Part III Category Theory

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1 Definitions and Examples

Definition (Category)**.** *A category $\mathcal{C}$ consists of*

a. *a collection* $\mathrm{ob}\,\mathcal{C}$ *of* ***objects*** A*,* B*,* C*, ...*

b. *a collection* $\mathrm{mor}\,\mathcal{C}$ *of* ***morphisms*** f*,* g*,* h*, ...*

c. *two operations* dom, cod *from morphisms to objects. We write* $f : A \to B$ *or* $A \xrightarrow{f} B$ *to mean 'f is a morphism and* $\mathrm{dom}\, f = A$ *and* $\mathrm{cod}\, f = B$*'*

d. *an operation assigning to each object* A *a morphism* $1_A : A \to A$

e. *a partial binary operation* $(f, g) \mapsto gf$*, s.t.* gf *is defined* $\iff$ $\mathrm{dom}\, g = \mathrm{cod}\, f$*, and then* $gf : \mathrm{dom}\, f \to \mathrm{cod}\, g$

satisfying

f. $f1_A = f$ *and* $1_B f = f$ $\forall f : A \to B$

g. $h(fg) = (hg)f$ *whenever* gf *and* hg *are defined*

Definition (Functor)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A* ***functor*** $\mathcal{C} \to \mathcal{D}$ *consists of*

a. *a mapping* $A \to FA$ *from* $\mathrm{ob}\,\mathcal{C}$ *to* $\mathrm{ob}\,\mathcal{D}$

b. *a mapping* $f \to Ff$ *from* $\mathrm{mor}\,\mathcal{C}$ *to* $\mathrm{mor}\,\mathcal{D}$

satisfying $\operatorname{dom} Ff = F\operatorname{dom} f$, $\operatorname{cod} Ff = F\operatorname{cod} f$ *for all* f, $F(1_A) = 1_{FA}$ *for all* A, *and* $F(gf) = (Fg)(Ff)$ *whenever* gf *is defined.*

Definition. *By a* ***contravariant functor*** $\mathcal{C} \to \mathcal{D}$ *we mean a functor* $\mathcal{C} \to \mathcal{D}^{op}$ *(or equivalently* $\mathcal{C}^{op} \to \mathcal{D}$*). A functor* $\mathcal{C} \to \mathcal{D}$ *is sometimes said to be* ***covariant****.*

Definition (Natural transformation)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories and* $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ *two functors. A* ***natural transformation*** $\alpha : F \to G$ *assigns to each* $A \in \operatorname{ob} \mathcal{C}$ *a morphism* $\alpha_A : FA \to GA$ *in* $\mathcal{D}$*, such that*

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given $\alpha : F \to G$ and $\beta : G \to H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A-component of a natural transformation $\beta\alpha : F \to H$.

Definition. *Given categories* $\mathcal{C}, \mathcal{D}$*, we write* $[\mathcal{C}, \mathcal{D}]$ *for the category of all functors* $\mathcal{C} \to \mathcal{D}$ *and natural transformations between them.*

Lemma 1. *Given* $F, G : \mathcal{C} \to \mathcal{D}$ *and* $\alpha : F \to G$*,* α *is an isomorphism in* $[\mathcal{C}, \mathcal{D}] \iff$ *each* α_A *is an isomorphism in* $\mathcal{D}$*.*

Definition (Faithful and full)**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor.*

a. *We say that* F *is* ***faithful*** *if, given* $f, g \in \operatorname{mor} \mathcal{C}$*, the equations* $\operatorname{dom} f = \operatorname{dom} g$, $\operatorname{cod} f = \operatorname{cod} g$ *and* $Ff = Fg$ *imply* $f = g$*.*

b. F *is* ***full*** *if, given any* $g : FA \to FB$ *in* $\mathcal{D}$*, there exists* $f : A \to B$ *in* $\mathcal{C}$ *with* $Ff = g$*.*

c. *We say a subcategory* $\mathcal{C}'$ *of* $\mathcal{C}$ *is* ***full*** *if the inclusion* $\mathcal{C}' \hookrightarrow \mathcal{C}$ *is a full functor.*

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition (Equivalence of categories)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. An* ***equivalence*** *between* $\mathcal{C}$ *and* $\mathcal{D}$ *is a pair of functors* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ *together with natural isomorphisms* $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$*. We write* $\mathcal{C} \simeq \mathcal{D}$ *to mean that* $\mathcal{C}$ *and* $\mathcal{D}$ *are equivalent.*

We say a property P *of categories is* ***categorical*** *if whenever* $\mathcal{C}$ *has* P *and* $\mathcal{C} \simeq \mathcal{D}$ *then* $\mathcal{D}$ *has* P*.*

For example, being a groupoid is a categorical property, but being a group is not.

Definition (Slice category)**.** *Given an object B of a category $\mathcal{C}$, define the **slice category** $\mathcal{C}/B$ to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \to (A' \xrightarrow{f'} B)$ are morphisms $h : A \to A'$ making*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commute.