

Part III Category Theory

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1 Definitions and Examples

Definition 1.1 (Category). A category \mathcal{C} consists of

- a. a collection $\text{ob } \mathcal{C}$ of **objects** A, B, C, \dots
- b. a collection $\text{mor } \mathcal{C}$ of **morphisms** f, g, h, \dots
- c. two operations dom, cod from morphisms to objects. We write $f : A \rightarrow B$ or $A \xrightarrow{f} B$ to mean ' f is a morphism and $\text{dom } f = A$ and $\text{cod } f = B$ '
- d. an operation assigning to each object A a morphism $1_A : A \rightarrow A$
- e. a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff \text{dom } g = \text{cod } f$, and then $gf : \text{dom } f \rightarrow \text{cod } g$

satisfying

- f. $f1_A = f$ and $1_B f = f \ \forall f : A \rightarrow B$

g. $h(fg) = (hg)f$ whenever gf and hg are defined

Definition 1.2 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A **functor** $\mathcal{C} \rightarrow \mathcal{D}$ consists of

- a. a mapping $A \rightarrow FA$ from $\text{ob } \mathcal{C}$ to $\text{ob } \mathcal{D}$
- b. a mapping $f \rightarrow Ff$ from $\text{mor } \mathcal{C}$ to $\text{mor } \mathcal{D}$

satisfying $\text{dom } Ff = F\text{dom } f$, $\text{cod } Ff = F\text{cod } f$ for all f , $F(1_A) = 1_{FA}$ for all A , and $F(gf) = (Fg)(Ff)$ whenever gf is defined.

Definition 1.3. By a **contravariant functor** $\mathcal{C} \rightarrow \mathcal{D}$ we mean a functor $\mathcal{C} \rightarrow \mathcal{D}^{op}$ (or equivalently $\mathcal{C}^{op} \rightarrow \mathcal{D}$). A functor $\mathcal{C} \rightarrow \mathcal{D}$ is sometimes said to be **covariant**.

Definition 1.4 (Natural transformation). Let \mathcal{C} and \mathcal{D} be two categories and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ two functors. A **natural transformation** $\alpha : F \rightarrow G$ assigns to each $A \in \text{ob } \mathcal{C}$ a morphism $\alpha_A : FA \rightarrow GA$ in \mathcal{D} , such that

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A -component of a natural transformation $\beta\alpha : F \rightarrow H$.

Definition 1.5. Given categories \mathcal{C}, \mathcal{D} , we write $[\mathcal{C}, \mathcal{D}]$ for the category of all functors $\mathcal{C} \rightarrow \mathcal{D}$ and natural transformations between them.

Lemma 1.6. *Given $F, G : \mathcal{C} \rightarrow \mathcal{D}$ and $\alpha : F \rightarrow G$, α is an isomorphism in $[\mathcal{C}, \mathcal{D}] \iff$ each α_A is an isomorphism in \mathcal{D} .*

Definition 1.7 (Faithful and full). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- a. We say that F is **faithful** if, given $f, g \in \text{mor } \mathcal{C}$, the equations $\text{dom } f = \text{dom } g$, $\text{cod } f = \text{cod } g$ and $Ff = Fg$ imply $f = g$.
- b. F is **full** if, given any $g : FA \rightarrow FB$ in \mathcal{D} , there exists $f : A \rightarrow B$ in \mathcal{C} with $Ff = g$.
- c. We say a subcategory \mathcal{C}' of \mathcal{C} is **full** if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition 1.8 (Equivalence of categories). Let \mathcal{C} and \mathcal{D} be categories. An **equivalence** between \mathcal{C} and \mathcal{D} is a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$, $\beta : FG \rightarrow 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to mean that \mathcal{C} and \mathcal{D} are equivalent.

We say a property P of categories is **categorical** if whenever \mathcal{C} has P and $\mathcal{C} \simeq \mathcal{D}$ then \mathcal{D} has P .

For example, being a groupoid is a categorical property, but being a group is not.

Definition 1.9 (Slice category). Given an object B of a category \mathcal{C} , define the **slice category** \mathcal{C}/B to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \rightarrow (A' \xrightarrow{f'} B)$ are morphisms $h : A \rightarrow A'$ making

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ & \searrow f & \swarrow f' \\ & B & \end{array}$$

commute.

Lemma 1.10. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and **essentially surjective**, i.e. for every $B \in \text{ob } \mathcal{D}$, there exists $A \in \text{ob } \mathcal{C}$ s.t. $FA \cong B$.*

Definition 1.11. a. A **skeleton** of a category \mathcal{C} is a full subcategory \mathcal{C}' containing exactly one object from each isomorphism class of objects of \mathcal{C} .

b. We say \mathcal{C} is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism f in \mathcal{C} satisfies $\text{dom } f = \text{cod } f$.

For example, \mathbf{Mat}_K is skeletal. The full subcategory of standard vector spaces K^n is a skeleton of $\mathbf{fd Mod}_K$.

Remark 1.12. The following statements are each equivalent to the Axiom of Choice:

1. Every small category has a skeleton
2. Any small category is equivalent to each of its skeletons
3. Any two skeletons of a given small category are isomorphic

Definition 1.13. Let $f : A \rightarrow B$ be a morphism in a category \mathcal{C} .

- a. f is a **monomorphism** if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightharpoonup B$ if f is monic.

- b. Dually, f is an **epimorphism** if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.
- c. \mathcal{C} is a **balanced** category if every $f \in \text{mor } \mathcal{C}$ which is both monic and epic is an isomorphism.

2 The Yoneda Lemma

Definition 2.1. A category \mathcal{C} is **locally small** if, for any two objects A, B of \mathcal{C} , the morphism $A \rightarrow B$ are parametrised by a set $\mathcal{C}(A, B)$.

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$: if $g : B \rightarrow B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \rightarrow B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \rightarrow \mathbf{Set}$.

Lemma 2.2 (Yoneda). *Let \mathcal{C} be a locally small category, $A \in \text{ob } \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathbf{Set}$. Then*

- i. *There is a bijection between natural transformations $\mathcal{C}(A, -) \rightarrow F$ and elements of FA .*
- ii. *Moreover, this bijection is natural in both A and F .*

Proof. Bijection: given $\alpha : \mathcal{C}(A, -) \rightarrow F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.

Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$ is natural: given $g : B \rightarrow C$, we have

$$\begin{aligned} \Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\ &= F(gf)(x) \\ &= (Fg)(Ff)(x) \\ &= (Fg)\Psi(x)_B(f) \end{aligned}$$

$\Phi\Psi(x) = x$ since $F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$ since, for any $f : A \rightarrow B$,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\ &= Ff(\alpha_A(1_A)) \\ &= \alpha_B(\mathcal{C}(A, f)(1_A)) \\ &= \alpha_B(f) \end{aligned}$$

□

Corollary 2.3. *The mapping $A \rightarrow \mathcal{C}(A, -)$ is a full and faithful functor $\mathcal{C}^{op} \rightarrow [\mathcal{C}, \mathbf{Set}]$.*

Proof. Given two objects A, B , 2.2(i) gives us a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$ (by taking $F : C \mapsto \mathcal{C}(B, C)$). We need to show this is functorial, but given $f \in \mathcal{C}(B, A)$, $\Psi(F)_A$ sends 1_A to $\mathcal{C}(B, f)(1_A) = f$, so it's the natural transformation $g \mapsto gf$.

Hence, given $e : C \rightarrow B$, $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$ \square

We call this functor the **Yoneda embedding**. Hence any locally small category \mathcal{C} is equivalent to a full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$.

Definition 2.4. A functor $\mathcal{C} \rightarrow \mathbf{Set}$ is **representable** if it's isomorphic to $\mathcal{C}(A, -)$ for some A .

A **representation** of $F : \mathcal{C} \rightarrow \mathbf{Set}$ is a pair (A, x) where $A \in \text{ob } \mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$ is an isomorphism. We also call x a **universal element** of F .

Corollary 2.5 ('Representations are unique up to unique isomorphism'). *If (A, x) and (B, y) are both representations of $F : \mathcal{C} \rightarrow \mathbf{Set}$, then there's a unique isomorphism $f : A \rightarrow B$ s.t. $Ff(x) = y$.*

Definition 2.6 (Product and coproduct). Given two objects A, B of a locally small category \mathcal{C} , we define their **product** to be a representation of the functor

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{op} \rightarrow \mathbf{Set}$$

i.e. an object $A \times B$ equipped with morphisms $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$ s.t. given any pair $(f : C \rightarrow A, g : C \rightarrow B)$, there exists a unique $h : C \rightarrow A \times B$ s.t. $\pi_1 h = f$ and $\pi_2 h = g$.

More generally, we can define the product $\prod_{i \in I} A_i$ of a family $\{A_i \mid i \in I\}$ of objects, or the product of the empty family, i.e. a **terminal object** 1 s.t. for every A there's a unique $A \rightarrow 1$.

Dualizing, we get the notion of **coproduct** or **sum**.

Definition 2.7 (Equaliser and coequaliser). Given a parallel pair $f, g : A \rightrightarrows B$ in a locally small category \mathcal{C} , the assignment $C \mapsto FC = \{h : C \rightarrow A \mid fh = gh\}$ is a subfunctor F of $\mathcal{C}(-, A)$. A representation of F is called an **equaliser** of (f, g) .

In elementary terms, it's an object E equipped with $e : E \rightarrow A$ s.t. $fe = ge$, s.t. any h with $fh = gh$ factors uniquely as $h = ek$

Dually, we have the notion of **coequaliser**, i.e. a morphism $q : B \rightarrow Q$ satisfying $qf = qg$, and universal among such.

Definition 2.8. a. We say a monomorphism is **regular** if it occurs as an equaliser (dually, regular epimorphism).

b. We say $f : A \rightarrow B$ is a **split monomorphism** if there exists $g : B \rightarrow A$ with $gf = 1_A$.

Every split monomorphism is regular: if $gf = 1_A$, f is an equaliser of $(1_B, fg)$ [see sheet 1, q2].

Definition 2.9. Let \mathcal{C} be a (locally small) category, \mathcal{G} a collection of objects of \mathcal{C} .

a. Say \mathcal{G} is a **separating family** if the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ are jointly faithful, i.e. if given $f, g : A \rightrightarrows B$ with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \rightarrow A$ with $fh \neq gh$.

b. Say \mathcal{G} is a **detecting family** if the $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ jointly reflect isomorphisms, i.e. if given $f : A \rightarrow B$ s.t. every $g : G \rightarrow B$ with $G \in \mathcal{G}$ factors uniquely through f , f is an isomorphism.

Lemma 2.10. i. If \mathcal{C} is balanced, then any separating family is detecting

ii. If \mathcal{C} has equalisers, then every detecting family is separating

Definition 2.11. An object P is **projective** if $\mathcal{C}(P, -)$ preserves epimorphisms, i.e. if given

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ A & \xrightarrow{e} & B \end{array}$$

there exists $g : P \rightarrow A$ with $eg = f$.

Dually, P is **injective** in \mathcal{C} if it's projective in \mathcal{C}^{op} .

If P satisfies this property $\forall e$ in some class \mathcal{E} of epimorphisms, we call it \mathcal{E} -projective.

Corollary 2.12. Representable functors are (pointwise) projective in $[\mathcal{C}, \mathbf{Set}]$

Proof. Given

$$\begin{array}{ccc} & \mathcal{C}(A, -) & \\ & \downarrow \beta & \\ F & \xrightarrow{\alpha} & G \end{array}$$

β corresponds to some $y \in GA$. α_A is surjective, so $\exists x \in FA$ with $\alpha_A(x) = y$. x corresponds to $\gamma : \mathcal{C}(A, -) \rightarrow F$ with $\alpha\gamma = \beta$. \square

3 Adjunctions

Definition 3.1 (D.M. Khan, 1958). Let \mathcal{C} and \mathcal{D} be categories and $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ be two functors. An **adjunction** between F and G is a bijection between morphisms $FA \rightarrow B$ in \mathcal{D} and morphisms $A \rightarrow GB$ in \mathcal{C} , which is natural in A and B .

(If \mathcal{C} and \mathcal{D} are locally small, this says that $(A, B) \rightarrow \mathcal{D}(FA, B)$ and $(A, B) \rightarrow \mathcal{C}(A, GB)$ are naturally isomorphic functors $\mathcal{C}^{op} \times \mathcal{D} \rightarrow \mathbf{Set}$).

We say F is **left adjoint** to G , or G is **right adjoint** to F , and write $F \dashv G$.

Theorem 3.2. Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor. Given $A \in \text{ob } \mathcal{C}$, let $(A \downarrow G)$ be the category whose objects are pairs (B, f) with $B \in \text{ob } \mathcal{D}$, $f : A \rightarrow GB$ and whose morphisms $(B, f) \rightarrow (B', f')$ are morphisms $g : B \rightarrow B'$ in \mathcal{D} such that

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & \searrow f' & \downarrow Gg \\ & & GB' \end{array}$$

commutes. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A .

Proof. First suppose G has a left adjoint F . Let $\eta_A : A \rightarrow GFA$ be the morphism corresponding to $1_{FA} : FA \rightarrow FA$. The pair (FA, η_A) is an object of $(A \downarrow G)$. We'll show it's initial.

Given $g : FA \rightarrow B$, the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$ must correspond to $FA \xrightarrow{1} FA \xrightarrow{g} B$ under the adjunction.

So, for any object (B, f) of $(A \downarrow G)$, the unique morphism $(FA, \eta_A) \rightarrow (B, f)$ in $(A \downarrow G)$ is the morphism $FA \rightarrow B$ corresponding to f .

Conversely, suppose we're given an initial object (FA, η_A) of $(A \downarrow G)$ for each G . Given $f : A \rightarrow A'$, the composite $A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA'$ is an object of $(A \downarrow G)$, so there's a unique morphism $Ff : FA \rightarrow FA'$ making

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow GFf \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commute.

$f \mapsto Ff$ is functorial: given $f' : A' \rightarrow A''$, then $(Ff')(Ff)$ and $F(f'f)$ are both morphisms $(FA, \eta_A) \rightarrow (FA'', \eta_{A''} f'f)$ in $(A \downarrow G)$, so they're equal.

Finally, given $f : A \rightarrow GB$, the morphism $g : FA \rightarrow B$ corresponding to it is the unique morphism $(FA, \eta_A) \rightarrow (B, f)$ in $(A \downarrow G)$.

The naturality of this bijection is given by naturality of η , and naturality in B is immediate. \square

Corollary 3.3. *If F, F' are both left-adjoint to G , then there's a canonical natural isomorphism $F \rightarrow F'$.*

Proof. For each A , (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \rightarrow (F'A, \eta'_A)$.

α is natural: given $f : A \rightarrow A'$, $\alpha_{A'}f$ and $(Ff)\alpha_A$ are both morphisms $(FA, \eta_A) \rightarrow (F'A', \eta'_{A'})$ in $(A \downarrow G)$. So they're equal. \square

Lemma 3.4. *Given $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D} \xrightleftharpoons[K]{H} \mathcal{E}$, if $F \dashv G$ and $H \dashv K$ then $HF \dashv GK$.*

Proof. We have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

which are natural in A and C . \square

Corollary 3.5. *Given a commutative square*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow G & & \downarrow H \\ \mathcal{E} & \xrightarrow{K} & \mathcal{F} \end{array}$$

of categories and functors, suppose all the functors in the diagram have left adjoints. Then the

diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{\quad} & \mathcal{C} \end{array}$$

of left adjoints commutes up to natural isomorphism.

Given $F \dashv G$, we have a natural transformation $\eta : 1_{\mathcal{C}} \rightarrow GF$ defined as in 3.2. We call η the **unit** of the adjunction.

Dually, we have $\epsilon : FG \rightarrow 1_{\mathcal{D}}$, the **counit**. $\epsilon_B : FGB \rightarrow B$ corresponds to $1_{GB} : GB \rightarrow GB$.

Theorem 3.6. *Suppose we're given $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \rightarrow GF$ and $\epsilon : FG \rightarrow 1_{\mathcal{D}}$ such that*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \epsilon_F \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & \searrow 1_G & \downarrow G\epsilon \\ & & G \end{array}$$

*commute. (We say η and ϵ satisfy the **triangular identities**).*

Proof. Given $F \dashv G$, we define η and ϵ as already described. Since $\epsilon_{FA} : FGFA \rightarrow FA$ corresponds to 1_{GFA} , the composite $\epsilon_{FA}(F\eta_A)$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$, so it must be 1_{FA} .

Similarly for the other identity.

Conversely, given η and ϵ satisfying the Δ^r identities, we map $f : A \rightarrow GB$ to the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$ and $g : FA \rightarrow B$ to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$.

We have

$$\begin{aligned}\Phi(A \xrightarrow{f} GB) &= FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B \\ \Psi(FA \xrightarrow{g} B) &= A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB\end{aligned}$$

So

$$\begin{aligned}\Psi\Phi(f) &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f\end{aligned}$$

And dually $\Phi\Psi(g) = g$.

Naturality of Φ in A is immediate from its definition, and naturality in B follows from that of ϵ . \square

Lemma 3.7. *Suppose given $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ and natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$, $\beta : FG \rightarrow 1_{\mathcal{D}}$. Then there exist natural isomorphisms α' , β' which additionally satisfy the triangular identities. In particular $(F \dashv G)$.*

Proof. We define $\alpha' = \alpha$ and take β' to be the composite

$$FG \xrightarrow{\beta_{FG}^{-1}} FGFG \xrightarrow{F\alpha_G^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that, since $\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \downarrow \beta_{FG} & & \downarrow \beta \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$ commutes and β is monic, we have $FG\beta = \beta_{FG}G$.

Similarly, $GF\alpha = \alpha_{GF} : GF \rightarrow GFGF$.

Now

$$\begin{aligned}\beta'_F \circ F\alpha' &= F \xrightarrow{F\alpha} FGF \xrightarrow{\beta_{FGF}^{-1}} FGFGF \xrightarrow{F\alpha_{GF}^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{\beta_F^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{FGF\alpha^{-1}} FGF \xrightarrow{\beta_F} F \\ &= 1_F\end{aligned}$$

and

$$\begin{aligned}G\beta' \circ \alpha'_G &= G \xrightarrow{\alpha_G} GFG \xrightarrow{GFG\beta^{-1}} GFGFG \xrightarrow{G\alpha_G^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{G\beta^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha_{GFG}^{-1}} GFG \xrightarrow{\beta_F} G \\ &= 1_G\end{aligned}$$

□

Lemma 3.8. Suppose $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$, $(F \dashv G)$ is an adjunction with counit ϵ . Then

i. ϵ is (pointwise) epic $\iff G$ is faithful

ii. ϵ is an isomorphism $\iff G$ is full and faithful

Proof. i. Given $g : B \rightarrow B'$, the morphism $Gg : GB \rightarrow GB'$ corresponds to

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$$

So, for fixed B , composition with ϵ_B is injective on morphisms $B \rightarrow B'$
 $\iff (g \mapsto Gg)$ is injective on morphisms $B \rightarrow B'$.

Hence G is faithful $\iff \epsilon_B$ is epic $\forall B$.

ii. Similarly, ϵ_B is 0 $\forall B \implies G$ is bijective on morphisms with given domain and codomain, i.e. G is full and faithful.

Conversely, if G is full and faithful, 1_{FGB} factors uniquely as

$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} FGB$, so ϵ_B is split monic. But it's epic by (i), hence an isomorphism.

□

Definition 3.9. i. A **reflection** is an adjunction satisfying the conditions of 3.8(ii).

ii. A **reflective** subcategory of \mathcal{C} is a full subcategory \mathcal{C}' for which the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint.

Dually, **coreflection** and **coreflective** subcategory.

4 Limits

Definition 4.1. a. Let J be a category (almost always small, often finite).

A **diagram of shape J** in a category \mathcal{C} is a functor $D : J \rightarrow \mathcal{C}$.

E.g. if J is the finite category $\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$, a diagram of shape J is a

commutative square. If J is the category $\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$, a diagram of shape

J is a not-necessarily-commutative square.

The objects $D(j)$, $j \in \text{ob } J$ are called **vertices** of D , and the morphisms $D(\alpha)$, $\alpha \in \text{mor } J$ are called **edges** of D .

- b. Let $D : J \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} . A **cone over D** is a pair $(A, (\lambda_j \mid j \in \text{ob } J))$ where $\lambda_j : A \rightarrow D(j) \forall j$, and

$$\begin{array}{ccc} & A & \\ \lambda_j \swarrow & & \searrow \lambda_{j'} \\ D(j) & \xrightarrow{D(\alpha)} & D(j') \end{array} \quad \text{commutes for}$$

each $\alpha : j \rightarrow j'$ in J .

A is called the **apex** of the cone, and the λ_j are its **legs**.

Equivalently, λ is a natural transformation $\Delta A \rightarrow D$, where ΔA is the **constant diagram** with all vertices A and all edges 1_A .

A **morphism** $f : (A, (\lambda_j)) \rightarrow (B, (\mu_j))$ of cones over D is a morphism

$$f : A \rightarrow B \text{ s.t. } \begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \lambda_j & \swarrow \mu_j \\ & D(j) & \end{array} \text{ commutes for each } j. \text{ We have a category}$$

Cone(D) of cones over D .

Note that $A \mapsto \Delta A$ is a functor $\mathcal{C} \rightarrow [J, \mathcal{C}]$ and **Cone**(D) is in fact the category $(\Delta \downarrow D)$.

A **cocone over D** is a cone over $D : J^{op} \rightarrow \mathcal{C}^{op}$. We write **Cocone**(D) for the category of cocones over D .

Definition 4.2. i. A **limit** (resp. **colimit**) for a diagram $D : J \rightarrow \mathcal{C}$ is a terminal object of **Cone**(D) (respectively an initial object of **Cocone**(D)).

- ii. We say \mathcal{C} has limits (resp. colimits) of shape J if $\Delta : \mathcal{C} \rightarrow [J, \mathcal{C}]$ has a right (resp. left) adjoint.

(This is equivalent to making a choice of limit (resp. colimit) for every diagram of shape J).

Definition 4.3 (Pullback). Let J be $\begin{array}{ccc} & \cdot & \\ & \downarrow & \\ \cdot & \longrightarrow & \cdot \end{array}$. A diagram of shape J looks

$$\text{like } \begin{array}{ccc} & A & \\ & \downarrow f & \\ B & \xrightarrow{g} & C \end{array} \text{ A cone over it consists of } \begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & \searrow l & \\ C & & B \end{array} \text{ satisfying } fh =$$

$l = gk$. Equivalently, it's a pair $\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & & \\ C & & \end{array}$ completing the diagram to a commutative square.

A universal such pair is called a **pullback** (or **fibre product**); in **Set** it can be defined as $\{(a, b) \in A \times B \mid f(a) = g(b)\}$. A colimit of shape J^{op} is called a **pushout**.

Theorem 4.4. *Let \mathcal{C} be a category.*

i. *If \mathcal{C} has equalisers and all finite (resp. all small) products, then \mathcal{C} has all finite (resp. all small) limits.*

ii. *If \mathcal{C} has pullbacks and a terminal object, then \mathcal{C} has all finite limits.*

Proof. i. Given $D : J \rightarrow \mathcal{C}$, first form the products

$$P = \prod_{j \in \text{ob } J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \text{mor } J} D(\text{cod } \alpha)$$

Define $P \xrightarrow[f]{f} Q$ by $\pi_\alpha f = \pi_{\text{cod } \alpha} : P \rightarrow D(\text{cod } \alpha)$ and $\pi_\alpha g = D(\alpha) \circ \pi_{\text{dom } \alpha} : P \rightarrow D(\text{dom } \alpha) \rightarrow D(\text{cod } \alpha)$, and let $e : E \rightarrow P$ be the equaliser of (f, g) .

Claim $(E, (\pi_j e \mid j \in \text{ob } J))$ is a limit cone for D . It is a cone since, for any $\alpha : j \rightarrow j'$, $D(\alpha)\pi_j e = \pi_{\alpha} g e = \pi_\alpha f e = \pi_{j'} e$.

Given any cone $(C, (\lambda_j \mid j \in \text{ob } J))$, the λ_j define a unique $\lambda : C \rightarrow P$, and $f\lambda = g\lambda$ since $\pi_\alpha f\lambda = \pi_\alpha g\lambda \forall \alpha$. So λ factors uniquely through e .

ii. Let 1 be a terminal object of \mathcal{C} . For any pair of objects (A, B) the pullback

of $\begin{array}{ccc} & A & \\ & \downarrow & \\ B & \longrightarrow & 1 \end{array}$ has the universal property of a product $A \times B$, so \mathcal{C} has binary products. Then we can define any finite product $\prod_{i=1}^n A_i$ as $((A_1 \times A_2) \times A_3) \times \dots \times A_n$.

So we need to show \mathcal{C} has equalisers. Given $A \xrightarrow[f]{f} B$, consider the

pullback of $\begin{array}{ccc} & B & \\ & \downarrow (1_A, f) & \\ A & \xrightarrow{(1_A, g)} & A \times B \end{array}$

It consists of $\begin{array}{ccc} P & \xrightarrow{h} & B \\ \downarrow k & & \\ A & & \end{array}$ satisfying $1_A h = 1_A k$ and $fh = gk$, and universal among such.

But this forces $h = k$, and h has the universal property of an equaliser for (f, g) . So by (i), \mathcal{C} has all finite limits.

□

Definition 4.5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- a. We say F **preserves** limits of shape J if, given $D : J \rightarrow \mathcal{C}$ and a limit cone $(L, (\lambda_j : j \in \text{ob } J))$ for D , the cone $(FL, (F\lambda_j : j \in \text{ob } J))$ is a limit for $FD : J \rightarrow \mathcal{D}$.
- b. We say F **reflects** limits of shape J if, given $D : J \rightarrow \mathcal{C}$ and a cone $(L, (\lambda_j))$ such that $(FL, (F\lambda_j))$ is a limit for FD , then $(L, (\lambda_j))$ is a limit for D .
- c. We say F **creates** limits of shape J if, given $D : J \rightarrow \mathcal{C}$ and a limit $(M, (\mu_j))$ for FD , there exists a cone (L, λ_j) over D whose image is isomorphic to $(M, (\mu_j))$, and any such cone is a limit for D .

Lemma 4.6. Suppose \mathcal{D} has limits of shape J . Then $[\mathcal{C}, \mathcal{D}]$ has limits of shape J , and they're constructed pointwise (i.e. the forgetful functor $[\mathcal{C}, \mathcal{D}] \rightarrow \mathcal{D}^{\text{ob } \mathcal{C}}$ creates them).

Proof. Consider a functor $D : J \times \mathcal{C} \rightarrow \mathcal{D}$. For each $A \in \text{ob } \mathcal{C}$, let $(LA, (\lambda_{j,A} : LA \rightarrow D(j, A) \mid j \in \text{ob } J))$ be a limit for the diagram $D(-, A) : J \rightarrow \mathcal{D}$.

Given any $f : A \rightarrow B$ in \mathcal{C} , the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, so they induce a unique $Lf : LA \rightarrow LB$ such that

$$\begin{array}{ccc} LA & \xrightarrow{Lf} & LB \\ \downarrow \lambda_{j,A} & & \downarrow \lambda_{j,B} \\ D(j, A) & \xrightarrow{D(j,f)} & D(j, B) \end{array}$$

commutes for all j . Uniqueness assures $L(gf) = L(g)L(f)$, so L is a functor $\mathcal{C} \rightarrow \mathcal{D}$, and the $\lambda_{j,-}$ are natural transformations $L \rightarrow D(j, -)$.

Suppose we're given any cone over D in $[\mathcal{C}, \mathcal{D}]$ with apex M and legs $\mu_j : M \rightarrow D(j, -)$. Then $(MA, (\mu_{j,A} : MA \rightarrow D(j, A) \mid j \in \text{ob } J))$ is a cone over $D(-, A)$ in \mathcal{D} , so we get a unique $\nu_A : MA \rightarrow LA$ s.t. $\lambda_{j,A}\nu_A = \mu_{j,A}$ for all j .

Uniqueness tells us that

$$\begin{array}{ccc} MA & \xrightarrow{Mf} & MB \\ \downarrow \nu_A & & \downarrow \nu_B \\ LA & \xrightarrow{Lf} & LB \end{array}$$

commutes for all $f \in \text{mor } \mathcal{C}$, so $\nu : M \rightarrow L$ in $[\mathcal{C}, \mathcal{D}]$, so it's the unique factorisation of the $\mu_{j,-}$ through the $\lambda_{j,-}$. \square

Lemma 4.7. A morphism $f : A \rightarrow B$ is monic \iff

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback.

Proof. f is monic \iff any cone (g, h) over (f, f) has $g = h \iff (g, h)$ factors uniquely through $(1_A, 1_A)$. \square

Hence, provided \mathcal{D} has pullbacks, a morphism $\alpha : F \rightarrow G$ in $[\mathcal{C}, \mathcal{D}]$ is monic $\iff \alpha_A : FA \rightarrow GA$ is monic for each A .

Theorem 4.8. *If $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint, then G preserves all limits which exist in \mathcal{D} .*

Proof. Suppose \mathcal{C} and \mathcal{D} both have limits of shape J and let $(F \dashv G)$. The diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \Delta & & \downarrow \Delta \\ [J, \mathcal{C}] & \xrightarrow{[J, F]} & [J, \mathcal{D}] \end{array}$$

commutes and $[J, F]$ has a right adjoint $[J, G]$. So by 3.5 the diagram of right adjoints

$$\begin{array}{ccc} [J, \mathcal{D}] & \xrightarrow{[J, G]} & [J, \mathcal{C}] \\ \downarrow \lim_J & & \downarrow \lim_J \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

commutes up to isomorphism, i.e. G preserves limits of shape J . \square

Proof. Let $D : J \rightarrow \mathcal{D}$ be a diagram with limit $(L, (\lambda_j \mid j \in \text{ob } J))$. Given a cone $(A, (\mu_j : A \rightarrow GD(j) \mid j \in \text{ob } J))$ in \mathcal{C} , we get a cone $(FA, (\bar{\mu}_j : FA \rightarrow D(j) \mid j \in \text{ob } J))$ in \mathcal{D} , and hence a unique $\bar{\nu} : FA \rightarrow L$ such that $\lambda_j \bar{\nu} = \bar{\mu}_j$ for all j .

Then $\nu : A \rightarrow GL$ is the unique morphism such that $(G\lambda_j)\nu = \mu_j \forall j$. \square

The ‘primeval’ Adjoint Functor Theorem says that if \mathcal{D} has and $G : \mathcal{D} \rightarrow \mathcal{C}$ preserves all limits, then G has a left adjoint.

This depends on two ideas:

Lemma 4.9. \mathcal{C} has an initial object $\iff 1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ has a limit.

Proof. Suppose \mathcal{C} has an initial object 0 . The morphisms $(0 \rightarrow A \mid A \in \text{ob } \mathcal{C})$ form a cone over $1_{\mathcal{C}}$. If we had another, say $(L, (\lambda_A \mid A \in \text{ob } \mathcal{C}))$, then $\lambda_0 : L \rightarrow 0$ would make

$$\begin{array}{ccc}
L & \xrightarrow{\lambda_0} & 0 \\
& \searrow \lambda_A & \swarrow \\
& & A
\end{array}$$

commute for all A , and it's the only morphism which does.

Conversely, suppose $(I, (\lambda_A : I \rightarrow A \mid A \in \text{ob } \mathcal{C}))$ is a limit for $1_{\mathcal{C}}$.

If $f : I \rightarrow A$, then

$$\begin{array}{ccc}
I & \xrightarrow{\lambda_I} & I \\
& \searrow \lambda_A & \swarrow f \\
& & A
\end{array}$$

commutes. In particular, $\lambda_A \lambda_I = \lambda_A$ for all A , so $\lambda_I = 1_I$ since both are factorisations of the limit cone through itself. So $f = \lambda_A$, and hence I is initial. \square

Lemma 4.10. *Suppose \mathcal{D} has and $G : \mathcal{D} \rightarrow \mathcal{C}$ preserves limits of shape J . Then, for each $A \in \text{ob } \mathcal{C}$, $(A \downarrow G)$ has limits of shape J and the forgetful functor $(A \downarrow G) \rightarrow \mathcal{D}$ creates them.*

Proof. Suppose given $D : J \rightarrow (A \downarrow G)$. Write $D(j)$ as $(UD(j), f_j : A \rightarrow GUD(j))$ for each j . Let $(L, (\lambda_j \mid j \in \text{ob } J))$ be a limit for UD , then $(GL, (G\lambda_j \mid j \in \text{ob } J))$ is a limit for GUD . But the f_j form a cone over GUD with apex A , so there's a unique $h : A \rightarrow GL$ such that

$$\begin{array}{ccc}
A & \xrightarrow{h} & GL \\
& \searrow f_j & \downarrow G\lambda_j \\
& & GUD(j)
\end{array}$$

commutes for all j . So there's a unique lifting of the cone over D in $(A \downarrow G)$.

Suppose we're given a cone $((B, g), (\mu_j \mid j \in \text{ob } J))$ over D . Then

$$\begin{array}{ccc}
A & \xrightarrow{g} & GB \\
& \searrow h & \downarrow G_k \\
& & GL
\end{array}$$

commutes since both ways round are factorisations of $(f_j \mid j \in \text{ob } J)$ through the limit GL . \square

Combining 4.10 and 4.9 with 3.2, we've proved the primeval Adjoint Functor Theorem. However, this requires \mathcal{D} to have limits for diagrams 'as big as \mathcal{D} itself', and the only such categories are preorders (c.f. Q6, sheet 2).

In practice, the most we can hope for is that \mathcal{D} has all small limits. We call such a \mathcal{D} **complete**.

Theorem 4.11 (General Adjoint Functor Theorem). *Suppose that \mathcal{D} is complete and locally small. Then a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if it preserves all small limits and satisfies the ‘solution set condition’: for any $A \in \text{ob } \mathcal{C}$, there is a set $\{f_i : A \rightarrow GB_i \mid i \in I\}$ of objects of $(A \downarrow G)$ such that any $h : A \rightarrow GC$ factors as*

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GC$$

for some $i \in I$ and $g : B_i \rightarrow C$.

Proof. If G has a left adjoint, then it preserves small limits by 4.8, and $\{\eta_A : A \rightarrow GFA\}$ is a singleton solution set at A .

Conversely, each $(A \downarrow G)$ is complete by 4.10, and locally small since it admits a faithful functor to \mathcal{D} . So we need to show: if \mathcal{A} is complete and locally small, and has a weakly initial set of objects $\{S_i \mid i \in I\}$, then \mathcal{A} has an initial object.

First form $P = \prod_{i \in I} S_i$: then P is weakly initial.

Now form the limit $I \xrightarrow{a} P$ of the diagram $P \rightrightarrows P$ whose edges are all morphism $P \rightarrow P$ in \mathcal{A} .

Claim I is initial: it’s weakly initial since it admits a morphism to P .

Suppose we had $I \xrightarrow[f]{g} A$. Let $b : E \rightarrow I$ be an equaliser for (f, g) : then there exists $c : P \rightarrow E$.

Now $P \xrightarrow{c} E \xrightarrow{b} I \xrightarrow{a} P$ is an edge of the diagram whose limit is I , but so is 1_P ; so $abca = 1_P a = a$. But a is monic, so $bca = 1_I$. So b is (split) epic, and $f = g$. So all the $(A \downarrow G)$ have initial objects, hence by 3.2 G has a left adjoint. \square

The Special Adjoint Functor Theorem imposes additional conditions on \mathcal{C} and \mathcal{D} which ensure that every functor $\mathcal{D} \rightarrow \mathcal{C}$ preserving small limits has a left adjoint.

Definition 4.12. a. A **subobject** of an object A is a monomorphism $A' \rightarrowtail A$. We write $\mathbf{Sub}_{\mathcal{C}}(A)$ for the full subcategory of \mathcal{C}/A whose objects are subobjects of A : note that this category is a preorder.

b. We say \mathcal{C} is **well-powered** if each $\mathbf{Sub}_{\mathcal{C}}(A)$ is equivalent to a small category, i.e. up to isomorphism each object has only a set of subobjects.

Dually, \mathcal{C} is **well-copowered** if \mathcal{C}^{op} is well-powered.

Lemma 4.13. *Suppose given a pullback*

$$\begin{array}{ccc}
P & \xrightarrow{k} & A \\
\downarrow h & & \downarrow f \\
B & \xrightarrow{g} & C
\end{array}$$

with f monic. Then h is monic.

Proof. Suppose $D \xrightarrow[x]{y} P$ satisfy $hx = hy$. Then $fkx = fky = ghx = ghy$ and f is monic so $kx = ky$.

Now $x = y$ since both are factorisations of the same cone through the pull-back. \square

Theorem 4.14 (Special Adjoint Functor Theorem). *Suppose both \mathcal{C} and \mathcal{D} are locally small, and \mathcal{D} is complete, well-powered and has a separating set. Then $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint $\iff G$ preserves all small limits.*

Proof. The forward implication is 4.8 again.

Conversely, we first show that $(A \downarrow G)$ has the properties we've assumed for \mathcal{D} : it's complete by 4.10, and locally small as in 4.11. It's well-powered since subobjects of (B, f) in $(A \downarrow G)$ are in bijection with subobjects $B' \rightarrowtail B$ such that f factors through $GB' \rightarrowtail GB$.

It has a coseparating set: if $\{S_i \mid i \in I\}$ is a coseparating set for \mathcal{D} , then $\{(S_i, f) \mid i \in I, f : A \rightarrow GS_i\}$ is a coseparating set for $(A \downarrow G)$, since if $(B, f) \xrightarrow[g']{g} (B', f')$ satisfies $g \neq g'$, there exists $h : B' \rightarrow S_i$ for some i with $hg \neq hg'$, and then h is a morphism $(B', f') \rightarrow (S_i, (Gh)f')$ in $(A \downarrow G)$.

Now we show that if \mathcal{A} is complete, locally small and well-powered and has a coseparating set, then it has an initial object.

First form $P = \prod_{i \in I} S_i$, where $\{S_i \mid i \in I\}$ is a coseparating set.

Consider the diagram

$$\begin{array}{ccc}
P' & & \\
\searrow & & \\
P'' & \xrightarrow{\quad} & P \\
\vdots & & \nearrow \\
P^{(n)} & &
\end{array}$$

whose edges are a representative set of subobjects of P .

Form its limit

$$\begin{array}{ccc}
& & P' \\
I & \xrightarrow{\quad} & P'' \\
& \searrow & \vdots \\
& & P^{(n)}
\end{array}$$

by the argument of 4.13 the legs $I \rightarrow P^{(-)}$ are monic, so $I \rightarrowtail P$ is monic and it's the least subobject of P .

Hence in particular I has no proper subobjects, so any two maps $I \rightrightarrows A$ must be equal, since their equaliser is an isomorphism.

Now given $A \in \mathcal{A}$, form the product $Q = \prod_{i,f:A \rightarrow S_i} S_i$. The canonical morphism $h : A \rightarrow Q$ defined by $\pi_{i,f}h = f$ is monic since the S_i form a coseparating set.

We also have $k : P \rightarrow Q$ defined by $\pi_{i,f}k = \pi_i$, and we can form the pullback

$$\begin{array}{ccccc}
I & \longrightarrow & B & \xrightarrow{m} & A \\
& \searrow & \downarrow l & & \downarrow h \\
& & P & \xrightarrow{k} & Q
\end{array}$$

By 4.13 l is monic and hence isomorphic to an edge of the diagram defining I , so $I \rightarrowtail P$ factors through it. So there exists a morphism $I \rightarrow A$, hence I is initial. \square

5 Monads

Suppose given an adjunction $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$, $F \dashv G$. How much of this can we describe purely in terms of \mathcal{C} ?

We have the composite $T = GF : \mathcal{C} \rightarrow \mathcal{C}$, and the unit $\eta : 1_{\mathcal{C}} \rightarrow T$. We also have $G\epsilon_F : GF GF \rightarrow GF$, which we'll denote $\mu : TT \rightarrow T$.

These satisfy the commutative diagrams

$$\begin{array}{ccc}
T & \xrightarrow{T\eta} & TT \xleftarrow{\eta_T} T \\
\textcircled{1} \searrow & 1_T \downarrow & \mu \downarrow \\
& T & \swarrow 1_T \textcircled{2}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
TTT & \xrightarrow{T\mu} & TT \\
\downarrow \mu_T & \textcircled{3} & \downarrow \mu \\
TT & \xrightarrow{\mu} & T
\end{array}$$

from the \triangle^r identities and naturality of ϵ .

Definition 5.1. A **monad** $\mathbb{T} = (T, \eta, \mu)$ on a category \mathcal{C} consists of a functor $T : \mathcal{C} \rightarrow \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \rightarrow T$, $\mu : TT \rightarrow T$ satisfying the commutative diagrams $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$.

Definition 5.2. Let \mathbb{T} be a monad on \mathcal{C} . A \mathbb{T} -**algebra** is a pair (A, α) where $A \in \text{ob } \mathcal{C}$, and $\alpha : TA \rightarrow A$ satisfies

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow 1_A & \downarrow \alpha \\ & & A \end{array} \quad \text{and} \quad \begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ \downarrow \mu_A & \textcircled{5} & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A \end{array}$$

A **homomorphism** $f : (A, \alpha) \rightarrow (B, \beta)$ of \mathbb{T} -algebras is a morphism $f : A \rightarrow B$ such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow \alpha & \textcircled{6} & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

commutes. We write $\mathcal{C}^{\mathbb{T}}$ for the category of \mathbb{T} -algebras.

Lemma 5.3. *The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ has a left adjoint F , and the adjunction $(F \dashv G)$ induces the monad \mathbb{T} .*

Proof. We define $FA = (TA, \mu_A)$ (which is an algebra by ② and ③), and $F(A \xrightarrow{f} B) = Tf$ (which is a homomorphism by naturality of μ).

Clearly $GF = T$ and $\eta : 1_{\mathcal{C}} \rightarrow GF$.

We define $\epsilon : FG \rightarrow 1_{\mathcal{C}^{\mathbb{T}}}$ by $\epsilon_{(A, \alpha)} = \alpha : (TA, \mu_A) \rightarrow (A, \alpha)$ (which is a homomorphism by ⑤).

The triangular identities for η and ϵ follow from ④ and ①, so $(F \dashv G)$.

Finally, $G_{\epsilon_{FA}} = \mu_A$ by the definitions of FA and ϵ , so the adjunction induces \mathbb{T} . \square

Note that if $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ induces \mathbb{T} , then so does $\mathcal{C} \xrightleftharpoons[G/\mathcal{D}']{F} \mathcal{D}'$ where \mathcal{D}' is the full subcategory of objects of the form FA . So in seeking to construct \mathcal{D} , we may require F to be bijective on objects. But then morphisms $FA \rightarrow FB$ in \mathcal{D} correspond bijectively to morphisms $A \rightarrow GFB = TB$ in \mathcal{C} .

Definition 5.4. Given a monad \mathbb{T} on \mathcal{C} , the **Kleisi category** $\mathcal{C}_{\mathbb{T}}$ is defined by: $\text{ob } \mathcal{C}_{\mathbb{T}} = \text{ob } \mathcal{C}$, morphisms $A \rightarrow B$ in $\mathcal{C}_{\mathbb{T}}$ are morphisms $A \rightarrow TB$ in \mathcal{C} , the identity $A \rightarrow A$ is $A \xrightarrow{\eta_A} TA$, and the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} C$.

We check

$$\begin{aligned} A \xrightarrow{1_A} A \xrightarrow{f} B &= A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} B \\ &= A \xrightarrow{f} TA \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} B \\ &= f \text{ by } \textcircled{2} \end{aligned}$$

$$\begin{aligned}
A \xrightarrow{f} B \xrightarrow{1_B} B &= A \xrightarrow{f} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} B \\
&= f \text{ by } \textcircled{1}
\end{aligned}$$

Given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

$$\begin{aligned}
(hg)f &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TT^h} TT^hD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TT^h} TT^hD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD \text{ by } \textcircled{3} \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{T^h} TTD \xrightarrow{\mu_D} TD \\
&= h(gf)
\end{aligned}$$

Lemma 5.5. *There exists an adjunction $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{C}_{\mathbb{T}}$ inducing \mathbb{T} .*

Proof. We define $FA = A$ and $F(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. This clearly preserves identities, and

$$\begin{aligned}
(Fg)(Ff) &= A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC \\
&= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \text{ by } \textcircled{1} \text{ and naturality of } \eta \\
&= F(gf)
\end{aligned}$$

We define $GA = TA$ and $G(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. G preserves identities by $\textcircled{1}$ and

$$\begin{aligned}
G(A \xrightarrow{f} B \xrightarrow{g} C) &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC \text{ by } \textcircled{3} \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \text{ by naturality of } \mu \\
&= (Gg)(Gf)
\end{aligned}$$

Clearly $GFA = TA$ and

$$\begin{aligned}
GF(A \xrightarrow{f} B) &= TA \xrightarrow{Tf} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} TB \\
&= Tf \text{ by } \textcircled{1}
\end{aligned}$$

so $GF = T$ and $\eta : 1_{\mathcal{C}} \rightarrow GF$.

We define $FGA \xrightarrow{\epsilon_A} A$ to be $TA \xrightarrow{\eta_{TA}} TA$. To verify naturality of ϵ , consider

$$\begin{array}{ccc}
FGA & \xrightarrow{FGf} & FGB \\
\downarrow \epsilon_A & & \downarrow \epsilon_B \\
A & \xrightarrow{f} & B
\end{array}$$

The top and right edges yield

$$TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}} TTB \xrightarrow{1_{TTB}} TTB \xrightarrow{\mu_B} TB$$

and the left and bottom yield

$$TA \xrightarrow{1_{TA}} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$$

For the \triangle^r identities,

$$GA \xrightarrow{\eta_{GA}} GF GA \xrightarrow{G\epsilon_A} GA = TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = 1_{TA}$$

and

$$\begin{aligned} FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\epsilon_{FA}} FA &= A \xrightarrow{\eta_A} TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA \\ &= A \xrightarrow{\eta_A} TA (= FA \xrightarrow{1_{FA}} FA) \end{aligned}$$

Finally, $G\epsilon_{FA} = TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = \mu_A$, so the adjunction induces the monad \mathbb{T} . \square

Theorem 5.6. *Given a monad \mathbb{T} on \mathcal{C} , let $\mathbf{Adj}(\mathbb{T})$ be the category whose objects are adjunctions $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ inducing \mathbb{T} , and whose morphisms $(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}) \rightarrow (\mathcal{C} \xrightleftharpoons[G]{F'} \mathcal{D}')$ are functors $K : \mathcal{D} \rightarrow \mathcal{D}'$ satisfying $KF = F'$ and $G'K = G$.*

Then the Kleisi category $\mathcal{C}_{\mathbb{T}}$ is initial in $\mathbf{Adj}(\mathbb{T})$, and the Eilenberg-Moore category $\mathcal{C}^{\mathbb{T}}$ is terminal.

Proof. Given $(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D})$ in $\mathbf{Adj}(\mathbb{T})$, we define the **Eilenberg-Moore comparison functor** $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\epsilon_B)$ (note that $G\epsilon_B$ is an algebra structure on GB : the unit condition ④ follows from a \triangle^r identity, and ⑤ follows from the naturality of ϵ).

$K(B \xrightarrow{g} B') = Gg : (GB, G\epsilon_B) \rightarrow (GB', G\epsilon_{B'})$ (a homomorphism since ϵ is natural).

It's clear that K is a functor, that $G^{\mathbb{T}}K = G$ and that $KFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$ and $KF(A \xrightarrow{f} B) = Tf = F^{\mathbb{T}}f$.

Uniqueness: suppose \bar{K} also satisfies $G^{\mathbb{T}}\bar{K} = G$ and $\bar{K}F = F^{\mathbb{T}}$. Then $\bar{K}B$ is of the form (GB, β_B) for some algebra structure β_B , and that $\beta_{FA} = \mu_A = G\epsilon_{FA}$ for all A .

Given any B , consider the diagram

$$\begin{array}{ccc} GFGFGB & \xrightarrow{GF G\epsilon_B} & GFGB \\ \downarrow \mu_{GB} & & \downarrow \beta_B \\ GFGB & \xrightarrow{G\epsilon_B} & GB \end{array}$$

which must commute, since $G\epsilon_B$ is an algebra homomorphism. But it would also commute with $G\epsilon_B$ in place of β_B , and $GFGE_B$ is (split) epic, so $\beta_B = G\epsilon_B$.

For the **Kleisi comparison functor** $K : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$, we define $KA = FA$, $K(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB$.

To verify this is functorial, consider

$$\begin{aligned} K(A \xrightarrow{f} B \xrightarrow{g} C) &= FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{FG\epsilon_{FC}} FGFC \xrightarrow{\epsilon_{FC}} FC \\ &= FA \xrightarrow{Ff} FGFB \xrightarrow{FGFg} FGFGFC \xrightarrow{\epsilon_{FGFC}} FGFC \xrightarrow{\epsilon_{FC}} FC \\ &= FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB \xrightarrow{Fg} FGFC \xrightarrow{\epsilon_{FC}} FC \\ &= (Kg)(Kf) \end{aligned}$$

$$GKA = GFA = TA = G_{\mathbb{T}}A$$

$$GK(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB = G_{\mathbb{T}}(f)$$

$$\text{And } KF_{\mathbb{T}}A = FA,$$

$$KF_{\mathbb{T}}(A \xrightarrow{f} B) = \begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ & \searrow 1_{FB} & \downarrow \epsilon_{FB} \\ & & FB \end{array}$$

So K is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Uniqueness: suppose \bar{K} is any other morphism $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$ in $\mathbf{Adj}(\mathbb{T})$. Then $\bar{K}A = FA = KA$ for all A ; since \bar{K} commutes with both the F s and the G s, we have $\bar{K}(\epsilon_A) = \epsilon_{FA}$.

We can write $A \xrightarrow{f} B$ as $A \xrightarrow{F_{\mathbb{T}}f} F_{\mathbb{T}}G_{\mathbb{T}} \xrightarrow{\epsilon_B} B$, so $\bar{K}(f) = \bar{K}(\epsilon_B)Ff = K(f)$. \square

The Kleisi category $\mathcal{C}_{\mathbb{T}}$ inherits coproducts from \mathcal{C} if \mathcal{C} has them, but it has few other limits or colimits in general.

Theorem 5.7. *i. The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ creates all limits which exist in \mathcal{C} .*

ii. If $T : \mathcal{C} \rightarrow \mathcal{C}$ preserves colimits of shape J , then $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ creates them.

Proof. i. Let $D : J \rightarrow \mathcal{C}^{\mathbb{T}}$ be a diagram, write $D(j) = (GD(j), \delta_j)$.

Let $(L, (\lambda_j : L \rightarrow GD(j)))$ be a limit for GD . The composites $TL \xrightarrow{T\lambda_j} TGD(j) \xrightarrow{\delta_j} GD(j)$ form a cone over GD , since the edges of GD are algebra homomorphisms.

So they induce a unique $l : TL \rightarrow L$ such that

$$\begin{array}{ccc} TL & \xrightarrow{T\lambda_j} & TGD(j) \\ \downarrow l & & \downarrow \delta_j \\ L & \xrightarrow{\lambda_j} & GD(j) \end{array}$$

commutes for each j .

l is an algebra structure: $l\eta_L = l_L$ since both are factorisations of (λ_j) through itself, and $lTl = l\mu_L$ since they're factorisations of the same cone through L .

So $((L, l), (\lambda_j))$ is the unique lifting of $(L, (\lambda_j))$ to a cone over D in $\mathcal{C}^{\mathbb{T}}$.

Any cone over D in $\mathcal{C}^{\mathbb{T}}$ factors uniquely through L , and the factorisation is an algebra homomorphism.

- ii. Similarly, given $D : J \rightarrow \mathcal{C}^{\mathbb{T}}$ as before and a colimit $(L, (\lambda_j : GD(j) \rightarrow L))$ for GD , we get a unique $l : TL \rightarrow L$ making

$$\begin{array}{ccc} TGD(j) & \xrightarrow{T\lambda_j} & TL \\ \downarrow \delta_j & & \downarrow l \\ GD(j) & \xrightarrow{\lambda_j} & L \end{array}$$

commute, since $(TL, (T\lambda_j))$ is a colimit. The rest of the proof is similar to (i). □

Definition 5.8. An adjunction $(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D})$, $(F \dashv G)$, is **monadic** if the comparison functor $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ is part of an equivalence, where \mathbb{T} is the monad induced by $(F \dashv G)$. We also say $G : \mathcal{D} \rightarrow \mathcal{C}$ is monadic if it has a left adjoint and the adjunction is monadic.

Note that K preserves all limits which exist in \mathcal{D} , since G preserves them and $G^{\mathbb{T}}$ creates them.

Lemma 5.9. Suppose given $(\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D})$, $(F \dashv G)$ inducing a monad \mathbb{T} on \mathcal{C} .

Suppose, for each \mathbb{T} -algebra (A, α) , the pair $FGFA \xrightleftharpoons[\epsilon_{FA}]{F\alpha} FA$ has a coequaliser in \mathcal{D} . Then $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ has a left adjoint L .

Proof. We define $L(A, \alpha) = \text{coeq}(FGFA \xrightleftharpoons[\epsilon_{FA}]{F\alpha} FA)$.

Given $(A, \alpha) \xrightarrow{f} (B, \beta)$, we get

$$\begin{array}{ccccc} FGFA & \xrightleftharpoons[\epsilon_{FA}]{F\alpha} & FA & \longrightarrow & L(A, \alpha) \\ \downarrow FGFf & & \downarrow Ff & & \downarrow Lf \\ FGFB & \xrightleftharpoons[\epsilon_{FB}]{F\beta} & FB & \longrightarrow & L(B, \beta) \end{array}$$

So $\exists! Lf$ making the right hand square commute. Uniqueness ensures L is functorial.

Morphisms $L(A, \alpha) \rightarrow B$ in \mathcal{D} correspond bijectively to morphisms $f : FA \rightarrow B$ such that $f(F\alpha) = f(\epsilon_{FA})$ and hence to morphisms $\bar{f} : A \rightarrow GB$ such that $f\bar{\alpha} = Gf|_{GFA} = G\epsilon_B \circ GF\bar{f}$, i.e. to algebra homomorphisms $(A, \alpha) \rightarrow (GB, G\epsilon_B) = KB$.

So $(L \dashv K)$. \square

Definition 5.10. a. A parallel pair $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$ is **reflexive** if $\exists r : B \rightarrow A$ such that $fr = gr = 1_B$.

Note that $FGFA \begin{smallmatrix} \xrightarrow{F\alpha} \\ \xrightarrow{\epsilon_{FA}} \end{smallmatrix} FA$ is reflexive, with common splitting $FA \xrightarrow{F\eta_A} FGFA$.

A **reflexive coequaliser** is the coequaliser of a reflexive pair.

b. A **split coequaliser diagram** is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{h} & C \\ & \searrow g & \swarrow s & & \\ & & & & \\ & \swarrow t & \searrow & & \end{array}$$

satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$.

If these equations hold, h is a coequaliser of f and g : given $k : B \rightarrow D$ with $kf = kg$, we have $k = kgt = kft = ksh$, so k factors through h , uniquely since h is split epic.

Note that **any** functor preserves split equalisers.

c. Given $G : \mathcal{D} \rightarrow \mathcal{C}$, a pair $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$ in \mathcal{D} is **G-split** if there exists a split coequaliser

$$\begin{array}{ccccc} GA & \xrightarrow{Gf} & GB & \xrightarrow{\quad} & C \\ & \searrow Gg & \swarrow s & & \\ & & & & \\ & \swarrow t & \searrow & & \end{array}$$

in \mathcal{C} .

The pair $FGFA \begin{smallmatrix} \xrightarrow{F\alpha} \\ \xrightarrow{\epsilon_{FA}} \end{smallmatrix} FA$ of 5.9 is G -split:

$$\begin{array}{ccccc} GFGFA & \xrightarrow{Gf\alpha} & GFA & \xrightarrow{\alpha} & A \\ & \searrow G\epsilon_{FA} & \swarrow \eta_A & & \\ & & & & \\ & \swarrow \eta_{GFA} & \searrow & & \end{array}$$

is a split coequaliser diagram.

Theorem 5.11 (Precise Monadicity Theorem). $G : \mathcal{D} \rightarrow \mathcal{C}$ is monadic \iff G has a left adjoint, and c creates coequalisers of G -split pairs.

Theorem 5.12 (Crude Monadicity Theorem). Suppose $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint, that \mathcal{D} has and G preserves reflexive coequalisers, and G reflects isomorphisms. Then G is monadic.

Proof. For the forward implication in 5.11, it's enough to show that $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ creates coequalisers of $G^{\mathbb{T}}$ -split pairs. This follows from 5.7, given that T and TT both preserve split coequalisers.

Conversely in either case, $K : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ has a left adjoint L by 5.9. Now $LKB = \text{coeq}(FGFGB \xrightarrow{FG\epsilon_B} FGB)$ and the counit $LKB \rightarrow B$ is the factorisation of $FGB \xrightarrow{\epsilon_{FGB}} B$ through this coequaliser.

$$\text{But } \begin{array}{c} FGFGB \rightrightarrows FGB \longrightarrow GB \\ \eta_{FGGB} \quad \eta_{GB} \end{array} \text{ is a split coequaliser diagram.}$$

So either set of hypotheses ensures that $LKB \rightarrow B$ is an isomorphism.

$KL(A, \alpha) = K(\text{coeq}(FGFA \xrightarrow{F\alpha} FA))$. Either hypothesis implies that $G = G^{\mathbb{T}}K$ preserves this coequaliser, but

$$\begin{array}{c} FGFGFA \xrightarrow{GF\alpha} GFA \xrightarrow{\alpha} A \\ \eta_{GF\alpha} \quad \eta_A \end{array}$$

is a split coequaliser, so $GL(A, \alpha) \cong A$.

The unit $(A, \alpha) \rightarrow KL(A, \alpha)$ is mapped to this isomorphism by $G^{\mathbb{T}}$, so it's an isomorphism in $\mathcal{C}^{\mathbb{T}}$. \square

Remark 5.13. Let $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ be an adjunction, and suppose \mathcal{D} has reflexive coequalisers. The **monadic tower** of $(F \dashv G)$ is the diagram

$$\begin{array}{ccc} & & (\mathcal{C}^{\mathbb{T}})^{\mathbb{S}} \\ & \nearrow & \nearrow \\ \mathcal{D} & \xrightarrow{K} & \mathcal{C}^{\mathbb{T}} \\ \uparrow F & \uparrow F^{\mathbb{T}} & \uparrow G^{\mathbb{T}} \\ \mathcal{C} & \xrightarrow{L} & \mathcal{C}^{\mathbb{T}} \end{array}$$

where \mathbb{T} is the monad induced by $(F \dashv G)$, K is the Eilenberg-Moore comparison functor, $(L \dashv K)$ (5.9), \mathbb{S} is the monad induced by $(L \dashv K)$, etc.

We say $(F \dashv G)$ has **monadic length** n if we reach an equivalence after n steps. **Top** \rightarrow **Set** has monadic length ∞ .

6 Regular Categories

Definition 6.1. The **image** of a morphism $A \xrightarrow{f} B$ is the smallest subobject of B through which f factors, if this exists.

We say \mathcal{C} **has images** if every $f \in \text{mor } \mathcal{C}$ has an image.

$A \xrightarrow{f} B$ is a **cover** if its image is 1_B , i.e. it doesn't factor through any proper subobject of B . We write $A \xrightarrow{f} B$ to indicate that f is a cover.

Lemma 6.2. *If \mathcal{C} has finite limits, then covers in \mathcal{C} coincide with strong epimorphisms.*

Proof. Recall that f is strong epic if and only if given $(*)$ $\begin{array}{ccc} A & \xrightarrow{g} & C \\ \downarrow f & \nearrow l & \downarrow k \\ B & \xrightarrow{h} & D \end{array}$ with k monic, $\exists l : B \rightarrow C$ with $kl = h$ and $lf = g$.

Being a cover is the special case of this condition with $h = 1_B$, so strong epimorphisms are covers.

Conversely, if f is a cover then it's epic, since if $gf = hf$ then f factors through the equaliser of g and h , so this must be an isomorphism. Given $(*)$,

we can form the pullback $\begin{array}{ccc} P & \xrightarrow{n} & C \\ \downarrow m & & \downarrow k \\ B & \xrightarrow{h} & D \end{array}$, and m is monic by 4.13.

f factors through m , so m is an isomorphism and $B \xrightarrow{nm^{-1}} C$ is the diagonal fill in for $(*)$. \square

It follows that image factorisation is functorial: given

$$\begin{array}{ccccc} A & & \xrightarrow{f} & & B \\ & \searrow & & \nearrow & \\ & I & & & \\ & \downarrow & & & \\ & I' & & & \\ & \nearrow & & \searrow & \\ A' & & \xrightarrow{f'} & & B' \end{array}$$

g h

we get a unique $I \rightarrow I'$ making everything commute. So image factorisation defines a functor $[\mathbf{2}, \mathcal{C}] \rightarrow [\mathbf{3}, \mathcal{C}]$.

Definition 6.3. A **regular category** is a category which preserves finite limits and images, in which strong epimorphisms are stable under pullback.

A **regular functor** is one which preserves finite limits and strong epimorphisms.

Theorem 6.4. *In a regular category, the strong epimorphisms coincide with regular epimorphisms.*

Proof. Regular \implies strong is true in general (see sheet 1). Suppose $A \xrightarrow{f} B$ is strong epic. Let $R \rightrightarrows A$ be the **kernel-pair** of f , i.e. the pullback of f against itself. Certainly $fa = fb$.

Suppose $A \xrightarrow{g} C$ with $ga = gb$. Form the image

$$A \implies I \xrightarrow{(k,l)} B \times C$$

of (f, g) .

Claim k is an isomorphism: given this, the composite $B \xrightarrow{k^{-1}} I \xrightarrow{l} C$ satisfies $lk^{-1}f = lk^{-1}kh = lh = g$, and it's unique since f is epic.

We know k is strong epic, since $kh = f$ is strong epic. So we need to show it's monic.

Suppose $D \rightrightarrows$ satisfy $kx = ky$. Form the pullback

$$\begin{array}{ccc} F & \xrightarrow{m} & D \\ \downarrow (n,p) & & \downarrow (x,y) \\ A \times A & \xrightarrow{1 \times h} & A \times I \xrightarrow{h \times 1} I \times I \end{array}$$

then $h \times 1$ and $1 \times h$ are strong epimorphisms, and hence so is m .

Now $fn = khn = kxm = kym = khp = fp$, so (n, p) factors through (a, b) , say by $E \xrightarrow{q} R$. Since $kha = khb$ and $lha = lhb$, and (k, l) is monic, we have $ha = hb$. So $xm = hn = haq = hbq = hp = ym$. But m is epic, so $x = y$. \square

Remark 6.5. In many textbooks, regular categories are defined as categories with (some) finite limits, in which every morphism factors as a regular epimorphism and a monomorphism, and regular epimorphisms are stable under pullback.

Note that if $A \xrightarrow{f} B$ has kernel pair $R \rightrightarrows A$ and image factorisation $A \xrightarrow{g} I \hookrightarrow B$, then (a, b) is also the kernel-pair of g since h is monic. So g may be obtained as the coequaliser of the kernel-pair of f , and f as the factorisation of f through this.

Definition 6.6. Let $R \rightrightarrows A$ be a parallel pair in a category with finite limits.

- (a, b) is a **relation** if $R \xrightarrow{(a,b)} A \times A$ is monic.
- (a, b) is **reflexive** if there exists $A \xrightarrow{r} R$ with $ar = br = 1_A$.
- (a, b) is **symmetric** if there exists $R \xrightarrow{s} R$ with $as = b$, $bs = a$.

- d. (a, b) is **transitive** if, given the pullback
- $$\begin{array}{ccc} T & \xrightarrow{q} & R \\ \downarrow p & & \downarrow a \\ R & \xrightarrow{b} & A \end{array}, \text{ there exists } t : T \rightarrow R \text{ such that } at = ap \text{ and } bt = bq.$$

- e. (a, b) is an **equivalence relation** if all of (a-d) hold.

The kernel-pair of any $A \xrightarrow{f} B$ is an equivalence relation.

(a, b) is an **effective** equivalence relation if it occurs as a kernel pair, \mathcal{C} is an **effective** regular category if all equivalence relations in \mathcal{C} are effective (aka 'Barr-exact').

Definition 6.7. The **support** $\sigma(A)$ of an object A in a regular category is the image of $A \rightarrow 1$. A is **well-supported** if $\sigma(A) \cong 1$, i.e. if $A \rightarrow 1$ is strong epic.

\mathcal{C} is **totally supported** if all its objects are well-supported. (E.g. **Gp** and **ApGp** are totally supported, since any $A \rightarrow 1$ is split epic).

An object 0 in a regular category is **strict** if any morphism $A \rightarrow 0$ is an isomorphism. (This implies that 0 is initial: $0 \times A \xrightarrow{\pi_1} 0$ is an isomorphism, so $0 \xrightarrow{\pi_1^{-1}} 0 \times A \xrightarrow{\pi_2} A$ exists for any A , but given $0 \xrightleftharpoons[g]{f} A$, the equaliser must be an isomorphism.)

\mathcal{C} is **almost totally supported** if every object of \mathcal{C} is either well-supported or strict (e.g. **Set**).

Theorem 6.8 (Barr's Embedding Theorem). *Let \mathcal{C} be a small regular category. Then there exists a small category \mathcal{D} and a full and faithful regular functor $\mathcal{C} \rightarrow [\mathcal{D}, \mathbf{Set}]$. Moreover, if \mathcal{C} is almost totally supported then \mathcal{D} can be taken to be a monoid.*

We'll prove the most important part of this: \mathcal{C} has an isomorphism reflecting regular functor to be a power of **Set**. We follow a proof due to F. Borceaux:

Theorem 6.9. *Let \mathcal{C} be a small a.t.s. regular category. Then there exists an isomorphism reflecting regular functor $F : \mathcal{C} \rightarrow \mathbf{Set}$.*

Proof. We construct F as the colimit in $[\mathcal{C}, \mathbf{Set}]$ of a diagram of representable functors: explicitly, J will be a meet-semilattice and $D : J \rightarrow \mathcal{C}$ a diagram such that each $D(j)$ is well-supported and each $D(j' \rightarrow j)$ is a strong epimorphism $D(j') \rightarrow D(j)$.

Then $F = \text{colim}(J^{\circ} p \xrightarrow{D} \mathcal{C}^{\circ} p \xrightarrow{Y} [\mathcal{C}, \mathbf{Set}])$.

Explicitly, elements of FA are represented by morphisms $D(j) \xrightarrow{f} A$ for some j , where $f \sim f'$ if and only if

$$\begin{array}{ccccc}
& & D(j) & & \\
& \nearrow & & \searrow f & \\
D(j \wedge j') & & & & A \\
& \searrow & & \nearrow f' & \\
& & D(j') & &
\end{array}$$

commutes.

F preserves finite products: $F1 = \{*\}$, and if $D(j) \xrightarrow{f} A$, $D(j') \xrightarrow{g} B$ represent elts of FA and FB , $D(j \wedge j') \rightarrow D(j) \xrightarrow{f} A$ and $D(j \wedge j') \rightarrow D(j') \xrightarrow{g} B$ induce an element of $F(A \times B)$, mapping to the given element of $FA \times FB$.

Hence $F(A \times B) \rightarrow FA \times FB$ is surjective, and it's easily seen to be injective.

F preserves equalisers: notes that if 0 exists in \mathcal{C} , then $F0 = \emptyset$. If

$$E \xrightarrow{e} A \rightrightarrows^f_g B$$

is an equaliser diagram in \mathcal{C} and E is well-supported, then the equaliser of $FA \rightrightarrows FB$ consists of morphisms $D(j) \rightarrow A$ having equal composites with f and g (and hence factoring through E). And if $E = 0$ then the equaliser of $FA \rightrightarrows FB$ is \emptyset .

Now assume that, for every strong epimorphism $A \xrightarrow{f} D(j)$ in \mathcal{C} , there exists $j' \leq j$ such that $D(j' \rightarrow j) = f$.

Then F preserves strong epimorphisms: given $A \xrightarrow{f} B$ and a morphism $D(j) \xrightarrow{g}$ representing an element of FB , form the pullback

$$\begin{array}{ccc}
D(j') & \xrightarrow{h} & A \\
\downarrow & & \Downarrow f \\
D(j) & \xrightarrow{g} & B
\end{array}$$

then h represents an element of FA whose image under Ff is g . So Ff is surjective.

Assume every well-supported object of \mathcal{C} occurs as $D(j)$ for some j . Then F preserves properness of subobjects: the element of FA represented by $D(j) \xrightarrow{1} A$ can't be in the image of $FA' \rightarrow FA$ for any proper subobject $A' \rightarrow A$ (if it

were, we'd have

$$\begin{array}{ccc}
D(j') & \longrightarrow & A' \\
\Downarrow & \nearrow \text{dashed} & \downarrow \\
D(j) & \xrightarrow{1} & A
\end{array}
\quad)$$

Since F preserves equalisers, it follows that it's faithful. Hence F reflects monomorphisms, and so the argument above shows that it reflects isomorphisms.

We'll construct J as the union $\bigcup_{n=0}^{\infty} J_n$ of an increasing sequence of sub-semilattices J_n : $J_0 = \{1\}$ and $D(1) = 1$, the terminal object of \mathcal{C} . Objects of $J_1 \setminus J_0$ are non-empty finite sets $\{A_1, A_2, \dots, A_n\}$ of well-supported objects

of \mathcal{C} , ordered by \supseteq (so $j \wedge j' = j \cup j'$), and $D(\{A_1, A_2, \dots, A_n\}) = \prod_{i=1}^n A_i$. (Note that is well-supported: if A and B are well supported, we have a pullback

$$\begin{array}{ccc} A \times B & \Longrightarrow & A \\ \Downarrow & & \Downarrow \\ B & \Longrightarrow & 1 \end{array}$$

If $j' \supseteq j$, $D(j \rightarrow j')$ is the product projection $\prod_{A \in j'} \rightarrow \prod_{A \in j} A$.

An object of $J_2 \setminus J_1$ is a pair $(j_1, \{f_1, \dots, f_n\})$ where $j_1 \in J_1 \setminus J_0$ and $\{f_1, \dots, f_n\}$ is a non-empty finite set of strong epimorphisms with codomain $D(j_1)$. (Equivalently, well-supported objects of $\mathcal{C}/D(j_1)$).

The meet of $(j_1, \{f_1, \dots, f_n\})$ and $(j'_1, \{g_1, \dots, g_m\})$ has first coordinate $j_1 \wedge j'_1$, and then we take the union of the sets of strong epimorphisms obtained from pulling back the f_i and g_j along $D(j_1) \wedge j'_1 \rightarrow D(j_1)$ and $D(j_1 \wedge j'_1) \rightarrow D(j'_1)$.

We define $D((j_1, \{f_1, \dots, f_n\}))$ to be the domain of the object $\prod_{i=1}^n f_i$ of $\mathcal{C}/D(j_1)$, and $D(j_2 \wedge j'_2)$ is the composite of the appropriate product projection in $\mathcal{C}/D(j_1 \wedge j'_1)$ with the appropriate pullback of $D(j_1 \wedge j'_1) \rightarrow D(j_1)$.

Similarly, object of $J_3 \setminus J_2$ are pairs $(j_2, \{h_1, \dots, h_n\})$ where $j_2 \in J_2 \setminus J_1$ and the h_i are strong epimorphisms with codomain $D(j_2)$, and so on.

Now we've satisfied the condition for F to preserve strong epimorphisms: if $A = D(j)$ where $j \in J_n \setminus J_{n-1}$, and $B \xrightarrow{g} A$ is strong epic, then $B = D((j, \{g\}))$ and $g = D((j, \{g\}) \rightarrow j)$. \square

Remark 6.10. If we define M to be the monoid of endomorphisms of $F : \mathcal{C} \rightarrow \mathbf{Set}$ in $[\mathcal{C}, \mathbf{Set}]$, then M acts on every FA , so we can regard F as taking values in $[M, \mathbf{Set}]$. As such, it's still regular and faithful, but also full.

Given a general regular category \mathcal{C} and $S \rightarrowtail 1$ in \mathcal{C} , we write \mathcal{C}_S for the full subcategory of \mathcal{C} on objects A with $\sigma(A) \cong S$. This is closed in \mathcal{C} under non-empty finite products, images and pullbacks of strong epimorphisms (if we're

$$\text{given } \begin{array}{ccc} P & \xrightarrow{h} & A \\ \Downarrow_k & & \Downarrow_f \\ B & \xrightarrow{g} & C \end{array} \text{ with } f \text{ strong epic then } k \text{ is strong epic, so } \sigma(P) = \sigma(B).$$

It doesn't have all finite limits, but if $D : J \rightarrow \mathcal{C}_S$ is a finite diagram whose limit in \mathcal{C} is not in \mathcal{C}_S , then there are no cones over D in \mathcal{C}_S .

Definition 6.11. Given a category \mathcal{C} , let \mathcal{C}^+ denote the category whose objects are those of \mathcal{C} plus a new object 0 , with one morphism $0 \rightarrow A$ and no morphisms $A \rightarrow 0$ for each $A \in \text{ob } \mathcal{C}$.

In \mathcal{C}_S^+ , every finite diagram has a limit: if it lies in \mathcal{C}_S and has a limit there, that is its limit in \mathcal{C}_S^+ , otherwise its limit is the unique cone with apex 0 .

\mathcal{C}_S^+ is regular: the new morphisms $0 \rightarrow A$ are monic, and hence their own images, and strong epimorphisms are still stable under pullback.

It's almost totally supported: note that \mathcal{C}/S may be identified with the full subcategory of \mathcal{C} on objects with support $\leq S$, so its well-supported objects are those of \mathcal{C}_S .

We have a functor $E : \mathcal{C}/S \rightarrow \mathcal{C}_S^+$ sending all objects of \mathcal{C}_S to themselves, and everything else to 0. E is regular! And we have a regular functor $(-) \times S : \mathcal{C} \rightarrow \mathcal{C}/S$ (it preserves finite limits because it's right adjoint to the forgetful functor, and images because they're stable under pullback along $S \rightarrow 1$).

Theorem 6.12. *For every small regular category \mathcal{C} , there exists a set I and an isomorphism reflecting regular functor $\mathcal{C} \rightarrow \mathbf{Set}^I$.*

Proof. Take $I = \mathbf{Sub}_{\mathcal{C}}(1)$, and for each $S \in \mathcal{I}$ consider the composite

$$G_S \xrightarrow{(-) \times S} \mathcal{C}/S \xrightarrow{E} \mathcal{C}_S^+ \xrightarrow{F_S} \mathbf{Set}$$

where F_S is defined as in 6.9. The G_S are all regular, and they jointly reflect isomorphisms: if $A \xrightarrow{f} B$ is not an isomorphism in \mathcal{C} , let $S = \sigma(B)$. Then $E(f)$ is either f itself (if $\sigma(A) = S$) or $0 \rightarrow B$ (otherwise). In either case it's not an isomorphism, so its image under F_S is not an isomorphism. \square