Part III Combinatorics

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Contents

1 Introduction

Let $X, Y, \ldots$ be sets

Definition. *We call $\mathcal{A} \subset \mathcal{P}(X)$ a **set system** or **family of sets**. $\mathcal{A}$ is naturally identified with a bipartite graph $G_{\mathcal{A}}(U, W)$ with $U = \mathcal{A}$, $W = \bigcup_{A \in \mathcal{A}} A$ or $W = X$. Indeed, $Ax \in E(G_{\mathcal{A}}) \iff x \in A$.*

Definition. *Given $\mathcal{A} \in \mathcal{P}(X)$, a **set of distinct representatives** (SDR) is an injection $f : \mathcal{A} \to X$ s.t. $f(A) \in A\ \forall A \in \mathcal{A}$. In its bipartite graph, an SDR corresponds to a complete matching $U \to W$.*

Theorem 1 (Hall, 1935)**.** *A set system $\mathcal{A}$ has an SDR if $\forall \mathcal{A}' \subset \mathcal{A}$, $|\cup_{A \in \mathcal{A}'} A| \geq |\mathcal{A}|'$.*

Theorem 1'. *A bipartite graph $G(U, W)$ has a complete matching $U \to W$ if $\forall S \subset U$, $|\Gamma(S)| \geq |S|$*

Corollary 2. *Suppose $G(U, W)$ bipartite, $d(u) \geq d(w)\ \forall u \in U,\ w \in W$. Then $\exists$ a complete matching $U \to W$.*

Definition. *A bipartite graph $G(U, W)$ is (r, s)**-regular** if $d(u) = r$ and $d(w) = s\ \forall u \in U$, $w \in W$.*

Instant from Cor 2: if $G(U, W)$ is (r, s)-regular then $\exists$ a complete matching from U to W if $|U| \leq |W|$.

Corollary 3. *Let $0 \leq i, j \leq n$, $\binom{n}{i} \leq \binom{n}{j}$. Then $\exists$ a complete matching $f : [n]^{(i)} \to [n]^{(j)}$ s.t. $f(A) \subset A$ if $j \leq i$, and $f(A) \supset A$ if $i \leq j$.*

Theorem 4. *Let $G = G(U, W)$ be a connected (r, s)-regular graph. Then for $\emptyset \neq A \subset U$,*

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

Also, equality holds iff $A = U$.

The **cube** $Q^n \cong \mathcal{P}(n) \cong [2]^n$ = set of all 0, 1 sequences of length n. Q^n is also a graph: AB is an edge if $|A \triangle B| = 1$. It is also a poset: $A < B$ if $A \subset B$.

Q^n has a natural orientation: $\overrightarrow{AB}$ if $A = B \cup \{a\}$.

$$\begin{tikzcd}
 & 123 \arrow[dl] \arrow[d] \arrow[dr] & \\
12 \arrow[d] \arrow[dr] & 13 \arrow[dl] \arrow[dr] & 23 \arrow[dl] \arrow[d] \\
1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\
 & \emptyset &
\end{tikzcd}$$

The order on $Q^n \cong \mathcal{P}(n)$ is induced by this oriented graph.

2 Sperner Systems

Definition. *A set system $\mathcal{A} \subset \mathcal{P}(n)$ is **Sperner** if $A, B \in \mathcal{A}$, $A \neq B \implies A \not\subset B$*

Theorem 1 (Sperner, 1928)**.** *If $\mathcal{A} \subset \mathcal{P}(n)$ is Sperner then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Definition. *The **weight** $w(A)$ of a set $A \in \mathcal{P}(n)$ is $w(A) = \frac{1}{\binom{n}{|A|}}$*

Theorem 2. *Let $\mathcal{A}$ be a Sperner system on X, $|X| = n$. Then*

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A) \leq 1$$