305 Lecture 18 - Building Complicated Truth Tables

Brian Weatherson

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Plan

This lecture is about how to build more complicated truth tables than we have looked at so far.

Associated Reading

No new reading; we're just going to do a more complicated example than the book goes over.

The Example

We are going to work out the truth table for this sentence:

$$(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$$

How Many Rows

- How many rows should there be in the truth table?

How Many Rows

- How many rows should there be in the truth table?
- There are three (3) atomic sentences, so there should be $2^3 = 8$ rows.

Laying Out the Rows

- The convention for these is a bit odd.
- Here’s one way to think about it.
- For the left-most column you fill the first half of the rows with T and then the second half of the rows with F.

First Column

A	B	C	$(A \vee \neg B) \to (B \to (A \wedge C))$
T			
T			
T			
T			
F			
F			
F			
F			

Second Column

- Then the second column has one quarter T, followed by one quarter F, followed by one quarter T, followed by one quarter F.
- In this case that means we alternate every two rows.

Second Column

A	B	C	$(A \vee \sim B) \rightarrow (B \rightarrow (A \& C))$
T	T		
T	T		
T	F		
T	F		
F	T		
F	T		
F	F		
F	F		

Third Column

- From now on you do half as many rows between changes.
- In this table we did 4 rows with one value then 4 of another for column 1, 2 with one value then 2 with another for column 2, and now alternate every row for column 3.
- It's helpful to know the full algorithm in case you ever have to do this with 5 or more variables.
- But I won't do that in this course.

Third Column

A	B	C	$(A \vee \sim B) \rightarrow (B \rightarrow (A \,\&\, C))$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Parsing the Sentence

Now we need to go back to our sentence.

$$(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$$

What is its **main connective**?

Parsing the Sentence

Now we need to go back to our sentence.

$$(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$$

What is its **main connective**?

- It's the first $\rightarrow$. The sentence is of the form $D \rightarrow E$, where D is $(A \vee \neg B)$ and E is $(B \rightarrow (A \wedge C))$

Building Up

So eventually, we will have the truth value for the whole sentence under the first $\rightarrow$.

- But that's some distance away.
- While that's where we want to get to, we have to build from the inside out.
- The first thing to do is to repeat the values for the atomic sentences.

Atomic Replicator

A	B	C	$(A$	$\lor$	$\neg$	$B)$	$\to$	$(B$	$\to$	$(A$	$\land$	$C))$
T	T	T	T			T		T		T		T
T	T	F	T			T		T		T		F
T	F	T	T			F		F		T		T
T	F	F	T			F		F		T		F
F	T	T	F			T		T		F		T
F	T	F	F			T		T		F		F
F	F	T	F			F		F		F		T
F	F	F	F			F		F		F		F

What can we fill in immediately?

Next Steps

- We have enough on the table to include the values for $\neg B$.
- And we have enough on the table to include the values for the $A \land C$ on the far right.
- We'll do these in order.

Negation

Everywhere B is T, $\neg B$ is F, so let's include all of those.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T		F	T		T		T		T
T	T	F	T		F	T		T		T		F
T	F	T	T			F		F		T		T
T	F	F	T			F		F		T		F
F	T	T	F		F	T		T		F		T
F	T	F	F		F	T		T		F		F
F	F	T	F			F		F		F		T
F	F	F	F			F		F		F		F

Negation (cont)

Everywhere B is F, $\neg B$ is T, so let's include all of those.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T		F	T		T		T		T
T	T	F	T		F	T		T		T		F
T	F	T	T		T	F		F		T		T
T	F	F	T		T	F		F		T		F
F	T	T	F		F	T		T		F		T
F	T	F	F		F	T		T		F		F
F	F	T	F		T	F		F		F		T
F	F	F	F		T	F		F		F		F

If A, C are both T, so is $A \wedge C$. So let's include those.

A	B	C	$(A$	$\vee$	$\neg$	$B)$	$\to$	$(B$	$\to$	$(A$	$\wedge$	$C))$
T	T	T	T		F	T		T		T	T	T
T	T	F	T		F	T		T		T		F
T	F	T	T		T	F		F		T	T	T
T	F	F	T		T	F		F		T		F
F	T	T	F		F	T		T		F		T
F	T	F	F		F	T		T		F		F
F	F	T	F		T	F		F		F		T
F	F	F	F		T	F		F		F		F

Conjunction (cont)

And $A \wedge C$ is false everywhere else.

A	B	C	(	A	$\vee$	$\neg$	B	)	$\to$	(	B	$\to$	(	A	$\wedge$	C	))
T	T	T		T		F	T				T			T	T	T	
T	T	F		T		F	T				T			T	F	F	
T	F	T		T		T	F				F			T	T	T	
T	F	F		T		T	F				F			T	F	F	
F	T	T		F		F	T				T			F	F	T	
F	T	F		F		F	T				T			F	F	F	
F	F	T		F		T	F				F			F	F	T	
F	F	F		F		T	F				F			F	F	F	

Complex Disjunction

- The next step is combining the values of A and $\neg B$ to get the value of $A \vee \neg B$.
- The main thing to remember here is what your inputs are.
- In this case it's not too confusing; it's the values immediately to either side of the $\vee$.
- But that won't be the general case.

Disjunction

When A is T, so is $A \vee \neg B$.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T		T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F		F	T		T		F	F	T
F	T	F	F		F	T		T		F	F	F
F	F	T	F		T	F		F		F	F	T
F	F	F	F		T	F		F		F	F	F

Disjunction (cont)

And when $\neg B$ is T, so is $A \vee \neg B$.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T		T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F		F	T		T		F	F	T
F	T	F	F		F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Disjunction (part III)

Otherwise, $A \vee \neg B$ is F.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T		T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Conditional

- Now we have to do $B \longrightarrow (A \land C)$.
- We have to remember the table for $\longrightarrow$ - TFTT.
- And we have to remember that what's on the right-hand side of this conditional is a complex sentence: $A \land C$.

Conditional Terminology

- It's helpful to recall our distinctive terminology for conditionals
- We'll often call the left-hand side of a conditional the **antecedent**. ('Ante' for before, if that helps.)

Conditional Terminology

- It’s helpful to recall our distinctive terminology for conditionals
- We’ll often call the left-hand side of a conditional the **antecedent**. (‘Ante’ for before, if that helps.)
- And we’ll call the right-hand side the **consequent** (i.e., what comes after).
- We won’t have fancy distinct terminology for the left-hand and right-hand sides of other sentences, because they are symmetric.

Row 1

B is T, $A \land C$ is T, so this is T $\to$ T, i.e., T.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 2

B is T, $A \wedge C$ is F, so this is T $\to$ F, i.e., F.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 3

B is F, $A \wedge C$ is T, so this is F $\longrightarrow$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 4

B is F, $A \wedge C$ is F, so this is F $\longrightarrow$ F, i.e., T.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 5

B is T, $A \wedge C$ is F, so this is T $\longrightarrow$ F, i.e., F.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Row 6

B is T, $A \wedge C$ is F, so this is T $\rightarrow$ F, i.e., F.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Rows 7 and 8

B is F, $A \wedge C$ is F, so this is F $\longrightarrow$ F, i.e., F.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

Almost Done

- Now we just need to put the two parts together.
- We have a conditional whose left-hand side, the antecedent, is $A \vee \neg B$.
- And the right-hand side, the consequent, is $B \longrightarrow (A \wedge C)$.
- In each row we've computed the truth values for the antecedent and consequent.
- Now it's a matter of just looking up how they combine.
- Remember that the truth table for $\longrightarrow$ is TFTT.
- To help, I've started by bolding the columns for the antecedent and consequent. (Not actually bold, but something more visible on projector.)

Getting There

Bolding the two relevant columns.

A	B	C	(A	$\lor$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\land$	C))
T	T	T	T	**T**	F	T		T	**T**	T	T	T
T	T	F	T	**T**	F	T		T	**F**	T	F	F
T	F	T	T	**T**	T	F		F	**T**	T	T	T
T	F	F	T	**T**	T	F		F	**T**	T	F	F
F	T	T	F	**F**	F	T		T	**F**	F	F	T
F	T	F	F	**F**	F	T		T	**F**	F	F	F
F	F	T	F	**F**	T	F		F	**T**	F	F	T
F	F	F	F	**F**	T	F		F	**T**	F	F	F

Row 1

That's T $\to$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T		T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Row 2

That's T $\to$ F, i.e., F.

A	B	C	(A	$\vee$	$\neg$	B)	$\to$	(B	$\to$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Row 3

That's T $\to$ T, i.e., T.

A	B	C	(A	$\lor$	$\neg$	B)	$\to$	(B	$\to$	(A	$\land$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F		F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Row 4

That's also T $\rightarrow$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T		T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Rows 5 and 6

That's F $\longrightarrow$ F, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\longrightarrow$	(B	$\longrightarrow$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F		F	$\mathbb{T}$	F	F	F

Rows 7 and 8

That's F $\rightarrow$ T, i.e., T.

A	B	C	(A	$\vee$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\wedge$	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	F

Summing Up

It's true everywhere except when A, B are both T, and C is F.

A	B	C	(A	∨	¬	B)	→	(B	→	(A	∧	C))
T	T	T	T	$\mathbb{T}$	F	T	T	T	$\mathbb{T}$	T	T	T
T	T	F	T	$\mathbb{T}$	F	T	F	T	$\mathbb{F}$	T	F	F
T	F	T	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	T	T
T	F	F	T	$\mathbb{T}$	T	F	T	F	$\mathbb{T}$	T	F	F
F	T	T	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	T
F	T	F	F	$\mathbb{F}$	F	T	T	T	$\mathbb{F}$	F	F	F
F	F	T	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	T
F	F	F	F	$\mathbb{F}$	T	F	T	F	$\mathbb{T}$	F	F	F

For Next Time

We'll finish our discussion of truth tables with discussion of what we can do with truth tables.