Part III Local Fields

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1 Basic Theory

Definition (Absolute value)**.** *Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ s.t.*

i. $|x| = 0 \iff x = 0$

ii. $|xy| = |x|\,|y| \quad \forall x, y \in K$

iii. $|x + y| \leq |x| + |y|$

Definition (Valued field)**.** *A **valued field** is a field with an absolute value.*

Definition (Equivalence of absolute values)**.** *Let K be a field and let $|\cdot|$, $|\cdot|^{'}$ be absolute values on K. We say that $|\cdot|$ and $|\cdot|^{'}$ are **equivalent** if the associated metrics induce the same topology.*

Definition (Non-archimedean absolute value)**.** *An absolute value $|\cdot|$ on a field K is called **non-archimedean** if $|x + y| \leq \max(|x|, |y|)$ (the **strong triangle inequality**).*

*Metrics s.t. $d(x, z) \leq \max(d(x, y), d(y, z))$ are called **ultrametrics**.*

Assumption: unless otherwise mentioned, all absolute values will be non-archimedean. These metrics are weird!

Proposition 1. *Let K be a valued field. Then $\mathcal{O} = \{x \mid |x| \leq 1\}$ is an open subring of K, called the **valuation ring** of K. $\forall r \in (0,1]$, $\{x \mid x < r\}$ and $\{x \mid x \leq r\}$ are open ideals of $\mathcal{O}$.*

Moreover, $\mathcal{O}^x = \{x \mid |x| = 1\}$.

Proposition 2. *Let K be a valued field.*

i. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) is Cauchy

Assume that K is complete

ii. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) converges

iii. Let $\sum_{n=0}^{\infty} y_n$ be a series in K. If $y_n \to 0$, then $\sum_{n=0}^{\infty} y_n$ converges

Definition. *Let $R \subseteq S$ be rings. Then $s \in S$ is **integral over R** if $\exists$ monic $f(x) \in R[x]$ s.t. $f(s) = 0$.*

Proposition 3. *Let $R \subseteq S$ be rings. Then $s_1, \dots, s_n \in S$ are all integral over $R \iff R[s_1, \dots, s_n] \subseteq S$ is a finitely generated R-module.*

Corollary 4. *let $R \subseteq S$ be rings. If $s_1, s_2 \in S$ are integral over R, then $s_1 + s_2$ and $s_1 s_2$ are integral over R. In particular, the set $\tilde{R} \subseteq S$ of all elements in S integral over R is a ring, called the **integral closure** of R in S.*

Definition. *Let R be a ring. A topology on R is called a **ring topology** on R if addition and multiplication are continuous maps $R \times R \to R$. A ring with a ring topology is called a **topological ring**.*

Definition. *Let R be a ring, $I \subseteq R$ an ideal. A subset $U \subseteq R$ is called **I-adically open** if $\forall x \in U$ $\exists n \geq 1$ s.t. $x + I^n \subseteq U$.*

Proposition 5. *The set of all I-adically open sets form a topology on R, called the **I-adic topology**.*

Definition. *Let $R_1, R_2, \dots$ be topological rings with continuous homomorphisms $f_n : R_{n+1} \to R_n$ $\forall n \geq 1$. The **inverse limit** of the R_i is the ring*

$$\begin{aligned} \varprojlim_n R_n &= \left\{ (x_n) \in \prod_n R_n \,\middle|\, f_n(x_{n+1}) = x_n \forall n \geq 1 \right\} \\ &\subseteq \prod_n R_n \end{aligned}$$

Proposition 6. *The inverse limit topology is a ring topology.*

Definition. *Let R be a ring, I an ideal. The **I-adic completion** of R is the topological ring $\varprojlim_n R/I^n$ (R/I^n has the discrete topology, and $R/I^{n+1} \to R/I^n$ is the natural map).*

*There exists a map $\nu : R \to \varprojlim R/I^n$, $r \mapsto (r \mod I^n)_n$ This map is a continuous ring homomorphism when R is given the I-adic topology. We say that R is **I-adically complete** if ν is a bijection.*

*If $I = xR$ then we often call the I-adic topology the **x-adic topology**.*

1.1 The p-adic Numbers

Let p be a prime number throughout.

If $x \in \mathbb{Q} \backslash \{0\}$ then $\exists!$ representation $x = p^n \frac{a}{b}$, where $n \in \mathbb{Z}$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{>0}$ and $(a, p) = (b, p) = (a, b) = 1$.

We define the **p-adic absolute value** on $\mathbb{Q}$ to be the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-n} & \text{if } x = p^n \frac{a}{b} \ (\neq 0) \text{ as before} \end{cases}$$

Then $|\cdot|_p$ is an absolute value.

Definition. *The **p-adic numbers** $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$.*

*The valuation ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is called the **p-adic integers**.*

Proposition 7. *$\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ inside $\mathbb{Q}_p$.*

Proposition 8. *The non-zero ideals of $\mathbb{Z}_p$ are $p_n\mathbb{Z}_p$ for $n \geq 0$. Moreover, $\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p/p^n\mathbb{Z}_p$*

Corollary 9. *$\mathbb{Z}_p$ is a PID with a unique prime element p (up to units).*

Proposition 10. *The topology on $\mathbb{Z}$ induced by $|\cdot|_p$ is the p-adic topology.*

Proposition 11. *$\mathbb{Z}_p$ is p-adically complete and is (isomorphic to) the p-adic completion of $\mathbb{Z}$.*

Corollary 12. *Every $a \in \mathbb{Z}_p$ has a unique expansion*

$$a = \sum_{i=0}^{\infty} a_i p^i$$

with $a_i \in \{0, 1, \ldots, p-1\}$