Part III Category Theory

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1 Definitions and Examples

Definition 1.1 (Category)**.** *A category $\mathcal{C}$ consists of*

a. a collection $\mathrm{ob}\,\mathcal{C}$ *of* ***objects*** *A, B, C, ...*

b. a collection $\mathrm{mor}\,\mathcal{C}$ *of* ***morphisms*** *f, g, h, ...*

c. two operations dom, cod *from morphisms to objects. We write $f : A \to B$ or $A \xrightarrow{f} B$ to mean 'f is a morphism and* $\mathrm{dom}\, f = A$ *and* $\mathrm{cod}\, f = B$*'*

d. an operation assigning to each object A a morphism $1_A : A \to A$

e. a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff$ $\mathrm{dom}\, g = \mathrm{cod}\, f$*, and then gf :* $\mathrm{dom}\, f \to \mathrm{cod}\, g$

satisfying

f. $f1_A = f$ and $1_B f = f$ $\forall f : A \to B$

g. $h(fg) = (hg)f$ whenever gf and hg are defined

Definition 1.2 (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor*** *$\mathcal{C} \to \mathcal{D}$ consists of*

a. a mapping $A \to FA$ from $\mathrm{ob}\,\mathcal{C}$ *to* $\mathrm{ob}\,\mathcal{D}$

b. a mapping $f \to Ff$ from $\mathrm{mor}\,\mathcal{C}$ *to* $\mathrm{mor}\,\mathcal{D}$

satisfying $\operatorname{dom} Ff = F\operatorname{dom} f$, $\operatorname{cod} Ff = F\operatorname{cod} f$ *for all* f, $F(1_A) = 1_{FA}$ *for all* A, *and* $F(gf) = (Fg)(Ff)$ *whenever* gf *is defined.*

Definition 1.3. *By a **contravariant functor*** $\mathcal{C} \to \mathcal{D}$ *we mean a functor* $\mathcal{C} \to \mathcal{D}^{op}$ *(or equivalently* $\mathcal{C}^{op} \to \mathcal{D}$*). A functor* $\mathcal{C} \to \mathcal{D}$ *is sometimes said to be **covariant**.*

Definition 1.4 (Natural transformation)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories and* $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ *two functors. A **natural transformation*** $\alpha : F \to G$ *assigns to each* $A \in \operatorname{ob}\mathcal{C}$ *a morphism* $\alpha_A : FA \to GA$ *in* $\mathcal{D}$*, such that*

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given $\alpha : F \to G$ and $\beta : G \to H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A-component of a natural transformation $\beta\alpha : F \to H$.

Definition 1.5. *Given categories* $\mathcal{C}, \mathcal{D}$*, we write* $[\mathcal{C}, \mathcal{D}]$ *for the category of all functors* $\mathcal{C} \to \mathcal{D}$ *and natural transformations between them.*

Lemma 1.6. *Given* $F, G : \mathcal{C} \to \mathcal{D}$ *and* $\alpha : F \to G$, α *is an isomorphism in* $[\mathcal{C}, \mathcal{D}]$ $\iff$ *each* α_A *is an isomorphism in* $\mathcal{D}$*.*

Definition 1.7 (Faithful and full)**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor.*

a. *We say that* F *is **faithful** if, given* $f, g \in \operatorname{mor}\mathcal{C}$*, the equations* $\operatorname{dom} f = \operatorname{dom} g$, $\operatorname{cod} f = \operatorname{cod} g$ *and* $Ff = Fg$ *imply* $f = g$*.*

b. F *is **full** if, given any* $g : FA \to FB$ *in* $\mathcal{D}$*, there exists* $f : A \to B$ *in* $\mathcal{C}$ *with* $Ff = g$*.*

c. *We say a subcategory* $\mathcal{C}'$ *of* $\mathcal{C}$ *is **full** if the inclusion* $\mathcal{C}' \hookrightarrow \mathcal{C}$ *is a full functor.*

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition 1.8 (Equivalence of categories)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. An **equivalence** between* $\mathcal{C}$ *and* $\mathcal{D}$ *is a pair of functors* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ *together with natural isomorphisms* $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$*. We write* $\mathcal{C} \simeq \mathcal{D}$ *to mean that* $\mathcal{C}$ *and* $\mathcal{D}$ *are equivalent.*

We say a property P *of categories is **categorical** if whenever* $\mathcal{C}$ *has* P *and* $\mathcal{C} \simeq \mathcal{D}$ *then* $\mathcal{D}$ *has* P*.*

For example, being a groupoid is a categorical property, but being a group is not.

Definition 1.9 (Slice category)**.** *Given an object B of a category $\mathcal{C}$, define the **slice category** $\mathcal{C}/B$ to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \to (A' \xrightarrow{f'} B)$ are morphisms $h : A \to A'$ making*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commute.

Lemma 1.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and **essentially surjective**, i.e. for every $B \in \operatorname{ob}\mathcal{D}$, there exists $A \in \operatorname{ob}\mathcal{C}$ s.t. $FA \cong B$.*

Definition 1.11.

a. *A **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ containing exactly one object from each isomorphism class of objects of $\mathcal{C}$.*

b. *We say $\mathcal{C}$ is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism f in $\mathcal{C}$ satisfies $\operatorname{dom} f = \operatorname{cod} f$.*

For example, $\mathbf{Mat}_K$ is skeletal. The full subgategory of standard vector spaces K^n is a skeleton of $\mathbf{fd\ Mod}_K$.

Remark 1.12. *The following statements are each equivalent to the Axiom of Choice:*

1. *Every small category has a skeleton*
2. *Any small category is equivalent to each of its skeletons*
3. *Any two skeletons of a given small category are isomorphic*

Definition 1.13. *Let $f : A \to B$ be a morphism in a category $\mathcal{C}$.*

a. *f is a **monomorphism** if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightarrowtail B$ if f is monic.*

b. *Dually, f is an **epimorphism** if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.*

c. *$\mathcal{C}$ is a **balanced** category if every $f \in \operatorname{mor}\mathcal{C}$ which is both monic and epic is an isomorphism.*

2 The Yoneda Lemma

Definition 2.1. *A category $\mathcal{C}$ is* ***locally small*** *if, for any two objects A, B of $\mathcal{C}$, the morphism $A \to B$ are parametrised by a set $\mathcal{C}(A, B)$.*

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$: if $g : B \to B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \to \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \to B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \to \mathbf{Set}$.

Lemma 2.2 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, $A \in \mathrm{ob}\,\mathcal{C}$ and $F : \mathcal{C} \to$* ***Set****. Then*

i. There is a bijection between natural transformations $\mathcal{C}(A, -) \to F$ and elements of FA.

ii. Moreover, this bijection is natural in both A and F.

Proof. Bijection: given $\alpha : \mathcal{C}(A, -) \to F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.

Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \to F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$ is natural: given $g : B \to C$, we have

$$\begin{aligned}
\Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\
&= F(gf)(x) \\
&= (Fg)(Ff)(x) \\
&= (Fg)\Psi(x)_B(f)
\end{aligned}$$

$\Phi\Psi(x) = x$ since $F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$ since, for any $f : A \to B$,

$$\begin{aligned}
\Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\
&= Ff(\alpha_A(1_A)) \\
&= \alpha_B(\mathcal{C}(A, f)(1_A)) \\
&= \alpha_B(f)
\end{aligned}$$

□

Corollary 2.3. *The mapping $A \to \mathcal{C}(A, -)$ is a full and faithful functor $\mathcal{C}^{op} \to [\mathcal{C},$* ***Set****$]$.*

Proof. Given two objects A, B, 2.2(i) gives us a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ (by taking $F : C \mapsto \mathcal{C}(B, C)$). We need to show this is functorial, but given $f \in \mathcal{C}(B, A)$, $\Psi(F)_A$ sends 1_A to $\mathcal{C}(B, f)(1_A) = f$, so it's the natural transformation $g \mapsto gf$.

Hence, given $e : C \to B$, $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$ □

We call this functor the **Yoneda embedding**. Hence any locally small category $\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$.

Definition 2.4. *A functor $\mathcal{C} \to$ **Set** is **representable** if it's isomorphic to $\mathcal{C}(A, -)$ for some A.*

*A **representation** of $F : \mathcal{C} \to$ **Set** is a pair (A, x) where $A \in \mathrm{ob}\,\mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, -) \to F$ is an isomorphism. We also call x a **universal element** of F.*

Corollary 2.5 ('Representations are unique up to unique isomorphism')**.** *If (A, x) and (B, y) are both representations of $F : \mathcal{C} \to$ **Set**, then there's a unique isomorphism $f : A \to B$ s.t $Ff(x) = y$.*