

Part III Combinatorics

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1 Introduction

Let X, Y, \dots be sets

Definition. We call $\mathcal{A} \subset \mathcal{P}(X)$ a **set system** or **family of sets**. \mathcal{A} is naturally identified with a bipartite graph $G_{\mathcal{A}}(U, W)$ with $U = \mathcal{A}$, $W = \bigcup_{A \in \mathcal{A}} A$ or $W = X$. Indeed, $Ax \in E(G_{\mathcal{A}}) \iff x \in A$.

Definition. Given $\mathcal{A} \subset \mathcal{P}(X)$, a **set of distinct representatives (SDR)** is an injection $f : \mathcal{A} \rightarrow X$ s.t. $f(A) \in A \forall A \in \mathcal{A}$. In its bipartite graph, an SDR corresponds to a complete matching $U \rightarrow W$.

Theorem 1 (Hall, 1935). A set system \mathcal{A} has an SDR if $\forall \mathcal{A}' \subset \mathcal{A}$, $|\bigcup_{A \in \mathcal{A}'} A| \geq |\mathcal{A}'|$.

Theorem 1'. A bipartite graph $G(U, W)$ has a complete matching $U \rightarrow W$ if $\forall S \subset U$, $|\Gamma(S)| \geq |S|$

Corollary 2. Suppose $G(U, W)$ bipartite, $d(u) \geq d(w) \forall u \in U, w \in W$. Then \exists a complete matching $U \rightarrow W$.

Definition. A bipartite graph $G(U, W)$ is **(r, s) -regular** if $d(u) = r$ and $d(w) = s \forall u \in U, w \in W$.

Instant from Cor 2: if $G(U, W)$ is (r, s) -regular then \exists a complete matching from U to W if $|U| \leq |W|$.

Corollary 3. Let $0 \leq i, j \leq n$, $\binom{n}{i} \leq \binom{n}{j}$. Then \exists a complete matching $f : [n]^{(i)} \rightarrow [n]^{(j)}$ s.t. $f(A) \subset A$ if $j \leq i$, and $f(A) \supset A$ if $i \leq j$.

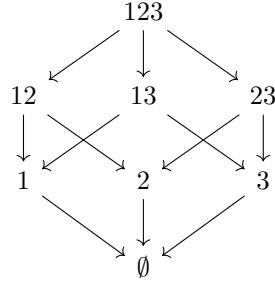
Theorem 4. Let $G = G(U, W)$ be a connected (r, s) -regular graph. Then for $\emptyset \neq A \subset U$,

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

Also, equality holds iff $A = U$.

The **cube** $Q^n \cong \mathcal{P}(n) \cong [2]^n$ = set of all 0, 1 sequences of length n . Q^n is also a graph: AB is an edge if $|A \triangle B| = 1$. It is also a poset: $A < B$ if $A \subset B$.

Q^n has a natural orientation: \overrightarrow{AB} if $A = B \cup \{a\}$.



The order on $Q^n \cong \mathcal{P}(n)$ is induced by this oriented graph.

2 Sperner Systems

Definition. A set system $\mathcal{A} \subset \mathcal{P}(n)$ is **Sperner** if $A, B \in \mathcal{A}$, $A \neq B \implies A \not\subset B$

Theorem 1 (Sperner, 1928). If $\mathcal{A} \subset \mathcal{P}(n)$ is Sperner then

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Definition. The **weight** $w(A)$ of a set $A \in \mathcal{P}(n)$ is $w(A) = \frac{1}{\binom{n}{|A|}}$

Theorem 2. Let \mathcal{A} be a Sperner system on X , $|X| = n$. Then

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A) \leq 1$$

Corollary 3. If $\mathcal{A} \subset \mathcal{P}(n)$ is a Sperner system then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, with equality $\iff \mathcal{A}$ is $X^{[n/2]}$ or $X^{[n/2]}$.

Definition. $\mathcal{A} \in \mathcal{P}(n)$ is ***k-Sperner*** if it does not contain

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{k+1}$$

Note that Sperner = 1-Sperner.

Corollary 4 (Erdős, 1945). *If $\mathcal{A} \subset \mathcal{P}(n)$ is *k-Sperner* then $|\mathcal{A}|$ is at most the sum of the *k* largest binomial coefficients.*

Theorem 5 (Erdős, 1945). *Let $x_1, \dots, x_n \in \mathbb{R}$, $x_i \geq 1$. Then the number of sums $\sum_1^n \pm x_i$ in an open interval *J* of length $2k$ is at most the sum of the *k* largest binomial coefficients.*

Definition. A chain $A_0 \subset A_1 \subset \cdots \subset A_k$ is ***symmetric*** if $|A_{i+1}| = |A_i| + 1 \ \forall i$ and $|A_0| + |A_k| = n$.

Theorem 6 (Kleitman and Katona). $\mathcal{P}(n)$ has a decomposition into symmetric chains.

Take such a partition $\mathcal{P}(n) = \bigcup_{i=1}^k C_i$, $j = \lfloor \frac{n}{2} \rfloor$. There is one chain of length $n+1$, $n-1$ chains of length $n-1$, etc: there are $\binom{n}{i} - \binom{n}{i-1}$ chains of length $n+1-2i$.

Let E be a normed space, let $x_1, \dots, x_n \in E$, $\|x_i\| \geq 1 \ \forall i$, for $A \in \mathcal{P}(n)$ let $x_A = \sum_{i \in A} x_i$.

Conjecture (Erdős, 1945). *If $\mathcal{A} \in \mathcal{P}(n)$ s.t. $\|x_A - x_B\| < 1$ then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$*

Definition. Call $\mathcal{D} \in \mathcal{P}(n)$ ***scattered*** if $\|x_A - x_B\| \geq 1 \ \forall A, B \in \mathcal{D}$. Call a partition $\mathcal{P}(n) = \bigcup_{i=1}^s \mathcal{D}_i$ ***symmetric*** if there are precisely $\binom{n}{i} - \binom{n}{i-1}$ sets \mathcal{D}_i of cardinality $n+1-2i$.

Theorem 7. (Kleitman, 1970) *E, $(x_i)_1^n$ as before. Then $\mathcal{P}(n)$ has a symmetric partition into scattered sets.*

Theorem 8. (Kleitman, 1970) *If $\mathcal{A} \in \mathcal{P}(n)$ s.t. $\|x_A - x_B\| < 1$ then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$*

3 The Kruskal-Katona Theorem

We know: if $\mathcal{A} \subset X^{(r)}$ then $\partial\mathcal{A}$ (the **lower shadow** of \mathcal{A}), defined by

$$\partial\mathcal{A} = \{B \in X^{(r-1)} \mid B \subset A \text{ for some } A \in \mathcal{A}\}$$

satisfies

$$\begin{aligned} |\partial\mathcal{A}| &\geq |\mathcal{A}| \frac{\binom{n}{r-1}}{\binom{n}{r}} \\ &= |\mathcal{A}| \frac{r}{n-r+1} \end{aligned}$$

with equality $\iff \mathcal{A}$ is \emptyset or $X^{(r)}$.

What about in between? What is $\mathcal{B} \in X^{(r)}$ s.t. $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$?

$\exists \mathcal{B}_1, \mathcal{B}_2, \dots \in X^{(r)}$ s.t. $|\mathcal{B}_m| = m$ and $|\partial\mathcal{B}_m| \leq |\partial\mathcal{A}| \ \forall \mathcal{A} \subset X^{(r)}$ where $|\mathcal{A}| = m$.

Incredibly luckily, we have a sequence of nested extremal sets. Equivalently, \exists total order on $X^{(r)}$ s.t. the first m sets form \mathcal{B}_m .