Part III Local Fields

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Michaelmas 2016
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1 Basic Theory

Definition (Absolute value)**.** *Let K be a field. An* ***absolute value*** *on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ s.t.*

i. $|x| = 0 \iff x = 0$

ii. $|xy| = |x|\,|y| \quad \forall x, y \in K$

iii. $|x + y| \leq |x| + |y|$

Definition (Valued field)**.** *A* ***valued field*** *is a field with an absolute value.*

Definition (Equivalence of absolute values)**.** *Let K be a field and let $|\cdot|$, $|\cdot|'$ be absolute values on K. We say that $|\cdot|$ and $|\cdot|'$ are* ***equivalent*** *if the associated metrics induce the same topology.*

Definition (Non-archimedean absolute value)**.** *An absolute value $|\cdot|$ on a field K is called* ***non-archimedean*** *if $|x + y| \leq \max(|x|, |y|)$ (the* ***strong triangle inequality****).*

Metrics s.t. $d(x, z) \leq \max(d(x, y), d(y, z))$ are called ***ultrametrics****.*

Assumption: unless otherwise mentioned, all absolute values will be non-archimedean. These metrics are weird!

Proposition 1. *Let K be a valued field. Then $\mathcal{O} = \{x \mid |x| \leq 1\}$ is an open subring of K, called the* ***valuation ring*** *of K. $\forall r \in (0, 1]$, $\{x \mid x < r\}$ and $\{x \mid x \leq r\}$ are open ideals of $\mathcal{O}$.*

Moreover, $\mathcal{O}^x = \{x \mid |x| = 1\}$.

Proposition 2. *Let K be a valued field.*

i. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) is Cauchy

Assume that K is complete

ii. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) converges

iii. Let $\sum_{n=0}^{\infty} y_n$ be a series in K. If $y_n \to 0$, then $\sum_{n=0}^{\infty} y_n$ converges

Definition. *Let $R \subseteq S$ be rings. Then $s \in S$ is **integral over R** if $\exists$ monic $f(x) \in R[x]$ s.t. $f(s) = 0$.*

Proposition 3. *Let $R \subseteq S$ be rings. Then $s_1, \ldots, s_n \in S$ are all integral over $R \iff R[s_1, \ldots, s_n] \subseteq S$ is a finitely generated R-module.*

Corollary 4. *let $R \subseteq S$ be rings. If $s_1, s_2 \in S$ are integral over R, then $s_1 + s_2$ and $s_1 s_2$ are integral over R. In particular, the set $\tilde{R} \subseteq S$ of all elements in S integral over R is a ring, called the **integral closure** of R in S.*

Definition. *Let R be a ring. A topology on R is called a **ring topology** on R if addition and multiplication are continuous maps $R \times R \to R$. A ring with a ring topology is called a **topological ring**.*

Definition. *Let R be a ring, $I \subseteq R$ an ideal. A subset $U \subseteq R$ is called **I-adically open** if $\forall x \in U\ \exists n \geq 1$ s.t. $x + I^n \subseteq U$.*

Proposition 5. *The set of all I-adically open sets form a topology on R, called the **I-adic topology**.*

Definition. *Let $R_1, R_2, \ldots$ be topological rings with continuous homomorphisms $f_n : R_{n+1} \to R_n\ \forall n \geq 1$. The **inverse limit** of the R_i is the ring*

$$\begin{aligned}\varprojlim_n R_n &= \left\{ (x_n) \in \prod_n R_n \,|\, f_n(x_{n+1}) = x_n \forall n \geq 1 \right\} \\ &\subseteq \prod_n R_n\end{aligned}$$

Proposition 6. *The inverse limit topology is a ring topology.*

Definition. *Let R be a ring, I an ideal. The **I-adic completion** of R is the topological ring $\varprojlim_n R/I^n$ (R/I^n has the discrete topology, and $R/I^{n+1} \to R/I^n$ is the natural map).*