

Part III Category Theory

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Contents

1	Definitions and Examples	1
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1 Definitions and Examples

Definition (Category). A category \mathcal{C} consists of

- a collection $\text{ob } \mathcal{C}$ of **objects** A, B, C, \dots
- a collection $\text{mor } \mathcal{C}$ of **morphisms** f, g, h, \dots
- two operations dom, cod from morphisms to objects. We write $f : A \rightarrow B$ or $A \xrightarrow{f} B$ to mean ' f is a morphism and $\text{dom } f = A$ and $\text{cod } f = B$ '
- an operation assigning to each object A a morphism $1_A : A \rightarrow A$
- a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff \text{dom } g = \text{cod } f$, and then $gf : \text{dom } f \rightarrow \text{cod } g$

Definition (Functor). Let \mathcal{C} and \mathcal{D} be categories. A **functor** $\mathcal{C} \rightarrow \mathcal{D}$ consists of

- a mapping $A \rightarrow FA$ from $\text{ob } \mathcal{C}$ to $\text{ob } \mathcal{D}$
- a mapping $f \rightarrow Ff$ from $\text{mor } \mathcal{C}$ to $\text{mor } \mathcal{D}$

satisfying $\text{dom } Ff = F\text{dom } f$, $\text{cod } Ff = F\text{cod } f$ for all f , $F(1_A) = 1_{FA}$ for all A , and $F(gf) = (Fg)(Ff)$ whenever gf is defined.

Definition. By a **contravariant functor** $\mathcal{C} \rightarrow \mathcal{D}$ we mean a functor $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$ (or equivalently $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$). A functor $\mathcal{C} \rightarrow \mathcal{D}$ is sometimes said to be **covariant**.

Definition (Natural transformation). Let \mathcal{C} and \mathcal{D} be two categories and $F, G : \mathcal{C} \Rightarrow \mathcal{D}$ two functors. A **natural transformation** $\alpha : F \rightarrow G$ assigns to each $A \in \text{ob } \mathcal{C}$ a morphism $\alpha_A : FA \rightarrow GA$ in \mathcal{D} , such that

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A -component of a natural transformation $\beta \alpha : F \rightarrow H$.

Definition. Given categories \mathcal{C}, \mathcal{D} , we write $[\mathcal{C}, \mathcal{D}]$ for the category of all functors $\mathcal{C} \rightarrow \mathcal{D}$ and natural transformations between them.

Lemma 1. Given $F, G : \mathcal{C} \rightarrow \mathcal{D}$ and $\alpha : F \rightarrow G$, α is an isomorphism in $[\mathcal{C}, \mathcal{D}] \iff$ each α_A is an isomorphism in \mathcal{D} .

Definition (Faithful and full). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- a. We say that F is **faithful** if, given $f, g \in \text{mor } \mathcal{C}$, the equations $\text{dom } f = \text{dom } g$, $\text{cod } f = \text{cod } g$ and $Ff = Fg$ imply $f = g$.
- b. F is **full** if, given any $g : FA \rightarrow FB$ in \mathcal{D} , there exists $f : A \rightarrow B$ in \mathcal{C} with $Ff = g$.
- c. We say a subcategory \mathcal{C}' of \mathcal{C} is **full** if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition (Equivalence of categories). Let \mathcal{C} and \mathcal{D} be categories. An **equivalence** between \mathcal{C} and \mathcal{D} is a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$, $\beta : FG \rightarrow 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to mean that \mathcal{C} and \mathcal{D} are equivalent.

We say a property P of categories is **categorical** if whenever \mathcal{C} has P and $\mathcal{C} \simeq \mathcal{D}$ then \mathcal{D} has P .

For example, being a groupoid is a categorical property, but being a group is not.

Definition (Slice category). Given an object B of a category \mathcal{C} , define the **slice category** \mathcal{C}/B to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \rightarrow (A' \xrightarrow{f'} B)$ are morphisms $h : A \rightarrow A'$ making

$$\begin{array}{ccc}
 A & \xrightarrow{h} & A' \\
 & \searrow f & \swarrow f' \\
 & B &
 \end{array}$$

commute.