CS70–Fall 2011 — Solutions to Homework 8

1. **Marbles**

   (a) Let BL be the event that we get a black marble. Therefore the probability that B happens is,

   $$\text{P(BL)} = \tfrac{1}{2} \cdot (\tfrac{1}{4} + \tfrac{2}{6}) = \tfrac{7}{24}$$

   (b) Let A be the event that we select Box A, and B be the event that we select Box B. Also let W be the event that we get a white marble. So, probability of W is

   $$P(W) = 1 - P(BL) = \frac{17}{24}$$

   As,

   $$P(W) \cdot P(A|W) = P(A) \cdot P(W|A)$$
   $$P(A|W) = \frac{P(A) \cdot P(W|A)}{P(W)} = \frac{9}{17}$$

   This is the probability that given a white marble, it is from Box A.

2. **Find the Probabilities**

   Note that $P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A) = \frac{1}{10}$

   (a) $P(B) = \frac{1}{4}$

   (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{11}{20}$

   (c) No, as $\text{P(A)} \cdot \text{P(B)} \neq \text{P(A} \cap \text{B)}$

   (d) No, as $\text{P(A} \cap \text{B)} \neq 0$

3. **Count the Square-Frees**

   We compute the number of positive integers strictly less than 201 that are not square-free. Let A_2 (respectively, $A_3, A_5, A_7, A_{11}, A_{13}$) be the set of multiples of 22 (respectively, $32, 52, 72, 112, 132$) less than 201. The union of these sets in the set of all numbers less than 201 which are not square-free, i.e., have some square divisor in them. (Why only primes? Why stop at 13?).
   The cardinality of this set can be computed by the Inclusion-Exclusion Principle. Since $2^2 \cdot 11^2 \geq 201$ and $3^2 \cdot 5^2 \geq 201$, it follows that $A_i \cap A_j = \emptyset$ unless $\{i, j\} = \{2, 3\}, \{2, 5\}, or\ \{2, 7\}$. Since $2^2 \cdot 3^3 \cdot 5^2 \geq 201$, it follows that $A_i \cap A_j \cap A_k = \emptyset\ \forall i, j, k$. So:

   $|A_2 \cup A_3 \cup A_5 \cup A_7 \cup A_{11} \cup A_{13}| = |A_2| + |A_3| + |A_5| + |A_7| + |A_{11}| + |A_{13}| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_2 \cap A_7|$
   $= \lfloor \frac{200}{2^2} \rfloor + \lfloor \frac{200}{3^2} \rfloor + \lfloor \frac{200}{5^2} \rfloor + \lfloor \frac{200}{7^2} \rfloor + \lfloor \frac{200}{11^2} \rfloor + \lfloor \frac{200}{13^2} \rfloor - \lfloor \frac{200}{2^2 \cdot 3^2} \rfloor - \lfloor \frac{200}{2^2 \cdot 5^2} \rfloor - \lfloor \frac{200}{2^2 \cdot 7^2} \rfloor$
   $= 50 + 22 + 8 + 4 + 1 + 1 - 5 - 2 - 1 = 78$

   Therefore, the number of square-free positive integers strictly less than 201 is $200 - 78 = 122$

4. **Independence**

   (a) True. A and B must be independent.

   $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$

   $= 1 - P(A \cap B) = 1 - (P(A) + P(B) - P(A \cap B))$

   $= 1 - P(A) - P(B) + P(A) \cdot P(B)$

   $= (1 - P(A)) \cdot (1 - P(B)) = P(\overline{A}) \cdot P(\overline{B})$

   (b) True. A and $\overline{B}$ must be independent.

   $P(A \cap \overline{B}) = P(A - (A \cap B))$

   $= P(A) - P(A \cap B) = P(A) - P(A) \cdot P(B)$

   $= P(A) \cdot (1 - P(B)) = P(A) \cdot P(\overline{B})$

(c) False in general. If $0 < P(A) < 1$, then $P(A \cap \overline{A}) = 0$ but $P(A) \cdot P(\overline{A}) > 0$, so $P(A \cap \overline{A}) \neq P(A) \cdot P(\overline{A})$ therefore A and $\overline{A}$ are not independent in this case.

(d) True. To give one example, if $P(A) = P(B) = 0$, then $P(A \cap B) = 0 = 0 \text{ x } 0 = P(A) \cdot P(B)$, so A and B are independent in this case.

5. **Conditional Independence**

(a) $C = A \cap B$

(b) Given that C is the event that Alekh plays volleyball, with A being the event that he has no homework and B being the event that the weather is nice.
Note that Alekh only plays volleyball when the weather is nice. So $P(B|C) = 1$. Another way to look at this is,

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{P(C)}{P(C)} = 1$$

(c) A and B are conditionally independent given C, since $P(A \cap C) = P(A|C) = P(B|C) = 1$. On the other hand, A and B are not conditionally independent given $\overline{C}$: $P(A \cap C|\overline{C}) = 0$, but $P(A|\overline{C}) = P(A \cap \overline{B})/P(\overline{C})$ and $P[B|\overline{C}]$ are both nonzero, so $P(A \cdot \overline{C}) \cdot P(B|\overline{C}) \neq 0$.