Part III Combinatorics

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Contents

1 Introduction

Let $X, Y, \dots$ be sets

Definition. *We call $\mathcal{A} \subset \mathcal{P}(X)$ a **set system** or **family of sets**. $\mathcal{A}$ is naturally identified with a bipartite graph $G_{\mathcal{A}}(U, W)$ with $U = \mathcal{A}$, $W = \bigcup_{A \in \mathcal{A}} A$ or $W = X$. Indeed, $Ax \in E(G_{\mathcal{A}}) \iff x \in A$.*

Definition. *Given $\mathcal{A} \in \mathcal{P}(X)$, a **set of distinct representatives** (SDR) is an injection $f : \mathcal{A} \to X$ s.t. $f(A) \in A\ \forall A \in \mathcal{A}$. In its bipartite graph, an SDR corresponds to a complete matching $U \to W$.*

Theorem 1 (Hall, 1935)**.** *A set system $\mathcal{A}$ has an SDR if $\forall \mathcal{A}' \subset \mathcal{A}$, $|\cup_{A \in \mathcal{A}'} A| \geq |\mathcal{A}|'$.*

Theorem 1'. *A bipartite graph $G(U, W)$ has a complete matching $U \to W$ if $\forall S \subset U$, $|\Gamma(S)| \geq |S|$*

Corollary 2. *Suppose $G(U, W)$ bipartite, $d(u) \geq d(w)\ \forall u \in U,\ w \in W$. Then $\exists$ a complete matching $U \to W$.*

Definition. *A bipartite graph $G(U, W)$ is (r, s)**-regular** if $d(u) = r$ and $d(w) = s\ \forall u \in U,\ w \in W$.*

Instant from Cor 2: if $G(U,W)$ is (r,s)-regular then $\exists$ a complete matching from U to W if $|U| \leq |W|$.

Corollary 3. *Let $0 \leq i,j \leq n$, $\binom{n}{i} \leq \binom{n}{j}$. Then $\exists$ a complete matching $f : [n]^{(i)} \to [n]^{(j)}$ s.t. $f(A) \subset A$ if $j \leq i$, and $f(A) \supset A$ if $i \leq j$.*

Theorem 4. *Let $G = G(U,W)$ be a connected (r,s)-regular graph. Then for $\emptyset \neq A \subset U$,*

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

Also, equality holds iff $A = U$.

The **cube** $Q^n \cong \mathcal{P}(n) \cong [2]^n$ = set of all 0, 1 sequences of length n. Q^n is also a graph: AB is an edge if $|A \bigtriangleup B| = 1$. It is also a poset: $A < B$ if $A \subset B$.

Q^n has a natural orientation: $\overrightarrow{AB}$ if $A = B \cup \{a\}$.

$$\begin{tikzcd}
 & 123 \arrow[dl] \arrow[d] \arrow[dr] & \\
12 \arrow[d] \arrow[dr] & 13 \arrow[dl] \arrow[dr] & 23 \arrow[dl] \arrow[d] \\
1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\
 & \emptyset &
\end{tikzcd}$$

The order on $Q^n \cong \mathcal{P}(n)$ is induced by this oriented graph.

2 Sperner Systems

Definition. *A set system $\mathcal{A} \subset \mathcal{P}(n)$ is **Sperner** if $A, B \in \mathcal{A}$, $A \neq B \implies A \not\subset B$*

Theorem 1 (Sperner, 1928)**.** *If $\mathcal{A} \subset \mathcal{P}(n)$ is Sperner then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Definition. *The **weight** $w(A)$ of a set $A \in \mathcal{P}(n)$ is $w(A) = \frac{1}{\binom{n}{|A|}}$*

Theorem 2. *Let $\mathcal{A}$ be a Sperner system on X, $|X| = n$. Then*

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A) \leq 1$$

Corollary 3. *If $\mathcal{A} \in \mathcal{P}(n)$ is a Sperner system then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, with equality $\iff$ $\mathcal{A}$ is $X^{\lfloor n/2 \rfloor}$ or $X^{\lceil n/2 \rceil}$.*

Definition. *$\mathcal{A} \in \mathcal{P}(n)$ is **k-Sperner** if it does not contain*

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{k+1}$$

Note that Sperner = 1-Sperner.

Corollary 4 (Erdős, 1945)**.** *If $\mathcal{A} \subset \mathcal{P}(n)$ is k-Sperner then $|\mathcal{A}|$ is at most the sum of the k largest binomial coefficients.*

Theorem 5 (Erdős, 1945)**.** *Let $x_1, \ldots, x_n \in \mathbb{R}$, $x_i \geq 1$. Then the number of sums $\sum_1^n \pm x_i$ in an open interval J of length $2k$ is at most the sum of the k largest binomial coefficients.*

Definition. *A chain $A_o \subset A_1 \subset \cdots \subset A_k$ is **symmetric** if $|A_{i+1}| = |A_i| + 1 \ \forall i$ and $|A_o| + |A_k| = n$.*

Theorem 6 (Kleitman and Katona)**.** *$\mathcal{P}(n)$ has a decomposition into symmetric chains.*

Take such a partition $\mathcal{P}(n) = \bigcup_{i=1}^k \mathcal{C}_i$, $j = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. There is one chain of length $n+1$, $n-1$ chains of length $n-1$, etc: there are $\binom{n}{i} - \binom{n}{i-1}$ chains of length $n+1-2i$.

Let E be a normed space, let $x_1, \ldots, x_n \in E$, $||x_i|| \geq 1 \ \forall i$, for $A \in \mathcal{P}(n)$ let $x_A = \sum i \in A x_i$.

Conjecture (Erdős, 1945)**.** *If $\mathcal{A} \in \mathcal{P}(n)$ s.t. $||x_A - x_B|| < 1$ then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$*

Definition. *Call $\mathcal{D} \in \mathcal{P}(n)$ **scattered** if $||x_A - x_B|| \geq 1 \ \forall A, B \in \mathcal{D}$. Call a partition $\mathcal{P}(n) = \bigcup_{i=1}^s \mathcal{D}_i$ **symmetric** if there are precisely $\binom{n}{i} - \binom{n}{i-1}$ sets $\mathcal{D}_i$ of cardinality $n+1-2i$.*

Theorem 7. *(Kleitman, 1970) $E, (x_i)_1^n$ as before. Then $\mathcal{P}(n)$ has a symmetric partition into scattered sets.*

Theorem 8. *(Kleitman, 1970) If $\mathcal{A} \in \mathcal{P}(n)$ s.t. $||x_A - x_B|| < 1$ then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$*

3 The Kruskal-Katona Theorem

We know: if $\mathcal{A} \subset X^{(r)}$ then $\partial\mathcal{A}$ (the **lower shadow** of $\mathcal{A}$), defined by

$$\partial\mathcal{A} = \{B \in X^{(r-1)} \mid B \subset A \text{ for some } A \in \mathcal{A}\}$$

satisfies

$$|\partial\mathcal{A}| \geq |\mathcal{A}| \frac{\binom{n}{r-1}}{\binom{n}{r}} = |\mathcal{A}| \frac{r}{n-r+1}$$

with equality $\iff$ $\mathcal{A}$ is $\emptyset$ or $X^{(r)}$.

What about in between? What is $\mathcal{B} \in X^{(r)}$ s.t. $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$?

$\exists \mathcal{B}_1, \mathcal{B}_2, \cdots \in X^{(r)}$ s.t. $|\mathcal{B}_m| = m$ and $|\partial\mathcal{B}_m| \leq |\partial\mathcal{A}|$ $\forall \mathcal{A} \subset X^{(r)}$ where $|\mathcal{A}| = m$.

Incredibly luckily, we have a sequence of nested extremal sets. Equivalently, $\exists$ total order on $X^{(r)}$ s.t. the first m sets form $\mathcal{B}_m$.

Definition. *Define the* ***colex*** *total order on $X^{(r)}$ by $A < B$ if $\max(A\Delta B) \in B$.*

Aim: given m and r, would like to find $\mathcal{B} \subset X^{(r)}$, $|\mathcal{B}| = m$ s.t. $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$ $\forall \mathcal{A} \subset X^{(r)}$, $|\mathcal{A}| = m$.

Define $\mathcal{B}^{(r)}(m_r, \ldots, m_s)$, $m_r > m_{r-1} > \cdots > m_s \geq s$ as follows:

$$\begin{aligned}\mathcal{B}^{(r)} = [m_r]^{(r)} &\cup ([m_{r-1}]^{(r-1)} + \{m_r + 1\}) \\ &\cup ([m_{r-2}]^{(r-2)} + \{m_{r-1} + 1, m_r + 1\}) \\ &\cup \ldots \\ &\cup ([m_s]^{(s)} + \{m_{s+1} + 1, m_{s+2} + 1, \ldots, m_r + 1\})\end{aligned}$$

Set $b^{(r)}(m_r, \ldots, m_s) = \left|\mathcal{B}^{(r)}(m_r, \ldots, m_s)\right| = \sum_{j=s}^{r} \binom{m_j}{j}$.

$$\partial\mathcal{B}^{(r)}(m_r, \ldots, m_s) = \mathcal{B}^{(r-1)}(m_r, \ldots, m_s)$$

This has cardinality $b^{(r-1)}(m_r, \ldots, m_s) = \sum_{j=s}^{r} \binom{m_j}{j-1}$.

Lemma 1. *For $l, r \in \mathbb{N}$ $\exists!$ $m_r > \cdots > m_s$ s.t. $l = \sum_{j=s}^{r} \binom{m_j}{j}$; the initial segment of $X^{(r)}$ in colex, consisting of l sets, is $\mathcal{B}^{(r)}(m_r, \ldots, m_s)$.*

Definition. *Let $i \neq j \in X$, $A \in \mathcal{P}(X)$. Define the* ***ij-compression***

$$A_{ij} = C_{ij}(A) = \begin{cases} (A\backslash\{j\}) \cup \{i\} & \text{if } j \in A,\ i \notin A \\ A & \text{otherwise} \end{cases}$$

Given $\mathcal{A} \subset \mathcal{P}(n), A \in \mathcal{A}$

$$C_{i,j,\mathcal{A}}(A) = \begin{cases} A_{ij} & \text{if } A_{ij} \notin \mathcal{A} \\ A & \text{otherwise} \end{cases}$$

Also,

$$\begin{aligned} C_{ij}(\mathcal{A}) &= \{C_{i,j,\mathcal{A}} \,|\, A \in \mathcal{A}\} \\ &= \{C_{ij}(A) \,|\, A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \,|\, C_{ij}(A) \in \mathcal{A}\} \end{aligned}$$

For $\mathcal{A} \in X^{(r)}$,

$$\begin{aligned} \mathcal{A}_{ij} &= \{A \in \mathcal{A} \,|\, \{i,j\} \subset A\} \\ \mathcal{A}_i &= \{A \in \mathcal{A} \,|\, i \in A,\, j \notin A\} \\ \mathcal{A}_\emptyset &= \{A \in \mathcal{A} \,|\, A \cap \{i,j\} = \emptyset\} \\ \mathcal{A}_j &= \{A \in \mathcal{A} \,|\, i \notin A,\, j \in A\} \end{aligned}$$

$C_{ij} : \mathcal{A} \mapsto C_{ij}(\mathcal{A})$ keeps $\mathcal{A}_\emptyset \cup \mathcal{A}_i \cup \mathcal{A}_{ij}$ fixed, and maps $\mathcal{A}_j$ into sets like those in $\mathcal{A}_i$.

Lemma 2. *For $\mathcal{A} \subset X^{(r)}$, $\partial C_{ij}(\mathcal{A}) \subseteq C_{ij}(\partial \mathcal{A})$. In particular, the cardinality decreases.*

Proof. Let $B \in \partial C_{ij}(\mathcal{A})$ and let $A \in \mathcal{A}$ s.t. $B \subset C_{i,j,\mathcal{A}}(A)$.

i. Suppose B meets $\{i,j\}$ in 0 or 2 elements. Then $B \subset A$ so $B \in \partial A$ and $B \in C_{ij}(\partial \mathcal{A})$

ii. Suppose $i \in B$, $j \notin B$. Then either B or $(B \backslash \{i\}) \cup \{j\}$ belongs to $\partial \mathcal{A}$, so $B \in C_{ij}(\partial \mathcal{A})$.

iii. Suppose $j \in B$, $i \notin B$. Then both B and $(B \backslash \{j\}) \cup \{i\}$ belong to $\partial \mathcal{A}$, so both belong to $C_{ij}(\partial \mathcal{A})$.

□

Definition. *Call $\mathcal{A} \subset X^{(r)}$* ***left-compressed*** *if $C_{ij}(\mathcal{A}) = \mathcal{A}$ $\forall i < j$.*

Lemma 3. *Let $\mathcal{A} \subset X^{(r)}$. Then $\exists$ a left-compressed family $\mathcal{B} \subset X^r$ s.t. $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial \mathcal{B}| \leq |\partial \mathcal{A}|$.*

Proof. Define $\mathcal{A}_0 = \mathcal{A}, \mathcal{A}_1, \ldots$ as follows: having reached $\mathcal{A}_k$, if $\mathcal{A}_k$ is not left-compressed, pick $i < j$ s.t. $C_{ij}(\mathcal{A}_k) \neq \mathcal{A}_k$, and set $\mathcal{A}_{k+1} = C_{ij}(\mathcal{A}_k)$

This sequence has to end because

$$\sum_{A \in \mathcal{A}_{k+1}} \sum_{a \in A} a < \sum_{A \in \mathcal{A}_k} \sum_{a \in A} a$$

let $\mathcal{A}_l$ be the last term: this will do for $\mathcal{B}$. □

Theorem 4 (Kruskal-Katona, 1963 and 1968)**.** *Let $\mathcal{A} \subset X^{(r)}$, $m = |\mathcal{A}|$. Then*

$$\begin{aligned} |\partial \mathcal{A}| &\geq \left|\partial \mathcal{B}_m^{(r)}\right| \\ &= \left|\partial \mathcal{B}^{(r)}(m_r, m_{r-1}, \dots, m_s)\right| \\ &= b^{(r-1)}(m_r, \dots, m_s) \end{aligned}$$

Proof. Induction on r and then m (or on $r+m$). $r = 1$✓ $m = 1$✓

Induction step: we may assume that $\mathcal{A}$ is left-compressed. Set $Y = X \backslash \{1\}$. Then $\mathcal{A} = (\mathcal{A}_1 + \{1\}) \cup \mathcal{A}_0$, where $\mathcal{A}_1 \subset Y^{(r-1)}$, $\mathcal{A}_0 \subset Y^{(r)}$.

$m = |\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1|$, $\partial \mathcal{A}_0 \subset \mathcal{A}_1$, $\partial(\mathcal{A}_1 + \{1\}) = \mathcal{A}_1 \cup (\partial \mathcal{A}_1 + \{1\})$.

In particular, $|\partial \mathcal{A}| = |\mathcal{A}_1| + |\partial \mathcal{A}_1|$.

For $\mathcal{A} = \mathcal{B}^{(r)}(m_r, \dots, m_s)$,

$$|\mathcal{A}_1| = b^{(r-1)}(m_r - 1, \dots, m_s - 1)$$

$$|\mathcal{A}_0| = b^{(r)}(m_r - 1, \dots, m_s - 1)$$

Suppose $|\mathcal{A}_0| > b^{(r)}(m_r - 1, \dots, m_s - 1)$. Then by the induction hypothesis, $|\partial \mathcal{A}_0| \geq b^{(r-1)}(m_r - 1, \dots, m_s - 1)$. Hence $|\mathcal{A}_1| \geq b^{(r-1)}(m_r - 1, \dots, m_s - 1)$ and so $|\partial \mathcal{A}| \geq b^{(r-1)}(m_r, \dots, m_s)$.

But if $|\mathcal{A}_0| \leq b^{(r)}(m_r - 1, \dots, m_s - 1)$, $|\mathcal{A}_1|$ is again $\geq b^{(r-1)}(m_r - 1, \dots, m_s - 1)$. Done as before. □

Soft version:

Theorem 5 (Lovász, 1979)**.** *If $\mathcal{A} \subset X^{(r)}$ satisfies $|\mathcal{A}| = \binom{X}{r}$ then $|\partial \mathcal{A}| \geq \binom{X}{r-1}$.*

Proof. Induction on r and $m = |\mathcal{A}|$. As before, $\mathcal{A}_0, \mathcal{A}_1$. Note that $\mathcal{A}_1 \geq \binom{X-1}{r-1}$ since otherwise $\mathcal{A}_0 > \binom{X-1}{r}$. But then $|\partial \mathcal{A}_0| \geq \binom{X-1}{r-1}$, contradicting the fact that $\partial \mathcal{A}_0 \subset \mathcal{A}_1$.

But if $|\mathcal{A}_1| \geq \binom{X-1}{r-1}$ then

$$|\mathcal{A}_1| + |\partial \mathcal{A}_1| \geq \binom{X-1}{r-1} + \binom{X-1}{r-2} = \binom{X}{r-1}$$

□

Definition. *Define the **uniform probability measure** on $X^{(r)}$, $|X| = n$ as $\mathbb{P}_{n,r}(A) = \frac{1}{\binom{n}{r}}$, and for $\mathcal{A} \subset X^{(r)}$, $\mathbb{P}_{n-r}(\mathcal{A}) = \frac{|\mathcal{A}|}{\binom{n}{r}}$.*

Definition. *$\mathcal{A} \subset \mathcal{P}(n)$ is **monotone decreasing** if $A \subset B \in \mathcal{A} \implies A \in \mathcal{A}$.*