Part III Local Fields

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Contents

1 Basic Theory

Definition 1 (Absolute value)**.** Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ s.t.

i. $|x| = 0 \iff x = 0$

ii. $|xy| = |x|\,|y| \quad \forall x, y \in K$

iii. $|x + y| \leq |x| + |y|$

Definition 2 (Valued field)**.** A **valued field** is a field with an absolute value.

Definition 3 (Equivalence of absolute values)**.** Let K be a field and let $|\cdot|$, $|\cdot|^{'}$ be absolute values on K. We say that $|\cdot|$ and $|\cdot|^{'}$ are **equivalent** if the associated metrics induce the same topology.

Definition 6 (Non-archimedean absolute value)**.** An absolute value $|\cdot|$ on a field K is called **non-archimedean** if $|x + y| \leq \max(|x|, |y|)$ (the **strong triangle inequality**).

Metrics s.t. $d(x, z) \leq \max(d(x, y), d(y, z))$ are called **ultrametrics**.

Assumption: unless otherwise mentioned, all absolute values will be non-archimedean. These metrics are weird!

Proposition 7. *Let K be a valued field. Then $\mathcal{O} = \{x \mid |x| \leq 1\}$ is an open subring of K, called the **valuation ring** of K. $\forall r \in (0, 1]$, $\{x \,|\, x < r\}$ and $\{x \,|\, x \leq r\}$ are open ideals of $\mathcal{O}$.*

Moreover, $\mathcal{O}^x = \{x \mid |x| = 1\}$.

Proposition 8. *Let K be a valued field.*

i. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) is Cauchy

Assume that K is complete

ii. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) converges

iii. Let $\sum_{n=0}^{\infty} y_n$ be a series in K. If $y_n \to 0$, then $\sum_{n=0}^{\infty} y_n$ converges

Definition 9. Let $R \subseteq S$ be rings. Then $s \in S$ is **integral over R** if $\exists$ monic $f(x) \in R[x]$ s.t. $f(s) = 0$.

Proposition 10. *Let $R \subseteq S$ be rings. Then $s_1, \ldots, s_n \in S$ are all integral over $R \iff R[s_1, \ldots, s_n] \subseteq S$ is a finitely generated R-module.*

Corollary 11. *let $R \subseteq S$ be rings. If $s_1, s_2 \in S$ are integral over R, then $s_1 + s_2$ and $s_1 s_2$ are integral over R. In particular, the set $\tilde{R} \subseteq S$ of all elements in S integral over R is a ring, called the **integral closure** of R in S.*

Definition 12. Let R be a ring. A topology on R is called a **ring topology** on R if addition and multiplication are continuous maps $R \times R \to R$. A ring with a ring topology is called a **topological ring**.

Definition 13. Let R be a ring, $I \subseteq R$ an ideal. A subset $U \subseteq R$ is called **I-adically open** if $\forall x \in U\ \exists n \geq 1$ s.t. $x + I^n \subseteq U$.

Proposition 14. *The set of all I-adically open sets form a topology on R, called the **I-adic topology**.*

Definition 15. Let $R_1, R_2, \ldots$ be topological rings with continuous homomorphisms $f_n : R_{n+1} \to R_n\ \forall n \geq 1$. The **inverse limit** of the R_i is the ring

$$\begin{aligned} \varprojlim_n R_n &= \left\{ (x_n) \in \prod_n R_n \mid f_n(x_{n+1}) = x_n \forall n \geq 1 \right\} \\ &\subseteq \prod_n R_n \end{aligned}$$

Proposition 16. *The inverse limit topology is a ring topology.*

Definition 17. Let R be a ring, I an ideal. The **I-adic completion** of R is the topological ring $\varprojlim_n R/I^n$ (R/I^n has the discrete topology, and $R/I^{n+1} \to R/I^n$ is the natural map).

There exists a map $\nu : R \to \varprojlim R/I^n$, $r \mapsto (r \mod I^n)_n$ This map is a continuous ring homomorphism when R is given the I-adic topology. We say that R is **I-adically complete** if ν is a bijection.

If $I = xR$ then we often call the I-adic topology the **x-adic topology**.

1.1 The p-adic Numbers

Let p be a prime number throughout.

If $x \in \mathbb{Q}\backslash\{0\}$ then $\exists!$ representation $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{>0}$ and $(a, p) = (b, p) = (a, b) = 1$.

We define the **p-adic absolute value** on $\mathbb{Q}$ to be the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-n} & \text{if } x = p^n \frac{a}{b}\ (\neq 0) \text{ as before} \end{cases}$$

Then $|\cdot|_p$ is an absolute value.

Definition 18. The **p-adic numbers** $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$.

The valuation ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is called the **p-adic integers**.

Proposition 19. *$\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ inside $\mathbb{Q}_p$.*

Proposition 20. *The non-zero ideals of $\mathbb{Z}_p$ are $p_n\mathbb{Z}_p$ for $n \geq 0$. Moreover, $\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p/p^n\mathbb{Z}_p$*

Corollary 21. *$\mathbb{Z}_p$ is a PID with a unique prime element p (up to units).*

Proposition 22. *The topology on $\mathbb{Z}$ induced by $|\cdot|_p$ is the p-adic topology.*

Proposition 23. *$\mathbb{Z}_p$ is p-adically complete and is (isomorphic to) the p-adic completion of $\mathbb{Z}$.*

Corollary 24. *Every $a \in \mathbb{Z}_p$ has a unique expansion*

$$a = \sum_{i=0}^{\infty} a_i p^i$$

with $a_i \in \{0, 1, \ldots, p-1\}$

Every $a \in \mathbb{Q}_p^\times$ has a unique expansion

$$a = \sum_{i=n} \infty a_i p^i$$

$n \in \mathbb{Z}$, $n = -\log_p |a|_p$, $a_n \neq 0$.

1.2 Valued Fields

Definition 25. Let K be a field. A **valuation** on K is a function $v : K \to \mathbb{R} \cup \{\infty\}$ s.t.

i. $v(x) = \infty \iff x = 0$

ii. $v(xy) = v(x) + v(y)$

iii. $v(x + y) \geq \min(v(x), v(y))$

$\forall x, y \in K$.

Here we use the conventions $r + \infty = \infty$, $r \leq \infty$ $\forall r \in \mathbb{R} \cup \{\infty\}$. v a valuation $\implies$ if $|x| = c^{-v(x)}$, $c \in \mathbb{R}_{>1}$, then $|\cdot|$ is an absolute value. Conversely, if $|\cdot|$ is an absolute value then $v(x) = -\log_c |x|$.

Let K be a valued field.

- $\mathcal{O} = \mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$ is the **valuation ring**

- $\mathfrak{m} = \mathfrak{m}_K = \{x \in K \mid |x| < 1\}$ is the **maximal ideal**
- $k = k_K = \mathcal{O}/\mathfrak{m}$ is the **residue field**

If K is a valued field and $F(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ is a polynomial, we say that F is **primitive** if $\max_i |a_i| = 1$ ($\implies F \in \mathcal{O}[x]$).

Theorem 26 (Hensel's Lemma)**.** *Assume that K is complete and that $F \in K[x]$ is primitive. Put $f = F \mod \mathfrak{m} \in k[x]$. If $\exists$ factorisation $f(x) = g(x)h(x)$ with $(g,h) = 1$, then $\exists$ factorisation $F(x) = G(x)H(x)$ in $\mathcal{O}[x]$ with $g \equiv G$, $h \equiv H \mod \mathfrak{m}$ and $\deg g = \deg G$.*

Proof. Put $d = \deg F$, $m = \deg g$, so $\deg h \leq d - m$. Pick lifts $G_0, H_0 \in \mathcal{O}[x]$ of g, h with $\deg G_0 = \deg g$, $\deg H_0 \leq d - m$.

$(g,h) = 1 \implies \exists A, B \in \mathcal{O}[x]$ s.t. $AG_0 + BH_0 \equiv 1 \mod \mathfrak{m}$.

Pick $\pi \in \mathfrak{m}$ s.t. $F - G_0 H_0 \equiv AG_0 + BH_0 - 1 \mod \pi$.

Want to find $G = G_0 + \pi P_1 + \pi^2 P_2 + \ldots$, $H = H_0 + \pi Q_1 + \pi^2 Q_2 + \cdots \in \mathcal{O}[x]$ with $P_i, Q_i \in \mathcal{O}[x]$, $\deg P_i < m$, $\deg Q_i \leq d - m$.

Define

$$\begin{aligned} G_{n-1} &= G_0 + \pi P_1 + \cdots + \pi^{n-1} P_{n-1} \\ H_{n-1} &= H_0 + \pi Q_1 + \cdots + \pi^{n-1} Q_{n-1} \end{aligned}$$

We want $F \equiv G_{n-1} H_{n-1} \mod \pi^n$, then take the limit.

Induction on n: $n = 1$ ✓

Assume we have G_{n-1}, H_{n-1}, $G_n = G_{n-1} + \pi^n P_n$, $H_n = H_{n-1} + \pi^n Q_n$. Expanding $F - H_n G_n$, we want

$$F - G_{n-1} H_{n-1} \equiv \pi^n (G_{n-1} Q_n + H_{n-1} P_n) \mod \pi^{n+1}$$

and divide by π^n

$$G_{n-1} Q_n + H_{n-1} P_n = \frac{1}{\pi^n} (F - G_{n-1} H_{n-1}) \mod \pi$$

Let $F_n := F - G_{n-1} H_{n-1}$. $AG_o + BH_0 \equiv 1 \mod \pi \implies F_n \equiv AG_0 F_n + BH_0 F_n \mod \pi$.

Write $BF_n = QG_0 + P_n$ with $\deg P_n < \deg G_0$, $P_n \in \mathcal{O}[x]$

$$\implies G_0(AF_n + H_0 Q) + H_0 P_n \equiv F_n \mod \pi$$

Now omit all coefficients from $AF_n + H_0 Q$ divisible by π to get Q_n. ☐

Corollary 27. *Let $F(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$, K complete, $a_0 a_n \neq 0$. If F is irreducible, then $|a_i| \leq \max(|a_0|, |a_n|)$ $\forall i$.*

Corollary 28. *$F \in \mathcal{O}[x]$ monic, K complete. If $F \mod \mathfrak{m}$ has a simple root $\bar{\alpha} \in k$, then F has a (unique) simple root $\alpha \in \mathcal{O}$ lifting $\bar{\alpha}$.*

Useful fact: let K be a valued field, $x, y \in K$. $|x| > |y| \implies |x + y| = |x|$. More generally, if we have a convergent series $\sum_{i=0}^{\infty} x_i$ and the non-zero $|x_i|$ are distinct, then $|x| = \max |x_i|$.

Theorem 29. *Let K be a complete valued field and let L/K be a finite extension. Then the absolute value $|\cdot|$ on K has a unique extension to an absolute value $|\cdot|_L$ on L, given by*

$$|\alpha|_L = \sqrt[n]{|N_{L/K}(\alpha)|},\ n = [L : K]$$

and L is complete w.r.t. $|\cdot|_L$.

Corollary 30. *Let K be a complete valued field. If M/K is an algebraic extension of K, then $|\cdot|$ extends uniquely to an absolute value on M.*

Corollary 31. *In the setting of Theorem 16, if $\sigma \in \mathrm{Aut}(L/K)$ then $|\sigma(\alpha)|_L = |\alpha|_L\ \forall \alpha \in L$*

Definition 32. Let K be a valued field and V a vector space over K. A **norm** on V is a function $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ such that

i. $||x|| = 0 \iff x = 0$

ii. $||\lambda x|| = |\lambda|\, ||x||\ \forall \lambda \in K, x \in V$

iii. $||x + y|| \leq \max(||x||, ||y||)\ \forall x, y, \in V$

Two norms $||\cdot||, ||\cdot||'$ are **equivalent** if they induce the same topology on V $\iff \exists C, D > 0$ s.t. $C\, ||x|| \leq ||x||' \leq D\, ||x||\ \forall x \in V$.

Proposition 33. *Let K be a complete valued field and V a finite dimensional K-vector space. Let $x_1, \ldots, x_n$ be a basis of V, then if $x = \sum a_i x_i \in V$,*

$$||x||_{\max} = \max_i |a_i|$$

defines a norm on V, and V is complete w.r.t $||\cdot||_{\max}$.

Moreover, if $||\cdot||$ is any norm on V, then $||\cdot||$ is equivalent to $||\cdot||_{\max}$ and hence V is complete w.r.t $||\cdot||$.

Lemma 34. *Let K be a valued field. Then $\mathcal{O}_K$ is integrally closed in K.*

Corollary 35. *Let K be a complete valued field, L/K finite. Equip L with $|\cdot|_L$ extending $|\cdot|$ on K. Then $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ inside L.*

1.3 Newton Polygons

Definition. $S \subset \mathbb{R}^2$ is **lower convex** if

i. $(x, y) \in S \implies (x, z) \in S \ \forall z \geq y$

ii. S is convex

Given any $T \subset \mathbb{R}^2$, there exists a minimal lower convex $LCH(T) \supseteq T$ ($LCH(T) = \bigcap_{T \subset S', S' \text{lower convex}} S'$).

Definition. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ where K is a valued field, v a valuation on K.

Define the **Newton polygon** of f as $LCH\left(\left\{(i, v(a_i)) \,\middle|\, \begin{array}{c} i = 0, 1, \ldots, n \\ a_i \neq 0 \end{array}\right\}\right)$.

Definition. The horizontal length of a line segment is called the **multiplicity**. Line segments have a **slope**.

Theorem 36. *Let K be a complete valued field, v a valuation on K, $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$. Let L be the splitting field of f over K, equipped with the unique extension w of v.*

If $(r, v(a_r)) \to (s, v(a_s))$ is a line segment of the Newton polygon of f with slope $-m \in \mathbb{R}$, then f has precisely $s - r$ roots of valuation m.

Proof. Dividing by a_n only shifts the NP vertically, so wlog $a_n = 1$.

Number the roots of f s.t.

$$\begin{array}{ccccccc} v(\alpha_1) & = & \ldots & = & v(\alpha_{s_1}) & = & m_1 \\ v(\alpha_{s_1+1}) & = & \ldots & = & v(\alpha_{s_2}) & = & m_2 \\ \vdots & & & & \vdots & & \vdots \\ v(\alpha_{s_t+1}) & = & \ldots & = & v(\alpha_{s_1}) & = & m_{t+1} \end{array}$$

where $m_1 < m_2 < \cdots < m_{t+1}$, and the α_i are the roots of f with multiplicity.

$v(a_n) = v(1) = 0$

$v(a_{n-1}) = v(\sum_i a_i) \geq \min_i v(\alpha_i) = m_1$

$v(a_{n-2}) \geq \min_{i \neq j} v(\alpha_i \alpha_j) = 2m_1$

$v(a_{n-s_1}) = v(\sum_{i_1, \ldots i_{s_1} \text{distinct}} \alpha_{i_1} \ldots \alpha_{i_{s_1}}) = s_1 m_1$

$v(a_{n-s_1-1}) \geq \min v(\alpha_{i_1} \ldots \alpha_{i_{s_1+1}}) = s_1 m_1 + m_2$

$\vdots$

$v(a_{n-s_2}) = \min v(\alpha_{i_1} \ldots \alpha_{i_{s_2}}) = s_1 m_1 + (s_2 - s_1) m_2$

etc. Drawing the lines between the points $(n, 0)$, $(n - s_1, s_1 m_1)$, $\ldots$ gives the NP of f.

The first line segment has length $n-(n-s_1) = s_1$ and slope $\frac{0-s_1m_1}{n-(n-s_1)} = -m_1$. For $k \geq 2$, the kth line segment has length $(n-s_{k-1})-(n-s_k) = s_k - s_{k-1}$ and slope

$$\frac{(s_1m_1 + \sum_{i=1}^{k-2}(s_{i+1}-s_i)m_{i+1}) - (s_1m_1 + \sum_{i=1}^{k-1}(s_{i+1}-s_i)m_{i+1})}{(n-s_{k-1})-(n-s_k)}$$
$$= \frac{-(s_k-s_{k-1})m_k}{s_k-s_{k-1}} = -m_k$$

□

Corollary 37. *If f is irreducible, then the NP has a single line segment.*

Proof. we need to show that all roots have the same valuation. Let α, β be roots in the splitting field L. Then $\exists \sigma \in \mathrm{Aut}(L/K)$ s.t. $\sigma(\alpha) = \beta$. So $v(\alpha) = v(\sigma(\alpha)) = v(\beta)$ by Corollary 30. □

Definition 38. Let K be a valued field with valuation v. K is a **discretely valued field** (DVF) if $v(K^\times) \subset \mathbb{R}$ is a discrete subgroup of $\mathbb{R}$ ($\iff$ $v(K^\times)$ is infinite cyclic).

Definition 39. A complete DVF with finite residue field is called a **local field**.

Let K be a DVF. $\pi \in K$ is called a **uniformiser** if $v(\pi) > 0$ and $v(\pi)$ generates $v(K^\times)$ ($\iff$ $v(\pi)$ has minimal positive valuation).

Proposition 40. *Let K be a DVF, uniformiser π. Let $S \subset \mathcal{O}_K$ be a set of coset representatives of $\mathcal{O}_k/\mathfrak{m}_K = k_K$ containing 0. Then*

1. *The non-zero ideals of $\mathcal{O}_K$ are $\pi^n\mathcal{O}_K$, $n \geq 0$*
2. *$\mathcal{O}_K$ is a PID with unique prime π (up to units), $\mathfrak{m}_K = \pi\mathcal{O}_K$*
3. *The topology on $\mathcal{O}_K$ induced by $|\cdot|$ is the π-adic topology*
4. *If K is complete, then $\mathcal{O}_K$ is π-adically complete*
5. *If K is complete, then any $x \in K$ can be written uniquely as*
$$x = \sum_{n \gg -\infty}^{\infty} a_n \pi^n$$
with $a_n \in S$ and $|x| = |pi|^{-\inf\{n \mid a_n \neq 0\}}$
6. *The completion $\hat{K}$ of K is a DVF, π is a uniformiser and*
$$\mathcal{O}_K/\pi^n\mathcal{O}_K \xrightarrow{\sim} \mathcal{O}_{\hat{K}}/\pi^n\mathcal{O}_{\hat{K}}$$
via the natural map.

Proof. The same as for $\mathbb{Q}_p$ and $\mathbb{Z}_p$ (use π instead of p). Note that $\left|\hat{K}\right| = |K|$ by Ex 9, sheet 1 ($\implies \hat{K}$ is a DVF). $\square$

Proposition 41. *Let K be a DVF. Then K is a local field $\iff \mathcal{O}_K$ is compact*

Proof. $\mathcal{O}_K$ compact $\implies \pi^{-n}\mathcal{O}_K$ is compact $\forall n \geq 0$ (π uniformiser).

$\mathcal{O}_K \cong \pi^{-n}\mathcal{O}_K \implies K = \bigcup_{n\geq 0}^{\infty} \pi^{-n}\mathcal{O}_K$ is complete.

Also $\mathcal{O}_K \twoheadrightarrow k_K$ and this map is continuous when k_K is given the discrete topology. So k_K is compact and discrete $\implies k_K$ finite.

Conversely, we seek to prove that K local $\implies \mathcal{O}_K$ is sequentially compact ($\iff$ compact). Note that $\mathcal{O}_K/\pi^n\mathcal{O}_K$ is finite $\forall n \geq 0$ (induction and $\pi^{n-1}\mathcal{O}_K/\pi^n\mathcal{O}_K \cong \mathcal{O}_K/\pi\mathcal{O}_K$).

Let (x_i) be a sequence in $\mathcal{O}_K$. $\exists$ a subsequence (x_{1i}) which is constant modulo π. Keep going: choose a subsequence $(x_{n+1,i})$ of (x_{ni}) s.t. $(x_{n+1,i})$ is constant mod π^{n+1}.

Then $(x_{ii})_{i=1}^{\infty}$ converges: it's Cauchy since $|x_{ii} - x_{jj}| \leq |\pi|^j \ \forall j \leq i$, and K is complete. $\square$

Definition 42. A ring R is called a **discrete valuation ring** (DVR) if it is a PID with a unique prime element (up to units).

Proposition 43. *R is a DVR $\iff R \cong \mathcal{O}_K$ for some DVF K.*

Proof. The reverse implication is contained in Proposition 42.

Suppose R is a DVR, π prime. $\forall x \in R\backslash\{0\}$, $\exists! u \in R^\times$, $n \in \mathbb{Z}_{\geq 0}$ such that $x = \pi^n u$ by uniqueness of prime factorisation.

Define $v(x) = \begin{cases} n & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

v defines a discrete valuation of $R \implies v$ extends uniquely to $K = \operatorname{Frac}(R)$. It remains to show that $R = \mathcal{O}_K$. First, note that $K = R[\frac{1}{\pi}]$. Any non-zero element looks like $\pi^n u$, $u \in R^\times$, $n \in \mathbb{Z}$, so it is invertible.

Then $v(\pi^n u) = n \in \mathbb{Z}_{\geq 0} \iff \pi^n u \in R$

$\therefore R = \mathcal{O}_K$. $\square$

Definition 44. Let K be a valued field with residue field k_K. K has **equal characteristic** if $\operatorname{char} K = \operatorname{char} k_K$, **mixed characteristic** otherwise ($\implies \operatorname{char} K = 0, \operatorname{char} k_K > 0$).

Definition 45. Let R be a ring of characteristic p. R is **perfect** if the Frobenius map $x \mapsto x^p$ is an automorphism of R.

Theorem 46. *Let K be a complete DVF of equal characteristic p and assume that k_K is perfect. Then $K \cong k_K[[T]]$ (as DVFs).*

Corollary 47. *Let K be a local field of equal characteristic p. Have $k_K \cong \mathbb{F}_q$ for some q a power of p, and $K \cong \mathbb{F}_q((T))$.*

Definition 48. Let K be a DVF. The **normalised valuation** v_K on K is the unique valuation on K in the given equivalence class s.t. $v_K(\pi) = 1$ for any uniformiser π.

Lemma 49. *Let R be a ring and let $x \in R$. Assume that R is x-adically complete and that R/xR is perfect of characteristic p.*

Then $\exists!$ map $[-] : R/xR \to R$ such that

$$[a] \equiv a \mod x$$
$$[ab] = [a][b] \ \forall a, b \in R/xR$$

Moreover if R has characteristic p, then $[-]$ is a ring homomorphism.

Proof. Let $a \in R/xR$. $\exists!$ $a^{p^{-n}} \in R/xR \ \forall n \geq 0$ since R/xR is perfect. Now lift arbitrarily: take $\alpha_n \in R$ such that $\alpha_n \equiv a^{p^{-n}} \mod x$.

Put $\beta_n = \alpha_n^{p^n}$.

Claim: $\lim_{n\to\infty} \beta_n$ exists and is independent of choices. Call this $[a]$.

Note that if the limit exists no matter how the α_n are chosen, then it is independent of the choices.

Want to prove $\beta_{n+1} - \beta_n \to 0$ x-adically.

$$\beta_{n+1} - \beta_n = (\alpha_{n+1}^p)^{p^n} - (\alpha_n)^{p^n}$$

$$\alpha_{n+1}^p \equiv (a^{p^{-n-1}})^p \equiv a^{p^{-n}} \equiv \alpha_n \mod x$$

The binomial theorem, R/xR characteristic p and induction $\implies$

$$(\alpha_{n+1}^p)^{p^n} \equiv \alpha_n^{p^n} \mod x^{n+1}$$

i.e. $\beta_{n+1} - \beta_n \equiv 0 \mod x^{n+1}$ so $\lim_{n\to\infty} \beta_n$ exists.

Multiplicativity: if $b \in R/xR$, with $\gamma_n \in R$ lifting $b^{p^{-n}}$ $\forall n \geq 0$, then $\alpha_n\gamma_n$ lifts $(ab)^{p^{-n}} = a^{p^{-n}} b^{p^{-n}}$

$$\implies [ab] = \lim_{n\to\infty} \alpha_n^{p^n} \lim_{n\to\infty} \gamma_n^{p^n} = [a][b]$$

$[a] \equiv a \mod x$:

$$\lim_{n\to\infty} \alpha_n^{p^n} \equiv \lim_{n\to\infty} (a^{p^{-n}})^{p^n} \equiv \lim_{n\to\infty} a \equiv a \mod x$$

Uniqueness: let $\phi : R/xR \to R$ be another map with these properties.

$$[a] = \lim_{n\to\infty} \phi(a^{p^{-n}})^{p^n} = \lim_{n\to\infty} \phi(a) = \phi(a)$$

since $\phi(a^{p^{-n}}) \equiv a^{p^{-n}} \mod x$ and ϕ is multiplicative.

Finally, if R has characteristic p, then $\alpha_n + \gamma_n$ lifts $a^{p^{-n}} + b^{p^{-n}} - (a+b)p^{-n}$, so

$$[a+b] = \lim_{n\to\infty} (\alpha_n + \gamma_n)^{p^n} = \lim_{n\to\infty} \alpha_n^{p^n} + \gamma_n^{p^n} = [a] + [b]$$

So $[-]$ is additive and multiplicative and (check!) $[1] = 1$, so it's a homomorphism. $\square$

Definition 50. $[-] : R/xR \to R$ is called the **Teichmüller map/lift** and $[x]$ is called the **Teichmüller lift/representative** of x.

Proof of Theorem 48. K is a complete DVF. We want to prove that $\mathcal{O}_K \cong k_K[[T]]$.

$\mathcal{O}_K$ char $p \implies [-] : k_K \hookrightarrow \mathcal{O}_K$ is an injective ring homomorphism.

Choose a uniformiser $\pi \in \mathcal{O}_K$. Then $k_K = \mathcal{O}/\pi\mathcal{O}_K$, $\mathcal{O}_K$ π-adically complete. Now define

$$k_K[[T]] \to \mathcal{O}_K$$
$$\sum_{n=0}^{\infty} a_n T^n \mapsto \sum_{n=0}^{\infty} [a_n]\pi^n$$

It's a bijection by one of the basic properties of complete DVFs, check it's a homomorphism. $\square$

Fact: let F be a field of characteristic p. Then F is perfect $\iff$ every finite extension of F is separable.

$\mathbb{F}_q$ is perfect for every $q = p^n$.

1.4 *Witt Vectors*

Definition 51. Let A be a ring. A is called a **strict p-ring** if A is p-torsionfree, p-adically complete and A/pA is perfect.

Proposition 52. *Let $X = \{x_i \mid i \in I\}$ be a set. Let*

$$\begin{aligned} B &= \mathbb{Z}[x_i^{p^{-\infty}} \mid i \in I] \\ &= \bigcup_{n=0}^{\infty} \mathbb{Z}[x_i^{p^{-n}} \mid i \in I] \end{aligned}$$

(Note that $\mathbb{Z}[x_i \mid i \in I] \subseteq \mathbb{Z}[x_i^{p^{-1}} \mid i \in I] \subseteq \dots$) and let A be the p-adic completion of B. Then A is a strict p-ring, and $A/pA \cong \mathbb{F}_p[x_i^{p^{-\infty}} \mid i \in I]$ (think of as 'universal perfect rings').

Lemma 53. *Let A and B be strict p-rings and let $f : A/pA \to B/pB$ be a ring homomorphism. Then $\exists!$ homomorphism $F : A \to B$ such that $f \equiv F \mod p$.*

F is explicitly given by $F(\sum_{n=0}^{\infty}[a_n]p^n) = \sum_{n=0}^{\infty}[f(a_n)]p^n$.

Theorem 54. *Let R be a perfect ring. Then $\exists!$ (up to isomorphism) strict p-ring $W(R)$ (called the* ***Witt vectors*** *of R) such that $W(R)/pW(R) \cong R$. Moreover, if R' is another perfect ring the reduction mod p map gives a bijection*

$$Hom_{Ring}(W(R), W(R')) \xrightarrow{\sim} Hom_{Ring}(R, R')$$

Proposition 55. *A complete DVR A of mixed characteristic with perfect residue field and such that p is a uniformiser is the same as a strict p-ring A such that A/pA is a field.*

Definition 56. Let R be a mixed characteristic DVR with normalised valuation v_R. The integer $v_R(p)$ where p is the characteristic of the residue field of R is called the **absolute ramification index** of R.

Corollary 57. *Let R be a CDVR of mixed characteristic with absolute ramification index 1 and perfect residue field k. Then $R \cong W(k)$.*

Lemma 53'. *Let A be a strict p-ring and let B be a p-adically complete ring. If $f : A/pA \to B/pB$ is a ring homomorphism, then $\exists!$ ring homomorphism $F : A \to B$ with $f \equiv F \mod p$.*

Theorem 58. *Let R be a CDVR of mixed characteristic with perfect residue field k and uniformiser π. Then R is finite over $W(k)$.*

Corollary 59. *Let K be a mixed characteristic local field. Then K is a finite extension of $\mathbb{Q}_p$.*

2 Some p-adic Analysis

Recall the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{x!}$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Proposition 60. *Let K be a complete valued field with absolute value $|\cdot|$, and assume that $K \supseteq \mathbb{Q}_p$, $|\cdot|\,|_{\mathbb{Q}_p} = |\cdot|_p$. Then $\exp(x)$ converges for $|x| < p^{-\frac{1}{p-1}}$ and $\log(1+x)$ converges for $|x| < 1$, and they define continuous maps*

$$\exp : \left\{x \in K \mid |x| < p^{-\frac{1}{p-1}}\right\} \to \mathcal{O}_K$$

$$\log : \{x \in K \mid |x| < 1\} \to K$$

Proof. $v = -\log_p |\cdot|$, this extends v_p.

log: $v(n) \leq \log_p n \implies$

$$v(\frac{x^n}{n}) \geq n \cdot v(x) - \log_p n \to \infty$$

if $v(x) > 0$.

exp: $v(n!) = \frac{n - s_p(n)}{p-1}$. Then

$$v(\frac{x^n}{n!}) \geq n \cdot v(x) - \frac{n}{p-1} = n(v(x) - \frac{1}{p-1}) \geq 0$$

and $\to \infty$ as $n \to \infty$ if $v(x) > \frac{1}{p-1}$.

For continuity, we use uniform convergence as in the real case. $\square$

Lemma 53". *Let A be a strict p-ring, B a ring with element $x \in B$ such that B is x-adically complete and B/xB is perfect of characteristic p. If $f : A/pA \to B/pB$ is a ring homomorphism, then $\exists!$ ring homomorphism $F : A \to B$ with $f \equiv F \mod p$.*

Let $n \geq 1$.

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}$$

is a polynomial in x, and so defines a continuous function $\mathbb{Z}_p \to \mathbb{Q}_p$, $x \mapsto \binom{x}{n}$.

Since $\binom{x}{n} \in \mathbb{Z}$ if $x \in \mathbb{Z}_{\geq 0}$, by the density of $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}_p$ we must have $\binom{x}{n} \in \mathbb{Z}_p \forall\, x \in \mathbb{Z}_p$.

When $n = 0$, set $\binom{x}{0} = 1 \forall\, x \in \mathbb{Z}_p$.

2.1 Mahler's Theorem

Theorem 61 (Mahler)**.** *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a continuous function. Then $\exists$ a unique sequence $(a_n)_{n \geq 0}$ with $a_n \in \mathbb{Q}_p$, $a_n \to 0$ such that*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \ \forall x \in \mathbb{Z}_p$$

and $\sup_{x \in \mathbb{Z}_p} |f(x)|_p = \max_{n=0,1,\dots} |a_n|_p$.

Let $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) = \{f : \mathbb{Z}_p \to \mathbb{Q}_p \text{ cts}\}$. This is a $\mathbb{Q}_p$-vector space.

If $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$, set $||f|| = \sup_{x \in \mathbb{Z}_p} |f(x)|_p$. $\mathbb{Z}_p$ compact $\implies$ f is bounded, so the supremum exists and is attained.

Let c_0 denote the set of sequences $(a_n)_{n=0}^{\infty}$ in $\mathbb{Q}_p$ such that $a_n \to 0$. This is a $\mathbb{Q}_p$-vector space, with a norm $||(a_n)|| = \max_{n=0,1,\dots} |a_n|_p$, and c_0 is complete w.r.t $||\cdot||$.

Define $\triangle : \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$ by $\triangle f(x) = f(x+1) - f(x)$. By induction,

$$\triangle^n f(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+n-i)$$

Note that $\triangle$ defines a linear operator on $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$, and

$$|\triangle f(x)|_p = |f(x+1) - f(x)|_p \leq ||f|| \implies ||\triangle f|| \leq ||f|| \text{ or } ||\triangle|| \leq 1$$

Definition 62. Let $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. The **nth Mahler coefficient** $a_n(f) \in \mathbb{Q}_p$ is defined by

$$a_n(f) = \triangle^n f(0) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n-i)$$

Lemma 63. *Let $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. Then $\exists k \geq 1$ such that $\left|\left|\triangle^{p^k} f\right|\right| \leq \frac{1}{p} ||f||$.*

Proof. If $f = 0$ there's nothing to prove, so wlog $||f|| = 1$ (by scaling). Then we want to show that $\triangle^{p^k} f(x) \equiv 0 \mod p \ \forall x \in \mathbb{Z}_p$, some $k \geq 1$.

$$\triangle^{p^k} f(x) = \sum_{i=0}^{p^k} (-1)^i \binom{p^k}{i} f(x + p^k - i) \equiv f(x+p^k) - f(x) \mod p$$

because $\binom{p^k}{i} \equiv 0 \mod p$ for $i = 1, 2, \ldots, p^k - 1$ and $(-1)^{p^k} \equiv -1 \mod p$.

Now $\mathbb{Z}_p$ compact $\implies$ f is uniformly continuous, so $\exists k$ such that $|x-y|_p \leq p^{-k} \implies |f(x) - f(y)|_p \leq \frac{1}{p} \ \forall x, y \in \mathbb{Z}_p$. Take this k, and we're done. ☐

Proposition 64. *The map $f \mapsto (a_n(f))_{n=0}^{\infty}$ defines an injective norm-decreasing linear map $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \to c_0$.*

Proof. First we prove that $a_n(f) \to 0$. We have $|a_n(f)|_p \leq ||\triangle^n f||$, so it suffices to prove that $||\triangle^n f|| \to 0$. Since $||\triangle|| \leq 1$, $||\triangle^n f||$ is monotonically decreasing, so it suffices to find a subsequence $\to 0$.

Apply Lemma 63 repeatedly to get $k_1, k_2, \ldots$ such that

$$\left|\left|\triangle^{p^{k_1 + \cdots + k_n}} f\right|\right| \leq \frac{1}{p^n} ||f||$$

This gives the desired subsequence.

Note that $|a_n(f)|_p \leq ||\triangle^n f|| \leq ||\triangle||$, so $||(a_n(f))_n|| = \max_{n=0,1,\ldots} |a_n(f)|_p \leq ||f||$, so the map is norm-decreasing. Linearity follows from the linearity of $\triangle$.

Injectivity: assume $a_n(f) = 0 \ \forall n \geq 0$. Then $a_0(f) = f(0) = 0$, and by induction $f(n) = \triangle^n f(0) = a_n(f) = 0 \ \forall n \geq 0$. So $f = 0$ by continuity since $\mathbb{Z}_{\geq 0} \subseteq \mathbb{Z}_p$ is dense. ☐

We will prove that the linear maps

$$\begin{aligned} f &\mapsto (a_n(f)) \\ \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) &\rightleftarrows c_0 \\ f_a(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} &\leftarrow\!\shortmid (a_n) = a \end{aligned}$$

are mutual inverses and norm-preserving.

Lemma 65. *We have* $\binom{x}{n} + \binom{x}{n-1} = \binom{x+1}{n}$ $\forall n \in \mathbb{Z}_{\geq 1}$ *and* $x \in \mathbb{Z}_p$.

Proof 1. True when $x \in \mathbb{Z}_{\geq n}$, and then the lemma follows by the density of $\mathbb{Z}_{\geq n} \subset \mathbb{Z}_p$ and continuity. $\square$

Proof 2. True when $x \in \mathbb{Z}_{\geq n}$, and both sides are polynomials which agree on an infinite set of points $\implies$ equal as elements of $\mathbb{Q}[x]$. Now evaluate. $\square$

Now let $a = (a_n)_{n=0}^{\infty} \in c_0$. Define $f_a : \mathbb{Z}_p \to \mathbb{Q}_p$,

$$f_a(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

This is a uniformly convergent series, so $f_a \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$.

Proposition 66. $a \mapsto f_a$ *defines a norm-decreasing linear map* $c_0 \to \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. *Moreover,* $a_n(f_a) = a_n$ $\forall n \geq 0$.

Proof. Linearity is clear.

Norm decreasing:

$$\begin{aligned} |f_a(x)|_p &= \left| \sum_{n=0}^{\infty} a_n \binom{x}{n} \right| \\ &\leq \sup_n |a_n|_p \left| \binom{x}{n} \right|_p \\ &\leq \sup_n |a_n|_p = ||a|| \;\; \forall x \in \mathbb{Z}_p \end{aligned}$$

$\implies ||f_a|| \leq ||a||$.

Inverses: $\forall k \in \mathbb{Z}_{\geq 0}$ define $a^{(k)} = (a_k, a_{k+1}, a_{k+2}, \dots)$

$$\begin{aligned} \triangle f_a(x) &= f_a(x+1) - f_a(x) \\ &= \sum_{n=1}^{\infty} a_n \left(\binom{x+1}{n} - \binom{x}{n} \right) \\ &= \sum_{n=1}^{\infty} a_n \binom{x}{n-1} \text{ by Lemma 65} \\ &= \sum_{n=0}^{\infty} a_{n+1} \binom{x}{n} = f_{a^{(1)}}(x) \end{aligned}$$

Iterating, $\triangle^k f_a = f_{a^{(k)}} \implies$

$$a_n(f_a) = \triangle^n f_a(0) = f_{a^{(n)}}(0) = a_n$$

$\square$

Summing up:

$$F(f) = (a_n(f))$$
$$V = \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \overset{F}{\underset{G}{\rightleftarrows}} c_0 = W$$
$$G(a) = f_a$$

We know: F is injective and norm-decreasing, $FG = id_W$ and G is norm-decreasing.

Lemma 67. *In this situation, $GF = \mathrm{id}_V$ and F and G are norm-preserving.*

Proof. Let $v \in V$. Then $F(v - GFv) = Fv - Fv = 0 \implies v = GFv$ since F is injective. So $GF = \mathrm{id}_V$.

Norm-preserving: $v \in V$, have $||Fv|| \leq ||v||$, but also $||Fv|| \geq ||GFv|| = ||v||$, so F is norm preserving. Same proof for G. $\square$

This finishes the proof of Mahler's Theorem.

3 Ramification Theory for Local Fields

The characteristic of the residue field of any local field from now on will be p (unless stated otherwise).

3.1 More on Finite Extensions

Recall: let R be a PID and let M be a f.g. R-module. Assume that M is torsion free. Then $\exists! n \geq 0$ such that $M \cong R^n$. Moreover, if $N \subseteq M$ is a submodule, then N is finitely generated and $N \cong R^m$, with $m \leq n$.

Proposition 68. *Let K be a local field, L/K finite of degree n. Then $\mathcal{O}_L$ is a finite, free $\mathcal{O}_K$-module of rank n (i.e. $\mathcal{O}_L \cong \mathcal{O}_K^n$ as $\mathcal{O}_K$-modules), and k_L/k_K is an extension of degree $\leq n$. Moreover, L is a local field.*

Proof. Choose a K-basis $\alpha_1, \ldots, \alpha_n$ of L. Let $||\cdot||$ denote the maximum norm $||\sum_{i=1}^n x_i \alpha_i|| = \max_{i=1,\ldots,n} |x_i|$ on L as in Proposition 33. $||\cdot||$ is equivalent to $|\cdot|$ (the extended absolute value on L) as K-norms, so $\exists r > s > 0$ such that

$$M = \{x \in L \mid ||x|| \leq s\} \subseteq \mathcal{O}_L \subseteq N = \{x \in L \mid ||x|| \leq r\}$$

Increasing r and decreasing s as necessary wlog $r = |a|$, $s = |b|$ for some $a, b \in K^\times$. Then

$$M = \bigoplus_{i=1}^{n} \mathcal{O}_K b\alpha_i \subseteq \mathcal{O}_L \subseteq N = \bigoplus_{i=1}^{n} \mathcal{O}_K a\alpha_i$$

$\implies \mathcal{O}_L$ is f.g. and free of rank n over $\mathcal{O}_K$.

Since $\mathfrak{m}_K = \mathfrak{m}_L \cap \mathcal{O}_K$, we have a natural injection

$$k_K = \mathcal{O}_K/\mathfrak{m}_K \hookrightarrow \mathcal{O}_L/\mathfrak{m}_L = k_L$$

Since $\mathcal{O}_L$ is generated over $\mathcal{O}_K$ by n elements, k_L is generated by n elements over k_K, i.e. $[k_L : k_K] \leq n$.

L a local field: k_L/k_K is finite and k_K finite $\implies$ k_L is a finite field. L is complete by Theorem 29.

Let v_K be the normalised valuation on K, w the extension of v_K to L. Then $w(\alpha) = \frac{1}{n} v_K(N_{L/K}(\alpha))$, so

$$w(L^\times) \subseteq \frac{1}{n} v(K^\times) = \frac{1}{n}\mathbb{Z}$$

$\implies$ it's discrete. □

Definition 69. Let L/K be a finite extension of local fields. The **inertia degree** of L/K is

$$f_{L/K} = [k_L : k_K]$$

Let v_L be the normalised valuation on L and π_K a uniformiser of K. The integer

$$e_{L/K} = v_L(\pi_K)$$

is called the **ramification index** of L/K.

Theorem 70. *Let L/K be a finite extension of local fields. Then $[L : K] = e_{L/K} f_{L/K}$ and $\exists \alpha \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.*

Proof. Write $e = e_{L/K}$, $f = f_{L/K}$.

k_L/k_K is separable, so $\exists \bar{\alpha} \in k_L$ such that $k_L = k_K(\bar{\alpha})$. Let $\bar{f}(x) \in k_K[x]$ be the minimal polynomial of $\bar{\alpha}$ over k_K, and let $f \in \mathcal{O}_K[x]$ be a monic lift of $\bar{f}$ with $\deg f = \deg \bar{f}$.

Claim: $\exists \alpha \in \mathcal{O}_L$ lifting $\bar{\alpha}$ and such that $v_L(f(\alpha)) = 1$ (always ≥ 1).

Let $\beta \in \mathcal{O}_L$ be any lift of $\bar{\alpha}$. If $v(f(\beta)) = 1$, then set $\alpha = \beta$. If not, set $\alpha = \beta + \pi_L$ (π_L uniformiser of L).

$f(\alpha) = f(\beta + \pi_L) = f(\beta) + f'(\beta)\pi_L + b\pi_L^2$ for some $b \in \mathcal{O}_L$ (Taylor expanding around β).

Since $v_L(f(\beta)) \geq 2$ and $v_L(f'(\beta)) = 0$, we have $v_L(f(\alpha)) = 1$. Put $\pi = f(\alpha)$ (uniformiser of L).

We claim that $\alpha^i\pi^j$, $i = 0, \dots, f-1$, $j = 0, \dots, e-1$ are an $\mathcal{O}_K$-basis of $\mathcal{O}_L$.

Linear independence: assume $\sum_{i,j} a_{ij}\alpha^i\pi^j = 0$ for some $a_{ij} \in K$, not all 0. Put $s_j = \sum_{i=0}^{f-1} a_{ij}\alpha^i \ \forall j$. $1, \alpha, \dots, \alpha^{f-1}$ are linearly independent over K since there reductions are linearly independent over k_K. So $\exists j$ such that $s_j \neq 0$.

Claim: $e | v_L(s_j)$ if $s_j \neq 0$.

Let k be such that $|a_{kj}|$ is maximal, then $a_{kj}^{-1}s_j = \sum_{i=0}^{f-1} a_{kj}^{-1}a_{ij}\alpha^i \implies a_{kj}^{-1}s_k \not\equiv 0 \mod \pi_L$ because $1, \bar{\alpha}, \dots, \bar{\alpha}^{f-1}$ are linearly independent over k_K.

$$\begin{aligned}\implies v_L(a_{kj}^{-1}s_j) = 0 \implies v_L(s_j) = v_L(a_{kj}) &= v_L(a_{kj}^{-1}s_j)\\ &\in v_L(K^\times)\\ &= ev_L(L^\times) = e\mathbb{Z}\end{aligned}$$

Now write $\sum_{i,j} a_{ij}\alpha^i\pi^j = \sum_{j=0}^{e-1} s_j\pi^j = 0$. If $s_j \neq 0$, we have $v_L(s_j\pi^j) = v_L(s_j) + j \in j + e\mathbb{Z}$.

$\implies$ no two non-zero terms in $\sum_{j=0}^{e-1} s_j\pi^j$ have the same valuation.

$\implies \sum_{j=0}^{e-1} s_j\pi^j \neq 0$, which is a contradiction.

Claim $\mathcal{O}_L = \oplus_{i,j}\alpha^i\pi^j$.

Set $M = \oplus_{i,j}\alpha^i\pi^j$ and $N = \oplus_{i=0}^{f-1}\mathcal{O}_K\alpha^i$. Then $M = N + \pi + N + \dots + \pi^{e-1}N$. Since $1, \bar{\alpha}, \dots, \bar{\alpha}^{f-1}$ span k_L over k_K we must have $\mathcal{O}_L = N + \pi\mathcal{O}_L$.

$$\begin{aligned}\text{Iterate: } \mathcal{O}_L &= N + \pi(N + \pi\mathcal{O}_L)\\ &= N + \pi N + \pi^2\mathcal{O}_L\\ &= \dots\\ &= N + \pi N + \dots + \pi^{e-1}N + \pi^e\mathcal{O}_L\\ &= M + \pi_K\mathcal{O}_L \ (\pi_K \text{ uniformiser of } K)\end{aligned}$$

Iterate: $\mathcal{O}_L = M + \pi_K^n\mathcal{O}_L \ \forall n \geq 1 \implies M$ is dense in $\mathcal{O}_L$. But M is the closed unit ball in $V = \oplus_{ij}K\alpha^i\pi^j \subseteq L$ w.r.t the maximum norm on V w.r.t the basis $\alpha^i\pi^j$.

Proposition 33 and Theorem 29 $\implies M$ is complete both w.r.t the maximum norm and $|\cdot|$ on L.

$\implies M \subseteq L$ is closed.

$\implies M = \mathcal{O}_L$.

Finally, since $\alpha^i\pi^j = \alpha^i f(\alpha)^j$ is a polynomial in α, have $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. □

Corollary 71. *Let $M/L/K$ be finite extensions of local fields. Then $f_{M/K} = f_{L/K}f_{M/L}$ and $e_{M/K} = e_{L/K}e_{M/L}$.*

Proof. $[k_M : k_K] = [k_M : k_L][k_L : k_K]$ by multiplicativity of degrees.

$e_{M/L}e_{L/K} = \frac{[M:L]}{f_{M/L}}\frac{[L:K]}{f_{L/K}} = \frac{[M:K]}{f_{M/K}} = e_{M/K}$. ☐

Definition 72. Let L/K be a finite extension of local fields. L/K is **unramified** if $e_{L/K} = 1$ (or $f_{L/K} = [L : K]$), and **totally ramified** if $f_{L/K} = 1$.

Theorem 73. *Let K be a local field. For each finite extension l/k_K there is a **unique** (up to isomorphism) finite unramified extension L/K with $k_L \cong l$ over k_K.*

Moreover, L/K is Galois with $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(l/k_K)$.

Proof. Existence: let $\bar{\alpha}$ be a primitive element of l/k_K with minimal polynomial $\bar{f} \in k_K[x]$. Take a monic lift $f \in \mathcal{O}_K[x]$ of $\bar{f}$ ($\deg f = \deg \bar{f}$).

Put $L = K(\alpha)$ where α is a root of f. $\bar{f}$ irreducible $\implies$ f irreducible $\implies$ $[L : K] = [l : k_K]$.

Moreover, k_L contains a root of $\bar{f}$ (the reduction of α). So $l \hookrightarrow k_L$ over $k_K \implies [L : K] \geq [k_L : k_K] = [L : K]$.

$\implies L/K$ is unramified and $k_L \cong l$ over k_K. ☐

Uniqueness and Galois property follows from:

Lemma 74. *Let L/K be a finite unramified extension of local fields and let M/K be a finite extension. Then there is a natural bijection*

$$\mathrm{Hom}_{K-alg}(L, M) \xrightarrow{\sim} \mathrm{Hom}_{k_K-alg}(k_L, k_M)$$

($\varphi : L \to M$ restricts to $\varphi : \mathcal{O}_L \to \mathcal{O}_M$, then take reductions).

Proof. By uniqueness of extended absolute values (Theorem 29) any K-algebra homomorphism $\phi : L \to M$ is an isometry for the extended absolute values.

Thus $\varphi(\mathcal{O}_L) \subseteq \mathcal{O}_M$, $\varphi(\mathfrak{m}_L) \subseteq \varphi(\mathfrak{m}_M)$ so we get the induced k_K-algebra homomorphism $\bar{\varphi} : k_L \to k_M$. This gives

$$\mathrm{Hom}_{K-alg}(L, M) \to \mathrm{Hom}_{k_K-alg}(k_L, k_M)$$

Bijectivity: let $\bar{\alpha} \in k_L$ be a primitive element over k_K, $\bar{f} \in k_K[x]$ its minimal polynomial, $f \in \mathcal{O}_K[x]$ a monic lift of $\bar{f}$ and $\alpha \in \mathcal{O}_L$ the unique root of f which lifts to $\bar{\alpha}$ (Hensel's Lemma).

Then $k_L = k_L(\bar{\alpha})$ and $L = K(\alpha)$.

$$\begin{array}{ccccc} \varphi & \mathrm{Hom}_{K-alg}(L, M) & \longrightarrow & \mathrm{Hom}_{k_K}(k_L, k_M) & \hat{\varphi} \\ \big\downarrow & \wr\big\downarrow & & \wr\big\downarrow & \big\downarrow \\ \varphi(\alpha) & \{x \in M \mid f(x) = 0\} & \longrightarrow & \{\bar{x} \in k_M \mid \bar{f}(\bar{x}) = 0\} & \bar{\varphi}(\bar{\alpha}) \end{array}$$

This is a bijection by Hensel's Lemma, since $\bar{f}$ is separable. ☐

Proof of 73 cont. Uniqueness: $k_L \cong k_M$ over k_K, L/K, M/K unramified. Then $\bar{\phi}$ lifts to a K-embedding $\phi : L \hookrightarrow M$ and $[L : K] = [M : K] \implies \phi$ an isomorphism.

Galois: $|\mathrm{Aut}_K(L)| = |\mathrm{Aut}_{k_K}(k_L)| = [k_L : k_K] = [L : K] \implies L/K$ Galois.

Also, $\mathrm{Aut}_K(L) \to \mathrm{Aut}_{k_K}(k_L)$ is really a homomorphism (so an isomorphism). ☐

Proposition 75. *Let K be a local field, L/K finite unramified, M/K finite. Say $L, M \subset$ fixed algebraic closure $\bar{K}$ of K. Then LM/M is unramified. Any subextension of L/K is unramified over K. If M/K is unramified, then LM/K is unramified.*

Proof. Let $\hat{\alpha}$ be a primitive element of k_L/k_K, $\bar{f} \in k_K[x]$ the minimal polynomial of $\hat{\alpha}$, $f \in \mathcal{O}_K[x]$ a monic lift of $\bar{f}$, $\alpha \in \mathcal{O}_L$ the unique root of f lifting $\hat{\alpha}$. Then $L = K(\alpha)$ so $LM = M(\alpha)$.

Let $\bar{g}$ be the minimal polynomial of $\bar{\alpha}$ over k_M. Then $\bar{g}|\bar{f} \implies f = gh$ in $\mathcal{O}_M[x]$ by Hensel's Lemma. g monic, lifts $\bar{g} \implies g(\alpha) = 0$ and g irreducible in $M[x]$.

So g is the minimal polynomial of α over $M \implies$

$$[LM : M] = \deg g = \deg \bar{g} \leq [k_{LM} : k_M] \leq [LM : M]$$

$\implies$ have equalities, LM/M unramified.

The second claim follows from the multiplicativity of $f_{L/K}$ and $e_{L/K}$ (Corollary 71), as does the third ($[LM : K] = [LM : M][M : K] = f_{LM/M} f_{M/K} = f_{LM/K} \implies LM/K$ unramified). ☐

Corollary 76. *Let K be a local field, L/K finite. Then $\exists$ a unique maximal subfield $K \subseteq T \subseteq L$ such that T/K is unramified. Moreover, $[T : K] = f_{L/K}$.*

Proof. Existence: T is the composite of all unramified subextensions of L/K (use Proposition 75).

Have $[T : K] = f_{T/K} \leq f_{L/K}$ by Corollary 71.

Let T'/K be the unique unramified extension with residue field extension k_L/k_K. Then $id : k_{T'} = k_L \to k_L$ lifts to a K-embedding $T' \overset{\varphi}{\hookrightarrow} L$, by Lemma 74.

Then $[T : K] \geq [\varphi(T') : K] = f_{L/K} \implies [T : K] = f_{L/K}$. ☐

3.2 Totally Ramified Extensions

Recall

Theorem 77 (Eisenstein's Criterion)**.** *Let K be a local field, $f(x) = x^n + \cdots + a_0 \in \mathcal{O}_K[x]$, π_K uniformiser of K. If $\pi_K | a_{n-1}, \ldots, a_0$ and $\pi_K^2 \nmid a_0$, then f is irreducible.*

Note that if L/K finite, v_K a normalised valuation on K and w the unique extension of v_K to L. Then $e_{L/K}^{-1} = w(\pi_L) = \min_{x \in \mathfrak{m}_L} w(x)$.

A polynomial $f(x) \in \mathcal{O}_K[x]$ satisfying the assumptions of Eisenstein's criterion is called an **Eisenstein polynomial**.

Proposition 78. *Let L/K be a totally ramified extension of local fields. Then $L = K(\pi_L)$ and the minimal polynomial of π_L over K is Eisenstein.*

Conversely, if $L = K(\alpha)$ and the minimal polynomial of α over K is Eisenstein, then L/K is totally ramified and α is a uniformiser of L.

Proof. First part: $n = [L : K]$, v_K a normalised valuation on K and w the unique extension of v_K to L. Then

$$[K(\pi_L) : K]^{-1} \leq e_{K(\pi_L)/K}^{-1} = \min_{x \in \mathfrak{m}_{K(\pi_L)}} w(x) \leq \frac{1}{n}$$

$\implies [K(\pi_L) : K] \geq [L : K] \implies L = K(\pi)$.

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathcal{O}_K[x]$ be the minimal polynomial of π_L over K.

$$\pi_L^n = -(a_0 + a_1(\pi_L) + \cdots + a_{n-1}\pi_L^{n-1})$$

So $1 = w(\pi_L^n) = w(a_0 + a_1\pi_L + \cdots + a_{n-1}\pi_L^{n-1}) = \min_{i=0,1,\ldots,n-1}(v_K(a_i) + \frac{i}{n})$ $\implies v_K(a_i) \geq 1 \ \forall i$ and $v_K(a_0) = 1$, so f is Eisenstein.

Converse: $L = K(\alpha)$, $n = [L : K]$. Let $g(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathcal{O}_K[x]$ be the minimal polynomial of α. g irreducible $\implies$ all roots have the same valuation, so

$$1 = w(b_0) = n \cdot w(\alpha) \implies w(\alpha) = \frac{1}{n}$$

$\implies e_{L/K}^{-1} = \min_{x \in \mathfrak{M}_L} w(x) \leq \frac{1}{n} = [L : K]^{-1}$

$\implies [L : K] = e_{L/K} = n$, so L/K is totally ramified and α is a uniformiser. ∎

We've show that if L/K is a totally ramified extension of local fields, then $L = K(\pi_L)$. In fact, $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$ (see proof of Theorem 70).

3.3 The Unit Group $\mathcal{O}_K^\times$

Let K be a local field. For each $s \in \mathbb{Z}_{\geq 1}$, set

$$U_K^{(s)} = U^{(s)} = 1 + \pi_K^S \mathcal{O}_K$$

where π_K is a uniformiser of K. Put $U_K = U_K^{(0)} = U^{(0)} = O_K^\times$.

Proposition 79. *We have* $U_K/U_K^{(1)} \cong (k_K^\times, \cdot)$ *and* $U_K^{(s)}/U_K^{(s+1)} \cong (k_K, +)$.

Proof. We have a surjective homomorphism $\mathcal{O}_K^\times \to k_K^\times$ which is just reduction mod π_K, and the kernel is $1 + \pi_K \mathcal{O}_K = U_K^{(1)}$.

For the second part, define a surjection

$$\begin{aligned} U_K^{(s)} &\to k_K \\ 1 + \pi_K^s x &\mapsto x \mod \pi_K \end{aligned}$$

This is a group homomorphism: writing $\pi = \pi_K$,

$$(1 + \pi^S x)(1 + \pi^s y) = 1 + \pi^s(x + y + \pi^s xy) \mapsto x + y + \pi^s xy \equiv x + y \mod \pi$$

The kernel is $1 + \pi^{s+1}\mathcal{O}_K = U_K^{s+1}$. □

3.4 The Inertia Group

Proposition 80. *If L/K is a finite Galois extension of local fields, then $\exists$ a surjective homomorphism* $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_L)$.

Proof. Lemma 74 gives us a homomorphism

$$\begin{tikzcd} \mathrm{Aut}_K(L) \arrow[r] \arrow[d, equal] & \mathrm{Aut}_{k_K}(k_L) \arrow[d, equal] \\ \mathrm{Gal}(L/K) & \mathrm{Gal}(k_L/k_K) \end{tikzcd}$$

Let T/K be the maximal unramified subextension of L/K.

$$\begin{tikzcd} \mathrm{Gal}(L/K) \arrow[r] \arrow[d] & \mathrm{Gal}(k_L/k_K) \arrow[d, equal, "(k_T = k_L)"] \\ \mathrm{Gal}(T/K) \arrow[r, "\sim"] & \mathrm{Gal}(k_T/k_K) \end{tikzcd}$$

$\implies$ surjectivity. □

Definition 81. In the setting of proposition 80, the kernel $I(L/K) = \mathrm{Gal}(L/T)$ of $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$ is called the **inertia group** of L/K (Trivial $\iff$ L/K unramified).

The field T is (sometimes) called the **inertial field** of L/K.

Lemma 82. *Let L/K be a finite Galois extension of local fields. Let $x \in k_L$ and $\sigma \in \mathrm{Gal}(L/K)$ with image $\bar{\sigma} \in \mathrm{Gal}(k_L/k_K)$. Then*

$$[\bar{\sigma}(x)] = \sigma([x])$$

In particular, $\sigma([x]) = [x]\ \forall x \in k_L \iff \sigma \in I(L/K)$.

Proof. The map

$$\begin{aligned} x &\mapsto \sigma^{-1}([\bar{\sigma}(x)]) \\ k_L &\to \mathcal{O}_L \end{aligned}$$

is multiplicative and $\sigma^{-1}([\bar{\sigma}(x)]) \equiv x \mod \pi_L$
$\implies \sigma^{-1}([\bar{\sigma}(x)]) = [x]$ by uniqueness of $[-]$. $\square$

3.5 Higher Ramification Groups

Let L/K be a finite Galois extension of local fields, v_L a normalised valuation on L.

Definition 83. Let $s \in \mathbb{R}_{\geq -1}$. Define the **s-th ramification group** of L/K by

$$G_s(L/K) = \{\sigma \in \mathrm{Gal}(L/K) \mid v_L(\sigma(x) - x) \geq s + 1\ \forall x \in \mathcal{O}_L\}$$

We could have defined these only for $s \in \mathbb{Z}_{\geq -1}$. Note that $G_{-1}(L/K) = \mathrm{Gal}(L/K)$, $G_0(L/K) = I(L/K)$.

Proposition 84. *Notation as above, π_L a uniformiser of L. Then $G_{s+1}(L/K)$ is a normal subgroup of $G_s(L/K)$ $\forall s \in \mathbb{Z}_{s\geq 0}$ and the map*

$$\begin{aligned} \frac{G_s(L/K)}{G_{s+1}(L/K)} &\to \frac{U_L^{(s)}}{U_L^{(s+1)}} \\ \sigma &\mapsto \frac{\sigma(\pi_L)}{\pi_L} \end{aligned}$$

is a well-defined injective group homomorphism, independent of the choice of π_L.

Proof. Define $\phi : G_s(L/K) \to \frac{U_L^{(s)}}{U_L^{(s+1)}}$ by $\sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L}$. $\sigma \in G_s(L/K)$, $\sigma(\pi_L) = \pi_L + \pi_L^{s+1}x$ for some $x \in \mathcal{O}_L \implies$

$$\frac{\sigma(\pi_L)}{\pi_L} = 1 + \pi_L^s x \in U_L^s$$

Now let $u \in \mathcal{O}_L^\times$. Then $\sigma(u) = u + \pi_L^{s+1} y$ for some $y \in \mathcal{O}_L$, so

$$\begin{aligned}\frac{\sigma(\pi_L u)}{\pi_L u} &= \frac{(\pi_L + \pi_L^{s+1}x)(u + \pi_L^{s+1}y)}{\pi_L u} \\ &= (1 + \pi_L^s x)(1 + \pi_L^{s+1} u^{-1} y) \\ &\equiv (1 + \pi_L^s x) = \frac{\sigma(\pi_L)}{\pi_L} \mod U_L^{(s+1)}\end{aligned}$$

So ϕ is independent of the choice of π_L.

It's a homomorphism:

$$\begin{aligned}\phi(\sigma\tau) &= \frac{\sigma(\tau(\pi_L))}{\pi_L} \\ &= \frac{\sigma(\tau(\pi_L))}{\tau(\pi_L)} \frac{\tau(\pi_L)}{\pi_L} \\ &\equiv \frac{\sigma(\pi_L)}{\pi_L} \frac{\tau(\pi_L)}{\pi_L} = \phi(\sigma)\phi(\tau) \mod U_L^{s+1}\end{aligned}$$

We have

$$\begin{aligned}\operatorname{Ker}\phi &= \{\sigma \in G_s(L/K) \mid v_L(\sigma(\pi_L) - \pi_L) \geq s+2\} \\ &\subseteq \{\sigma \in G_s(L/K) \mid v_L(\sigma(z) - z) \geq s+2 \ \forall \in \mathcal{O}_L\} \\ &= G_{s+1}(L/K)\end{aligned}$$

Conversely, let $x \in \mathcal{O}_L$ and write $x = \sum_{n=0}^\infty [x_n]\pi_L^n$, $x_n \in k_L$. Write $\sigma(\pi_L) = \pi_L + \pi_L^{s+2} y$, $y \in \mathcal{O}_L$. Let $\sigma \in \operatorname{Ker}\phi \subseteq I(L/K)$.

By Lemma 82,

$$\begin{aligned}\sigma(x) - x &= \sum_{n=1}^\infty [x_n]((\pi_L + \pi_L^{s+2}y)^n - \pi_L^n) \\ &= \pi_L^{s+2} y \sum_{n=1}^\infty [x_n]((\pi_L + \pi_L^{s+2}y)^{n-1} + (\pi_L + \pi_L^{s+2}y)^{n-2}\pi_L + \cdots + \pi_L^n)\end{aligned}$$

so $v_L(\sigma(x) - x) \geq s+2$, so $\sigma \in G_{s+1}(L/K)$. □

Corollary 85. $\mathrm{Gal}(L/K)$ *is soluble.*

Proof. Note that $\bigcap_s G_s(L/K) = \{id\}$, so $(G_s(L/K))_{s \in \mathbb{Z}_{\geq -1}}$ is a subnormal series of $\mathrm{Gal}(L/K)$ and $\frac{G_s(L/K)}{G_{s+1}(L/K)}$ is abelian. □

Let L/K be a finite Galois extension of local fields. Then $G_1(L/K)$ is a p-group (since $\frac{G_s(L/K)}{G_{s+1}(L/K)} \hookrightarrow k_L \ \forall s \in \mathbb{Z}_{\geq} 1$) and $\frac{G_0(L/K)}{G_1(L/K)} \hookrightarrow k_L^\times$, which has order prime to p.

$\implies G_1(L/K)$ is the unique Sylow p-subgroup of $G_0(L/K)$.

$G_1(L/K)$ is called the **wild inertia group** and $\frac{G_0(L/K)}{G_1(L/K)}$ is called the **tame quotient**.

Proposition 86. *Let $M/L/K$ be finite extensions of local fields, M/K Galois. Then $G_s(M/K) \cap \mathrm{Gal}(M/L) = G_s(M/L)$.*

Proof.

$$\begin{aligned} G_s(M/L) &= \{\sigma \in \mathrm{Gal}(M/L) \,|\, v_M(\sigma(x) - x) \geq s+1\} \\ &= G_s(M/K) \cap \mathrm{Gal}(M/L) \end{aligned}$$

□

3.6 Quotients

Let L/K be a finite Galois extension of local fields. Pick $\alpha \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$. set $i_{L/K}(\sigma) = v_L(\sigma(\alpha) - \alpha)$ for $\sigma \in \mathrm{Gal}(L/K)$.

If $g(x) = \sum_{i=0}^m b_i x^i \in \mathcal{O}_K[x]$, then

$$v_L(\sigma(g(\alpha)) - g(\alpha)) = v_L\left(\sum_{i=1}^m b_i(\sigma(\alpha)^i - \alpha^i)\right) \geq v_L(\sigma(\alpha) - \alpha)$$

$\implies$ $i_{L/K}(\sigma)$ is independent of α, and

$$G_s(L/K) = \{\sigma \in \mathrm{Gal}(L/K) \,|\, i_{L/K}(\sigma) \geq s+1\}$$

Proposition 87. *Let $M/L/K$ be finite extension of local fields, M/K and L/K Galois. Then*

$$(*) \qquad i_{L/K}(\sigma) = e_{M/L}^{-1} \sum_{\substack{\tau \in \mathrm{Gal}(M/K) \\ \tau|_L = \sigma}} i_{M/K}(\tau) \ \forall \sigma \in \mathrm{Gal}(L/K)$$

Proof. If $\sigma = 1$, both sides $= \infty$. Assume $\sigma \neq 1$. Let $\mathcal{O}_M = \mathcal{O}_K[\alpha]$, $\mathcal{O}_L = \mathcal{O}_K[\beta]$, $\alpha \in \mathcal{O}_M$, $\beta \in \mathcal{O}_L$.

$\implies e_{M/L} i_{L/K}(\sigma) = e_{M/L} v_L(\sigma(\beta) - \beta) = v_M(\sigma(\beta) - \beta)$.

$\tau \in \mathrm{Gal}(M/K) \implies i_{M/K}(\tau) = v_M(\tau(\alpha) - \alpha)$.

Fix τ such that $\tau|_L = \sigma$. Set $H = \mathrm{Gal}(M/L)$. Then

$$(\text{RHS of } *) \cdot e_{M/L} = \sum_{g \in H} (\tau(g(\alpha)) - \alpha) = v_M(\prod_{g \in H} (\tau(g(\alpha)) - \alpha))$$

Set $b = \sigma(\beta) - \beta = \tau(\beta) - \beta$ and $a = \prod_{g \in H}(\tau(g(\alpha))) - \alpha)$. We want to prove $v_M(b) = v_M(a)$.

General observation: let $z \in \mathcal{O}_L$, write $z = \sum_{i=0}^h z_i \beta^i$, $z_i \in \mathcal{O}_K$. Then $\tau(z) - z = \sum_{i=1}^n z_i(\tau(\beta)^i - \beta^i)$ is divisible by $\tau(\beta) - \beta = b$.

Now let $F(x) \in \mathcal{O}_L[x]$ be the minimal polynomial of α over L. Explicitly, $F(x) = \prod_{g \in H}(x - g(\alpha))$.

We have $(\tau F)(x) = \prod_{g \in H}(x - \tau(g(\alpha)))$ [τF is the polynomial obtained from F by applying τ to all coefficients], then all coefficients of $\tau F - F$ are of the form $\tau(z) - z$ for some $z \in \mathcal{O}_L \implies$ they are divisible by b.

$$\implies b | (\tau F - F)(a) = \pm a \implies b | a$$

Conversely, pick $f \in \mathcal{O}_K[x]$ such that $f(\alpha) = \beta$. Since $f(\alpha) - \beta = 0$, $f(x) - \beta = F(x)h(x)$ for some $h(x) \in \mathcal{O}_L[x]$.

Then $(f - \tau(\beta))(x) = (\tau F - \tau(\beta))(x) = (\tau F)(x)(\tau(h))(x)$. Set $x = \alpha$: $-b = \beta - \tau(\beta) = (\pm a)\tau h(\alpha) \implies a | b$. $\square$

Let L/K be a finite Galois extension of local fields. Define $\eta_{L/K} : [-1, \infty) \to [-1, \infty)$ by

$$\eta_{L/K}(s) = \int_0^s \frac{dx}{|G_0(L/K) : G_x(L/K)|}$$

When $-1 \le x < 0$, our convention is that $\frac{1}{|G_0 L/K : G_x L/K|} = |G_x(L/K) : G_0(L/K)|$ which is just $= 1$ when $-1 < x < 0$.

$\implies \eta_{L/K}(s) = s$ if $-1 \le s \le 0$.

Proposition 88. *Let $G = \mathrm{Gal}(L/K)$. Then $\eta_{L/K}(s) = \left(e_{L/K}^{-1} \sum_{\sigma \in G} \min(i_{L/K}(\sigma), s+1)\right) - 1$, for $s \in [-1, \infty)$.*

Proof. Let RHS $= \theta(s)$. Look at $s \mapsto \min(i_{L/K}, s+1)$.

$\implies \theta(s)$ is piecewise linear and break points are integers (same for $\eta_{L/K}$).

Have

$$\theta(0) = \frac{\#\{\sigma \in G \mid i_{L/K}(\sigma) \ge 1\}}{e_{L/K}} - 1 = 0 = \eta_{L/K}(0)$$

If $s \in [-1, \infty) \backslash \mathbb{Z}$,

$$\theta'(s) = e_{L/K}^{-1} \#\{\sigma \in G \mid i_{L/K}(\sigma) \ge s+1\} = \frac{1}{|G_0 L/K : G_s L/L|} = \eta'_{L/K}(s)$$

$\implies \theta(s) = \eta_{L/K}(s)$. $\square$

Theorem 89 (Herbrand)**.** *Let $M/L/K$ be finite extensions of local fields, M/K and L/K Galois. Set $H = \mathrm{Gal}(M/L)$ and $t = \eta_{L/K}(s)$, $s \in [-1, \infty)$.*

Then $\frac{G_s(M/K)H}{H} = G_t(L/K)$.

Proof. Put $G = \mathrm{Gal}(M/K)$. Choose $\tau \in G$ such that $i_{M/K}(\tau) \ge i_{M/K}(\tau g)$ for all $g \in H$. Put $m = i_{M/K}(\tau)$, $\sigma = \tau|_L$.

Claim: $i_{L/K}(\sigma) - 1 = \eta_{M/L}(m-1)$.

If $g \in G_{m-1}(M/L) \leq H$, then $i_{M/K}(g) \geq m$, so

$$\begin{aligned} i_{M/K}(\tau g) &= v_M(\tau g(\alpha) - \alpha) \\ &= v_M(\tau g(\alpha) - g(\alpha) + g(\alpha) - \alpha) \\ &\geq \min(v_M(\tau g(\alpha) - g(\alpha)), v_M(g(\alpha) - \alpha)) \\ &= \min(i_{M/K}(\tau g), i_{M/K}(g)) = m \end{aligned}$$

If $g \in H \backslash G_{m-1}(M/L)$, then $i_{M/K}(g) < m$ and $i_{M/K}(\tau g) = i_{M/K}(g)$. In either case, $i_{M/K}(\tau g) = \min(m, i_{M/K}(g))$. By Proposition 87, $i_{L/K}(\sigma) = e_{M/L}^{-1} \sum_{g \in H} \min(m, i_{M/K}(g))$.

By Proposition 88,

$$\eta_{M/L}(m-1) = \left(e_{M/L}^{-1} \sum_{g \in H} \min(i_{M/K}, m) \right) - 1 = i_{L/K}(\sigma) - 1$$

This proves the claim.

Now

$$\begin{aligned} \sigma \in \frac{G_s(M/K)H}{H} &\iff \tau \in G_s(M/K) \iff i_{M/K}(\tau) - 1 \geq s \\ &\iff \eta_{M/L}(i_{M/K}(\tau) - 1) \geq \eta_{M/L}(s) = t \text{ since } \eta_{M/L} \text{ strictly increasing} \\ &\iff i_{L/K}(\sigma) - 1 \geq t \iff \sigma \in G_t(L/K) \end{aligned}$$

□

Let L/K be a Galois extension of local fields. $\eta_{L/K} : [-1, \infty) \to [-1, \infty)$ is continuous, strictly increasing, $\eta_{L/K}(-1) = -1$ and $\eta_{L/K}(s) \to \infty$ as $s \to \infty$, so it is invertible. Set $\chi_{L/K} = \eta_{L/K}^{-1}$.

Definition 90. L/K as before. The **upper numbering** of the ramification groups of L/K is defined by

$$G^t(L/K) = G_{\chi_{L/K}(t)}(L/K)$$

for $t \in [-1, \infty)$. The previous numbering is called the **lower numbering**.

Lemma 91. *Let $M/L/K$ be finite extension of local fields, M/K and L/K Galois. Then $\eta_{M/K} = \eta L/K \circ \eta M/L$, hence $\chi_{M/K} = \chi_{M/L} \circ \chi_{L/K}$.*

Proof. Let $s \in [-1, \infty)$, set $t = \eta_{M/L}(s)$ and $H = \mathrm{Gal}(M/L)$.

By Theorem 89,

$$\begin{aligned} G_t(L/K) &\cong \frac{G_s(M/K)H}{H} \\ &\cong \frac{G_s(M/K)}{H \cap G_s(M/K)} \\ &= \frac{G_s(M/K)}{G_s(M/L)} \end{aligned}$$

Thus

$$\frac{\#G_s(M/K)}{e_{M/K}} = \frac{\#G_t(L/K)}{e_{L/K}} \cdot \frac{\#G_s(M/L)}{e_{M/L}}$$

so

$$\begin{aligned} \eta'_{M/K}(s) &= \frac{\#G_s(M/K)}{e_{M/K}} \\ &= \frac{\#G_t(L/K)}{e_{L/K}} \cdot \frac{\#G_s(M/L)}{e_{M/L}} \\ &= \eta'_{L/K}(t)\eta'_{M/L}(s) = (\eta_{L/K} \circ \eta_{M/L})'(s) \end{aligned}$$

whenever these derivatives make sense.

Since $\eta_{L/K}(\eta_{M/L}(0)) = \eta_{L/K}(0) = 0 = \eta_{M/K}(0)$, we get $\eta_{M/K} = \eta_{L/K} \circ \eta_{M/L}$. □

Corollary 92. *Keep the notation of Lemma 91 and its proof. Let $t \in [-1, \infty)$. Then*

$$\frac{G^t(M/K)H}{H} = G^t(L/K)$$

Proof. Put $s = \chi_{L/K}(t)$. Then, by Theorem 89 and Lemma 91,

$$\begin{aligned} \frac{G^t(M/K)H}{H} &\overset{\text{def}}{=} \frac{G_{\chi_{M/K}(t)}(M/K)H}{H} \\ &\overset{89}{=} G_{\eta_{M/L}(\chi_{M/K}(t))}(L/K) \\ &\overset{91}{=} G_s(L/K) \overset{\text{def}}{=} G^t(L/K) \end{aligned}$$

□

4 Local Class Field Theory

This is the study of abelian extensions (i.e. extensions with abelian Galois groups) of local fields.

4.1 Infinite Galois Theory

Definition 93. Let L/K be an algebraic field extension. We say that L/K is **seperable** if, for every $\alpha \in L$, the minimal polynomial $f_\alpha \in K[x]$ is seperable. We say L/K is **normal** if f_α splits into linear factors in $L[x]$ for every $\alpha \in L$.

L/K is **Galois** if it is normal and seperable. If so, we write $\mathrm{Gal}(L/K) = \mathrm{Aut}_K(L)$.

Definition 94. Let M/K be a Galois extension. $U \subseteq \mathrm{Gal}(M/K)$ is open if for every $\sigma \in U$, $\exists L/K$ a finite subextension of M/K such that $\sigma\,\mathrm{Gal}(M/K) \subseteq U$.

These sets form the open sets of a topology on $\mathrm{Gal}(M/K)$ called the **Krull topology**. $G = \mathrm{Gal}(M/K)$ is a topological group w.r.t. the Krull topology.

Proposition 95. *Let M/K be a Galois extension. Then $\mathrm{Gal}(M/K)$ is compact and Hausdorff, and if $U \subseteq \mathrm{Gal}(M/K)$ is an open subset such that $1 \in U$, then there exists an open normal subgroup $N \subseteq \mathrm{Gal}(M/K)$ such that $N \subseteq U$.*

Remarks.

1. When M/K is finite, the Krull topology is discrete.
2. Topological groups with the properties in Proposition 95 are called **profinite**.
3. Last part: by definition, $\exists L/K$ a finite subextension of M/K such that $\mathrm{Gal}(M/L) \subseteq U$. Let L' be the Galois closure of L over K, then $\mathrm{Gal}(M/L') \subseteq \mathrm{Gal}(M/L) \subseteq U$, and $\mathrm{Gal}(M/L')$ is open and normal.

Definition 96. Let I be a set with a partial order $\leq$. We say that I is a **directed system** if $\forall i, j \in I\ \exists I$ such that $I \leq k$ and $j \leq k$.

Definition 97. Let I be a directed system. An **inverse system** (of topological groups) indexed by I is a collection of topological groups G_i, $i \in I$ and continuous homomorphisms $f_{ij} : G_j \to G_i\ \forall i, j \in I$ with $i \leq j$ such that

1. $f_{ii} = id_{G_i}$
2. $f_{ik} = f_{ij} \circ f_{jk}$ when $i \leq j \leq k$

We define the **inverse limit** of the system (G_i, f_{ij}) to be

$$\varprojlim_{i\in I} G_i = \left\{ (g_i) \in \prod_{i\in I} G_i \mid f_{ij}(g_j) = g_i\ \forall i \leq j \right\} \subseteq \prod_{i\in I} G_i$$

It's a group under coordinate-wise multiplication and a topological space when given the subspace topology of the product topology on $\prod_{i\in} G_i$. This makes $\varprojlim_{i\in I} G_i$ into a topological group.

Proposition 98. *Let M/K be a Galois extension. The set I of finite Galois subextensions L/K of M/K is a directed system under inclusion. If $L, L' \in I$ with $L \subseteq L'$, then we have a map $\cdot|_L^{L'} : \mathrm{Gal}(L'/K) \to \mathrm{Gal}(L/K)$. Then $(\mathrm{Gal}(L/K), \cdot|_L^{L'})_{L\in I,\, L\subseteq L'}$ is an inverse system, and the map*

$$\mathrm{Gal}(M/K) \to \varprojlim_{L\in I} \mathrm{Gal}(L/K)$$
$$\sigma \mapsto (\sigma|_L)_{L\in I}$$

is an isomorphism of topological groups.

Theorem 99 (Fundamental Theorem of Galois Theory)**.** *Let M/K be Galois. The map $L \mapsto \mathrm{Gal}(M/L)$ defines a bijection between subextensions L/K of M/K and closed subgroups of $\mathrm{Gal}(N/K)$, with inverse $H \mapsto M^H$.*

Moreover, L/K is finite $\iff$ $\mathrm{Gal}(M/L)$ is open, and L/K Galois $\iff$ $\mathrm{Gal}(M/L)$ is normal, and then

$$\frac{\mathrm{Gal}(M/K)}{\mathrm{Gal}(M/L)} \xrightarrow[\sim]{\sigma \mapsto \sigma|_L} \mathrm{Gal}(L/K)$$

and $Gal(M/L)$ is closed.

4.2 Unramified Extensions and Weil Groups

Definition 100. Let K be a local field, M/K an algebraic extension. M/K is **unramified** (or **totally ramified**) if L/K is unramified (or totally ramified) for every finite subextension L/K of M/K.

In general, an algebraic extension M/K has a maximal unramified subextension $T = T_{M/K}/K$, which is Galois.

If L/K is a finite unramified extension of local fields with $q = \#k_K$, then $\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(k_L/k_K) \ni x \mapsto x^q$, so $\mathrm{Gal}(L/K)$ is cyclic with a canonical generator $\mathrm{Frob}_{L/K}$, which is a lift of $x \mapsto x^q$. This is called the (arithmetic) **Frobenius element** of L/K.

Frob is compatible in towers: if $M/L/K$ are finite unramified extensions of local fields, then $\mathrm{Frob}_{M/K}|_L = \mathrm{Frob}_{L/K}$ ($x \mapsto x^q$ on k_M restricts to $x \mapsto x^q$ on k_L, $q = \#k_K$).

$\implies$ for M/K an arbitrary unramified extension, we get

$$\mathrm{Frob}_{L/K} \in \varprojlim_{\substack{L/K \\ \text{finite subexts} \\ \text{of } M/K}} \mathrm{Gal}(L/K) \cong \mathrm{Gal}(M/K)$$

so we get an element $\mathrm{Frob}_{M/K} \in \mathrm{Gal}(M/K)$. It is the unique lift of $x \mapsto x^{\#k_K}$ on k_M/k_K.

Remarks. Let K be a local field, M/K unramified.

$$\begin{tikzcd} \mathrm{Gal}(M/K) \arrow[r, "\text{red.}"] \arrow[d, "\wr"'] & \mathrm{Gal}(k_M/k_K) \arrow[d, "\wr"'] \\ \varprojlim \mathrm{Gal}(L/K) \arrow[r, "\text{red.}"] & \varprojlim \mathrm{Gal}(k_L/k_K) \end{tikzcd}$$

$\implies \mathrm{Gal}(M/K) \xrightarrow{\sim} \varprojlim \mathrm{Gal}(k_L/k_K)$

Note that finite subextensions of M/K biject with finite subextensions of k_M/k_K. So $\mathrm{Frob}_{M/K}$ is the unique lift of $x \mapsto x^{\#k_K}$ on k_M.

Definition 101. Let K be a local field, M/K Galois, $T = T_{M/K}/K$ the maximal unramified subextension of M/K. The **Weil Group** $W(M/K)$ of M/K is

$$W(M/K) = \{\sigma \in \mathrm{Gal}\, M/K \mid \sigma|_T = \mathrm{Frob}^n_{T/K},\ \text{some } n \in \mathbb{Z}\}$$

We define a topology on $W(M/K)$ by saying that U is open $\iff \forall \sigma \in U\ \exists L/T$ a finite extension such that $\sigma\,\mathrm{Gal}(L/T) \subset U$.

$$\begin{tikzcd}
\mathrm{Gal}(M/T) \arrow[r] \arrow[dr] & W(M/K) \arrow[r] \arrow[d, hook] & \mathrm{Frob}^{\mathbb{Z}}_{T/K} \arrow[d, hook] \\
& \mathrm{Gal}(M/K) \arrow[r] & \mathrm{Gal}(T/K)
\end{tikzcd}$$

Discrete topology on $\mathrm{Frob}^{\mathbb{Z}}_{T/K} \rightsquigarrow$ topology of $W(M/K)$.

Proposition 102. *Let K be a local field, M/K Galois. Then $W(M/K)$ is dense in $\mathrm{Gal}(M/K)$. If L/K is a finite subextension of M/K, then $W(M/L) = W(M/K) \cap \mathrm{Gal}(M/L)$. If L/K is also Galois, then $\frac{W(M/K)}{W(M/L)} \xrightarrow{\sim} \mathrm{Gal}(L/K)$, via restriction.*

Proof. Density: need to sho that, for every finite Galois subextension L/K of M/K, $W(M/K)$ surjects onto $\mathrm{Gal}(L/K)$ (via restriction).

Let $T = T_{M/K}$, then $T_{L/K} = T \cap L$. Then

$$\begin{tikzcd}
\mathrm{Gal}(M/T) \arrow[r] \arrow[d, two heads] & W(M/K) \arrow[r] \arrow[d] & \mathrm{Frob}^{\mathbb{Z}}_{T/K} \arrow[r, phantom, "\cong"] \arrow[d, two heads] & (x \mapsto x^{\#k_K})^{\mathbb{Z}} \arrow[d, two heads] \\
\mathrm{Gal}(T/L \cap T) \arrow[r] & \mathrm{Gal}(L/K) \arrow[r] & \mathrm{Gal}(T \cap L/K) \arrow[r, phantom, "\cong"] & \langle x \mapsto x^{\#k_K} \rangle
\end{tikzcd}$$

Chasing the diagram implies surjectivity in the middle.

Second part: let L be as in the first part. $LT_{M/K} \subseteq T_{M/L}$.

$$\begin{tikzcd}
\mathrm{Frob}^{\mathbb{Z}}_{T_{M/K}/K} \arrow[r, phantom, "\subseteq"] & \mathrm{Gal}(T_{M/K}/K) \arrow[r, phantom, "\cong"] & \mathrm{Gal}(k_M/k_K) \\
\mathrm{Frob}^{\mathbb{Z}}_{T_{M/L}/L} \arrow[r, phantom, "\subseteq"] \arrow[u, hook] & \mathrm{Gal}(T_{M/L}/L) \arrow[r, phantom, "\cong"] \arrow[u, hook] & \mathrm{Gal}(k_M/k_L) \arrow[u, hook]
\end{tikzcd}$$

$\implies \mathrm{Frob}^{\mathbb{Z}}_{T_{M/L}/L} = \mathrm{Frob}^{\mathbb{Z}}_{T_{M/K}/K} \cap \mathrm{Gal}(T_{M/L}/L)$ (and $T_{M/L} = L \cdot T_{M/K}$).

If $\sigma \in \mathrm{Gal}(M/L)$, then

$$\begin{aligned}
\sigma \in W(M/K) &\iff \sigma|_{T_{M/L}} \in \mathrm{Frob}^{\mathbb{Z}}_{T_{M/L}/L} \\
&\overset{\text{above}}{\iff} \sigma|_{T_{M/K}} \in \mathrm{Frob}^{\mathbb{Z}}_{T_{M/K}/K} \\
&\iff \sigma \in W(M/K)
\end{aligned}$$

Third part: now L/K is Galois as well.

$\mathrm{Gal}(M/L)$ is normal in $\mathrm{Gal}(M/K) \implies W(M/L)$ is normal in $W(M/K)$ by the second part.

$$\begin{aligned}\frac{W(M/K)}{W(M/L)} &= \frac{W(M/K)}{W(M/K)\cap \mathrm{Gal}(M/K)} \\ &\cong \frac{W(M/K)\,\mathrm{Gal}(M/L)}{\mathrm{Gal}(M/L)} \\ &= \frac{\mathrm{Gal}(M/K)}{\mathrm{Gal}(M/L)} \\ &\cong \mathrm{Gal}(L/K)\end{aligned}$$

Since $W(M/K)\,\mathrm{Gal}(M/L) = \mathrm{Gal}(M/K)$ by density (first part). $\square$

4.3 Main Theorems of Local Class Field Theory

Let K be a local field. A Galois extension L/K is called **abelian** if $\mathrm{Gal}(L/K)$ is abelian.

Fix an algebraic closure $\bar{K}$ of K, and all algebraic extensions considered are subfields of $\bar{K}$. Let K^{sep} be the separable closure of K inside $\bar{K}$.

If L/K and M/K are Galois, then LM/K is Galois and

$$\begin{aligned}\mathrm{Gal}(LM/K) &\hookrightarrow \mathrm{Gal}(L/K) \times \mathrm{Gal}(M/K) \\ \sigma &\mapsto (\sigma|_L,\, \sigma_M)\end{aligned}$$

In particular, L/K and M/K abelian $\implies LM/K$ is abelian.

$\implies \exists$ maximal abelian extension K^{ab} of K.

Notes that $K^{ur} := T_{K^{sep}/K} \subseteq K^{ab}$. Put $\mathrm{Frob}_K = \mathrm{Frob}_{K^{ur}/K}$.

Theorem 103 (Local Artin Reciprocity)**.** *There exists a unique topological isomorphism* $\mathrm{Art}_L : K^\times \xrightarrow{\sim} W(K^{ab}/K)$*, characterised by*

1. $\mathrm{Art}_K(\pi_K)|_{K^{ur}} = \mathrm{Frob}_K$ *(π_K any uniformiser)*
2. $\mathrm{Art}_K(N_{L/K}(x))|_L = id_L\ \forall L/K$ *finite abelian,* $x \in L^\times$

Moreover, if M/K is finite, then $\mathrm{Art}_M(x)|_{K^{ab}} = \mathrm{Art}_K(N_{M/K}(x))\ \forall x \in M^\times$*, and* Art_K *induces an isomorphism*

$$\frac{K^\times}{N_{M/K}(M^\times)} \xrightarrow{\sim} \mathrm{Gal}((M \cap K^{ab})/K)$$

Write $N(L/K) = N_{L/K}(L^\times)$ for L/K finite.

Theorem 104. *L/K finite $\implies N(L/K) = N((L\cap K^{ab})/K)$, and $[K^\times : N(L/K)] \leq [L:K]$ with equality $\iff L/K$ abelian.*

Proof. Put $M = L \cap K^{ab}$. Have

$$\frac{K^\times}{N(L/K)} \xrightarrow[\mathrm{Art}_K]{\sim} \mathrm{Gal}(M/K) \xleftarrow[\mathrm{Art}_K]{\sim} \frac{K^\times}{N(M/K)}$$

Since $N(L/K) \subseteq N(M/K)$, we are done. □

Theorem 105. *Let L/K be a finite extension, M/K abelian. Then $N(L/K) \subseteq N(M/K) \iff M \subseteq L$.*

Proof. By Theorem 104, wlog L/K abelian (replace it with $L \cap K^{ab}$). $\impliedby$ is clear. Assume that $N(L/K) \subseteq N(M/K)$ and let $\sigma \in \mathrm{Gal}(K^{ab}/L)$.

Then $W(K^{ab}/L) = \mathrm{Art}_K(N(L/K)) \subseteq \mathrm{Art}_K(N(M/K)) \implies \exists m \in M^\times$ such that $\sigma = \mathrm{Art}_K(N_{M/K}(x))$.

Then $\sigma|_M = id_M$ by Theorem 103. □

Theorem 106. *let L/K, M/K be finite abelian extensions of a local field K. Then $N(LM/K) = N(L/K) \cap N(M/K)$ and $N(L\cap M/K) = N(L/K)\cdot N(M/K)$.*

Theorem 107 (Existence Theorem)**.** *For every open subgroup $H \subseteq K^\times$ of finite index, $\exists!$ L/K finite abelian such that $H = N(L/K)$.*

Summary:

$$\left\{\begin{array}{c}\text{Open finite}\\ \text{index subgroups}\\ \text{of } K^\times\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Finite abelian}\\ \text{extensions}\\ L/K\end{array}\right\}$$

$$H \longmapsto (K^{ab})^{\mathrm{Art}_K(H)}$$

$$N(L/K) \longleftarrow\!\shortmid\; L/K$$

Goal for the rest of the course: indicate how one can explicitly construct the field K^{ab} and Art_K (Lubin-Tate theory).

Lemma 108. *Let L/K be a finite abelian extension. Then*

$$e_{L/K} = (\mathcal{O}_L^\times : N_{L/K}(\mathcal{O}_L^\times))$$

Proof. Let $x \in L^\times$, w valuation on L extending v_K. $n = [L:K]$.

$$v_K(N_{L/K}(x)) = nw(x) = f_{L/K}v_L(x)$$

Thus

$$\frac{K^\times}{N(L/K)} \overset{v_K}{\twoheadrightarrow} \frac{\mathbb{Z}}{f_{L/K}(\mathbb{Z})}$$

Kernel $= \frac{\mathcal{O}_K^\times N(L/K)}{N(L/K)} \cong \frac{\mathcal{O}_K^\times}{\mathcal{O}_K^\times \cap N(L/K)} = \frac{\mathcal{O}_K^\times}{N_{L/K}(\mathcal{O}_L^\times)}$

$$\implies n \overset{\text{LCFT}}{=} (K^\times : N(L/K)) = f_{L/K}(\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times))$$
$$\implies (\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times)) = e_{L/K}$$

□

Corollary 109. *L/K finite abelian. Then L/K unramified $\implies N_{L/K}(\mathcal{O}_L^\times) = \mathcal{O}_K^\times$.*

Fix a uniformiser π_K. $K^\times \cong \langle \pi_K \rangle \times \mathcal{O}_K^\times$ (topologically as well). To construct K^{ab}, we need extensions with norm groups $\langle \pi_K^m \rangle \times U_K^{(n)}$ for all $m, n \in \mathbb{Z}_{\geq 0}$. Suffices to consider $\langle \pi_K^m \rangle \times \mathcal{O}_K^\times$ and $\langle \pi_K \rangle \times U_K^{(n)}$.

By Lemma 108, $\langle \pi_K^m \rangle \times \mathcal{O}_K^\times$ is the norm group of the unique unramified extension of degree m. So we need to focus on $\langle \pi_K \rangle \times U_K^{(n)}$ (note the groups depend on the choice of π_K).

$K = \mathbb{Q}_p$, $\pi_K = p$, ζ_{p^n} a primitive root of 1:

$L_n = \mathbb{Q}_p(\zeta^n)$ is the field with norm group $\langle p \rangle \times (1 + p^n\mathbb{Z}_p)$.

Put $\mathbb{Q}_p(\zeta_{p^\infty}) = \bigcup_{n=1}^\infty \mathbb{Q}_p(\zeta_{p^n})$. We have

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \arrow[r, "\sim"] & \varprojlim_n \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) & (\sigma_m, \sigma_m(\zeta_{p^n}) = \zeta_{p^n}^m) \\
\mathbb{Z}_p^\times \arrow[u, "\mathrm{Art}_{\mathbb{Q}_p}|_{\mathbb{Z}_p^\times}"] \arrow[r, "\sim"] & \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times \arrow[u] & m \arrow[u, mapsto]
\end{tikzcd}$$

Explicitly, if $m \in \mathbb{Z}_p^\times$, $m = a_0 + a_1 p + \dots$, $a_i \in \{0, \dots, p-1\}$, $a_0 \neq 0$ then $\mathrm{Art}_{\mathbb{Q}_p}(m) = \sigma_m$,

$$\begin{aligned}
\sigma_m(\zeta_{p^n}) &= \zeta_{p^n}^{a_0 + a_1 p + \dots + a_{n-1}p^{n-1}} \\
&= \lim_{k\to\infty} \zeta_{p^n}^{a_0 + a_1 p + \dots + a_k p^k} \text{ `='} \zeta_{p^n}^m
\end{aligned}$$

for all m, n.

$$\begin{tikzcd}
\mathbb{Q}_p^\times \arrow[r, "\sim", "\mathrm{Art}_{\mathbb{Q}_p}"'] \arrow[d, phantom, "\cong" rotate=90] & W(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) = W(\mathbb{Q}_p^{ur} \cdot \mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \arrow[d, "\wr"] & \sigma \arrow[d, mapsto] \\
\langle p \rangle \times \mathbb{Z}_p^\times \arrow[r, "\sim"] & W(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) & (\sigma|_{\mathbb{Q}_p^{ur}}, \sigma|_{\mathbb{Q}_p(\zeta_{p^\infty})})
\end{tikzcd}$$

$$\langle p^n, m \rangle \longmapsto (\mathrm{Frob}_{\mathbb{Q}_p}^n, \sigma_m)$$

Theorem 110 (Local Kronecker-Weber Theorem)**.**

$$\mathbb{Q}_p^{ab} = \bigcup_{n \in \mathbb{Z}_{\geq 1}} \mathbb{Q}_p(\zeta_n)$$

(Since $\mathbb{Q}_p^{ur} = \bigcup_{\substack{n \in \mathbb{Z}_{\geq 1} \\ (n,p)=1}} \mathbb{Q}_p(\zeta_n)$, *Q2 sheet 3).*

Definition 111. Let K be a local field, M/K a Galois extension. Define, for $s \in \mathbb{R}_{\geq -1}$,

$$G^s(M/K) = \{\sigma \in \mathrm{Gal}(M/K) \,|\, \sigma|_L \in G^s(L/K) \ {}_{\substack{\forall L/K \text{ finite Galois} \\ \text{subextensions of } M/K}}\}$$

Note that $G^s(M/K) = \varprojlim_{L/K} G^s(L/K)$.

$K = \mathbb{Q}_p$, write $\mathbb{Q}_{p^n}$ for the unramified extension of degree n of $\mathbb{Q}_p$.

Q11 on sheet 3 $\implies$

$$G^s(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p) = \begin{cases} \mathrm{Gal}(\mathbb{Q}_{p^n}(\zeta_{p^m})/\mathbb{Q}_p) & s = -1 \\ \mathrm{Gal}(\mathbb{Q}_{p^n}(\zeta_{p^m})/\mathbb{Q}_{p^n}) \cong \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p) & -1 < s \leq 0 \\ \mathrm{Gal}(\mathbb{Q}_{p^n}(\zeta_{p^m})/\mathbb{Q}_{p^n}(\zeta_{p^k})) & k-1 < s \leq k,\ k = 1, \ldots, m-1 \\ 1 & s > m-1 \end{cases}$$

Which corresponds to

$$\begin{cases} \langle p \rangle \times \mathbb{Z}_p^\times / \langle p^n \rangle \times (1 + p^m \mathbb{Z}_p) & s = -1 \\ \langle p^n \rangle \times \mathbb{Z}_p^\times / \langle p^n \rangle \times (1 + p^m \mathbb{Z}_p) & -1 < s \leq 0 \\ \langle p^n \rangle \times (1 + p^k \mathbb{Z}_p) / \langle p^n \rangle \times (1 + p^m \mathbb{Z}_p) & k-1 < s \leq k,\ k = 1, \ldots, m-1 \\ 1 & s > m-1 \end{cases}$$

under $\mathrm{Art}_{\mathbb{Q}_p}$.

Theorem 112. $G^s(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) = \mathrm{Art}_{\mathbb{Q}_p}(1 + p^n \mathbb{Z}_p)$ $\big(= \mathrm{Art}_{\mathbb{Q}_p}(U^{(n)})\big)$ *where* $n - 1 < s \leq n$, $n \in \mathbb{Z}_{\geq 0}$.

Corollary 113. *Let* $L/\mathbb{Q}_p$ *be a finite abelian extension. Then*

$$G^s(L/\mathbb{Q}_p) = \mathrm{Art}_{\mathbb{Q}_p}\left(\frac{N(L/\mathbb{Q}_p)(1 + p^n \mathbb{Z}_p)}{N(L/\mathbb{Q}_p)}\right)$$

for $n - 1 < s \leq n$.

($\mathrm{Art}_{\mathbb{Q}_p} : \frac{\mathbb{Q}_p^\times}{N(L/\mathbb{Q}_p)} \xrightarrow{\sim} \mathrm{Gal}(L/\mathbb{Q}_p)$*)*

It follows that $L \subseteq \mathbb{Q}_{p^n}(\zeta_{p^m})$ for some $n \iff G^s(L/\mathbb{Q}_p) = 1\ \forall s > m - 1$.

4.4 Formal Groups

Let R be a ring.

Write

$$R[[X_1, \ldots, X_n]] = \left\{ \sum_{k_1,\ldots,k_n \in \mathbb{Z}_{\geq 0}} a_{k_1,\ldots,k_n} X_1^{k_1} \ldots X_n^{k_n} \mid a_{k_1,\ldots,k_n} \in R \right\}$$

the ring of formal power series in n variables over R.

Definition 114. A (one-dimensional, commutative) **formal group** over R is a power series $F(X,Y) \in R[X,Y]$ such that

1. $F(X,Y) = X + Y \mod (X^2, XY, Y^2)$
2. $F(X,Y) = F(Y,X)$ (commutativity)
3. $F(X, F(Y,Z)) = F(F(X,Y),Z)$ (associativity)

If F is a formal group over $\mathcal{O}_K$, K a complete valued field, then $F(x,y)$ converges for all $x, y \in \mathfrak{m}_K$, so $\mathfrak{m}_K$ becomes a (semi)group under the multiplication

$$(x,y) \mapsto F(x,y) \in \mathfrak{m}_K$$

For example,

1. $\hat{\mathbb{G}}_a(X,Y) = X + Y$, the formal additive group
2. $\hat{\mathbb{G}}_m(X,Y) = X + Y + XY$, the formal multiplicative group

Note that $X + Y + XY = (1+X)(1+Y) - 1$. If K is a complete valued field then

$$\mathfrak{m}_K \xrightarrow{\sim} 1 + \mathfrak{m}_K$$
$$x \mapsto 1 + x$$

and the rule $(x,y) \in \mathfrak{m}_K^2 \mapsto x + y + xy \in \mathfrak{m}_K$ is just the usual multiplication on $1 + \mathfrak{m}_K$ transported to $\mathfrak{m}_K$ via the bijection above.

Lemma 115. *Let R be a ring and F a formal group over R. Then*

1. *$F(X,0) = X$ (existence of identity)*
2. *$\exists i(X) \in XR[[X]]$ such that $F(X, i(X)) = 0$ (inverses)*

Proof. Example sheet 4 $\square$

Definition 116. Let R be a ring, F, G formal groups over R. A **homomorphism** $f : F \to G$ is an element $f \in R[[X]]$ such that $f(X) \equiv 0 \mod X$ and

$$f(F(X,Y)) = G(f(X), f(Y))$$

The endomorphisms $f : F \to F$ form a ring $\mathrm{End}_R(F)$ with addition $+_F$ given by

$$(f +_F g)(X) = F(f(X), g(X))$$

and multiplication

$$(f \circ g)(X) = f(g(X))$$

Definition 117. Let $\mathcal{O}$ be a ring. A **formal $\mathcal{O}$-module** F is a formal group F with a ring homomorphism

$$\begin{aligned} \mathcal{O} &\to \mathrm{End}_{\mathcal{O}}(F) \\ a &\mapsto [a]_F \end{aligned}$$

such that

$$[a]_F(X) \equiv aX \mod X^2$$

Now let K be a local field, $q = \#k_K$ and $\pi \in \mathcal{O}_K$ a uniformiser.

Definition 118. A **Lubin-Tate module** over $\mathcal{O}_K$ with respect to π is a formal $\mathcal{O}_K$-module F such that $[\pi]_F(X) \equiv X^q \mod \pi$

Think of this condition as 'uniformiser $\iff$ Frobenius'.

$\hat{\mathbb{G}}_m$ is a Lubin-Tate $\mathbb{Z}_p$-module with respect to p. If $a \in \mathbb{Z}_p$, define

$$[a]_{\hat{\mathbb{G}}_m}(X) = (1+X)^a - 1 = \sum_{n=1}^{\infty} \binom{a}{n} X^n$$

Note that $(1+X)^a - 1 \equiv aX \mod X^2$. That $a \mapsto [a]_F$ is a ring homomorphism follows from the identities

$$((1+X)^a)^b = (1+X)^{ab}$$

$$(1+X)^a(1+X)^b = (1+X)^{a+b}$$

So $\hat{\mathbb{G}}_m$ is a formal $\mathbb{Z}_p$-module, and

$$[p]_{\hat{\mathbb{G}}_m}(X) = \sum_{n=1}^{p} \binom{p}{n} X^n \equiv X^p \mod p$$

So $\hat{\mathbb{G}}_m$ is a Lubin-Tate $\mathbb{Z}_p$-module for p.

Definition 119. A **Lubin-Tate series** for π is a power series $e(X) \in \mathcal{O}_K[[X]]$ such that $e(X) \equiv \pi X \mod X^2$, and $e(X) \equiv X^q \mod \pi$. We denote the set of Lubin-Tate series for π by $\mathcal{E}_\pi$.

Inside $\mathcal{E}_\pi$ we have the polynomials

$$uX^q + \pi(a_{q-1}X^{q-1} + \cdots + a_2X^2) + \pi X$$

with $u \in U_K^{(1)}$ and $a_2, \ldots, a_{q-1} \in \mathcal{O}_K$. These are called **Lubin-Tate polynomials**.

For example, $X^q + \pi X$.

If $K = \mathbb{Q}_p$, $\pi = p$ then $(1+X)^p - 1$ is a Lubin-Tate polynomial.

Note that, by definition, if F is a Lubin-Tate $\mathcal{O}_K$-module for π, then $[\pi]_F$ is a Lubin-Tate series for π.

Proposition 120. *Let $e_1, e_2 \in \mathcal{E}_\pi$ and a linear form $L(X_1, \ldots, X_n) = \sum_{i=1}^n a_i X_i$, $a_i \in \mathcal{O}_K$. Then $\exists!$ power series $F(X_1, \ldots, X_n) \in \mathcal{O}_K[[X_1, \ldots, X_n]]$ such that*

$$F(X_1, \ldots, X_n) \equiv L(X_1, \ldots, X_n) \mod (X_1, \ldots, X_n)^2$$

$$e_1(F(X_1, \ldots, X_n)) = F(e_2(X_1), \ldots, e_2(X_n))$$

Now let $e, e_1, e_2 \in \mathcal{E}_\pi$ and $a \in \mathcal{O}_K$. Proposition 120 $\implies$ $\exists!$ $F_e(X,Y) \in \mathcal{O}_K[X,Y]$ and $[a]_{e_1,e_2}(X) \in \mathcal{O}_K[[X]]$ such that

$$F_e(X,Y) \equiv X + Y \mod (X,Y)^2,\ e(F_e(X,Y)) = F_e(e(X), e(Y))$$

$$[a]_{e_1,e_2}(X) \equiv aX \mod X^2,\ e_1([a]_{e_1,e_2}(X)) = [a]_{e_1,e_2}(e_2(X))$$

If $e_1 = e_2 = e$, write $[a]_e = [a]_{e,e}$.

Theorem 121. *The Lubin-Tate $\mathcal{O}_K$-modules for π are precisely the series F_e for $e \in \mathcal{E}_\pi$, with formal $\mathcal{O}_K$-module structure given by $a \mapsto [a]_e$.*

Moreover, if $e_1, e_2 \in \mathcal{E}_\pi$ and $a \in \mathcal{O}_K$, then $[a]_{e_1,e_2}$ is a homomorphism $F_{e_2} \to F_{e_1}$. If $a \in \mathcal{O}_K^\times$, then it is an isomorphism with inverse $[a^{-1}]_{e_2,e_1}$.

Proof (sketch). If F is a Lubin-Tate $\mathcal{O}_K$-module for π, then $e = [\pi]_F \in \mathcal{E}_\pi$ and F satisfies the properties that characterise F_e, so Proposition 120 $\implies$ $F = F_e$.

For the remaining parts, one has to verify

1. $F_e(X,Y) = F_e(Y,X)$
2. $F_e(X, F_e(Y,Z)) = F_e(F_E(X,Y), Z)$

3. $[a]_{e_1,e_2}(F_{e_2}(X,Y)) = F_{e_1}([a]_{e_1,e_2}(X),[a]_{e_1,e_2}(Y))$

4. $[ab]_{e_1,e_3}(X) = [a]_{e_1,e_2}([b]_{e_2,e_3}(X))$

5. $[a+b]_{e_1,e_2}(X) = F_{e_1}([a]_{e_1,e_2}(X),[b]_{e_1,e_2}(X))$

6. $[\pi]_e(X) = e(X)$

for all $e, e_1, e_2, e_3 \in \mathcal{E}_\pi$ and $a, b \in \mathcal{O}_K$.

The proof of these all follow the same pattern: show that LHS and RHS satisfy the same 'characterising properties' in Proposition 120 and use uniqueness. □

4.5 Lubin-Tate Extensions

Recall $\bar{K}$, a fixed algebraic closure of K. Let $\bar{\mathfrak{m}} = \mathfrak{m}_{\bar{K}}$, the maximal ideal in $\mathcal{O}_{\bar{K}}$.

Proposition 122. *If F is a formal $\mathcal{O}_K$-module, then $\bar{\mathfrak{m}}$ becomes an $\mathcal{O}_K$-module under the operations $+_F, \cdot$.*

$$x +_F y = F(x,y) \quad x, y \in \bar{\mathfrak{m}}$$

$$a \cdot x = [a]_F(x) \quad a \in \mathcal{O}_K,\, x \in \bar{\mathfrak{m}}$$

which we denote $\mathfrak{m}_F$.

Proof. Note that if $x, y \in \bar{\mathfrak{m}}$, then $F(x,y)$ is a series in $K(x,y) \subseteq \bar{K}$ with coefficients of absolute value < 1 and $\to 0$, so it converges to an element in $\mathfrak{m}_{K(x,y)} \subseteq \bar{\mathfrak{m}}$. The rest follows from the definitions. □

Let F be a Lubin-Tate $\mathcal{O}_K$-module for π.

Definition 123. Let $n \geq 1$. The group $F(n)$ of $\pi^{\mathbf{n}}$**-division points** of F is defined to be

$$\begin{aligned} F(n) &= \{x \in \bar{\mathfrak{m}}_F \mid \pi^n \cdot x = 0\} \\ &= \ker[\pi^n]_F \end{aligned}$$

For example, $F = \hat{\mathbb{G}}_m$, $K = \mathbb{Q}_p$, $\pi = p$:

$$p^n \cdot x = (1+x)^{p^n} - 1,\, x \in \bar{\mathfrak{m}}_{\hat{\mathbb{G}}_m}$$

So $\hat{\mathbb{G}}_m(n) = \{\zeta_{p^n}^i - 1 \mid i = 0, 1, \ldots, p^n - 1\}$, $\zeta_{p^n} \in \mathbb{Q}_p$ primitive p^n-th root.

So $\hat{\mathbb{G}}_m(n)$ generates $\mathbb{Q}_p(\zeta_{p^n})$.

Lemma 124. *Let $e(X) = X^q + \pi X$, $f_n(X) = (e \circ \cdots \circ e)(X)$ (composed n times).*

Then f_n has no repeated roots.

Proof. Let $x \in \bar{K}$.

Claim: if $|f_i(x)| < 1$ for $i = 0, \ldots, n-1$ then $f_n'(x) \neq 0$.

Induction on n. $n = 1$: assume $|X| < 1$, then

$$\begin{aligned} f_1'(X) &= e'(X) \\ &= qX^{q-1} + \pi \\ &= \pi(1 + \frac{q}{\pi}X^{q-1}) \neq 0 \end{aligned}$$

since $\left|1 + \frac{1}{\pi}X^{q-1}\right| < 1$.

Induction step:

$$\begin{aligned} f_{n+1}'(X) &= (qf_n(X)^{q-1} + \pi)f_n'(X) \\ &= \pi(1 + \frac{q}{\pi}f_n(X)^{q-1})f_n'(X) \end{aligned}$$

By induction $f_n'(X) \neq 0$, and by assumption $|f_n(X)| < 1$, so the same argument works.

We now prove the lemma by showing that if $f_n(X) = 0$, then $|f_i(X)| < 1 \ \forall i = 0, 1, \ldots, n-1$. By induction,

$$f_n(X) = X^{q^n} + \pi g_n(X)$$

for some $g_n \in \mathcal{O}_K[X]$.

It follows that if $f_n(X) = 0$, then $|X| < 1 \implies |f_i(X)| < 1 \ \forall i$. $\square$

Proposition 125. *$F(n)$ is a free $\mathcal{O}_K/\pi^n\mathcal{O}_K$-module of rank 1.*

Proof. By Theorem 121 all Lubin-Tate modules for π are isomorphic $\implies$ all the $\mathcal{O}_K$-modules $F(n)$ are isomorphic. By definition $\pi^n \cdot F(n) = 0$, so $F(n)$ is an $\mathcal{O}_K/\pi^n\mathcal{O}_K$-module.

Choose $F = F_e$, $e(X) = X^q + \pi X$. $F(n)$ consists of the roots of the polynomial $f_n(X) = e^n(X)$, which os of degree q^n and has no repeated roots (Lemma 124).

So $\#F(n) = q^n$.

If $\lambda_n \in F(n) \backslash F(n-1)$, then we have a homomorphism

$$\begin{aligned} \mathcal{O}_K &\to F(n) \\ a &\mapsto a \cdot \lambda_n \end{aligned}$$

with kernel $\pi^n\mathcal{O}_K$ by choice of λ_n. By counting we get an $\mathcal{O}_K$-module isomorphism $\mathcal{O}_K/\pi^n\mathcal{O}_K \xrightarrow{\sim} F(n)$ as desired. $\square$

Corollary 126. *We have isomorphisms*

$$\mathcal{O}_K/\pi^n\mathcal{O}_K \cong \mathrm{End}_{\mathcal{O}_K}(F(n))$$

$$U_K/U_K^{(n)} \cong \mathrm{Aut}_{\mathcal{O}_K}(F(n))$$

Given a Lubin-Tate $\mathcal{O}_K$-module F for π, consider $L_{n,\pi} = L_n = K(F(n))$ of π^n-division points of F. We have inclusions $F(n) \subseteq F(n+1)$ $\forall n$, so $L_n \subseteq L_{n+1}$. The field L_n only depends on π and **not** on F. To see this, let G be another Lubin-Tate $\mathcal{O}_K$-module, and let $f : F \to G$ be an isomorphism of formal $\mathcal{O}_K$-modules.

Then $G(n) = f(F(n)) \subseteq K(F(n)) \implies K(G(n)) \subseteq K(F(n))$. By symmetry, $K(G(n)) = K(F(n))$.

Theorem 127. *L_n/K is a totally ramified abelian extension of degree $q^{n-1}(q-1)$ with Galois group* $\mathrm{Gal}(L_n/K) \cong \mathrm{Aut}_{\mathcal{O}_K}(F(n)) \cong U_K/U_K^{(n)}$*.*

Here $\forall\sigma \in \mathrm{Gal}(L_n/K)$ $\exists!\, u \in U_K/U_K^{(n)}$ such that $\sigma(\lambda) = [u]_F(\lambda)$ $\forall\lambda \in F(n)$.

Moreover, if $F = F_e$, where $e(X) = X^q + \pi(a_{q-1}X^{q-1} + \cdots + a_2X^2) + \pi X$, and $\lambda_n \in F_n \backslash F_{n-1}$, then λ_n is a uniformiser of L_n and

$$\phi_n(X) = \frac{e^n(X)}{e^{n-1}(X)} = X^{q^{n-1}(q-1)} + \cdots + \pi$$

is the minimal polynomial of λ_n. In particular, $N_{L_n/K}(-\lambda_n) = \pi$.

Proof. If $e(X) = X^q + \pi(a_{q-1}X^{q-1} + \cdots + a_2X^2) + \pi X$, set $F = F_e$.

Then $\phi_n(X) = \frac{e^n(X)}{e^{n-1}(X)} = e^{n-1}(X)^{q-1} + \pi(a_{q-1}e^{n-1}(X)^{q-2} + \cdots + a_2e^{n-1}(X)) + \pi$ is an Eisenstein polynomial of degree $q^{n-1}(q-1)$. If $\lambda_n \in F(n)\backslash F(n-1)$ then λ_n is a root of $\phi_n(X)$, so $K(\lambda_n)/K$ is totally ramified of degree $q^{n-1}(q-1)$ and λ_n is a uniformiser, and $N_{K(\lambda_n)/K}(-\lambda_n) = \pi$.

Now let $\sigma \in \mathrm{Gal}(L_n/K)$. σ induces a permutation of $F(n)$, which is $\mathcal{O}_K$-linear:

$$\sigma(x) +_F \sigma(y) = F(\sigma(x), \sigma(y)) = \sigma(F(x, y)) = \sigma(x +_F y)$$

$$\sigma(a \cdot x) = \sigma([a]_F(X)) = [a]_F(\sigma(x)) = a \cdot \sigma(x)$$

for all $x, y \in \mathfrak{m}_{L_n}$ and $a \in \mathcal{O}_K$.

So we have an injection $\mathrm{Gal}(L_n/K) \hookrightarrow \mathrm{Aut}_{G_K}(F(n)) \cong U_K/U_K^{(n)}$ of groups. Since

$$\#(U_K/U_K^{(n)})) = q^{n-1}(q-1) = [K(\lambda_n) : K] \leq [L_n : K] = \#\,\mathrm{Gal}(L_n/K)$$

we must have equality and $\mathrm{Gal}(L_n/K) \xrightarrow{\sim} U_K/U_K^{(n)}$, moreover $K(\lambda_n) = L_n$. $\square$

$K = \mathbb{Q}_p$, $\pi = p$, recall that $\hat{\mathbb{G}}_m(n) = \{\zeta_{p^n}^i - 1 \,|\, i = 0, \dots, p^n - 1\}$, ζ_{p^n} primitive p^n-th root of 1. The theorem gives $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$, given by, if $a \in \mathbb{Z}_{\geq 0}$, $(a,p) = 1$ then

$$\begin{aligned}\sigma_a(\zeta_{p^n}^i - 1) &= [a]_{\hat{\mathbb{G}}_m(n)}(\zeta_{p^n}^i - 1)\\ &= (1 + (\zeta_{p^n}^i - 1))^a - 1\\ &= \zeta_{P^n}^{ia} - 1\end{aligned}$$

so this agrees with the isomorphism $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$ constructed by hand.

Back to the general situation: set $L_\infty = \bigcup_{n=1}^\infty L_n$, we have

$$\begin{aligned}\mathrm{Gal}(L_\infty/K) &\xrightarrow{\sim} \varprojlim_n \mathrm{Gal}(L_n/K) \xrightarrow{\sim} \lim_n U_K/U_K^{(n)} \cong U_K\\ \sigma &\longmapsto (\sigma|_{L_n})_n\end{aligned}$$

This is $\mathrm{Art}_K|_{L_\infty}$.

Theorem 128 (Generalised Local Kronecker-Weber Theorem)**.**

$$K^{ab} = K^{ur} \cdot L_\infty \ \forall \pi$$

Theorem 129.

$$N(L_n/K) = \langle \pi \rangle \times U_K^{(n)}$$

Full Artin map for K:

$$\begin{tikzcd} K^\times \arrow[r, "\sim", "\mathrm{Art}_K"'] \arrow[d, phantom, "\cong" rotate=90] & W(K^{ab}/K) \arrow[d, "\wr"'] & \sigma \arrow[d, mapsto] \\ \langle\pi\rangle \times U_K \arrow[r, "\sim"] & W(K^{ur}/K) \times \mathrm{Gal}(L_\infty/K) & (\sigma|_{K^{ur}}, \sigma|_{L^\infty}) \end{tikzcd}$$

$$(\pi^m, u) \longmapsto (\mathrm{Frob}_K^m, \sigma_u)$$

where $\sigma_u(\lambda) = [u]_F(\lambda)$ for all $\lambda \in \bigcup_{n=1}^\infty F(n)$.

Lemma 130. *The following diagram commutes ($m \geq n$)*

$$\begin{tikzcd} \mathrm{Gal}(L_m/K) \arrow[r, "\sim", "127"'] \arrow[d, "restriction"'] & U_K/U_K^{(m)} \arrow[d, "quotient"] \\ \mathrm{Gal}(L_n/K) \arrow[r, "\sim", "127"'] & U_K/U_K^{(n)} \end{tikzcd}$$

Proof. Let $u \in U_K$, $\sigma = \sigma_u \in \mathrm{Gal}(L_m/K)$. Then $\sigma_u(\lambda) = [u]_F(\lambda)$ for all $\lambda \in F(m) \implies \sigma_u(\lambda) = [u]_F(\lambda)$ for all $\lambda \in F(n) \subseteq F(m)$

So $\sigma_u|_{L_n}$ corresponds to u under $\mathrm{Gal}(L_n/K) \cong U_K/U_K^{(n)}$. $\square$

Corollary 131. *If $m \geq n$, then under the isomorphism* $\mathrm{Gal}(L_m/K) \cong U_K/U_K^{(m)}$ *we have* $\mathrm{Gal}(L_m/L_n) \cong U_K^{(n)}/U_K^{(m)}$.

Proof. Look at the kernels of the vertical maps in the diagram in Lemma 130. □

4.6 Ramification Groups of L_n/K

Theorem 132.

$$G_s(L_n/K) = \begin{cases} \mathrm{Gal}(L_n/L) & -1 \leq s \leq 0 \\ \mathrm{Gal}(L_n/L_k) & q^{k-1} < s \leq q^k - 1,\ k = 1, \ldots, n-1 \\ 1 & s > q^{n-1} - 1 \end{cases}$$

Proof. By Corollary 131, $\mathrm{Gal}(L_n/L_k) \cong U_K^{(k)}/U_K^{(n)}$ under $\mathrm{Gal}(L_n/K) \cong U_K/U_K^{(n)}$.

In particular, $G_1(L_n/K)$ is a Sylow p-subgroup of $\mathrm{Gal}(L_n/K)$, so we must have $G_1(L_n/K) \cong U_K^{(1)}/U_K^{(n)}$.

$$\implies G_1(L_n/K) = \mathrm{Gal}(L_n/L_1)$$

$$\implies G_s(L_n/K) = \mathrm{Gal}(L_n/L_1) \text{ for } 0 < s \leq 1$$

Let $\sigma = \sigma_u \in G_1(L_n/K)$, $u \in U_K^{(1)}/U_K^{(n)}$.

Write $u = 1 + \epsilon\pi^k$, $\epsilon \in U_K$, some $k = k(u) \geq 1$. Let $\lambda \in F(n) \backslash F(n-1)$ (F a choice of Lubin-Tate $\mathcal{O}_K$-module for π), λ is a uniformiser of L_n and $\mathcal{O}_{L_n} = \mathcal{O}_K[\lambda]$.

We have

$$\begin{aligned} \sigma_u(\lambda) &= [u]_F(\lambda) \\ &= [1 + \epsilon\pi^k]_F(\lambda) \\ &= F(\lambda, [\epsilon\pi^k]_F(\lambda)) \end{aligned}$$

If $k \geq n$, $\sigma = 1$ so $v_{L_n}(\sigma(\lambda) - \lambda) = \infty$. If $k < n$, then $[\epsilon\pi^k]_F(\lambda) = [\epsilon]_F([\pi^k]_F(\lambda)) \in F(n-k) \backslash F(n-k-1)$ so $[\epsilon\pi^k]_F(\lambda)$ is a uniformiser of L_{n-k}.

L_n/L_{n-k} is totally ramified of degree q^k, so $[\epsilon\pi^k]_F(\lambda) = \epsilon_0\lambda^{q^k}$, $\epsilon_0 \in \mathcal{O}_{L_n}^\times$.

Recall that $F(X, 0) = X$, $F(0, Y) = Y$, so

$$F(X, Y) = X + Y + XYG(X, Y),\ G(X, Y) \in \mathcal{O}_K[[X, Y]]$$

So

$$\begin{aligned} \sigma(\lambda) - \lambda &= F(\lambda, [\epsilon\pi^k]_F(\lambda)) - \lambda \\ &= F(\lambda, \epsilon_0\lambda^{q^k}) - \lambda \\ &= \lambda + \epsilon_0\lambda^{q^k} + \epsilon_0\lambda^{q^k+1}G(\lambda, \epsilon_0^{q^k}) - \lambda \\ &= \epsilon_0\lambda^{q^k} + \epsilon_0\lambda^{q^k+1}G(\lambda, \epsilon_0^{q^k}) \end{aligned}$$

$\implies v_{L_n}(\sigma(\lambda) - \lambda) = q^k$

So $i_{L_n/K}(\sigma_u) \geq s + 1 \iff q^{k(u)} - 1 \geq s$

$$\implies G_s(L_n/K) = \{\sigma_u \in G_1(L_n/K) \,|\, q^{k(u)} - 1 \geq s\}$$
$$= \begin{cases} \mathrm{Gal}(L_n/L_k) & q^{k-1} - 1 < s \leq q^k - 1 \text{ for } k = 1, \ldots, n-1 \\ 1 & s > q^{n-1} - 1 \end{cases}$$

□

Corollary 133.

$$G^t(L_n/K) = \begin{cases} \mathrm{Gal}(L_n/K) & -1 \leq t \leq 0 \\ \mathrm{Gal}(L_n/L_k) & k-1 < t \leq k, \;\; k = 1, 2, \ldots, n-1 \\ 1 & t > n-1 \end{cases}$$

Proof. Invert:

$$\chi_{L_n/K}(t) = \begin{cases} t & -1 \leq t \leq 0 \\ q^{q-1}(q-1)(t-(k-1)) + q^{k-1} - 1 & k-1 < t \leq k, \;\; k = 1, 2, \ldots, n-1 \\ q^{q-1}(q-1)(t-(n-1)) + q^{n-1} - 1 & t > n-1 \end{cases}$$

$$\eta_{L_n/K}(s) = \begin{cases} s & -1 \leq s \leq 0 \\ (k-1) + \frac{s-(q^{k-1})}{q^{k-1}(q-1)} & q^{k-1} - 1 \leq s \leq q^{k-1} - 1 \\ (n-1) + \frac{s-(q^{n-1})}{q^{n-1}(q-1)} & s \geq q^{n-1} - 1 \end{cases}$$

$\implies G^t(L_n/K) = G_{\chi_{L_n/K}(t)}(L_n/K)$ is as claimed. □

In other words,

$$G^t(L_n/K) = \begin{cases} \mathrm{Gal}(L_n/L_{\lceil t \rceil}) & -1 < t \leq n \\ 1 & t \geq n \end{cases}$$

where $\lceil t \rceil$ = smallest integer m such that $t \leq m$ (here $L_0 = K$). So

$$\mathrm{Art}_K^{-1}(G^t(L_n/K)) = \begin{cases} U_K^{(\lceil t \rceil)}/U_K^{(n)} & -1 \leq t \leq n \\ 1 & t \geq n \end{cases}$$

Corollary 134. *When $t > -1$, $G^t(K^{ab}/K) = \mathrm{Gal}(K^{ab}/K^{ur} \cdot L_{\lceil t \rceil})$ and $\mathrm{Art}_K^{-1}(G^t(K^{ab}/K) = U_K^{(\lceil t \rceil)}$.*

Proof. Recall from examples class:

Lemma 135. *If L/K is a finite unramified extension and M/K is a finite totally ramified extension, then LM/L is totally ramified and*

$$\begin{aligned}\mathrm{Gal}(LM/L) &\cong \mathrm{Gal}(M/K)\\ \sigma &\mapsto \sigma|_M\end{aligned}$$

and $G^t(LM/L) \cong G^t(M/K)$ via this isomorphism ($t > -1$).

Proof cont. Let K_M/K be the unramified extension of degree m. By the Lemma and Corollary 133,

$$G^t(K_mL_n/K) \cong G^t(L_n/K) = \begin{cases}\mathrm{Gal}(L_n/L_{\lceil t\rceil}) & 1 < t \leq n\\ 1 & t \geq n\end{cases}$$

$$\implies G^t(K_mL_n/K) = \begin{cases}\mathrm{Gal}(K_mL_n/K_mL_{\lceil t\rceil}) & -1 < t \leq n\\ 1 & t \leq n\end{cases}$$

$$\begin{aligned}\implies G^t(K^{ab}/K) &= G^t(K^{ur}L_\infty/K)\\ &= \varprojlim_{m,n} G^t(K_mL_n/K)\\ &= \varprojlim_{\substack{m,n\\ n\geq\lceil t\rceil}} \mathrm{Gal}(K_mL_n/K_mL_{\lceil t\rceil})\\ &= \mathrm{Gal}(K^{ur}L_\infty/K^{ur}L_{\lceil t\rceil}) = \mathrm{Gal}(K^{ab}/K^{ur}L_{\lceil t\rceil})\end{aligned}$$

and

$$\begin{aligned}\mathrm{Art}_K^{-1}(\mathrm{Gal}(K^{ab}/K^{ur}L_{\lceil t\rceil})) &= \mathrm{Art}_K^{-1}\left(\varprojlim_{\substack{m,n\\ n\geq\lceil t\rceil}} \mathrm{Gal}(K_mL_n/K_mL_{\lceil t\rceil})\right)\\ &= \varprojlim_{\substack{m,n\\ n\geq\lceil t\rceil}} \mathrm{Art}_K^{-1}\,\mathrm{Gal}(K_mL_n/K_mL_{\lceil t\rceil})\\ &= \varprojlim_{\substack{m,n\\ n\geq\lceil t\rceil}} U_K^{(\lceil t\rceil)}/U_K^{(\lceil t\rceil)} = U^{(\lceil t\rceil)}\end{aligned}$$

□

Corollary 136. *Let M/K be a finite abelian extension. Then, under $\mathrm{Art}_K : \frac{K^\times}{N(M/K)} \xrightarrow{\sim} \mathrm{Gal}(M/K)$,*

$$G^t(M/K) = \mathrm{Art}_K\left(\frac{U_K^{(\lceil t\rceil)}N(M/K)}{N(M/K)}\right) \qquad (t > 1)$$

Proof.

$$\begin{aligned} G^t(M/K) &= \frac{G^t(K^{ab}/K)G(K^{ab}/M)}{G(K^{ab}/M)} \\ &= \mathrm{Art}_K\left(\frac{U_K^{(\lceil t \rceil)}N(M/K)}{N(M/K)}\right) \end{aligned}$$

□