Part III Combinatorics

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Contents

1 Introduction

Let $X, Y, \dots$ be sets

Definition. We call $\mathcal{A} \subset \mathcal{P}(X)$ a **set system** or **family of sets**. $\mathcal{A}$ is naturally identified with a bipartite graph $G_{\mathcal{A}}(U, W)$ with $U = \mathcal{A}$, $W = \bigcup_{A \in \mathcal{A}} A$ or $W = X$. Indeed, $Ax \in E(G_{\mathcal{A}}) \iff x \in A$.

Definition. Given $\mathcal{A} \in \mathcal{P}(X)$, a **set of distinct representatives** (SDR) is an injection $f : \mathcal{A} \to X$ s.t. $f(A) \in A \ \forall A \in \mathcal{A}$. In its bipartite graph, an SDR corresponds to a complete matching $U \to W$.

Theorem 1 (Hall, 1935). *A set system $\mathcal{A}$ has an SDR if $\forall \mathcal{A}' \subset \mathcal{A}$, $|\cup_{A \in \mathcal{A}'} A| \geq |\mathcal{A}|'$.*

Theorem 1'. *A bipartite graph $G(U, W)$ has a complete matching $U \to W$ if $\forall S \subset U$, $|\Gamma(S)| \geq |S|$*

Corollary 2. *Suppose $G(U, W)$ bipartite, $d(u) \geq d(w) \ \forall u \in U, \ w \in W$. Then $\exists$ a complete matching $U \to W$.*

Definition. A bipartite graph $G(U, W)$ is (r, s)**-regular** if $d(u) = r$ and $d(w) = s \ \forall u \in U$, $w \in W$.

Instant from Cor 2: if $G(U, W)$ is (r, s)-regular then $\exists$ a complete matching from U to W if $|U| \leq |W|$.

Corollary 3. *Let $0 \leq i, j \leq n$, $\binom{n}{i} \leq \binom{n}{j}$. Then $\exists$ a complete matching $f : [n]^{(i)} \to [n]^{(j)}$ s.t. $f(A) \subset A$ if $j \leq i$, and $f(A) \supset A$ if $i \leq j$.*

Theorem 4. *Let $G = G(U, W)$ be a connected (r, s)-regular graph. Then for $\emptyset \neq A \subset U$,*

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

Also, equality holds iff $A = U$.

The **cube** $Q^n \cong \mathcal{P}(n) \cong [2]^n =$ set of all 0, 1 sequences of length n. Q^n is also a graph: AB is an edge if $|A \triangle B| = 1$. It is also a poset: $A < B$ if $A \subset B$.

Q^n has a natural orientation: $\overrightarrow{AB}$ if $A = B \cup \{a\}$.

$$\begin{tikzcd}
& 123 \arrow[dl] \arrow[d] \arrow[dr] & \\
12 \arrow[d] \arrow[dr] & 13 \arrow[dl] \arrow[dr] & 23 \arrow[dl] \arrow[d] \\
1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\
& \emptyset &
\end{tikzcd}$$

The order on $Q^n \cong \mathcal{P}(n)$ is induced by this oriented graph.

2 Sperner Systems

Definition. A set system $\mathcal{A} \subset \mathcal{P}(n)$ is **Sperner** if $A, B \in \mathcal{A}$, $A \neq B \implies A \not\subset B$

Theorem 1 (Sperner, 1928)**.** *If $\mathcal{A} \subset \mathcal{P}(n)$ is Sperner then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Definition. The **weight** $w(A)$ of a set $A \in \mathcal{P}(n)$ is $w(A) = \frac{1}{\binom{n}{|A|}}$

Theorem 2. *Let $\mathcal{A}$ be a Sperner system on X, $|X| = n$. Then*

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A) \leq 1$$

Corollary 3. *If $\mathcal{A} \in \mathcal{P}(n)$ is a Sperner system then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, with equality $\iff$ $\mathcal{A}$ is $X^{\lfloor n/2 \rfloor}$ or $X^{\lceil n/2 \rceil}$.*

Definition. $\mathcal{A} \in \mathcal{P}(n)$ is **k-Sperner** if it does not contain

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{k+1}$$

Note that Sperner = 1-Sperner.

Corollary 4 (Erdős, 1945)**.** *If $\mathcal{A} \subset \mathcal{P}(n)$ is k-Sperner then $|\mathcal{A}|$ is at most the sum of the k largest binomial coefficients.*

Theorem 5 (Erdős, 1945)**.** *Let $x_1, \ldots, x_n \in \mathbb{R}$, $x_i \geq 1$. Then the number of sums $\sum_1^n \pm x_i$ in an open interval J of length $2k$ is at most the sum of the k largest binomial coefficients.*

Definition. A chain $A_o \subset A_1 \subset \cdots \subset A_k$ is **symmetric** if $|A_{i+1}| = |A_i| + 1 \ \forall i$ and $|A_o| + |A_k| = n$.

Theorem 6 (Kleitman and Katona)**.** *$\mathcal{P}(n)$ has a decomposition into symmetric chains.*

Take such a partition $\mathcal{P}(n) = \bigcup_{i=1}^k \mathcal{C}_i$, $j = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. There is one chain of length $n+1$, $n-1$ chains of length $n-1$, etc: there are $\binom{n}{i} - \binom{n}{i-1}$ chains of length $n+1-2i$.

Let E be a normed space, let $x_1, \ldots, x_n \in E$, $||x_i|| \geq 1 \ \forall i$, for $A \in \mathcal{P}(n)$ let $x_A = \sum i \in A x_i$.

Conjecture (Erdős, 1945)**.** *If* $\mathcal{A} \in \mathcal{P}(n)$ *s.t.* $||x_A - x_B|| < 1$ *then* $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Definition. Call $\mathcal{D} \in \mathcal{P}(n)$ **scattered** if $||x_A - x_B|| \geq 1$ $\forall A, B \in \mathcal{D}$. Call a partition $\mathcal{P}(n) = \bigcup_{i=1}^{s} \mathcal{D}_i$ **symmetric** if there are precisely $\binom{n}{i} - \binom{n}{i-1}$ sets $\mathcal{D}_i$ of cardinality $n + 1 - 2i$.

Theorem 7. *(Kleitman, 1970)* $E, (x_i)_1^n$ *as before. Then* $\mathcal{P}(n)$ *has a symmetric partition into scattered sets.*

Theorem 8. *(Kleitman, 1970) If* $\mathcal{A} \in \mathcal{P}(n)$ *s.t.* $||x_A - x_B|| < 1$ *then* $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

3 The Kruskal-Katona Theorem

We know: if $\mathcal{A} \subset X^{(r)}$ then $\partial\mathcal{A}$ (the **lower shadow** of $\mathcal{A}$), defined by

$$\partial\mathcal{A} = \{B \in X^{(r-1)} \mid B \subset A \text{ for some } A \in \mathcal{A}\}$$

satisfies

$$\begin{aligned} |\partial\mathcal{A}| &\geq |\mathcal{A}| \frac{\binom{n}{r-1}}{\binom{n}{r}} \\ &= |\mathcal{A}| \frac{r}{n-r+1} \end{aligned}$$

with equality $\iff$ $\mathcal{A}$ is $\emptyset$ or $X^{(r)}$.

What about in between? What is $\mathcal{B} \in X^{(r)}$ s.t. $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$?

$\exists \mathcal{B}_1, \mathcal{B}_2, \dots \in X^{(r)}$ s.t. $|\mathcal{B}_m| = m$ and $|\partial\mathcal{B}_m| \leq |\partial\mathcal{A}|$ $\forall \mathcal{A} \subset X^{(r)}$ where $|\mathcal{A}| = m$.

Incredibly luckily, we have a sequence of nested extremal sets. Equivalently, $\exists$ total order on $X^{(r)}$ s.t. the first m sets form $\mathcal{B}_m$.

Definition. Define the **colex** total order on $X^{(r)}$ by $A < B$ if $\max(A \Delta B) \in B$.

Aim: given m and r, would like to find $\mathcal{B} \subset X^{(r)}$, $|\mathcal{B}| = m$ s.t. $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$ $\forall \mathcal{A} \subset X^{(r)}$, $|\mathcal{A}| = m$.

Define $\mathcal{B}^{(r)}(m_r, \dots, m_s)$, $m_r > m_{r-1} > \dots > m_s \geq s$ as follows:

$$\begin{aligned} \mathcal{B}^{(r)} = [m_r]^{(r)} &\cup ([m_{r-1}]^{(r-1)} + \{m_r + 1\}) \\ &\cup ([m_{r-2}]^{(r-2)} + \{m_{r-1} + 1, m_r + 1\}) \\ &\cup \dots \\ &\cup ([m_s]^{(s)} + \{m_{s+1} + 1, m_{s+2} + 1, \dots, m_r + 1\}) \end{aligned}$$

Set $b^{(r)}(m_r, \dots, m_s) = \left|\mathcal{B}^{(r)}(m_r, \dots, m_s)\right| = \sum_{j=s}^{r} \binom{m_j}{j}$.

$$\partial\mathcal{B}^{(r)}(m_r, \dots, m_s) = \mathcal{B}^{(r-1)}(m_r, \dots, m_s)$$

This has cardinality $b^{(r-1)}(m_r, \dots, m_s) = \sum_{j=s}^{r} \binom{m_j}{j-1}$.

Lemma 1. *For $l, r \in \mathbb{N}$ $\exists!$ $m_r > \dots > m_s$ s.t. $l = \sum_{j=s}^{r} \binom{m_j}{j}$; the initial segment of $X^{(r)}$ in colex, consisting of l sets, is $\mathcal{B}^{(r)}(m_r, \dots, m_s)$.*

Definition. Let $i \neq j \in X$, $A \in \mathcal{P}(X)$. Define the **ij-compression**

$$A_{ij} = C_{ij}(A) = \begin{cases} (A \backslash \{j\}) \cup \{i\} & \text{if } j \in A,\, i \notin A \\ A & \text{otherwise} \end{cases}$$

Given $\mathcal{A} \subset \mathcal{P}(n), A \in \mathcal{A}$

$$C_{i,j,\mathcal{A}}(A) = \begin{cases} A_{ij} & \text{if } A_{ij} \notin \mathcal{A} \\ A & \text{otherwise} \end{cases}$$

Also,

$$\begin{aligned} C_{ij}(\mathcal{A}) &= \{C_{i,j,\mathcal{A}} \,|\, A \in \mathcal{A}\} \\ &= \{C_{ij}(A) \,|\, A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \,|\, C_{ij}(A) \in \mathcal{A}\} \end{aligned}$$

For $\mathcal{A} \in X^{(r)}$,

$$\begin{aligned} \mathcal{A}_{ij} &= \{A \in \mathcal{A} \,|\, \{i,j\} \subset A\} \\ \mathcal{A}_{i} &= \{A \in \mathcal{A} \,|\, i \in A,\, j \notin A\} \\ \mathcal{A}_{\emptyset} &= \{A \in \mathcal{A} \,|\, A \cap \{i,j\} = \emptyset\} \\ \mathcal{A}_{j} &= \{A \in \mathcal{A} \,|\, i \notin A,\, j \in A\} \end{aligned}$$

$C_{ij} : \mathcal{A} \mapsto C_{ij}(\mathcal{A})$ keeps $\mathcal{A}_\emptyset \cup \mathcal{A}_i \cup \mathcal{A}_{ij}$ fixed, and maps $\mathcal{A}_j$ into sets like those in $\mathcal{A}_i$.

Lemma 2. *For $\mathcal{A} \subset X^{(r)}$, $\partial C_{ij}(\mathcal{A}) \subseteq C_{ij}(\partial\mathcal{A})$. In particular, the cardinality decreases.*

Proof. Let $B \in \partial C_{ij}(\mathcal{A})$ and let $A \in \mathcal{A}$ s.t. $B \subset C_{i,j,\mathcal{A}}(A)$.

i. Suppose B meets $\{i,j\}$ in 0 or 2 elements. Then $B \subset A$ so $B \in \partial A$ and $B \in C_{ij}(\partial\mathcal{A})$

ii. Suppose $i \in B$, $j \notin B$. Then either B or $(B \backslash \{i\}) \cup \{j\}$ belongs to $\partial\mathcal{A}$, so $B \in C_{ij}(\partial\mathcal{A})$.

iii. Suppose $j \in B$, $i \notin B$. Then both B and $(B \backslash \{j\}) \cup \{i\}$ belong to $\partial\mathcal{A}$, so both belong to $C_{ij}(\partial\mathcal{A})$.

$\square$

Definition. Call $\mathcal{A} \subset X^{(r)}$ **left-compressed** if $C_{ij}(\mathcal{A}) = \mathcal{A}\ \forall i < j$.

Lemma 3. *Let $\mathcal{A} \subset X^{(r)}$. Then $\exists$ a left-compressed family $\mathcal{B} \subset X^r$ s.t. $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$.*

Proof. Define $\mathcal{A}_0 = \mathcal{A}, \mathcal{A}_1, \dots$ as follows: having reached $\mathcal{A}_k$, if $\mathcal{A}_k$ is not left-compressed, pick $i < j$ s.t. $C_{ij}(\mathcal{A}_k) \neq \mathcal{A}_k$, and set $\mathcal{A}_{k+1} = C_{ij}(\mathcal{A}_k)$

This sequence has to end because

$$\sum_{A \in \mathcal{A}_{k+1}} \sum_{a \in A} a < \sum_{A \in \mathcal{A}_k} \sum_{a \in A} a$$

let $\mathcal{A}_l$ be the last term: this will do for $\mathcal{B}$. $\square$

Theorem 4 (Kruskal-Katona, 1963 and 1968)**.** *Let $\mathcal{A} \subset X^{(r)}$, $m = |\mathcal{A}|$. Then*

$$\begin{aligned} |\partial\mathcal{A}| &\geq \left|\partial\mathcal{B}_m^{(r)}\right| \\ &= \left|\partial\mathcal{B}^{(r)}(m_r, m_{r-1}, \dots, m_s)\right| \\ &= b^{(r-1)}(m_r, \dots, m_s) \end{aligned}$$

Proof. Induction on r and then m (or on $r+m$). $r = 1$✓ $m = 1$✓

Induction step: we may assume that $\mathcal{A}$ is left-compressed. Set $Y = X \backslash \{1\}$. Then $\mathcal{A} = (\mathcal{A}_1 + \{1\}) \cup \mathcal{A}_0$, where $\mathcal{A}_1 \subset Y^{(r-1)}$, $\mathcal{A}_0 \subset Y^{(r)}$.

$m = |\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1|$, $\partial\mathcal{A}_0 \subset \mathcal{A}_1$, $\partial(\mathcal{A}_1 + \{1\}) = \mathcal{A}_1 \cup (\partial\mathcal{A}_1 + \{1\})$.

In particular, $|\partial\mathcal{A}| = |\mathcal{A}_1| + |\partial\mathcal{A}_1|$.

For $\mathcal{A} = \mathcal{B}^{(r)}(m_r, \dots, m_s)$,

$$|\mathcal{A}_1| = b^{(r-1)}(m_r - 1, \dots, m_s - 1)$$

$$|\mathcal{A}_0| = b^{(r)}(m_r - 1, \dots, m_s - 1)$$

Suppose $|\mathcal{A}_0| > b^{(r)}(m_r - 1, \dots, m_s - 1)$. Then by the induction hypothesis, $|\partial\mathcal{A}_0| \geq b^{(r-1)}(m_r - 1, \dots, m_s - 1)$. Hence $|\mathcal{A}_1| \geq b^{(r-1)}(m_r - 1, \dots, m_s - 1)$ and so $|\partial\mathcal{A}| \geq b^{(r-1)}(m_r, \dots, m_s)$.

But if $|\mathcal{A}_0| \leq b^{(r)}(m_r - 1, \dots, m_s - 1)$, $|\mathcal{A}_1|$ is again $\geq b^{(r-1)}(m_r - 1, \dots, m_s - 1)$. Done as before. $\square$

Soft version:

Theorem 5 (Lovász, 1979)**.** *If $\mathcal{A} \subset X^{(r)}$ satisfies $|\mathcal{A}| = \binom{X}{r}$ then $|\partial\mathcal{A}| \geq \binom{X}{r-1}$.*

Proof. Induction on r and $m = |\mathcal{A}|$. As before, $\mathcal{A}_0, \mathcal{A}_1$. Note that $\mathcal{A}_1 \geq \binom{X-1}{r-1}$ since otherwise $\mathcal{A}_0 > \binom{X-1}{r}$. But then $|\partial \mathcal{A}_0| \geq \binom{X-1}{r-1}$, contradicting the fact that $\partial \mathcal{A}_0 \subset \mathcal{A}_1$.

But if $|\mathcal{A}_1| \geq \binom{X-1}{r-1}$ then

$$|\mathcal{A}_1| + |\partial \mathcal{A}_1| \geq \binom{X-1}{r-1} + \binom{X-1}{r-2} = \binom{X}{r-1}$$

$\square$

Definition. Define the **uniform probability measure** on $X^{(r)}$, $|X| = n$ as $\mathbb{P}_{n,r}(A) = \frac{1}{\binom{n}{r}}$, and for $\mathcal{A} \subset X^{(r)}$, $\mathbb{P}_{n,r}(\mathcal{A}) = \frac{|\mathcal{A}|}{\binom{n}{r}}$.

Definition. $\mathcal{A} \subset \mathcal{P}(n)$ is **monotone decreasing** if $A \subset B \in \mathcal{A} \implies A \in \mathcal{A}$.

Theorem 6. *If $1 \leq s < r \leq n$, $\mathcal{A} \subset \mathcal{P}(n)$ decreasing, then $\mathbb{P}_s(\mathcal{A})^r \geq \mathbb{P}_r(\mathcal{A})^s$.*

[$\mathbb{P}_k(\mathcal{A}) = \mathbb{P}_k(\mathcal{A}_k)$, $\mathcal{A}_k = \mathcal{A} \cap X^{(k)}$]

Proof. $\mathbb{P}_k(\mathcal{A}) = \frac{|\mathcal{A}_k|}{\binom{n}{k}}$, if $|\mathcal{A}_r| = \binom{X}{r}$ then we know $|\mathcal{A}_s| \geq \binom{X}{s}$. Hence, the inequality holds if

$$\prod_{i=0}^{s-1} \left(\frac{X-i}{n-i}\right)^r \geq \prod_{i=0}^{r-1} \left(\frac{X-i}{n-i}\right)^s$$

since $\frac{\binom{X}{r}}{\binom{n}{r}} = \prod_{i=0}^{r-1} \frac{X-i}{n-i}$.

But this is

$$\prod_{i=0}^{s-1} \left(\frac{X-i}{n-i}\right)^{r-s} \geq \prod_{i=s}^{r-1} \left(\frac{X-i}{n-i}\right)^s$$

Every factor on the left is larger than every factor on the right:

$$\frac{X-i}{n-i} > \frac{X-j}{n-j}$$

for $i \leq s-1$, $j \geq s$. $\square$

Definition (Erdős and Rényi, 1960)**.** Given an increasing family ('property of sets') $\mathcal{A}(n) \subset \mathcal{P}(n)$, a function $k^*(n)$ is a **threshold function** for $\mathcal{A}(n)$ if $\mathbb{P}_{k(n)}(\mathcal{A}(n)) \to 0$ if $\frac{k}{k^*} \to 0$, and $\mathbb{P}_{k(n)}(\mathcal{A}(n)) \to 1$ if $\frac{k}{k^*} \to 1$.

Erdős and Rényi: for many monotone increasing graph properties, $\exists$ a threshold.

Corollary 7. *Let $\mathcal{A} \subset \mathcal{P}(n)$, $k_1 < k < k_2$*

i. If $\mathcal{A}$ is decreasing, $\mathbb{P}_{k_2}(\mathcal{A})^{k/k_2} \leq \mathcal{P}_k(\mathcal{A}) \leq \mathcal{P}_{k_1}(\mathcal{A})^{k/k_1}$

ii. If $\mathcal{A}$ is increasing, $(1 - \mathbb{P}_{k_2}(\mathcal{A}))^{k/k_2} \leq 1 - \mathcal{P}_k(\mathcal{A}) \leq (1 - \mathcal{P}_{k_1}(\mathcal{A}))^{k/k_1}$

Proof. i. This is precisely Theorem 6

ii. Set $\mathcal{A}^c = \mathcal{P}(n)\backslash\mathcal{A}$. Then $\mathcal{A}^c$ is decreasing and

$$\mathbb{P}_k(\mathcal{A}^c) = 1 - \mathbb{P}_k(\mathcal{A})$$

Apply (i) to $\mathcal{A}^c$.

$\square$

Theorem 8. *Every monotone increasing function has a threshold.*

Proof. We may assume $\mathcal{A}$ is non-trivial. Set $k^*(n) = \max\left\{k \,|\, \mathbb{P}_k(\mathcal{A}) \leq \frac{1}{2}\right\}$.

Then, for $k < k^*$,

$$\mathbb{P}_k(\mathcal{A}) \leq 1 - (1 - \mathbb{P}_{k*}(\mathcal{A}))^{k/k^*} \leq 1 - 2^{-k/k^*}$$

For $k > k^* + 1$,

$$\mathbb{P}_k(\mathcal{A}) \geq 1 - (1 - \mathbb{P}_{k*}(\mathcal{A}))^{k/(k^*+1)} \geq 1 - 2^{-k/(k^*+1)}$$

$\square$

This is essentially best possible, but only for lop-sided systems $\mathcal{A}$.

Definition. $\mathcal{A} \subset \mathcal{P}(n)$ is **symmetric** if $\forall x, y, \in X$ $\exists$ a permutation π of X mapping x onto y, keeping $\mathcal{A}$ invariant.

Definition. Another measure on $\mathcal{P}(n)$: the **binomial measure**. Let $0 < p < 1$.

$$\mathbb{P}_{n,p}(A) = \mathbb{P}_p(A) = p^{|A|}(1-p)^{n-|A|}$$

$\mathbb{P}_{n,p}$ is very similar to $\mathbb{P}_{n,k}$ for $k \sim pn$.

Theorem 9 (Friedgut and Kaloi, 1996)**.** *There is an absolute constant* $c_0 > 0$ *s.t. if* $\mathcal{A} \subset \mathcal{P}(n)$ *is a symmetric increasing family and* $\mathbb{P}_p(\mathcal{A}) > \epsilon > 0$ *then* $\mathbb{P}_{p'}(\mathcal{A}) > 1 - \epsilon$ *provided* $p' \geq p + c_0 \frac{\log 1/\epsilon}{\log n}$

4 Intersecting Families

Definition. $\mathcal{A} \subset \mathcal{P}(n)$ is **intersecting** if $A \cap B \neq \emptyset$ $\forall A, B \in \mathcal{A}$.

Suppose $\mathcal{A} \subset X^{(r)}$. If $r > \frac{n}{2}$, $\mathcal{A}$ is intersecting. If $r = \frac{n}{2}$, we can take families of size $\frac{1}{2}\binom{n}{r}$. $r < \frac{n}{2}$?

Let

$$X_x^{(r)} = \{A \in X^{(r)} \,|\, x \in A\}$$

for any $x \in X$.

Theorem 1 (Erdős, Ko and Rado 1961)**.** *Let $n > 2r \geq 4$ and let $\mathcal{A} \subset X^{(r)}$ be an intersecting family. Then $|\mathcal{A}| \leq \binom{n-1}{r-1}$ with equality $\iff$ $\mathcal{A} = X_x^{(r)}$.*

Proof. We may assume $|\mathcal{A}| \geq \binom{n-1}{r-1}$. Take $\mathcal{B} = \{X \backslash A \,|\, A \in \mathcal{A}\} \subset X^{(n-r)}$. For $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \not\subset B$.

Let $\mathcal{C} = \partial \ldots \partial \mathcal{B}$ (shadow $n - r$ times). Then $\mathcal{C} \subset X^{(r)}$ and $\mathcal{C} \cap \mathcal{A} = \emptyset$, $\therefore |\mathcal{A}| + |\mathcal{C}| \leq \binom{n}{r}$.

By Kruskal-Katona, since $|B| \geq \binom{n-1}{r-1} = \binom{n-1}{n-r}$, have $|\mathcal{C}| \geq \binom{n-1}{r}$.

Hence $|\mathcal{A}| \leq \binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1}$. □

Definition. We call $\mathcal{A}$ **l-intersecting** if $|A \cap B| \geq l$ $\forall A, B \in \mathcal{A}$.

Let
$$\mathcal{F}_0 = \{A \in X^{(r)} \,|\, A \supset [l]\}$$

Lemma 2. *Let $2 \leq l < r$ and $n \geq \frac{4}{3} l r^3$. Let $\mathcal{A} \subset X^{(r)}$ be l-intersecting, **not** fixed by an l-set (i.e. $\mathcal{A} \not\subset \mathcal{F}' \cong \mathcal{F}_0$). Then*
$$|\mathcal{A}| \leq (r-l)\binom{n-l-1}{r-l-1} + \sum_{t=1}^{t_0} \binom{l}{t}\binom{r-l}{t}^2\binom{n-l-2t}{r-l-t}$$
where $t_0 = \min\{l, r-l\}$.

Proof. We may assume $\mathcal{A}$ is maximal l-intersecting. So $\exists A_1, A_2 \in \mathcal{A}$ s.t. $A_1 \cap A_2 = B$, $|B| = l$.

Let $\mathcal{A}_t = \{A \in \mathcal{A} \,|\, |B \backslash A| = t\}$.

$|\mathcal{A}_0| \leq (r-l)\binom{n-l-1}{r-l-1}$

$|\mathcal{A}_t| \leq \binom{l}{t}\binom{r-l}{t}^2\binom{n-l-2t}{r-l-t}$ □

Theorem 3. *Suppose $2 \leq l < r < n$ and $n \geq \frac{3}{2} l r^3$. Let $\mathcal{A} \subset X^{(r)}$ be l-intersecting. Then $|\mathcal{A}| \leq \binom{n-l}{r-l}$ and equality holds only if*
$$\mathcal{A} \cong \{A \in X^{(r)} \,|\, A \supset L\}$$
for some $L \in X^{(l)}$.

Proof. Suppose $\mathcal{A}$ is not fixed by an l-set. Then by Lemma 2,
$$\begin{aligned} |A| &\leq (r-l)\binom{n-l-1}{r-l-1} + \sum_{t=1}^{t_0} \binom{l}{t}\binom{r-l}{t}^2\binom{n-l-t}{r-l-t} \\ &= (r-l)\binom{n-l-1}{r-l-1} + \sum_{t=1}^{t_0} S_t \end{aligned}$$

Note

$$\frac{S_{t+1}}{S_t} = \frac{l-t}{t+1}\frac{(r-l-t)^2}{(t+1)^2}\frac{r-l-t}{n-l-t}$$
$$\leq \frac{lr^3}{(t+1)^3 n} \leq \frac{2}{3(t+1)^3} \leq \frac{1}{12}$$

Thus

$$\frac{|\mathcal{A}|}{\binom{n-l}{r-l}} \leq (r-l)\frac{r-l}{n-l} + \frac{12}{11}l(r-l)^2\frac{r-l}{n-l}$$
$$= (1+\frac{12}{11}l(r-l))\frac{(r-l)^2}{n-l}$$
$$< \frac{3}{2}l\frac{r^3}{n} \leq 1$$

If $r = l+2$ then $<$. ◻

Suppose $\mathcal{P}(X) \supset \mathcal{A}$ is intersecting. $\mathcal{A} \leq 2^{n-1}$. Binomial probability measure:

$$\mathbb{P}_p(A) = p^{|A|}(1-p)^{n-|A|}$$
$$\mathbb{P}_p(\mathcal{A}) = \sum_{A \in \mathcal{A}} \mathbb{P}_p(A)$$

$\mathcal{A}$ intersecting $\implies \mathbb{P}_{\frac{1}{2}}(\mathcal{A}) \leq \frac{1}{2}$.

Theorem 4. *Let $0 < p \leq \frac{1}{2}$ and let $\mathcal{A} \subset \mathcal{P}(X)$ be intersecting. Then $\mathbb{P}_p(\mathcal{A}) \leq p$.*

Proof. Set $N_k = |\mathcal{A}_k|$. $A \in \mathcal{A} \implies A^c = X \backslash A \notin \mathcal{A}$.

Hence $N_k + N_{n-k} \leq \binom{n}{k}$. Also, for $k \leq \frac{n}{2}$, $p^k(1-p)^{n-k} \geq p^{n-k}(1-p)^k$, so

$$N_k p^k(1-p)^{n-k} + N_{n-k}p^{n-k}(1-p)^k \leq \binom{n-1}{k-1}p^k(1-p)^{n-k} + \left(\binom{n}{k} - \binom{n-1}{k-1}\right)p^{n-k}(1-p)^k$$
$$\leq \binom{n-1}{k-1}p^k(1-p)^{n-k} + \binom{n-1}{n-k-1}p^{n-k}(1-p)^k$$

Thus

$$\mathbb{P}_p(\mathcal{A}) = \sum_{k=1}^{n} p^k(1-p)^{n-k}$$
$$\leq p\sum k = 1^n \binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} = p$$

◻

Definition. $\mathcal{A} \subset \mathcal{P}(X)$ is **k-wise-intersecting** if $A_1 \cap \cdots \cap A_k \neq \emptyset \ \forall A_i \in \mathcal{A}$.

Theorem 5. *Let $ks \geq n$, let $\mathcal{A} \subset X^{(s)}$ be such that X is **not** the union of k sets from $\mathcal{A}$. Then $|\mathcal{A}| \leq \binom{n-1}{s}$.*

Proof. Apply Katona's circle method. Let Π be the set of all $(n-1)!$ cyclic orders on X. For $\pi \in \Pi$, let $\mathcal{A}_\pi = \{A \in \mathcal{A} \,|\, A \text{ is a } \pi\text{-arc}\}$.

Claim: $|\mathcal{A}_\pi| \leq n - s$.

Proof of claim: we may assume $X = \mathbb{Z}_n$ is given by π; we may assume one of the arcs in $\mathcal{A}_\pi$ ends in n. Associate with each arc its end point, except for the one ending in n, to which we associate all $ks - n + 1$ numbers in $[n, ks]$.

Thus, if $l = |\mathcal{A}_\pi|$, and L is the set of elements associated with our arcs, then $|L| = l + (ks - n)$.

For $1 \leq i \leq s$, let $K_i = \{i, i+s, i+2s, \ldots, i+(k-1)s\}$. Then $K_1, \ldots, K_s$ partition $[ks]$ into s sets of k elements each. Can $K_i \subset L$ happen? No, as then the corresponding k arcs would cover X.

Hence, $|L \cap K_i| \leq k - 1 \; \forall i$, so $l + ks - n = |L| \leq (k-1)s$, i.e. $l \leq n - s$. ✓

Double counting:

$$\begin{aligned} s!(n-s)!\,|\mathcal{A}| &= \sum_{A \in \mathcal{A}} |\{\pi \in \Pi : A \text{ is a } \pi\text{-arc}\}| \\ &= \sum_{\pi \in \Pi} |\mathcal{A}_\pi| \leq (n-1)!(n-s) \end{aligned}$$

□

Corollary 6 (Equivalent to Theorem 5)**.** *Let $2 \leq k, r < n$, $kr \leq (k-1)n$. Let $\mathcal{A} \subset X^{(r)}$ be k-wise intersecting. Then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

Proof. Note that $\mathcal{A}^c = \{X \backslash A \,|\, A \in \mathcal{A}\} \subset X^{(s)}$, $s = n - r$, satisfies the conditions of Theorem 5, so $|\mathcal{A}| = |\mathcal{A}^c| \leq \binom{n-1}{s} = \binom{n-1}{r-1}$. □

Theorem 7. *Let $2 \leq k, r < n$, $kr \leq (k-1)n$; let $\mathcal{A} \subset X^{(\leq r)}$ be a k-wise intersecting Sperner family. Then*

$$\sum_{j=1}^{n} |\mathcal{A}_j| \bigg/ \binom{n-1}{j-1} = \sum_{A \in \mathcal{A}} \binom{n-1}{|A|-1}^{-1} \leq 1$$

Proof. Set $l = \min\{j \,|\, \mathcal{A})j \neq \emptyset\}$, $m = \max\{j \,|\, \mathcal{A}_j \neq \emptyset\}$.

Induction on $m - l$: $m = l$ is exactly Corollary 6.

Induction step: $m - l \geq 1$. Let $\mathcal{A}_l^+$ be the upper shadow of $\mathcal{A}_l$ at level $l+1$. Then $\mathcal{A}' = (\mathcal{A} \backslash \mathcal{A}_l) \cup \mathcal{A}_l^+$ is again $k - wise$ intersecting Sperner, with a smaller difference $m - l$. Thus, we're done if

$$|\mathcal{A}_l^+| \bigg/ \binom{n-1}{l} \geq |\mathcal{A}_l| \bigg/ \binom{n-1}{l-1}$$

$\mathcal{A}_l^+$ is the cardinality of the lower shadow of $\mathcal{A}_l^c$. Set $|\mathcal{A}_l| = \binom{x}{n-l}$. Then, by the weak Kruskal-Katona theorem, $|\mathcal{A}_l^+| \geq \binom{x}{n-l-1}$. We know $\binom{x}{n-l} \geq \binom{n-1}{l-1} = \binom{n-1}{n-l}$, so $x \leq n - 1$.

Would like:

$$\binom{x}{n-l}\Big/\binom{n-1}{l-1} \le \binom{x}{n-l-1}\Big/\binom{n-1}{l}$$
$$\binom{x}{n-l}\Big/\binom{n-1}{n-l} \overset{?}{\le} \binom{x}{n-l-1}\Big/\binom{n-1}{n-l-1}$$
$$x-(n-l)+1 \overset{?}{\le} n-(n-l) = l$$
$$x \le n-1 \checkmark$$

□

5 Correlation Inequalities

Let $0 < p < 1$, $\mathcal{G}(n,p)$ the probability space of all $2^{\binom{n}{2}}$ graphs on $[n]$ such that $\mathbb{P}_p(G_{n,p} = H) = p^{e(H)}(1-p)^{\binom{n}{2}-e(H)}$.

This is really the weighted cube Q_p^n. $\mathbf{p} = (p_1, \dots, p_n)$, random subset of $X = [n]$: $\mathbb{P}_{\mathbf{p}}(A) = \prod_{i\in A} p_i \prod_{i\notin A}(1-p_i)$. For $\mathcal{G}(n,p)$, consider $Q_{\mathbf{p}}^{\binom{n}{2}}$.

Theorem 1. *Let $A, B \in Q_{\mathbf{p}}^n$. If both are up-sets or both are down-sets, then $\mathbb{P}_{\mathbf{p}}(A\cap B) \ge \mathbb{P}_{\mathbf{p}}(A)\mathbb{P}_{\mathbf{p}}(B)$. IF one is an up-set and the other is a down-set, then the inequality reverses.*

Proof. Induction on n. $n = 1$: ✓. Let $n \ge 1$.

Let $A_i = \{\mathbf{x} \in \{0,1\}^{n-1} \,|\, (x_1, \dots, x_{n-1}, i) \in A\}$, similary B_i.

Then $\mathbb{P}_{\mathbf{p}}(A) = (1-p_n)\mathbb{P}_{\mathbf{p}'}(A_0) + p_n\mathbb{P}_{\mathbf{p}'}(A_1)$ $\;(\mathbf{p}' = (p_1, p_2, \dots, p_{n-1}))$

Also $(*):\; (\mathbb{P}_{\mathbf{p}'}(A_1) - \mathbb{P}_{\mathbf{p}'}(A_0))(\mathbb{P}_{\mathbf{p}'}(B_1) - \mathbb{P}_{\mathbf{p}'}(B_0)) \ge 0 - (*)$ since both are up/down sets.

$$\begin{aligned}
\mathbb{P}_{\mathbf{p}}(A\cap B) &= (1-p_n)\mathbb{P}_{\mathbf{p}'}(A_0\cap B_0) + p_n\mathbb{P}_{\mathbf{p}'}(A_1\cap B_1)\\
&\ge (1-p_n)\mathbb{P}_{\mathbf{p}'}(A_0)\mathbb{P}_{\mathbf{p}'}(B_0) + p_n\mathbb{P}_{\mathbf{p}'}(A_1)\mathbb{P}_{\mathbf{p}'}(B_1) \text{ by induction}\\
&\overset{?}{\ge} ((1-p_n)\mathbb{P}(A_0) + p_n\mathbb{P}(A_1))((1-p_n)\mathbb{P}(B_0) + p_n\mathbb{P}(B_1))
\end{aligned}$$

This holds if $\mathbb{P}(A_0)\mathbb{P}(B_0) - \mathbb{P}(A_0)\mathbb{P}(B_1) - \mathbb{P}(A_1)\mathbb{P}(B_0) + \mathbb{P}(A_1)\mathbb{P}(B_1) \ge 0$, which is exactly $(*)$.

If A is an up-set, B a down-set then

$$\begin{aligned}
\mathbb{P}(A\cap B) &= \mathbb{P}(A) - \mathbb{P}(A\cap B^c)\\
&\le \mathbb{P}(A) - \mathbb{P}(B)(1-\mathbb{P}(B))\\
&= \mathbb{P}(A)\mathbb{P}(B)
\end{aligned}$$

□

Definition. Let $A, B \in Q^n = \{0,1\}^n$.

$$A \Box B = \{z \in Q^n \mid \exists \text{ disjoint } I, J \in [n] \text{ s.t. } x|I = z|I \implies x \in A, \\ y|J = z|J \implies y \in B\}$$

If A and B are increasing then

$$A \Box B = \{x + y \mid x \in A,\ y \in B\}$$

$$\mathcal{A} \Box \mathcal{B} = \{A \cup B \mid A \cap B = \emptyset,\ A \in \mathcal{A},\ B \in \mathcal{B}\}$$

Theorem 2. *If A and B are up-sets in $Q^n_{\mathbf{p}}$, then*

$$\mathbb{P}_{\mathbf{p}}(A \Box B) \leq \mathbb{P}_{\mathbf{p}}(A)\mathbb{P}_{\mathbf{p}}(B)$$

Proof. Put $C = A \Box B$. Induction on n: $n = 0$ ✓. So let $n \geq 1$.

Let $C_0 = A_0 \Box B_0$, $C_1 = (A_0 \Box B_1) \cup (A_1 \Box B_0) \subseteq A_1 \Box B_1$. Then we have $C_0 \subset (A_0 \Box B_1) \cap (A_1 \Box B_0)$.

$\mathbb{P}_{\mathbf{p}'}(C_0) \leq \mathbb{P}_{\mathbf{p}'}(A_0)\mathbb{P}_{\mathbf{p}'}(B_0)$, $\mathbb{P}(C_1) \leq \mathbb{P}(A_1)\mathbb{P}(B_1)$.

$$\begin{aligned}
\mathbb{P}(C_0) + \mathbb{P}(C_1) &\leq \mathbb{P}((A_0 \Box B_1) \cap (A_1 \Box B_0)) + \mathbb{P}((A_0 \Box B_1) \cup (A_1 \Box B_0)) \\
&= \mathbb{P}(A_0 \Box B_1) + \mathbb{P}(A_1 \Box B_0) \\
&\leq \mathbb{P}(A_0)\mathbb{P}(B_1) + \mathbb{P}(A_1)\mathbb{P}(B_0)
\end{aligned}$$

Multiply then by $(1-p_n)^2$, p_n^2, $p_n(1-p_n)$ and add them:

$$\mathbb{P}(C_0)((1-p_n)^2 + (1-p_n)p_n) + \mathbb{P}(C_1)(p_n^2 + p_n(1-p_n)) \leq \mathbb{P}_{\mathbf{p}}(A)\mathbb{P}_{\mathbf{p}}(B)$$

Obtain $\mathbb{P}(C) \leq \mathbb{P}(A)\mathbb{P}(B)$. $\square$

The full Van den Berg - Kesten conjecture that $\mathbb{P}(A \Box B) \leq \mathbb{P}(A)\mathbb{P}(B)$ was proved by Reimer.

Theorem 3 (Ahlswede-Daykin Four Functions Theorem)**.** *let $\alpha, \beta, \gamma, \delta : \mathcal{P}(X) \to \mathbb{R}^+$. Suppose $\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)\ \forall A, B \subset X$.*

Then $\alpha(\mathcal{A})\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \vee \mathcal{B})\delta(\mathcal{A} \wedge \mathcal{B})$ where $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$, $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$

Proof. Induction on n. $n = 1 : \alpha : \{\emptyset, \{1\}\} \to \mathbb{R}$, etc. $\alpha_0, \alpha_1, \beta_0, \beta_1, \ldots$

The conditions become

$$\alpha_0\beta_0 \leq \gamma_0\delta_0$$

$$\alpha_1\beta_0 \leq \gamma_1\delta_0$$

$$\alpha_0\beta_1 \leq \gamma_1\delta_0$$

$$\alpha_1\beta_1 \leq \gamma_1\delta_1$$

We need

$$(\alpha_0 + \alpha_1)(\beta_0 + \beta_1) \overset{?}{\leq} (\gamma_0 + \gamma_1)(\delta_0 + \delta_1) \qquad (*)$$

We may assume all are > 0, also $\gamma_0 = \frac{\alpha_0\beta_0}{\delta_0}$, $\delta_1 = \frac{\alpha_1\beta_1}{\gamma_1}$.

(*) becomes

$$\begin{aligned}\alpha_0\beta_1 + \alpha_1\beta_0 &\overset{?}{\leq} \gamma_0\delta_1 + \gamma_1\delta_0 \\ &= \frac{\alpha_0\beta_0}{\delta_0}\frac{\alpha_1\beta_1}{\gamma_1} + \gamma_1\delta_0\end{aligned}$$

$$\alpha_0\beta_1\gamma_1\delta_0 + \alpha_1\beta_0\gamma_1\delta_0 \leq \alpha_0\beta_0\alpha_1\beta_1 + (\gamma_1\delta_0)^2$$

$$(\gamma_1\delta_0 - \alpha_1\beta_0)(\gamma_1\delta_0 - \alpha_0\beta_1) \geq 0 \qquad \checkmark$$

$X = [n], Y = [n] - \{n\}$. We may assume supp $\alpha = \mathcal{A}$, supp $\beta = \mathcal{B}$, supp $\gamma = \mathcal{A} \wedge \mathcal{B}$, supp $\delta = \mathcal{A} \vee \mathcal{B}$.

Need: $\alpha(\mathcal{P})\beta(\mathcal{P}) \leq \gamma(\mathcal{P})\delta(\mathcal{P})$.

Define $\alpha', \beta', \gamma', \delta' : \mathcal{P}(Y) \to \mathbb{R}^+$ by $\alpha'(E) = \alpha(E) + \alpha(E \cup \{n\})$, $\beta'(E) = \beta(E) + \beta(E \cup \{n\}), \dots$

Need $\alpha'(\mathcal{P}(Y))\beta'(\mathcal{P}(Y)) \leq \gamma'(\mathcal{P}(Y))\delta'(\mathcal{P}(Y))$. By induction this holds if it holds $\forall A, B \subset Y$.

Suffices to prove that

$$(\alpha(A) + \alpha(A \cup \{n\})(\beta(B) + \beta(B \cup \{n\}) \leq (\gamma(C) + \gamma(C \cup \{n\})(\delta(D) + \delta(D \cup \{n\})$$

Since we know the four function theorem for $n = 1$, this holds if $\tilde{\alpha} : \{\emptyset, 1\} \to \mathbb{R}^+$, $\tilde{\alpha}_0 = \alpha(A), \tilde{\alpha}_1 = \alpha(A \cup \{n\})$, $\tilde{\beta}_0 = \beta(B), \dots$ satisfy our four conditions. But these are exactly the inequalities satisfied by $\alpha, \beta, \gamma and \delta$. $\square$

Definition. A **lattice** L is a poset with finite meets and joins. L is a **distributive lattice** if $\forall x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (equivalently $x \wedge (y \vee z)$).

Prime example: $\mathcal{P}(X)$. Every finite distributive lattice is a sublattice of $\mathcal{P}(X)$.

Corollary 4. *If L is a distributive lattice and $\alpha, \beta, \gamma, \delta : L \to \mathbb{R}^+$ then $\alpha(A)\beta(B) \leq \gamma(A \wedge B)\delta(A \vee B)$ $\forall A, B \subset L \iff$ it holds $\forall A, B$ singletons.*

Definition. A probability measure $\mu : L \to \mathbb{R}^+$ is **log-supermodular** if

$$\mu(x)\mu(y) \leq \mu(x \wedge y)\mu(x \vee y)$$

Corollary 5 (FKG inequality). *If $\mu : L \to \mathbb{R}^+$ is a log-supermodular probability measure and f, g are increasing, non-negative functions then*

$$\int f\, d\mu \int g\, d\mu \leq \int fg\, d\mu$$

6 Isoperimetric Inequalities

6.1 Vertex Isoperimetric Inequality

Let $A \subset Q^n \cong \mathcal{P}(X)$; $A \leftrightarrow \mathcal{A} \subset \mathcal{P}(X)$; $x, y \in A$, $x, y \leftrightarrow A, B \subset X$.

Let $N(A) = A \cup \{x \in Q^n \,|\, x \notin A, \exists y \in A \text{ s.t. } xy \in E\}$, the neighbourhood of A.

Given $a \geq 1$, which set $A \in X^{(a)}$ minimise $|N(A)|$ over $X^{(a)}$?

Definition. The **simplicial order** on $Q_n \cong \mathcal{P}(X)$ is given by $A < B$ if $|A| < |B|$ or $|A| = |B|$ and $\min(A \triangle B) \in A$.

$n = 4:$ $\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, \ldots$

$N(\{\emptyset, 1, 2, 3, 4, 12, 13\}) = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134\}$

If A is an initial segment in the simplicial order then $N(A)$ is also an initial segment.

Definition. For $S \subset Q^n$ and direction i, $1 \leq i \leq n$, the **i-sections** of S are

$$S_-^{(i)} = \{x \in S \,|\, i \notin x\} \subset \mathcal{P}([n] - \{i\})$$

$$S_+^{(i)} = \{x - \{i\} \,|\, x \in S,\, i \in x\} \subset \mathcal{P}([n] - \{i\})$$

Theorem 1. *Let $A \subset Q^n$ and let B be the initial segment of length $|A|$ in the simplicial order on Q^n. Then $|N(A)| \geq |N(B)|$. In particular, $|A| = \sum_{n=0}^{r} \binom{n}{i} \implies |N(A)| \geq \sum_{n=0}^{r+1} \binom{n}{i}$.*

Proof. Let $C_i(A)$ be obtained from A by replacing each i-segment by the initial segment of $\mathcal{P}([n] - \{i\})$ of the same size. Let $C_-^{(i)}(A)$ be the initial segment of length $\left|A_-^{(i)}\right|$ in the simplicial order on $\mathcal{P}([n] - \{i\})$, similarly $C_+^{(i)}(A)$.

$C_i(A)$ is given by

$$C_i(A)_-^{(i)} = C_-^{(i)}(A)$$
$$C_i(A)_+^{(i)} = C_+^{(i)}(A)$$

Then $|C_i(A)| = |A|$.

$|N(S)| = \left|N(S_-^{(i)}) \cup S_+^{(i)}\right| + \left|N(S_+^{(i)}) \cup S_-^{(i)}\right|$

Induction on n. $n = 1:$ ✓

Note

$$\begin{aligned} |N(C_i(A))| &= \left|N(C_-^{(i)}(A)) \cup C_+^{(i)}(A)\right| + \left|N(C_+^{(i)}(A)) \cup C_-^{(i)}(A)\right| \\ &= \left|N(A_-^{(i)}) \cup A_+^{(i)}\right| + \left|N(A_+^{(i)}) \cup A_-^{(i)}\right| \end{aligned}$$

Thus $|N(C_i(A))| \leq |N(A)|$.

We may compress A in any direction. Compress while the set moves. This ends, since the elements of A move closer to the beginning. We end with a compressed set A: $C_i(A) = A \;\forall i$.

Define $A^{(n)}_{vc,exc} \subset Q^n$:

$$\begin{aligned} n = 2k+1: &\qquad A^{(n)}_{vx,exc} = (X^{(\leq k)} - \text{last}) \cup \text{next} \\ n = 2k: &\qquad A^{(n)}_{vx,exc} = (\text{half - last}) \cup \text{next} \end{aligned}$$

Lemma 2. *Let $A \subset Q^n$ be i-compressed $\forall i$, but not an initial segment. Then $A = A^{(n)}_{vx,exc}$.*

Proof. $\exists x \in Q^n \backslash A$, $y \in A$, $x < y$. Since A is compressed for $1 \leq i \leq n$, $i \in x \iff i \notin y$. Indeed, if $i \in x \cap y$ or $i \notin x \cup y$ then the i-compression would move A.

Hence $y = x^c$, so A has only one 'gap' (x), and that is followed by a single element (y).

Thus A is an initial segment minus its last element, followed by the complement of this element.

In the simplicial order, x is followed by x^c

$$\begin{aligned} n = 2k+1: &\qquad x = \{k+2,\, k+3,\, 2k+1\};\; y = \{1,\, 2,\, \ldots,\, k+1\} \\ n = 2k: &\qquad x = \{1,\, k+2,\, \ldots,\, 2k\};\; y = \{2,\, 3,\, \ldots,\, k+1\} \end{aligned}$$

□

Proof of Theorem 1 cont. We may assume A is compressed. Hence either A is an initial segment, so we're done, or else $A = A^n_{vx,exc}$. But this set has too large a neighbourhood. □

6.2 Edge Isoperimetric Inequality

$A \subset Q^n$, $\partial_e A = \{xy \in E \,|\, x \in A,\, y \notin A\}$.

Binary order on $\mathcal{P}(n)$: $A < B$ if $\max(A \triangle B) \in B$.

Theorem 3. *Let $A \subset Q^n$ and let B be the initial segment of length $|A|$ in the binary order. Then $|\partial_e A| \geq |\partial_e B|$. In particular, if $|A| = 2^k$ then $|\partial_e A| \geq 2^{k(n-k)}$.*

Proof. The 1-codimensional compression of A in the direction i, $C_i(A)$ is the set B such that

- $B_-^{(i)}$ is the initial segment in $\mathcal{P}([n]\backslash\{i\})$ in the binary order of length $\left|A_-^{(i)}\right|$
- $B_+^{(i)}$ similarly

Note:

$$\begin{aligned} |\partial_e(A)| &= \left|\partial_e A_-^{(i)}\right| + \left|\partial_e A_+^{(i)}\right| + \left|A_-^{(i)} \triangle A_+^{(i)}\right| \\ &\geq \left|\partial_e B_-^{(i)}\right| + \left|\partial_e B_+^{(i)}\right| + \left|B_-^{(i)} \triangle B_+^{(i)}\right| = |\partial_e(B)| \end{aligned}$$

Compress until our set moves. This stops: we get a compressed set A.

Lemma 4. *If $A \subset Q^n$ is compressed (in the binary order) but not an initial segment, then $A =$ (half - last) $\cup$ next, e.g. $(\mathcal{P}(n-1)\backslash\{1, 2, \ldots, n-1\}) \cup \{\{n\}\}$*

Proof. $\exists x \in Q^n\backslash A$, $y \in A$, $x < y$. As before, $y = x^c$. Then $y = \{n\}$, $x = \{1, 2, \ldots, n-1\}$. □

Proof of Theorem 3 cont. We may assume A is compressed, so it is either an initial segment or our exceptional set $A^n_{edge,exc}$. But $\left|\partial_e A^n_{edge,exc}\right|$ is too large. □

7 Intersecting Families II

Modular intersections: $L = \{l_1, \ldots, l_s\}$, $\mathcal{A} \subset \mathcal{P}(n)$, $|A \cap B| \in L \ \forall A, B \in \mathcal{A}$.

Theorem 1 (Ray-Chaudhuri-Wilson, 1975)**.** *Let p be a prime and $L = \{l_1, \ldots, l_s\}$ a set of s integers. Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathcal{P}(n)$ be a set system such that*

- $|A_i| \notin L \mod p$
- $|A_i \cap A_j| \in L \mod p$

Then $m = |\mathcal{A}| \leq \sum_{i=0}^{s} \binom{n}{i}$.

Remark. For $\mathcal{A} = [n]^{(s)}$ and $L = \{0, 1, \ldots, s-1\}$ we have equality

Proof. We work with the polynomial ring $\mathbb{F}_p[X] = \mathbb{F}_p[X_1, \ldots, X_n]$ as a vector space. For $A \in \mathcal{P}(n)$, write $v_A \in \mathbb{F}_p^n$ for the characteristic function of A: $v_A = (v_1, \ldots, v_n)$, $v_i = \begin{cases} 1 & i \in A \\ 0 & \text{otherwise} \end{cases}$

For $A \in \mathcal{P}(n)$, define $f_A(X) = f_A^{(L)}(X) = \prod_{h=1}^s (\langle X, v_A \rangle - l_h)$.

Thus $f_A(X) = \prod_{h=1}^s (\sum_{i \in A} X_i - l_h)$.

For $f \in \mathbb{F}_p[X]$, its multilinear form $\tilde{f}(X)$ is obtained from f by replacing each exponent ≥ 1 with 1. Then $\mathbb{F}_p[X] \to M[X]$, $f \mapsto \tilde{f}$ is a linear map. If $v \in \{0,1\}^n \subset \mathbb{F}_p^n$, then $f(v) = \tilde{f}(v)$.

To avoid clutter, we'll write f_i for f_{A_i}, v_i for v_{A_i}. Note that

$$\begin{aligned} \tilde{f}_i(v_j) &= f_i(v_j) \\ &= \prod_{h=1} s(|A_i \cap A_j| - l_h) \\ &= \begin{cases} c_i \neq 0 & j = i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence, $\{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m\}$ is independent. Indeed, if $\sum_i \lambda_i \tilde{f}_i = 0 \in \mathbb{F}_p[X]$ then $(\sum_i \lambda_i \tilde{f}_i)(v_j) = \lambda_j c_j = 0$, so $\lambda_j = 0$. But $\tilde{f}_i \in M_{n,s}$ = vector space of multilinear polynomials of degree $\leq s$.

Hence $m = |\mathcal{A}| \leq \dim M_{n,s} = \sum_{i=0}^s \binom{n}{i}$. □

Theorem 2 (RW 1975)**.** *Let p be a prime, $L = \{l_1, \ldots, l_s\}$ a set of s integers and let $\mathcal{A} = A_1, \ldots, A_m \subset [n]^{(r)}$ be such that $r \notin L \mod p$ and $|A_i \cap A_j| \in L$ for all $i \neq j$. Then $|\mathcal{A}| \leq \binom{n}{s}$.*

Proof. Proceed as before: we get $\tilde{f}_1, \ldots, \tilde{f}_m \in M_{n,s}$. For $I \in [n]^{(\leq s-1)}$, define $p_I(X) = (\prod_{i \in I} X_i)(\sum_1^n X_i - r)$. Then $p_I(v_L) = 0$. Let $\tilde{p}_I$ be the multilinear form of p_I.

Claim: $\{\tilde{f}_i \mid 1 \leq i \leq m\} \cup \{\tilde{p}_I \mid I \in [n]^{(\leq s-1)}\}$ is an independent set in $M_{n,s}$.

Indeed, suppose $F = \sum_1^m \lambda_i \tilde{f}_i + \sum_{|I| \leq s-1} \mu_I \tilde{p}_I = 0$.

Evaluating F at v_h (characteristic function of A_h) the second sum is 0, so $F(v_h) = \lambda_h c_h$ so $\lambda_h = 0$. Thus every λ_i is 0,

$$G = \sum_{|I| \leq s-1} \mu_I \tilde{p}_I = 0$$

Let $I_1, I_2, \ldots, I_t$ be an enumeration of $[n]^{(\leq s-1)}$ such that if $i < j$ then $|I_i| \leq |I_j|$.

Writing w_i for the characteristic vector of I_i,

$$\tilde{p}_{I_i}(w_j) = \begin{cases} |I_i| - r \neq 0 & j = i \\ 0 & j < i \end{cases}$$

Hence $\{\tilde{p}_I \mid I \in [n]^{(\leq s-1)}\}$ is an independent set in $M_{n,s}$. The claim is proved.

Therefore

$$\mathcal{A} + \sum_{i=0}^{s-1} \binom{n}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$$
$$\implies \mathcal{A} \leq \binom{n}{s}$$

□

Frankl (1981): how large can a 3-wise intersecting family be?

Conjecture: $\mathcal{A} = o(2^n)$.

Definition. $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{P}(n)$ are **cross-intersecting** if $A \cap B \neq \emptyset$ $\forall A \in \mathcal{A}$, $B \in \mathcal{B}$.

Definition. $\mathcal{A}$ is symmetric if its automorphism group is transitive on $[n]$.

Theorem 3. *If $\mathcal{A} \subset \mathcal{P}(n)$ is a 3-wise intersecting symmetric family then $\mathcal{A} = o(2^n)$; in fact, $\mathcal{A} \leq \frac{2^n}{n^{1/8}}$ if n is large.*

Proof. Let $J(\mathcal{A}) = \{A \cap B \mid A, B \in \mathcal{A}\}$. $\mathcal{A}$ and $J(\mathcal{A})$ are cross-intersecting.

$$\mathbb{P}_{\frac{1}{2}}(\mathcal{A}) = \delta \implies \mathbb{P}_{\frac{1}{4}}(J(\mathcal{A})) \geq \delta^2$$
$$\mathbb{P}_{\frac{1}{4}}(J(\mathcal{A})) \geq \delta^2 \implies \mathbb{P}_{\frac{3}{4}} \leq 1 - \delta^2$$
$$\mathbb{P}_{\frac{1}{2}}(\mathcal{A}) > \delta^2 \implies \mathbb{P}_{\frac{1}{2}+\epsilon}(\mathcal{A}) > 1 - \delta^2$$

Friedgut and Kalai: If $\mathcal{A}$ is increasing and symmetric then

$$\mathbb{P}_p(\mathcal{A}) > \epsilon \implies \mathbb{P}_q(\mathcal{A}) > 1 - \epsilon$$

where $q = \min\{1, p + \frac{\log(\frac{1}{\epsilon})}{\log n}\}$

We may assume $\mathcal{A}$ is increasing.

Lemma 4. $\mathbb{P}_{\frac{1}{2}}(\mathcal{A}) = \delta \implies \mathcal{P}_{\frac{1}{4}}(J(\mathcal{A})) \geq \delta^2$.

Proof. Set $N_j = \left|J(\mathcal{A}) \cap [n]^{(j)}\right|$.

Define

$$F : \mathcal{A} \times \mathcal{A} \to J(\mathcal{A})$$
$$(A, B) \mapsto A \cap B$$

Then $\forall C \in J(\mathcal{A})$, $\left|F^{-1}(C)\right| \leq 3^{n-j} \qquad (j = |C|)$
$|\mathcal{A}|^2 \leq \sum N_j 3^{n-j}$, hence

$$\begin{aligned}
\mathbb{P}_{\frac{1}{4}} &= \sum_{j=1}^{n} \left(\frac{1}{4}\right)^{j} \left(\frac{3}{4}\right)^{n-j} N_j \\
&= 2^{-2n} \sum N_j 3^{n-j} \\
&\geq \mathbb{P}_{\frac{1}{2}}(\mathcal{A})^2 = \delta^2
\end{aligned}$$

□

Lemma 5. *If $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting then $\mathbb{P}_p(\mathcal{A}) + \mathbb{P}_{1-p}(\mathcal{B}) \leq 1$.*

Proof. $\mathcal{B}^c = \{[n]\backslash B \mid B \in \mathcal{B}\}$. Then $\mathcal{A} \cap \mathcal{B}^c = \emptyset$.

Also,

$$\begin{aligned}
\mathbb{P}_p(\mathcal{B}^c) &= \mathbb{P}_{1-p}(\mathcal{B}) \\
\implies \mathbb{P}_p(\mathcal{A}) + \mathbb{P}_{1-p}(\mathcal{B}) &= \mathbb{P}_p(\mathcal{A}) + \mathbb{P}_p(\mathcal{B}^c) \leq 1
\end{aligned}$$

□

Lemma 6. *If $\mathcal{A}$ is an increasing, intersecting family then $\mathbb{P}_{\frac{1}{2}} = \delta \implies \mathbb{P}_q(\mathcal{A}) \geq 1 - \delta^2$, where $q = \frac{1}{2} + \frac{\log(1/\delta^2)}{\log n}$.*

Proof. $\mathbb{P}_{\frac{1}{2}} > \delta^2$, apply FK. □

Proof of Theorem 3 cont. We may assume $\mathcal{A}$ is increasing, set $J(\mathcal{A})$ as before. Suppose $\mathbb{P}_{\frac{1}{2}}(\mathcal{A}) = \delta$. Then $\mathbb{P}_{\frac{1}{4}}(J(\mathcal{A})) \geq \delta^2$.

Hence by Lemma 4, $\mathbb{P}_{\frac{1}{4}}(J(\mathcal{A})) + \mathbb{P}_{\frac{3}{4}}(\mathcal{A}) \leq 1$, so $\mathbb{P}_{\frac{3}{4}}(\mathcal{A}) \leq 1 - \delta^2$. But $\mathbb{P}_{\frac{1}{2}}(\mathcal{A}) = \delta > \delta^2$, so for $q = \frac{1}{2} + \frac{2\log(1/\delta)}{\log n}$, $\mathbb{P}_q(\mathcal{A}) \geq 1 - \delta^2$.

Hence $q > \frac{3}{4}$, $\frac{2\log(1/\delta)}{\log n} > \frac{1}{4}$, $\delta < n^{-1/8}$. □