305 Lecture 11.3 - Two Basic Results

Brian Weatherson

Plan

- To go over two fundamental results in modal logic.

Associated Reading

- Boxes and Diamonds, section 3.4.

Duality

These two claims are equivalent.

1. $\Box A$
2. $\neg \Diamond \neg A$

Duality

These two claims are equivalent.

1. $\square A$
2. $\neg \Diamond \neg A$

From 1 to 2: If $\square A$ is true at x, then A is true for all y such that xRy. That means there is no y such that xRy and A is not true. That means there is no y such that xRy and $\neg A$ is true. That means $\Diamond\neg A$ is not true at w. That means $\neg \Diamond \neg A$ is true at x.

Duality

These two claims are equivalent.

1. $\Box A$
2. $\neg \Diamond \neg A$

Duality

These two claims are equivalent.

1. $\Box A$
2. $\neg \Diamond \neg A$

From 2 to 1: If $\neg \Diamond \neg A$ is true at x, then $\Diamond\neg A$ is not true at w. So there is no world y such that xRy and $\neg A$ is true at y. So at all worlds y such that xRy, $\neg A$ is not true. So at all worlds y such that xRy, A is true. So $\Box A$ is true at x.

Duality

These two claims are also equivalent, though I will not prove this.

1. $\Diamond A$
2. $\neg \Box \neg A$

Big Picture

These claims are both logically true.

1. $\Box\neg A \leftrightarrow \neg\Diamond A$
2. $\Diamond\neg A \leftrightarrow \neg\Box A$

To move a negation sign outside of a modal operator, either $\Box$ or $\Diamond$, you have to rotate the operator by 45 degrees.

Normality

This sentence is also true no matter what the model looks like, and no matter what sentence A is.

$$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

- Assume it is false at w.
- So $\Box(A \rightarrow B)$ is true at w and $(\Box A \rightarrow \Box B)$ is false at w.
- So $\Box A$ is true at w and $\Box B$ is false at w.
- So at every where y such that wRy, A must be true (since $\Box A$ is true at w), and $A \rightarrow B$ must be true (since $\Box(A \rightarrow B)$ is true at w).
- If A and $A \rightarrow B$ are true at y, B must be true at y as well.
- But this implies that B is true all y such that wRy, contradicting the assumption that $\Box B$ is false at w.

Normality

This principle has a very important role in the history of modal logics.

$$\Box(A \to B) \to (\Box A \to \Box B)$$

Having this be an axiom is one of two conditions on what have come to be called **normal** modal logics.

For Next Time

We’ll talk about what it is for a sentence to be true on a whole model.