305 Lecture 19 - Truth Tables and Validity

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July 13, 2020

Plan

This lecture finishes our discussion of truth tables by looking at the role truth tables play in determining validity.

The Rules

- An argument is **invalid** if there is a row on the truth table where all the premises are true and the conclusion is false. (Roughly!)
- It is **valid** if all the rows where the premises are all true, the conclusion is true as well.

Another Relevance Failure

Is this argument valid?

$$A \vDash B \vee \neg B$$

Another Relevance Failure

Is this argument valid?

$$A \vDash B \vee \neg B$$

Yes!

- There is no line where the conclusion is false.
- So there are no lines where the premise is true and the conclusion false.
- So it is not invalid.
- So it is valid.

Terminology

Say a **valuation** is a function v from sentences to $\{T, F\}$ satisfying these constraints.

1. $v(\neg A) = T$ if $v(A) = F$, and $v(\neg A) = F$ otherwise.
2. $v(A \vee B) = T$ if $V(A) = T$ or $v(B) = T$, and $v(A \vee B) = F$ otherwise.
3. $v(A \wedge B) = T$ if $V(A) = T$ and $v(B) = T$, and $v(A \wedge B) = F$ otherwise.
4. $v(A \to B) = T$ if $V(A) = F$ or $v(B) = T$, and $v(A \to B) = F$ otherwise.

Restating

- An argument is valid relative to a class of valuations V iff any valuation $v \in V$ that makes all the premises T also makes the conclusion T.

Restating

- An argument is valid relative to a class of valuations V iff any valuation $v \in V$ that makes all the premises T also makes the conclusion T.
- An argument is truth functionally valid when the class V is the class of valuations satisfying the constraints on the previous slide.

Very Technical Terminology

- I'll use $\Gamma \vDash A$ to mean that the argument with premises Γ and conclusion A is valid in this sense - i.e., all valuations that make all of Γ come out T also make A come out T.
- The double bar in $\vDash$ is to represent that this is a kind of validity defined in terms of valuations (or, as we'll start calling them, models), and not proofs.
- For purposes of 305, the difference between $\vdash$ and $\vDash$ is not important, and if this is the last logic/mathematical philosophy course you plan to take, you don't have to worry about this.
- But I like being pedantic even when it isn't relevant to the course.

Closure

If $\Gamma \vDash A$ and $\Gamma \vDash A \rightarrow B$ then $\Gamma \vDash B$

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If $\Gamma \models A$ and $\Gamma \models A \to B$ then $\Gamma \models B$.

Proof: Assume this is false. So Assume that $\Gamma \not\models B$. So there is a valuation function v that makes everything in Γ come out T and B come out F

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If $\Gamma \vDash A$ and $\Gamma \vDash A \rightarrow B$ then $\Gamma \vDash B$.

Proof: Assume this is false. So Assume that $\Gamma \nvDash B$. So there is a valuation function v that makes everything in Γ come out T and B come out F. Either $v(A) = \mathrm{T}$ or $v(A) = \mathrm{F}$. If $v(A) = \mathrm{T}$, then $v(A \rightarrow B) = \mathrm{F}$, contradicting $\Gamma \vDash A \rightarrow B$

Closure

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Proof: Assume this is false. So Assume that $\Gamma \nvDash B$. So there is a valuation function v that makes everything in Γ come out T and B come out F. Either $v(A) = \text{T}$ or $v(A) = \text{F}$. If $v(A) = \text{T}$, then $v(A \rightarrow B) = \text{F}$, contradicting $\Gamma \vDash A \rightarrow B$. If $v(A) = \text{F}$, then v is a counterexample to $\Gamma \vDash A$, but we know $\Gamma \vDash A$ is true. Either way, such a v cannot exist, so $\Gamma \vDash B$ is true.

Monotony

If $\Gamma \vDash A$, and $\Gamma \subset \Delta$, then $\Delta \vDash A$.

That is, adding premises can't turn an argument from being valid to invalid.

Monotony Proof

- Assume that for all $B \in \Delta, v(B) = \mathsf{T}$.
- We need to prove that $v(A) = \mathsf{T}$.

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- Assume $C \in \Gamma$.
- Then $C \in \Delta$, since $\Gamma \subset \Delta$.

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- So by hypothesis, $v(C) = \mathsf{T}$, since everything in Δ is T.

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- So v is such that everything in Γ is T.

Monotony Proof

- Assume that for all $B \in \Delta$, $v(B) = \mathsf{T}$.
- We need to prove that $v(A) = \mathsf{T}$.
- Assume $C \in \Gamma$.
- Then $C \in \Delta$, since $\Gamma \subset \Delta$.
- So by hypothesis, $v(C) = \mathsf{T}$, since everything in Δ is T.
- So v is such that everything in Γ is T.
- And since $\Gamma \vDash A$, that implies $v(A) = \mathsf{T}$, as required.

Monotony Commentary

- This idea, that adding premises doesn't destroy validity, only works for logical arguments.
- It isn't true for good arguments in general.

Tweety the First

1. Tweety is a bird.

Tweety flies.

That’s a perfectly good, though not logically valid, argument.

Tweety the Second

1. Tweety is a bird.
2. Tweety is black and white, lives in Antarctica, and lays large eggs.

Tweety flies.

That's not a very good argument!

Transitivity

If $\Gamma \vDash A$ and $\Delta \cup A \vDash B$ then $\Gamma \cup \Delta \vDash B$

If some premises entail A, and some other premises plus A entail B, then the two sets of premises between them entail B.

Transitivity

If $\Gamma \vDash A$ and $\Delta \cup A \vDash B$ then $\Gamma \cup \Delta \vDash B$

If some premises entail A, and some other premises plus A entail B, then the two sets of premises between them entail B. This is crucial for being able to chain together lines of reasoning.

Transitivity Proof

- Assume that for all $C \in \Gamma \cup \Delta, v(C) = \mathsf{T}$.
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Transitivity Proof

- Assume that for all $C \in \Gamma \cup \Delta$, $v(C) = \mathsf{T}$.
- We need to prove $v(B) = \mathsf{T}$.
- Since everything in Γ is T according to v, and $\Gamma \vDash A$, it follows that $v(A) = \mathsf{T}$.
- Since everything in Δ is T according to v, and A is T according to v, and $\Delta \cup A \vDash B$, it follows that $v(B) = \mathsf{T}$, as required.

Deduction Theorem

This is why we define $\rightarrow$ the way we do.

$$\Gamma \vDash A \rightarrow B \text{ if and only if } \Gamma \cup A \vDash B.$$

Note that there are two claims here - one each direction. We need to prove each.

Deduction Theorem Left-to-Right

- Assume $\Gamma \models A \rightarrow B$, and prove $\Gamma \cup A \models B$.
- So assume $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and $v(A) = \mathsf{T}$, and aim to prove $v(B) = \mathsf{T}$.

Deduction Theorem Left-to-Right

- Assume $\Gamma \vDash A \rightarrow B$, and prove $\Gamma \cup A \vDash B$.
- So assume $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and $v(A) = \mathsf{T}$, and aim to prove $v(B) = \mathsf{T}$.
- Since $\Gamma \vDash A \rightarrow B$ and $v(C) = \mathsf{T}$ for all $C \in \Gamma$, it follows that $v(A \rightarrow B) = \mathsf{T}$.

Deduction Theorem Left-to-Right

- Assume $\Gamma \vDash A \to B$, and prove $\Gamma \cup A \vDash B$.
- So assume $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and $v(A) = \mathsf{T}$, and aim to prove $v(B) = \mathsf{T}$.
- Since $\Gamma \vDash A \to B$ and $v(C) = \mathsf{T}$ for all $C \in \Gamma$, it follows that $v(A \to B) = \mathsf{T}$.
- Since $v(A \to B) = \mathsf{T}$ and $v(A) = \mathsf{T}$, it must be that $v(B) = \mathsf{T}$, since that's the only line on the truth table where $A \to B$ and A are both T.

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \mathsf{T}$.

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \mathsf{T}$.
- Either $v(A) = \mathsf{T}$ or $v(A) = \mathsf{F}$. Take each case in turn.

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \text{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \text{T}$.
- Either $v(A) = \text{T}$ or $v(A) = \text{F}$. Take each case in turn.
- If $v(A) = \text{T}$, then since $v(C) = \text{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \text{T}$

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \rightarrow B$.
- So assume $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and prove $v(A \rightarrow B) = \mathsf{T}$.
- Either $v(A) = \mathsf{T}$ or $v(A) = \mathsf{F}$. Take each case in turn.
- If $v(A) = \mathsf{T}$, then since $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \mathsf{T}$, so $v(A \rightarrow B) = \mathsf{T}$

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \rightarrow B$.
- So assume $v(C) = \text{T}$ for all $C \in \Gamma$, and prove $v(A \rightarrow B) = \text{T}$.
- Either $v(A) = \text{T}$ or $v(A) = \text{F}$. Take each case in turn.
- If $v(A) = \text{T}$, then since $v(C) = \text{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \text{T}$, so $v(A \rightarrow B) = \text{T}$.
- If $v(A) = \text{F}$, it follows directly that $v(A \rightarrow B) = \text{T}$

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \mathsf{T}$.
- Either $v(A) = \mathsf{T}$ or $v(A) = \mathsf{F}$. Take each case in turn.
- If $v(A) = \mathsf{T}$, then since $v(C) = \mathsf{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \mathsf{T}$, so $v(A \to B) = \mathsf{T}$.
- If $v(A) = \mathsf{F}$, it follows directly that $v(A \to B) = \mathsf{T}$.
- Either way, $v(A \to B) = \mathsf{T}$ as required.

Deduction Theorem Comments

- This is a striking result.
- It shows that proving $A \longrightarrow B$ is just the same as proving B, assuming you're allowed to add A as an extra assumption.
- And that's a good thing, intuitively. That is how we prove conditionals.
- But it only works if you have the (very odd looking) truth table that we're using for $\longrightarrow$.
- This is the main reason for thinking, despite it's odd appearance, that this truth table is the right one for $\longrightarrow$.

For Next Time

On Wednesday we will start working on a different way to analyse arguments: truth trees.