Part III Algebraic Geometry

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Contents

1 Sheaves

Definition (Presheaf)**.** *Let X be a topological space. A* ***presheaf*** *$\mathcal{F}$ consists of a collection of abelian groups, $\mathcal{F}(U)$, where $U \subseteq X$ are the open subsets of X s.t. $\mathcal{F}(\emptyset) = 0$.*

$\exists$ a homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$, $s \mapsto s|_V$ for each inclusion $V \subseteq U$ of open sets. $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map. If $W \subseteq V \subseteq U$ are open sets then $\forall s \in \mathcal{F}(U)$, $(s|_V)|_W = s|_W$.

Definition (Sheaf)**.** *A* ***sheaf*** *$\mathcal{F}$ is a presheaf s.t. if $U = \bigcup U_i$, U, U_i open and if $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j$ then $\exists! s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$ $\forall i$.*

Definition (Stalk)**.** *Let X be a topological space, $\mathcal{F}$ a presheaf, $x \in X$. Define the* ***stalk*** *of $\mathcal{F}$ at x by $\mathcal{F}_x = \lim_{U \ni x} \mathcal{F}(U)$.*

More explicitly, each element of $\mathcal{F}_x$ is given by a pair (U, s) where $x \in U$ open, $s \in \mathcal{F}(U)$ subject to the condition

$$(U, s) = (V, t) \text{ if } \exists x \in W \subseteq U \cap V \text{ s.t. } s|_W = t|_W$$

Definition (Morphism)**.** *Let X be a topological space, $\mathcal{F}$, $\mathcal{G}$ presheaves. A* ***morphism*** *$\varphi : \mathcal{F} \to \mathcal{G}$ is given by a collection of homomorphisms $\mathcal{F}(U) \overset{\varphi(U)}{\to} \mathcal{G}$ s.t. if $V \subseteq U$, the diagram*

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}(V) \arrow[r, "\varphi_V"] & \mathcal{G}(V)
\end{tikzcd}$$

commutes. We say φ is an ***isomorphism*** *if it has an inverse.*

Definition. *Let X be a topological space, $\mathcal{F}$ a presheaf. Then $\exists$ a sheaf $\mathcal{F}^+$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}^+$ s.t. if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism into a sheaf $\mathcal{G}$, then φ factors uniquely*

$$\begin{tikzcd}
& \mathcal{F}^+ \arrow[d] \\
\mathcal{F} \arrow[ur, "\alpha"] \arrow[r, "\varphi"'] & \mathcal{G}
\end{tikzcd}$$

*for some morphism $\mathcal{F}^+ \to \mathcal{G}$. We call $\mathcal{F}^+$ the sheaf **associated** to $\mathcal{F}$.*

$\mathcal{F}^+$ is constructed as follows:

$$\mathcal{F}^+(U) := \left\{ \textit{functions } s : U \to \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \forall x \in U,\ s(x) \in \mathcal{F}_x,\ \exists x \in W \subseteq V \textit{ and} \\ t \in \mathcal{F}(W) \textit{ s.t. } s(y) = (V, t) \in \mathcal{F}_y\ \forall y \in W \end{array} \right\}$$

Definition (Kernel and Image)**.** *Let X be a topological space, $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ a morphism of presheaves. The **kernel** of φ, denoted $Ker\varphi$, is defined by*

$$(Ker\varphi)(U) = Ker(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

*The **presheaf image** of φ, denoted $Im(\varphi^{pre})$ is defined by*

$$(Im\varphi^{pre})(U) = Im(\varphi_U)$$

Now assume $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Define the kernel of $\varphi = Ker\varphi$ as above, which is a sheaf. Define the image of φ by $Im(\varphi^{pre})^+$, denoted $Im\varphi$.