

# Part III Combinatorics

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## 1 Introduction

Let  $X, Y, \dots$  be sets

**Definition.** We call  $\mathcal{A} \subset \mathcal{P}(X)$  a **set system** or **family of sets**.  $\mathcal{A}$  is naturally identified with a bipartite graph  $G_{\mathcal{A}}(U, W)$  with  $U = \mathcal{A}$ ,  $W = \bigcup_{A \in \mathcal{A}} A$  or  $W = X$ . Indeed,  $Ax \in E(G_{\mathcal{A}}) \iff x \in A$ .

**Definition.** Given  $\mathcal{A} \in \mathcal{P}(X)$ , a **set of distinct representatives (SDR)** is an injection  $f : \mathcal{A} \rightarrow X$  s.t.  $f(A) \in A \forall A \in \mathcal{A}$ . In its bipartite graph, an SDR corresponds to a complete matching  $U \rightarrow W$ .

**Theorem 1** (Hall, 1935). A set system  $\mathcal{A}$  has an SDR if  $\forall \mathcal{A}' \subset \mathcal{A}$ ,  $|\bigcup_{A \in \mathcal{A}'} A| \geq |\mathcal{A}'|$ .

**Theorem 1'.** A bipartite graph  $G(U, W)$  has a complete matching  $U \rightarrow W$  if  $\forall S \subset U$ ,  $|\Gamma(S)| \geq |S|$

**Corollary 2.** Suppose  $G(U, W)$  bipartite,  $d(u) \geq d(w) \forall u \in U, w \in W$ . Then  $\exists$  a complete matching  $U \rightarrow W$ .

**Definition.** A bipartite graph  $G(U, W)$  is  **$(r, s)$ -regular** if  $d(u) = r$  and  $d(w) = s \forall u \in U, w \in W$ .

Instant from Cor 2: if  $G(U, W)$  is  $(r, s)$ -regular then  $\exists$  a complete matching from  $U$  to  $W$  if  $|U| \leq |W|$ .

**Corollary 3.** Let  $0 \leq i, j \leq n$ ,  $\binom{n}{i} \leq \binom{n}{j}$ . Then  $\exists$  a complete matching  $f : [n]^{(i)} \rightarrow [n]^{(j)}$  s.t.  $f(A) \subset A$  if  $j \leq i$ , and  $f(A) \supset A$  if  $i \leq j$ .

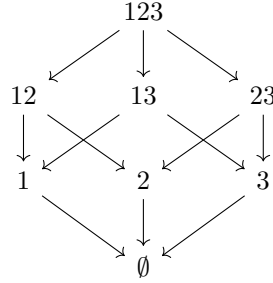
**Theorem 4.** Let  $G = G(U, W)$  be a connected  $(r, s)$ -regular graph. Then for  $\emptyset \neq A \subset U$ ,

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

Also, equality holds iff  $A = U$ .

The **cube**  $Q^n \cong \mathcal{P}(n) \cong [2]^n$  = set of all 0, 1 sequences of length  $n$ .  $Q^n$  is also a graph:  $AB$  is an edge if  $|A \triangle B| = 1$ . It is also a poset:  $A < B$  if  $A \subset B$ .

$Q^n$  has a natural orientation:  $\overrightarrow{AB}$  if  $A = B \cup \{a\}$ .



The order on  $Q^n \cong \mathcal{P}(n)$  is induced by this oriented graph.

## 2 Sperner Systems

**Definition.** A set system  $\mathcal{A} \subset \mathcal{P}(n)$  is **Sperner** if  $A, B \in \mathcal{A}$ ,  $A \neq B \implies A \not\subset B$

**Theorem 1** (Sperner, 1928). If  $\mathcal{A} \subset \mathcal{P}(n)$  is Sperner then

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

**Definition.** The **weight**  $w(A)$  of a set  $A \in \mathcal{P}(n)$  is  $w(A) = \frac{1}{\binom{n}{|A|}}$

**Theorem 2.** Let  $\mathcal{A}$  be a Sperner system on  $X$ ,  $|X| = n$ . Then

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A) \leq 1$$

**Corollary 3.** If  $\mathcal{A} \subset \mathcal{P}(n)$  is a Sperner system then  $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ , with equality  $\iff \mathcal{A}$  is  $X^{[n/2]}$  or  $X^{[n/2]}$ .

**Definition.**  $\mathcal{A} \subset \mathcal{P}(n)$  is **k-Sperner** if it does not contain

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{k+1}$$

Note that Sperner = 1-Sperner.

**Corollary 4** (Erdős, 1945). *If  $\mathcal{A} \subset \mathcal{P}(n)$  is  $k$ -Sperner then  $|\mathcal{A}|$  is at most the sum of the  $k$  largest binomial coefficients.*

**Theorem 5** (Erdős, 1945). *Let  $x_1, \dots, x_n \in \mathbb{R}$ ,  $x_i \geq 1$ . Then the number of sums  $\sum_1^n \pm x_i$  in an open interval  $J$  of length  $2k$  is at most the sum of the  $k$  largest binomial coefficients.*

**Definition.** *A chain  $A_o \subset A_1 \subset \dots \subset A_k$  is **symmetric** if  $|A_{i+1}| = |A_i| + 1 \ \forall i$  and  $|A_o| + |A_k| = n$ .*

**Theorem 6** (Kleitman and Katona).  *$\mathcal{P}(n)$  has a decomposition into symmetric chains*