Part III Local Fields

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Contents

1 Basic Theory

Definition 1 (Absolute value)**.** *Let K be a field. An* ***absolute value*** *on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ s.t.*

i. $|x| = 0 \iff x = 0$

ii. $|xy| = |x|\,|y| \quad \forall x, y \in K$

iii. $|x + y| \leq |x| + |y|$

Definition 2 (Valued field)**.** *A* ***valued field*** *is a field with an absolute value.*

Definition 3 (Equivalence of absolute values)**.** *Let K be a field and let $|\cdot|$, $|\cdot|^{'}$ be absolute values on K. We say that $|\cdot|$ and $|\cdot|^{'}$ are* ***equivalent*** *if the associated metrics induce the same topology.*

Definition 6 (Non-archimedean absolute value)**.** *An absolute value $|\cdot|$ on a field K is called* ***non-archimedean*** *if $|x+y| \leq \max(|x|, |y|)$ (the* ***strong triangle inequality****).*

Metrics s.t. $d(x,z) \leq \max(d(x,y), d(y,z))$ are called ***ultrametrics****.*

Assumption: unless otherwise mentioned, all absolute values will be non-archimedean. These metrics are weird!

Proposition 7. *Let K be a valued field. Then $\mathcal{O} = \{x \mid |x| \leq 1\}$ is an open subring of K, called the* ***valuation ring*** *of K. $\forall r \in (0,1]$, $\{x \mid x < r\}$ and $\{x \mid x \leq r\}$ are open ideals of $\mathcal{O}$.*

Moreover, $\mathcal{O}^x = \{x \mid |x| = 1\}$.

Proposition 8. *Let K be a valued field.*

i. *Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) is Cauchy*

Assume that K is complete

ii. *Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) converges*

iii. *Let $\sum_{n=0}^{\infty} y_n$ be a series in K. If $y_n \to 0$, then $\sum_{n=0}^{\infty} y_n$ converges*

Definition 9. *Let $R \subseteq S$ be rings. Then $s \in S$ is* ***integral over R*** *if $\exists$ monic $f(x) \in R[x]$ s.t. $f(s) = 0$.*

Proposition 10. *Let $R \subseteq S$ be rings. Then $s_1, \ldots, s_n \in S$ are all integral over $R \iff R[s_1, \ldots, s_n] \subseteq S$ is a finitely generated R-module.*

Corollary 11. *let $R \subseteq S$ be rings. If $s_1, s_2 \in S$ are integral over R, then $s_1 + s_2$ and $s_1 s_2$ are integral over R. In particular, the set $\tilde{R} \subseteq S$ of all elements in S integral over R is a ring, called the* ***integral closure*** *of R in S.*

Definition 12. *Let R be a ring. A topology on R is called a* ***ring topology*** *on R if addition and multiplication are continuous maps $R \times R \to R$. A ring with a ring topology is called a* ***topological ring****.*

Definition 13. *Let R be a ring, $I \subseteq R$ an ideal. A subset $U \subseteq R$ is called* ***I-adically open*** *if $\forall x \in U$ $\exists n \geq 1$ s.t. $x + I^n \subseteq U$.*

Proposition 14. *The set of all I-adically open sets form a topology on R, called the* ***I-adic topology****.*

Definition 15. *Let $R_1, R_2, \ldots$ be topological rings with continuous homomorphisms $f_n : R_{n+1} \to R_n$ $\forall n \geq 1$. The **inverse limit** of the R_i is the ring*

$$\begin{aligned}\varprojlim_n R_n &= \left\{ (x_n) \in \prod_n R_n \mid f_n(x_{n+1}) = x_n \forall n \geq 1 \right\} \\ &\subseteq \prod_n R_n\end{aligned}$$

Proposition 16. *The inverse limit topology is a ring topology.*

Definition 17. *Let R be a ring, I an ideal. The **I-adic completion** of R is the topological ring $\varprojlim_n R/I^n$ (R/I^n has the discrete topology, and $R/I^{n+1} \to R/I^n$ is the natural map).*

*There exists a map $\nu : R \to \varprojlim R/I^n$, $r \mapsto (r \mod I^n)_n$ This map is a continuous ring homomorphism when R is given the I-adic topology. We say that R is **I-adically complete** if ν is a bijection.*

*If $I = xR$ then we often call the I-adic topology the **x-adic topology**.*

1.1 The p-adic Numbers

Let p be a prime number throughout.

If $x \in \mathbb{Q}\backslash\{0\}$ then $\exists!$ representation $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{>0}$ and $(a,p) = (b,p) = (a,b) = 1$.

We define the **p-adic absolute value** on $\mathbb{Q}$ to be the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-n} & \text{if } x = p^n \frac{a}{b} \ (\neq 0) \text{ as before} \end{cases}$$

Then $|\cdot|_p$ is an absolute value.

Definition 18. *The **p-adic numbers** $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$.*

*The valuation ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is called the **p-adic integers**.*

Proposition 19. *$\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ inside $\mathbb{Q}_p$.*

Proposition 20. *The non-zero ideals of $\mathbb{Z}_p$ are $p_n\mathbb{Z}_p$ for $n \geq 0$. Moreover, $\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p/p^n\mathbb{Z}_p$*

Corollary 21. *$\mathbb{Z}_p$ is a PID with a unique prime element p (up to units).*

Proposition 22. *The topology on $\mathbb{Z}$ induced by $|\cdot|_p$ is the p-adic topology.*

Proposition 23. *$\mathbb{Z}_p$ is p-adically complete and is (isomorphic to) the p-adic completion of $\mathbb{Z}$.*

Corollary 24. *Every $a \in \mathbb{Z}_p$ has a unique expansion*

$$a = \sum_{i=0}^{\infty} a_i p^i$$

with $a_i \in \{0, 1, \ldots, p-1\}$

Every $a \in \mathbb{Q}_p^\times$ has a unique expansion

$$a = \sum_{i=n} \infty a_i p^i$$

$n \in \mathbb{Z}$, $n = -\log_p |a|_p$, $a_n \neq 0$.

1.2 Valued Fields

Definition 25. *Let K be a field. A* ***valuation*** *on K is a function $v : K \to \mathbb{R} \cup \{\infty\}$ s.t.*

i. $v(x) = \infty \iff x = 0$

ii. $v(xy) = v(x) + v(y)$

iii. $v(x + y) \geq \min(v(x), v(y))$

$\forall x, y \in K$.

Here we use the conventions $r + \infty = \infty$, $r \leq \infty \ \forall r \in \mathbb{R} \cup \{\infty\}$. v a valuation $\implies$ if $|x| = c^{-v(x)}$, $c \in \mathbb{R}_{>1}$, then $|\cdot|$ is an absolute value. Conversely, if $|\cdot|$ is an absolute value then $v(x) = -\log_c |x|$.

Let K be a valued field.

- $\mathcal{O} = \mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$ is the **valuation ring**
- $\mathfrak{m} = \mathfrak{m}_K = \{x \in K \mid |x| < 1\}$ is the **maximal ideal**
- $k = k_K = \mathcal{O}/\mathfrak{m}$ is the **residue field**

If K is a valued field and $F(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ is a polynomial, we say that F is **primitive** if $\max_i |a_i| = 1$ ($\implies F \in \mathcal{O}[x]$).

Theorem 26 (Hensel's Lemma)**.** *Assume that K is complete and that $F \in K[x]$ is primitive. Put $f = F \mod \mathfrak{m} \in k[x]$. If $\exists$ factorisation $f(x) = g(x)h(x)$ with $(g, h) = 1$, then $\exists$ factorisation $F(x) = G(x)H(x)$ in $\mathcal{O}[x]$ with $g \equiv G$, $h \equiv H \mod \mathfrak{m}$ and $\deg g = \deg G$.*

Proof. Put $d = \deg F$, $m = \deg g$, so $\deg h \leq d - m$. Pick lifts $G_0, H_0 \in \mathcal{O}[x]$ of g, h with $\deg G_0 = \deg g$, $\deg H_0 \leq d - m$.

$(g, h) = 1 \implies \exists A, B \in \mathcal{O}[x]$ s.t. $AG_0 + BH_0 \equiv 1 \mod \mathfrak{m}$.

Pick $\pi \in \mathfrak{m}$ s.t. $F - G_0 H_0 \equiv AG_0 + BH_0 - 1 \mod \pi$.

Want to find $G = G_0 + \pi P_1 + \pi^2 P_2 + \dots$, $H = H_0 + \pi Q_1 + \pi^2 Q_2 + \dots \in \mathcal{O}[x]$ with $P_i, Q_i \in \mathcal{O}[x]$, $\deg P_i < m$, $\deg Q_i \leq d - m$.

Define

$$G_{n-1} = G_0 + \pi P_1 + \dots + \pi^{n-1} P_{n-1}$$
$$H_{n-1} = H_0 + \pi Q_1 + \dots + \pi^{n-1} Q_{n-1}$$

We want $F \equiv G_{n-1} H_{n-1} \mod \pi^n$, then take the limit.

Induction on n: $n = 1$ ✓

Assume we have G_{n-1}, H_{n-1}, $G_n = G_{n-1} + \pi^n P_n, H_n = H_{n-1} + \pi^n Q_n$. Expanding $F - H_n G_n$, we want

$$F - G_{n-1} H_{n-1} \equiv \pi^n (G_{n-1} Q_n + H_{n-1} P_n) \mod \pi^{n+1}$$

and divide by π^n

$$G_{n-1} Q_n + H_{n-1} P_n = \frac{1}{\pi^n} (F - G_{n-1} H_{n-1}) \mod \pi$$

Let $F_n := F - G_{n-1} H_{n-1}$. $AG_o + BH_0 \equiv 1 \mod \pi \implies F_n \equiv AG_0 F_n + BH_0 F_n \mod \pi$.

Write $BF_n = QG_0 + P_n$ with $\deg P_n < \deg G_0$, $P_n \in \mathcal{O}[x]$

$$\implies G_0 (AF_n + H_0 Q) + H_0 P_n \equiv F_n \mod \pi$$

Now omit all coefficients from $AF_n + H_0 Q$ divisible by π to get Q_n. □

Corollary 27. *Let $F(x) = a_0 + a_1 x + \dots + a_n x^n \in K[x]$, K complete, $a_0 a_n \neq 0$. If F is irreducible, then $|a_i| \leq \max(|a_0|, |a_n|)$ $\forall i$.*

Corollary 28. *$F \in \mathcal{O}[x]$ monic, K complete. If $F \mod \mathfrak{m}$ has a simple root $\bar{\alpha} \in k$, then F has a (unique) simple root $\alpha \in \mathcal{O}$ lifting $\bar{\alpha}$.*

Useful fact: let K be a valued field, $x, y \in K$. $|x| > |y| \implies |x + y| = |x|$. More generally, if we have a convergent series $\sum_{i=0}^{\infty} x_i$ and the non-zero $|x_i|$ are distinct, then $|x| = \max |x_i|$.

Theorem 29. *Let K be a complete valued field and let L/K be a finite extension. Then the absolute value $|\cdot|$ on K has a unique extension to an absolute value $|\cdot|_L$ on L, given by*

$$|\alpha|_L = \sqrt[n]{\left|N_{L/K}(\alpha)\right|},\ n = [L : K]$$

and L is complete w.r.t. $|\cdot|_L$.

Corollary 30. *Let K be a complete valued field. If M/K is an algebraic extension of K, then $|\cdot|$ extends uniquely to an absolute value on M.*

Corollary 31. *In the setting of Theorem 16, if $\sigma \in \mathrm{Aut}(L/K)$ then $|\sigma(\alpha)|_L = |\alpha|_L \ \forall \alpha \in L$*

Definition 32. *Let K be a valued field and V a vector space over K. A* ***norm*** *on V is a function $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ such that*

i. $||x|| = 0 \iff x = 0$

ii. $||\lambda x|| = |\lambda| \, ||x|| \ \forall \lambda \in K, x \in V$

iii. $||x + y|| \leq \max(||x|| , ||y||) \ \forall x, y, \in V$

Two norms $||\cdot|| , ||\cdot||'$ are ***equivalent*** *if they induce the same topology on V $\iff \exists C, D > 0$ s.t. $C \, ||x|| \leq ||x||' \leq D \, ||x|| \ \forall x \in V$.*

Proposition 33. *Let K be a complete valued field and V a finite dimensional K-vector space. Let $x_1, \dots, x_n$ be a basis of V, then if $x = \sum a_i x_i \in V$,*

$$||x||_{\max} = \max_i |a_i|$$

defines a norm on V, and V is complete w.r.t $||\cdot||_{\max}$.

Moreover, if $||\cdot||$ is any norm on V, then $||\cdot||$ is equivalent to $||\cdot||_{\max}$ and hence V is complete w.r.t $||\cdot||$.

Lemma 34. *Let K be a valued field. Then $\mathcal{O}_K$ is integrally closed in K.*

Corollary 35. *Let K be a complete valued field, L/K finite. Equip L with $|\cdot|_L$ extending $|\cdot|$ on K. Then $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ inside L.*

1.3 Newton Polygons

Definition. *$S \subset \mathbb{R}^2$ is* ***lower convex*** *if*

i. $(x, y) \in S \implies (x, z) \in S \ \forall z \geq y$

ii. *S is convex*

Given any $T \subset \mathbb{R}^2$, there exists a minimal lower convex $LCH(T) \supseteq T$ ($LCH(T) = \bigcap_{T \subset S', S' \text{lower convex}} S'$).

Definition. *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ where K is a valued field, v a valuation on K.*

Define the ***Newton polygon*** *of f as* $LCH\left(\left\{ (i, v(a_i)) \;\middle|\; \begin{array}{c} i = 0, 1, \dots, n \\ a_i \neq 0 \end{array} \right\}\right)$.

Definition. *The horizontal length of a line segment is called the* ***multiplicity****. Line segments have a* ***slope****.*

Theorem 36. *Let K be a complete valued field, v a valuation on K, $f(x) = a_0 + a_1x + \cdots + a_nx^n \in K[x]$. Let L be the splitting field of f over K, equipped with the unique extension w of v.*

If $(r, v(a_r)) \to (s, v(a_s))$ is a line segment of the Newton polygon of f with slope $-m \in \mathbb{R}$, then f has precisely $s - r$ roots of valuation m.

Proof. Dividing by a_n only shifts the NP vertically, so wlog $a_n = 1$.

Number the roots of f s.t.

$$\begin{array}{ccccccc} v(\alpha_1) & = & \dots & = & v(\alpha_{s_1}) & = & m_1 \\ v(\alpha_{s_1+1}) & = & \dots & = & v(\alpha_{s_2}) & = & m_2 \\ \vdots & & & & \vdots & & \vdots \\ v(\alpha_{s_t+1}) & = & \dots & = & v(\alpha_{s_1}) & = & m_{t+1} \end{array}$$

where $m_1 < m_2 < \cdots < m_{t+1}$, and the α_i are the roots of f with multiplicity.

$v(a_n) = v(1) = 0$
$v(a_{n-1}) = v(\sum_i a_i) \geq \min_i v(\alpha_i) = m_1$
$v(a_{n-2}) \geq \min_{i \neq j} v(\alpha_i\alpha_j) = 2m_1$
$v(a_{n-s_1}) = v(\sum_{i_1,\dots i_{s_1} \text{distinct}} \alpha_{i_1} \dots \alpha_{i_{s_1}}) = s_1m_1$
$v(a_{n-s_1-1}) \geq \min v(\alpha_{i_1} \dots \alpha_{i_{s_1+1}}) = s_1m_1 + m_2$
$\vdots$
$v(a_{n-s_2}) = \min v(\alpha_{i_1} \dots \alpha_{i_{s_2}}) = s_1m_1 + (s_2 - s_1)m_2$

etc. Drawing the lines between the points $(n, 0)$, $(n - s_1, s_1m_1)$, $\dots$ gives the NP of f.

The first line segment has length $n-(n-s_1) = s_1$ and slope $\frac{0-s_1m_1}{n-(n-s_1)} = -m_1$. For $k \geq 2$, the kth line segment has length $(n - s_{k-1}) - (n - s_k) = s_k - s_{k-1}$ and slope

$$\frac{(s_1m_1 + \sum_{i=1}^{k-2}(s_{i+1} - s_i)m_{i+1}) - (s_1m_1 + \sum_{i=1}^{k-1}(s_{i+1} - s_i)m_{i+1})}{(n - s_{k-1}) - (n - s_k)}$$

$$= \frac{-(s_k - s_{k-1})m_k}{s_k - s_{k-1}} = -m_k$$

□

Corollary 37. *If f is irreducible, then the NP has a single line segment.*

Proof. we need to show that all roots have the same valuation. Let α, β be roots in the splitting field L. Then $\exists \sigma \in \operatorname{Aut}(L/K)$ s.t. $\sigma(\alpha) = \beta$. So $v(\alpha) = v(\sigma(\alpha)) = v(\beta)$ by Corollary 30. □

Definition 38. *Let K be a valued field with valuation v. K is a* ***discretely valued field*** *(DVF) if $v(K^\times) \subset \mathbb{R}$ is a discrete subgroup of $\mathbb{R}$ ($\iff v(K^\times)$ is infinite cyclic).*

Definition 39. *A complete DVF with finite residue field is called a* ***local field****.*

Let K be a DVF. $\pi \in K$ is called a **uniformiser** if $v(\pi) > 0$ and $v(\pi)$ generates $v(K^\times)$ ($\iff v(\pi)$ has minimal positive valuation).

Proposition 40. *Let K be a DVF, uniformiser π. Let $S \subset \mathcal{O}_K$ be a set of coset representatives of $\mathcal{O}_k/\mathfrak{m}_K = k_K$ containing 0. Then*

1. *The non-zero ideals of $\mathcal{O}_K$ are $\pi^n\mathcal{O}_K$, $n \geq 0$*
2. *$\mathcal{O}_K$ is a PID with unique prime π (up to units), $\mathfrak{m}_K = \pi\mathcal{O}_K$*
3. *The topology on $\mathcal{O}_K$ induced by $|\cdot|$ is the π-adic topology*
4. *If K is complete, then $\mathcal{O}_K$ is π-adically complete*
5. *If K is complete, then any $x \in K$ can be written uniquely as*
$$x = \sum_{n \gg -\infty}^{\infty} a_n \pi^n$$
with $a_n \in S$ and $|x| = |pi|^{-\inf\{n \mid a_n \neq 0\}}$
6. *The completion $\hat{K}$ of K is a DVF, π is a uniformiser and*
$$\mathcal{O}_K/\pi^n\mathcal{O}_K \xrightarrow{\sim} \mathcal{O}_{\hat{K}}/\pi^n\mathcal{O}_{\hat{K}}$$
via the natural map.

Proof. The same as for $\mathbb{Q}_p$ and $\mathbb{Z}_p$ (use π instead of p). Note that $\left|\hat{K}\right| = |K|$ by Ex 9, sheet 1 ($\implies \hat{K}$ is a DVF). $\square$

Proposition 41. *Let K be a DVF. Then K is a local field $\iff \mathcal{O}_K$ is compact*

Proof. $\mathcal{O}_K$ compact $\implies \pi^{-n}\mathcal{O}_K$ is compact $\forall n \geq 0$ (π uniformiser).

$\mathcal{O}_K \cong \pi^{-n}\mathcal{O}_K \implies K = \bigcup_{n\geq 0}^{\infty} \pi^{-n}\mathcal{O}_K$ is complete.

Also $\mathcal{O}_K \twoheadrightarrow k_K$ and this map is continuous when k_K is given the discrete topology. So k_K is compact and discrete $\implies k_K$ finite.

Conversely, we seek to prove that K local $\implies \mathcal{O}_K$ is sequentially compact ($\iff$ compact). Note that $\mathcal{O}_K/\pi^n\mathcal{O}_K$ is finite $\forall n \geq 0$ (induction and $\pi^{n-1}\mathcal{O}_K/\pi^n\mathcal{O}_K \cong \mathcal{O}_K/\pi\mathcal{O}_K$).

Let (x_i) be a sequence in $\mathcal{O}_K$. $\exists$ a subsequence (x_{1i}) which is constant modulo π. Keep going: choose a subsequence $(x_{n+1,i})$ of (x_{ni}) s.t. $(x_{n+1,i})$ is constant mod π^{n+1}.

Then $(x_{ii})_{i=1}^{\infty}$ converges: it's Cauchy since $|x_{ii} - x_{jj}| \leq |\pi|^j \;\; \forall j \leq i$, and K is complete. $\square$

Definition 42. *A ring R is called a **discrete valuation ring** (DVR) if it is a PID with a unique prime element (up to units).*

Proposition 43. *R is a DVR $\iff R \cong \mathcal{O}_K$ for some DVF K.*

Proof. The reverse implication is contained in Proposition 42.

Suppose R is a DVR, π prime. $\forall x \in R \backslash \{0\}$, $\exists! u \in R^{\times}$, $n \in \mathbb{Z}_{\geq 0}$ such that $x = \pi^n u$ by uniqueness of prime factorisation.

Define $v(x) = \begin{cases} n & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

v defines a discrete valuation of $R \implies v$ extends uniquely to $K = \operatorname{Frac}(R)$. It remains to show that $R = \mathcal{O}_K$. First, note that $K = R[\frac{1}{\pi}]$. Any non-zero element looks like $\pi^n u$, $u \in R^{\times}$, $n \in \mathbb{Z}$, so it is invertible.

Then $v(\pi^n u) = n \in \mathbb{Z}_{\geq 0} \iff \pi^n u \in R$

$\therefore R = \mathcal{O}_K$. $\square$

Definition 44. *Let K be a valued field with residue field k_K. K has **equal characteristic** if $\operatorname{char} K = \operatorname{char} k_K$, **mixed characteristic** otherwise ($\implies$ $\operatorname{char} K = 0, \operatorname{char} k_K > 0$).*

Definition 45. *Let R be a ring of characteristic p. R is **perfect** if the Frobenius map $x \mapsto x^p$ is an automorphism of R.*

Theorem 46. *Let K be a complete DVF of equal characteristic p and assume that k_K is perfect. Then $K \cong k_K[[T]]$ (as DVFs).*

Corollary 47. *Let K be a local field of equal characteristic p. Have $k_K \cong \mathbb{F}_q$ for some q a power of p, and $K \cong \mathbb{F}_q((T))$.*

Definition 48. *Let K be a DVF. The **normalised valuation** v_K on K is the unique valuation on K in the given equivalence class s.t. $v_K(\pi) = 1$ for any uniformiser π.*

Lemma 49. *Let R be a ring and let $x \in R$. Assume that R is x-adically complete and that R/xR is perfect of characteristic p.*

Then $\exists!$ map $[-] : R/xR \to R$ such that

$$[a] \equiv a \mod x$$
$$[ab] = [a][b] \ \forall a, b \in R/xR$$

Moreover if R has characteristic p, then $[-]$ is a ring homomorphism.

Proof. Let $a \in R/xR$. $\exists!\ a^{p^{-n}} \in R/xR\ \forall n \geq 0$ since R/xR is perfect. Now lift arbitrarily: take $\alpha_n \in R$ such that $\alpha_n \equiv a^{p^{-n}} \mod x$.

Put $\beta_n = \alpha_n^{p^n}$.

Claim: $\lim_{n\to\infty} \beta_n$ exists and is independent of choices. Call this $[a]$.

Note that if the limit exists no matter how the α_n are chosen, then it is independent of the choices.

Want to prove $\beta_{n+1} - \beta_n \to 0$ x-adically.

$$\beta_{n+1} - \beta_n = (\alpha_{n+1}^p)^{p^n} - (\alpha_n)^{p^n}$$

$$\alpha_{n+1}^p \equiv (a^{p^{-n-1}})^p \equiv a^{p^{-n}} \equiv \alpha_n \mod x$$

The binomial theorem, R/xR characteristic p and induction $\implies$

$$(\alpha_{n+1}^p)^{p^n} \equiv \alpha_n^{p^n} \mod x^{n+1}$$

i.e. $\beta_{n+1} - \beta_n \equiv 0 \mod x^{n+1}$ so $\lim_{n\to\infty} \beta_n$ exists.

Multiplicativity: if $b \in R/xR$, with $\gamma_n \in R$ lifting $b^{p^{-n}}$ $\forall n \geq 0$, then $\alpha_n\gamma_n$ lifts $(ab)^{p^{-n}} = a^{p^{-n}} b^{p^{-n}}$

$$\implies [ab] = \lim_{n\to\infty} \alpha_n^{p^n} \lim_{n\to\infty} \gamma_n^{p^n} = [a][b]$$

$[a] \equiv a \mod x$:

$$\lim_{n\to\infty} \alpha_n^{p^n} \equiv \lim_{n\to\infty} (a^{p^{-n}})^{p^n} \equiv \lim_{n\to\infty} a \equiv a \mod x$$

Uniqueness: let $\phi : R/xR \to R$ be another map with these properties.

$$[a] = \lim_{n\to\infty} \phi(a^{p^{-n}})^{p^n} = \lim_{n\to\infty} \phi(a) = \phi(a)$$

since $\phi(a^{p^{-n}}) \equiv a^{p^{-n}} \mod x$ and ϕ is multiplicative.

Finally, if R has characteristic p, then $\alpha_n + \gamma_n$ lifts $a^{p^{-n}} + b^{p^{-n}} - (a+b)p^{-n}$, so

$$[a+b] = \lim_{n\to\infty} (\alpha_n + \gamma_n)^{p^n} = \lim_{n\to\infty} \alpha_n^{p^n} + \gamma_n^{p^n} = [a] + [b]$$

So $[-]$ is additive and multiplicative and (check!) $[1] = 1$, so it's a homomorphism. $\square$

Definition 50. *$[-] : R/xR \to R$ is called the **Teichmüller map/lift** and $[x]$ is called the **Teichmüller lift/representative** of x.*

Proof of Theorem 48. K is a complete DVF. We want to prove that $\mathcal{O}_K \cong k_K[[T]]$.

$\mathcal{O}_K$ char $p \implies [-] : k_K \hookrightarrow \mathcal{O}_K$ is an injective ring homomorphism.

Choose a uniformiser $\pi \in \mathcal{O}_K$. Then $k_K = \mathcal{O}/\pi\mathcal{O}_K$, $\mathcal{O}_K$ π-adically complete. Now define

$$\begin{aligned} k_K[[T]] &\to \mathcal{O}_K \\ \sum_{n=0}^{\infty} a_n T^n &\mapsto \sum_{n=0}^{\infty} [a_n]\pi^n \end{aligned}$$

It's a bijection by one of the basic properties of complete DVFs, check it's a homomorphism. $\square$

Fact: let F be a field of characteristic p. Then F is perfect $\iff$ every finite extension of F is separable.

$\mathbb{F}_q$ is perfect for every $q = p^n$.

1.4 *Wiff Vectors*

Definition 51. *Let A be a ring. A is called a **strict p-ring** if A is p-torsionfree, p-adically complete and A/pA is perfect.*

Proposition 52. *Let $X = \{x_i \mid i \in I\}$ be a set. Let*

$$\begin{aligned} B &= \mathbb{Z}[x_i^{p^{-\infty}} \mid i \in I] \\ &= \bigcup_{n=0}^{\infty} \mathbb{Z}[x_i^{p^{-n}} \mid i \in I] \end{aligned}$$

(Note that $\mathbb{Z}[x_i \mid i \in I] \subseteq \mathbb{Z}[x_i^{p^{-1}} \mid i \in I] \subseteq \dots$) and let A be the p-adic completion of B. Then A is a strict p-ring, and $A/pA \cong \mathbb{F}_p[x_i^{p^{-\infty}} \mid i \in I]$ (think of as 'universal perfect rings').

Lemma 53. *Let A and B be strict p-rings and let $f : A/pA \to B/pB$ be a ring homomorphism. Then $\exists!$ homomorphism $F : A \to B$ such that $f \equiv F \mod p$.*

F is explicitly given by $F(\sum_{n=0}^{\infty}[a_n]p^n) = \sum_{n=0}^{\infty}[f(a_n)]p^n$.

Theorem 54. *Let R be a perfect ring. Then $\exists!$ (up to isomorphism) strict p-ring $W(R)$ (called the **Wiff vectors** of R) such that $W(R)/pW(R) \cong R$. Moreover, if R' is another perfect ring the reduction mod p map gives a bijection*

$$Hom_{Ring}(W(R), W(R')) \xrightarrow{\sim} Hom_{Ring}(R, R')$$

Proposition 55. *A complete DVR A of mixed characteristic with perfect residue field and such that p is a uniformiser is the same as a strict p-ring A such that A/pA is a field.*

Definition 56. *Let R be a mixed characteristic DVR with normalised valuation v_R. The integer $v_R(p)$ where p is the characteristic of the residue field of R is called the **absolute ramification index** of R.*

Corollary 57. *Let R be a CDVR of mixed characteristic with absolute ramification index 1 and perfect residue field k. Then $R \cong W(k)$.*

Lemma 53'. *Let A be a strict p-ring and let B be a p-adically complete ring. If $f : A/pA \to B/pB$ is a ring homomorphism, then $\exists!$ ring homomorphism $F : A \to B$ with $f \equiv F \mod p$.*

Theorem 58. *Let R be a CDVR of mixed characteristic with perfect residue field k and uniformiser π. Then R is finite over $W(k)$.*

Corollary 59. *Let K be a mixed characteristic local field. Then K is a finite extension of $\mathbb{Q}_p$.*

2 Some p-adic Analysis

Recall the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{x!}$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Proposition 60. *Let K be a complete valued field with absolute value $|\cdot|$, and assume that $K \supseteq \mathbb{Q}_p$, $|\cdot|\,|_{\mathbb{Q}_p} = |\cdot|_p$. Then $\exp(x)$ converges for $|x| < p^{-\frac{1}{p-1}}$ and $\log(1+x)$ converges for $|x| < 1$, and they define continuous maps*

$$\exp : \left\{ x \in K \mid |x| < p^{-\frac{1}{p-1}} \right\} \to \mathcal{O}_K$$

$$\log : \{ x \in K \mid |x| < 1 \} \to K$$

Proof. $v = -\log_p |\cdot|$, this extends v_p.

log: $v(n) \leq \log_p n \implies$

$$v(\frac{x^n}{n}) \geq n \cdot v(x) - \log_p n \to \infty$$

if $v(x) > 0$.

exp: $v(n!) = \frac{n-s_p(n)}{p-1}$. Then

$$v(\frac{x^n}{n!}) \geq n \cdot v(x) - \frac{n}{p-1} = n(v(x) - \frac{1}{p-1}) \geq 0$$

and $\to \infty$ as $n \to \infty$ if $v(x) > \frac{1}{p-1}$.

For continuity, we use uniform convergence as in the real case. $\square$

Lemma 53". *Let A be a strict p-ring, B a ring with element $x \in B$ such that B is x-adically complete and B/xB is perfect of characteristic p. If $f : A/pA \to B/pB$ is a ring homomorphism, then $\exists!$ ring homomorphism $F : A \to B$ with $f \equiv F \mod p$.*

Let $n \geq 1$.

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}$$

is a polynomial in x, and so defines a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$, $x \mapsto \binom{x}{n}$.

Since $\binom{x}{n} \in \mathbb{Z}$ if $x \in \mathbb{Z}_{\geq 0}$, by the density of $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}_p$ we must have $\binom{x}{n} \in \mathbb{Z}_p \forall\ x \in \mathbb{Z}_p$.

When $n = 0$, set $\binom{x}{0} = 1 \forall\ x \in \mathbb{Z}_p$.

2.1 Mahler's Theorem

Theorem 61 (Mahler)**.** *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a continuous function. Then $\exists$ a unique sequence $(a_n)_{n\geq 0}$ with $a_n \in \mathbb{Q}_p$, $a_n \to 0$ such that*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \ \forall x \in \mathbb{Z}_p$$

and $\sup_{x\in\mathbb{Z}_p} |f(x)|_p = \max_{n=0,1,\dots} |a_n|_p$.

Let $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) = \{f : \mathbb{Z}_p \to \mathbb{Q}_p \text{ cts}\}$. This is a $\mathbb{Q}_p$-vector space.

If $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$, set $||f|| = \sup_{x\in\mathbb{Z}_p} |f(x)|_p$. $\mathbb{Z}_p$ compact $\implies f$ is bounded, so the supremum exists and is attained.

Let c_0 denote the set of sequences $(a_n)_{n=0}^{\infty}$ in $\mathbb{Q}_p$ such that $a_n \to 0$. This is a $\mathbb{Q}_p$-vector space, with a norm $||(a_n)|| = \max_{n=0,1,\dots} |a_n|_p$, and c_0 is complete w.r.t $||\cdot||$.

Define $\triangle : \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$ by $\triangle f(x) = f(x+1) - f(x)$. By induction,

$$\triangle^n f(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+n-i)$$

Note that $\triangle$ defines a linear operator on $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$, and

$$|\triangle f(x)|_p = |f(x+1) - f(x)|_p \leq ||f|| \implies ||\triangle f|| \leq ||f|| \text{ or } ||\triangle|| \leq 1$$

Definition 62. *Let $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. The* ***nth Mahler coefficient*** $a_n(f) \in \mathbb{Q}_p$ *is defined by*

$$a_n(f) = \triangle^n f(0) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n-i)$$

Lemma 63. *Let $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. Then $\exists k \geq 1$ such that $\left|\left|\triangle^{p^k} f\right|\right| \leq \frac{1}{p} \, ||f||$.*

Proof. If $f = 0$ there's nothing to prove, so wlog $||f|| = 1$ (by scaling). Then we want to show that $\triangle^{p^k} f(x) \equiv 0 \mod p$ $\forall x \in \mathbb{Z}_p$, some $k \geq 1$.

$$\triangle^{p^k} f(x) = \sum_{i=0}^{p^k} (-1)^i \binom{p^k}{i} f(x + p^k - i) \equiv f(x + p^k) - f(x) \mod p$$

because $\binom{p^k}{i} \equiv 0 \mod p$ for $i = 1, 2, \ldots, p^k - 1$ and $(-1)^{p^k} \equiv -1 \mod p$.

Now $\mathbb{Z}_p$ compact $\implies$ f is uniformly continuous, so $\exists k$ such that $|x - y|_p \leq p^{-k} \implies |f(x) - f(y)|_p \leq \frac{1}{p}$ $\forall x, y \in \mathbb{Z}_p$. Take this k, and we're done. □

Proposition 64. *The map $f \mapsto (a_n(f))_{n=0}^{\infty}$ defines an injective norm-decreasing linear map $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \to c_0$.*

Proof. First we prove that $a_n(f) \to 0$. We have $|a_n(f)|_p \leq ||\triangle^n f||$, so it suffices to prove that $||\triangle^n f|| \to 0$. Since $||\triangle|| \leq 1$, $||\triangle^n f||$ is monotonically decreasing, so it suffices to find a subsequence $\to 0$.

Apply Lemma 63 repeatedly to get $k_1, k_2, \ldots$ such that

$$\left|\left|\triangle^{p^{k_1 + \cdots + k_n}} f\right|\right| \leq \frac{1}{p^n} \, ||f||$$

This gives the desired subsequence.

Note that $|a_n(f)|_p \leq ||\triangle^n f|| \leq ||\triangle||$, so $||(a_n(f))_n|| = \max_{n=0,1,\ldots} |a_n(f)|_p \leq ||f||$, so the map is norm-decreasing. Linearity follows from the linearity of $\triangle$.

Injectivity: assume $a_n(f) = 0$ $\forall n \geq 0$. Then $a_0(f) = f(0) = 0$, and by induction $f(n) = \triangle^n f(0) = a_n(f) = 0$ $\forall n \geq 0$. So $f = 0$ by continuity since $\mathbb{Z}_{\geq 0} \subseteq \mathbb{Z}_p$ is dense. □

We will prove that the linear maps

$$\begin{array}{c} f \mapsto (a_n(f)) \\ \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \rightleftarrows c_0 \\ f_a(x) = \displaystyle\sum_{n=0}^{\infty} a_n \binom{x}{n} \leftarrow\!\shortmid (a_n) = a \end{array}$$

are mutual inverses and norm-preserving.

Lemma 65. *We have $\binom{x}{n} + \binom{x}{n-1} = \binom{x+1}{n}$ $\forall n \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{Z}_p$.*

Proof 1. True when $x \in \mathbb{Z}_{\geq n}$, and then the lemma follows by the density of $\mathbb{Z}_{\geq n} \subset \mathbb{Z}_p$ and continuity. $\square$

Proof 2. True when $x \in \mathbb{Z}_{\geq n}$, and both sides are polynomials which agree on an infinite set of points $\implies$ equal as elements of $\mathbb{Q}[x]$. Now evaluate. $\square$

Now let $a = (a_n)_{n=0}^{\infty} \in c_0$. Define $f_a : \mathbb{Z}_p \to \mathbb{Q}_p$,

$$f_a(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

This is a uniformly convergent series, so $f_a \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$.

Proposition 66. *$a \mapsto f_a$ defines a norm-decreasing linear map $c_0 \to \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. Moreover, $a_n(f_a) = a_n$ $\forall n \geq 0$.*

Proof. Linearity is clear.

Norm decreasing:

$$\begin{aligned} |f_a(x)|_p &= \left| \sum_{n=0}^{\infty} a_n \binom{x}{n} \right| \\ &\leq \sup_n |a_n|_p \left| \binom{x}{n} \right|_p \\ &\leq \sup_n |a_n|_p = ||a|| \;\; \forall x \in \mathbb{Z}_p \end{aligned}$$

$\implies ||f_a|| \leq ||a||$.

Inverses: $\forall k \in \mathbb{Z}_{\geq 0}$ define $a^{(k)} = (a_k, a_{k+1}, a_{k+2}, \dots)$

$$\begin{aligned} \triangle f_a(x) &= f_a(x+1) - f_a(x) \\ &= \sum_{n=1}^{\infty} a_n \left(\binom{x+1}{n} - \binom{x}{n} \right) \\ &= \sum_{n=1}^{\infty} a_n \binom{x}{n-1} \text{ by Lemma 65} \\ &= \sum_{n=0}^{\infty} a_{n+1} \binom{x}{n} = f_{a^{(1)}}(x) \end{aligned}$$

Iterating, $\triangle^k f_a = f_{a^{(k)}} \implies$

$$a_n(f_a) = \triangle^n f_a(0) = f_{a^{(n)}}(0) = a_n$$

$\square$

Summing up:

$$F(f) = (a_n(f))$$
$$V = \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \underset{G}{\overset{F}{\rightleftarrows}} c_0 = W$$
$$G(a) = f_a$$

We know: F is injective and norm-decreasing, $FG = id_W$ and G is norm-decreasing.

Lemma 67. *In this situation, $GF = \mathrm{id}_V$ and F and G are norm-preserving.*

Proof. Let $v \in V$. Then $F(v - GFv) = Fv - Fv = 0 \implies v = GFv$ since F is injective. So $GF = \mathrm{id}_V$.

Norm-preserving: $v \in V$, have $||Fv|| \leq ||v||$, but also $||Fv|| \geq ||GFv|| = ||v||$, so F is norm preserving. Same proof for G. $\square$

This finishes the proof of Mahler's Theorem.

3 Ramification Theory for Local Fields

The characteristic of the residue field of any local field from now on will be p (unless stated otherwise).

3.1 More on Finite Extensions

Recall: let R be a PID and let M be a f.g. R-module. Assume that M is torsion free. Then $\exists! n \geq 0$ such that $M \cong R^n$. Moreover, if $N \subseteq M$ is a submodule, then N is finitely generated and $N \cong R^m$, with $m \leq n$.

Proposition 68. *Let K be a local field, L/K finite of degree n. Then $\mathcal{O}_L$ is a finite, free $\mathcal{O}_K$-module of rank n (i.e. $\mathcal{O}_L \cong \mathcal{O}_K^n$ as $\mathcal{O}_K$-modules), and k_L/k_K is an extension of degree $\leq n$. Moreover, L is a local field.*

Proof. Choose a K-basis $\alpha_1, \ldots, \alpha_n$ of L. Let $||\cdot||$ denote the maximum norm $||\sum_{i=1}^n x_i\alpha_i|| = \max_{i=1,\ldots,n} |x_i|$ on L as in Proposition 33. $||\cdot||$ is equivalent to $|\cdot|$ (the extended absolute value on L) as K-norms, so $\exists r > s > 0$ such that

$$M = \{x \in L \mid ||x|| \leq s\} \subseteq \mathcal{O}_L \subseteq N = \{x \in L \mid ||x|| \leq r\}$$

Increasing r and decreasing s as necessary wlog $r = |a|$, $s = |b|$ for some $a, b \in K^\times$. Then

$$M = \bigoplus_{i=1}^n \mathcal{O}_K b\alpha_i \subseteq \mathcal{O}_L \subseteq N = \bigoplus_{i=1}^n \mathcal{O}_K a\alpha_i$$

$\implies \mathcal{O}_L$ is f.g. and free of rank n over $\mathcal{O}_K$.

Since $\mathfrak{m}_K = \mathfrak{m}_L \cap \mathcal{O}_K$, we have a natural injection

$$k_K = \mathcal{O}_K/\mathfrak{m}_K \hookrightarrow \mathcal{O}_L/\mathfrak{m}_L = k_L$$

Since $\mathcal{O}_L$ is generated over $\mathcal{O}_K$ by n elements, k_L is generated by n elements over k_K, i.e. $[k_L : k_K] \leq n$.

L a local field: k_L/k_K is finite and k_K finite $\implies$ k_L is a finite field. L is complete by Theorem 29.

Let v_K be the normalised valuation on K, w the extension of v_K to L. Then $w(\alpha) = \frac{1}{n} v_K(N_{L/K}(\alpha))$, so

$$w(L^\times) \subseteq \frac{1}{n} v(K^\times) = \frac{1}{n}\mathbb{Z}$$

$\implies$ it's discrete. $\square$

Definition 69. *Let L/K be a finite extension of local fields. The* ***inertia degree*** *of L/K is*

$$f_{L/K} = [k_L : k_K]$$

Let v_L be the normalised valuation on L and π_K a uniformiser of K. The integer

$$e_{L/K} = v_L(\pi_K)$$

is called the ***ramification index*** *of L/K.*

Theorem 70. *Let L/K be a finite extension of local fields. Then $[L : K] = e_{L/K} f_{L/K}$ and $\exists \alpha \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.*

Proof. Write $e = e_{L/K}$, $f = f_{L/K}$.

k_L/k_K is separable, so $\exists \bar{\alpha} \in k_L$ such that $k_L = k_K(\bar{\alpha})$. Let $\bar{f}(x) \in k_K[x]$ be the minimal polynomial of $\bar{\alpha}$ over k_K, and let $f \in \mathcal{O}_K[x]$ be a monic lift of $\bar{f}$ with $\deg f = \deg \bar{f}$.

Claim: $\exists \alpha \in \mathcal{O}_L$ lifting $\bar{\alpha}$ and such that $v_L(f(\alpha)) = 1$ (always ≥ 1).

Let $\beta \in \mathcal{O}_L$ be any lift of $\bar{\alpha}$. If $v(f(\beta)) = 1$, then set $\alpha = \beta$. If not, set $\alpha = \beta + \pi_L$ (π_L uniformiser of L).

$f(\alpha) = f(\beta + \pi_L) = f(\beta) + f'(\beta)\pi_L + b\pi_L^2$ for some $b \in \mathcal{O}_L$ (Taylor expanding around β).

Since $v_L(f(\beta)) \geq 2$ and $v_L(f'(\beta)) = 0$, we have $v_L(f(\alpha)) = 1$. Put $\pi = f(\alpha)$ (uniformiser of L).

We claim that $\alpha^i \pi^j$, $i = 0, \dots, f-1$, $j = 0, \dots, e-1$ are an $\mathcal{O}_K$-basis of $\mathcal{O}_L$.

Linear independence: assume $\sum_{i,j} a_{ij}\alpha^i\pi^j = 0$ for some $a_{ij} \in K$, not all 0. Put $s_j = \sum_{i=0}^{f-1} a_{ij}\alpha^i$ $\forall j$. $1, \alpha, \dots, \alpha^{f-1}$ are linearly independent over K since there reductions are linearly independent over k_K. So $\exists j$ such that $s_j \neq 0$.

Claim: $e|v_L(s_j)$ if $s_j \neq 0$.

Let k be such that $|a_{kj}|$ is maximal, then $a_{kj}^{-1}s_j = \sum_{i=0}^{f-1} a_{kj}^{-1}a_{ij}\alpha^i \implies a_{kj}^{-1}s_k \not\equiv 0 \mod \pi_L$ because $1, \bar{\alpha}, \ldots, \bar{\alpha}^{f-1}$ are linearly independent over k_K.

$$\begin{aligned} \implies v_L(a_{kj}^{-1}s_j) = 0 \implies v_L(s_j) = v_L(a_{kj}) &= v_L(a_{kj}^{-1}s_j) \\ &\in v_L(K^\times) \\ &= ev_L(L^\times) = e\mathbb{Z} \end{aligned}$$

Now write $\sum_{i,j} a_{ij}\alpha^i\pi^j = \sum_{j=0}^{e-1} s_j\pi^j = 0$. If $s_j \neq 0$, we have $v_L(s_j\pi^j) = v_L(s_j) + j \in j + e\mathbb{Z}$.

$\implies$ no two non-zero terms in $\sum_{j=0}^{e-1} s_j\pi^j$ have the same valuation.

$\implies \sum_{j=0}^{e-1} s_j\pi^j \neq 0$, which is a contradiction.

Claim $\mathcal{O}_L = \oplus_{i,j}\alpha^i\pi^j$.

Set $M = \oplus_{i,j}\alpha^i\pi^j$ and $N = \oplus_{i=0}^{f-1}\mathcal{O}_K\alpha^i$. Then $M = N + \pi + N + \cdots + \pi^{e-1}N$. Since $1, \bar{\alpha}, \ldots, \bar{\alpha}^{f-1}$ span k_L over k_K we must have $\mathcal{O}_L = N + \pi\mathcal{O}_L$.

$$\begin{aligned} \text{Iterate: } \mathcal{O}_L &= N + \pi(N + \pi\mathcal{O}_L) \\ &= N + \pi N + \pi^2\mathcal{O}_L \\ &= \ldots \\ &= N + \pi N + \cdots + \pi^{e-1}N + \pi^e\mathcal{O}_L \\ &= M + \pi_K\mathcal{O}_L \ (\pi_K \text{ uniformiser of } K) \end{aligned}$$

Iterate: $\mathcal{O}_L = M + \pi_K^n\mathcal{O}_L \ \forall n \geq 1 \implies M$ is dense in $\mathcal{O}_L$. But M is the closed unit ball in $V = \oplus_{ij}K\alpha^i\pi^j \subseteq L$ w.r.t the maximum norm on V w.r.t the basis $\alpha^i\pi^j$.

Proposition 33 and Theorem 29 $\implies M$ is complete both w.r.t the maximum norm and $|\cdot|$ on L.

$\implies M \subseteq L$ is closed.

$\implies M = \mathcal{O}_L$.

Finally, since $\alpha^i\pi^j = \alpha^i f(\alpha)^j$ is a polynomial in α, have $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. □

Corollary 71. *Let $M/L/K$ be finite extensions of local fields. Then $f_{M/K} = f_{L/K}f_{M/L}$ and $e_{M/K} = e_{L/K}e_{M/L}$.*

Proof. $[k_M : k_K] = [k_M : k_L][k_L : k_K]$ by multiplicativity of degrees.

$e_{M/L}e_{L/K} = \frac{[M:L]}{f_{M/L}}\frac{[L:K]}{f_{L/K}} = \frac{[M:K]}{f_{M/K}} = e_{M/K}$. □

Definition 72. *Let L/K be a finite extension of local fields. L/K is **unramified** if $e_{L/K} = 1$ (or $f_{L/K} = [L : K]$), and **totally ramified** if $fL/K = 1$.*

Theorem 73. *Let K be a local field. For each finite extension l/k_K there is a* ***unique*** *(up to isomorphism) finite unramified extension L/K with $k_L \cong l$ over k_K.*

Moreover, L/K is Galois with $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(l/k_K)$.

Proof. Existence: let $\bar{\alpha}$ be a primitive element of l/k_K with minimal polynomial $\bar{f} \in k_K[x]$. Take a monic lift $f \in \mathcal{O}_K[x]$ of $\bar{f}$ ($\deg f = \deg \bar{f}$).

Put $L = K(\alpha)$ where α is a root of f. $\bar{f}$ irreducible $\implies$ f irreducible $\implies$ $[L : K] = [l : k_K]$.

Moreover, k_L contains a root of $\bar{f}$ (the reduction of α). So $l \hookrightarrow k_L$ over $k_K \implies [L : K] \geq [k_L : k_K] = [L : K]$.

$\implies$ L/K is unramified and $k_L \cong l$ over k_K. ☐

Uniqueness and Galois property follows from:

Lemma 74. *Let L/K be a finite unramified extension of local fields and let M/K be a finite extension. Then there is a natural bijection*

$$\mathrm{Hom}_{K-alg}(L, M) \xrightarrow{\sim} \mathrm{Hom}_{k_K-alg}(k_L, k_M)$$

($\varphi : L \to M$ restricts to $\varphi : \mathcal{O}_L \to \mathcal{O}_M$, then take reductions).

Proof. By uniqueness of extended absolute values (Theorem 29) any K-algebra homomorphism $\phi : L \to M$ is an isometry for the extended absolute values.

Thus $\varphi(\mathcal{O}_L) \subseteq \mathcal{O}_M$, $\varphi(\mathfrak{m}_L) \subseteq \varphi(\mathfrak{m}_M)$ so we get the induced k_K-algebra homomorphism $\bar{\varphi} : k_L \to k_M$. This gives

$$\mathrm{Hom}_{K-alg}(L, M) \to \mathrm{Hom}_{k_K-alg}(k_L, k_M)$$

Bijectivity: let $\bar{\alpha} \in k_L$ be a primitive element over k_K, $\bar{f} \in k_K[x]$ its minimal polynomial, $f \in \mathcal{O}_K[x]$ a monic lift of $\bar{f}$ and $\alpha \in \mathcal{O}_L$ the unique root of f which lifts to $\bar{\alpha}$ (Hensel's Lemma).

Then $k_L = k_L(\bar{\alpha})$ and $L = K(\alpha)$.

$$\begin{tikzcd}
\varphi \arrow[d, mapsto] & \mathrm{Hom}_{K-alg}(L, M) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}_{k_K}(k_L, k_M) \arrow[d, "\wr"'] & \hat{\varphi} \arrow[d, mapsto] \\
\varphi(\alpha) & \{x \in M \mid f(x) = 0\} \arrow[r] & \{\bar{x} \in k_M \mid \bar{f}(\bar{x}) = 0\} & \bar{\varphi}(\bar{\alpha})
\end{tikzcd}$$

This is a bijection by Hensel's Lemma, since $\bar{f}$ is seperable. ☐

Proof of 73 cont. Uniqueness: $k_L \cong k_M$ over k_K, L/K, M/K unramified. Then $\bar{\phi}$ lifts to a K-embedding $\phi : L \hookrightarrow M$ and $[L : K] = [M : K] \implies \phi$ an isomorphism.

Galois: $|\mathrm{Aut}_K(L)| = |\mathrm{Aut}_{k_K}(k_L)| = [k_L : k_K] = [L : K] \implies L/K$ Galois.

Also, $\mathrm{Aut}_K(L) \to \mathrm{Aut}_{k_K}(k_L)$ is really a homomorphism (so an isomorphism). ☐

Proposition 75. *Let K be a local field, L/K finite unramified, M/K finite. Say $L, M \subset$ fixed algebraic closure $\bar{K}$ of K. Then LM/M is unramified. Any subextension of L/K is unramified over K. If M/K is unramified, then LM/K is unramified.*

Proof. Let $\hat{\alpha}$ be a primitive element of k_L/k_K, $\bar{f} \in k_K[x]$ the minimal polynomial of $\hat{\alpha}$, $f \in \mathcal{O}_K[x]$ a monic lift of $\bar{f}$, $\alpha \in \mathcal{O}_L$ the unique root of f lifting $\hat{\alpha}$. Then $L = K(\alpha)$ so $LM = M(\alpha)$.

Let $\bar{g}$ be the minimal polynomial of $\bar{\alpha}$ over k_M. Then $\bar{g}|\bar{f} \implies f = gh$ in $\mathcal{O}_M[x]$ by Hensel's Lemma. g monic, lifts $\bar{g} \implies g(\alpha) = 0$ and g irreducible in $M[x]$.

So g is the minimal polynomial of α over $M \implies$

$$[LM : M] = \deg g = \deg \bar{g} \le [k_{LM} : k_M] \le [LM : M]$$

$\implies$ have equalities, LM/M unramified.

The second claim follows from the multiplicativity of $f_{L/K}$ and $e_{L/K}$ (Corollary 71), as does the third ($[LM : K] = [LM : M][M : K] = f_{LM/M} f_{M/K} = f_{LM/K} \implies LM/K$ unramified). ☐

Corollary 76. *Let K be a local field, L/K finite. Then $\exists$ a unique maximal subfield $K \subseteq T \subseteq L$ such that T/K is unramified. Moreover, $[T : K] = f_{L/K}$.*

Proof. Existence: T is the composite of all unramified subextensions of L/K (use Proposition 75).

Have $[T : K] = f_{T/K} \le f_{L/K}$ by Corollary 71.

Let T'/K be the unique unramified extension with residue field extension k_L/k_K. Then $id : k_{T'} = k_L \to k_L$ lifts to a K-embedding $T' \overset{\varphi}{\hookrightarrow} L$, by Lemma 74.

Then $[T : K] \ge [\varphi(T') : K] = f_{L/K} \implies [T : K] = f_{L/K}$. ☐

3.2 Totally Ramified Extensions

Recall

Theorem 77 (Eisenstein's Criterion)**.** *Let K be a local field, $f(x) = x^n + \cdots + a_0 \in \mathcal{O}_K[x]$, π_K uniformiser of K. If $\pi_K | a_{n-1}, \ldots, a_0$ and $\pi_K^2 \nmid a_0$, then f is irreducible.*

Note that if L/K finite, v_K a normalised valuation on K and w the unique extension of v_K to L. Then $e_{L/K}^{-1} = w(\pi_L) = \min_{x\in\mathfrak{m}_L} w(x)$.

A polynomial $f(x) \in \mathcal{O}_K[x]$ satisfying the assumptions of Eisenstein's criterion is called an **Eisenstein polynomial**.

Proposition 78. *Let L/K be a totally ramified extension of local fields. Then $L = K(\pi_L)$ and the minimal polynomial of π_L over K is Eisenstein.*

Conversely, if $L = K(\alpha)$ and the minimal polynomial of α over K is Eisenstein, then L/K is totally ramified and α is a uniformiser of L.

Proof. First part: $n = [L : K]$, v_K a normalised valuation on K and w the unique extension of v_K to L. Then

$$[K(\pi_L) : K]^{-1} \leq e_{K(\pi_L)/K}^{-1} = \min_{x\in\mathfrak{m}_{K(\pi_L)}} w(x) \leq \frac{1}{n}$$

$\implies [K(\pi_L) : K] \geq [L : K] \implies L = K(\pi)$.

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathcal{O}_K[x]$ be the minimal polynomial of π_L over K.

$$\pi_L^n = -(a_0 + a_1(\pi_L) + \cdots + a_{n-1}\pi_L^{n-1})$$

So $1 = w(\pi_L^n) = w(a_0 + a_1\pi_L + \cdots + a_{n-1}\pi_L^{n-1}) = \min_{i=0,1,\ldots,n-1}(v_K(a_i) + \frac{i}{n})$ $\implies v_K(a_i) \geq 1\ \forall i$ and $v_K(a_0) = 1$, so f is Eisenstein.

Converse: $L = K(\alpha)$, $n = [L : K]$. Let $g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathcal{O}_K[x]$ be the minimal polynomial of α. g irreducible $\implies$ all roots have the same valuation, so

$$1 = w(b_0) = n \cdot w(\alpha) \implies w(\alpha) = \frac{1}{n}$$

$\implies e_{L/K}^{-1} = \min_{x\in\mathfrak{M}_L} w(x) \leq \frac{1}{n} = [L : K]^{-1}$

$\implies [L : K] = e_{L/K} = n$, so L/K is totally ramified and α is a uniformiser.

□