

Part III Category Theory

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1 Definitions and Examples

Definition 1.1 (Category). *A category \mathcal{C} consists of*

- a. a collection $\text{ob } \mathcal{C}$ of **objects** A, B, C, \dots*
- b. a collection $\text{mor } \mathcal{C}$ of **morphisms** f, g, h, \dots*
- c. two operations dom, cod from morphisms to objects. We write $f : A \rightarrow B$ or $A \xrightarrow{f} B$ to mean ' f is a morphism and $\text{dom } f = A$ and $\text{cod } f = B$ '*
- d. an operation assigning to each object A a morphism $1_A : A \rightarrow A$*
- e. a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff \text{dom } g = \text{cod } f$, and then $gf : \text{dom } f \rightarrow \text{cod } g$*

satisfying

- f. $f1_A = f$ and $1_B f = f \ \forall f : A \rightarrow B$*
- g. $h(fg) = (hg)f$ whenever gf and hg are defined*

Definition 1.2 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A **functor** $\mathcal{C} \rightarrow \mathcal{D}$ consists of

- a. a mapping $A \rightarrow FA$ from $\text{ob } \mathcal{C}$ to $\text{ob } \mathcal{D}$
- b. a mapping $f \rightarrow Ff$ from $\text{mor } \mathcal{C}$ to $\text{mor } \mathcal{D}$

satisfying $\text{dom } Ff = F\text{dom } f$, $\text{cod } Ff = F\text{cod } f$ for all f , $F(1_A) = 1_{FA}$ for all A , and $F(gf) = (Fg)(Ff)$ whenever gf is defined.

Definition 1.3. By a **contravariant functor** $\mathcal{C} \rightarrow \mathcal{D}$ we mean a functor $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$ (or equivalently $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$). A functor $\mathcal{C} \rightarrow \mathcal{D}$ is sometimes said to be **covariant**.

Definition 1.4 (Natural transformation). Let \mathcal{C} and \mathcal{D} be two categories and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ two functors. A **natural transformation** $\alpha : F \rightarrow G$ assigns to each $A \in \text{ob } \mathcal{C}$ a morphism $\alpha_A : FA \rightarrow GA$ in \mathcal{D} , such that

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes.

We can compose natural transformations: given $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$, the mapping $A \mapsto \beta_A \alpha_A$ is the A -component of a natural transformation $\beta\alpha : F \rightarrow H$.

Definition 1.5. Given categories \mathcal{C}, \mathcal{D} , we write $[\mathcal{C}, \mathcal{D}]$ for the category of all functors $\mathcal{C} \rightarrow \mathcal{D}$ and natural transformations between them.

Lemma 1.6. Given $F, G : \mathcal{C} \rightarrow \mathcal{D}$ and $\alpha : F \rightarrow G$, α is an isomorphism in $[\mathcal{C}, \mathcal{D}] \iff$ each α_A is an isomorphism in \mathcal{D} .

Definition 1.7 (Faithful and full). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- a. We say that F is **faithful** if, given $f, g \in \text{mor } \mathcal{C}$, the equations $\text{dom } f = \text{dom } g$, $\text{cod } f = \text{cod } g$ and $Ff = Fg$ imply $f = g$.
- b. F is **full** if, given any $g : FA \rightarrow FB$ in \mathcal{D} , there exists $f : A \rightarrow B$ in \mathcal{C} with $Ff = g$.
- c. We say a subcategory \mathcal{C}' of \mathcal{C} is **full** if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full functor.

For example, **Gp** is a full subcategory of the category **Mon** of monoids, but **Mon** is a non-full subcategory of the category **Sgp** of semigroups.

Definition 1.8 (Equivalence of categories). Let \mathcal{C} and \mathcal{D} be categories. An **equivalence** between \mathcal{C} and \mathcal{D} is a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$, $\beta : FG \rightarrow 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ to mean that \mathcal{C} and \mathcal{D} are equivalent.

We say a property P of categories is **categorical** if whenever \mathcal{C} has P and $\mathcal{C} \simeq \mathcal{D}$ then \mathcal{D} has P .

For example, being a groupoid is a categorical property, but being a group is not.

Definition 1.9 (Slice category). Given an object B of a category \mathcal{C} , define the **slice category** \mathcal{C}/B to have morphisms $A \xrightarrow{f} B$ as objects, and morphisms $(A \xrightarrow{f} B) \rightarrow (A' \xrightarrow{f'} B)$ are morphisms $h : A \rightarrow A'$ making

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ & \searrow f & \swarrow f' \\ & B & \end{array}$$

commute.

Lemma 1.10. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D} \iff F$ is full, faithful and **essentially surjective**, i.e. for every $B \in \text{ob } \mathcal{D}$, there exists $A \in \text{ob } \mathcal{C}$ s.t. $FA \cong B$.

Definition 1.11. a. A **skeleton** of a category \mathcal{C} is a full subcategory \mathcal{C}' containing exactly one object from each isomorphism class of objects of \mathcal{C} .

b. We say \mathcal{C} is **skeletal** if it's a skeleton of itself. Equivalently, any isomorphism f in \mathcal{C} satisfies $\text{dom } f = \text{cod } f$.

For example, \mathbf{Mat}_K is skeletal. The full subcategory of standard vector spaces K^n is a skeleton of $\mathbf{fd Mod}_K$.

Remark 1.12. The following statements are each equivalent to the Axiom of Choice:

1. Every small category has a skeleton
2. Any small category is equivalent to each of its skeletons
3. Any two skeletons of a given small category are isomorphic

Definition 1.13. Let $f : A \rightarrow B$ be a morphism in a category \mathcal{C} .

- a. f is a **monomorphism** if, given $g, h : D \rightrightarrows A$, the equation $fg = fh$ implies $g = h$. We write $A \rightarrowtail B$ if f is monic.

- b. Dually, f is an **epimorphism** if, given $k, l : B \rightrightarrows C$, $kf = lf$ implies $k = l$. We write $A \twoheadrightarrow B$ if f is epic.
- c. \mathcal{C} is a **balanced** category if every $f \in \text{mor } \mathcal{C}$ which is both monic and epic is an isomorphism.

2 The Yoneda Lemma

Definition 2.1. A category \mathcal{C} is **locally small** if, for any two objects A, B of \mathcal{C} , the morphism $A \rightarrow B$ are parametrised by a set $\mathcal{C}(A, B)$.

Given local smallness, $B \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$: if $g : B \rightarrow B'$, the mapping $f \mapsto gf : \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, B')$ is functorial since $h(gf) = (hg)f$ for any $h : B' \rightarrow B''$.

Similarly, $A \mapsto \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}^{op} \rightarrow \mathbf{Set}$.

Lemma 2.2 (Yoneda). Let \mathcal{C} be a locally small category, $A \in \text{ob } \mathcal{C}$ and $F : \mathcal{C} \rightarrow \mathbf{Set}$. Then

- i. There is a bijection between natural transformations $\mathcal{C}(A, -) \rightarrow F$ and elements of FA .
- ii. Moreover, this bijection is natural in both A and F .

Proof. Bijection: given $\alpha : \mathcal{C}(A, -) \rightarrow F$, define $\Phi(\alpha) = \alpha_A(1_A) \in FA$.

Given $x \in FA$, define $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$$

$\Psi(x)$ is natural: given $g : B \rightarrow C$, we have

$$\begin{aligned} \Psi(x)_C(\mathcal{C}(A, g)(f)) &= \Psi(x)_C(gf) \\ &= F(gf)(x) \\ &= (Fg)(Ff)(x) \\ &= (Fg)\Psi(x)_B(f) \end{aligned}$$

$\Phi\Psi(x) = x$ since $F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$ since, for any $f : A \rightarrow B$,

$$\begin{aligned} \Psi\Phi(\alpha)_B(f) &= Ff(\Phi(\alpha)) \\ &= Ff(\alpha_A(1_A)) \\ &= \alpha_B(\mathcal{C}(A, f)(1_A)) \\ &= \alpha_B(f) \end{aligned}$$

□

Corollary 2.3. *The mapping $A \rightarrow \mathcal{C}(A, -)$ is a full and faithful functor $\mathcal{C}^{op} \rightarrow [\mathcal{C}, \mathbf{Set}]$.*

Proof. Given two objects A, B , 2.2(i) gives us a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$ (by taking $F : C \mapsto \mathcal{C}(B, C)$). We need to show this is functorial, but given $f \in \mathcal{C}(B, A)$, $\Psi(F)_A$ sends 1_A to $\mathcal{C}(B, f)(1_A) = f$, so it's the natural transformation $g \mapsto gf$.

Hence, given $e : C \rightarrow B$, $\Psi(fe)(g) = g(fe) = (gf)(e) = \Psi(e)\Psi(f)g$ \square

We call this functor the **Yoneda embedding**. Hence any locally small category \mathcal{C} is equivalent to a full subcategory of $[\mathcal{C}^{op}, \mathbf{Set}]$.

Definition 2.4. *A functor $\mathcal{C} \rightarrow \mathbf{Set}$ is **representable** if it's isomorphic to $\mathcal{C}(A, -)$ for some A .*

*A **representation** of $F : \mathcal{C} \rightarrow \mathbf{Set}$ is a pair (A, x) where $A \in \text{ob } \mathcal{C}$, $x \in FA$ and $\Psi(x) : \mathcal{C}(A, -) \rightarrow F$ is an isomorphism. We also call x a **universal element** of F .*

Corollary 2.5 ('Representations are unique up to unique isomorphism'). *If (A, x) and (B, y) are both representations of $F : \mathcal{C} \rightarrow \mathbf{Set}$, then there's a unique isomorphism $f : A \rightarrow B$ s.t. $Ff(x) = y$.*

Definition 2.6 (Product and coproduct). *Given two objects A, B of a locally small category \mathcal{C} , we define their **product** to be a representation of the functor*

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B) : \mathcal{C}^{op} \rightarrow \mathbf{Set}$$

i.e. an object $A \times B$ equipped with morphisms $\pi_1 : A \times B \rightarrow A$, $\pi_2 : A \times B \rightarrow B$ s.t. given any pair $(f : C \rightarrow A, g : C \rightarrow B)$, there exists a unique $h : C \rightarrow A \times B$ s.t. $\pi_1 h = f$ and $\pi_2 h = g$.

*More generally, we can define the product $\prod_{i \in I} A_i$ of a family $\{A_i \mid i \in I\}$ of objects, or the product of the empty family, i.e. a **terminal object** 1 s.t. for every A there's a unique $A \rightarrow 1$.*

*Dualizing, we get the notion of **coproduct** or **sum**.*

Definition 2.7 (Equaliser and coequaliser). *Given a parallel pair $f, g : A \rightrightarrows B$ in a locally small category \mathcal{C} , the assignment $C \mapsto FC = \{h : C \rightarrow A \mid fh = gh\}$ is a subfunctor F of $\mathcal{C}(-, A)$. A representation of F is called an **equaliser** of (f, g) .*

In elementary terms, it's an object E equipped with $e : E \rightarrow A$ s.t. $fe = ge$, s.t. any h with $fh = gh$ factors uniquely as $h = ek$

*Dually, we have the notion of **coequaliser**, i.e. a morphism $q : B \rightarrow Q$ satisfying $qf = qg$, and universal among such.*

Definition 2.8. a. We say a monomorphism is **regular** if it occurs as an equaliser (dually, regular epimorphism).

b. We say $f : A \rightarrow B$ is a **split monomorphism** if there exists $g : B \rightarrow A$ with $gf = 1_A$.

Every split monomorphism is regular: if $gf = 1_A$, f is an equaliser of $(1_B, fg)$ [see sheet 1, q2].

Definition 2.9. Let \mathcal{C} be a (locally small) category, \mathcal{G} a collection of objects of \mathcal{C} .

a. Say \mathcal{G} is a **separating family** if the functors $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ are jointly faithful, i.e. if given $f, g : A \rightrightarrows B$ with $f \neq g$, there exists $G \in \mathcal{G}$ and $h : G \rightarrow A$ with $fh \neq gh$.

b. Say \mathcal{G} is a **detecting family** if the $\mathcal{C}(G, -)$, $G \in \mathcal{G}$ jointly reflect isomorphisms, i.e. if given $f : A \rightarrow B$ s.t. every $g : G \rightarrow B$ with $G \in \mathcal{G}$ factors uniquely through f , f is an isomorphism.

Lemma 2.10. i. If \mathcal{C} is balanced, then any separating family is detecting

ii. If \mathcal{C} has equalisers, then every detecting family is separating

Definition 2.11. An object P is **projective** if $\mathcal{C}(P, -)$ preserves epimorphisms, i.e. if given

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ A & \xrightarrow{e} & B \end{array}$$

there exists $g : P \rightarrow A$ with $eg = f$.

Dually, P is **injective** in \mathcal{C} if it's projective in \mathcal{C}^{op} .

If P satisfies this property $\forall e$ in some class \mathcal{E} of epimorphisms, we call it \mathcal{E} -projective.

Corollary 2.12. Representable functors are (pointwise) projective in $[\mathcal{C}, \mathbf{Set}]$

Proof. Given

$$\begin{array}{ccc} & \mathcal{C}(A, -) & \\ & \downarrow \beta & \\ F & \xrightarrow{\alpha} & G \end{array}$$

β corresponds to some $y \in GA$. α_A is surjective, so $\exists x \in FA$ with $\alpha_A(x) = y$. x corresponds to $\gamma : \mathcal{C}(A, -) \rightarrow F$ with $\alpha\gamma = \beta$. \square

3 Adjunctions

Definition 3.1 (D.M. Khan, 1958). Let \mathcal{C} and \mathcal{D} be categories and $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ be two functors. An **adjunction** between F and G is a bijection between morphisms $FA \rightarrow B$ in \mathcal{D} and morphisms $A \rightarrow GB$ in \mathcal{C} , which is natural in A and B .

(If \mathcal{C} and \mathcal{D} are locally small, this says that $(A, B) \rightarrow \mathcal{D}(FA, B)$ and $(A, B) \rightarrow \mathcal{C}(A, GB)$ are naturally isomorphic functors $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$).

We say F is **left adjoint** to G , or G is **right adjoint** to F , and write $F \dashv G$.

Theorem 3.2. Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor. Given $A \in \text{ob } \mathcal{C}$, let $(A \downarrow G)$ be the category whose objects are pairs (B, f) with $B \in \text{ob } \mathcal{D}$, $f : A \rightarrow GB$ and whose morphisms $(B, f) \rightarrow (B', f')$ are morphisms $g : B \rightarrow B'$ in \mathcal{D} such that

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & \searrow f' & \downarrow Gg \\ & & GB' \end{array}$$

commutes. Then specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A .

Proof. First suppose G has a left adjoint F . Let $\eta_A : A \rightarrow GFA$ be the morphism corresponding to $1_{FA} : FA \rightarrow FA$. The pair (FA, η_A) is an object of $(A \downarrow G)$. We'll show it's initial.

Given $g : FA \rightarrow B$, the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$ must correspond to $FA \xrightarrow{1} FA \xrightarrow{g} B$ under the adjunction.

So, for any object (B, f) of $(A \downarrow G)$, the unique morphism $(FA, \eta_A) \rightarrow (B, f)$ in $(A \downarrow G)$ is the morphism $FA \rightarrow B$ corresponding to f .

Conversely, suppose we're given an initial object (FA, η_A) of $(A \downarrow G)$ for each G . Given $f : A \rightarrow A'$, the composite $A \xrightarrow{f} A' \xrightarrow{\eta_{A'}} GFA'$ is an object of $(A \downarrow G)$, so there's a unique morphism $Ff : FA \rightarrow FA'$ making

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow GFf \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commute.

$f \mapsto Ff$ is functorial: given $f' : A' \rightarrow A''$, then $(Ff')(Ff)$ and $F(f'f)$ are both morphisms $(FA, \eta_A) \rightarrow (FA'', \eta_{A''}f'f)$ in $(A \downarrow G)$, so they're equal.

Finally, given $f : A \rightarrow GB$, the morphism $g : FA \rightarrow B$ corresponding to it is the unique morphism $(FA, \eta_A) \rightarrow (B, f)$ in $(A \downarrow G)$.

The naturality of this bijection is given by naturality of η , and naturality in B is immediate. \square

Corollary 3.3. *If F, F' are both left-adjoint to G , then there's a canonical natural isomorphism $F \rightarrow F'$.*

Proof. For each A , (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$, so there's a unique isomorphism $\alpha_A : (FA, \eta_A) \rightarrow (F'A, \eta'_A)$.

α is natural: given $f : A \rightarrow A'$, $\alpha_{A'}f$ and $(Ff)\alpha_A$ are both morphisms $(FA, \eta_A) \rightarrow (F'A', \eta'_{A'})$ in $(A \downarrow G)$. So they're equal. \square

Lemma 3.4. *Given $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D} \xrightleftharpoons[K]{H} \mathcal{E}$, if $F \dashv G$ and $H \dashv K$ then $HF \dashv GK$.*

Proof. We have bijections

$$\mathcal{E}(HFA, C) \cong \mathcal{D}(FA, KC) \cong \mathcal{C}(A, GKC)$$

which are natural in A and C . \square

Corollary 3.5. *Given a commutative square $\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow G & & \downarrow H \\ \mathcal{E} & \xrightarrow{K} & \mathcal{F} \end{array}$ of categories and functors, suppose all the functors in the diagram have left adjoints. Then the diagram $\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{C} \end{array}$ of left adjoints commutes up to natural isomorphism.*

Given $F \dashv G$, we have a natural transformation $\eta : 1_{\mathcal{C}} \rightarrow GF$ defined as in 3.2. We call η the **unit** of the adjunction.

Dually, we have $\epsilon : FG \rightarrow 1_{\mathcal{D}}$, the **counit**. $\epsilon_B : FGB \rightarrow B$ corresponds to $1_{GB} : GB \rightarrow GB$.

Theorem 3.6. *Suppose we're given $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \rightarrow GF$ and $\epsilon : FG \rightarrow 1_{\mathcal{D}}$ such that*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \epsilon_F \\ & & F \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & \searrow 1_G & \downarrow G\epsilon \\ & & G \end{array}$$

*commute. (We say η and ϵ satisfy the **triangular identities**).*

Proof. Given $F \dashv G$, we define η and ϵ as already described. Since $\epsilon_{FA} : FGFA \rightarrow FA$ corresponds to 1_{GFA} , the composite $\epsilon_{FA}(F\eta_A)$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$, so it must be 1_{FA} .

Similarly for the other identity.

Conversely, given η and ϵ satisfying the \triangle^r identities, we map $f : A \rightarrow GB$ to the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$ and $g : FA \rightarrow B$ to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$.

We have

$$\begin{aligned}\Phi(A \xrightarrow{f} GB) &= FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B \\ \Psi(FA \xrightarrow{g} B) &= A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB\end{aligned}$$

So

$$\begin{aligned}\Psi\Phi(f) &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= f\end{aligned}$$

And dually $\Phi\Psi(g) = g$.

Naturality of Φ in A is immediate from its definition, and naturality in B follows from that of ϵ . \square

Lemma 3.7. Suppose given $\mathcal{C} \xrightleftharpoons[F]{F} \mathcal{D}$ and natural isomorphisms $\alpha : 1_{\mathcal{C}} \rightarrow GF$, $\beta : FG \rightarrow 1_{\mathcal{D}}$. Then there exist natural isomorphisms α', β' which additionally satisfy the triangular identities. In particular $(F \dashv G)$.

Proof. We define $\alpha' = \alpha$ and take β' to be the composite

$$FG \xrightarrow{\beta_{FG}^{-1}} FGFG \xrightarrow{F\alpha_G^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that, since $\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \downarrow \beta_{FG} & & \downarrow \beta \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$ commutes and β is monic, we have $FG\beta = \beta_{FG}G$.

Similarly, $GF\alpha = \alpha_{GF} : GF \rightarrow GFGF$.

Now

$$\begin{aligned}\beta'_F \circ F\alpha' &= F \xrightarrow{F\alpha} FGF \xrightarrow{\beta_{FGF}^{-1}} FGFGF \xrightarrow{F\alpha_{GF}^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{\beta_F^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{FGF\alpha^{-1}} FGF \xrightarrow{\beta_F} F \\ &= 1_F\end{aligned}$$

and

$$\begin{aligned}
G\beta' \circ \alpha'_G &= G \xrightarrow{\alpha_G} GFG \xrightarrow{GFG\beta^{-1}} GFGFG \xrightarrow{GF\alpha_G^{-1}} GFG \xrightarrow{G\beta} G \\
&= G \xrightarrow{G\beta^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha_{GFG}^{-1}} GFG \xrightarrow{\beta_F} G \\
&= 1_G
\end{aligned}$$

□

Lemma 3.8. Suppose $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$, $(F \dashv G)$ is an adjunction with counit ϵ . Then

- i. ϵ is (pointwise) epic $\iff G$ is faithful
- ii. ϵ is an isomorphism $\iff G$ is full and faithful

Proof. i. Given $g : B \rightarrow B'$, the morphism $Gg : GB \rightarrow GB'$ corresponds to

$$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} B'$$

So, for fixed B , composition with ϵ_B is injective on morphisms $B \rightarrow B'$
 $\iff (g \mapsto Gg)$ is injective on morphisms $B \rightarrow B'$.

Hence G is faithful $\iff \epsilon_B$ is epic $\forall B$.

- ii. Similarly, ϵ_B is 0 $\forall B \implies G$ is bijective on morphisms with given domain and codomain, i.e. G is full and faithful.

Conversely, if G is full and faithful, 1_{FGB} factors uniquely as

$FGB \xrightarrow{\epsilon_B} B \xrightarrow{g} FGB$, so ϵ_B is split monic. But it's epic by (i), hence an isomorphism.

□

Definition 3.9. i. A **reflection** is an adjunction satisfying the conditions of 3.8(ii).

- ii. A **reflective** subcategory of \mathcal{C} is a full subcategory \mathcal{C}' for which the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint.

Dually, **coreflection** and **coreflective** subcategory.

4 Limits

Definition 4.1. a. Let J be a category (almost always small, often finite). A **diagram of shape J** in a category \mathcal{C} is a functor $D : J \rightarrow \mathcal{C}$.

E.g. if J is the finite category $\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$, a diagram of shape J is a

commutative square. If J is the category $\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$, a diagram of shape J is a not-necessarily-commutative square.

The objects $D(j)$, $j \in \text{ob } J$ are called **vertices** of D , and the morphisms $D(\alpha)$, $\alpha \in \text{mor } J$ are called **edges** of D .

b. Let $D : J \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} . A **cone over D** is a pair $(A, (\lambda_j \mid j \in \text{ob } J))$ where $\lambda_j : A \rightarrow D(j) \forall j$, and

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \lambda_{j'} \\ D(j) & \xrightarrow{D(\alpha)} & D(j') \end{array} \quad \text{commutes for}$$

each $\alpha : j \rightarrow j'$ in J .

A is called the **apex** of the cone, and the λ_j are its **legs**.

Equivalently, λ is a natural transformation $\Delta A \rightarrow D$, where ΔA is the **constant diagram** with all vertices A and all edges 1_A .

A **morphism** $f : (A, (\lambda_j)) \rightarrow (B, (\mu_j))$ of cones over D is a morphism

$$f : A \rightarrow B \text{ s.t. } \begin{array}{ccc} A & \xrightarrow{f} & B \\ \searrow \lambda_j & & \swarrow \mu_j \\ & D(j) & \end{array} \quad \text{commutes for each } j. \text{ We have a category}$$

Cone(D) of cones over D .

Note that $A \mapsto \Delta A$ is a functor $\mathcal{C} \rightarrow [J, \mathcal{C}]$ and **Cone**(D) is in fact the category $(\Delta \downarrow D)$.

A **cocone over $D : J \rightarrow \mathcal{C}$** is a cone over $D : J^{op} \rightarrow \mathcal{C}^{op}$. We write **Cocone**(D) for the category of cocones over D .

Definition 4.2. i. A **limit** (resp. **colimit**) for a diagram $D : J \rightarrow \mathcal{C}$ is a terminal object of **Cone**(D) (respectively an initial object of **Cocone**(D)).

ii. We say \mathcal{C} has limits (resp. colimits) of shape J if $\Delta : \mathcal{C} \rightarrow [J, \mathcal{C}]$ has a right (resp. left) adjoint.

(This is equivalent to making a choice of limit (resp. colimit) for every diagram of shape J).

Definition 4.3 (Pullback). Let J be $\begin{array}{ccc} & \cdot & \\ & \downarrow & \\ \cdot & \longrightarrow & \cdot \end{array}$. A diagram of shape J looks like

$$\begin{array}{ccc}
& A & D \xrightarrow{h} A \\
& \downarrow f & \downarrow k \searrow l \\
B \xrightarrow{g} C & & C \quad B
\end{array}
\quad \text{satisfying } fh = l = gk.$$

Equivalently, it's a pair $\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & & \\ C & & \end{array}$ completing the diagram to a commutative square.

A universal such pair is called a **pullback** (or **fibre product**); in **Set** it can be defined as $\{(a, b) \in A \times B \mid f(a) = g(b)\}$. A colimit of shape J^{op} is called a **pushout**.

Theorem 4.4. *Let \mathcal{C} be a category.*

- i. *If \mathcal{C} has equalisers and all finite (resp. all small) products, then \mathcal{C} has all finite (resp. all small) limits.*
- ii. *If \mathcal{C} has pullbacks and a terminal object, then \mathcal{C} has all finite limits.*

Proof. i. Given $D : J \rightarrow \mathcal{C}$, first form the products

$$P = \prod_{j \in \text{ob } J} D(j) \quad \text{and} \quad Q = \prod_{\alpha \in \text{mor } J} D(\text{cod } \alpha)$$

Define $P \xrightarrow[g]{f} Q$ by $\pi_\alpha f = \pi_{\text{cod } \alpha} : P \rightarrow D(\text{cod } \alpha)$ and $\pi_\alpha g = D(\alpha) \circ \pi_{\text{dom } \alpha} : P \rightarrow D(\text{dom } \alpha) \rightarrow D(\text{cod } \alpha)$, and let $e : E \rightarrow P$ be the equaliser of (f, g) .

Claim $(E, (\pi_j e \mid j \in \text{ob } J))$ is a limit cone for D . It is a cone since, for any $\alpha : j \rightarrow j'$, $D(\alpha)\pi_j e = \pi_{\alpha} g e = \pi_\alpha f e = \pi_{j'} e$.

Given any cone $(C, (\lambda_j \mid j \in \text{ob } J))$, the λ_j define a unique $\lambda : C \rightarrow P$, and $f\lambda = g\lambda$ since $\pi_\alpha f\lambda = \pi_\alpha g\lambda \forall \alpha$. So λ factors uniquely through e .

- ii. Let 1 be a terminal object of \mathcal{C} . For any pair of objects (A, B) the pullback

$$\begin{array}{ccc}
& A & \\
& \downarrow & \\
B & \longrightarrow & 1
\end{array}$$

has the universal property of a product $A \times B$, so \mathcal{C} has binary products. Then we can define any finite product $\prod_{i=1}^n A_i$ as $((A_1 \times A_2) \times A_3) \times \dots \times A_n$.

So we need to show \mathcal{C} has equalisers. Given $A \xrightarrow[g]{f} B$, consider the

$$\begin{array}{ccc}
& B & \\
& \downarrow (1_A, f) & \\
A & \xrightarrow{(1_A, g)} & A \times B
\end{array}$$

It consists of $\begin{array}{ccc} P & \xrightarrow{h} & B \\ \downarrow k & & \\ A & & \end{array}$ satisfying $1_A h = 1_A k$ and $fh = gk$, and universal among such.

But this forces $h = k$, and h has the universal property of an equaliser for (f, g) . So by (i), \mathcal{C} has all finite limits. □

Definition 4.5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- a. We say F **preserves** limits of shape J if, given $D : J \rightarrow \mathcal{C}$ and a limit cone $(L, (\lambda_j : j \in \text{ob } J))$ for D , the cone $(FL, (F\lambda_j : j \in \text{ob } J))$ is a limit for $FD : J \rightarrow \mathcal{D}$.
- b. We say F **reflects** limits of shape J if, given $D : J \rightarrow \mathcal{C}$ and a cone $(L, (\lambda_j))$ such that $(FL, (F\lambda_j))$ is a limit for FD , then $(L, (\lambda_j))$ is a limit for D .
- c. We say F **creates** limits of shape J if, given $D : J \rightarrow \mathcal{C}$ and a limit $(M, (\mu_j))$ for FD , there exists a cone (L, λ_j) over D whose image is isomorphic to $(M, (\mu_j))$, and any such cone is a limit for D .

Lemma 4.6. Suppose \mathcal{D} has limits of shape J . Then $[\mathcal{C}, \mathcal{D}]$ has limits of shape J , and they're constructed pointwise (i.e. the forgetful functor $[\mathcal{C}, \mathcal{D}] \rightarrow \mathcal{D}^{\text{ob } \mathcal{C}}$ creates them).

Proof. Consider a functor $D : J \times \mathcal{C} \rightarrow \mathcal{D}$. For each $A \in \text{ob } \mathcal{C}$, let $(LA, (\lambda_{j,A} : LA \rightarrow D(j, A) \mid j \in \text{ob } J))$ be a limit for the diagram $D(-, A) : J \rightarrow \mathcal{D}$.

Given any $f : A \rightarrow B$ in \mathcal{C} , the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, so they induce a unique $Lf : LA \rightarrow LB$ such that

$$\begin{array}{ccc} LA & \xrightarrow{Lf} & LB \\ \downarrow \lambda_{j,A} & & \downarrow \lambda_{j,B} \\ D(j, A) & \xrightarrow{D(j,f)} & D(j, B) \end{array}$$

commutes for all j . Uniqueness assures $L(gf) = L(g)L(f)$, so L is a functor $\mathcal{C} \rightarrow \mathcal{D}$, and the $\lambda_{j,-}$ are natural transformations $L \rightarrow D(j, -)$.

Suppose we're given any cone over D in $[\mathcal{C}, \mathcal{D}]$ with apex M and legs $\mu_j : M \rightarrow D(j, -)$. Then $(MA, (\mu_{j,A} : MA \rightarrow D(j, A) \mid j \in \text{ob } J))$ is a cone over $D(-, A)$ in \mathcal{D} , so we get a unique $\nu_A : MA \rightarrow LA$ s.t. $\lambda_{j,A}\nu_A = \mu_{j,A}$ for all j .

Uniqueness tells us that

$$\begin{array}{ccc} MA & \xrightarrow{Mf} & MB \\ \downarrow \nu_A & & \downarrow \nu_B \\ LA & \xrightarrow{Lf} & LB \end{array}$$

commutes for all $f \in \text{mor } \mathcal{C}$, so $\nu : M \rightarrow L$ in $[\mathcal{C}, \mathcal{D}]$, so it's the unique factorisation of the $\mu_{j,-}$ through the $\lambda_{j,-}$. \square

Lemma 4.7. *A morphism $f : A \rightarrow B$ is monic \iff*

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback.

Proof. f is monic \iff any cone (g, h) over (f, f) has $g = h \iff (g, h)$ factors uniquely through $(1_A, 1_A)$. \square

Hence, provided \mathcal{D} has pullbacks, a morphism $\alpha : F \rightarrow G$ in $[\mathcal{C}, \mathcal{D}]$ is monic $\iff \alpha_A : FA \rightarrow GA$ is monic for each A .

Theorem 4.8. *If $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint, then G preserves all limits which exist in \mathcal{D} .*

Proof. Suppose \mathcal{C} and \mathcal{D} both have limits of shape J and let $(F \dashv G)$. The diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \Delta & & \downarrow \Delta \\ [J, \mathcal{C}] & \xrightarrow{[J, F]} & [J, \mathcal{D}] \end{array}$$

commutes and $[J, F]$ has a right adjoint $[J, G]$. So by 3.5 the diagram of right adjoints

$$\begin{array}{ccc} [J, \mathcal{D}] & \xrightarrow{[J, G]} & [J, \mathcal{C}] \\ \downarrow \lim_J & & \downarrow \lim_J \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

commutes up to isomorphism, i.e. G preserves limits of shape J . \square

Proof. Let $D : J \rightarrow \mathcal{D}$ be a diagram with limit $(L, (\lambda_j \mid j \in \text{ob } J))$. Given a cone $(A, (\mu_j : A \rightarrow GD(j) \mid j \in \text{ob } J))$ in \mathcal{C} , we get a cone $(FA, (\bar{\mu}_j : FA \rightarrow D(j) \mid j \in \text{ob } J))$ in \mathcal{D} , and hence a unique $\bar{\nu} : FA \rightarrow L$ such that $\lambda_j \bar{\nu} = \bar{\mu}_j$ for all j .

Then $\nu : A \rightarrow GL$ is the unique morphism such that $(G\lambda_j)\nu = \mu_j \forall j$. \square

The ‘primeval’ Adjoint Functor Theorem says that if \mathcal{D} has and $G : \mathcal{D} \rightarrow \mathcal{C}$ preserves all limits, then G has a left adjoint.

This depends on two ideas:

Lemma 4.9. \mathcal{C} has an initial object $\iff 1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ has a limit.

Proof. Suppose \mathcal{C} has an initial object 0 . The morphisms $(0 \rightarrow A \mid A \in \text{ob } \mathcal{C})$ form a cone over $1_{\mathcal{C}}$. If we had another, say $(L, (\lambda_A \mid A \in \text{ob } \mathcal{C}))$, then $\lambda_0 : L \rightarrow 0$ would make

$$\begin{array}{ccc} L & \xrightarrow{\lambda_0} & 0 \\ & \searrow \lambda_A & \swarrow \\ & A & \end{array}$$

commute for all A , and it’s the only morphism which does.

Conversely, suppose $(I, (\lambda_A : I \rightarrow A \mid A \in \text{ob } \mathcal{C}))$ is a limit for $1_{\mathcal{C}}$.

If $f : I \rightarrow A$, then

$$\begin{array}{ccc} I & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \swarrow f \\ & A & \end{array}$$

commutes. In particular, $\lambda_A \lambda_I = \lambda_A$ for all A , so $\lambda_I = 1_I$ since both are factorisations of the limit cone through itself. So $f = \lambda_A$, and hence I is initial. \square

Lemma 4.10. Suppose \mathcal{D} has and $G : \mathcal{D} \rightarrow \mathcal{C}$ preserves limits of shape J . Then, for each $A \in \text{ob } \mathcal{C}$, $(A \downarrow G)$ has limits of shape J and the forgetful functor $(A \downarrow G) \rightarrow \mathcal{D}$ creates them.

Proof. Suppose given $D : J \rightarrow (A \downarrow G)$. Write $D(j)$ as $(UD(j), f_j : A \rightarrow GUD(j))$ for each j . Let $(L, (\lambda_j \mid j \in \text{ob } J))$ be a limit for UD , then $(GL, (G\lambda_j \mid j \in \text{ob } J))$ is a limit for GUD . But the f_j form a cone over GUD with apex A , so there’s a unique $h : A \rightarrow GL$ such that

$$\begin{array}{ccc} A & \xrightarrow{h} & GL \\ & \searrow f_j & \downarrow G\lambda_j \\ & & GUD(j) \end{array}$$

commutes for all j . So there’s a unique lifting of the cone over D in $(A \downarrow G)$.

Suppose we’re given a cone $((B, g), (\mu_j \mid j \in \text{ob } J))$ over D . Then

$$\begin{array}{ccc} A & \xrightarrow{g} & GB \\ & \searrow h & \downarrow G_k \\ & & GL \end{array}$$

commutes since both ways round are factorisations of $(f_j \mid j \in \text{ob } J)$ through the limit GL . \square

Combining 4.10 and 4.9 with 3.2, we've proved the primeval Adjoint Functor Theorem. However, this requires \mathcal{D} to have limits for diagrams 'as big as \mathcal{D} itself', and the only such categories are preorders (c.f. Q6, sheet 2).

In practice, the most we can hope for is that \mathcal{D} has all small limits. We call such a \mathcal{D} **complete**.

Theorem 4.11 (General Adjoint Functor Theorem). *Suppose that \mathcal{D} is complete and locally small. Then a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if it preserves all small limits and satisfies the 'solution set condition': for any $A \in \text{ob } \mathcal{C}$, there is a set $\{f_i : A \rightarrow GB_i \mid i \in I\}$ of objects of $(A \downarrow G)$ such that any $h : A \rightarrow GC$ factors as*

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GC$$

for some $i \in I$ and $g : B_i \rightarrow C$.

Proof. If G has a left adjoint, then it preserves small limits by 4.8, and $\{\eta_A : A \rightarrow GFA\}$ is a singleton solution set at A .

Conversely, each $(A \downarrow G)$ is complete by 4.10, and locally small since it admits a faithful functor to \mathcal{D} . So we need to show: if \mathcal{A} is complete and locally small, and has a weakly initial set of objects $\{S_i \mid i \in I\}$, then \mathcal{A} has an initial object.

First form $P = \prod_{i \in I} S_i$: then P is weakly initial.

Now form the limit $I \xrightarrow{a} P$ of the diagram $P \rightrightarrows P$ whose edges are all morphism $P \rightarrow P$ in \mathcal{A} .

Claim I is initial: it's weakly initial since it admits a morphism to P .

Suppose we had $I \xrightarrow[f]{f} A$. Let $b : E \rightarrow I$ be an equaliser for (f, g) : then there exists $c : P \rightarrow E$.

Now $P \xrightarrow{c} E \xrightarrow{b} I \xrightarrow{a} P$ is an edge of the diagram whose limit is I , but so is 1_P ; so $abca = 1_P a = a$. But a is monic, so $bca = 1_I$. So b is (split) epic, and $f = g$. So all the $(A \downarrow G)$ have initial objects, hence by 3.2 G has a left adjoint. \square

The Special Adjoint Functor Theorem imposes additional conditions on \mathcal{C} and \mathcal{D} which ensure that every functor $\mathcal{D} \rightarrow \mathcal{C}$ preserving small limits has a left adjoint.

Definition 4.12. *a. A **subobject** of an object A is a monomorphism $A' \rightarrow A$. We write $\mathbf{Sub}_{\mathcal{C}}(A)$ for the full subcategory of \mathcal{C}/A whose objects are subobjects of A : note that this category is a preorder.*

- b. We say \mathcal{C} is **well-powered** if each $\mathbf{Sub}_{\mathcal{C}}(A)$ is equivalent to a small category, i.e. up to isomorphism each object has only a set of subobjects.

Dually, \mathcal{C} is **well-copowered** if \mathcal{C}^{op} is well-powered.

Lemma 4.13. Suppose given a pullback

$$\begin{array}{ccc} P & \xrightarrow{k} & A \\ \downarrow h & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

with f monic. Then h is monic.

Proof. Suppose $D \xrightarrow[x]{y} P$ satisfy $hx = hy$. Then $fkx = fky = ghx = ghy$ and f is monic so $kx = ky$.

Now $x = y$ since both are factorisations of the same cone through the pullback. \square

Theorem 4.14 (Special Adjoint Functor Theorem). Suppose both \mathcal{C} and \mathcal{D} are locally small, and \mathcal{D} is complete, well-powered and has a separating set. Then $G : \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint $\iff G$ preserves all small limits.

Proof. The forward implication is 4.8 again.

Conversely, we first show that $(A \downarrow G)$ has the properties we've assumed for \mathcal{D} : it's complete by 4.10, and locally small as in 4.11. It's well-powered since subobjects of (B, f) in $(A \downarrow G)$ are in bijection with subobjects $B' \rightarrowtail B$ such that f factors through $GB' \rightarrowtail GB$.

It has a coseparating set: if $\{S_i \mid i \in I\}$ is a coseparating set for \mathcal{D} , then $\{(S_i, f) \mid i \in I, f : A \rightarrow GS_i\}$ is a coseparating set for $(A \downarrow G)$, since if $(B, f) \xrightarrow[g']{g} (B', f')$ satisfies $g \neq g'$, there exists $h : B' \rightarrow S_i$ for some i with $hg \neq hg'$, and then h is a morphism $(B', f') \rightarrow (S_i, (Gh)f')$ in $(A \downarrow G)$.

Now we show that if \mathcal{A} is complete, locally small and well-powered and has a coseparating set, then it has an initial object.

First form $P = \prod_{i \in I} S_i$, where $\{S_i \mid i \in I\}$ is a coseparating set.

Consider the diagram

$$\begin{array}{ccc} P' & & \\ & \searrow & \\ P'' & \xrightarrow{\quad} & P \\ & \nearrow & \\ \vdots & & \\ P^{(n)} & & \end{array}$$

whose edges are a representative set of subobjects of P .

Form its limit

$$\begin{array}{ccc}
& & P' \\
I & \xrightarrow{\quad} & P'' \\
& \searrow & \vdots \\
& & P^{(n)}
\end{array}$$

by the argument of 4.13 the legs $I \rightarrow P^{(-)}$ are monic, so $I \rightarrowtail P$ is monic and it's the least subobject of P .

Hence in particular I has no proper subobjects, so any two maps $I \rightrightarrows A$ must be equal, since their equaliser is an isomorphism.

Now given $A \in \mathcal{A}$, form the product $Q = \prod_{i,f:A \rightarrow S_i} S_i$. The canonical morphism $h : A \rightarrow Q$ defined by $\pi_{i,f}h = f$ is monic since the S_i form a coseparating set.

We also have $k : P \rightarrow Q$ defined by $\pi_{i,f}k = \pi_i$, and we can form the pullback

$$\begin{array}{ccccc}
I & \longrightarrow & B & \xrightarrow{m} & A \\
& \searrow & \downarrow l & & \downarrow h \\
& & P & \xrightarrow{k} & Q
\end{array}$$

By 4.13 l is monic and hence isomorphic to an edge of the diagram defining I , so $I \rightarrowtail P$ factors through it. So there exists a morphism $I \rightarrow A$, hence I is initial. \square

5 Monads

Suppose given an adjunction $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$, $F \dashv G$. How much of this can we describe purely in terms of \mathcal{C} ?

We have the composite $T = GF : \mathcal{C} \rightarrow \mathcal{C}$, and the unit $\eta : 1_{\mathcal{C}} \rightarrow T$. We also have $G\epsilon_F : GF GF \rightarrow GF$, which we'll denote $\mu : TT \rightarrow T$.

These satisfy the commutative diagrams

$$\begin{array}{ccc}
T & \xrightarrow{T\eta} & TT \xleftarrow{\eta_T} T \\
\textcircled{1} \searrow 1_T & & \downarrow \mu \\
& & T
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
TTT & \xrightarrow{T\mu} & TT \\
\downarrow \mu_T & \textcircled{3} & \downarrow \mu \\
TT & \xrightarrow{\mu} & T
\end{array}$$

from the \triangle^r identities and naturality of ϵ .

Definition 5.1. A *monad* $\mathbb{T} = (T, \eta, \mu)$ on a category \mathcal{C} consists of a functor $T : \mathcal{C} \rightarrow \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \rightarrow T$, $\mu : TT \rightarrow T$ satisfying the commutative diagrams $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$.

Definition 5.2. Let \mathbb{T} be a monad on \mathcal{C} . A \mathbb{T} -**algebra** is a pair (A, α) where $A \in \text{ob } \mathcal{C}$, and $\alpha : TA \rightarrow A$ satisfies

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ \textcircled{4} \searrow 1_A & & \downarrow \alpha \\ & & A \end{array} \quad \text{and} \quad \begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ \downarrow \mu_A & \textcircled{5} & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A \end{array}$$

A **homomorphism** $f : (A, \alpha) \rightarrow (B, \beta)$ of \mathbb{T} -algebras is a morphism $f : A \rightarrow B$ such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow \alpha & \textcircled{6} & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

commutes. We write $\mathcal{C}^{\mathbb{T}}$ for the category of \mathbb{T} -algebras.

Lemma 5.3. The forgetful functor $G : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ has a left adjoint F , and the adjunction $(F \dashv G)$ induces the monad \mathbb{T} .

Proof. We define $FA = (TA, \mu_A)$ (which is an algebra by ② and ③), and $F(A \xrightarrow{f} B) = Tf$ (which is a homomorphism by naturality of μ).

Clearly $GF = T$ and $\eta : 1_{\mathcal{C}} \rightarrow GF$.

We define $\epsilon : FG \rightarrow 1_{\mathcal{C}^{\mathbb{T}}}$ by $\epsilon_{(A, \alpha)} = \alpha : (TA, \mu_A) \rightarrow (A, \alpha)$ (which is a homomorphism by ⑤).

The triangular identities for η and ϵ follow from ④ and ①, so $(F \dashv G)$.

Finally, $G_{\epsilon_{FA}} = \mu_A$ by the definitions of FA and ϵ , so the adjunction induces \mathbb{T} . \square

Note that if $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ induces \mathbb{T} , then so does $\mathcal{C} \xrightleftharpoons[G/\mathcal{D}']{F} \mathcal{D}'$ where \mathcal{D}' is the full subcategory of objects of the form FA . So in seeking to construct \mathcal{D} , we may require F to be bijective on objects. But then morphisms $FA \rightarrow FB$ in \mathcal{D} correspond bijectively to morphisms $A \rightarrow GFB = TB$ in \mathcal{C} .

Definition 5.4. Given a monad \mathbb{T} on \mathcal{C} , the **Kleisi category** $\mathcal{C}_{\mathbb{T}}$ is defined by: $\text{ob } \mathcal{C}_{\mathbb{T}} = \text{ob } \mathcal{C}$, morphisms $A \rightarrow B$ in $\mathcal{C}_{\mathbb{T}}$ are morphisms $A \rightarrow TB$ in \mathcal{C} , the identity $A \rightarrow A$ is $A \xrightarrow{\eta_A} TA$, and the composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} C$.

We check

$$\begin{aligned} A \xrightarrow{1_A} A \xrightarrow{f} B &= A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} B \\ &= A \xrightarrow{f} TA \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} B \\ &= f \text{ by } \textcircled{2} \end{aligned}$$

$$\begin{aligned}
A \xrightarrow{f} B \xrightarrow{1_B} B &= A \xrightarrow{f} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} B \\
&= f \text{ by } \textcircled{1}
\end{aligned}$$

Given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

$$\begin{aligned}
(hg)f &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TT^h} TT^hD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TT^h} TT^hD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD \text{ by } \textcircled{3} \\
&= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{T^h} TTD \xrightarrow{\mu_D} TD \\
&= h(gf)
\end{aligned}$$

Lemma 5.5. *There exists an adjunction $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{C}_{\mathbb{T}}$ inducing \mathbb{T} .*

Proof. We define $FA = A$ and $F(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. This clearly preserves identities, and

$$\begin{aligned}
(Fg)(Ff) &= A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC \\
&= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \text{ by } \textcircled{1} \text{ and naturality of } \eta \\
&= F(gf)
\end{aligned}$$

We define $GA = TA$ and $G(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. G preserves identities by $\textcircled{1}$ and

$$\begin{aligned}
G(A \xrightarrow{f} B \xrightarrow{g} C) &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC \text{ by } \textcircled{3} \\
&= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \text{ by naturality of } \mu \\
&= (Gg)(Gf)
\end{aligned}$$

Clearly $GFA = TA$ and

$$\begin{aligned}
GF(A \xrightarrow{f} B) &= TA \xrightarrow{Tf} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} TB \\
&= Tf \text{ by } \textcircled{1}
\end{aligned}$$

so $GF = T$ and $\eta : 1_{\mathcal{C}} \rightarrow GF$.

□