Part III Combinatorics

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Contents

1 Introduction

Let $X, Y, \ldots$ be sets

Definition. *We call $\mathcal{A} \subset \mathcal{P}(X)$ a **set system** or **family of sets**. $\mathcal{A}$ is naturally identified with a bipartite graph $G_{\mathcal{A}}(U, W)$ with $U = \mathcal{A}$, $W = \bigcup_{A \in \mathcal{A}} A$ or $W = X$. Indeed, $Ax \in E(G_{\mathcal{A}}) \iff x \in A$.*

Definition. *Given $\mathcal{A} \in \mathcal{P}(X)$, a **set of distinct representatives** (SDR) is an injection $f : \mathcal{A} \to X$ s.t. $f(A) \in A \ \forall A \in \mathcal{A}$. In its bipartite graph, an SDR corresponds to a complete matching $U \to W$.*

Theorem 1 (Hall, 1935)**.** *A set system $\mathcal{A}$ has an SDR if $\forall \mathcal{A}' \subset \mathcal{A}$, $|\cup_{A \in \mathcal{A}'} A| \geq |\mathcal{A}|'$.*

Theorem 1'. *A bipartite graph $G(U, W)$ has a complete matching $U \to W$ if $\forall S \subset U$, $|\Gamma(S)| \geq |S|$*

Corollary 2. *Suppose $G(U,W)$ bipartite, $d(u) \geq d(w)$ $\forall u \in U,\ w \in W$. Then $\exists$ a complete matching $U \to W$.*

Definition. *A bipartite graph $G(U,W)$ is (r,s)-**regular** if $d(u) = r$ and $d(w) = s$ $\forall u \in U,\ w \in W$.*

Instant from Cor 2: if $G(U,W)$ is (r,s)-regular then $\exists$ a complete matching from U to W if $|U| \leq |W|$.

Corollary 3. *Let $0 \leq i,j \leq n$, $\binom{n}{i} \leq \binom{n}{j}$. Then $\exists$ a complete matching $f : [n]^{(i)} \to [n]^{(j)}$ s.t. $f(A) \subset A$ if $j \leq i$, and $f(A) \supset A$ if $i \leq j$.*

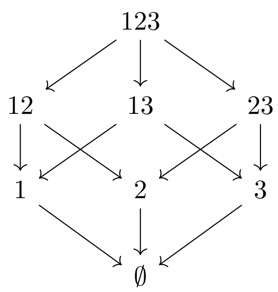
Theorem 4. *Let $G = G(U,W)$ be a connected (r,s)-regular graph. Then for $\emptyset \neq A \subset U$,*

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

Also, equality holds iff $A = U$.

The **cube** $Q^n \cong \mathcal{P}(n) \cong [2]^n$ = set of all 0, 1 sequences of length n. Q^n is also a graph: AB is an edge if $|A \bigtriangleup B| = 1$. It is also a poset: $A < B$ if $A \subset B$.

Q^n has a natural orientation: $\overrightarrow{AB}$ if $A = B \cup \{a\}$.



The order on $Q^n \cong \mathcal{P}(n)$ is induced by this oriented graph.

2 Sperner Systems

Definition. *A set system $\mathcal{A} \subset \mathcal{P}(n)$ is **Sperner** if $A, B \in \mathcal{A}$, $A \neq B \implies A \not\subset B$*

Theorem 1 (Sperner, 1928)**.** *If $\mathcal{A} \subset \mathcal{P}(n)$ is Sperner then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Definition. *The **weight** $w(A)$ of a set $A \in \mathcal{P}(n)$ is $w(A) = \frac{1}{\binom{n}{|A|}}$*

Theorem 2. *Let $\mathcal{A}$ be a Sperner system on X, $|X| = n$. Then*

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A) \leq 1$$

Corollary 3. *If $\mathcal{A} \in \mathcal{P}(n)$ is a Sperner system then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, with equality $\iff \mathcal{A}$ is $X^{\lfloor n/2 \rfloor}$ or $X^{\lceil n/2 \rceil}$.*

Definition. *$\mathcal{A} \in \mathcal{P}(n)$ is **k-Sperner** if it does not contain*

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{k+1}$$

Note that Sperner = 1-Sperner.

Corollary 4 (Erdős, 1945)**.** *If $\mathcal{A} \subset \mathcal{P}(n)$ is k-Sperner then $|\mathcal{A}|$ is at most the sum of the k largest binomial coefficients.*

Theorem 5 (Erdős, 1945)**.** *Let $x_1, \ldots, x_n \in \mathbb{R}$, $x_i \geq 1$. Then the number of sums $\sum_1^n \pm x_i$ in an open interval J of length $2k$ is at most the sum of the k largest binomial coefficients.*

Definition. *A chain $A_o \subset A_1 \subset \cdots \subset A_k$ is **symmetric** if $|A_{i+1}| = |A_i| + 1 \ \forall i$ and $|A_o| + |A_k| = n$.*

Theorem 6 (Kleitman and Katona)**.** *$\mathcal{P}(n)$ has a decomposition into symmetric chains.*

Take such a partition $\mathcal{P}(n) = \bigcup_{i=1}^k \mathcal{C}_i$, $j = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. There is one chain of length $n+1$, $n-1$ chains of length $n-1$, etc: there are $\binom{n}{i} - \binom{n}{i-1}$ chains of length $n + 1 - 2i$.

Let E be a normed space, let $x_1, \ldots, x_n \in E$, $||x_i|| \geq 1 \ \forall i$, for $A \in \mathcal{P}(n)$ let $x_A = \sum i \in A x_i$.

Conjecture (Erdős, 1945)**.** *If $\mathcal{A} \in \mathcal{P}(n)$ s.t. $||x_A - x_B|| < 1$ then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$*

Definition. *Call $\mathcal{D} \in \mathcal{P}(n)$ **scattered** if $||x_A - x_B|| \geq 1 \ \forall A, B \in \mathcal{D}$. Call a partition $\mathcal{P}(n) = \bigcup_{i=1}^s \mathcal{D}_i$ **symmetric** if there are precisely $\binom{n}{i} - \binom{n}{i-1}$ sets $\mathcal{D}_i$ of cardinality $n + 1 - 2i$.*

Theorem 7. *(Kleitman, 1970) $E, (x_i)_1^n$ as before. Then $\mathcal{P}(n)$ has a symmetric partition into scattered sets.*

Theorem 8. *(Kleitman, 1970) If $\mathcal{A} \in \mathcal{P}(n)$ s.t. $||x_A - x_B|| < 1$ then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$*

3 The Kruskal-Katona Theorem

We know: if $\mathcal{A} \subset X^{(r)}$ then $\partial\mathcal{A}$ (the **lower shadow** of $\mathcal{A}$), defined by

$$\partial\mathcal{A} = \{B \in X^{(r-1)} \mid B \subset A \text{ for some } A \in \mathcal{A}\}$$

satisfies

$$\begin{aligned} |\partial\mathcal{A}| &\geq |\mathcal{A}| \frac{\binom{n}{r-1}}{\binom{n}{r}} \\ &= |\mathcal{A}| \frac{r}{n-r+1} \end{aligned}$$

with equality $\iff$ $\mathcal{A}$ is $\emptyset$ or $X^{(r)}$.

What about in between? What is $\mathcal{B} \in X^{(r)}$ s.t. $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$?

$\exists \mathcal{B}_1, \mathcal{B}_2, \cdots \in X^{(r)}$ s.t. $|\mathcal{B}_m| = m$ and $|\partial\mathcal{B}_m| \leq |\partial\mathcal{A}|$ $\forall \mathcal{A} \subset X^{(r)}$ where $|\mathcal{A}| = m$.

Incredibly luckily, we have a sequence of nested extremal sets. Equivalently, $\exists$ total order on $X^{(r)}$ s.t. the first m sets form $\mathcal{B}_m$.

Definition. *Define the **colex** total order on $X^{(r)}$ by $A < B$ if* $\max(A \Delta B) \in B$.

Aim: given m and r, would like to find $\mathcal{B} \subset X^{(r)}$, $|\mathcal{B}| = m$ s.t. $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$ $\forall \mathcal{A} \subset X^{(r)}$, $|\mathcal{A}| = m$.

Define $\mathcal{B}^{(r)}(m_r, \dots, m_s)$, $m_r > m_{r-1} > \cdots > m_s \geq s$ as follows:

$$\begin{aligned} \mathcal{B}^{(r)} = [m_r]^{(r)} &\cup ([m_{r-1}]^{(r-1)} + \{m_r + 1\}) \\ &\cup ([m_{r-2}]^{(r-2)} + \{m_{r-1} + 1, m_r + 1\}) \\ &\cup \dots \\ &\cup ([m_s]^{(s)} + \{m_{s+1} + 1, m_{s+2} + 1, \dots, m_r + 1\}) \end{aligned}$$

Set $b^{(r)}(m_r, \dots, m_s) = \left|\mathcal{B}^{(r)}(m_r, \dots, m_s)\right| = \sum_{j=s}^{r} \binom{m_j}{j}$.

$$\partial\mathcal{B}^{(r)}(m_r, \dots, m_s) = \mathcal{B}^{(r-1)}(m_r, \dots, m_s)$$

This has cardinality $b^{(r-1)}(m_r, \dots, m_s) = \sum_{j=s}^{r} \binom{m_j}{j-1}$.

Lemma 1. *For $l, r \in \mathbb{N}$ $\exists!$ $m_r > \cdots > m_s$ s.t. $l = \sum_{j=s}^{r} \binom{m_j}{j}$; the initial segment of $X^{(r)}$ in colex, consisting of l sets, is $\mathcal{B}^{(r)}(m_r, \dots, m_s)$.*

Definition. *Let $i \neq j \in X$, $A \in \mathcal{P}(X)$. Define the **ij-compression***

$$A_{ij} = C_{ij}(A) = \begin{cases} (A \backslash \{j\}) \cup \{i\} & \text{if } j \in A,\ i \notin A \\ A & \text{otherwise} \end{cases}$$

Given $\mathcal{A} \subset \mathcal{P}(n), A \in \mathcal{A}$

$$C_{i,j,\mathcal{A}}(A) = \begin{cases} A_{ij} & \textit{if } A_{ij} \notin \mathcal{A} \\ A & \textit{otherwise} \end{cases}$$

Also,

$$\begin{aligned} C_{ij}(\mathcal{A}) &= \{C_{i,j,\mathcal{A}} \,|\, A \in \mathcal{A}\} \\ &= \{C_{ij}(A) \,|\, A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \,|\, C_{ij}(A) \in \mathcal{A}\} \end{aligned}$$

For $\mathcal{A} \in X^{(r)}$,

$$\begin{aligned} \mathcal{A}_{ij} &= \{A \in \mathcal{A} \,|\, \{i,j\} \subset A\} \\ \mathcal{A}_{i} &= \{A \in \mathcal{A} \,|\, i \in A,\, j \notin A\} \\ \mathcal{A}_{\emptyset} &= \{A \in \mathcal{A} \,|\, A \cap \{i,j\} = \emptyset\} \\ \mathcal{A}_{j} &= \{A \in \mathcal{A} \,|\, i \notin A,\, j \in A\} \end{aligned}$$

$C_{ij} : \mathcal{A} \mapsto C_{ij}(\mathcal{A})$ keeps $\mathcal{A}_\emptyset \cup \mathcal{A}_i \cup \mathcal{A}_{ij}$ fixed, and maps $\mathcal{A}_j$ into sets like those in $\mathcal{A}_i$.

Lemma 2. *For* $\mathcal{A} \subset X^{(r)}$, $\partial C_{ij}(\mathcal{A}) \subseteq C_{ij}(\partial \mathcal{A})$. *In particular, the cardinality decreases.*

Proof. Let $B \in \partial C_{ij}(\mathcal{A})$ and let $A \in \mathcal{A}$ s.t. $B \subset C_{i,j,\mathcal{A}}(A)$.

i. Suppose B meets $\{i,j\}$ in 0 or 2 elements. Then $B \subset A$ so $B \in \partial A$ and $B \in C_{ij}(\partial \mathcal{A})$

ii. Suppose $i \in B$, $j \notin B$. Then either B or $(B \backslash \{i\}) \cup \{j\}$ belongs to $\partial \mathcal{A}$, so $B \in C_{ij}(\partial \mathcal{A})$.

iii. Suppose $j \in B$, $i \notin B$. Then both B and $(B \backslash \{j\}) \cup \{i\}$ belong to $\partial \mathcal{A}$, so both belong to $C_{ij}(\partial \mathcal{A})$.

□

Definition. *Call* $\mathcal{A} \subset X^{(r)}$ ***left-compressed*** *if* $C_{ij}(\mathcal{A}) = \mathcal{A}\ \forall i < j$.

Lemma 3. *Let* $\mathcal{A} \subset X^{(r)}$. *Then* $\exists$ *a left-compressed family* $\mathcal{B} \subset X^r$ *s.t.* $|\mathcal{B}| = |\mathcal{A}|$ *and* $|\partial \mathcal{B}| \leq |\partial \mathcal{A}|$.

Proof. Define $\mathcal{A}_0 = \mathcal{A}, \mathcal{A}_1, \dots$ as follows: having reached $\mathcal{A}_k$, if $\mathcal{A}_k$ is not left-compressed, pick $i < j$ s.t. $C_{ij}(\mathcal{A}_k) \neq \mathcal{A}_k$, and set $\mathcal{A}_{k+1} = C_{ij}(\mathcal{A}_k)$

This sequence has to end because

$$\sum_{A\in\mathcal{A}_{k+1}}\sum_{a\in A} a < \sum_{A\in\mathcal{A}_k}\sum_{a\in A} a$$

let $\mathcal{A}_l$ be the last term: this will do for $\mathcal{B}$. $\square$

Theorem 4 (Kruskal-Katona, 1963 and 1968)**.** *Let $\mathcal{A} \subset X^{(r)}$, $m = |\mathcal{A}|$. Then*

$$\begin{aligned} |\partial\mathcal{A}| &\geq \left|\partial\mathcal{B}_m^{(r)}\right| \\ &= \left|\partial\mathcal{B}^{(r)}(m_r, m_{r-1}, \ldots, m_s)\right| \\ &= b^{(r-1)}(m_r, \ldots, m_s) \end{aligned}$$

Proof. Induction on r and then m (or on $r+m$). $r = 1$✓ $m = 1$✓

Induction step: we may assume that $\mathcal{A}$ is left-compressed. Set $Y = X\backslash\{1\}$. Then $\mathcal{A} = (\mathcal{A}_1 + \{1\}) \cup \mathcal{A}_0$, where $\mathcal{A}_1 \subset Y^{(r-1)}$, $\mathcal{A}_0 \subset Y^{(r)}$.

$m = |\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1|$, $\partial\mathcal{A}_0 \subset \mathcal{A}_1$, $\partial(\mathcal{A}_1 + \{1\}) = \mathcal{A}_1 \cup (\partial\mathcal{A}_1 + \{1\})$.

In particular, $|\partial\mathcal{A}| = |\mathcal{A}_1| + |\partial\mathcal{A}_1|$.

For $\mathcal{A} = \mathcal{B}^{(r)}(m_r, \ldots, m_s)$,

$$|\mathcal{A}_1| = b^{(r-1)}(m_r - 1, \ldots, m_s - 1)$$

$$|\mathcal{A}_0| = b^{(r)}(m_r - 1, \ldots, m_s - 1)$$

Suppose $|\mathcal{A}_0| > b^{(r)}(m_r - 1, \ldots, m_s - 1)$. Then by the induction hypothesis, $|\partial\mathcal{A}_0| \geq b^{(r-1)}(m_r - 1, \ldots, m_s - 1)$. Hence $|\mathcal{A}_1| \geq b^{(r-1)}(m_r - 1, \ldots, m_s - 1)$ and so $|\partial\mathcal{A}| \geq b^{(r-1)}(m_r, \ldots, m_s)$.

But if $|\mathcal{A}_0| \leq b^{(r)}(m_r - 1, \ldots, m_s - 1)$, $|\mathcal{A}_1|$ is again $\geq b^{(r-1)}(m_r - 1, \ldots, m_s - 1)$. Done as before. $\square$

Soft version:

Theorem 5 (Lovász, 1979)**.** *If $\mathcal{A} \subset X^{(r)}$ satisfies $|\mathcal{A}| = \binom{X}{r}$ then $|\partial\mathcal{A}| \geq \binom{X}{r-1}$.*

Proof. Induction on r and $m = |\mathcal{A}|$. As before, $\mathcal{A}_0, \mathcal{A}_1$. Note that $\mathcal{A}_1 \geq \binom{X-1}{r-1}$ since otherwise $\mathcal{A}_0 > \binom{X-1}{r}$. But then $|\partial\mathcal{A}_0| \geq \binom{X-1}{r-1}$, contradicting the fact that $\partial\mathcal{A}_0 \subset \mathcal{A}_1$.

But if $|\mathcal{A}_1| \geq \binom{X-1}{r-1}$ then

$$|\mathcal{A}_1| + |\partial\mathcal{A}_1| \geq \binom{X-1}{r-1} + \binom{X-1}{r-2} = \binom{X}{r-1}$$

$\square$

Definition. *Define the **uniform probability measure** on $X^{(r)}$, $|X| = n$ as $\mathbb{P}_{n,r}(A) = \frac{1}{\binom{n}{r}}$, and for $\mathcal{A} \subset X^{(r)}$, $\mathbb{P}_{n,r}(\mathcal{A}) = \frac{|\mathcal{A}|}{\binom{n}{r}}$.*

Definition. $\mathcal{A} \subset \mathcal{P}(n)$ *is* ***monotone decreasing*** *if* $A \subset B \in \mathcal{A} \implies A \in \mathcal{A}$.

Theorem 6. *If* $1 \leq s < r \leq n$, $\mathcal{A} \subset \mathcal{P}(n)$ *decreasing, then* $\mathbb{P}_s(\mathcal{A})^r \geq \mathbb{P}_r(\mathcal{A})^s$.
[$\mathbb{P}_k(\mathcal{A}) = \mathbb{P}_k(\mathcal{A}_k)$, $\mathcal{A}_k = \mathcal{A} \cap X^{(k)}$*]*

Proof. $\mathbb{P}_k(\mathcal{A}) = \frac{|\mathcal{A}_k|}{\binom{n}{k}}$, if $|\mathcal{A}_r| = \binom{X}{r}$ then we know $|\mathcal{A}_s| \geq \binom{X}{s}$. Hence, the inequality holds if

$$\prod_{i=0}^{s-1} \left(\frac{X-i}{n-i}\right)^r \geq \prod_{i=0}^{r-1} \left(\frac{X-i}{n-i}\right)^s$$

since $\frac{\binom{X}{r}}{\binom{n}{r}} = \prod_{i=0}^{r-1} \frac{X-i}{n-i}$.

But this is

$$\prod_{i=0}^{s-1} \left(\frac{X-i}{n-i}\right)^{r-s} \geq \prod_{i=s}^{r-1} \left(\frac{X-i}{n-i}\right)^s$$

Every factor on the left is larger than every factor on the right:

$$\frac{X-i}{n-i} > \frac{X-j}{n-j}$$

for $i \leq s-1$, $j \geq s$. □

Definition (Erdős and Rényi, 1960)**.** *Given an increasing family ('property of sets')* $\mathcal{A}(n) \subset \mathcal{P}(n)$*, a function* $k^*(n)$ *is a* ***threshold function*** *for* $\mathcal{A}(n)$ *if* $\mathbb{P}_{k(n)}(\mathcal{A}(n)) \to 0$ *if* $\frac{k}{k^*} \to 0$*, and* $\mathbb{P}_{k(n)}(\mathcal{A}(n)) \to 1$ *if* $\frac{k}{k^*} \to 1$.

Erdős and Rényi: for many monotone increasing graph properties, $\exists$ a threshold.

Corollary 7. *Let* $\mathcal{A} \subset \mathcal{P}(n)$, $k_1 < k < k_2$

i. If $\mathcal{A}$ *is decreasing,* $\mathbb{P}_{k_2}(\mathcal{A})^{k/k_2} \leq \mathcal{P}_k(\mathcal{A}) \leq \mathcal{P}_{k_1}(\mathcal{A})^{k/k_1}$

ii. If $\mathcal{A}$ *is increasing,* $(1 - \mathbb{P}_{k_2}(\mathcal{A}))^{k/k_2} \leq 1 - \mathcal{P}_k(\mathcal{A}) \leq (1 - \mathcal{P}_{k_1}(\mathcal{A}))^{k/k_1}$

Proof. i. This is precisely Theorem 6

ii. Set $\mathcal{A}^c = \mathcal{P}(n) \backslash \mathcal{A}$. Then $\mathcal{A}^c$ is decreasing and

$$\mathbb{P}_k(\mathcal{A}^c) = 1 - \mathbb{P}_k(\mathcal{A})$$

Apply (i) to $\mathcal{A}^c$.

□

Theorem 8. *Every monotone increasing function has a threshold.*

Proof. We may assume $\mathcal{A}$ is non-trivial. Set $k^*(n) = \max\left\{k \,|\, \mathbb{P}_k(\mathcal{A}) \le \frac{1}{2}\right\}$.

Then, for $k < k^*$,

$$\mathbb{P}_k(\mathcal{A}) \le 1 - (1 - \mathbb{P}_{k*}(\mathcal{A}))^{k/k^*} \le 1 - 2^{-k/k^*}$$

For $k > k^* + 1$,

$$\mathbb{P}_k(\mathcal{A}) \ge 1 - (1 - \mathbb{P}_{k*}(\mathcal{A}))^{k/(k^*+1)} \ge 1 - 2^{-k/(k^*+1)}$$

□

This is essentially best possible, but only for lop-sided systems $\mathcal{A}$.

Definition. *$\mathcal{A} \subset \mathcal{P}(n)$ is **symmetric** if $\forall x, y, \in X$ $\exists$ a permutation π of X mapping x onto y, keeping $\mathcal{A}$ invariant.*

Definition. *Another measure on $\mathcal{P}(n)$: the **binomial measure**. Let $0 < p < 1$.*

$$\mathbb{P}_{n,p}(A) = \mathbb{P}_p(A) = p^{|A|}(1-p)^{n-|A|}$$

$\mathbb{P}_{n,p}$ is very similar to $\mathbb{P}_{n,k}$ for $k \sim pn$.

Theorem 9 (Friedgut and Kaloi, 1996)**.** *There is an absolute constant $c_0 > 0$ s.t. if $\mathcal{A} \subset \mathcal{P}(n)$ is a symmetric increasing family and $\mathbb{P}_p(\mathcal{A}) > \epsilon > 0$ then $\mathbb{P}_{p'}(\mathcal{A}) > 1 - \epsilon$ provided $p' \ge p + c_0 \frac{\log 1/\epsilon}{\log n}$*

4 Intersecting Families

Definition. *$\mathcal{A} \subset \mathcal{P}(n)$ is **intersecting** if $A \cap B \neq \emptyset$ $\forall A, B \in \mathcal{A}$.*

Suppose $\mathcal{A} \subset X^{(r)}$. If $r > \frac{n}{2}$, $\mathcal{A}$ is intersecting. If $r = \frac{n}{2}$, we can take families of size $\frac{1}{2}\binom{n}{r}$. $r < \frac{n}{2}$?

Let

$$X_x^{(r)} = \{A \in X^{(r)} \,|\, x \in A\}$$

for any $x \in X$.

Theorem 1 (Erdős, Ko and Rado 1961)**.** *Let $n > 2r \ge 4$ and let $\mathcal{A} \subset X^{(r)}$ be an intersecting family. Then $|\mathcal{A}| \le \binom{n-1}{r-1}$ with equality $\iff$ $\mathcal{A} = X_x^{(r)}$.*

Proof. We may assume $|\mathcal{A}| \ge \binom{n-1}{r-1}$. Take $\mathcal{B} = \{X \backslash A \,|\, A \in \mathcal{A}\} \subset X^{(n-r)}$. For $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \not\subset B$.

Let $\mathcal{C} = \partial \ldots \partial \mathcal{B}$ (shadow $n - r$ times). Then $\mathcal{C} \subset X^{(r)}$ and $\mathcal{C} \cap \mathcal{A} = \emptyset$, $\therefore |\mathcal{A}| + |\mathcal{C}| \le \binom{n}{r}$.

By Kruskal-Katona, since $|B| \ge \binom{n-1}{r-1} = \binom{n-1}{n-r}$, have $|\mathcal{C}| \ge \binom{n-1}{r}$.

Hence $|\mathcal{A}| \le \binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1}$. □

Definition. *We call $\mathcal{A}$* ***l-intersecting*** *if $|A \cap B| \geq l \ \forall A, B \in \mathcal{A}$.*

Let

$$\mathcal{F}_0 = \{A \in X^{(r)} \mid A \supset [l]\}$$

Lemma 2. *Let $2 \leq l < r$ and $n \geq \frac{4}{3}lr^3$. Let $\mathcal{A} \subset X^{(r)}$ be l-intersecting,* ***not*** *fixed by an l-set (i.e. $\mathcal{A} \not\subset \mathcal{F}' \cong \mathcal{F}_0$). Then*

$$|\mathcal{A}| \leq (r-l)\binom{n-l-1}{r-l-1} + \sum_{t=1}^{t_0} \binom{l}{t}\binom{r-l}{t}^2\binom{n-l-2t}{r-l-t}$$

where $t_0 = \min\{l, r-l\}$.

Proof. We may assume $\mathcal{A}$ is maximal l-intersecting. So $\exists A_1, A_2 \in \mathcal{A}$ s.t. $A_1 \cap A_2 = B$, $|B| = l$.

Let $\mathcal{A}_t = \{A \in \mathcal{A} \mid |B \backslash A| = t\}$.

$|\mathcal{A}_0| \leq (r-l)\binom{n-l-1}{r-l-1}$

$|\mathcal{A}_t| \leq \binom{l}{t}\binom{r-l}{t}^2\binom{n-l-2t}{r-l-t}$ □

Theorem 3. *Suppose $2 \leq l < r < n$ and $n \geq \frac{3}{2}lr^3$. Let $\mathcal{A} \subset X^{(r)}$ be l-intersecting. Then $|\mathcal{A}| \leq \binom{n-l}{r-l}$ and equality holds only if*

$$\mathcal{A} \cong \{A \in X^{(r)} \mid A \supset L\}$$

for some $L \in X^{(l)}$.

Proof. Suppose $\mathcal{A}$ is not fixed by an l-set. Then by Lemma 2,

$$\begin{aligned} |A| &\leq (r-l)\binom{n-l-1}{r-l-1} + \sum_{t=1}^{t_0} \binom{l}{t}\binom{r-l}{t}^2\binom{n-l-t}{r-l-t} \\ &= (r-l)\binom{n-l-1}{r-l-1} + \sum_{t=1}^{t_0} S_t \end{aligned}$$

Note

$$\begin{aligned} \frac{S_{t+1}}{S_t} &= \frac{l-t}{t+1}\frac{(r-l-t)^2}{(t+1)^2}\frac{r-l-t}{n-l-t} \\ &\leq \frac{lr^3}{(t+1)^3 n} \leq \frac{2}{3(t+1)^3} \leq \frac{1}{12} \end{aligned}$$

Thus

$$\begin{aligned} \frac{|\mathcal{A}|}{\binom{n-l}{r-l}} &\leq (r-l)\frac{r-l}{n-l} + \frac{12}{11}l(r-l)^2\frac{r-l}{n-l} \\ &= (1 + \frac{12}{11}l(r-l))\frac{(r-l)^2}{n-l} \\ &< \frac{3}{2}l\frac{r^3}{n} \leq 1 \end{aligned}$$

If $r = l + 2$ then $<$. □

Suppose $\mathcal{P}(X) \supset \mathcal{A}$ is intersecting. $\mathcal{A} \leq 2^{n-1}$. Binomial probability measure:

$$\mathbb{P}_p(A) = p^{|A|}(1-p)^{n-|A|}$$

$$\mathbb{P}_p(\mathcal{A}) = \sum_{A \in \mathcal{A}} \mathbb{P}_p(A)$$

$\mathcal{A}$ intersecting $\implies \mathbb{P}_{\frac{1}{2}}(\mathcal{A}) \leq \frac{1}{2}$.

Theorem 4. *Let $0 < p \leq \frac{1}{2}$ and let $\mathcal{A} \subset \mathcal{P}(X)$ be intersecting. Then $\mathbb{P}_p(\mathcal{A}) \leq p$.*

Proof. Set $N_k = |\mathcal{A}_k|$. $A \in \mathcal{A} \implies A^c = X \backslash A \notin \mathcal{A}$.

Hence $N_k + N_{n-k} \leq \binom{n}{k}$. Also, for $k \leq \frac{n}{2}$, $p^k(1-p)^{n-k} \geq p^{n-k}(1-p)^k$, so

$$\begin{aligned} N_k p^k(1-p)^{n-k} + N_{n-k}p^{n-k}(1-p)^k &\leq \binom{n-1}{k-1}p^k(1-p)^{n-k} + \left(\binom{n}{k} - \binom{n-1}{k-1}\right)p^{n-k}(1-p)^k \\ &\leq \binom{n-1}{k-1}p^k(1-p)^{n-k} + \binom{n-1}{n-k-1}p^{n-k}(1-p)^k \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{P}_p(\mathcal{A}) &= \sum_{k=1}^{n} p^k(1-p)^{n-k} \\ &\leq p \sum k = 1^n \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} = p \end{aligned}$$

□

Definition. *$\mathcal{A} \subset \mathcal{P}(X)$ is **k-wise-intersecting** if $A_1 \cap \cdots \cap A_k \neq \emptyset$ $\forall A_i \in \mathcal{A}$.*

Theorem 5. *Let $ks \geq n$, let $\mathcal{A} \subset X^{(s)}$ be such that X is **not** the union of k sets from $\mathcal{A}$. Then $|\mathcal{A}| \leq \binom{n-1}{s}$.*

Proof. Apply Katona's circle method. Let Π be the set of all $(n-1)!$ cyclic orders on X. For $\pi \in \Pi$, let $\mathcal{A}_\pi = \{A \in \mathcal{A} \,|\, A \text{ is a } \pi\text{-arc}\}$.

Claim: $|\mathcal{A}_\pi| \leq n - s$.

Proof of claim: we may assume $X = \mathbb{Z}_n$ is given by π; we may assume one of the arcs in $\mathcal{A}_\pi$ ends in n. Associate with each arc its end point, except for the one ending in n, to which we associate all $ks - n + 1$ numbers in $[n, ks]$.

Thus, if $l = |\mathcal{A}_\pi|$, and L is the set of elements associated with our arcs, then $|L| = l + (ks - n)$.

For $1 \leq i \leq s$, let $K_i = \{i, i+s, i+2s, \ldots, i+(k-1)s\}$. Then $K_1, \ldots, K_s$ partition $[ks]$ into s sets of k elements each. Can $K_i \subset L$ happen? No, as then the corresponding k arcs would cover X.

Hence, $|L \cap K_i| \leq k - 1$ $\forall i$, so $l + ks - n = |L| \leq (k-1)s$, i.e. $l \leq n - s$. ✓

Double counting:

$$\begin{aligned} s!(n-s)!\,|\mathcal{A}| &= \sum_{A \in \mathcal{A}} |\{\pi \in \Pi : A \text{ is a } \pi\text{-arc}\}| \\ &= \sum_{\pi \in \Pi} |\mathcal{A}_\pi| \leq (n-1)!(n-s) \end{aligned}$$

□

Corollary 6 (Equivalent to Theorem 5)**.** *Let $2 \leq k, r < n$, $kr \leq (k-1)n$. Let $\mathcal{A} \subset X^{(r)}$ be k-wise intersecting. Then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

Proof. Note that $\mathcal{A}^c = \{X \backslash A \mid A \in \mathcal{A}\} \subset X^{(s)}$, $s = n-r$, satisfies the conditions of Theorem 5, so $|\mathcal{A}| = |\mathcal{A}^c| \leq \binom{n-1}{s} = \binom{n-1}{r-1}$. □

Theorem 7. *Let $2 \leq k, r < n$, $kr \leq (k-1)n$; let $\mathcal{A} \subset X^{(\leq r)}$ be a k-wise intersecting Sperner family. Then*

$$\sum_{j=1}^{n} |\mathcal{A}_j| \bigg/ \binom{n-1}{j-1} = \sum_{A \in \mathcal{A}} \binom{n-1}{|A|-1}^{-1} \leq 1$$

Proof. Set $l = \min\{j \mid \mathcal{A})j \neq \emptyset\}$, $m = \max\{j \mid \mathcal{A}_j \neq \emptyset\}$.

Induction on $m-l$: $m = l$ is exactly Corollary 6.

Induction step: $m - l \geq 1$. Let $\mathcal{A}_l^+$ be the upper shadow of $\mathcal{A}_l$ at level $l+1$. Then $\mathcal{A}' = (\mathcal{A} \backslash \mathcal{A}_l) \cup \mathcal{A}_l^+$ is again $k - wise$ intersecting Sperner, with a smaller difference $m - l$. Thus, we're done if

$$|\mathcal{A}_l^+| \bigg/ \binom{n-1}{l} \geq |\mathcal{A}_l| \bigg/ \binom{n-1}{l-1}$$

$\mathcal{A}_l^+$ is the cardinality of the lower shadow of $\mathcal{A}_l^c$. Set $|\mathcal{A}_l| = \binom{x}{n-l}$. Then, by the weak Kruskal-Katona theorem, $\left|\mathcal{A}_l^+\right| \geq \binom{x}{n-l-1}$. We know $\binom{x}{n-l} \geq \binom{n-1}{l-1} = \binom{n-1}{n-l}$, so $x \leq n-1$.

Would like:

$$\begin{gathered} \binom{x}{n-l} \bigg/ \binom{n-1}{l-1} \leq \binom{x}{n-l-1} \bigg/ \binom{n-1}{l} \\ \binom{x}{n-l} \bigg/ \binom{n-1}{n-l} \overset{?}{\leq} \binom{x}{n-l-1} \bigg/ \binom{n-1}{n-l-1} \\ x - (n-l) + 1 \overset{?}{\leq} n - (n-l) = l \\ x \leq n-1 \checkmark \end{gathered}$$

□

5 Correlation Inequalities

Let $0 < p < 1$, $\mathcal{G}(n,p)$ the probability space of all $2^{\binom{n}{2}}$ graphs on $[n]$ such that $\mathbb{P}_p(G_{n,p} = H) = p^{e(H)}(1-p)^{\binom{n}{2}-e(H)}$.

This is really the weighted cube Q_p^n. $\mathbf{p} = (p_1, \ldots, p_n)$, random subset of $X = [n]$: $\mathbb{P}_{\mathbf{p}}(A) = \prod_{i \in A} p_i \prod_{i \notin A}(1-p_i)$. For $\mathcal{G}(n,p)$, consider $Q_{\mathbf{p}}^{\binom{n}{2}}$.

Theorem 1. *Let $A, B \in Q_{\mathbf{p}}^n$. If both are up-sets or both are down-sets, then $\mathbb{P}_{\mathbf{p}}(A \cap B) \geq \mathbb{P}_{\mathbf{p}}(A)\mathbb{P}_{\mathbf{p}}(B)$. IF one is an up-set and the other is a down-set, then the inequality reverses.*

Proof. Induction on n. $n = 1$: ✓. Let $n \geq 1$.

Let $A_i = \{\mathbf{x} \in \{0,1\}^{n-1} \,|\, (x_1, \ldots, x_{n-1}, i) \in A\}$, similary B_i.

Then $\mathbb{P}_{\mathbf{p}}(A) = (1-p_n)\mathbb{P}_{\mathbf{p}'}(A_0) + p_n\mathbb{P}_{\mathbf{p}'}(A_1)$ $\;(\mathbf{p}' = (p_1, p_2, \ldots, p_{n-1}))$

Also $(*)$: $(\mathbb{P}_{\mathbf{p}'}(A_1) - \mathbb{P}_{\mathbf{p}'}(A_0))(\mathbb{P}_{\mathbf{p}'}(B_1) - \mathbb{P}_{\mathbf{p}'}(B_0)) \geq 0 - (*)$ since both are up/down sets.

$$
\begin{aligned}
\mathbb{P}_{\mathbf{p}}(A \cap B) &= (1-p_n)\mathbb{P}_{\mathbf{p}'}(A_0 \cap B_0) + p_n\mathbb{P}_{\mathbf{p}'}(A_1 \cap B_1) \\
&\geq (1-p_n)\mathbb{P}_{\mathbf{p}'}(A_0)\mathbb{P}_{\mathbf{p}'}(B_0) + p_n\mathbb{P}_{\mathbf{p}'}(A_1)\mathbb{P}_{\mathbf{p}'}(B_1) \text{ by induction} \\
&\overset{?}{\geq} ((1-p_n)\mathbb{P}(A_0) + p_n\mathbb{P}(A_1))((1-p_n)\mathbb{P}(B_0) + p_n\mathbb{P}(B_1))
\end{aligned}
$$

This holds if $\mathbb{P}(A_0)\mathbb{P}(B_0) - \mathbb{P}(A_0)\mathbb{P}(B_1) - \mathbb{P}(A_1)\mathbb{P}(B_0) + \mathbb{P}(A_1)\mathbb{P}(B_1) \geq 0$, which is exactly $(*)$.

If A is an up-set, B a down-set then

$$
\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(A) - \mathbb{P}(A \cap B^c) \\
&\leq \mathbb{P}(A) - \mathbb{P}(B)(1 - \mathbb{P}(B)) \\
&= \mathbb{P}(A)\mathbb{P}(B)
\end{aligned}
$$

□

Definition. *Let $A, B \in Q^n = \{0,1\}^n$.*

$$
\begin{aligned}
A \square B = \{z \in Q^n \,|\, \exists \textit{ disjoint } I, J \in [n] \textit{ s.t. } x|I = z|I \implies x \in A, \\
y|J = z|J \implies y \in B\}
\end{aligned}
$$

If A and B are increasing then

$$A \square B = \{x + y \,|\, x \in A,\, y \in B\}$$

$$\mathcal{A} \square \mathcal{B} = \{A \cup B \,|\, A \cap B = \emptyset,\, A \in \mathcal{A},\, B \in \mathcal{B}\}$$

Theorem 2. *If A and B are up-sets in $Q^n_{\mathbf{p}}$, then*

$$\mathbb{P}_{\mathbf{p}}(A\Box B) \leq \mathbb{P}_{\mathbf{p}}(A)\mathbb{P}_{\mathbf{p}}(B)$$

Proof. Put $C = A\Box B$. Induction on n: $n = 0$ ✓. So let $n \geq 1$.

Let $C_0 = A_0\Box B_0$, $C_1 = (A_0\Box B_1) \cup (A_1\Box B_0) \subseteq A_1\Box B_1$. Then we have $C_0 \subset (A_0\Box B_1) \cap (A_1\Box B_0)$.

$\mathbb{P}_{\mathbf{p}'}(C_0) \leq \mathbb{P}_{\mathbf{p}'}(A_0)\mathbb{P}_{\mathbf{p}'}(B_0)$, $\mathbb{P}(C_1) \leq \mathbb{P}(A_1)\mathbb{P}(B_1)$.

$$\begin{aligned}\mathbb{P}(C_0) + \mathbb{P}(C_1) &\leq \mathbb{P}((A_0\Box B_1) \cap (A_1\Box B_0)) + \mathbb{P}((A_0\Box B_1) \cup (A_1\Box B_0)) \\ &= \mathbb{P}(A_0\Box B_1) + \mathbb{P}(A_1\Box B_0) \\ &\leq \mathbb{P}(A_0)\mathbb{P}(B_1) + \mathbb{P}(A_1)\mathbb{P}(B_0)\end{aligned}$$

Multiply then by $(1-p_n)^2$, p_n^2, $p_n(1-p_n)$ and add them:

$$\mathbb{P}(C_0)((1-p_n)^2 + (1-p_n)p_n) + \mathbb{P}(C_1)(p_n^2 + p_n(1-p_n)) \leq \mathbb{P}_{\mathbf{p}}(A)\mathbb{P}_{\mathbf{p}}(B)$$

Obtain $\mathbb{P}(C) \leq \mathbb{P}(A)\mathbb{P}(B)$. □

The full Van den Berg - Kesten conjecture that $\mathbb{P}(A\Box B) \leq \mathbb{P}(A)\mathbb{P}(B)$ was proved by Reimer.

5.1 The Ahlswede-Daykin Four Functions Theorem

Theorem 3. *let $\alpha, \beta, \gamma, \delta : \mathcal{P}(X) \to \mathbb{R}^+$. Suppose $\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$ $\forall A, B \subset X$.*

Then $\alpha(\mathcal{A})\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \vee \mathcal{B})\delta(\mathcal{A} \wedge \mathcal{B})$ where $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$, $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$