : FGB \to B$ corresponds to $1_{GB} : GB \to GB$.

Theorem 3.6. *Suppose we're given $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\epsilon : FG \to 1_{\mathcal{D}}$ such that*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

commute. (We say η and ϵ satisfy the ***triangular identities****).*

Proof. Given $F \dashv G$, we define η and ϵ as already described. Since $\epsilon_{FA} : FGFA \to FA$ corresponds to 1_{GFA}, the composite $\epsilon_{FA}(F\eta_A)$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$, so it must be 1_{FA}.

Similarly for the other identity.

Conversely, given η and ϵ satisfying the $\triangle^r$ identities, we map $f : A \to GB$ to the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$ and $g : FA \to B$ to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$.

We have

$$\Phi(\ A \xrightarrow{f} GB\ ) = \ FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$$
$$\Psi(\ FA \xrightarrow{g} B\ ) = \ A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$$

So

$$\begin{aligned}\Psi\Phi(f) &= \ A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= \ A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f\end{aligned}$$

And dually $\Phi\Psi(g) = g$.

Naturality of Φ in A is immediate from its definition, and naturality in B follows from that of ϵ. ☐

Lemma 3.7. *Suppose given* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *and natural isomorphisms* $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. *Then there exist natural isomorphisms* α', β' *which additionally satisfy the triangular identities. In particular* $(F \dashv G)$.

Proof. We define $\alpha' = \alpha$ and take β' to be the composite

$$FG \xrightarrow{\beta^{-1}_{FG}} FGFG \xrightarrow{F^{-1}_{\alpha_G}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that, since
$$\begin{tikzcd} FGFG \arrow[r, "FG\beta"] \arrow[d, "\beta_{FG}"'] & FG \arrow[d, "\beta"] \\ FG \arrow[r, "\beta"] & 1_{\mathcal{D}} \end{tikzcd}$$
commutes and β is monic, we have $FG\beta = \beta_F G$.

Similarly, $GF\alpha = \alpha_{GF} : GF \to GFGF$.

Now

$$\begin{aligned}\beta'_F \circ F_{\alpha'} &= \ F \xrightarrow{F\alpha} FGF \xrightarrow{\beta^{-1}_{FGF}} FGFGF \xrightarrow{F\alpha^{-1}_{GF}} FGF \xrightarrow{\beta_F} F \\ &= \ F \xrightarrow{\beta^{-1}_F} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{FGF\alpha^{-1}} FGF \xrightarrow{\beta_F} F \\ &= 1_F\end{aligned}$$

and

$$\begin{aligned}
G\beta' \circ \alpha'_G = {} & G \xrightarrow{\alpha_G} GFG \xrightarrow{GFG\beta^{-1}} GFGFG \xrightarrow{GF_{\alpha_G}^{-1}} GFG \xrightarrow{G\beta} G \\
= {} & G \xrightarrow{G\beta^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha_{GFG}^{-1}} GFG \xrightarrow{\beta_F} G \\
= {} & 1_G
\end{aligned}$$

□

Lemma 3.8. *Suppose* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *, $(F \dashv G)$ is an adjunction with counit ϵ. Then*

i. ϵ is (pointwise) epic $\iff$ G is faithful

ii. ϵ is an isomorphism $\iff$ G is full and faithful

Proof. i. Given $g : B \to B'$, the morphism $Gg : GB \to GB'$ corresponds to

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$$

So, for fixed B, composition with ϵ_B is injective on morphisms $B \to B'$ $\iff$ $(g \mapsto Gg)$ is injective on morphisms $B \to B'$.

Hence G is faithful $\iff$ ϵ_B is epic $\forall B$.

ii. Similarly, ϵ_B is 0 $\forall B \implies G$ is bijective on morphisms with given domain and codomain, i.e. G is full and faithful.

Conversely, if G is full and faithful, 1_{FGB} factors uniquely as $FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} FGB$, so ϵ_B is split monic. But it's epic by (i), hence an isomorphism.

□

Definition 3.9. *i. A* ***reflection*** *is an adjunction satisfying the conditions of 3.8(ii).*

ii. A ***reflective*** *subcategory of $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ for which the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint.*

Dually, ***coreflection*** *and* ***coreflective*** *subcategory.*

4 Limits

Definition 4.1. *a. Let J be a category (almost always small, often finite). A* ***diagram of shape J*** *in a category $\mathcal{C}$ is a functor $D : J \to \mathcal{C}$.*

E.g. if J is the finite category $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ *, a diagram of shape J is a commutative square. If J is the category* $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ *, a diagram of shape J is a not-necessarily-commutative square.*

The objects $D(j)$, $j \in \operatorname{ob} J$ are called ***vertices*** *of D, and the morphisms $D(\alpha)$, $\alpha \in \operatorname{mor} J$ are called* ***edges*** *of D.*

b. *Let $D : J \to \mathcal{C}$ be a diagram in $\mathcal{C}$. A* ***cone over D*** *is a pair $(A, (\lambda_j \mid j \in \operatorname{ob} J))$ where $\lambda_j : A \to D(j)$ $\forall j$, and* $\begin{tikzcd} & A \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"] & & D(j') \end{tikzcd}$ *commutes for each $\alpha : j \to j'$ in J.*

A is called the ***apex*** *of the cone, and the λ_j are its* ***legs****.*

Equivalently, λ is a natural transformation $\triangle A \to D$, where $\triangle A$ is the ***constant diagram*** *with all vertices A and all edges 1_A.*

A ***morphism*** *$f : (A, (\lambda_j)) \to (B, (\mu_j))$ of cones over D is a morphism $f : A \to B$ s.t.* $\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\lambda_j"'] & & B \arrow[dl, "\mu_j"] \\ & D(j) & \end{tikzcd}$ *commutes for each j. We have a category* ***Cone****(D) of cones over D.*

Note that $A \mapsto \triangle A$ is a functor $\mathcal{C} \to [J, \mathcal{C}]$ and ***Cone****(D) is in fact the category $(\triangle \downarrow D)$.*

A cocone over $D : J \to \mathcal{C}$ is a cone over $D : J^{op} \to \mathcal{C}^{op}$. We write ***Cocone***$(D)$ *for the category of cocones over D.*

Definition 4.2. i. *A* ***limit*** *(resp.* ***colimit****) for a diagram $D : J \to \mathcal{C}$ is a terminal object of* ***Cone****(D) (respectively an initial object of* ***Cocone****(D)).*

ii. *We say $\mathcal{C}$ has limits (resp. colimits) of shape J if $\triangle : \mathcal{C} \to [J, \mathcal{C}]$ has a right (resp. left) adjoint.*

(This is equivalent to making a choice of limit (resp. colimit) for every diagram of shape J).

Definition 4.3 (Pullback)**.** *Let J be* $\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ *. A diagram of shape J looks like*

$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$. *A cone over it consists of* $\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, "l"] & A \\ C & B \end{tikzcd}$ *satisfying* $fh = l = gk$.

Equivalently, it's a pair $\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] & A \\ C & \end{tikzcd}$ *completing the diagram to a commutative square.*

A universal such pair is called a ***pullback*** *(or* ***fibre product****); in* ***Set*** *it can be defined as* $\{(a,b) \in A \times B \mid f(a) = g(b)\}$. *A colimit of shape* J^{op} *is called a* ***pushout***.

Theorem 4.4. *Let* $\mathcal{C}$ *be a category.*

i. If $\mathcal{C}$ *has equalisers and all finite (resp. all small) products, then* $\mathcal{C}$ *has all finite (resp. all small) limits.*

ii. If $\mathcal{C}$ *has pullbacks and a terminal object, then* $\mathcal{C}$ *has all finite limits.*

Proof. i. Given $D : J \to \mathcal{C}$, first form the products

$$P = \prod_{j \in \mathrm{ob}\, J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \mathrm{mor}\, J} D(\operatorname{cod} \alpha)$$

Define $P \underset{g}{\overset{f}{\rightrightarrows}} Q$ by $\pi_\alpha f = \pi_{\operatorname{cod}\alpha} : P \to D(\operatorname{cod}\alpha)$ and $\pi_\alpha g = D(\alpha) \circ \pi_{\operatorname{dom}\alpha} : P \to D(\operatorname{dom}\alpha) \to D(\operatorname{cod}\alpha)$, and let $e : E \to P$ be the equaliser of (f, g).

Claim $(E, (\pi_j e \mid j \in \mathrm{ob}\, J))$ is a limit cone for D. It is a cone since, for any $\alpha : j \to j'$, $D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e$.

Given any cone $(C, (\lambda_j \mid j \in \mathrm{ob}\, J))$, the λ_j define a unique $\lambda : C \to P$, and $f\lambda = g\lambda$ since $\pi_\alpha f \lambda = \pi_\alpha g \lambda\ \forall \alpha$. So λ factors uniquely through e.

ii. Let 1 be a terminal object of $\mathcal{C}$. For any pair of objects (A, B) the pullback of $\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$ has the universal property of a product $A \times B$, so $\mathcal{C}$ has binary products. Then we can define any finite product $\prod_{i=1}^n A_i$ as $(((A_1 \times A_2) \times A_3) \times \dots) \times A_n$.

So we need to show $\mathcal{C}$ has equalisers. Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$, consider the pullback of $\begin{tikzcd} & B \arrow[d, "(1_A, f)"] \\ A \arrow[r, "(1_A, g)"] & A \times B \end{tikzcd}$.

It consists of $\begin{array}{ccc} P & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \\ A & & \end{array}$ satisfying $1_A h = 1_A k$ and $fh = gk$, and universal among such.

But this forces $h = k$, and h has the universal property of an equaliser for (f, g). So by (i), $\mathcal{C}$ has all finite limits.

□

Definition 4.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

a. *We say F* ***preserves*** *limits of shape J if, given $D : J \to \mathcal{C}$ and a limit cone $(L, (\lambda_j : j \in \mathrm{ob}\, J))$ for D, the cone $(FL, (F\lambda_j : j \in \mathrm{ob}\, J))$ is a limit for $FD : J \to \mathcal{D}$.*

b. *We say F* ***reflects*** *limits of shape J if, given $D : J \to \mathcal{C}$ and a cone $(L, (\lambda_j))$ such that $(FL, (F\lambda_j))$ is a limit for FD, then $(L, (\lambda_j))$ is a limit for D.*

c. *We say F* ***creates*** *limits of shape J if, given $D : J \to \mathcal{C}$ and a limit $(M, (\mu_j))$ for FD, there exists a cone (L, λ_j) over D whose image is isomorphic to $(M, (\mu_j))$, and any such cone is a limit for D.*

Lemma 4.6. *Suppose $\mathcal{D}$ has limits of shape J. Then $[\mathcal{C}, \mathcal{D}]$ has limits of shape J, and they're constructed pointwise (i.e. the forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\mathrm{ob}\,\mathcal{C}}$ creates them).*

Proof. Consider a functor $D : J \times \mathcal{C} \to \mathcal{D}$. For each $A \in \mathrm{ob}\,\mathcal{C}$, let $(LA, (\lambda_{j,A} : LA \to D(j, A) \mid j \in \mathrm{ob}\, J))$ be a limit for the diagram $D(-, A) : J \to \mathcal{D}$.

Given any $f : A \to B$ in $\mathcal{C}$, the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, so they induce a unique $Lf : LA \to LB$ such that

$$\begin{tikzcd} LA \arrow[r, "Lf"] \arrow[d, "\lambda_{j,A}"] & LB \arrow[d, "\lambda_{j,B}"] \\ D(j, A) \arrow[r, "D(j,f)"] & D(j, B) \end{tikzcd}$$

commutes for all j. Uniqueness assures $L(gf) = L(g)L(f)$, so L is a functor $\mathcal{C} \to \mathcal{D}$, and the $\lambda_{j,-}$ are natural transformations $L \to D(j, -)$.

Suppose we're given any cone over D in $[\mathcal{C}, \mathcal{D}]$ with apex M and legs $\mu_j : M \to D(j, -)$. Then $(MA, (\mu_{j,A} : MA \to D(j, A) \mid j \in \mathrm{ob}\, J))$ is a cone over $D(-, A)$ in $\mathcal{D}$, so we get a unique $\nu_A : MA \to LA$ s.t. $\lambda_{j,A}\nu_A = \mu_{j,A}$ for all j.

Uniqueness tells us that

$$\begin{array}{ccc} MA & \xrightarrow{Mf} & MB \\ \Big\downarrow{\scriptstyle \nu_A} & & \Big\downarrow{\scriptstyle \nu_B} \\ LA & \xrightarrow{Lf} & LB \end{array}$$

commutes for all $f \in \operatorname{mor} \mathcal{C}$, so $\nu : M \to L$ in $[\mathcal{C}, \mathcal{D}]$, so it's the unique factorisation of the $\mu_{j,-}$ through the $\lambda_{j,-}$. $\square$

Lemma 4.7. *A morphism $f : A \to B$ is monic $\iff$*

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \Big\downarrow{\scriptstyle 1_A} & & \Big\downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback.

Proof. f is monic $\iff$ any cone (g,h) over (f, f) has $g = h$ $\iff$ (g, h) factors uniquely through $(1_A, 1_A)$. $\square$

Hence, provided $\mathcal{D}$ has pullbacks, a morphism $\alpha : F \to G$ in $[\mathcal{C}, \mathcal{D}]$ is monic $\iff$ $\alpha_A : FA \to GA$ is monic for each A.

Theorem 4.8. *If $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, then G preserves all limits which exist in $\mathcal{D}$.*

Proof. Suppose $\mathcal{C}$ and $\mathcal{D}$ both have limits of shape J and let $(F \dashv G)$. The diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \Big\downarrow{\scriptstyle \triangle} & & \Big\downarrow{\scriptstyle \triangle} \\ [J, \mathcal{C}] & \xrightarrow{[J,F]} & [J, \mathcal{D}] \end{array}$$

commutes and $[J, F]$ has a right adjoint $[J, G]$. So by 3.5 the diagram of right adjoints

$$\begin{array}{ccc} [J, \mathcal{D}] & \xrightarrow{[J,G]} & [J, \mathcal{C}] \\ \Big\downarrow{\scriptstyle \lim_J} & & \Big\downarrow{\scriptstyle \lim_J} \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

commutes up to isomorphism, i.e. G preserves limits of shape J. $\square$

Proof. Let $D : J \to \mathcal{D}$ be a diagram with limit $(L, (\lambda_j \mid j \in \operatorname{ob} J))$. Given a cone $(A, (\mu_j : A \to GD(j) \mid j \in \operatorname{ob} J))$ in $\mathcal{C}$, we get a cone $(FA, (\bar{\mu_j} : FA \to D(j) \mid j \in \operatorname{ob} J))$ in $\mathcal{D}$, and hence a unique $\bar{\nu} : FA \to L$ such that $\lambda_j \bar{\nu} = \bar{\mu_j}$ for all j.

Then $\nu : A \to GL$ is the unique morphism such that $(G\lambda_j)\nu = \mu_j \forall j$. $\square$

The 'primeval' Adjoint Functor Theorem says that if $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits, then G has a left adjoint.

This depends on two ideas:

Lemma 4.9. *$\mathcal{C}$ has an initial object $\iff 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ has a limit.*

Proof. Suppose $\mathcal{C}$ has an initial object 0. The morphisms $(0 \to A \mid A \in \operatorname{ob}\mathcal{C})$ form a cone over $1_{\mathcal{C}}$. If we had another, say $(L, (\lambda_A \mid A \in \operatorname{ob}\mathcal{C}))$, then $\lambda_0 : L \to 0$ would make

$$\begin{tikzcd} L \arrow[r, "\lambda_0"] \arrow[dr, "\lambda_A"'] & 0 \arrow[d] \\ & A \end{tikzcd}$$

commute for all A, and it's the only morphism which does.

Conversely, suppose $(I, (\lambda_A : I \to A \mid A \in \operatorname{ob}\mathcal{C}))$ is a limit for $1_{\mathcal{C}}$.

If $f : I \to A$, then

$$\begin{tikzcd} I \arrow[r, "\lambda_I"] \arrow[dr, "\lambda_A"'] & I \arrow[d, "f"] \\ & A \end{tikzcd}$$

commutes. In particular, $\lambda_A \lambda_I = \lambda_A$ for all A, so $\lambda_I = 1_I$ since both are factorisations of the limit cone through itself. So $f = \lambda_A$, and hence I is initial. □

Lemma 4.10. *Suppose $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then, for each $A \in \operatorname{ob}\mathcal{C}$, $(A \downarrow G)$ has limits of shape J and the forgetful functor $(A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Suppose given $D : J \to (A \downarrow G)$. Write $D(j)$ as $(UD(j), f_j : A \to GUD(j))$ for each j. Let $(L, (\lambda_j \mid j \in \operatorname{ob} J))$ be a limit for UD, then $(GL, (G\lambda_j \mid j \in \operatorname{ob} J))$ is a limit for GUD. But the f_j form a cone over GUD with apex A, so there's a unique $h : A \to GL$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f_j"'] & GL \arrow[d, "G\lambda_j"] \\ & GUD(j) \end{tikzcd}$$

commutes for all j. So there's a unique lifting of the cone over D in $(A \downarrow G)$.

Suppose we're given a cone $((B, g), (\mu_j \mid j \in \operatorname{ob} J))$ over D. Then

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "h"'] & GB \arrow[d, "G_k"] \\ & GL \end{tikzcd}$$

commutes since both ways round are factorisations of $(f_j \mid j \in \mathrm{ob}\, J)$ through the limit GL. $\square$

Combining 4.10 and 4.9 with 3.2, we've proved the primeval Adjoint Functor Theorem. However, this requries $\mathcal{D}$ to have limits for diagrams 'as big as $\mathcal{D}$ itself', and the only such categories are preorders (c.f. Q6, sheet 2).

In practice, the most we can hope for is that $\mathcal{D}$ has all small limits. We call such a $\mathcal{D}$ **complete**.

Theorem 4.11 (General Adjoint Functor Theorem)**.** *Suppose that $\mathcal{D}$ is complete and locally small. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if it preserves all small limits and satisfies the 'solution set condition': for any $A \in \mathrm{ob}\,\mathcal{C}$, there is a set $\{f_i : A \to GB_i \mid i \in I\}$ of objects of $(A \downarrow G)$ such that any $h : A \to GC$ factors as*

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} Gc$$

for some $i \in I$ and $g : B_i \to C$.

Proof. If G has a left adjoint, then it preserves small limits by 4.8, and $\{\eta_A : A \to GFA\}$ is a singleton solution set at A.

Conversely, each $(A \downarrow G)$ is complete by 4.10, and locally small since it admits a faithful functor to $\mathcal{D}$. So we need to show: if $\mathcal{A}$ is complete and locally small, and has a weakly initial set of objects $\{S_i \mid i \in I\}$, then $\mathcal{A}$ has an initial object.

First form $P = \prod_{i \in I} S_i$: then P is weakly initial.

Now form the limit $I \xrightarrow{a} P$ of the diagram $P \rightrightarrows P$ whose edges are all morphism $P \to P$ in $\mathcal{A}$.

Claim I is initial: it's weakly initial since it admits a morphism to P.

Suppose we had $I \underset{g}{\overset{f}{\rightrightarrows}} A$. Let $b : E \to I$ be an equaliser for (f, g): then there exists $c : P \to E$.

Now $P \xrightarrow{c} E \xrightarrow{b} I \xrightarrow{a} P$ is an edge of the diagram whose limit is I, but so is 1_P; so $abca = 1_P a = a$. But a is monic, so $bca = 1_I$. So b is (split) epic, and $f = g$. So all the $(A \downarrow G)$ have initial objects, hence by 3.2 G has a left adjoint. $\square$

The Special Adjoint Functor Theorem imposes additional conditions on $\mathcal{C}$ and $\mathcal{D}$ which ensure that every functor $\mathcal{D} \to \mathcal{C}$ preserving small limits has a left adjoint.

Definition 4.12. *a. A **subobject** of an object A is a monomorphism $A' \rightarrowtail A$. We write $\boldsymbol{Sub}_{\mathcal{C}}(A)$ for the full subcategory of $\mathcal{C}/A$ whose objects are subobjects of A: note that this category is a preorder.*

*b. We say $\mathcal{C}$ is **well-powered** if each $\boldsymbol{Sub}_{\mathcal{C}}(A)$ is equivalent to a small category, i.e. up to isomorphism each object has only a set of subobjects.*

*Dually, $\mathcal{C}$ is **well-copowered** if $\mathcal{C}^{op}$ is well-powered.*