305 Lecture 5.5 - Strategies 2: Working Forwards

Brian Weatherson

Plan

This lecture discusses strategies for constructing proofs that involve working forwards.

Associated Reading

forall x, section 17.2.

Working Forwards

Big Idea: Plan to use the Elimination rules on the connectives in the premises.

Simple Illustration: And

When one of the premises is of the form $X \wedge y$, you'll almost certainly need to apply $\wedge$E to get X and Y.

Slightly Trickier: If

- When one of the premises is of the form $X \rightarrow Y$, you'll almost certainly need to apply $\rightarrow$E.

Slightly Trickier: If

- When one of the premises is of the form $X \rightarrow Y$, you'll almost certainly need to apply $\rightarrow$E.
- And that means you'll need X.

Slightly Trickier: If

- When one of the premises is of the form $X \to Y$, you'll almost certainly need to apply $\to$E.
- And that means you'll need X.
- But in practice it's hard to tell in advance whether you'll prove X, or have it as the start of a subproof, or something else.

Working forward from Or

When one of the premises is $X \vee Y$ there is a clear(ish) strategy.

1. Find a target conclusion C.
2. Do a subproof from X to C.
3. Do a subproof from Y to C.
4. Conclude C by $\vee$E.

Working forward from Or

Why clear-ish?

- Because it isn’t always true that the target here should be the conclusion of the whole argument.
- Sometimes it is optimal to do a step or two of working backwards first.
- But if you want a simple rule to go by, the best is to do what’s on the previous slide with C as the conclusion of the whole argument.

$A \vee B, B \rightarrow C \vdash A \vee C$

No Rule Found

```
1. A \/ B          :PR     +
2. B -> C          :PR     +
3.                         ⚠
4.                         ⚠
5.                         ⚠
6.                         ⚠
7.                         ⚠
8.                         ⚠
9. A \/ C                  ⚠
```

1.	$A \vee B$	PR
2.	$B \rightarrow C$	PR
3.	""	
4.	""	
5.	""	
6.	""	
7.	""	
8.	""	
9.	$A \vee C$	

Write Out Premises and Conclusion

$A \vee B, B \rightarrow C \vdash A \vee C$

No Rule Found

```
 1. A \/ B          :PR     +
 2. B -> C          :PR     +
 3.     A           :AS     +
 4.                         ⚠
 5.                         ⚠
 6.     A \/ C              ⚠
 7. --
 8.     B           :AS     +
 9.                         ⚠
10.                         ⚠
11.                         ⚠
12.                         ⚠
13.     A \/ C              ⚠
14. A \/ C                  ⚠
```

Expand

1.	$A \vee B$	PR
2.	$B \rightarrow C$	PR
3.	A	AS
4.	""	
5.	""	
6.	$A \vee C$	
8.	B	AS
9.	""	
10.	""	
11.	""	
12.	""	
13.	$A \vee C$	
14.	$A \vee C$	

Set Up $\vee$E

$A \vee B, B \rightarrow C \vdash A \vee C$

No Rule Found

```
 1. A \/ B          :PR    +
 2. B -> C          :PR    +
 3.     A           :AS    +
 4.                        ⚠
 5.                        ⚠
 6.     A \/ C             ⚠
 7. --
 8.     B           :AS    +
 9.                        ⚠
10.                        ⚠
11.                        ⚠
12.                        ⚠
13.     A \/ C             ⚠
14. A \/ C                 ⚠
```

Expand

1.	$A \vee B$	PR
2.	$B \rightarrow C$	PR
3.	A	AS
4.	""	
5.	""	
6.	$A \vee C$	
8.	B	AS
9.	""	
10.	""	
11.	""	
12.	""	
13.	$A \vee C$	
14.	$A \vee C$	

Note what happens on line 7

$A \vee B, B \rightarrow C \vdash A \vee C$

No Rule Found

```
 1. A \/ B          :PR          +
 2. B -> C          :PR          +
 3.     A           :AS          +
 4.                              ⚠
 5.                              ⚠
 6.     A \/ C                   ⚠
 7. --
 8.     B           :AS          +
 9.                              ⚠
10.                              ⚠
11.                              ⚠
12.                              ⚠
13.     A \/ C                   ⚠
14. A \/ C                       ⚠
```

Expand

1.	$A \vee B$	PR
2.	$B \rightarrow C$	PR
3.	A	AS
4.	""	
5.	""	
6.	$A \vee C$	
8.	B	AS
9.	""	
10.	""	
11.	""	
12.	""	
13.	$A \vee C$	
14.	$A \vee C$	

There are indents on all the blank lines

$A \lor B, B \to C \vdash A \lor C$

No Rule Found

```
 1. A \/ B          :PR        +
 2. B -> C          :PR        +
 3.     A           :AS        +
 4.     A \/ C      :\/I 3     +
 5. --
 6.     B           :AS        +
 7.                            ⚠
 8.                            ⚠
 9.                            ⚠
10.                            ⚠
11.     A \/ C                 ⚠
12. A \/ C                     ⚠
```

Expand

1.	$A \lor B$	PR
2.	$B \to C$	PR
3.	A	AS
4.	$A \lor C$	$\lor$I 3
6.	B	AS
7.	""	
8.	""	
9.	""	
10.	""	
11.	$A \lor C$	
12.	$A \lor C$	

The left-hand subproof

$A \vee B, B \to C \vdash A \vee C$

No Rule Found

```
1. A \/ B          :PR         +
2. B -> C          :PR         +
3.     A           :AS         +
4.     A \/ C      :\/I 3      +
5. --
6.     B           :AS         +
7.     C           :->E 2, 6   +
8.     A \/ C      :\/I 7      +
9. A \/ C                      ⚠
```

Expand

1.	$A \vee B$	PR
2.	$B \to C$	PR
3.	$\quad A$	AS
4.	$\quad A \vee C$	$\vee$I 3
6.	$\quad B$	AS
7.	$\quad C$	$\to$E 2, 6
8.	$\quad A \vee C$	$\vee$I 7
9.	$A \vee C$	

The right-hand subproof

$A \vee B, B \rightarrow C \vdash A \vee C$

$A \vee B, B \rightarrow C \vdash A \vee C$

```
1. A \/ B           :PR                  +
2. B -> C           :PR                  +
3.     A            :AS                  +
4.     A \/ C       :\/I 3               +
5. --
6.     B            :AS                  +
7.     C            :->E 2, 6            +
8.     A \/ C       :\/I 7               +
9. A \/ C           :\/E 1, 3-4, 6-8     +
```

1.	$A \vee B$	PR
2.	$B \rightarrow C$	PR
3.	A	AS
4.	$A \vee C$	∨I 3
6.	B	AS
7.	C	→E 2, 6
8.	$A \vee C$	∨I 7
9.	$A \vee C$	∨E 1, 3-4, 6-8

Finishing the proof - note the justifications

Proofs From Disjunctions

- That’s the basic structure.
- They are a bit of a pain; I’ve illustrated almost the easiest one I could find.
- But it’s really important to keep track of what your goal is at every point.
- For almost everyone, that’s impossible if you try to just start at line 1 and work to line 9.
- You have to bounce forward and backward in these proofs; just like I’ve done here.

Working Forward from Not

- It's going to be some kind of proof involving $\bot$.
- Whether that's Indirect Proof or $\neg$E isn't always clear, but that's going to be the structure.

A Simple Strategy

- If any of the premises is negated, then assume the opposite of the conclusion and try to derive $\perp$.
- If the conclusion is positive, its opposite is adding a negation.
- If the conclusion is already negated, its opposite is deleting the negation.

A More Complicated Strategy

- Sometimes the simple strategy won’t be optimal.
- Sometimes it will be quicker to do some working forward from the other premises, or backwards from the conclusion.
- But the simple strategy is going to work, even in those cases.

$\neg A \vdash \neg(A \land B)$

No Rule Found

```
1. ~A                 :PR      +
2.                             ⚠
3.                             ⚠
4.                             ⚠
5.                             ⚠
6.                             ⚠
7. ~(A /\ B)                   ⚠
```

1.	$\neg A$	PR
2.	""	
3.	""	
4.	""	
5.	""	
6.	""	
7.	$\neg(A \land B)$	

Premise and Conclusion

¬A ⊢ ¬(A ∧ B)

No Rule Found

```
1. ~A                 :PR        +
2.      A /\ B        :AS        +
3.                               ⚠
4.      !?                       ⚠
5. ~(A /\ B)          :~I        ?
```

1.	¬A	PR
2.	A ∧ B	AS
3.	""	
4.	⊥	
5.	¬(A ∧ B)	¬I

Set up ¬I - note how ⊥ is written

$\neg A \vdash \neg(A \land B)$

No Rule Found

```
1. ~A                   :PR              +
2.     A /\ B           :AS              +
3.     A                :/\E 2           +
4.     !?               :~E 1, 3         +
5. ~(A /\ B)            :~I              ?
```

Expand

1.	$\neg A$	PR
2.	$\quad A \land B$	AS
3.	$\quad A$	$\land$E 2
4.	$\quad \bot$	$\neg$E 1, 3
5.	$\neg(A \land B)$	$\neg$I

Derive the contradiction

$\neg A \vdash \neg(A \wedge B)$

$\neg A \vdash \neg(A \wedge B)$

```
1. ~A                   :PR          +
2.     A /\ B           :AS          +
3.     A                :/\E 2       +
4.     !?               :~E 1, 3     +
5. ~(A /\ B)            :~I 2-4      +
```

1.	$\neg A$	PR
2.	$\quad A \wedge B$	AS
3.	$\quad A$	$\wedge$E 2
4.	$\quad \bot$	$\neg$E 1, 3
5.	$\neg(A \wedge B)$	$\neg$I 2-4

Finish the proof

For Next Time

- I’ll end with two special techniques