

305 Lecture 09 - Nested Derivations

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Plan

We're going to go over what happens when there are multiple indirect derivations in a single argument.

Associated Reading

Carnap Book, chapter 5.

Using Conditionals

Sometimes we need to prove conditionals along the way of an argument.

- The big picture is that we can introduce 'Show' lines at any stage.
- These will introduce what I'll call a 'sub-proof'.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

There is a lot to unpack here, and we'll spend some time going over it all.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

The first line is simply the conclusion of the argument - that's nothing new.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

The next two lines are the premises of the argument - again nothing new here.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

The big new step is at line

4.

- This is the first use of 'Show' after line 1 we've seen.

Proving a Conditional

The way you prove a conditional that you need along the way is to:

1. Use 'Show' to say you're going to prove it.
2. Assume the antecedent.
3. Derive the consequent.
4. End the subproof with a CD statement.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

At line 4 we say what we're doing.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

At line 5 we assume the antecedent of the conditional - the left-hand side.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

From here we just start
using familiar rules.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

It turns out we just need
one step - MT gets from 2
and 5 to P .

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

Note that these are doubly indented.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

Every time we start a sub-proof, we indent by more spaces. I'm using four, though I don't think it insists.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

Now we've got from
antecedent to consequent,
so we can end.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R
2. $P \rightarrow Q$:PR
3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR
4. Show: $\neg Q \rightarrow \neg P$
5. $\neg Q$:AS
6. $\neg P$:MT 2, 5
7. :CD 6
8. R :MP 3, 4
9. :DD 8

At line 7 we record that we've got from $\neg Q$ to $\neg P$, so we can say we've shown that $\neg Q \rightarrow \neg P$.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

We are done with that part of the proof, so we can go back to normal indenting.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

Given $\neg Q \rightarrow \neg P$ we can
apply MP to line 3, and
that's what we do at line 8.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R
2. $P \rightarrow Q$:PR
3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR
4. Show: $\neg Q \rightarrow \neg P$
5. $\neg Q$:AS
6. $\neg P$:MT 2, 5
7. :CD 6
8. R :MP 3, 4
9. :DD 8

The next bit is one I didn't expect - we cite the 'show' line, not the CD line.

- That is, at line 8 we don't cite line 7 - just the 'show' at line 4.

Example

To prove: $P \rightarrow Q, (\neg Q \rightarrow \neg P) \rightarrow R \vdash R$

1. Show: R

2. $P \rightarrow Q$:PR

3. $(\neg Q \rightarrow \neg P) \rightarrow R$:PR

4. Show: $\neg Q \rightarrow \neg P$

5. $\neg Q$:AS

6. $\neg P$:MT 2, 5

7. :CD 6

8. R :MP 3, 4

9. :DD 8

And now we're done -
we've proven R as required.

A Big Restriction

When you do this kind of nesting, the ‘nested’ lines are not available for later reasoning.

- Anything between ‘Show’ and ‘CD’ is off-limits for later reasoning.
- That’s why we are indenting those lines - to say that they are all a bit of suppositional reasoning that is out-of-bounds once the supposition has been lifted.

A Bad Attempt at a Proof

To prove: $P \rightarrow (P \rightarrow Q), Q \rightarrow (P \rightarrow R) \vdash (P \rightarrow Q) \rightarrow R$

1. Show: $(P \rightarrow Q) \rightarrow R$

2. $P \rightarrow Q$:AS

3. $P \rightarrow (P \rightarrow Q)$:PR

4. $Q \rightarrow (P \rightarrow R)$:PR

5. Show: $P \rightarrow Q$

6. P :AS

7. $P \rightarrow Q$:MP 3, 6

8. Q :MP 6, 7

9. $P \rightarrow R$:MP 6, 7

10. :CD 8

11. $P \rightarrow R$:MP 4, 8

12. R :MP 6, 11

13. :CD 12

A Bad Attempt at a Proof

Three mistakes on previous slide.

1. No reason to try to show something that you already have.
2. At line 11, cite a line inside a subproof.
 - It's to prove something already seen but that's actually ok; getting something outside of the subproof could be useful. But it's an illegal step.
3. At line 12, cite a line inside a subproof.

Subproof

The Carnap book doesn't use the term 'subproof', but I find it useful.

- I mean the lines from one of these 'show' statements not on line 1 until the ':CD' that closes it off.

For Next Time

We'll start talking about how to do something that pop culture sometimes says is impossible: prove a negative.