

# Part III Category Theory

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## 1 Definitions and Examples

**Definition 1.1** (Category). *A category  $\mathcal{C}$  consists of*

- a. a collection  $\text{ob } \mathcal{C}$  of **objects**  $A, B, C, \dots$*
- b. a collection  $\text{mor } \mathcal{C}$  of **morphisms**  $f, g, h, \dots$*
- c. two operations  $\text{dom}, \text{cod}$  from morphisms to objects. We write  $f : A \rightarrow B$  or  $A \xrightarrow{f} B$  to mean ' $f$  is a morphism and  $\text{dom } f = A$  and  $\text{cod } f = B$ '*
- d. an operation assigning to each object  $A$  a morphism  $1_A : A \rightarrow A$*
- e. a partial binary operation  $(f, g) \mapsto gf$ , s.t.  $gf$  is defined  $\iff \text{dom } g = \text{cod } f$ , and then  $gf : \text{dom } f \rightarrow \text{cod } g$*

*satisfying*

- f.  $f1_A = f$  and  $1_B f = f \ \forall f : A \rightarrow B$*
- g.  $h(fg) = (hg)f$  whenever  $gf$  and  $hg$  are defined*

**Definition 1.2** (Functor). *Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A **functor**  $\mathcal{C} \rightarrow \mathcal{D}$  consists of*

a. a mapping  $A \rightarrow FA$  from  $\text{ob } \mathcal{C}$  to  $\text{ob } \mathcal{D}$

b. a mapping  $f \rightarrow Ff$  from  $\text{mor } \mathcal{C}$  to  $\text{mor } \mathcal{D}$

satisfying  $\text{dom } Ff = F\text{dom } f$ ,  $\text{cod } Ff = F\text{cod } f$  for all  $f$ ,  $F(1_A) = 1_{FA}$  for all  $A$ , and  $F(gf) = (Fg)(Ff)$  whenever  $gf$  is defined.

**Definition 1.3.** By a **contravariant functor**  $\mathcal{C} \rightarrow \mathcal{D}$  we mean a functor  $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$  (or equivalently  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ ). A functor  $\mathcal{C} \rightarrow \mathcal{D}$  is sometimes said to be **covariant**.

**Definition 1.4** (Natural transformation). Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories and  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  two functors. A **natural transformation**  $\alpha : F \rightarrow G$  assigns to each  $A \in \text{ob } \mathcal{C}$  a morphism  $\alpha_A : FA \rightarrow GA$  in  $\mathcal{D}$ , such that

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given  $\alpha : F \rightarrow G$  and  $\beta : G \rightarrow H$ , the mapping  $A \mapsto \beta_A \alpha_A$  is the  $A$ -component of a natural transformation  $\beta\alpha : F \rightarrow H$ .

**Definition 1.5.** Given categories  $\mathcal{C}, \mathcal{D}$ , we write  $[\mathcal{C}, \mathcal{D}]$  for the category of all functors  $\mathcal{C} \rightarrow \mathcal{D}$  and natural transformations between them.

**Lemma 1.6.** Given  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : F \rightarrow G$ ,  $\alpha$  is an isomorphism in  $[\mathcal{C}, \mathcal{D}] \iff$  each  $\alpha_A$  is an isomorphism in  $\mathcal{D}$ .

**Definition 1.7** (Faithful and full). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

- a. We say that  $F$  is **faithful** if, given  $f, g \in \text{mor } \mathcal{C}$ , the equations  $\text{dom } f = \text{dom } g$ ,  $\text{cod } f = \text{cod } g$  and  $Ff = Fg$  imply  $f = g$ .
- b.  $F$  is **full** if, given any  $g : FA \rightarrow FB$  in  $\mathcal{D}$ , there exists  $f : A \rightarrow B$  in  $\mathcal{C}$  with  $Ff = g$ .
- c. We say a subcategory  $\mathcal{C}'$  of  $\mathcal{C}$  is **full** if the inclusion  $\mathcal{C}' \hookrightarrow \mathcal{C}$  is a full functor.

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

**Definition 1.8** (Equivalence of categories). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. An **equivalence** between  $\mathcal{C}$  and  $\mathcal{D}$  is a pair of functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  together with natural isomorphisms  $\alpha : 1_{\mathcal{C}} \rightarrow GF$ ,  $\beta : FG \rightarrow 1_{\mathcal{D}}$ . We write  $\mathcal{C} \simeq \mathcal{D}$  to mean that  $\mathcal{C}$  and  $\mathcal{D}$  are equivalent.

We say a property  $P$  of categories is **categorical** if whenever  $\mathcal{C}$  has  $P$  and  $\mathcal{C} \simeq \mathcal{D}$  then  $\mathcal{D}$  has  $P$ .

For example, being a groupoid is a categorical property, but being a group is not.

**Definition 1.9** (Slice category). Given an object  $B$  of a category  $\mathcal{C}$ , define the **slice category**  $\mathcal{C}/B$  to have morphisms  $A \xrightarrow{f} B$  as objects, and morphisms  $(A \xrightarrow{f} B) \rightarrow (A' \xrightarrow{f'} B)$  are morphisms  $h : A \rightarrow A'$  making

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ & \searrow f & \swarrow f' \\ & B & \end{array}$$

commute.

**Lemma 1.10.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then  $F$  is part of an equivalence  $\mathcal{C} \simeq \mathcal{D} \iff F$  is full, faithful and **essentially surjective**, i.e. for every  $B \in \text{ob } \mathcal{D}$ , there exists  $A \in \text{ob } \mathcal{C}$  s.t.  $FA \cong B$ .

**Definition 1.11.** a. A **skeleton** of a category  $\mathcal{C}$  is a full subcategory  $\mathcal{C}'$  containing exactly one object from each isomorphism class of objects of  $\mathcal{C}$ .

b. We say  $\mathcal{C}$  is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism  $f$  in  $\mathcal{C}$  satisfies  $\text{dom } f = \text{cod } f$ .

For example,  $\mathbf{Mat}_K$  is skeletal. The full subcategory of standard vector spaces  $K^n$  is a skeleton of  $\mathbf{fd Mod}_K$ .

**Remark 1.12.** The following statements are each equivalent to the Axiom of Choice:

1. Every small category has a skeleton
2. Any small category is equivalent to each of its skeletons
3. Any two skeletons of a given small category are isomorphic

**Definition 1.13.** Let  $f : A \rightarrow B$  be a morphism in a category  $\mathcal{C}$ .

- a.  $f$  is a **monomorphism** if, given  $g, h : D \rightrightarrows A$ , the equation  $fg = fh$  implies  $g = h$ . We write  $A \twoheadrightarrow B$  if  $f$  is monic.

- b. Dually,  $f$  is an **epimorphism** if, given  $k, l : B \rightrightarrows C$ ,  $kf = lf$  implies  $k = l$ . We write  $A \twoheadrightarrow B$  if  $f$  is epic.
- c.  $\mathcal{C}$  is a **balanced** category if every  $f \in \text{mor } \mathcal{C}$  which is both monic and epic is an isomorphism.

## 2 The Yoneda Lemma

**Definition 2.1.** A category  $\mathcal{C}$  is **locally small** if, for any two objects  $A, B$  of  $\mathcal{C}$ , the morphism  $A \rightarrow B$  are parametrised by a set  $\mathcal{C}(A, B)$ .

Given local smallness,  $B \mapsto \mathcal{C}(A, B)$  becomes a functor  $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ : if  $g : B \rightarrow B'$ , the mapping  $f \mapsto gf : \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, B')$  is functorial since  $h(gf) = (hg)f$  for any  $h : B' \rightarrow B''$ .

Similarly,  $A \mapsto \mathcal{C}(A, B)$  becomes a functor  $\mathcal{C}^{op} \rightarrow \mathbf{Set}$ .

**Lemma 2.2** (Yoneda). Let  $\mathcal{C}$  be a locally small category,  $A \in \text{ob } \mathcal{C}$  and  $F : \mathcal{C} \rightarrow \mathbf{Set}$ . Then

- i. There is a bijection between natural transformations  $\mathcal{C}(A, -) \rightarrow F$  and elements of  $FA$ .
- ii. Moreover, this bijection is natural in both  $A$  and  $F$ .

*Proof.* Bijection: given  $\alpha : \mathcal{C}(A, -) \rightarrow F$ , define  $\Phi(\alpha) = \alpha_A(1_A) \in FA$ .

Given  $x \in FA$ , define  $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$  by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$  is natural: given  $g : B \rightarrow C$ , we have

$$\begin{aligned} \Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\ &= F(gf)(x) \\ &= (Fg)(Ff)(x) \\ &= (Fg)\Psi(x)_B(f) \end{aligned}$$

$\Phi\Psi(x) = x$  since  $F(1_A)(x) = x$ , and  $\Psi\Phi(\alpha) = \alpha$  since, for any  $f : A \rightarrow B$ ,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\ &= Ff(\alpha_A(1_A)) \\ &= \alpha_B(\mathcal{C}(A, f)(1_A)) \\ &= \alpha_B(f) \end{aligned}$$

□

**Corollary 2.3.** *The mapping  $A \rightarrow \mathcal{C}(A, -)$  is a full and faithful functor  $\mathcal{C}^{op} \rightarrow [\mathcal{C}, \mathbf{Set}]$ .*

*Proof.* Given two objects  $A, B$ , 2.2(i) gives us a bijection from  $\mathcal{C}(B, A)$  to the collection of natural transformations  $\mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$  (by taking  $F : C \mapsto \mathcal{C}(B, C)$ ). We need to show this is functorial, but given  $f \in \mathcal{C}(B, A)$ ,  $\Psi(F)_A$  sends  $1_A$  to  $\mathcal{C}(B, f)(1_A) = f$ , so it's the natural transformation  $g \mapsto gf$ .

Hence, given  $e : C \rightarrow B$ ,  $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$   $\square$

We call this functor the **Yoneda embedding**. Hence any locally small category  $\mathcal{C}$  is equivalent to a full subcategory of  $[\mathcal{C}^{op}, \mathbf{Set}]$ .

**Definition 2.4.** *A functor  $\mathcal{C} \rightarrow \mathbf{Set}$  is **representable** if it's isomorphic to  $\mathcal{C}(A, -)$  for some  $A$ .*

*A **representation** of  $F : \mathcal{C} \rightarrow \mathbf{Set}$  is a pair  $(A, x)$  where  $A \in \text{ob } \mathcal{C}$ ,  $x \in FA$  and  $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$  is an isomorphism. We also call  $x$  a **universal element** of  $F$ .*

**Corollary 2.5** ('Representations are unique up to unique isomorphism'). *If  $(A, x)$  and  $(B, y)$  are both representations of  $F : \mathcal{C} \rightarrow \mathbf{Set}$ , then there's a unique isomorphism  $f : A \rightarrow B$  s.t.  $Ff(x) = y$ .*

**Definition 2.6** (Product and coproduct). *Given two objects  $A, B$  of a locally small category  $\mathcal{C}$ , we define their **product** to be a representation of the functor*

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{op} \rightarrow \mathbf{Set}$$

*i.e. an object  $A \times B$  equipped with morphisms  $\pi_1 : A \times B \rightarrow A$ ,  $\pi_2 : A \times B \rightarrow B$  s.t. given any pair  $(f : C \rightarrow A, g : C \rightarrow B)$ , there exists a unique  $h : C \rightarrow A \times B$  s.t.  $\pi_1 h = f$  and  $\pi_2 h = g$ .*

*More generally, we can define the product  $\prod_{i \in I} A_i$  of a family  $\{A_i \mid i \in I\}$  of objects, or the product of the empty family, i.e. a **terminal object** 1 s.t. for every  $A$  there's a unique  $A \rightarrow 1$ .*

*Dualizing, we get the notion of **coproduct** or **sum**.*

**Definition 2.7** (Equaliser and coequaliser). *Given a parallel pair  $f, g : A \rightrightarrows B$  in a locally small category  $\mathcal{C}$ , the assignment  $C \mapsto FC = \{h : C \rightarrow A \mid fh = gh\}$  is a subfunctor  $F$  of  $\mathcal{C}(-, A)$ . A representation of  $F$  is called an **equaliser** of  $(f, g)$ .*

*In elementary terms, it's an object  $E$  equipped with  $e : E \rightarrow A$  s.t.  $fe = ge$ , s.t. any  $h$  with  $fh = gh$  factors uniquely as  $h = ek$*

*Dually, we have the notion of **coequaliser**, i.e. a morphism  $q : B \rightarrow Q$  satisfying  $qf = qg$ , and universal among such.*

**Definition 2.8.** a. We say a monomorphism is **regular** if it occurs as an equaliser (dually, regular epimorphism).

b. We say  $f : A \rightarrow B$  is a **split monomorphism** if there exists  $g : B \rightarrow A$  with  $gf = 1_A$ .

Every split monomorphism is regular: if  $gf = 1_A$ ,  $f$  is an equaliser of  $(1_B, fg)$  [see sheet 1, q2].

**Definition 2.9.** Let  $\mathcal{C}$  be a (locally small) category,  $\mathcal{G}$  a collection of objects of  $\mathcal{C}$ .

a. Say  $\mathcal{G}$  is a **separating family** if the functors  $\mathcal{C}(G, -)$ ,  $G \in \mathcal{G}$  are jointly faithful, i.e. if given  $f, g : A \rightrightarrows B$  with  $f \neq g$ , there exists  $G \in \mathcal{G}$  and  $h : G \rightarrow A$  with  $fh \neq gh$ .

b. Say  $\mathcal{G}$  is a **detecting family** if the  $\mathcal{C}(G, -)$ ,  $G \in \mathcal{G}$  jointly reflect isomorphisms, i.e. if given  $f : A \rightarrow B$  s.t. every  $g : G \rightarrow B$  with  $G \in \mathcal{G}$  factors uniquely through  $f$ ,  $f$  is an isomorphism.

**Lemma 2.10.** i. If  $\mathcal{C}$  is balanced, then any separating family is detecting

ii. If  $\mathcal{C}$  has equalisers, then every detecting family is separating

**Definition 2.11.** An object  $P$  is **projective** if  $\mathcal{C}(P, -)$  preserves epimorphisms, i.e. if given

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ A & \xrightarrow{e} & B \end{array}$$

there exists  $g : P \rightarrow A$  with  $eg = f$ .

Dually,  $P$  is **injective** in  $\mathcal{C}$  if it's projective in  $\mathcal{C}^{\text{op}}$ .

If  $P$  satisfies this property  $\forall e$  in some class  $\mathcal{E}$  of epimorphisms, we call it  $\mathcal{E}$ -projective.

**Corollary 2.12.** Representable functors are (pointwise) projective in  $[\mathcal{C}, \mathbf{Set}]$

*Proof.* Given

$$\begin{array}{ccc} & \mathcal{C}(A, -) & \\ & \downarrow \beta & \\ F & \xrightarrow{\alpha} & G \end{array}$$

$\beta$  corresponds to some  $y \in GA$ .  $\alpha_A$  is surjective, so  $\exists x \in FA$  with  $\alpha_A(x) = y$ .  $x$  corresponds to  $\gamma : \mathcal{C}(A, -) \rightarrow F$  with  $\alpha\gamma = \beta$ .  $\square$

### 3 Adjunctions

**Definition 3.1** (D.M. Khan, 1958). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  be two functors. An **adjunction** between  $F$  and  $G$  is a bijection between morphisms  $FA \rightarrow B$  in  $\mathcal{D}$  and morphisms  $A \rightarrow GB$  in  $\mathcal{C}$ , which is natural in  $A$  and  $B$ .

(If  $\mathcal{C}$  and  $\mathcal{D}$  are locally small, this says that  $(A, B) \rightarrow \mathcal{D}(FA, B)$  and  $(A, B) \rightarrow \mathcal{C}(A, GB)$  are naturally isomorphic functors  $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$ ).

We say  $F$  is **left adjoint** to  $G$ , or  $G$  is **right adjoint** to  $F$ , and write  $F \dashv G$ .

**Theorem 3.2.** Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor. Given  $A \in \text{ob } \mathcal{C}$ , let  $(A \downarrow G)$  be the category whose objects are pairs  $(B, f)$  with  $B \in \text{ob } \mathcal{D}$ ,  $f : A \rightarrow GB$  and whose morphisms  $(B, f) \rightarrow (B', f')$  are morphisms  $g : B \rightarrow B'$  in  $\mathcal{D}$  such that

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & \searrow f' & \downarrow Gg \\ & & GB' \end{array}$$

commutes. Then specifying a left adjoint for  $G$  is equivalent to specifying an initial object of  $(A \downarrow G)$  for each  $A$ .

*Proof.* First suppose  $G$  has a left adjoint  $F$ . Let  $\eta_A : A \rightarrow GFA$  be the morphism corresponding to  $1_{FA} : FA \rightarrow FA$ . The pair  $(FA, \eta_A)$  is an object of  $(A \downarrow G)$ . We'll show it's initial.

Given  $g : FA \rightarrow B$ , the composite  $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$  must correspond to  $FA \xrightarrow{1} FA \xrightarrow{g} B$  under the adjunction.

So, for any object  $(B, f)$  of  $(A \downarrow G)$ , the unique morphism  $(FA, \eta_A) \rightarrow (B, f)$  in  $(A \downarrow G)$  is the morphism  $FA \rightarrow B$  corresponding to  $f$ .

Conversely, suppose we're given an initial object  $(FA, \eta_A)$  of  $(A \downarrow G)$  for each  $G$ . Given  $f : A \rightarrow A'$ , the composite  $A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA'$  is an object of  $(A \downarrow G)$ , so there's a unique morphism  $Ff : FA \rightarrow FA'$  making

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow GFf \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commute.

$f \mapsto Ff$  is functorial: given  $f' : A' \rightarrow A''$ , then  $(Ff')(Ff)$  and  $F(f'f)$  are both morphisms  $(FA, \eta_A) \rightarrow (FA'', \eta_{A''}f'f)$  in  $(A \downarrow G)$ , so they're equal.

Finally, given  $f : A \rightarrow GB$ , the morphism  $g : FA \rightarrow B$  corresponding to it is the unique morphism  $(FA, \eta_A) \rightarrow (B, f)$  in  $(A \downarrow G)$ .

The naturality of this bijection is given by naturality of  $\eta$ , and naturality in  $B$  is immediate.  $\square$

**Corollary 3.3.** *If  $F, F'$  are both left-adjoint to  $G$ , then there's a canonical natural isomorphism  $F \rightarrow F'$ .*

*Proof.* For each  $A$ ,  $(FA, \eta_A)$  and  $(F'A, \eta'_A)$  are both initial in  $(A \downarrow G)$ , so there's a unique isomorphism  $\alpha_A : (FA, \eta_A) \rightarrow (F'A, \eta'_A)$ .

$\alpha$  is natural: given  $f : A \rightarrow A'$ ,  $\alpha_{A'}f$  and  $(Ff)\alpha_A$  are both morphisms  $(FA, \eta_A) \rightarrow (F'A', \eta'_{A'})$  in  $(A \downarrow G)$ . So they're equal.  $\square$

**Lemma 3.4.** *Given  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D} \xrightleftharpoons[K]{H} \mathcal{E}$ , if  $F \dashv G$  and  $H \dashv K$  then  $HF \dashv GK$ .*

*Proof.* We have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

which are natural in  $A$  and  $C$ .  $\square$

**Corollary 3.5.** *Given a commutative square  $\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow G & & \downarrow H \\ \mathcal{E} & \xrightarrow{K} & \mathcal{F} \end{array}$  of categories and functors, suppose all the functors in the diagram have left adjoints. Then the diagram  $\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{C} \end{array}$  of left adjoints commutes up to natural isomorphism.*

Given  $F \dashv G$ , we have a natural transformation  $\eta : 1_{\mathcal{C}} \rightarrow GF$  defined as in 3.2. We call  $\eta$  the **unit** of the adjunction.

Dually, we have  $\epsilon : FG \rightarrow 1_{\mathcal{D}}$ , the **counit**.  $\epsilon_B : FGB \rightarrow B$  corresponds to  $1_{GB} : GB \rightarrow GB$ .

**Theorem 3.6.** *Suppose we're given  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$ . Specifying an adjunction  $F \dashv G$  is equivalent to specifying natural transformations  $\eta : 1_{\mathcal{C}} \rightarrow GF$  and  $\epsilon : FG \rightarrow 1_{\mathcal{D}}$  such that*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \epsilon_F \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & \searrow 1_G & \downarrow G\epsilon \\ & & G \end{array}$$

*commute. (We say  $\eta$  and  $\epsilon$  satisfy the **triangular identities**).*



*Proof.* Given  $F \dashv G$ , we define  $\eta$  and  $\epsilon$  as already described. Since  $\epsilon_{FA} : FGFA \rightarrow FA$  corresponds to  $1_{GFA}$ , the composite  $\epsilon_{FA}(F\eta_A)$  corresponds to  $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$ , so it must be  $1_{FA}$ .

Similarly for the other identity.

Conversely, given  $\eta$  and  $\epsilon$  satisfying the  $\triangle^r$  identities, we map  $f : A \rightarrow GB$  to the composite  $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$  and  $g : FA \rightarrow B$  to the composite  $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$ .

We have

$$\begin{aligned}\Phi( A \xrightarrow{f} GB ) &= FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B \\ \Psi( FA \xrightarrow{g} B ) &= A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB\end{aligned}$$

So

$$\begin{aligned}\Psi\Phi(f) &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f\end{aligned}$$

And dually  $\Phi\Psi(g) = g$ .

Naturality of  $\Phi$  in  $A$  is immediate from its definition, and naturality in  $B$  follows from that of  $\epsilon$ .  $\square$

**Lemma 3.7.** Suppose given  $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$  and natural isomorphisms  $\alpha : 1_{\mathcal{C}} \rightarrow GF$ ,  $\beta : FG \rightarrow 1_{\mathcal{D}}$ . Then there exist natural isomorphisms  $\alpha', \beta'$  which additionally satisfy the triangular identities. In particular  $(F \dashv G)$ .

*Proof.* We define  $\alpha' = \alpha$  and take  $\beta'$  to be the composite

$$FG \xrightarrow{\beta_{FG}^{-1}} FGFG \xrightarrow{F\alpha_G^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that, since  $\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \downarrow \beta_{FG} & & \downarrow \beta \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$  commutes and  $\beta$  is monic, we have  $FG\beta = \beta_{FG}G$ .

Similarly,  $GF\alpha = \alpha_{GF} : GF \rightarrow GFGF$ .

Now

$$\begin{aligned}\beta'_F \circ F\alpha' &= F \xrightarrow{F\alpha} FGF \xrightarrow{\beta_{FGF}^{-1}} FGFGF \xrightarrow{F\alpha_{GF}^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{\beta_F^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{FGF\alpha^{-1}} FGF \xrightarrow{\beta_F} F \\ &= 1_F\end{aligned}$$

and

$$\begin{aligned}
G\beta' \circ \alpha'_G &= G \xrightarrow{\alpha_G} GFG \xrightarrow{GFG\beta^{-1}} GFGFG \xrightarrow{GF\alpha_G^{-1}} GFG \xrightarrow{G\beta} G \\
&= G \xrightarrow{G\beta^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha_{GFG}^{-1}} GFG \xrightarrow{\beta_F} G \\
&= 1_G
\end{aligned}$$

□

**Lemma 3.8.** Suppose  $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ ,  $(F \dashv G)$  is an adjunction with counit  $\epsilon$ . Then

- i.  $\epsilon$  is (pointwise) epic  $\iff G$  is faithful
- ii.  $\epsilon$  is an isomorphism  $\iff G$  is full and faithful

*Proof.* i. Given  $g : B \rightarrow B'$ , the morphism  $Gg : GB \rightarrow GB'$  corresponds to

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$$

So, for fixed  $B$ , composition with  $\epsilon_B$  is injective on morphisms  $B \rightarrow B'$   
 $\iff (g \mapsto Gg)$  is injective on morphisms  $B \rightarrow B'$ .

Hence  $G$  is faithful  $\iff \epsilon_B$  is epic  $\forall B$ .

- ii. Similarly,  $\epsilon_B$  is 0  $\forall B \implies G$  is bijective on morphisms with given domain and codomain, i.e.  $G$  is full and faithful.

Conversely, if  $G$  is full and faithful,  $1_{FGB}$  factors uniquely as

$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} FGB$ , so  $\epsilon_B$  is split monic. But it's epic by (i), hence an isomorphism.

□

**Definition 3.9.** i. A **reflection** is an adjunction satisfying the conditions of 3.8(ii).

- ii. A **reflective** subcategory of  $\mathcal{C}$  is a full subcategory  $\mathcal{C}'$  for which the inclusion  $\mathcal{C}' \hookrightarrow \mathcal{C}$  has a left adjoint.

Dually, **coreflection** and **coreflective** subcategory.

## 4 Limits

**Definition 4.1.** a. Let  $J$  be a category (almost always small, often finite). A **diagram of shape  $J$**  in a category  $\mathcal{C}$  is a functor  $D : J \rightarrow \mathcal{C}$ .

E.g. if  $J$  is the finite category  $\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$ , a diagram of shape  $J$  is a

commutative square. If  $J$  is the category  $\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$ , a diagram of shape  $J$  is a not-necessarily-commutative square.

The objects  $D(j)$ ,  $j \in \text{ob } J$  are called **vertices** of  $D$ , and the morphisms  $D(\alpha)$ ,  $\alpha \in \text{mor } J$  are called **edges** of  $D$ .

b. Let  $D : J \rightarrow \mathcal{C}$  be a diagram in  $\mathcal{C}$ . A **cone over  $D$**  is a pair  $(A, (\lambda_j \mid j \in \text{ob } J))$  where  $\lambda_j : A \rightarrow D(j) \forall j$ , and

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \lambda_{j'} \\ D(j) & \xrightarrow{D(\alpha)} & D(j') \end{array} \quad \text{commutes for}$$

each  $\alpha : j \rightarrow j'$  in  $J$ .

$A$  is called the **apex** of the cone, and the  $\lambda_j$  are its **legs**.

Equivalently,  $\lambda$  is a natural transformation  $\Delta A \rightarrow D$ , where  $\Delta A$  is the **constant diagram** with all vertices  $A$  and all edges  $1_A$ .

A **morphism**  $f : (A, (\lambda_j)) \rightarrow (B, (\mu_j))$  of cones over  $D$  is a morphism

$$f : A \rightarrow B \text{ s.t. } \begin{array}{ccc} A & \xrightarrow{f} & B \\ \searrow \lambda_j & & \swarrow \mu_j \\ & D(j) & \end{array} \quad \text{commutes for each } j. \text{ We have a category}$$

**Cone**( $D$ ) of cones over  $D$ .

Note that  $A \mapsto \Delta A$  is a functor  $\mathcal{C} \rightarrow [J, \mathcal{C}]$  and **Cone**( $D$ ) is in fact the category  $(\Delta \downarrow D)$ .

A **cocone over  $D : J \rightarrow \mathcal{C}$**  is a cone over  $D : J^{op} \rightarrow \mathcal{C}^{op}$ . We write **Cocone**( $D$ ) for the category of cocones over  $D$ .

**Definition 4.2.** i. A **limit** (resp. **colimit**) for a diagram  $D : J \rightarrow \mathcal{C}$  is a terminal object of **Cone**( $D$ ) (respectively an initial object of **Cocone**( $D$ )).

ii. We say  $\mathcal{C}$  has limits (resp. colimits) of shape  $J$  if  $\Delta : \mathcal{C} \rightarrow [J, \mathcal{C}]$  has a right (resp. left) adjoint.

(This is equivalent to making a choice of limit (resp. colimit) for every diagram of shape  $J$ ).

**Definition 4.3** (Pullback). Let  $J$  be  $\begin{array}{ccc} & \cdot & \\ & \downarrow & \\ \cdot & \longrightarrow & \cdot \end{array}$ . A diagram of shape  $J$  looks like

$$\begin{array}{ccc}
& A & D \xrightarrow{h} A \\
& \downarrow f & \downarrow k \searrow l \\
B \xrightarrow{g} C & & C \quad B
\end{array}
\quad \text{satisfying } fh = l = gk.$$

Equivalently, it's a pair  $\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & & \\ C & & \end{array}$  completing the diagram to a commutative square.

A universal such pair is called a **pullback** (or **fibre product**); in **Set** it can be defined as  $\{(a, b) \in A \times B \mid f(a) = g(b)\}$ . A colimit of shape  $J^{op}$  is called a **pushout**.

**Theorem 4.4.** Let  $\mathcal{C}$  be a category.

- i. If  $\mathcal{C}$  has equalisers and all finite (resp. all small) products, then  $\mathcal{C}$  has all finite (resp. all small) limits.
- ii. If  $\mathcal{C}$  has pullbacks and a terminal object, then  $\mathcal{C}$  has all finite limits.

*Proof.* i. Given  $D : J \rightarrow \mathcal{C}$ , first form the products

$$P = \prod_{j \in \text{ob } J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \text{mor } J} D(\text{cod } \alpha)$$

Define  $P \xrightarrow[g]{f} Q$  by  $\pi_\alpha f = \pi_{\text{cod } \alpha} : P \rightarrow D(\text{cod } \alpha)$  and  $\pi_\alpha g = D(\alpha) \circ \pi_{\text{dom } \alpha} : P \rightarrow D(\text{dom } \alpha) \rightarrow D(\text{cod } \alpha)$ , and let  $e : E \rightarrow P$  be the equaliser of  $(f, g)$ .

Claim  $(E, (\pi_j e \mid j \in \text{ob } J))$  is a limit cone for  $D$ . It is a cone since, for any  $\alpha : j \rightarrow j'$ ,  $D(\alpha)\pi_j e = \pi_{\alpha} g e = \pi_\alpha f e = \pi_{j'} e$ .

Given any cone  $(C, (\lambda_j \mid j \in \text{ob } J))$ , the  $\lambda_j$  define a unique  $\lambda : C \rightarrow P$ , and  $f\lambda = g\lambda$  since  $\pi_\alpha f\lambda = \pi_\alpha g\lambda \forall \alpha$ . So  $\lambda$  factors uniquely through  $e$ .

- ii. Let  $1$  be a terminal object of  $\mathcal{C}$ . For any pair of objects  $(A, B)$  the pullback

$$\begin{array}{ccc}
& A & \\
& \downarrow & \\
B & \longrightarrow & 1
\end{array}$$

has the universal property of a product  $A \times B$ , so  $\mathcal{C}$  has binary products. Then we can define any finite product  $\prod_{i=1}^n A_i$  as  $((A_1 \times A_2) \times A_3) \times \dots \times A_n$ .

So we need to show  $\mathcal{C}$  has equalisers. Given  $A \xrightarrow[g]{f} B$ , consider the

$$\begin{array}{ccc}
& B & \\
& \downarrow (1_A, f) & \\
A & \xrightarrow{(1_A, g)} & A \times B
\end{array}$$

It consists of  $\begin{array}{ccc} P & \xrightarrow{h} & B \\ \downarrow k & & \\ A & & \end{array}$  satisfying  $1_A h = 1_A k$  and  $fh = gk$ , and universal among such.

But this forces  $h = k$ , and  $h$  has the universal property of an equaliser for  $(f, g)$ . So by (i),  $\mathcal{C}$  has all finite limits. □

**Definition 4.5.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

- a. We say  $F$  **preserves** limits of shape  $J$  if, given  $D : J \rightarrow \mathcal{C}$  and a limit cone  $(L, (\lambda_j : j \in \text{ob } J))$  for  $D$ , the cone  $(FL, (F\lambda_j : j \in \text{ob } J))$  is a limit for  $FD : J \rightarrow \mathcal{D}$ .
- b. We say  $F$  **reflects** limits of shape  $J$  if, given  $D : J \rightarrow \mathcal{C}$  and a cone  $(L, (\lambda_j))$  such that  $(FL, (F\lambda_j))$  is a limit for  $FD$ , then  $(L, (\lambda_j))$  is a limit for  $D$ .
- c. We say  $F$  **creates** limits of shape  $J$  if, given  $D : J \rightarrow \mathcal{C}$  and a limit  $(M, (\mu_j))$  for  $FD$ , there exists a cone  $(L, \lambda_j)$  over  $D$  whose image is isomorphic to  $(M, (\mu_j))$ , and any such cone is a limit for  $D$ .

**Lemma 4.6.** Suppose  $\mathcal{D}$  has limits of shape  $J$ . Then  $[\mathcal{C}, \mathcal{D}]$  has limits of shape  $J$ , and they're constructed pointwise (i.e. the forgetful functor  $[\mathcal{C}, \mathcal{D}] \rightarrow \mathcal{D}^{\text{ob } \mathcal{C}}$  creates them).

*Proof.* Consider a functor  $D : J \times \mathcal{C} \rightarrow \mathcal{D}$ . For each  $A \in \text{ob } \mathcal{C}$ , let  $(LA, (\lambda_{j,A} : LA \rightarrow D(j, A) \mid j \in \text{ob } J))$  be a limit for the diagram  $D(-, A) : J \rightarrow \mathcal{D}$ .

Given any  $f : A \rightarrow B$  in  $\mathcal{C}$ , the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over  $D(-, B)$ , so they induce a unique  $Lf : LA \rightarrow LB$  such that

$$\begin{array}{ccc} LA & \xrightarrow{Lf} & LB \\ \downarrow \lambda_{j,A} & & \downarrow \lambda_{j,B} \\ D(j, A) & \xrightarrow{D(j,f)} & D(j, B) \end{array}$$

commutes for all  $j$ . Uniqueness assures  $L(gf) = L(g)L(f)$ , so  $L$  is a functor  $\mathcal{C} \rightarrow \mathcal{D}$ , and the  $\lambda_{j,-}$  are natural transformations  $L \rightarrow D(j, -)$ .

Suppose we're given any cone over  $D$  in  $[\mathcal{C}, \mathcal{D}]$  with apex  $M$  and legs  $\mu_j : M \rightarrow D(j, -)$ . Then  $(MA, (\mu_{j,A} : MA \rightarrow D(j, A) \mid j \in \text{ob } J))$  is a cone over  $D(-, A)$  in  $\mathcal{D}$ , so we get a unique  $\nu_A : MA \rightarrow LA$  s.t.  $\lambda_{j,A}\nu_A = \mu_{j,A}$  for all  $j$ .

Uniqueness tells us that

$$\begin{array}{ccc} MA & \xrightarrow{Mf} & MB \\ \downarrow \nu_A & & \downarrow \nu_B \\ LA & \xrightarrow{Lf} & LB \end{array}$$

commutes for all  $f \in \text{mor } \mathcal{C}$ , so  $\nu : M \rightarrow L$  in  $[\mathcal{C}, \mathcal{D}]$ , so it's the unique factorisation of the  $\mu_{j,-}$  through the  $\lambda_{j,-}$ .  $\square$

**Lemma 4.7.** *A morphism  $f : A \rightarrow B$  is monic  $\iff$*

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

*is a pullback.*

*Proof.*  $f$  is monic  $\iff$  any cone  $(g, h)$  over  $(f, f)$  has  $g = h \iff (g, h)$  factors uniquely through  $(1_A, 1_A)$ .  $\square$

Hence, provided  $\mathcal{D}$  has pullbacks, a morphism  $\alpha : F \rightarrow G$  in  $[\mathcal{C}, \mathcal{D}]$  is monic  $\iff \alpha_A : FA \rightarrow GA$  is monic for each  $A$ .

**Theorem 4.8.** *If  $G : \mathcal{D} \rightarrow \mathcal{C}$  has a left adjoint, then  $G$  preserves all limits which exist in  $\mathcal{D}$ .*

*Proof.* Suppose  $\mathcal{C}$  and  $\mathcal{D}$  both have limits of shape  $J$  and let  $(F \dashv G)$ . The diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \Delta & & \downarrow \Delta \\ [J, \mathcal{C}] & \xrightarrow{[J, F]} & [J, \mathcal{D}] \end{array}$$

commutes and  $[J, F]$  has a right adjoint  $[J, G]$ . So by 3.5 the diagram of right adjoints

$$\begin{array}{ccc} [J, \mathcal{D}] & \xrightarrow{[J, G]} & [J, \mathcal{C}] \\ \downarrow \lim_J & & \downarrow \lim_J \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

commutes up to isomorphism, i.e.  $G$  preserves limits of shape  $J$ .  $\square$

*Proof.* Let  $D : J \rightarrow \mathcal{D}$  be a diagram with limit  $(L, (\lambda_j \mid j \in \text{ob } J))$ . Given a cone  $(A, (\mu_j : A \rightarrow GD(j) \mid j \in \text{ob } J))$  in  $\mathcal{C}$ , we get a cone  $(FA, (\bar{\mu}_j : FA \rightarrow D(j) \mid j \in \text{ob } J))$  in  $\mathcal{D}$ , and hence a unique  $\bar{\nu} : FA \rightarrow L$  such that  $\lambda_j \bar{\nu} = \bar{\mu}_j$  for all  $j$ .

Then  $\nu : A \rightarrow GL$  is the unique morphism such that  $(G\lambda_j)\nu = \mu_j \forall j$ .  $\square$

The ‘primeval’ Adjoint Functor Theorem says that if  $\mathcal{D}$  has and  $G : \mathcal{D} \rightarrow \mathcal{C}$  preserves all limits, then  $G$  has a left adjoint.

This depends on two ideas:

**Lemma 4.9.**  *$\mathcal{C}$  has an initial object  $\iff 1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  has a limit.*

*Proof.* Suppose  $\mathcal{C}$  has an initial object  $0$ . The morphisms  $(0 \rightarrow A \mid A \in \text{ob } \mathcal{C})$  form a cone over  $1_{\mathcal{C}}$ . If we had another, say  $(L, (\lambda_A \mid A \in \text{ob } \mathcal{C}))$ , then  $\lambda_0 : L \rightarrow 0$  would make

$$\begin{array}{ccc} L & \xrightarrow{\lambda_0} & 0 \\ & \searrow \lambda_A & \swarrow \\ & A & \end{array}$$

commute for all  $A$ , and it’s the only morphism which does.

Conversely, suppose  $(I, (\lambda_A : I \rightarrow A \mid A \in \text{ob } \mathcal{C}))$  is a limit for  $1_{\mathcal{C}}$ .

If  $f : I \rightarrow A$ , then

$$\begin{array}{ccc} I & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \swarrow f \\ & A & \end{array}$$

commutes. In particular,  $\lambda_A \lambda_I = \lambda_A$  for all  $A$ , so  $\lambda_I = 1_I$  since both are factorisations of the limit cone through itself. So  $f = \lambda_A$ , and hence  $I$  is initial.  $\square$