305 Lecture 18 - Building Complicated Truth Tables

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Plan

This lecture is about how to build more complicated truth tables than we have looked at so far.

The Example

We are going to work out the truth table for this sentence:

$$(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$$

How Many Rows

- How many rows should there be in the truth table?

How Many Rows

- How many rows should there be in the truth table?
- There are three (3) atomic sentences, so there should be $2^3 = 8$ rows.

Laying Out the Rows

- The convention for these is a bit odd.
- Here’s one way to think about it.
- For the left-most column you fill the first half of the rows with T and then the second half of the rows with F.

First Column

A	B	C	$(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$
T			
T			
T			
T			
F			
F			
F			
F			

Second Column

- Then the second column has one quarter T, followed by one quarter F, followed by one quarter T, followed by one quarter F.
- In this case that means we alternate every two rows.

Second Column

A	B	C	$(A \vee \sim B) \rightarrow (B \rightarrow (A \mathbin{\&} C))$
T	T		
T	T		
T	F		
T	F		
F	T		
F	T		
F	F		
F	F		

Third Column

- From now on you do half as many rows between changes.
- In this table we did 4 rows with one value then 4 of another for column 1, 2 with one value then 2 with another for column 2, and now alternate every row for column 3.
- It's helpful to know the full algorithm in case you ever have to do this with 5 or more variables.
- But I won't do that in this course.

Third Column

A	B	C	$(A \vee \sim B) \rightarrow (B \rightarrow (A \,\&\, C))$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Parsing the Sentence

Now we need to go back to our sentence.

$$(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$$

What is its **main connective**?

Parsing the Sentence

Now we need to go back to our sentence.

$$(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$$

What is its **main connective**?

- It's the first $\rightarrow$. The sentence is of the form $D \rightarrow E$, where D is $(A \vee \neg B)$ and E is $(B \rightarrow (A \wedge C))$

Building Up

So eventually, we will have the truth value for the whole sentence under the first $\rightarrow$.

- But that's some distance away.
- While that's where we want to get to, we have to build from the inside out.
- The first thing to do is to repeat the values for the atomic sentences.

Atomic Replicator

A	B	C	$(A$	$\lor$	$\neg$	$B)$	$\to$	$(B$	$\to$	$(A$	$\land$	$C))$
T	T	T	T			T		T		T		T
T	T	F	T			T		T		T		F
T	F	T	T			F		F		T		T
T	F	F	T			F		F		T		F
F	T	T	F			T		T		F		T
F	T	F	F			T		T		F		F
F	F	T	F			F		F		F		T
F	F	F	F			F		F		F		F

What can we fill in immediately?

Next Steps

- We have enough on the table to include the values for $\neg B$.
- And we have enough on the table to include the values for the $A \wedge C$ on the far right.
- We'll do these in order.

Negation

Everywhere B is T, $\neg B$ is F, so let's include all of those.

A	B	C	$(A$	$\vee$	$\neg$	$B)$	$\to$	$(B$	$\to$	$(A$	$\wedge$	$C))$
T	T	T	T		F	T		T		T		T
T	T	F	T		F	T		T		T		F
T	F	T	T			F		F		T		T
T	F	F	T			F		F		T		F
F	T	T	F		F	T		T		F		T
F	T	F	F		F	T		T		F		F
F	F	T	F			F		F		F		T
F	F	F	F			F		F		F		F

Negation (cont)

Everywhere B is F, $\neg B$ is T, so let's include all of those.

A	B	C	(	A	$\vee$	$\neg$	B	)	$\to$	(	B	$\to$	(	A	$\wedge$	C	))
T	T	T		T		F	T				T			T		T	
T	T	F		T		F	T				T			T		F	
T	F	T		T		T	F				F			T		T	
T	F	F		T		T	F				F			T		F	
F	T	T		F		F	T				T			F		T	
F	T	F		F		F	T				T			F		F	
F	F	T		F		T	F				F			F		T	
F	F	F		F		T	F				F			F		F	

Conjunction

If A, C are both T, so is $A \wedge C$. So let's include those.

A	B	C	$(A$	$\vee$	$\neg$	$B)$	$\to$	$(B$	$\to$	$(A$	$\wedge$	$C))$
T	T	T	T		F	T		T		T	T	T
T	T	F	T		F	T		T		T		F
T	F	T	T		T	F		F		T	T	T
T	F	F	T		T	F		F		T		F
F	T	T	F		F	T		T		F		T
F	T	F	F		F	T		T		F		F
F	F	T	F		T	F		F		F		T
F	F	F	F		T	F		F		F		F

Conjunction (cont)

And $A \land C$ is false everywhere else.

A	B	C	(A	$\lor$	$\neg$	B)	$\to$	(B	$\to$	(A	$\land$	C))
T	T	T	T		F	T		T		T	T	T
T	T	F	T		F	T		T		T	F	F
T	F	T	T		T	F		F		T	T	T
T	F	F	T		T	F		F		T	F	F
F	T	T	F		F	T		T		F	F	T
F	T	F	F		F	T		T		F	F	F
F	F	T	F		T	F		F		F	F	T
F	F	F	F		T	F		F		F	F	F

Complex Disjunction

- The next step is combining the values of A and $\neg B$ to get the value of $A \vee \neg B$.
- The main thing to remember here is what your inputs are.
- In this case it's not too confusing; it's the values immediately to either side of the $\vee$.
- But that won't be the general case.

Disjunction

When A is T, so is $A \vee \neg B$.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T		T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F		F	T		T		F	F	T
F	T	F	F		F	T		T		F	F	F
F	F	T	F		T	F		F		F	F	T
F	F	F	F		T	F		F		F	F	F

Disjunction (cont)

And when $\neg B$ is T, so is $A \lor \neg B$.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T		T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F		F	T		T		F	F	T
F	T	F	F		F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Disjunction (part III)

Otherwise, $A \lor \neg B$ is F.

A	B	C	(A	$\lor$	$\neg$	B)	$\to$ (B	$\to$ (A	$\land$	C))
T	T	T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	T	T	F	F
T	F	T	T	T	T	F	F	T	T	T
T	F	F	T	T	T	F	F	T	F	F
F	T	T	F	F	F	T	T	F	F	T
F	T	F	F	F	F	T	T	F	F	F
F	F	T	F	T	T	F	F	F	F	T
F	F	F	F	T	T	F	F	F	F	F

Conditional

- Now we have to do $B \longrightarrow (A \wedge C)$.
- We have to remember the table for $\longrightarrow$ - TFTT.
- And we have to remember that what's on the right-hand side of this conditional is a complex sentence: $A \wedge C$.

Conditional Terminology

- It’s helpful to recall our distinctive terminology for conditionals
- We’ll often call the left-hand side of a conditional the **antecedent**.

  (‘Ante’ for before, if that helps.)

Conditional Terminology

- It’s helpful to recall our distinctive terminology for conditionals
- We’ll often call the left-hand side of a conditional the **antecedent**. (‘Ante’ for before, if that helps.)
- And we’ll call the right-hand side the **consequent** (i.e., what comes after).
- We won’t have fancy distinct terminology for the left-hand and right-hand sides of other sentences, because they are symmetric.

Row 1

B is T, $A \wedge C$ is T, so this is T $\to$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 2

B is T, $A \land C$ is F, so this is T $\to$ F, i.e., F.

A	B	C	(A	$\lor$	$\neg$	B)	$\to$	(B	$\to$	(A	$\land$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 3

B is F, $A \wedge C$ is T, so this is F $\rightarrow$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 4

B is F, $A \wedge C$ is F, so this is F $\longrightarrow$ F, i.e., T.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 5

B is T, $A \wedge C$ is F, so this is T $\rightarrow$ F, i.e., F.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 6

B is T, $A \wedge C$ is F, so this is T $\longrightarrow$ F, i.e., F.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Rows 7 and 8

B is F, $A \wedge C$ is F, so this is F $\longrightarrow$ F, i.e., F.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

Almost Done

- Now we just need to put the two parts together.
- We have a conditional whose left-hand side, the antecedent, is $A \vee \neg B$.
- And the right-hand side, the consequent, is $B \rightarrow (A \wedge C)$.
- In each row we've computed the truth values for the antecedent and consequent.
- Now it's a matter of just looking up how they combine.
- Remember that the truth table for $\rightarrow$ is TFTT.
- To help, I've started by bolding the columns for the antecedent and consequent. (Not actually bold, but something more visible on projector.)

Getting There

Bolding the two relevant columns.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	**T**	F	T		T	**T**	T	T	T
T	T	F	T	**T**	F	T		T	**F**	T	F	F
T	F	T	T	**T**	T	F		F	**T**	T	T	T
T	F	F	T	**T**	T	F		F	**T**	T	F	F
F	T	T	F	**F**	F	T		T	**F**	F	F	T
F	T	F	F	**F**	F	T		T	**F**	F	F	F
F	F	T	F	**F**	T	F		F	**T**	F	F	T
F	F	F	F	**F**	T	F		F	**T**	F	F	F

Row 1

That’s T $\rightarrow$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T		T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Row 2

That's T $\to$ F, i.e., F.

A	B	C	(A	$\lor$	$\neg$	B)	$\to$	(B	$\to$	(A	$\land$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Row 3

That's T $\rightarrow$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Row 4

That's also T $\rightarrow$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	**T**	F	T	T	T	**T**	T	T	T
T	T	F	T	**T**	F	T	F	T	**F**	T	F	F
T	F	T	T	**T**	T	F	T	F	**T**	T	T	T
T	F	F	T	**T**	T	F	T	F	**T**	T	F	F
F	T	T	F	**F**	F	T		T	**F**	F	F	T
F	T	F	F	**F**	F	T		T	**F**	F	F	F
F	F	T	F	**F**	T	F		F	**T**	F	F	T
F	F	F	F	**F**	T	F		F	**T**	F	F	F

Rows 5 and 6

That's F $\rightarrow$ F, i.e., T.

A	B	C	(A	$\lor$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\land$	C))
T	T	T	T	**T**	F	T	T	T	**T**	T	T	T
T	T	F	T	**T**	F	T	F	T	**F**	T	F	F
T	F	T	T	**T**	T	F	T	F	**T**	T	T	T
T	F	F	T	**T**	T	F	T	F	**T**	T	F	F
F	T	T	F	**F**	F	T	T	T	**F**	F	F	T
F	T	F	F	**F**	F	T	T	T	**F**	F	F	F
F	F	T	F	**F**	T	F		F	**T**	F	F	T
F	F	F	F	**F**	T	F		F	**T**	F	F	F

Rows 7 and 8

That's F $\to$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	F

Summing Up

It’s true everywhere except when A, B are both T, and C is F.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	F

For Next Time

We’ll finish our discussion of truth tables with discussion of what we can do with truth tables.