Part III Algebraic Geometry

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Contents

1 Sheaves

Definition (Presheaf)**.** *Let X be a topological space. A* ***presheaf*** *$\mathcal{F}$ consists of a collection of abelian groups, $\mathcal{F}(U)$, where $U \subseteq X$ are the open subsets of X s.t. $\mathcal{F}(\emptyset) = 0$.*

$\exists$ a homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$, $s \mapsto s|_V$ for each inclusion $V \subseteq U$ of open sets. $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map. If $W \subseteq V \subseteq U$ are open sets then $\forall s \in \mathcal{F}(U)$, $(s|_V)|_W = s|_W$.

Definition (Sheaf)**.** *A* ***sheaf*** *$\mathcal{F}$ is a presheaf s.t. if $U = \bigcup U_i$, U, U_i open and if $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j$ then $\exists! s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$ $\forall i$.*

Definition (Stalk)**.** *Let X be a topological space, $\mathcal{F}$ a presheaf, $x \in X$. Define the* ***stalk*** *of $\mathcal{F}$ at x by $\mathcal{F}_x = \lim_{U \ni x} \mathcal{F}(U)$.*

More explicitly, each element of $\mathcal{F}_x$ is given by a pair (U, s) where $x \in U$ open, $s \in \mathcal{F}(U)$ subject to the condition

$$(U, s) = (V, t) \text{ if } \exists x \in W \subseteq U \cap V \text{ s.t. } s|_W = t|_W$$

Definition (Morphism)**.** *Let X be a topological space, $\mathcal{F}$, $\mathcal{G}$ presheaves. A* ***morphism*** *$\varphi : \mathcal{F} \to \mathcal{G}$ is given by a collection of homomorphisms $\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)$ s.t. if $V \subseteq U$, the diagram*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

*commutes. We say φ is an **isomorphism** if it has an inverse.*

Remark. *$\forall x \in X$, any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, we get a homomorphism*

$$\begin{aligned} \varphi_x : \mathcal{F}_x &\to \mathcal{G}_x \\ (U, s) &\mapsto (U, \varphi_U(s)) \end{aligned}$$

Definition. *Let X be a topological space, $\mathcal{F}$ a presheaf. Then $\exists$ a sheaf $\mathcal{F}^+$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}^+$ s.t. if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism into a sheaf $\mathcal{G}$, then φ factors uniquely*

$$\begin{tikzcd} & \mathcal{F}^+ \arrow[d] \\ \mathcal{F} \arrow[ur, "\alpha"] \arrow[r, "\varphi"] & \mathcal{G} \end{tikzcd}$$

*for some morphism $\mathcal{F}^+ \to \mathcal{G}$. We call $\mathcal{F}^+$ the sheaf **associated** to $\mathcal{F}$.*

$\mathcal{F}^+$ is constructed as follows:

$$\mathcal{F}^+(U) := \left\{ \text{functions } s : U \to \bigsqcup_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \forall x \in U,\ s(x) \in \mathcal{F}_x,\ \exists x \in W \subseteq V \text{ and} \\ t \in \mathcal{F}(W) \text{ s.t. } s(y) = (V, t) \in \mathcal{F}_y \ \forall y \in W \end{array} \right\}$$

Definition (Kernel and Image)**.** *Let X be a topological space, $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ a morphism of presheaves. The **kernel** of φ, denoted $\operatorname{Ker}\varphi$, is defined by*

$$(\operatorname{Ker}\varphi)(U) = \operatorname{Ker}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$$

*The **presheaf image** of φ, denoted $\operatorname{Im}(\varphi^{pre})$ is defined by*

$$(\operatorname{Im}\varphi^{pre})(U) = \operatorname{Im}(\varphi_U)$$

Now assume $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Define the kernel of $\varphi = \operatorname{Ker}\varphi$ as above, which is a sheaf. Define the image of φ by $\operatorname{Im}(\varphi^{pre})^+$, denoted $\operatorname{Im}\varphi$.

2 Sheaves II

Theorem 1. *Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on a topological space X. Then*

i. φ is injective $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective $\forall x \in X$

ii. φ *is surjective* $\iff \varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ *is surjective* $\forall x \in X$

iii. φ *is an isomorphism* $\iff \varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ *is an isomorphism* $\forall x \in X$

Definition. *Let X be a topological space. A* ***complex of sheaves*** *is a sequence*

$$\cdots \to \mathcal{F}_{-2} \overset{\varphi_{-2}}{\to} \mathcal{F}_{-1} \overset{\varphi_{-1}}{\to} \mathcal{F}_0 \overset{\varphi_0}{\to} \mathcal{F}_1 \overset{\varphi_1}{\to} \mathcal{F}_2 \overset{\varphi_2}{\to} \dots$$

of sheaves s.t. $\operatorname{Im} \varphi_i \subseteq \operatorname{Ker} \varphi_{i+1}$ $\forall i$. *We say it is an* ***exact sequence*** *if* $\operatorname{Im} \varphi_i = \operatorname{Ker} \varphi_{i+1}$ $\forall i$. *An exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is called a ***short exact sequence****.*

Definition (Constant sheaf)**.** *Let X be a topological space and A an abelian group. Define a presheaf $\mathcal{F}$ by $\mathcal{F}(U) = A$ $\forall$ open $U \neq \emptyset$. We call $\mathcal{F}^+$ the* ***constant sheaf*** *associated to A.*

Definition (Direct image)**.** *Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. The* ***direct image*** *$f_*\mathcal{F}$ is defined by*

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$$

Definition (Skyscraper sheaf)**.** *Let X be a topological space, $x \in X$, A an abelian group. Define $\mathcal{F}$ by*

$$\mathcal{F} = \begin{cases} A \text{ if } x \in U \\ 0 \text{ if } x \notin U \end{cases}$$

We call $\mathcal{F}$ the ***skyscraper sheaf*** *associated to A at x.*

3 Basics of Commutative Algebra

Ref: Atiyah and Macdonald, *Introduction to Commutative Algebra.*

Convention: all rings in this course are commutative with 1 (unless ring $= 0$).

Definition. *Let A be a ring. A* ***prime ideal*** *is an ideal $P \trianglelefteq A$ s.t. $ab \in P \implies a \in P$ or $b \in P$.* ***maximal ideal*** *is an ideal $M \trianglelefteq A$ s.t. if $M \subseteq I \ntrianglelefteq A$, then $M = I$. (Maximal ideals are prime).*

A is an ***integral domain*** *if 0 is a prime ideal.*

A is a ***local ring*** *if it has a unique maximal ideal.*

A ***local homomorphism*** *is a homomorphism of local rings $\alpha : A \to B$ s.t. α(maximal ideal of A) $\subseteq$ maximal ideal of B.*

Definition (Radical). *Let A, be a ring, $I \trianglelefteq A$. The **radical** of I is defined as*

$$\sqrt{I} = \{a \in A \,|\, a^n \in I, \text{ some } n \in N\}$$

$\sqrt{I}$ is an ideal. Fact: $\sqrt{I} = \bigcap_{I \subseteq \text{ prime } P \trianglelefteq A} P$

Fact. *Let K be an algebraically closed field. Then $I \trianglelefteq K[t_1, \dots, t_n]$ is maximal $\iff I = \langle t_1 - a_1, \dots, t_n - a_n \rangle$ for some $a_1, \dots, a_n \in K$.*

Definition (Ring of fractions). *Let A be a ring. $S \subseteq A$ is a **multiplicatively closed set** if $1 \in S$ and $s, t \in S \implies st \in S$.*

Pick such a set. Let $S^{-1}A = \{\frac{a}{s} \,|\, a \in A,\, s \in S\}$ subject to $\frac{a}{s} = \frac{b}{t} \iff \exists u \in S$ s.t. $u(at - bs) = 0$.

$S^{-1}A$ is a ring: $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$, $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$.

*We call $S^{-1}A$ the **ring of fractions** of A w.r.t S.*

We have a homomoprhism $\alpha : A \to S^{-1}A$, $a \mapsto \frac{a}{1}$. If $s \in S$, then $\alpha(s) = \frac{s}{1}$ is a unit in $S^{-1}A$. Also, $\alpha a = 0 \iff \exists u \in S$ s.t. $ua = 0$.

If $I \trianglelefteq A$, then $S^{-1}I = \{\frac{a}{s} \in S^{-1}A \,|\, a \in I\}$ is an ideal of $S^{-1}A$.

Fact. *Every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some $I \trianglelefteq A$.*

$$\{P \trianglelefteq A \,|\, P \text{ prime},\, P \cap S = \emptyset\} \xleftrightarrow{1-1} \{\text{prime ideals of } S^{-1}A\}$$

Definition. *Let A be a ring, S a multiplicatively closed set, M an A-module. Let $S^{-1}M = \{\frac{m}{s} \,|\, m \in M,\, s \in S\}$ subject to: $\frac{m}{s} = \frac{n}{t} \iff \exists u \in S$ s.t. $u(tm - sn) = 0$. Thus $\frac{m}{s} + \frac{n}{t} = \frac{tm+sn}{st}$, $\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st}$, $\forall a \in A$, $m, n \in M$, $s, t \in S$. So $S^{-1}M$ is an $S^{-1}A$-module.*

If $M \to N$ is an A-homomorphism, then we get an $S^{-1}A$-homomorphism $S^{-1}M \to S^{-1}N$.

Fact. *$S^{-1}(M/K) = (S^{-1}M)/(S^{-1}K)$, $K \subseteq M$ a submodule.*

Definition (Direct product and direct sum). *Let A be a ring, $\{M_i\}_{i \in I}$ a family of A-modules. Define the **direct product***

$$\prod_{i \in I} M_i = \{(m_i)_{i \in I} \,|\, m_i \in M_i\}$$

This is an A-module: $(m_i) + (n_i) = (m_i + n_i)$, $a \cdot (m_i) = (am_i)$ $\forall m_i, n_i \in M_i$, $a \in A$.

*Also define the **direct sum***

$$\bigoplus_{i \in I} M_i = \left\{ (m_i) \in \prod_{i \in I} M_i \,|\, m_i = 0 \text{ for all but finitely many } i \right\}$$