CS70–Fall 2011 — Solutions to Homework 2

October 11, 2016

1. Practice Proving Propositions

   (a) **Claim:** For all natural numbers n, if n is even then $n^2 + 2011$ is odd.

   Proof: *by Induction*
   Let
   $$P(n) = n^2 + 2011 \qquad \forall n \in N$$

   Base Case:

   $$P(0) = 0^2 + 2011 = 2011$$

   which is odd. So the base case is true.

   Inductive Step:
   Assume $P(n)$ is true. Taking $P(n+2)$

   $$P(n+2) = (n+2)^2 + 2011$$
   $$P(n+2) = n^2 + 4n + 2015$$

   Take $K(q) = q^2 + 4q + 2015$. There is only one possibility

   i. q is even. If q is even, q^2 is also even. Moreover $4q$ becomes even. So we have

   $$K(q) = Even + Even + Odd$$

   $$K(q) = Even + Odd$$

   $$K(q) = Odd \checkmark$$

So, $P(n+2) = n^2 + 4n + 2015$ is odd. So, $P(n) \to P(n+2)$ is true. The claim is correct. ✓

(b) **Claim:** For all natural numbers n, $n^2 + 5n + 1$ is odd.
Proof: *by Induction*
Let

$$P(n) = n^2 + 5n + 1 \qquad \forall n \in N$$

Base Case:

$$P(0) = 0^2 + 5(0) + 1 = 1$$

which is odd. So the base case is true.

Inductive Step:
Assume $P(n)$ is true. Taking $P(n+1)$

$$P(n+1) = (n+1)^2 + 5(n+1) + 1$$
$$P(n+1) = n^2 + 7n + 7$$

Take $K(q) = q^2 + 7q + 7$. There are two possibilities

i. q is odd. If q is odd, q^2 is also odd. Morever $7q$ becomes odd. So we have

$$K(q) = Odd + Odd + Odd$$

Axiom: Adding an Odd number with an Odd number gives an even number.

$$K(q) = Even + Odd$$

Axiom: Adding an Even number with an Odd number gives an Odd number.

$$K(q) = Odd \checkmark$$

ii. q is even. If q is even, q^2 is also even. Moreover $7q$ becomes even. So we have

$$K(q) = Even + Even + Odd$$

$$K(q) = Even + Odd$$

$$K(q) = Odd \checkmark$$

So, $P(n+1) = n^2 + 7n + 7$ is odd. So, $P(n) \to P(n+1)$ is true. The claim is correct. ✓

(c) **Claim:** For all real numbers a, b, if $a + b \geq 2011$ then $a > 1005$ or $b > 1005$.

Proof: *Direct Proof*

Let

$$P(a,b) = (a + b \geq 2011)$$

and

$$Q(a,b) = (a > 1005 \text{ or } b > 1005)$$

We have to prove that $P(a,b) \to Q(a,b)$ is true. Just take a to be 1005, therefore for the implication to be true b has to be greater than 1005, because otherwise $Q(a,b)$ will be false, and so will the implication.

$$a + b \geq 2011$$

$$1005 + b \geq 2011$$

$$b \geq 1006$$

$$b > 1005$$

which implies that $Q(a,b)$ is true. ✓

(d) **Claim:** For all real numbers r, if r is irrational then $r/4$ is irrational.

Proof: *by Contraposition*

Let $P(r)$ be the statement that r is irrational, and $Q(r)$ be the statement that $r/4$ is irrational. Thus, we have to prove

$$P(r) \to Q(r)$$

By Contraposition,

$$\neg Q(r) \to \neg P(r)$$

.

Let $X(r)$ be the proposition that $r/4$ is rational, and $Y(r)$ be the proposition that r is rational. So we have to prove,

$$X(r) \to Y(r)$$

Given that $r/4$ is a rational number, we have

$$\left(\frac{r}{4}\right) = \left(\frac{p}{q}\right)$$

where $p, q \in Z$

$$r = \left(\frac{4p}{q}\right)$$

Axiom: *Given p is an integer, np is also integer where $n \in Z$*
Thus,

$$r = \left(\frac{m}{q}\right)$$

where $m = 4 \cdot p$, is rational. The claim is correct. ✓

(e) **Claim:** For all natural numbers n, $10n^2 > n!$.
Proof: *by Counterexample*
Let $P(n)$ be the proposition $10n^2 > n!$ where $n \in N$. We could simply iterate through a couple of elements in N until we find a $n \in N$ for which $P(n)$ doesn't hold.

i. $P(1) = (10 > 1)$ ✓
ii. $P(2) = (40 > 2)$ ✓
iii. $P(3) = (90 > 6)$ ✓
iv. $P(4) = (160 > 24)$ ✓
v. $P(5) = (250 > 120)$ ✓
vi. $P(6) = (360 > 720)$ ✗

At $n = 6$, we have a counterexample to the claim $P(n)$. Thus the claim is incorrect. ✗

2. Interesting Induction

(a) For $n \in N$ with $n \geq 2$, define s_n by

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

Claim: $s_n = 1/n$ for every natural number $n \geq 2$.
Proof: *by Induction*
Given

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

Base Case:

$$s_2 = \left(1 - \frac{1}{2}\right) = \left(\frac{1}{2}\right) \checkmark$$

Inductive Step:
Assume s_n to be true.

$$s_{n+1} = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right) \times \left(1 - \frac{1}{n+1}\right).$$

$$s_{n+1} = \left(\frac{1}{2}\right) \times \left(\frac{2}{3}\right) \times \cdots \times \left(\frac{n-1}{n}\right) \times \left(\frac{n}{n+1}\right).$$

$$s_{n+1} = \left(\frac{1 \cdot 2 \cdots (n-1) \cdot n}{2 \cdot 3 \cdots n \cdot (n+1)}\right).$$

$$s_{n+1} = \left(\frac{n!}{(n+1)!}\right)$$

$$s_{n+1} = \left(\frac{1}{n}\right) \checkmark$$

The claim is correct. ✓.

(b) Let $a_n = 3^{n+2} + 4^{2n+1}$. **Claim:** 13 divides a_n for every $n \in N$.
Proof: *by Induction*
Given

$$a_n = 3^{n+2} + 4^{2n+1}$$

Base Case:

$$a_0 = 3^2 + 4 = 13$$

$$13|a_0 = 13|13 \checkmark$$

Inductive Step: Assume a_n to be true.

$$a_{n+1} = 3^{n+3} + 4^{2n+3}$$

Using the Hint

$$a_{n+1} - 3a_n = (3^{n+3} + 4^{2n+3}) - (3^{n+2} + 4^{2n+1})$$

After solving algebraically

$$a_{n+1} - 3a_n = 52 \cdot 4^{2n}$$

$$13|(a_{n+1} - 3a_n) = 13|52 \cdot 4^{2n} \checkmark$$

This means that

$$(13|a_{n+1}) - (13|3a_n)$$

is true. Which tells us that $13|a_{n+1}$ is true. ✓

3. Proofs, Perhaps

   (a) This claim is false, and the proof is incorrect. The problem lies in the inductive step. The inductive hypothesis is

   $$P(k) = (k^2 < k)$$

   In Mathematical Induction, you assume $P(k)$ is true and prove that $P(k+1)$ is true, and therefore prove that

   $$P(k) \rightarrow P(k+1)$$

   is true $\forall k \in D$ where D is some domain. However in this proof, you prove something(the inductive hypothesis) that you already had assumed to be true, which is incorrect.

   (b) This claim is false, and the proof is incorrect. In the Inductive step, you have

   $$7^{k+1} = (7^k \cdot 7^k)/7^{k-1}$$

   However, when k = 0

   $$7^1 = (7^0 \cdot 7^0)/7^{-1}$$

   The claim only holds for $k \in N$, and $-1 \notin N$. Therefore, you can't substitute 1 in its place, and the proof breaks.

   (c) This claim is correct, and so is the proof.

   (d) This claim is false, and the proof is incorrect. The problem is in the claim that

   $$max[(a-1),(b-1)] = n$$

   because this doesn't hold hold for $a = b = 0$ because $-1 \notin N$.

4. Take the Tokens

   Claim: For all natural numbers k, if the pile starts with $4k+1$ tokens, then Thuc has a winning strategy.

   Proof: *by Induction*

   Let $P(k) = 4k + 1 \qquad \forall k \in N$

   Base Case:

$$P(0) = 4(0) + 1 = 1$$

   If there's only 1 token, then Tamara will end up picking it firstly, and thus Thuc wins.

   Inductive Step:

   Assume $P(k)$ is true. Taking $P(k+1)$

$$P(k+1) = 4(k+1) + 1$$
$$P(k+1) = (4k+1) + 4$$
$$P(k+1) = P(k) + 4$$

   Tamara has to go first, and can pick up either 1, 2 or 3 tiles. Thus, Thuc can afterwards pick

$$4 - \textit{No of tiles Tamara picked}$$

   to get the number of tiles in the form of $P(k) = 4k + 1$, under which condition Thuc knows he will win. ✓

5. Rigorous Recursion

Claim: For all inputs $n \in \mathbf{N}$, the value returned by the program is $G(n) = 3^n - 2^n$.

Proof: *by Induction*

Given $G(n) = 3^n - 2^n$

Base Case:

(a) *if (n = 0)*

$$G(0) = 3^0 - 2^0 = 0 \checkmark$$

(b) *if (n = 1)*

$$G(1) = 3^1 - 2^1 = 1 \checkmark$$

Inductive Step: Assume G(k) is true $\forall 0 \leq k \leq i$. Solving for $G(k+1)$

$$5G(k+1-1) - 6G(k+1-2) = 5G(k) - 6G(k-1)$$

$$= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$$

After algrebic manipulation

$$= 3^{k+1} - 2^{k+1} \checkmark$$

6. Coloring Countries (Some of the content here including pictures has been taken from `http://mathonline.wikidot.com/6-colour-theorem-for-planar-graphs`)

   (a) **Theorem:**

   Every possible map can be colored with six colors, in such a way that no two neighboring countries have the same color.

   Proof: *by Induction*

   Let $P(n)$ be the statement that the vertices in a planar graph can be coloured with 6 colors.

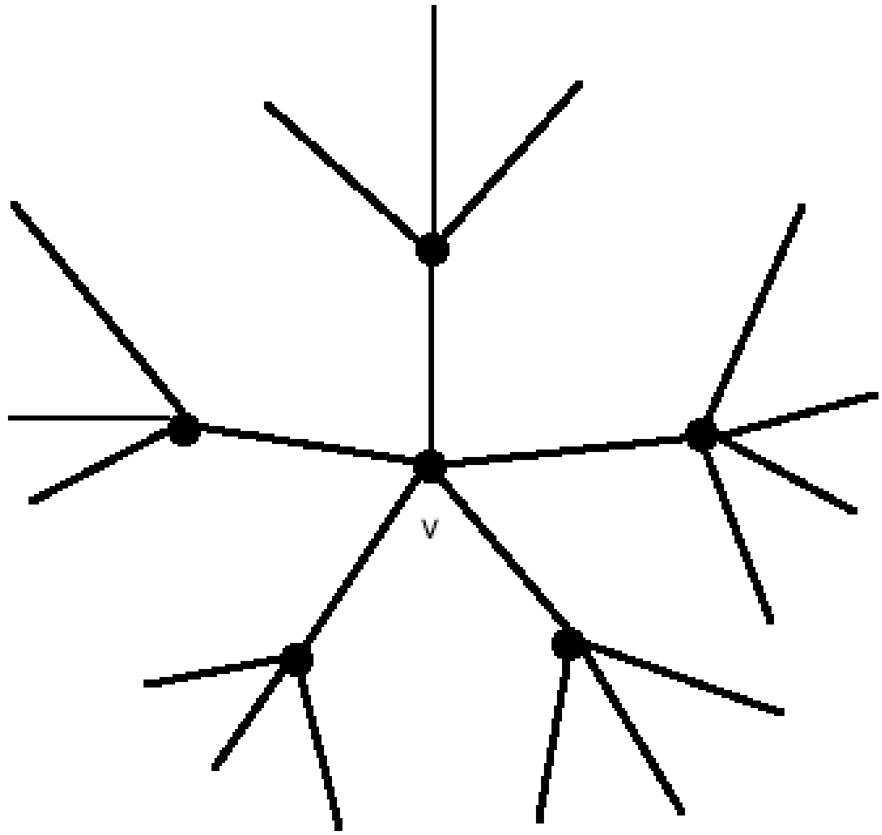
   Base Case:

   For $1 \leq n \leq 6$, $P(n)$ is correct because you can give each and every vertex a different color than its neighbor.

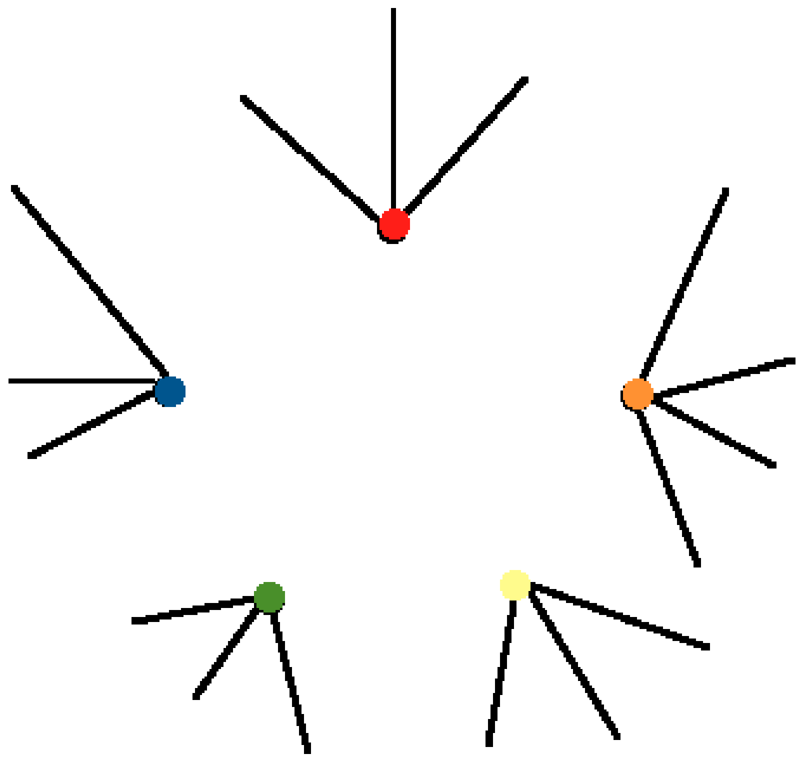
   Inductive Step:

   Assume that $P(n)$ is true, which means that if we have a planar graph with n vertices, we can obtain a good coloring. We want to prove that if we have $n+1$ vertices, the coloring is still possible.

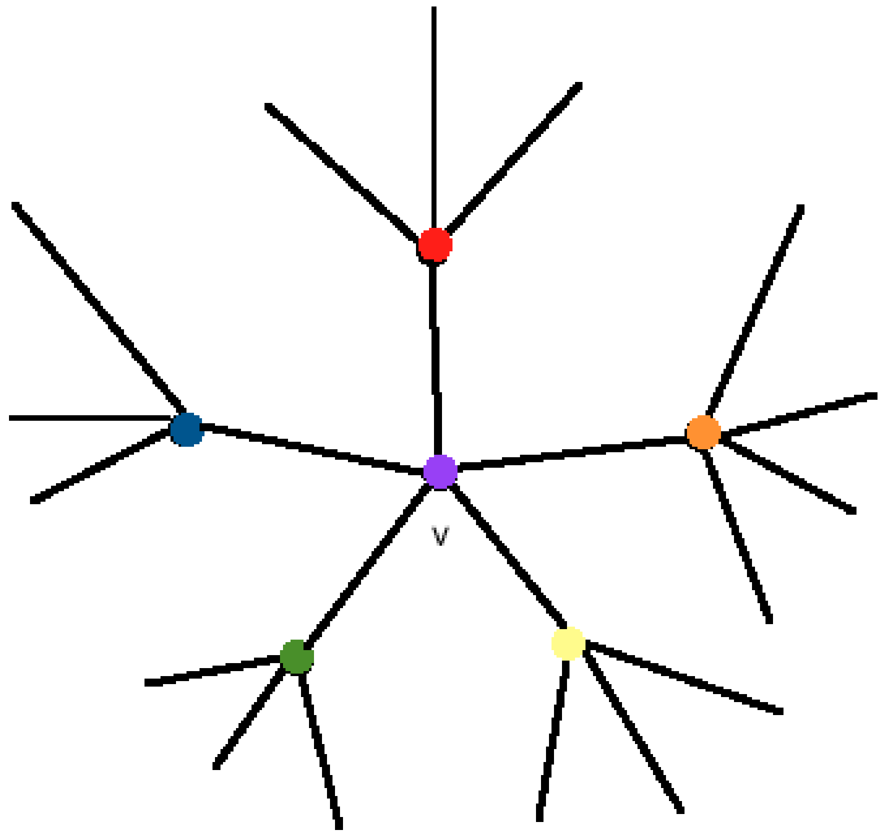
   Now, assume G is a planar graph with $n+1$ vertices. The lemma tells us that the degree of the vertex ≤ 5. Therefore, we would have a graph as shown in the figure below.



If we remove the vertex v, the number of vertices become n, for which we know that good coloring is possible, as in the following figure .



Adding the vertex v back, the figure becomes



Thus, a good coloring is possible. So $P(n+1)$ is true. ✓