

305 Lecture 5.4 - Strategies 1: Working Backwards

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Plan

This lecture discusses strategies for constructing proofs that involve working backwards.

Associated Reading

forall x , section 17.1.

Big Picture

- When you are given a proof to do, you are told what the intended conclusion is.
- That conclusion will usually have a connective in it.
- And when it does, it will often be good to aim to use the introduction rule for that connective to complete the proof.
- Thinking about how that could happen will often give us something to aim for.

- The strategies they discuss in chapter 17 apply recursively.
- Whenever we talk about a 'target' or a 'conclusion', that could be the conclusion of the whole argument, but it does not have to be.
- It could just be something else we've set as a target.

Working Backwards: And

The $\wedge I$ rule says

- From X , and Y , infer $X \wedge Y$.

So if the last line is a conjunction, one strategy is to aim to prove both parts.

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

No Rule Found				1.	$A \wedge B$	PR
				2.	C	PR
				<hr/>		
1.	$A \wedge B$:PR	+	3.	""	
2.	C	:PR	+	4.	""	
3.			Δ	5.	""	
4.			Δ	6.	""	
5.			Δ	7.	""	
6.			Δ	8.	""	
7.			Δ	9.	""	
8.			Δ	10.	""	
9.			Δ			
10.			Δ			
11.	$(A \wedge C) \wedge (B \wedge C)$: \wedge I	?	11.	$(A \wedge C) \wedge (B \wedge C)$	\wedge I

Writing out premises and conclusion

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

No Rule Found

1. $A \wedge B$:PR	+
2. C	:PR	+
3.		Δ
4.		Δ
5.		Δ
6. $A \wedge C$		Δ
7.		Δ
8.		Δ
9.		Δ
10. $B \wedge C$		Δ
11. $(A \wedge C) \wedge (B \wedge C)$: \wedge I	?

1.	$A \wedge B$	PR
2.	C	PR
3.	""	
4.	""	
5.	""	
6.	$A \wedge C$	
7.	""	
8.	""	
9.	""	
10.	$B \wedge C$	
11.	$(A \wedge C) \wedge (B \wedge C)$	\wedge I

Setting up \wedge introduction

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

No Rule Found

1. $A \wedge B$:PR	+
2. C	:PR	+
3. A		Δ
4. C		Δ
5. $A \wedge C$		Δ
6.		Δ
7.		Δ
8.		Δ
9. $B \wedge C$		Δ
10. $(A \wedge C) \wedge (B \wedge C)$: \wedge I	?

1.	$A \wedge B$	PR
2.	C	PR
3.	A	
4.	C	
5.	$A \wedge C$	
6.	""	
7.	""	
8.	""	
9.	$B \wedge C$	
10.	$(A \wedge C) \wedge (B \wedge C)$	\wedge I

Working backwards from $A \wedge C$

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

No Rule Found

1. $A \wedge B$:PR	+
2. C	:PR	+
3. A	: \wedge E 1	+
4. C	:R 2	+
5. $A \wedge C$: \wedge I 3,4	+
6.		Δ
7.		Δ
8.		Δ
9. $B \wedge C$		Δ
10. $(A \wedge C) \wedge (B \wedge C)$: \wedge I	?

1.	$A \wedge B$	PR
2.	C	PR
3.	A	\wedge E 1
4.	C	R 2
5.	$A \wedge C$	\wedge I 3, 4
6.	""	
7.	""	
8.	""	
9.	$B \wedge C$	
10.	$(A \wedge C) \wedge (B \wedge C)$	\wedge I

Filling in rules

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

No Rule Found

1. $A \wedge B$:PR	+
2. C	:PR	+
3. A	: $\wedge E$ 1	+
4. C	:R 2	+
5. $A \wedge C$: $\wedge I$ 3,4	+
6. B		Δ
7. C		Δ
8. $B \wedge C$		Δ
9. $(A \wedge C) \wedge (B \wedge C)$: $\wedge I$?

1.	$A \wedge B$	PR
2.	C	PR
3.	A	$\wedge E$ 1
4.	C	R 2
5.	$A \wedge C$	$\wedge I$ 3, 4
6.	B	
7.	C	
8.	$B \wedge C$	
9.	$(A \wedge C) \wedge (B \wedge C)$	$\wedge I$

Working backwards from $B \wedge C$

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

No Rule Found

1. $A \wedge B$:PR	+
2. C	:PR	+
3. A	: $\wedge E$ 1	+
4. C	:R 2	+
5. $A \wedge C$: $\wedge I$ 3,4	+
6. B	: $\wedge E$ 1	+
7. C	:R 2	+
8. $B \wedge C$: $\wedge I$ 6, 7	+
9. $(A \wedge C) \wedge (B \wedge C)$: $\wedge I$?

1.	$A \wedge B$	PR
2.	C	PR
3.	A	$\wedge E$ 1
4.	C	R 2
5.	$A \wedge C$	$\wedge I$ 3, 4
6.	B	$\wedge E$ 1
7.	C	R 2
8.	$B \wedge C$	$\wedge I$ 6, 7
9.	$(A \wedge C) \wedge (B \wedge C)$	$\wedge I$

Filling in line numbers for the second half

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

$A \wedge B, C \vdash (A \wedge C) \wedge (B \wedge C)$

1. $A \wedge B$:PR	+
2. C	:PR	+
3. A	: \wedge E 1	+
4. C	:R 2	+
5. $A \wedge C$: \wedge I 3,4	+
6. B	: \wedge E 1	+
7. C	:R 2	+
8. $B \wedge C$: \wedge I 6, 7	+
9. $(A \wedge C) \wedge (B \wedge C)$: \wedge I 5, 8	+

1.	$A \wedge B$	PR
2.	C	PR
3.	A	\wedge E 1
4.	C	R 2
5.	$A \wedge C$	\wedge I 3, 4
6.	B	\wedge E 1
7.	C	R 2
8.	$B \wedge C$	\wedge I 6, 7
9.	$(A \wedge C) \wedge (B \wedge C)$	\wedge I 5, 8

Filling in line numbers for the final line

$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$

No Rule Found

1. $A \rightarrow B$:PR	+
2.		Δ
3.		Δ
4.		Δ
5.		Δ
6.		Δ
7.		Δ
8. $(A \wedge C) \rightarrow (B \wedge C)$		Δ

1.	$A \rightarrow B$	PR
2.	""	
3.	""	
4.	""	
5.	""	
6.	""	
7.	""	
8.	$(A \wedge C) \rightarrow (B \wedge C)$	

Premise and conclusion

$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$

No Rule Found

1.	$A \rightarrow B$:PR	+
2.	$A \wedge C$:AS	+
3.			Δ
4.			Δ
5.			Δ
6.			Δ
7.			Δ
8.	$B \wedge C$		Δ
9.	$(A \wedge C) \rightarrow (B \wedge C)$	\rightarrow I 2-8	Δ

1.	$A \rightarrow B$	PR
2.	$A \wedge C$	AS
3.	""	
4.	""	
5.	""	
6.	""	
7.	""	
8.	$B \wedge C$	
9.	$(A \wedge C) \rightarrow (B \wedge C)$	\rightarrow I 2-8

Setting up \rightarrow I

$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$

No Rule Found

1.	$A \rightarrow B$:PR	+
2.	$A \wedge C$:AS	+
3.			Δ
4.	B		Δ
5.			Δ
6.	C	: \wedge E 2	?
7.	$B \wedge C$: \wedge I 4, 5	?
8.	$(A \wedge C) \rightarrow (B \wedge C)$: \rightarrow I 2-6	?

1.	$A \rightarrow B$	PR
2.	$A \wedge C$	AS
3.	""	
4.	B	
5.	""	
6.	C	\wedge E 2
7.	$B \wedge C$	\wedge I 4, 5
8.	$(A \wedge C) \rightarrow (B \wedge C)$	\rightarrow I 2-6

Setting up \wedge I

$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$

No Rule Found

1.	$A \rightarrow B$:PR	+
2.	$A \wedge C$:AS	+
3.	A	: \wedge E 2	+
4.	B	: \rightarrow E 1, 3	+
5.			Δ
6.	C		Δ
7.	$B \wedge C$: \wedge I	?
8.	$(A \wedge C) \rightarrow (B \wedge C)$: \rightarrow I 2-8	?

1.	$A \rightarrow B$	PR
2.	$A \wedge C$	AS
3.	A	\wedge E 2
4.	B	\rightarrow E 1, 3
5.	""	
6.	C	
7.	$B \wedge C$	\wedge I
8.	$(A \wedge C) \rightarrow (B \wedge C)$	\rightarrow I 2-8

Getting the first conjunct

$$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$$

No Rule Found

1.	$A \rightarrow B$:PR	+
2.	$A \wedge C$:AS	+
3.	A	: \wedge E 2	+
4.	B	: \rightarrow E 1, 3	+
5.	C	: \wedge E 2	+
6.	$B \wedge C$: \wedge I	?
7.	$(A \wedge C) \rightarrow (B \wedge C)$: \rightarrow I 2-8	?

1.	$A \rightarrow B$	PR
2.	$A \wedge C$	AS
3.	A	\wedge E 2
4.	B	\rightarrow E 1, 3
5.	C	\wedge E 2
6.	$B \wedge C$	\wedge I
7.	$(A \wedge C) \rightarrow (B \wedge C)$	\rightarrow I 2-8

Getting the second conjunct

$$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$$

No Rule Found

1.	$A \rightarrow B$:PR	+
2.	$A \wedge C$:AS	+
3.	A	: \wedge E 2	+
4.	B	: \rightarrow E 1, 3	+
5.	C	: \wedge E 2	+
6.	$B \wedge C$: \wedge I 4, 5	+
7.	$(A \wedge C) \rightarrow (B \wedge C)$: \rightarrow I 2-8	?

1.	$A \rightarrow B$	PR
2.	$A \wedge C$	AS
3.	A	\wedge E 2
4.	B	\rightarrow E 1, 3
5.	C	\wedge E 2
6.	$B \wedge C$	\wedge I 4, 5
7.	$(A \wedge C) \rightarrow (B \wedge C)$	\rightarrow I 2-8

Line numbers for the \wedge I step

$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$

$A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$

1.	$A \rightarrow B$:PR	+
2.	$A \wedge C$:AS	+
3.	A	: \wedge E 2	+
4.	B	: \rightarrow E 1, 3	+
5.	C	: \wedge E 2	+
6.	$B \wedge C$: \wedge I 4, 5	+
7.	$(A \wedge C) \rightarrow (B \wedge C)$: \rightarrow I 2-6	+

1.	$A \rightarrow B$	PR
2.	$A \wedge C$	AS
3.	A	\wedge E 2
4.	B	\rightarrow E 1, 3
5.	C	\wedge E 2
6.	$B \wedge C$	\wedge I 4, 5
7.	$(A \wedge C) \rightarrow (B \wedge C)$	\rightarrow I 2-6

Line numbers for the \rightarrow I step

$A \rightarrow B, \neg B \vdash \neg A$

No Rule Found

1. $A \rightarrow B$:PR	+
2. $\neg B$:PR	+
3.		Δ
4.		Δ
5.		Δ
6.		Δ
7.		Δ
8. $\neg A$		Δ

1.	$A \rightarrow B$	PR
2.	$\neg B$	PR
3.	""	
4.	""	
5.	""	
6.	""	
7.	""	
8.	$\neg A$	

Premises and Conclusion

$A \rightarrow B, \neg B \vdash \neg A$

No Rule Found

1.	$A \rightarrow B$:PR	+
2.	$\neg B$:PR	+
3.	A	:AS	+
4.			Δ
5.	$!?$: $\neg E$?
6.	$\neg A$: $\neg I$?

1.	$A \rightarrow B$	PR
2.	$\neg B$	PR
3.	A	AS
4.	""	
5.	\perp	$\neg E$
6.	$\neg A$	$\neg I$

Setting up $\neg I$

$A \rightarrow B, \neg B \vdash \neg A$

No Rule Found

1.	$A \rightarrow B$:PR	+
2.	$\neg B$:PR	+
3.	A	:AS	+
4.	B	: \rightarrow E 1, 3	+
5.	$!?$: \sim E 2, 4	+
6.	$\neg A$: \sim I	?

1.	$A \rightarrow B$	PR
2.	$\neg B$	PR
3.	A	AS
4.	B	\rightarrow E 1, 3
5.	\perp	\neg E 2, 4
6.	$\neg A$	\neg I

Getting the Contradiction

$A \rightarrow B, \neg B \vdash \neg A$

$\neg B, A \rightarrow B \vdash \neg A$

1.	$A \rightarrow B$:PR	+
2.	$\neg B$:PR	+
3.	A	:AS	+
4.	B	: $\rightarrow E$ 1, 3	+
5.	!?	: $\neg E$ 2, 4	+
6.	$\neg A$: $\neg I$ 3-5	+

1.	$A \rightarrow B$	PR
2.	$\neg B$	PR
3.	A	AS
4.	B	$\rightarrow E$ 1, 3
5.	\perp	$\neg E$ 2, 4
6.	$\neg A$	$\neg I$ 3-5

Finishing the Proof

Working Backwards

- What if the conclusion is a disjunction?
- Don't work backwards!

For Next Time

- We'll look at strategies that involve going forwards.