# 305 Lecture 35 - Updating on Multiple Data Points

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# Plan

- We will end the week by looking at some examples of updating on multiple data points.

# Associated Reading

Odds and Ends, Chapter 9

# Conditional Independence

In a lot of cases, the two data points we get will not be probabilistically independent, but they will be **conditionally independent**.

That is, if $B_1$ and $B_2$ are the data points, and $X$ is an arbitrary hypothesis (like $A, \neg A$), we will have

$$\Pr(B_1|X)\Pr(B_2|X) = \Pr(B_1 \wedge B_2|X)$$

# Biased Coins

Here is one kind of case where the happens.

- We have a bunch of biased coins. For each of them, there is a probability *p* of heads on an arbitrary flip, but we don't know what that is.

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- The results of two flips of the same coin are not indepdendent.
- If one flip lands heads, that is evidence of a bias towards heads, and hence it increases the probability of heads on the next flip.

# Biased Coins

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- We have a bunch of biased coins. For each of them, there is a probability $p$ of heads on an arbitrary flip, but we don't know what that is.
- The results of two flips of the same coin are not indepdendent.
- If one flip lands heads, that is evidence of a bias towards heads, and hence it increases the probability of heads on the next flip.
- But conditional on a hypothesis about the bias of the coin, the flips are independent.

# Skilled Activity

A perhaps more real-life case of this is skilled action, like shooting free throws.

- The success of one attempt is not independent of the success of the previous.
- But conditional on the skill of the actor, the attempts are (probably, more or less) independent.

# Sampling With Replacement

Drawing from a selection **with replacement** produces conditional independence.

- If I don't know how many black marbles are in an urn, then drawing a black marble **and replacing it** will be evidence that the next marble is black.
- But conditional on a hypothesis about the nature of the urn, the draws with replacement will be independent.

# Yesterday, Today, Tomorrow

This is a little off topic, but a lot of real world phenomena satisfy (roughly) the following condition.

- How things were yesterday is a good (probabilistic) guide to how things will be tomorrow.
- So how things will be tomorrow is not independent of how things were yesterday.

## Yesterday, Today, Tomorrow

This is a little off topic, but a lot of real world phenomena satisfy (roughly) the following condition.

- How things were yesterday is a good (probabilistic) guide to how things will be tomorrow.
- So how things will be tomorrow is not independent of how things were yesterday.
- But, conditional on how things are today, how things were yesterday and will be tomorrow are independent.
- Knowing how things were yesterday doesn't tell you any more about how things will be tomorrow once you know how things are today.

# Markov Chains

A chain of events where every event is probabilistically dependent on the previous one, but only on the previous one, is called a **Markov Chain**.

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A chain of events where every event is probabilistically dependent on the previous one, but only on the previous one, is called a **Markov Chain**.

- Lots of real world processes are (more or less) Markov Chains.
- Weather systems, for instance, are probably more or less Markov Chains.
- And lots of ecological models assume that animal populations are Markov Chains.
- And the core idea is just conditional independence.

# Conditional Independence

In cases where the data points $B_1$ and $B_2$ are independent, we have an easy story about how to work out the probabilities.

$$\Pr(B_1 \wedge B_2 | X) = \Pr(B_1 | X) \Pr(B_2 | X)$$

## Same Event

There is an even simpler formula where $B_1$ and $B_2$ are the 'same' event, like the coin landing heads both time, or the same color marble being drawn.

$$\Pr(B_1 \wedge B_2|X) = \Pr(B_1|X)^2$$

# An Example

There are two urns in front of us.

- One of them - urn A - has 4 red marbles, 3 green marbles, and 3 blue marbles.
- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

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- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

One of the urns will be selected at random, and then a marble drawn from it.

- If the marble is red, what is the probability that Urn A was selected?

# A Table

I'll just do the column for red marble selected.

| | Red |
|---|---|
| Urn A | $0.5 \times 0.4 = 0.2$ |
| Urn B | $0.5 \times 0.8 = 0.4$ |
| **Total** | $0.2 + 0.4 = 0.6$ |

# A Table

I'll just do the column for red marble selected.

| | Red |
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| Urn A | $0.5 \times 0.4 = 0.2$ |
| Urn B | $0.5 \times 0.8 = 0.4$ |
| **Total** | $0.2 + 0.4 = 0.6$ |

$$\Pr(A|Red) = \frac{\Pr(A \wedge Red)}{\Pr(Red)} = \frac{0.2}{0.6} = \frac{1}{3}$$

# Another Example

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One of the urns will be selected at random, and then two marbles drawn from it **with replacement**.

- If both draws are red, what is the probability that Urn A was selected?

# A Table

| | Red-Red |
|---|---|
| Urn A | $0.5 \times 0.4^2 = 0.08$ |
| Urn B | $0.5 \times 0.8^2 = 0.32$ |
| **Total** | $0.08 + 0.32 = 0.4$ |

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| Urn B | $0.5 \times 0.8^2 = 0.32$ |
| **Total** | $0.08 + 0.32 = 0.4$ |

$$\Pr(A \mid Red - Red) = \frac{\Pr(A \wedge Red - Red)}{\Pr(Red - Red)} = \frac{0.08}{0.4} = \frac{1}{5}$$

The probability of Urn A fell by a lot.

# Yet Another Example

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- One of them - urn A - has 4 red marbles, 3 green marbles, and 3 blue marbles.
- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

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One of the urns will be selected at random, and then two marbles drawn from it **with replacement**.

- If the first draw is red and the second green, what is the probability that Urn A was selected?

# A Table

| | Red-Green |
|---|---|
| Urn A | $0.5 \times 0.4 \times 0.3 = 0.06$ |
| Urn B | $0.5 \times 0.8 \times 0.1 = 0.04$ |
| **Total** | $0.06 + 0.04 = 0.1$ |

# A Table

| | Red-Green |
|---|---|
| Urn A | $0.5 \times 0.4 \times 0.3 = 0.06$ |
| Urn B | $0.5 \times 0.8 \times 0.1 = 0.04$ |
| **Total** | $0.06 + 0.04 = 0.1$ |

$$\Pr(A|Red-Green) = \frac{\Pr(A \wedge Red-Green)}{\Pr(Red-Green)} = \frac{0.06}{0.1} = \frac{3}{5}$$

The probability of Urn A rose by a lot.

# Dependent Events

# Dependence

What happens if the events $B_1$ and $B_2$ are dependent on one or other of the hypotheses?

- The typical case is that they will be dependent on none or all of the hypotheses.
- But it's possible in principle to have independence on some and dependence on others.
- And in that case we have to use the more complicated procedure I'm about to describe.

# Sampling Without Replacement

The paradigm example of conditional dependence is sampling **without replacement**.

- Assume you know which urn I’m using.
- Then the draws without replacement won’t be independent because every time you draw a marble, there are fewer marbles of that color to draw the next time.

# Example

Assume that I am using urn A. (Or assume that we are working out conditional probabilities conditional on urn A.)

- For the first draw, the probability of red is 4 in 10, or 0.4.
- Conditional on the first draw being red, the probability of the second draw being red is 3 in 9, or $\frac{1}{3}$.
- That's because there are now 9 marbles left, and 3 of them are red.

## Continuing the Example

So to work out the probability of some sequence of draws $D_1, D_2$ given a hypothesis $X$ about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2|X) = \Pr(D_1|X)\Pr(D_2|X \wedge D_1)$$

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So to work out the probability of some sequence of draws $D_1, D_2$ given a hypothesis $X$ about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

For example

$$\Pr(Red_1 \wedge Red_2 | A) = \Pr(Red_1 | A) \Pr(Red_2 | A \wedge Red_1) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

## Continuing the Example

So to work out the probability of some sequence of draws $D_1, D_2$ given a hypothesis $X$ about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

For example

$$\Pr(Red_1 \wedge Red_2 | B) = \Pr(Red_1 | B) \Pr(Red_2 | B \wedge Red_1) = \frac{8}{10} \times \frac{7}{9} = \frac{28}{45}$$

# Another Example

There are two urns in front of us.

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# Another Example

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- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

One of the urns will be selected at random, and then two marbles drawn from it **without replacement**.

- If both draws are red, what is the probability that Urn A was selected?

# A Table

| | Red-Red |
|---|---|
| Urn A | $0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$ |
| Urn B | $0.5 \times \frac{8}{10} \times \frac{7}{9} = \frac{14}{45}$ |
| **Total** | $\frac{1}{15} + \frac{14}{45} = \frac{17}{45}$ |

# A Table

| | Red-Red |
|---|---|
| Urn A | $0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$ |
| Urn B | $0.5 \times \frac{8}{10} \times \frac{7}{9} = \frac{14}{45}$ |
| **Total** | $\frac{1}{15} + \frac{14}{45} = \frac{17}{45}$ |

$$\Pr(A|Red-Red) = \frac{\Pr(A \wedge Red-Red)}{\Pr(Red-Red)} = \frac{\frac{1}{15}}{\frac{17}{45}} = \frac{3}{17}$$

The probability of Urn A fell by a bit more.

# Yet Another Example

There are two urns in front of us.

- One of them - urn A - has 4 red marbles, 3 green marbles, and 3 blue marbles.
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One of the urns will be selected at random, and then two marbles drawn from it **with replacement**.

- If the first draw is red and the second green, what is the probability that Urn A was selected?

# The General Conjunction Rule

To work out the probability of some sequence of draws $D_1, D_2$ given a hypothesis $X$ about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

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To work out the probability of some sequence of draws $D_1, D_2$ given a hypothesis $X$ about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

So in this case we get

$$\Pr(Red_1 \wedge Green_2 | A) = \Pr(Red_1 | A) \Pr(Green_2 | A \wedge Red_1) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

# The General Conjunction Rule

To work out the probability of some sequence of draws $D_1, D_2$ given a hypothesis $X$ about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

And for Urn B we get

$$\Pr(Red_1 \wedge Green_2 | B) = \Pr(Red_1 | B) \Pr(Green_2 | B \wedge Red_1) = \frac{8}{10} \times \frac{1}{9} = \frac{4}{45}$$

# A Table

| | Red-Green |
|---|---|
| Urn A | $0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$ |
| Urn B | $0.5 \times \frac{8}{10} \times \frac{1}{9} = \frac{2}{45}$ |
| **Total** | $\frac{1}{15} + \frac{2}{45} = \frac{5}{45}$ |

## A Table

| | Red-Green |
|---|---|
| Urn A | $0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$ |
| Urn B | $0.5 \times \frac{8}{10} \times \frac{1}{9} = \frac{2}{45}$ |
| **Total** | $\frac{1}{15} + \frac{2}{45} = \frac{5}{45}$ |

$$\Pr(A|Red - Green) = \frac{\Pr(A \wedge Red - Red)}{\Pr(Red - Red)} = \frac{\frac{1}{15}}{\frac{5}{45}} = \frac{3}{5}$$

Which, interestingly, is exactly the same as in the with replacement case.

# Last (Difficult) Example

- There are four urns in the room, three of type X, one of type Y.
- The type X urns have 4 blue marbles and 2 yellow marbles.
- The type Y urn has 5 blue marbles and 3 yellow marbles.
- One of the four urns was selected at random.
- Then two marbles were selected **without replacement** from the randomly selected urn.
- The first was blue, the second was yellow.
- A third marble is about to be selected.
- What is the probability that it is blue?

# The Table

| Urn | Blue-then-Yellow |
| --- | --- |
| Type X | $\frac{3}{4} \times \frac{4}{6} \times \frac{2}{5} = \frac{1}{5}$ |
| Type Y | |
| **Total** | |

$$\Pr(X \wedge Blue_1 \wedge Yellow_2) = \Pr(X) \times \Pr(Blue_1 | X) \times \Pr(Yellow_2 | X \wedge Blue_1)$$

# The Table

| Urn | Blue-then-Yellow |
| --- | --- |
| Type X | $\frac{3}{4} \times \frac{4}{6} \times \frac{2}{5} = \frac{1}{5}$ |
| Type Y | $\frac{1}{4} \times \frac{5}{8} \times \frac{3}{7} = \frac{15}{224}$ |
| **Total** | |

$$\Pr(Y \wedge Blue_1 \wedge Yellow_2) = \Pr(Y) \times \Pr(Blue_1|Y) \times \Pr(Yellow_2|Y \wedge Blue_1)$$

## The Table

| Urn | Blue-then-Yellow |
|---|---|
| Type X | $\frac{1}{5}$ |
| Type Y | $\frac{15}{224}$ |
| **Total** | $\frac{299}{1120}$ |

You should double check this, but I think

$$\frac{1}{5} + \frac{15}{224} = \frac{299}{1120}$$

So that's $\Pr(Blue_1 \wedge Yellow_2)$

# Conditional Probabilities

$$\Pr(X|Blue_1 \wedge Yellow_2) = \frac{\Pr(X \wedge Blue_1 \wedge Yellow_2)}{\Pr(Blue_1 \wedge Yellow_2)} = \frac{\frac{1}{5}}{\frac{299}{1120}} = \frac{224}{299}$$

$$\Pr(Y|Blue_1 \wedge Yellow_2) = \frac{\Pr(Y \wedge Blue_1 \wedge Yellow_2)}{\Pr(Blue_1 \wedge Yellow_2)} = \frac{\frac{15}{224}}{\frac{299}{1120}} = \frac{75}{299}$$

The probability of Y is ever so fractionally higher than when we started.

## Next Marble

- If X (and Blue-followed-by-Yellow), the probability of next marble being blue is $\frac{3}{4}$.
- If Y (and Blue-followed-by-Yellow), the probability of next marble being blue is $\frac{2}{3}$.

# Next Marble

- If X (and Blue-followed-by-Yellow), the probability of next marble being blue is $\frac{3}{4}$.
- If Y (and Blue-followed-by-Yellow), the probability of next marble being blue is $\frac{2}{3}$.
- So overall probability of next marble being blue is

$$\frac{224}{299} \times \frac{3}{4} + \frac{75}{299} \times \frac{2}{3} = \frac{218}{299} \approx 0.729$$

## General Strategy of Last Slide

- If there are two hypotheses X and Y, and you want to know the probability of some event E, it will be given by

$$\Pr(E) = \Pr(X)\Pr(E|X) + \Pr(Y)\Pr(E|Y)$$

And that generalises to the case where there are multiple hypotheses $H_1, \dots H_n$

$$\Pr(E) = \Pr(H_1)\Pr(E|H_1) + \cdots + \Pr(H_n)\Pr(E|H_n)$$

# For Next Time

Next week we will look at the use of probability in decision making, and in science.