Part III Category Theory

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1 Definitions and Examples

Definition (Category)**.** *A category $\mathcal{C}$ consists of*

a. a collection $\mathrm{ob}\,\mathcal{C}$ *of* ***objects*** *A, B, C, ...*

b. a collection $\mathrm{mor}\,\mathcal{C}$ *of* ***morphisms*** *f, g, h, ...*

c. two operations dom, cod *from morphisms to objects. We write $f : A \to B$ or $A \xrightarrow{f} B$ to mean 'f is a morphism and* $\mathrm{dom}\, f = A$ *and* $\mathrm{cod}\, f = B$*'*

d. an operation assigning to each object A a morphism $1_A : A \to A$

e. a partial binary operation $(f, g) \mapsto gf$, s.t. gf is defined $\iff$ $\mathrm{dom}\, g = \mathrm{cod}\, f$*, and then* $gf : \mathrm{dom}\, f \to \mathrm{cod}\, g$

Definition (Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor*** *$\mathcal{C} \to \mathcal{D}$ consists of*

a. a mapping $A \to FA$ from $\mathrm{ob}\,\mathcal{C}$ *to* $\mathrm{ob}\,\mathcal{D}$

b. a mapping $f \to Ff$ from $\mathrm{mor}\,\mathcal{C}$ *to* $\mathrm{mor}\,\mathcal{D}$

satisfying $\mathrm{dom}\, Ff = F\mathrm{dom}\, f$, $\mathrm{cod}\, Ff = F\mathrm{cod}\, f$ *for all f, $F(1_A) = 1_{FA}$ for all A, and $F(gf) = (Fg)(Ff)$ whenever gf is defined.*

Definition. *By a* ***contravariant functor*** *$\mathcal{C} \to \mathcal{D}$ we mean a functor $\mathcal{C} \to \mathcal{D}^{op}$ (or equivalently $\mathcal{C}^{op} \to \mathcal{D}$). A functor $\mathcal{C} \to \mathcal{D}$ is sometimes said to be* ***covariant****.*

Definition (Natural transformation)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two functors. A* ***natural transformation*** *$\alpha : F \to G$ assigns to each $A \in \operatorname{ob} \mathcal{C}$ a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$, such that*

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

commutes.