305 Lecture 3.6 - Rules for Truth Trees

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Plan

This lecture introduces the rules we use for building up truth trees.

Associated Reading

Boxes and Diamonds, sections 2.2-2.3.

What Rules Do

The rules tell you what new lines to write down given the lines you've already got.

- To some extent they simply have to be memorised.
- But hopefully they are all (except for the rules about $\to$) fairly intuitive.

Rules for ¬

1.	$\mathbb{T}\,\neg A$	
2.	$\mathbb{F}\,A$	$\neg\mathbb{T}$, 1

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- Note that the line numbers are just for illustration, and are arbitrary in two senses.
- First, you apply the rule wherever a sentence like ¬A appears, not just at line 1.
- Second, you don't need to apply the rules immediately, so the successor line could come later than 2.

Rule for true $\wedge$ sentence

1.	$\mathbb{T}\, A \wedge B$	
2.	$\mathbb{T}\, A$	$\wedge\mathbb{T}$, 1
3.	$\mathbb{T}\, B$	$\wedge\mathbb{T}$, 1

When you have a true $\wedge$ sentence, you can write down that the sentences either side of it are true.

Rule for true ∨ sentence

1.	$\mathbb{T}\, A \vee B$		
2.	$\mathbb{T}\, A$	$\mathbb{T}\, B$	$\vee\mathbb{T}, 1$

- When you have a true ∨ sentence, you create two **branches**.
- The way to read the tree is that at least one of the branches must be all true.

Rule for true $\vee$ sentence

$$
\begin{array}{llcl}
1. & & \mathbb{T}\, A \vee B & \\
2. & \mathbb{T}\, A & & \mathbb{T}\, B \qquad \vee\mathbb{T}, 1
\end{array}
$$

- When you have a true $\vee$ sentence, you create two **branches**.
- The way to read the tree is that at least one of the branches must be all true.
- The ‘trunk’ above the branching (in this case just line 1), is part of both branches.
- Branches are inclusive; you are saying that at least one branch is true, not that precisely one is.

Rule for false $\wedge$ sentence

1.	$\mathbb{F}\ A \wedge B$		
2.	$\mathbb{F}\ A$	$\mathbb{F}\ B$	$\wedge\mathbb{F}$, 1

- If an $\wedge$ sentence is false, then we know that one or other (or both) of the sides are false.
- So we create two branches, one where each side is false.

Rule for false ∨ sentence

1.	$\mathbb{F}\ A \vee B$	
2.	$\mathbb{F}\ A$	$\vee\mathbb{F}$, 1
3.	$\mathbb{F}\ B$	$\vee\mathbb{F}$, 1

When you have a false ∨ sentence, you know that each side is false, so you write down that they are both false.

Justifying the rule for false ∨ sentences

Recall the truth table for ∨

A	B	A	∨	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

- The only line where the whole sentence is $\mathbb{F}$ is line 4.
- So if a ∨ sentence is $\mathbb{F}$, we know that we're on line 4.
- And on line 4, both A and B are false.

Rule for false → sentence

1.	$\mathbb{F}\, A \to B$	
2.	$\mathbb{T}\, A$	$\to\mathbb{F}$, 1
3.	$\mathbb{F}\, B$	$\to\mathbb{F}$, 1

When you have a false → sentence, you know that the left side is true and the right side is false, so you write those things down.

Justifying the rule for false → sentences

Recall the truth table for →

A	B	A	→	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

- The only line where the whole sentence is $\mathbb{F}$ is line 2.
- So if a → sentence is $\mathbb{F}$, we know that we're on line 2.
- And on line 4,A is true and B are false.

Rule for true → sentence

$$
\begin{array}{lccr}
1. & \multicolumn{2}{c}{\mathbb{T}\, A \to B} & \\
2. & \mathbb{F}\, A & \mathbb{T}\, B & \to\mathbb{T}, 1
\end{array}
$$

- When you have a true → sentence, you create two **branches**.
- On the first, A is false. That covers lines 3 and 4 of the truth table.
- On the second, B is true. That covers lines 1 and 3 of the truth table.
- Between them, they cover lines 1, 3 and 4 of the truth table.
- And those are the lines where A → B is true.

Next Week

- We will look at some examples of truth trees.
- I find truth trees are much easier in practice than in theory, so if this was all a bit abstract, hopefully it will be more intelligible once we work through some examples.