Part III Category Theory

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Contents

1 Definitions and Examples

Definition 1.1 (Category)**.** A category $\mathcal{C}$ consists of

a. a collection $\operatorname{ob}\mathcal{C}$ of **objects** A, B, C, ...

b. a collection $\operatorname{mor}\mathcal{C}$ of **morphisms** f, g, h, ...

c. two operations dom, cod from morphisms to objects. We write $f : A \to B$ or $A \xrightarrow{f} B$ to mean 'f is a morphism and $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$'

d. an operation assigning to each object A a morphism $1_A : A \to A$

e. a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff$ $\operatorname{dom} g = \operatorname{cod} f$, and then $gf : \operatorname{dom} f \to \operatorname{cod} g$

satisfying

f. $f1_A = f$ and $1_B f = f$ $\forall f : A \to B$

g. $h(fg) = (hg)f$ whenever gf and hg are defined

Definition 1.2 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $\mathcal{C} \to \mathcal{D}$ consists of

a. a mapping $A \to FA$ from $\operatorname{ob}\mathcal{C}$ to $\operatorname{ob}\mathcal{D}$

b. a mapping $f \to Ff$ from $\operatorname{mor}\mathcal{C}$ to $\operatorname{mor}\mathcal{D}$

satisfying $\operatorname{dom} Ff = F\operatorname{dom} f$, $\operatorname{cod} Ff = F\operatorname{cod} f$ for all f, $F(1_A) = 1_{FA}$ for all A, and $F(gf) = (Fg)(Ff)$ whenever gf is defined.

Definition 1.3. By a **contravariant functor** $\mathcal{C} \to \mathcal{D}$ we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or equivalently $\mathcal{C}^{op} \to \mathcal{D}$). A functor $\mathcal{C} \to \mathcal{D}$ is sometimes said to be **covariant**.

Definition 1.4 (Natural transformation)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two functors. A **natural transformation** $\alpha : F \to G$ assigns to each $A \in \operatorname{ob}\mathcal{C}$ a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$, such that

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given $\alpha : F \to G$ and $\beta : G \to H$, the mapping $A \mapsto \beta_A\alpha_A$ is the A-component of a natural transformation $\beta\alpha : F \to H$.

Definition 1.5. Given categories $\mathcal{C}, \mathcal{D}$, we write $[\mathcal{C}, \mathcal{D}]$ for the category of all functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

Lemma 1.6. *Given $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \to G$, α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ $\iff$ each α_A is an isomorphism in $\mathcal{D}$.*

Definition 1.7 (Faithful and full)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

a. We say that F is **faithful** if, given $f, g \in \operatorname{mor}\mathcal{C}$, the equations $\operatorname{dom} f = \operatorname{dom} g$, $\operatorname{cod} f = \operatorname{cod} g$ and $Ff = Fg$ imply $f = g$.

b. F is **full** if, given any $g : FA \to FB$ in $\mathcal{D}$, there exists $f : A \to B$ in $\mathcal{C}$ with $Ff = g$.

c. We say a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is **full** if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition 1.8 (Equivalence of categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to mean that $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

We say a property P of categories is **categorical** if whenever $\mathcal{C}$ has P and $\mathcal{C} \simeq \mathcal{D}$ then $\mathcal{D}$ has P.

For example, being a groupoid is a categorical property, but being a group is not.

Definition 1.9 (Slice category)**.** Given an object B of a category $\mathcal{C}$, define the **slice category** $\mathcal{C}/B$ to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \to (A' \xrightarrow{f'} B)$ are morphisms $h : A \to A'$ making

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commute.

Lemma 1.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and* ***essentially surjective****, i.e. for every $B \in \operatorname{ob}\mathcal{D}$, there exists $A \in \operatorname{ob}\mathcal{C}$ s.t. $FA \cong B$.*

Definition 1.11.

a. A **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ containing exactly one object from each isomorphism class of objects of $\mathcal{C}$.

b. We say $\mathcal{C}$ is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism f in $\mathcal{C}$ satisfies $\operatorname{dom} f = \operatorname{cod} f$.

For example, $\mathbf{Mat}_K$ is skeletal. The full subgategory of standard vector spaces K^n is a skeleton of $\mathbf{fd}\ \mathbf{Mod}_K$.

Remark 1.12. The following statements are each equivalent to the Axiom of Choice:

1. Every small category has a skeleton
2. Any small category is equivalent to each of its skeletons
3. Any two skeletons of a given small category are isomorphic

Definition 1.13. Let $f : A \to B$ be a morphism in a category $\mathcal{C}$.

a. f is a **monomorphism** if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightarrowtail B$ if f is monic.

b. Dually, f is an **epimorphism** if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.

c. $\mathcal{C}$ is a **balanced** category if every $f \in \operatorname{mor} \mathcal{C}$ which is both monic and epic is an isomorphism.

2 The Yoneda Lemma

Definition 2.1. A category $\mathcal{C}$ is **locally small** if, for any two objects A, B of $\mathcal{C}$, the morphism $A \to B$ are parametrised by a set $\mathcal{C}(A, B)$.

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$: if $g : B \to B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \to \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \to B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \to \mathbf{Set}$.

Lemma 2.2 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob} \mathcal{C}$ and $F : \mathcal{C} \to$ **Set**. Then*

i. There is a bijection between natural transformations $\mathcal{C}(A, -) \to F$ and elements of FA.

ii. Moreover, this bijection is natural in both A and F.

Proof. Bijection: given $\alpha : \mathcal{C}(A, -) \to F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.

Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \to F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$ is natural: given $g : B \to C$, we have

$$\begin{aligned}
\Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\
&= F(gf)(x) \\
&= (Fg)(Ff)(x) \\
&= (Fg)\Psi(x)_B(f)
\end{aligned}$$

$\Phi\Psi(x) = x$ since $F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$ since, for any $f : A \to B$,

$$\begin{aligned}
\Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\
&= Ff(\alpha_A(1_A)) \\
&= \alpha_B(\mathcal{C}(A, f)(1_A)) \\
&= \alpha_B(f)
\end{aligned}$$

$\square$

Corollary 2.3. *The mapping $A \to \mathcal{C}(A, -)$ is a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C}, \textbf{\textit{Set}}]$.*

Proof. Given two objects A, B, 2.2(i) gives us a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ (by taking $F : C \mapsto \mathcal{C}(B, C)$). We need to show this is functorial, but given $f \in \mathcal{C}(B, A)$, $\Psi(F)_A$ sends 1_A to $\mathcal{C}(B, f)(1_A) = f$, so it's the natural transformation $g \mapsto gf$.

Hence, given $e : C \to B$, $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$ □

We call this functor the **Yoneda embedding**. Hence any locally small category $\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$.

Definition 2.4. A functor $\mathcal{C} \to \mathbf{Set}$ is **representable** if it's isomorphic to $\mathcal{C}(A, -)$ for some A.

A **representation** of $F : \mathcal{C} \to \mathbf{Set}$ is a pair (A, x) where $A \in \operatorname{ob}\mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, -) \to F$ is an isomorphism. We also call x a **universal element** of F.

Corollary 2.5 ('Representations are unique up to unique isomorphism')**.** *If (A, x) and (B, y) are both representations of $F : \mathcal{C} \to \textbf{\textit{Set}}$, then there's a unique isomorphism $f : A \to B$ s.t $Ff(x) = y$.*

Definition 2.6 (Product and coproduct)**.** Given two objects A, B of a locally small category $\mathcal{C}$, we define their **product** to be a representation of the functor

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{op} \to \mathbf{Set}$$

i.e. an object $A \times B$ equipped with morphisms $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ s.t. given any pair $(f : C \to A, g : C \to B)$, there exists a unique $h : C \to A \times B$ s.t. $\pi_1 h = f$ and $\pi_2 h = g$.

More generally, we can define the product $\prod_{i \in I} A_i$ of a family $\{A_i \mid i \in I\}$ of objects, or the product of the empty family, i.e. a **terminal object** 1 s.t. for every A there's a unique $A \to 1$.

Dualizing, we get the notion of **coproduct** or **sum**.

Definition 2.7 (Equaliser and coequaliser)**.** Given a parallel pair $f, g : A \rightrightarrows B$ in a locally small category $\mathcal{C}$, the assignment $C \mapsto FC = \{h : C \to A \mid fh = gh\}$ is a subfunctor F of $\mathcal{C}(-, A)$. A representation of F is called an **equaliser** of (f, g).

In elementary terms, it's an object E equipped with $e : E \to A$ s.t. $fe = ge$, s.t. any h with $fh = gh$ factors uniquely as $h = ek$

Dually, we have the notion of **coequaliser**, i.e. a morphism $q : B \to Q$ satisfying $qf = qg$, and universal among such.

Definition 2.8. a. We say a monomorphism is **regular** if it occurs as an equaliser (dually, regular epimorphism).

b. We say $f : A \to B$ is a **split monomorphism** if there exists $g : B \to A$ with $gf = 1_A$.

Every split monomorphism is regular: if $gf = 1_A$, f is an equaliser of $(1_B, fg)$ [see sheet 1, q2].

Definition 2.9. Let $\mathcal{C}$ be a (locally small) category, $\mathcal{G}$ a collection of objects of $\mathcal{C}$.

a. Say $\mathcal{G}$ is a **separating family** if the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ are jointly faithful, i.e. if given $f, g : A \rightrightarrows B$ with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \to A$ with $fh \neq gh$.

b. Say $\mathcal{G}$ is a **detecting family** if the $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ jointly reflect isomorphisms, i.e. if given $f : A \to B$ s.t. every $g : G \to B$ with $G \in \mathcal{G}$ factors uniquely through f, f is an isomorphism.

Lemma 2.10. *i. If $\mathcal{C}$ is balanced, then any separating family is detecting*

ii. If $\mathcal{C}$ has equalisers, then every detecting family is separating

Definition 2.11. An object P is **projective** if $\mathcal{C}(P, -)$ preserves epimorphisms, i.e. if given

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

there exists $g : P \to A$ with $eg = f$.

Dually, P is **injective** in $\mathcal{C}$ if it's projective in $\mathcal{C}^{op}$.

If P satisfies this property $\forall e$ in some class $\mathcal{E}$ of epimorphisms, we call it $\mathcal{E}$-projective.

Corollary 2.12. *Representable functors are (pointwise) projective in* $[\mathcal{C}, \textbf{\textit{Set}}]$

Proof. Given

$$\begin{tikzcd} & \mathcal{C}(A, -) \arrow[d, "\beta"] \\ F \arrow[r, two heads, "\alpha"] & G \end{tikzcd}$$

β corresponds to some $y \in GA$. α_A is surjective, so $\exists x \in FA$ with $\alpha_A(x) = y$. x corresponds to $\gamma : \mathcal{C}(A, -) \to F$ with $\alpha\gamma = \beta$. $\square$

3 Adjunctions

Definition 3.1 (D.M. Khan, 1958)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be two functors. An **adjunction** between F and G is a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and morphisms $A \to GB$ in $\mathcal{C}$, which is natural in A and B.

(If $\mathcal{C}$ and $\mathcal{D}$ are locally small, this says that $(A, B) \to \mathcal{D}(FA, B)$ and $(A, B) \to \mathcal{C}(A, GB)$ are naturally isomorphic functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$).

We say F is **left adjoint** to G, or G is **right adjoint** to F, and write $F \dashv G$.

Theorem 3.2. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Given $A \in \operatorname{ob}\mathcal{C}$, let $(A \downarrow G)$ be the category whose objects are pairs (B, f) with $B \in \operatorname{ob}\mathcal{D}$, $f : A \to GB$ and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ in $\mathcal{D}$ such that*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & GB \arrow[d, "Gg"] \\ & GB' \end{tikzcd}$$

commutes. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A.

Proof. First suppose G has a left adjoint F. Let $\eta_A : A \to GFA$ be the morphism corresponding to $1_{FA} : FA \to FA$. The pair (FA, η_A) is an object of $(A \downarrow G)$. We'll show it's initial.

Given $g : FA \to B$, the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$ must correspond to $FA \xrightarrow{1} FA \xrightarrow{g} B$ under the adjunction.

So, for any object (B, f) of $(A \downarrow G)$, the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$ is the morphism $FA \to B$ corresponding to f.

Conversely, suppose we're given an initial object (FA, η_A) of $(A \downarrow G)$ for each G. Given $f : A \to A'$, the composite $A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA'$ is an object of $(A \downarrow G)$, so there's a unique morphism $Ff : FA \to FA'$ making

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{a'}"] & GFA' \end{tikzcd}$$

commute.

$f \mapsto Ff$ is functorial: given $f' : A' \to A''$, then $(Ff')(Ff)$ and $F(f'f)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''} f'f)$ in $(A \downarrow G)$, so they're equal.

Finally, given $f : A \to GB$, the morphism $g : FA \to B$ corresponding to it is the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$.

The naturality of this bijection is given by naturality of η, and naturality in B is immediate. □

Corollary 3.3. *If F, F' are both left-adjoint to G, then there's a canonical natural isomorphism $F \to F'$.*

Proof. For each A, (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \to (F'A, \eta'_A)$.

α is natural: given $f : A \to A'$, $\alpha_{A'} f$ and $(Ff)\alpha_A$ are both morphisms $(FA, \eta_A) \to (F'A', \eta'_{A'} f)$ in $(A \downarrow G)$. So they're equal. □

Lemma 3.4. *Given* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$ *, if $F \dashv G$ and $H \dashv K$ then $HF \dashv GK$.*

Proof. We have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

which are natural in A and C. □

Corollary 3.5. *Given a commutative square* $\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{E} \arrow[r, "K"] & \mathcal{F} \end{tikzcd}$ *of categories and functors, suppose all the functors in the diagram have left adjoints. Then the diagram* $\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{C} \end{tikzcd}$ *of left adjoints commutes up to natural isomorphism.*

Given $F \dashv G$, we have a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ defined as in 3.2. We call η the **unit** of the adjunction.

Dually, we have $\epsilon : FG \to 1_{\mathcal{D}}$, the **counit**. $\epsilon_B : FGB \to B$ corresponds to $1_{GB} : GB \to GB$.

Theorem 3.6. *Suppose we're given $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\epsilon : FG \to 1_{\mathcal{D}}$ such that*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

*commute. (We say η and ϵ satisfy the **triangular identities**).*

Proof. Given $F \dashv G$, we define η and ϵ as already described. Since $\epsilon_{FA} : FGFA \to FA$ corresponds to 1_{GFA}, the composite $\epsilon_{FA}(F\eta_A)$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$, so it must be 1_{FA}.

Similarly for the other identity.

Conversely, given η and ϵ satisfying the $\triangle^r$ identities, we map $f : A \to GB$ to the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$ and $g : FA \to B$ to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$.

We have

$$\Phi(\; A \xrightarrow{f} GB \;) = \; FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$$
$$\Psi(\; FA \xrightarrow{g} B \;) = \; A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$$

So

$$\begin{aligned}\Psi\Phi(f) &= \; A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= \; A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f\end{aligned}$$

And dually $\Phi\Psi(g) = g$.

Naturality of Φ in A is immediate from its definition, and naturality in B follows from that of ϵ. ☐

Lemma 3.7. *Suppose given* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *and natural isomorphisms* $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. *Then there exist natural isomorphisms* α', β' *which additionally satisfy the triangular identities. In particular* $(F \dashv G)$.

Proof. We define $\alpha' = \alpha$ and take β' to be the composite

$$FG \xrightarrow{\beta^{-1}_{FG}} FGFG \xrightarrow{F^{-1}_{\alpha_G}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that, since
$$\begin{tikzcd} FGFG \arrow[r, "FG\beta"] \arrow[d, "\beta_{FG}"] & FG \arrow[d, "\beta"] \\ FG \arrow[r, "\beta"] & 1_{\mathcal{D}} \end{tikzcd}$$
commutes and β is monic, we have $FG\beta = \beta_F G$.

Similarly, $GF\alpha = \alpha_{GF} : GF \to GFGF$.

Now

$$\begin{aligned}\beta'_F \circ F_{\alpha'} &= \; F \xrightarrow{F\alpha} FGF \xrightarrow{\beta^{-1}_{FGF}} FGFGF \xrightarrow{F\alpha^{-1}_{GF}} FGF \xrightarrow{\beta_F} F \\ &= \; F \xrightarrow{\beta^{-1}_F} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{FGF\alpha^{-1}} FGF \xrightarrow{\beta_F} F \\ &= 1_F\end{aligned}$$

and

$$\begin{aligned}G\beta' \circ \alpha'_G &= \; G \xrightarrow{\alpha_G} GFG \xrightarrow{GFG\beta^{-1}} GFGFG \xrightarrow{GF^{-1}_{\alpha_G}} GFG \xrightarrow{G\beta} G \\ &= \; G \xrightarrow{G\beta^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha^{-1}_{GFG}} GFG \xrightarrow{\beta_F} G \\ &= 1_G\end{aligned}$$

$\square$

Lemma 3.8. *Suppose* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $(F \dashv G)$ *is an adjunction with counit* ϵ. *Then*

i. ϵ *is (pointwise) epic* $\iff$ G *is faithful*

ii. ϵ *is an isomorphism* $\iff$ G *is full and faithful*

Proof. i. Given $g : B \to B'$, the morphism $Gg : GB \to GB'$ corresponds to

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$$

So, for fixed B, composition with ϵ_B is injective on morphisms $B \to B'$ $\iff$ $(g \mapsto Gg)$ is injective on morphisms $B \to B'$.

Hence G is faithful $\iff$ ϵ_B is epic $\forall B$.

ii. Similarly, ϵ_B is 0 $\forall B$ $\implies$ G is bijective on morphisms with given domain and codomain, i.e. G is full and faithful.

Conversely, if G is full and faithful, 1_{FGB} factors uniquely as $FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} FGB$, so ϵ_B is split monic. But it's epic by (i), hence an isomorphism.

$\square$

Definition 3.9. i. A **reflection** is an adjunction satisfying the conditions of 3.8(ii).

ii. A **reflective** subcategory of $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ for which the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint.

Dually, **coreflection** and **coreflective** subcategory.

4 Limits

Definition 4.1. a. Let J be a category (almost always small, often finite). A **diagram of shape J** in a category $\mathcal{C}$ is a functor $D : J \to \mathcal{C}$.

E.g. if J is the finite category
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$
, a diagram of shape J is a commutative square. If J is the category
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$
, a diagram of shape J is a not-necessarily-commutative square.

The objects $D(j)$, $j \in \mathrm{ob}\, J$ are called **vertices** of D, and the morphisms $D(\alpha)$, $\alpha \in \mathrm{mor}\, J$ are called **edges** of D.

b. Let $D : J \to \mathcal{C}$ be a diagram in $\mathcal{C}$. A **cone over D** is a pair $(A, (\lambda_j \mid j \in \mathrm{ob}\, J))$ where $\lambda_j : A \to D(j)$ $\forall j$, and
$$\begin{tikzcd} & A \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"] & & D(j') \end{tikzcd}$$
commutes for each $\alpha : j \to j'$ in J.

A is called the **apex** of the cone, and the λ_j are its **legs**.

Equivalently, λ is a natural transformation $\triangle A \to D$, where $\triangle A$ is the **constant diagram** with all vertices A and all edges 1_A.

A **morphism** $f : (A, (\lambda_j)) \to (B, (\mu_j))$ of cones over D is a morphism $f : A \to B$ s.t.
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\lambda_j"] & & B \arrow[dl, "\mu_j"] \\ & D(j) & \end{tikzcd}$$
commutes for each j. We have a category **Cone**(D) of cones over D.

Note that $A \mapsto \triangle A$ is a functor $\mathcal{C} \to [J, \mathcal{C}]$ and **Cone**(D) is in fact the category $(\triangle \downarrow D)$.

A cocone over $D : J \to \mathcal{C}$ is a cone over $D : J^{op} \to \mathcal{C}^{op}$. We write **Cocone**$(D)$ for the category of cocones over D.

Definition 4.2. i. A **limit** (resp. **colimit**) for a diagram $D : J \to \mathcal{C}$ is a terminal object of **Cone**(D) (respectively an initial object of **Cocone**(D)).

ii. We say $\mathcal{C}$ has limits (resp. colimits) of shape J if $\triangle : \mathcal{C} \to [J, \mathcal{C}]$ has a right (resp. left) adjoint.

(This is equivalent to making a choice of limit (resp. colimit) for every diagram of shape J).

Definition 4.3 (Pullback)**.** Let J be
$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}.$$
A diagram of shape J looks like
$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}.$$
A cone over it consists of
$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, "l"] & A \\ C & B \end{tikzcd}$$
satisfying $fh = l = gk$. Equivalently, it's a pair
$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"] & A \\ C & \end{tikzcd}$$
completing the diagram to a commutative square.

A universal such pair is called a **pullback** (or **fibre product**); in **Set** it can be defined as $\{(a,b) \in A \times B \mid f(a) = g(b)\}$. A colimit of shape J^{op} is called a **pushout**.

Theorem 4.4. *Let $\mathcal{C}$ be a category.*

i. If $\mathcal{C}$ has equalisers and all finite (resp. all small) products, then $\mathcal{C}$ has all finite (resp. all small) limits.

ii. If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.

Proof. i. Given $D : J \to \mathcal{C}$, first form the products

$$P = \prod_{j \in \mathrm{ob}\, J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \mathrm{mor}\, J} D(\mathrm{cod}\, \alpha)$$

Define $P \underset{g}{\overset{f}{\rightrightarrows}} Q$ by $\pi_\alpha f = \pi_{\mathrm{cod}\, \alpha} : P \to D(\mathrm{cod}\, \alpha)$ and $\pi_\alpha g = D(\alpha) \circ \pi_{\mathrm{dom}\, \alpha} : P \to D(\mathrm{dom}\, \alpha) \to D(\mathrm{cod}\, \alpha)$, and let $e : E \to P$ be the equaliser of (f, g).

Claim $(E, (\pi_j e \mid j \in \mathrm{ob}\, J))$ is a limit cone for D. It is a cone since, for any $\alpha : j \to j'$, $D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e$.

Given any cone $(C, (\lambda_j \mid j \in \mathrm{ob}\, J))$, the λ_j define a unique $\lambda : C \to P$, and $f\lambda = g\lambda$ since $\pi_\alpha f \lambda = \pi_\alpha g \lambda$ $\forall \alpha$. So λ factors uniquely through e.

ii. Let 1 be a terminal object of $\mathcal{C}$. For any pair of objects (A, B) the pullback of

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$

has the universal property of a product $A \times B$, so $\mathcal{C}$ has binary products. Then we can define any finite product $\prod_{i=1}^n A_i$ as $(((A_1 \times A_2) \times A_3) \times \dots) \times A_n$.

So we need to show $\mathcal{C}$ has equalisers. Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$, consider the pullback of

$$\begin{tikzcd} & B \arrow[d, "(1_A, f)"] \\ A \arrow[r, "(1_A, g)"] & A \times B \end{tikzcd} .$$

It consists of

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "k"] & B \\ A & \end{tikzcd}$$

satisfying $1_A h = 1_A k$ and $fh = gk$, and universal among such.

But this forces $h = k$, and h has the universal property of an equaliser for (f, g). So by (i), $\mathcal{C}$ has all finite limits.

$\square$

Definition 4.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

a. We say F **preserves** limits of shape J if, given $D : J \to \mathcal{C}$ and a limit cone $(L, (\lambda_j : j \in \operatorname{ob} J))$ for D, the cone $(FL, (F\lambda_j : j \in \operatorname{ob} J))$ is a limit for $FD : J \to \mathcal{D}$.

b. We say F **reflects** limits of shape J if, given $D : J \to \mathcal{C}$ and a cone $(L, (\lambda_j))$ such that $(FL, (F\lambda_j))$ is a limit for FD, then $(L, (\lambda_j))$ is a limit for D.

c. We say F **creates** limits of shape J if, given $D : J \to \mathcal{C}$ and a limit $(M, (\mu_j))$ for FD, there exists a cone (L, λ_j) over D whose image is isomorphic to $(M, (\mu_j))$, and any such cone is a limit for D.

Lemma 4.6. *Suppose $\mathcal{D}$ has limits of shape J. Then $[\mathcal{C}, \mathcal{D}]$ has limits of shape J, and they're constructed pointwise (i.e. the forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\operatorname{ob} \mathcal{C}}$ creates them).*

Proof. Consider a functor $D : J \times \mathcal{C} \to \mathcal{D}$. For each $A \in \operatorname{ob} \mathcal{C}$, let $(LA, (\lambda_{j,A} : LA \to D(j, A) \,|\, j \in \operatorname{ob} J))$ be a limit for the diagram $D(-, A) : J \to \mathcal{D}$.

Given any $f : A \to B$ in $\mathcal{C}$, the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, so they induce a unique $Lf : LA \to LB$ such that

$$\begin{tikzcd} LA \arrow[r, "Lf"] \arrow[d, "\lambda_{j,A}"] & LB \arrow[d, "\lambda_{j,B}"] \\ D(j,A) \arrow[r, "D(j,f)"] & D(j,B) \end{tikzcd}$$

commutes for all j. Uniqueness assures $L(gf) = L(g)L(f)$, so L is a functor $\mathcal{C} \to \mathcal{D}$, and the $\lambda_{j,-}$ are natural transformations $L \to D(j, -)$.

Suppose we're given any cone over D in $[\mathcal{C}, \mathcal{D}]$ with apex M and legs $\mu_j : M \to D(j, -)$. Then $(MA, (\mu_{j,A} : MA \to D(j, A) \,|\, j \in \operatorname{ob} J))$ is a cone over $D(-, A)$ in $\mathcal{D}$, so we get a unique $\nu_A : MA \to LA$ s.t. $\lambda_{j,A}\nu_A = \mu_{j,A}$ for all j.

Uniqueness tells us that

$$\begin{tikzcd} MA \arrow[r, "Mf"] \arrow[d, "\nu_A"] & MB \arrow[d, "\nu_B"] \\ LA \arrow[r, "Lf"] & LB \end{tikzcd}$$

commutes for all $f \in \operatorname{mor} \mathcal{C}$, so $\nu : M \to L$ in $[\mathcal{C}, \mathcal{D}]$, so it's the unique factorisation of the $\mu_{j,-}$ through the $\lambda_{j,-}$. □

Lemma 4.7. *A morphism $f : A \to B$ is monic $\iff$*

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback.

Proof. f is monic $\iff$ any cone (g,h) over (f, f) has $g = h$ $\iff$ (g, h) factors uniquely through $(1_A, 1_A)$. $\square$

Hence, provided $\mathcal{D}$ has pullbacks, a morphism $\alpha : F \to G$ in $[\mathcal{C}, \mathcal{D}]$ is monic $\iff$ $\alpha_A : FA \to GA$ is monic for each A.

Theorem 4.8. *If $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, then G preserves all limits which exist in $\mathcal{D}$.*

Proof. Suppose $\mathcal{C}$ and $\mathcal{D}$ both have limits of shape J and let $(F \dashv G)$. The diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle \triangle} & & \downarrow{\scriptstyle \triangle} \\ [J, \mathcal{C}] & \xrightarrow{[J,F]} & [J, \mathcal{D}] \end{array}$$

commutes and $[J, F]$ has a right adjoint $[J, G]$. So by 3.5 the diagram of right adjoints

$$\begin{array}{ccc} [J, \mathcal{D}] & \xrightarrow{[J,G]} & [J, \mathcal{C}] \\ \downarrow{\scriptstyle \lim_J} & & \downarrow{\scriptstyle \lim_J} \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

commutes up to isomorphism, i.e. G preserves limits of shape J. $\square$

Proof. Let $D : J \to \mathcal{D}$ be a diagram with limit $(L, (\lambda_j \mid j \in \mathrm{ob}\, J))$. Given a cone $(A, (\mu_j : A \to GD(j) \mid j \in \mathrm{ob}\, J))$ in $\mathcal{C}$, we get a cone $(FA, (\bar{\mu_j} : FA \to D(j) \mid j \in \mathrm{ob}\, J))$ in $\mathcal{D}$, and hence a unique $\bar{\nu} : FA \to L$ such that $\lambda_j \bar{\nu} = \bar{\mu_j}$ for all j.

Then $\nu : A \to GL$ is the unique morphism such that $(G\lambda_j)\nu = \mu_j \forall j$. $\square$

The 'primeval' Adjoint Functor Theorem says that if $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits, then G has a left adjoint.

This depends on two ideas:

Lemma 4.9. *$\mathcal{C}$ has an initial object $\iff$ $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ has a limit.*

Proof. Suppose $\mathcal{C}$ has an initial object 0. The morphisms $(0 \to A \mid A \in \mathrm{ob}\, \mathcal{C})$ form a cone over $1_{\mathcal{C}}$. If we had another, say $(L, (\lambda_A \mid A \in \mathrm{ob}\, \mathcal{C}))$, then $\lambda_0 : L \to 0$ would make

$$\begin{tikzcd} L \arrow[r, "\lambda_0"] \arrow[dr, "\lambda_A"'] & 0 \arrow[d] \\ & A \end{tikzcd}$$

commute for all A, and it's the only morphism which does.

Conversely, suppose $(I, (\lambda_A : I \to A \mid A \in \mathrm{ob}\,\mathcal{C}))$ is a limit for $1_{\mathcal{C}}$.

If $f : I \to A$, then

$$\begin{tikzcd} I \arrow[r, "\lambda_I"] \arrow[dr, "\lambda_A"'] & I \arrow[d, "f"] \\ & A \end{tikzcd}$$

commutes. In particular, $\lambda_A \lambda_I = \lambda_A$ for all A, so $\lambda_I = 1_I$ since both are factorisations of the limit cone through itself. So $f = \lambda_A$, and hence I is initial. $\square$

Lemma 4.10. *Suppose $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then, for each $A \in \mathrm{ob}\,\mathcal{C}$, $(A \downarrow G)$ has limits of shape J and the forgetful functor $(A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Suppose given $D : J \to (A \downarrow G)$. Write $D(j)$ as $(UD(j), f_j : A \to GUD(j))$ for each j. Let $(L, (\lambda_j \mid j \in \mathrm{ob}\,J))$ be a limit for UD, then $(GL, (G\lambda_j \mid j \in \mathrm{ob}\,J))$ is a limit for GUD. But the f_j form a cone over GUD with apex A, so there's a unique $h : A \to GL$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f_j"'] & GL \arrow[d, "G\lambda_j"] \\ & GUD(j) \end{tikzcd}$$

commutes for all j. So there's a unique lifting of the cone over D in $(A \downarrow G)$.

Suppose we're given a cone $((B, g), (\mu_j \mid j \in \mathrm{ob}\,J))$ over D. Then

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "h"'] & GB \arrow[d, "G_k"] \\ & GL \end{tikzcd}$$

commutes since both ways round are factorisations of $(f_j \mid j \in \mathrm{ob}\,J)$ through the limit GL. $\square$

Combining 4.10 and 4.9 with 3.2, we've proved the primeval Adjoint Functor Theorem. However, this requries $\mathcal{D}$ to have limits for diagrams 'as big as $\mathcal{D}$ itself', and the only such categories are preorders (c.f. Q6, sheet 2).

In practice, the most we can hope for is that $\mathcal{D}$ has all small limits. We call such a $\mathcal{D}$ **complete**.

Theorem 4.11 (General Adjoint Functor Theorem)**.** *Suppose that $\mathcal{D}$ is complete and locally small. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if it preserves all small limits and satisfies the 'solution set condition': for any $A \in \operatorname{ob}\mathcal{C}$, there is a set $\{f_i : A \to GB_i \mid i \in I\}$ of objects of $(A \downarrow G)$ such that any $h : A \to GC$ factors as*

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} Gc$$

for some $i \in I$ and $g : B_i \to C$.

Proof. If G has a left adjoint, then it preserves small limits by 4.8, and $\{\eta_A : A \to GFA\}$ is a singleton solution set at A.

Conversely, each $(A \downarrow G)$ is complete by 4.10, and locally small since it admits a faithful functor to $\mathcal{D}$. So we need to show: if $\mathcal{A}$ is complete and locally small, and has a weakly initial set of objects $\{S_i \mid i \in I\}$, then $\mathcal{A}$ has an initial object.

First form $P = \prod_{i \in I} S_i$: then P is weakly initial.

Now form the limit $I \xrightarrow{a} P$ of the diagram $P \Rrightarrow P$ whose edges are all morphism $P \to P$ in $\mathcal{A}$.

Claim I is initial: it's weakly initial since it admits a morphism to P.

Suppose we had $I \underset{g}{\overset{f}{\rightrightarrows}} A$. Let $b : E \to I$ be an equaliser for (f, g): then there exists $c : P \to E$.

Now $P \xrightarrow{c} E \xrightarrow{b} I \xrightarrow{a} P$ is an edge of the diagram whose limit is I, but so is 1_P; so $abca = 1_P a = a$. But a is monic, so $bca = 1_I$. So b is (split) epic, and $f = g$. So all the $(A \downarrow G)$ have initial objects, hence by 3.2 G has a left adjoint. □

The Special Adjoint Functor Theorem imposes additional conditions on $\mathcal{C}$ and $\mathcal{D}$ which ensure that every functor $\mathcal{D} \to \mathcal{C}$ preserving small limits has a left adjoint.

Definition 4.12.

a. A **subobject** of an object A is a monomorphism $A' \rightarrowtail A$. We write $\mathbf{Sub}_{\mathcal{C}}(A)$ for the full subcategory of $\mathcal{C}/A$ whose objects are subobjects of A: note that this category is a preorder.

b. We say $\mathcal{C}$ is **well-powered** if each $\mathbf{Sub}_{\mathcal{C}}(A)$ is equivalent to a small category, i.e. up to isomorphism each object has only a set of subobjects.

   Dually, $\mathcal{C}$ is **well-copowered** if $\mathcal{C}^{op}$ is well-powered.

Lemma 4.13. *Suppose given a pullback*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "h"] & A \arrow[d, tail, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

with f monic. Then h is monic.

Proof. Suppose $D \underset{y}{\overset{x}{\rightrightarrows}} P$ satisfy $hx = hy$. Then $fkx = fky = ghx = ghy$ and f is monic so $kx = ky$.

Now $x = y$ since both are factorisations of the same cone through the pullback. □

Theorem 4.14 (Special Adjoint Functor Theorem)**.** *Suppose both $\mathcal{C}$ and $\mathcal{D}$ are locally small, and $\mathcal{D}$ is complete, well-powered and has a separating set. Then $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint $\iff$ G preserves all small limits.*

Proof. The forward implication is 4.8 again.

Conversely, we first show that $(A \downarrow G)$ has the properties we've assumed for $\mathcal{D}$: it's complete by 4.10, and locally small as in 4.11. It's well-powered since subobjects of (B, f) in $(A \downarrow G)$ are in bijection with subobjects $B' \rightarrowtail B$ such that f factors through $GB' \rightarrowtail GB$.

It has a coseparating set: if $\{S_i \,|\, i \in I\}$ is a coseparating set for $\mathcal{D}$, then $\{(S_i, f) \,|\, i \in I, f : A \to GS_i\}$ is a coseperating set for $(A \downarrow G)$, since if $(B, f) \underset{g'}{\overset{g}{\rightrightarrows}} (B', f')$ satisfies $g \neq g'$, there exists $h : B' \to S_i$ for some i with $hg \neq hg'$, and then h is a morphism $(B', f') \to (S_i, (Gh)f')$ in $(A \downarrow G)$.

Now we show that if $\mathcal{A}$ is complete, locally small and well-powered and has a coseperating set, then it has an initial object.

First form $P = \prod_{i \in I} S_i$, where $\{S_i \,|\, i \in I\}$ is a coseparating set.

Consider the diagram

$$\begin{tikzcd} P' \arrow[dr, tail] & \\ P'' \arrow[r, tail] & P \\ \vdots & \\ P^{(n)} \arrow[uur, tail] & \end{tikzcd}$$

whose edges are a representative set of subobjects of P.

Form its limit

$$\begin{tikzcd}
 & P' \\
I \arrow[ur] \arrow[r] \arrow[ddr] & P'' \\
 & \vdots \\
 & P^{(n)}
\end{tikzcd}$$

by the argument of 4.13 the legs $I \to P^{(-)}$ are monic, so $I \rightarrowtail P$ is monic and it's the least subobject of P.

Hence in particular I has no proper subobjects, so any two maps $I \underset{g}{\overset{f}{\rightrightarrows}} A$ must be equal, since their equaliser is an isomorphism.

Now given $A \in \mathcal{A}$, form the product $Q = \prod_{i,f:A\to S_i} S_i$. The canonical morphism $h : A \to Q$ defined by $\pi_{i,f}h = f$ is monic since the S_i form a coseparating set.

We also have $k : P \to Q$ defined by $\pi_{i,f}k = \pi_i$, and we can form the pullback

$$\begin{tikzcd}
I \arrow[r] \arrow[dr, tail] & B \arrow[r, "m"] \arrow[d, tail, "l"] & A \arrow[d, tail, "h"] \\
 & P \arrow[r, "k"] & Q
\end{tikzcd}$$

By 4.13 l is monic and hence isomorphic to an edge of the diagram defining I, so $I \rightarrowtail P$ factors through it. So there exists a morphism $I \to A$, hence I is initial. $\square$

5 Monads

Suppose given an adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $F \dashv G$. How much of this can we describe purely in terms of $\mathcal{C}$?

We have the composite $T = GF : \mathcal{C} \to \mathcal{C}$, and the unit $\eta : 1_{\mathcal{C}} \to T$. We also have $G\epsilon_F : GFGF \to GF$, which we'll denote $\mu : TT \to T$.

These satisfy the commutative diagrams

$$\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "\eta_T"'] \arrow[dl, "1_T"] \\
① & T & ②
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] \arrow[dr, phantom, "③"] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"] & T
\end{tikzcd}$$

from the $\triangle^r$ identities and naturality of ϵ.

Definition 5.1. A **monad** $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \to T$, $\mu : TT \to T$ satisfying the commutative diagrams ①, ② and ③.

Definition 5.2. Let $\mathbb{T}$ be a monad on $\mathcal{C}$. A **$\mathbb{T}$-algebra** is a pair (A, α) where $A \in \operatorname{ob} \mathcal{C}$, and $\alpha : TA \to A$ satisfies

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] \arrow[dr, phantom, "④" description, shift right=4ex] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "⑤"] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"] & A \end{tikzcd}$$

A **homomorphism** $f : (A, \alpha) \to (B, \beta)$ of $\mathbb{T}$-algebras is a morphism $f : A \to B$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"] \arrow[dr, phantom, "⑥"] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commutes. We write $\mathcal{C}^{\mathbb{T}}$ for the category of $\mathbb{T}$-algebras.

Lemma 5.3. *The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ has a left adjoint F, and the adjunction $(F \dashv G)$ induces the monad $\mathbb{T}$.*

Proof. We define $FA = (TA, \mu_A)$ (which is an algebra by ② and ③), and $F(A \xrightarrow{f} B) = Tf$ (which is a homomorphism by naturality of μ).

Clearly $GF = T$ and $\eta : 1_{\mathcal{C}} \to GF$.

We define $\epsilon : FG \to 1_{\mathcal{C}^{\mathbb{T}}}$ by $\epsilon_{(A,\alpha)} = \alpha : (TA, \mu_A) \to (A, \alpha)$ (which is a homomorphism by ⑤).

The triangular identities for η and ϵ follow from ④ and ①, so $(F \dashv G)$.

Finally, $G_{\epsilon_{FA}} = \mu_A$ by the definitions of FA and ϵ, so the adjunction incudes $\mathbb{T}$. $\square$

Note that if $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ induces $\mathbb{T}$, then so does $\mathcal{C} \underset{G/\mathcal{D}'}{\overset{F}{\rightleftarrows}} \mathcal{D}'$ where $\mathcal{D}'$ is the full subcategory of objects of the form FA. So in seeking to construct $\mathcal{D}$, we may require F to be bijective on objects. But then morphisms $FA \to FB$ in $\mathcal{D}$ correspond bijectively to morphisms $A \to GFB = TB$ in $\mathcal{C}$.

Definition 5.4. Given a monad $\mathbb{T}$ on $\mathcal{C}$, the **Kleisli category** $\mathcal{C}_{\mathbb{T}}$ is defined by: $\operatorname{ob} \mathcal{C}_{\mathbb{T}} = \operatorname{ob} \mathcal{C}$, morphisms $A \to B$ in $\mathcal{C}_{\mathbb{T}}$ are morphisms $A \to TB$ in $\mathcal{C}$, the identity $A \to A$ is $A \xrightarrow{\eta_A} TA$, and the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} C$.

We check

$$\begin{aligned} A \xrightarrow{1_A} A \xrightarrow{f} B &= A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} B \\ &= A \xrightarrow{f} TA \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} B \\ &= f \text{ by ②} \end{aligned}$$

$$
\begin{aligned}
A \xrightarrow{f} B \xrightarrow{1_B} B &= A \xrightarrow{f} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} B \\
&= f \text{ by } ①
\end{aligned}
$$

Given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

$$
\begin{aligned}
(hg)f &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD \text{ by } ③ \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{Th} TTD \xrightarrow{\mu_D} TD \\
&= h(gf)
\end{aligned}
$$

Lemma 5.5. *There exists an adjunction* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_{\mathbb{T}}$ *inducing* $\mathbb{T}$.

Proof. We define $FA = A$ and $F(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. This clearly preserves identities, and

$$
\begin{aligned}
(Fg)(Ff) &= A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC \\
&= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \text{ by } ① \text{ and naturality of } \eta \\
&= F(gf)
\end{aligned}
$$

We define $GA = TA$ and $G(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. G preserves identities by ① and

$$
\begin{aligned}
G(A \xrightarrow{f} B \xrightarrow{g} C) &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC \text{ by } ③ \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \text{ by naturality of } \mu \\
&= (Gg)(Gf)
\end{aligned}
$$

Clearly $GFA = TA$ and

$$
\begin{aligned}
GF(A \xrightarrow{f} B) &= TA \xrightarrow{Tf} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} TB \\
&= Tf \text{ by } ①
\end{aligned}
$$

so $GF = T$ and $\eta : 1_{\mathcal{C}} \to GF$.

We define $FGA \xrightarrow{\epsilon_A} A$ to be $TA \xrightarrow{\eta_{TA}} TA$. To verify naturality of ϵ, consider

$$
\begin{tikzcd}
FGA \arrow[r, "FGf"] \arrow[d, "\epsilon_A"] & FGB \arrow[d, "\epsilon_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

The top and right edges yield

$$TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}} TTB \xrightarrow{1_{TTB}} TTB \xrightarrow{\mu_B} TB$$

and the left and bottom yield

$$TA \xrightarrow{1_{TA}} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$$

For the $\triangle^r$ identities,

$$GA \xrightarrow{\eta_{GA}} GFGA \xrightarrow{G\epsilon_A} GA = TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = 1_{TA}$$

and

$$\begin{aligned} FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\epsilon_{FA}} FA &= A \xrightarrow{\eta_A} TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA \\ &= A \xrightarrow{\eta_A} TA \; (= FA \xrightarrow{1_{FA}} FA) \end{aligned}$$

Finally, $G\epsilon_{FA} = TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA = \mu_A$, so the adjunction induces the monad $\mathbb{T}$. $\square$

Theorem 5.6. *Given a monad $\mathbb{T}$ on $\mathcal{C}$, let **Adj**$(\mathbb{T})$ be the category whose objects are adjunctions $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ inducing $\mathbb{T}$, and whose morphisms $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}) \to (\mathcal{C} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{D}')$ are functors $K : \mathcal{D} \to \mathcal{D}'$ satisfying $KF = F'$ and $G'K = G$.*

*Then the Kleisi category $\mathcal{C}_{\mathbb{T}}$ is initial in **Adj**$(\mathbb{T})$, and the Eilenberg-Moore category $\mathcal{C}^{\mathbb{T}}$ is terminal.*

Proof. Given $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$ in **Adj**$(\mathbb{T})$, we define the **Eilenberg-Moore comparison functor** $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\epsilon_B)$ (note that $G\epsilon_B$ is an algebra structure on GB: the unit condition ④ follows from a $\triangle^r$ identity, and ⑤ follows from the naturality of ϵ).

$K(B \xrightarrow{g} B') = Gg : (GB, G\epsilon_B) \to (GB', G\epsilon_{B'})$ (a homomorphism since ϵ is natural).

It's clear that K is a functor, that $G^{\mathbb{T}}K = G$ and that $KFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$ and $KF(A \xrightarrow{f} B) = Tf = F^{\mathbb{T}}f$.

Uniqueness: suppose $\bar{K}$ also satisfies $G^{\mathbb{T}}\bar{K} = G$ and $\bar{K}F = F^{\mathbb{T}}$. Then $\bar{K}B$ is of the form (GB, β_B) for some algebra structure β_B, and that $\beta_{FA} = \mu_A = G\epsilon_{FA}$ for all A.

Given any B, consider the diagram

$$\begin{array}{ccc} GFGFGB & \xrightarrow{GFG\epsilon_B} & GFGB \\ \downarrow{\scriptstyle \mu_{GB}} & & \downarrow{\scriptstyle \beta_B} \\ GFGB & \xrightarrow{G\epsilon_B} & GB \end{array}$$

which must commute, since $G\epsilon_B$ is an algebra homomorphism. But it would also commute with $G\epsilon_B$ in place of β_B, and $GFG\epsilon_B$ is (split) epic, so $\beta_B = G\epsilon_B$.

For the **Kleisi comparison functor** $K : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$, we define $KA = FA$, $K(A \xrightarrow{f} B) = FA \overset{Ff}{\to} FGFB \overset{\epsilon_{FB}}{\to} FB$.

To verify this is functorial, consider

$$\begin{aligned}
K(A \xrightarrow{f} B \xrightarrow{g} C) &= FA \overset{Ff}{\to} FGFB \overset{FGFg}{\to} FGFGFC \overset{FG\epsilon_{FC}}{\to} FGFC \overset{\epsilon_{FC}}{\to} FC \\
&= FA \overset{Ff}{\to} FGFB \overset{FGFg}{\to} FGFGFC \overset{\epsilon_{FGFC}}{\to} FGFC \overset{\epsilon_{FC}}{\to} FC \\
&= FA \overset{Ff}{\to} FGFB \overset{\epsilon_{FB}}{\to} FB \overset{Fg}{\to} FGFC \overset{\epsilon_{FC}}{\to} FC \\
&= (Kg)(Kf)
\end{aligned}$$

$GKA = GFA = TA = G_{\mathbb{T}}A$

$GK(A \xrightarrow{f} B) = TA \overset{Tf}{\to} TTB \overset{\mu_B}{\to} TB = G_{\mathbb{T}}(f)$

And $KF_{\mathbb{T}}A = FA$,

$$KF_{\mathbb{T}}(A \overset{f}{\to} B) = \begin{tikzcd} FA \arrow[r, "Ff"] & FB \arrow[r, "F\eta_B"] \arrow[dr, "1_{FB}"] & FGFB \arrow[d, "\epsilon_{FB}"] \\ & & FB \end{tikzcd}$$

So K is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Uniqueness: suppose $\bar{K}$ is any other morphism $\mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ in $\mathbf{Adj}(\mathbb{T})$. Then $\bar{K}A = FA = KA$ for all A; since $\bar{K}$ commutes with both the Fs and the Gs, we have $\bar{K}(\epsilon_A) = \epsilon_{FA}$.

We can write $A \xrightarrow{f} B$ as $A \overset{F_{\mathbb{T}}f}{\to} F_{\mathbb{T}}G_{\mathbb{T}} \overset{\epsilon_B}{\to} B$, so $\bar{K}(f) = \bar{K}(\epsilon_B)Ff = K(f)$. □

The Kleisi category $\mathcal{C}_{\mathbb{T}}$ inherits coproducts from $\mathcal{C}$ if $\mathcal{C}$ has them, but it has few other limits or colimits in general.

Theorem 5.7. *i. The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates all limits which exist in $\mathcal{C}$.*

ii. If $T : \mathcal{C} \to \mathcal{C}$ preserves colimits of shape J, then $G : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates them.

Proof. i. Let $D : J \to \mathcal{C}^{\mathbb{T}}$ be a diagram, write $D(j) = (GD(j), \delta_j)$.

Let $(L, (\lambda_j : L \to GD(j)))$ be a limit for GD. The composites $TL \overset{T\lambda_j}{\to} TGD(j) \overset{\delta_j}{\to} GD(j)$ form a cone over GD, since the edges of GD are algebra homomorphisms.

So they induce a unique $l : TL \to L$ such taht

$$\begin{tikzcd} TL \arrow[r, "T\lambda_j"] \arrow[d, "l"] & TGD(j) \arrow[d, "\delta_j"] \\ L \arrow[r, "\lambda_j"] & GD(j) \end{tikzcd}$$

commutes for each j.

l is an algebra structure: $l\eta_L = l_L$ since both are factorisations of (λ_j) through itself, and $lTl = l\mu_L$ since they're factorisations of the same cone through L.

So $((L, l), (\lambda_j))$ is the unique lifting of $(L, (\lambda_j))$ to a cone over D in $\mathcal{C}^{\mathbb{T}}$.

Any cone over D in $\mathcal{C}^{\mathbb{T}}$ factors uniquely through L, and the factorisation is an algebra homomorphism.

ii. Similarly, given $D : J \to \mathcal{C}^{\mathbb{T}}$ as before and a colimit $(L, (\lambda_j : GD(j) \to L))$ for GD, we get a unique $l : TL \to L$ making

$$\begin{tikzcd} TGD(j) \arrow[r, "T\lambda_j"] \arrow[d, "\delta_j"] & TL \arrow[d, "l"] \\ GD(j) \arrow[r, "\lambda_j"] & L \end{tikzcd}$$

commute, since $(TL, (T\lambda_j))$ is a colimit. The rest of the proof is similar to (i).

$\square$

Definition 5.8. An adjunction $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$, $(F \dashv G)$, is **monadic** if the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is part of an equivalence, where $\mathbb{T}$ is the monad induced by $(F \dashv G)$. We also say $G : \mathcal{D} \to \mathcal{C}$ is monadic if it has a left adjoint and the adjunction is monadic.

Note that K preserves all limits which exist in $\mathcal{D}$, since G preserves them and $G^{\mathbb{T}}$ creates them.

Lemma 5.9. *Suppose given $(\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D})$, $(F \dashv G)$ inducing a monad $\mathbb{T}$ on $\mathcal{C}$. Suppose, for each $\mathbb{T}$-algebra (A, α), the pair $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ has a coequaliser in $\mathcal{D}$. Then $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ has a left adjoint L.*

Proof. We define $L(A, \alpha) = \mathrm{coeq}(FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA)$.

Given $(A, \alpha) \xrightarrow{f} (B, \beta)$, we get

$$\begin{tikzcd} FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] \arrow[d, "FGFf"] & FA \arrow[r] \arrow[d, "Ff"] & L(A, \alpha) \arrow[d, dashed, "Lf"] \\ FGFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\epsilon_{FB}"'] & FB \arrow[r] & L(B, \beta) \end{tikzcd}$$

So $\exists! Lf$ making the right hand square commute. Uniqueness ensures L is functorial.

Morphisms $L(A,\alpha) \to B$ in $\mathcal{D}$ correspond bijectively to morphisms $f : FA \to B$ such that $f(F\alpha) = f(\epsilon_{FA})$ and hence to morphisms $\bar{f} : A \to GB$ such that $f\bar{\alpha} = Gf|_{GFA} = G\epsilon_B \circ GF\bar{f}$, i.e. to algebra homomorphisms $(A,\alpha) \to (GB, G\epsilon_B) = KB$.

So $(L \dashv K)$. $\square$

Definition 5.10. a. A parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is **reflexive** if $\exists r : B \to A$ such that $fr = gr = 1_B$.

Note that $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ is reflexive, with common splitting $FA \overset{F\eta_A}{\to} FGFA$.

A **reflexive coequaliser** is the coequaliser of a reflexive pair.

b. A **split coequaliser diagram** is a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$.

If these equations hold, h **is** a coequaliser of f and g: given $k : B \to D$ with $kf = kg$, we have $k = kgt = kft = ksh$, so k factors through h, uniquely since h is split epic.

Note that **any** functor preserves split equalisers.

c. Given $G : \mathcal{D} \to \mathcal{C}$, a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{D}$ is **G-split** if there exists a split coequaliser

$$\begin{tikzcd} GA \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB \arrow[r] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

in $\mathcal{C}$.

The pair $FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$ of 5.9 is G-split:

$$\begin{tikzcd} GFGFA \arrow[r, shift left, "Gf\alpha"] \arrow[r, shift right, "G\epsilon_{FA}"'] & GFA \arrow[r, "\alpha"] \arrow[l, bend left=60, "\eta_{GFA}"] & A \arrow[l, bend left, "\eta_A"] \end{tikzcd}$$

is a split coequaliser diagram.

Theorem 5.11 (Precise Monadicity Theorem)**.** *$G : \mathcal{D} \to \mathcal{C}$ is monadic $\iff$ G has a left adjoint, and c reates coequalisers of G-split pairs.*

Theorem 5.12 (Crude Monadicity Theorem)**.** *Suppose $G : \mathcal{C} \to \mathcal{C}$ has a left adjoint, that $\mathcal{D}$ has and G preserves reflexive coequalisers, and G reflects isomorphisms. Then G is monadic.*

Proof. For the forward implication in 5.11, it's enough to show that $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates coequalisers of $G^{\mathbb{T}}$-split pairs. This follows from 5.7, given that T and TT both preserve split coequalisers.

Conversely in either case, $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ has a left adjoint L by 5.9. Now $LKB = \mathrm{coeq}(FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB)$ and the counit $LKB \to B$ is the factorisation of $FGB \underset{\epsilon_B}{\overset{\epsilon_{FGB}}{\rightrightarrows}} B$ through this coequaliser.

But
$$\begin{tikzcd} GFGFGB \arrow[r, shift left] \arrow[r, shift right] & GFGB \arrow[r] \arrow[l, bend left, "\eta_{GFGB}"] & GB \arrow[l, bend left, "\eta_{GB}"] \end{tikzcd}$$
is a split coequaliser diagram.

So either set of hypotheses ensures that $LKB \to B$ is an isomorphism.

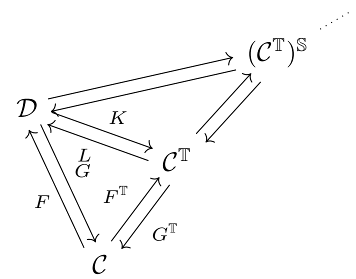
$KL(A, \alpha) = K(\mathrm{coeq}(FGFA \underset{\epsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA))$. Either hypothesis implies that $G = G^{\mathbb{T}}K$ preserves this coequaliser, but

$$\begin{tikzcd} GFGFGA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\epsilon_{FA}"'] & GFA \arrow[r, "\alpha"] \arrow[l, bend left, "\eta_{GFA}"] & A \arrow[l, bend left, "\eta_A"] \end{tikzcd}$$

is a split coequaliser, so $GL(A, \alpha) \cong A$.

The unit $(A, \alpha) \to KL(A, \alpha)$ is mapped to this isomorphism by $G^{\mathbb{T}}$, so it's an isomorphism in $\mathcal{C}^{\mathbb{T}}$. $\square$

Remark 5.13. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction, and suppose $\mathcal{D}$ has reflexive coequalisers. The **monadic tower** of $(F \dashv G)$ is the diagram



where $\mathbb{T}$ is the monad induced by $(F \dashv T)$, K is the Eilenberg-Moore comparison functor, $(L \dashv K)$ (5.9), $\mathbb{S}$ is the monad induced by $(L \dashv K)$, etc.

We say $(F \dashv G)$ has **monadic length** n if we reach an equivalence after n steps. **Top** $\to$ **Set** has monadic length ∞.

6 Regular Categories

Definition 6.1. The **image** of a morphism $A \xrightarrow{f}$ is the smallest subobject of B through which f factors, if this exists.

We say $\mathcal{C}$ **has images** if every $f \in \operatorname{mor} \mathcal{C}$ has an image.

$A \xrightarrow{f} B$ is a **cover** if its image is 1_B, i.e. it doesn't factor through any proper subobject of B. We write $A \overset{f}{\twoheadrightarrow} B$ to indicate that f is a cover.

Lemma 6.2. *If $\mathcal{C}$ has finite limits, then covers in $\mathcal{C}$ coincide with strong epimorphisms.*

Proof. Recall that f is strong epic if and only if given $(*)$
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, tail, "k"] \\ B \arrow[r, "h"'] \arrow[ur, dashed, "l"] & D \end{tikzcd}$$
with k monic, $\exists l : B \to C$ with $kl = h$ and $lf = g$.

Being a cover is the special case of this condition with $h = 1_B$, so strong epimorphisms are covers.

Conversely, if f is a cover then it's epic, since if $gf = hf$ then f factors through the equaliser of g and h, so this must be an isomorphism. Given $(*)$, we can form the pullback
$$\begin{tikzcd} P \arrow[r, "n"] \arrow[d, "m"'] & C \arrow[d, tail, "k"] \\ B \arrow[r, "h"'] & D \end{tikzcd}$$
, and m is monic by 4.13.

f factors through m, so m is an isomorphism and $B \xrightarrow{nm^{-1}} C$ is the diagonal fill in for $(*)$. ☐

It follows that image factorisation is functorial: given

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] \arrow[ddd, "g"'] & & B \arrow[ddd, "h"] \\ & I \arrow[ur, tail] \arrow[d, dashed] & \\ & I' \arrow[dr, tail] & \\ A' \arrow[ur, two heads] \arrow[rr, "f'"'] & & B' \end{tikzcd}$$

we get a unique $I \to I'$ making everything commute. So image factorisation defines a functor $[\mathbf{2}, \mathcal{C}] \to [\mathbf{3}, \mathcal{C}]$.

Definition 6.3. A **regular category** is a category which preserves finite limits and images, in which strong epimorphisms are stable under pullback.

A **regular functor** is one which preserves finite limits and strong epimorphisms.

Theorem 6.4. *In a regular category, the strong epimorphisms coincide with regular epimorphisms.*

Proof. Regular $\implies$ strong is true in general (see sheet 1). Suppose $A \overset{f}{\twoheadrightarrow} B$ is strong epic. Let $R \underset{b}{\overset{a}{\rightrightarrows}}$ be the **kernel-pair** of f, i.e. the pullback of f against itself. Certainly $fa = fb$.

Suppose $A \overset{g}{\to} C$ with $ga = gb$. Form the image

$$A \Longrightarrow I \overset{(k,l)}{\rightarrowtail} B \times C$$

of (f, g).

Claim k is an isomorphism: given this, the composite $B \overset{k^{-1}}{\to} I \overset{l}{\to} C$ satisfies $lk^{-1}f = lk^{-1}kh = lh = g$, and it's unique since f is epic.

We know k is strong epic, since $kh = f$ is strong epic. So we need to show it's monic.

Suppose $D \underset{y}{\overset{x}{\rightrightarrows}}$ satisfy $kx = ky$. Form the pullback

$$\begin{tikzcd}
F \arrow[rr, "m"] \arrow[d, "{(n,p)}"] & & D \arrow[d, "{(x,y)}"] \\
A \times A \arrow[r, Rightarrow, "1\times h"] & A \times I \arrow[r, Rightarrow, "h\times 1"] & I \times I
\end{tikzcd}$$

then $h \times 1$ and $1 \times h$ are strong epimorphisms, and hence so is m.

Now $fn = khn = kxm = kym = khp = fp$, so (n, p) factors through (a, b), say by $E \overset{q}{\to} R$. Since $kha = khb$ and $lha = lhb$, and (k, l) is monic, we have $ha = hb$. So $xm = hn = haq = hbq = hp = ym$. But m is epic, so $x = y$. $\square$

Remark 6.5. In many textbooks, regular categories are defined as categories with (some) finite limits, in which every morphism factors as a regular epimorphism and a monomorphism, and regular epimorphims are stable under pullback.

Note that if $A \overset{f}{\to} B$ has kernel pair $R \underset{b}{\overset{a}{\rightrightarrows}} A$ and image factorisation $A \overset{g}{\twoheadrightarrow} I \rightarrowtail B$, then (a, b) is also the kernel-pair of g since h is monic. So g may be obtained as the coequaliser of the kernel-pair of f, and f as the factorisation of f through this.

Definition 6.6. Let $R \underset{b}{\overset{a}{\rightrightarrows}} A$ be a parallel pair in a category with finite limits.

a. (a, b) is a **relation** if $R \overset{(a,b)}{\longrightarrow} A \times A$ is monic.

b. (a, b) is **reflexive** if there exists $A \overset{r}{\to} R$ with $ar = br = 1_A$.

c. (a, b) is **symmetric** if there exists $R \overset{s}{\to} R$ with $as = b$, $bs = a$.

d. (a, b) is **transitive** if, given the pullback $\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "p"] & R \arrow[d, "a"] \\ R \arrow[r, "b"] & A \end{tikzcd}$, there exists $t : T \to R$ such that $at = ap$ and $bt = bq$.

e. (a, b) is an **equivalence relation** if all of (a-d) hold.

The kernel-pair of any $A \xrightarrow{f} B$ is an equivalence relation.

(a, b) is an **effective** equivalence relation if it occurs as a kernel pair, $\mathcal{C}$ is an **effective** regular category if all equivalence relations in $\mathcal{C}$ are effective (aka 'Barr-exact').

Definition 6.7. The **support** σA of an object A in a regular category is the image of $A \to 1$. A is **well-supported** if $\sigma A \cong 1$, i.e. if $A \twoheadrightarrow 1$ is strong epic.

$\mathcal{C}$ is **totally supported** if all its objects are well-supported. (E.g. **Gp** and **ApGp** are totally supported, since any $A \to 1$ is split epic).

An object 0 in a regular category is **strict** if any morphism $A \to 0$ is an isomorphism. (This implies that 0 is initial: $O \times A \xrightarrow{\pi_1} 0$ is an isomorphism, so $0 \xrightarrow{\pi_1^{-1}} 0 \times A \xrightarrow{\pi_2} A$ exists for any A, but given $O \underset{g}{\overset{f}{\rightrightarrows}} A$, the equaliser must be an isomorphism.)

$\mathcal{C}$ is **almost totally supported** if every object of $\mathcal{C}$ is either well-supported or strict (e.g. **Set**).

Theorem 6.8 (Barr's Embedding Theorem)**.** *Let $\mathcal{C}$ be a small regular category. Then there exists a small category $\mathcal{D}$ and a full and faithful regular functor $\mathcal{C} \to [\mathcal{D}, \textbf{Set}]$. Moreover, if $\mathcal{C}$ is almost totally supported then $\mathcal{D}$ can be taken to be a monoid.*

We'll prove the most important part of this: $\mathcal{C}$ has an isomorphism reflecting regular functor to be a power of **Set**. We follow a proof due to F. Borceaux:

Theorem 6.9. *Let $\mathcal{C}$ be a small a.t.s. regular category. Then there exists an isomorphism reflecting regular functor $F : \mathcal{C} \to \textbf{Set}$.*

Proof. We construct F as the colimit in $[\mathcal{C}, \textbf{Set}]$ of a diagram of representable functors: explicitly, J will be a meet-semilattice and $D : J \to \mathcal{C}$ a diagram such that each $D(j)$ is well-supported and each $D(j' \to j)$ is a strong epimorphism $D(j') \twoheadrightarrow D(j)$.

Then $F = colim(J^op \xrightarrow{D} \mathcal{C}^op \xrightarrow{Y} [\mathcal{C}, \textbf{Set}])$.

Explicitly, elements of FA are represented by morphisms $D(j) \xrightarrow{f} A$ for some j, where $f \sim f'$ if and only if

$$\begin{tikzcd}
 & D(j) \arrow[dr, "f"] & \\
D(j \wedge j') \arrow[ur] \arrow[dr] & & A \\
 & D(j') \arrow[ur, "f'"] &
\end{tikzcd}$$

commutes.

F preserves finite products: $F1 = \{*\}$, and if $D(j) \xrightarrow{f} A$, $D(j') \xrightarrow{g} B$ represent elts of FA and FB, $D(j \wedge j') \to D(j) \xrightarrow{f} A$ and $D(j \wedge j') \to D(j) \xrightarrow{g} B$ induce an element of $F(A \times B)$, mapping to the given element of $FA \times FB$.

Hence $F(A \times B) \to FA \times FB$ is surjective, and it's easily seen to be injective.

F preserves equalisers: notes that if 0 exists in $\mathcal{C}$, then $F0 = \emptyset$. If $E \overset{e}{\rightarrowtail} A \underset{g}{\overset{f}{\rightrightarrows}} B$ is an equaliser diagram in $\mathcal{C}$ and E is well-supported, then the equaliser of $FA \rightrightarrows FB$ consists of morphisms $D(j) \to A$ having equal composites with f and g (and hence factoring through E). And if $E = 0$ then the equaliser of $FA \rightrightarrows FB$ is $\emptyset$.

Now assume that, for every strong epimorphism $A \xrightarrow{f} D(j)$ in $\mathcal{C}$, there exists $j' \leq j$ such that $D(j' \to j) = f$.

Then F preserves strong epimorphisms: given $A \overset{f}{\twoheadrightarrow} B$ and a morphism $D(j) \xrightarrow{g}$ representing an element of FB, form the pullback

$$\begin{tikzcd}
D(j') \arrow[r, "h"] \arrow[d] & A \arrow[d, Rightarrow, "f"] \\
D(j) \arrow[r, "g"] & B
\end{tikzcd}$$

then h represents an element of FA whose image under Ff is g. So Ff is surjective. $\square$