Part III Topics in Additive Combinatorics

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Contents

1 Discrete Fourier Analysis and Roth's Theorem

Let $N \in \mathbb{N}$, $\omega = e^{\frac{2\pi i}{N}}$. Write $\mathbb{Z}_N$ for the cyclic group of integers mod N. Use the notation $\mathbb{E}_x f(x)$ to stand for the average $N^{-1} \sum_{x \in \mathbb{Z}_N} f(x)$.

Definition (Discrete Fourier Transform)**.** *Given a function $f : \mathbb{Z}_N \to \mathbb{C}$, define its* ***discrete Fourier transform*** *$\hat{f}$ by the formula*

$$\hat{f}(r) = \mathbb{E}_x f(x) \omega^{-rx}$$

Definition (Convolution)**.** *We define the* ***convolution*** *$f * g$ of f and g by*

$$f * g(x) = \mathbb{E}_{y+z=x} f(y) g(z)$$

$$\hat{f} * \hat{g}(r) = \sum_{s+t=r} \hat{f}(s) \hat{g}(t)$$

We also define two inner products

$$\langle f, g \rangle = \mathbb{E}_x f(x) \overline{g(x)}$$

$$\langle \hat{f}, \hat{g} \rangle = \sum_r \hat{f}(r) \overline{\hat{g}(r)}$$

Have the following basic properties:

1. Parseval's Identity:
$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$$
for any $f, g : \mathbb{Z}_N \to \mathbb{C}$.

2. Convolution Law: for any $f, g : \mathbb{Z}_N \to \mathbb{C}$, $r \in \mathbb{Z}_N$
$$\widehat{f * g}(r) = \hat{f}(r) \hat{g}(r)$$

3. Inversion Formula: let $f : \mathbb{Z}_N \to \mathbb{C}$. Then

$$f(x) = \sum_r \hat{f}(r)\omega^{rx}$$

4. Dilation Rule: let a be invertible mod N and define $f_a(x)$ to be $f(a^{-1}x)$. Then

$$\hat{f_a}(r) = \hat{f}(ar)$$