Part III Combinatorics

Based on lectures by Prof B. Bollobás

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University of Cambridge

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1 Introduction

Let $X, Y, \ldots$ be sets

Definition. *We call $\mathcal{A} \subset \mathcal{P}(X)$ a **set system** or **family of sets**. $\mathcal{A}$ is naturally identified with a bipartite graph $G_{\mathcal{A}}(U, W)$ with $U = \mathcal{A}$, $W = \bigcup_{A \in \mathcal{A}} A$ or $W = X$. Indeed, $Ax \in E(G_{\mathcal{A}}) \iff x \in A$.*

Definition. *Given $\mathcal{A} \in \mathcal{P}(X)$, a **set of distinct representatives** (SDR) is an injection $f : \mathcal{A} \to X$ s.t. $f(A) \in A\ \forall A \in \mathcal{A}$. In its bipartite graph, an SDR corresponds to a complete matching $U \to W$.*

Theorem 1 (Hall, 1935)**.** *A set system $\mathcal{A}$ has an SDR if $\forall \mathcal{A}' \subset \mathcal{A}$, $|\cup_{A \in \mathcal{A}'} A| \geq |\mathcal{A}|'$.*

Theorem 1'. *A bipartite graph $G(U, W)$ has a complete matching $U \to W$ if $\forall S \subset U$, $|\Gamma(S)| \geq |S|$*

Corollary 2. *Suppose $G(U, W)$ bipartite, $d(u) \geq d(w)\ \forall u \in U,\ w \in W$. Then $\exists$ a complete matching $U \to W$.*