305 Lecture 11.6 - Frames and Axioms

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Plan

- To introduce the relationship between frames and axioms.

Associated Reading

- Boxes and Diamonds, section 4.1 and 4.2.

Two Way Equivalence

Some conditions on R are strongly tied to dedicated axioms:

- If the condition on R holds, then the axiom is guaranteed to hold.
- If the condition on R does not hold, then there is guaranteed to be a way to make the axiom fail.

We'll illustrate this with the following pair:

- R is transitive
- $\Box A \rightarrow \Box\Box A$

What Are Frames

- Frames are models without the valuations.
- They are the $\langle W, R \rangle$ part of $\langle W, R, V \rangle$.

From R to Axiom

In any frame where R is transitive, any instance of this schema is true at any point in any model on the frame.

- $\Box A \to \Box\Box A$

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- $\Box A \to \Box\Box A$

This is a really strong claim. It says you can pick any value you like for the following three things, and $\Box A \to \Box\Box A$ will be true

1. Any valuation function V you like.
2. Any point w in W that you like.
3. Any substitution instance for A that you like.

From Axiom to R

The claim here is in a sense weaker, but still interesting. It says that if the R in $\langle W, R \rangle$ is not transitive, then there is some way to make $\Box A \rightarrow \Box\Box A$ false.

From Axiom to R

The claim here is in a sense weaker, but still interesting. It says that if the R in $\langle W, R\rangle$ is not transitive, then there is some way to make $\Box A \rightarrow \Box\Box A$ false. That is, if you pick the right values for the following three things:

1. The point w in W;
2. The substitution instance for A
3. The valuation function V;

then $\Box A \rightarrow \Box\Box A$ will be false.

Picking the Substitution Instance

In this case, it isn't that hard to get all three.

1. Let w be a point in W such that for some x, y, wRx and xRy, but not wRy. (Since R is not transitive, we know such a point exists.)
2. Let A be p, i.e., an atomic sentence.
3. Let V make p true at z if and only if wRz

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Then at w, $\Box p$ will be true (since p is true everywhere that w can access), but $\Box\Box p$ will be false. That's because we know that p is false at the point y mentioned in 1.

Reflexivity

The same equivalence holds between

- R is reflexive, i.e., for all x, xRx.
- The axiom $\Box A \to A$ is valid on the frame.

Left-to-right If R is reflexive, then x itself is one of the accessible worlds, so if $\Box A$ is true at x, then A must also be true at x.

Right-to-left If R is not reflexive, then there is a point x such that $\neg xRx$. Let A be true everywhere but x. Then $\Box A$ will be true at x, since A is true at all the points it can ‘see’, but A is false.

Symmetry

The same equivalence holds between

- R is symmetric, i.e., for all x, y, $xRy \to yRx$.
- The axiom $A \to \Box\Diamond A$ is valid on the frame.

Left-to-right If R is symmetric, then everywhere x can see can also see x. So if A is true at x, then $\Diamond A$ is true at all those worlds. So $\Box\Diamond A$ is true at x.

Right-to-left If R is not symmetric, then there is a pair x, y such that $xRy \wedge \neg yRx$. Let A be true only at x. Since $\neg yRx$, $\Diamond A$ is false at y. And since xRy, $\Box\Diamond A$ is false at x.

For Next Time

We’ll talk about the philosophical significance of these results.