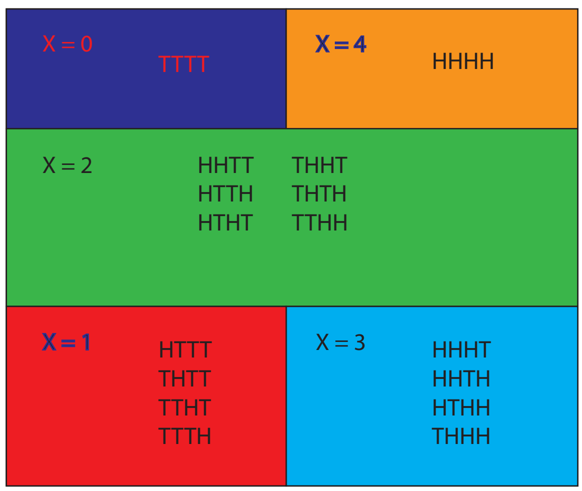
CS70–Fall 2011 — Solutions to Homework 9

2. Random Variables and Their Distributions

(a)

Outcome	P	Outcome	P
HHHH	p^4	THHH	$p^3(1-p)$
HHHT	$p^3(1-p)$	THHT	$p^2(1-p)^2$
HHTH	$p^3(1-p)$	THTH	$p^2(1-p)^2$
HHTT	$p^2(1-p)^2$	THTT	$p(1-p)^3$
HTHH	$p^3(1-p)$	TTHH	$p^2(1-p)^2$
HTHT	$p^2(1-p)^2$	TTHT	$p(1-p)^3$
HTTH	$p^2(1-p)^2$	TTTH	$p(1-p)^3$
HTTT	$p(1-p)^3$	TTTT	$(1-p)^4$

(b)



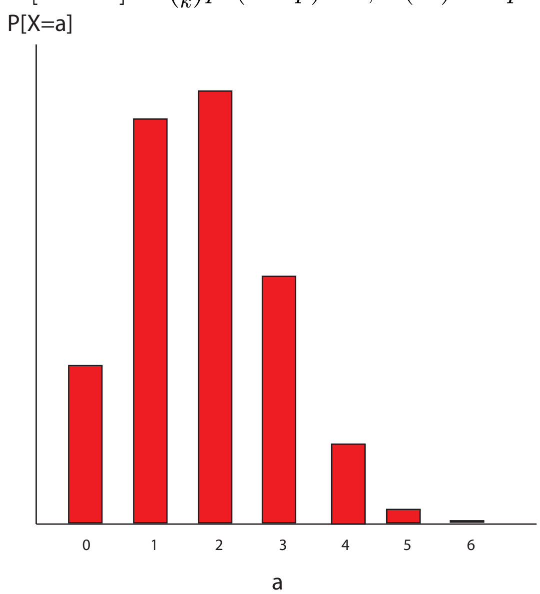
(c) $X = 1$ and $X = 2$ are disjoint (there is no way they can occur simultaneously), but they are not independent (since if $X = 1$, then $X = 2$ occurs with zero probability, but if we do not know $X = 1$, then $X = 2$ occurs with positive probability).

(d) $P[X = k] = p^k(1-p)^{4-k} \times \binom{4}{k}$
$E(X) = 4p(1-p)^3 + 12p^2(1-p)^2 + 12p^3(1-p) + 4p^4$

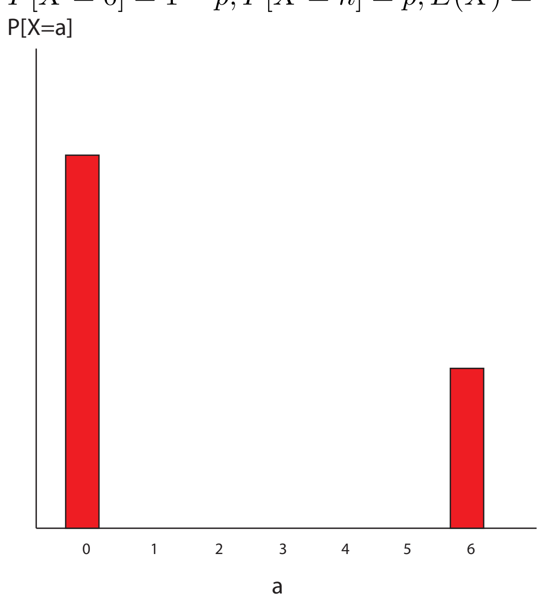
(e) $X = 0$ and E are disjoint (they can never happen at the same time). They are not independent. The events $X = 2$ and E are not disjoint (the sample point HTHT is an example), and independent, because
$P(X = 2 \wedge E) = \frac{3}{16} = \frac{3}{8} \times \frac{1}{2} = P(X = 2) \times P(E)$

3. Packets Over the Internet

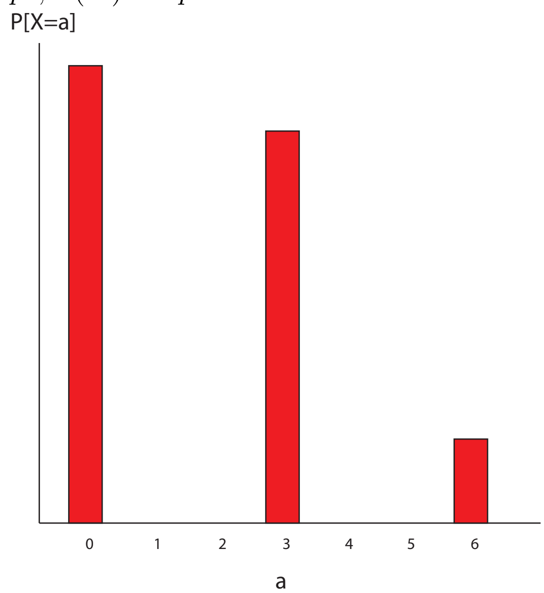
(a) Distribution
$P[X=k] = \binom{n}{k} p^k (1-p)^{n-k}, E(X) = np$



(b) Distribution
$P[X=0] = 1-p, P[X=n] = p, E(X) = np$



(c) Distribution
$P[X = 0] = (1-p)^2, P[X = \frac{n}{2}] = 2p(1-p), P[X = n] = p^2, E(X) = np$



There are valid reasons to prefer any of the three routing protocols, depending on your application's tolerance for dropped packets. If your application can afford to drop a few packets (e.g. streaming video), then you may prefer distribution (a). If losing any packets is as bad as losing all the packets, you may prefer (c). For full credit, any choice is acceptable so long as it is adequately justified.

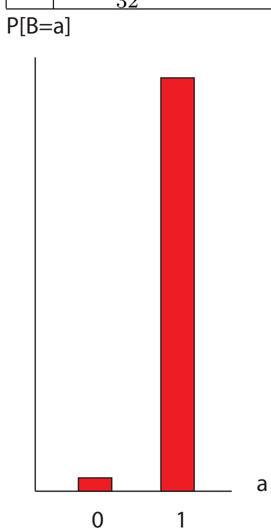
4. Family Planning

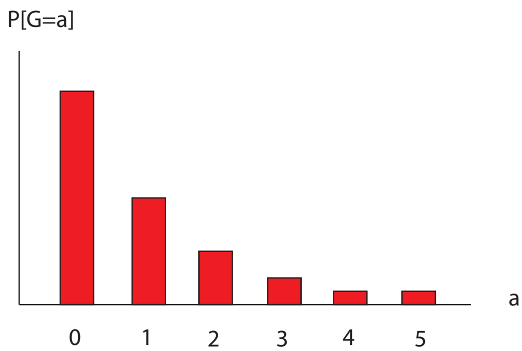
(a)

Outcome	P
b	$\frac{1}{2}$
gb	$\frac{1}{4}$
ggb	$\frac{1}{8}$
gggb	$\frac{1}{16}$
ggggb	$\frac{1}{32}$
ggggg	$\frac{1}{32}$

(b)

a	$P[G=a]$	$P[B=a]$
0	$\frac{1}{2}$	$\frac{1}{32}$
1	$\frac{1}{4}$	$\frac{31}{32}$
2	$\frac{1}{8}$	0
3	$\frac{1}{16}$	0
4	$\frac{1}{32}$	0
5	$\frac{1}{32}$	0





(c) $E(B) = 0 \times \frac{1}{32} + 1 \times \frac{31}{32} = \frac{31}{32}$
$E(G) = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{16} + 4 \times \frac{1}{32} + 5 \times \frac{1}{32} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{32} = \frac{31}{32}$

5. Games

Let random variable X be Alice's profit from a given round of the game. If Alice's number is higher than Bob's, $X = 3$. If Alice's number is less than or equal to Bob's, $X = -2$. This will be a good game for Alice if her expected profit is positive ($E(X) > 0$).
In the case of doubles, Bob wins. This occurs with probability $\frac{6}{36}$. In the set of outcomes other than doubles, half of the time Alice wins and half of the time Bob wins. Hence, Alice win's with probability $\frac{15}{36}$. Bob thus wins with probability $\frac{21}{36}$.
Hence, $E(X) = \frac{15}{36} \times (3) + \frac{21}{36} \times (-2) = \frac{15}{12} - \frac{14}{12} = \frac{1}{12}$. Since $E(X) > 0$, this is a good game for Alice.

6. Expectations

(a) We must show $\sum_{a\in\mathcal{A}} a \times P[X=a] = \sum_{\omega\in\Omega} X(\omega)\times P[\omega]$.
Let $\Omega_i \subseteq \Omega$ denote the subset of the sample space such that $(\forall\omega\in\Omega)((X(\omega)=i) \Leftrightarrow (\omega\in\Omega_i))$
Given this definition, $P[X=a] = \sum_{\omega\in\Omega_a} P[\omega]$, hence
$a\times P[X=a] = a\times \sum_{\omega\in\Omega_a} P[\omega] = \sum_{\omega\in\Omega_a} a\times P[\omega] = \sum_{\omega\in\Omega_a} X(\omega)\times P[\omega]$
Thus,
$\sum_{a\in\mathcal{A}} a\times P[X=a] = \sum_{a\in\mathcal{A}}\sum_{\omega\in\Omega_a} X(\omega)\times P[\omega]$
Since $\mathcal{A}$ denotes all possible values of the random variable X, and Ω_a denotes all sample outcomes such that $X=a$, clearly $\bigcup_{a\in\mathcal{A}} \Omega_a = \Omega$. There can be no intersection between Ω_i and Ω_j for $i \neq j$, since this would imply there existed an outcome $\omega*$ in the sample space such that $X(\omega*)=i$ and $X(\omega*)=j$, which is clearly impossible. Hence,
$\sum_{a\in\mathcal{A}}\sum_{\omega\in\Omega_a} = \sum_{\omega\in\Omega}$, and $\sum_{a\in\mathcal{A}} a\times P[X=a] = \sum_{\omega\in\Omega} X(\omega)\times P[\omega]$

(b) From Definition 13.1 (page 77 in the reader), a *random variable* X on a sample space Ω is a function that assigns to each sample point $\omega\in\Omega$ a real number $X(\omega)$. If X is a random variable defined on Ω, $Y=X^2$ is also a random variable on Ω, because it too assigns to each sample point $\omega\in\Omega$ a real number $Y(\omega) = (X(\omega))^2$.

(c) From the definition of expectation,
$\mathbb{E}(X^2) = \sum_{x\in\mathcal{X}} x\times P[X^2=x]$
where $\mathcal{X}$ is the range of values X^2 can take on.
$P[X^2=x] = P[(X=\sqrt{x})\vee(X=-\sqrt{x})] =$
$P[X=\sqrt{x}] + P[X=-\sqrt{x}] - P[X=0]$.
Now define $a=\sqrt{x}$ and $b=-\sqrt{x}$. Then
$\sum_{x\in\mathcal{X}} x\times P[X^2=x] = \sum_{a\in\mathcal{A}^+} a^2\times P[X=a] + \sum_{b\in\mathcal{A}^-} b^2\times P[X=b]$
where $\mathcal{A}^+$ denotes the possible positive values X can take on (including 0), and $\mathcal{A}^-$ denotes the possible negative values X can take on. $\mathcal{A}^+\cup\mathcal{A}^- = \mathcal{A}$, and they are disjoint, so
$\sum_{a\in\mathcal{A}^+} a^2\times P[X=a] + \sum_{b\in\mathcal{A}^-} b^2\times P[X=b] = \sum_{a\in\mathcal{A}} a^2\times P[X=a]$,

so

$$\mathbb{E}(X^2) = \sum_{a \in \mathcal{A}} a^2 \times P[X = a]$$

(d) $$\mathbb{E}(f(X)) = \sum_{a \in \mathcal{A}} f(a) \times P[X = a]$$