Part III Topics in Additive Combinatorics

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Contents

1 Discrete Fourier Analysis and Roth's Theorem

Let $N \in \mathbb{N}$, $\omega = e^{\frac{2\pi i}{N}}$. Write $\mathbb{Z}_N$ for the cyclic group of integers mod N. Use the notation $\mathbb{E}_x f(x)$ to stand for the average $N^{-1} \sum_{x \in \mathbb{Z}_N} f(x)$.

Definition (Discrete Fourier Transform)**.** *Given a function $f : \mathbb{Z}_N \to \mathbb{C}$, define its **discrete Fourier transform** $\hat{f}$ by the formula*

$$\hat{f}(r) = \mathbb{E}_x f(x) \omega^{-rx}$$

Definition (Convolution)**.** *We define the **convolution** $f * g$ of f and g by*

$$f * g(x) = \mathbb{E}_{y+z=x} f(y) g(z)$$

$$\hat{f} * \hat{g}(r) = \sum_{s+t=r} \hat{f}(s) \hat{g}(t)$$

We also define two inner products

$$\langle f, g \rangle = \mathbb{E}_x f(x) \overline{g(x)}$$

$$\langle \hat{f}, \hat{g} \rangle = \sum_r \hat{f}(r) \overline{\hat{g}(r)}$$

Have the following basic properties:

1. Parseval's Identity:
$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$$
for any $f, g : \mathbb{Z}_N \to \mathbb{C}$.

2. Convolution Law: for any $f, g : \mathbb{Z}_N \to \mathbb{C}$, $r \in \mathbb{Z}_N$
$$\widehat{f * g}(r) = \hat{f}(r)\hat{g}(r)$$

3. Inversion Formula: let $f : \mathbb{Z}_N \to \mathbb{C}$. Then
$$f(x) = \sum_r \hat{f}(r)\omega^{rx}$$

4. Dilation Rule: let a be invertible mod N and define $f_a(x)$ to be $f(a^{-1}x)$. Then
$$\hat{f_a}(r) = \hat{f}(ar)$$

If $A \subset \mathbb{Z}_N$, we shall write $A(x)$ for $\begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$. If $|A| = \alpha N$, then $\hat{A}(0) = \mathbb{E}_x A(x) = \alpha$.

We shall define $||f||_p$ to be $(\mathbb{E}_x |f(x)|^p)^{\frac{1}{p}}$ and $\left|\left|\hat{f}\right|\right|_p$ to be $\left(\sum_r \left|\hat{f}(x)\right|^p\right)^{\frac{1}{p}}$.

Then if $A \subset \mathbb{Z}_N$, $||A||_2^2 = \langle A, A \rangle = \alpha$. By Parseval, we get
$$\sum_r \left|\hat{A}(r)\right|^2 = \alpha \ \left(= \left|\left|\hat{A}\right|\right|_2^2\right)$$

Theorem 1 (Roth)**.** *For every $\delta > 0$ $\exists N$ s.t. every subset $A \subset [N]$ of size at least δN contains an arithmetic progression of length 3.*

Broad strategy: a density increment argument.

The idea is to show that if A has density α and contains no 3-AP then there is a reasonably long AP P s.t. $\frac{|A \cap P|}{|P|}$ is significantly larger than α. There we are either done or can pass to P and start again with a larger density. Then repeat, and eventually, since α can't exceed 1, we must get a 3-AP.

In order to use Fourier analysis, we want to think of A as a subset of $\mathbb{Z}_n$. For this purpose, define sets $B = C = A \cap \left[\frac{N}{3}, \frac{2N}{3}\right]$, and observe that if (x, y, z) is an AP in $A \times B \times C$ in $\mathbb{Z}_N$, then it also is in $[N]$.

Let α be the density of A. Assume that N is odd. If $|B| < \frac{\alpha N}{5}$ then one of $\left|A \cap \left[1, \frac{N}{3}\right]\right|$ and $\left|A \cap \left[\frac{2N}{3}, N\right]\right|$ is at least $\frac{2\alpha N}{5}$, so we get an interval in which A has density at least $\frac{6\alpha}{5}$, which is a very healthy density increment.

Otherwise, $|B| = |C| > \frac{\alpha N}{5}$, so let's assume that.

Define the **3-AP-density** of (A, B, C) to be $\mathbb{E}_{x+z=2y} A(x)B(y)C(z)$. This is the probability that a random (x, y, z) with $x + z = 2y$ lies in $A \times B \times C$.

$$\begin{aligned}
\mathbb{E}_{x+z=2y} A(x)B(y)C(z) &= \mathbb{E}_u \left(\mathbb{E}_{x+z=u} A(x)C(z)\right) B(u/2) \\
&= \mathbb{E}_u A * C(u) B_2(u) \\
&= \langle A * C, B_2 \rangle \\
&= \langle \widehat{A * C}, \hat{B}_2 \rangle \\
&= \langle \hat{A}\hat{C}, \hat{B}_2 \rangle \\
&= \sum_r \hat{A}(r)\hat{C}(r)\overline{\hat{B}_2(r)} \\
&= \sum_r \hat{A}(r)\hat{C}(r)\hat{B}(-2r) \\
&= \alpha\beta\gamma + \sum_{r \neq 0} \hat{A}(r)\hat{C}(r)\hat{B}(-2r)
\end{aligned}$$

where $\beta = \gamma =$ density of B (or C). Now

$$\begin{aligned}
\left|\sum_{r \neq 0} \hat{A}(r)\hat{B}(-2r)\hat{C}(r)\right| &\leq \max_{r \neq 0} \left|\hat{A}(r)\right| \sum_r \hat{B}(-2r)\hat{C}(r) \\
&\leq \max_{r \neq 0} \left|\hat{A}(r)\right| \left|\left|\hat{B}\right|\right|_2 \left|\left|\hat{C}\right|\right|_2 \text{ (Cauchy-Schwarz)} \\
&= \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} \max_{r \neq 0} \left|\hat{A}(r)\right|
\end{aligned}$$

Therefore, if $\max_{r \neq 0} \left|\hat{A}(r)\right| \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} \leq \frac{\alpha\beta\gamma}{2}$, i.e. $\max_{r \neq 0} \left|\hat{A}(r)\right| \leq \frac{1}{2}\alpha(\beta\gamma)^{\frac{1}{2}}$ then the 3-AP-density of (A, B, C) is at least $\frac{\alpha\beta\gamma}{2}$. Since $\beta\gamma \geq \frac{\alpha^2}{25}$, this tells us that we get 3-APs provided $\max_{r \neq 0} \left|\hat{A}(r)\right| \leq \frac{\alpha^2}{10}$ and $\frac{\alpha^3}{50} > \frac{1}{N}$ (ensures that the progression is non-trivial). So we may assume that $\exists r$ s.t. $\left|\hat{A}(r)\right| \geq \frac{\alpha^2}{10}$.

Lemma 2. *Let $\epsilon > 0$ and let $r \in \mathbb{Z}_N$. Then the set $[N]$ can be partitioned into arithmetic progressions of length at least $\frac{\epsilon}{8\pi} N^{\frac{1}{2}}$ on each of which the function $x \mapsto \omega^{rx}$ varies by at most ϵ.*

Proof. Let $m = \lfloor N^{\frac{1}{2}} \rfloor$. Of the numbers $1, \omega^r, \ldots, \omega^{mr}$ there must be two, say ω^{ur} and ω^{vr} with $u < v$, that differ by at most $\frac{2\pi}{m}$.

Let $t = v - u$ and note that $|\omega^{ur} - \omega^{vr}| = |1 - \omega^{tr}|$, so $|1 - \omega^{tr}| \leq \frac{2\pi}{m}$.

Note also that if $a < b$, then

$$\begin{aligned} \left|\omega^{btr} - \omega^{atr}\right| &\leq \sum_{j=1}^{b-a} \left|\omega^{(a+j)tr} - \omega^{(a+j-1)tr}\right| \\ &\leq (b-a)\frac{2\pi}{m} \end{aligned}$$

by the triangle inquality.

Now partition $[N]$ into congruence classes mod t, and partition each congruence class into 'intervals' of length at most $\frac{\epsilon m}{2\pi}$ and at least $\frac{\epsilon m}{4\pi}$. This is possible, since $t \leq m \leq \sqrt{N}$ (exercise). These progressions do the job, since $\frac{\epsilon m}{4\pi} \geq \frac{\epsilon N^{\frac{1}{2}}}{8\pi}$. □