Part III Local Fields

Based on lectures by Dr. C Johansson

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University of Cambridge

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1 Basic Theory

Definition (Absolute value)**.** *Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ s.t.*

i. $|x| = 0 \iff x = 0$

ii. $|xy| = |x||y| \quad \forall x, y \in K$

iii. $|x + y| \leq |x| + |y|$

Definition (Valued field)**.** *A **valued field** is a field with an absolute value.*

Definition (Equivalence of absolute values)**.** *Let K be a field and let $|\cdot|, |\cdot|'$ be absolute values on K. We say that $|\cdot|$ and $|\cdot|'$ are **equivalent** if the associated metrics induce the same topology.*

Definition (Non-archimedean absolute value)**.** *An absolute value $|\cdot|$ on a field K is called **non-archimedean** if $|x + y| \leq \max(|x|, |y|)$ (the **strong triangle inequality**).*

*Metrics s.t. $d(x, z) \leq \max(d(x, y), d(y, z))$ are called **ultrametrics**.*

Assumption: unless otherwise mentioned, all absolute values will be non-archimedean. These metrics are weird!

Proposition 1. *Let K be a valued field. Then $\mathcal{O} = \{x \mid |x| \leq 1\}$ is an open subring of K, called the **valuation ring** of K. $\forall r \in (0, 1]$, $\{x \mid x < r\}$ and $\{x \mid x \leq r\}$ are open ideals of $\mathcal{O}$.*

Moreover, $\mathcal{O}^x = \{x \mid |x| = 1\}$.

Proposition 2. *Let K be a valued field.*

i. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) is Cauchy

Assume that K is complete

ii. Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$ then (x_n) converges

iii. Let $\sum_{n=0}^{\infty} y_n$ be a series in K. If $y_n \to 0$, then $\sum_{n=0}^{\infty} y_n$ converges