305 Lecture 11.2 - Truth in Modal Logic

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Plan

- To talk about what a model for a modal logic is.

Associated Reading

- Boxes and Diamonds, section 3.3 and 3.4.

Worlds

We start with Leibniz's idea that necessity is truth in all possible worlds.

- Leibniz was interested in metaphysical necessity, so we'll have to qualify this a little, but it's a good idea.
- So instead of saying that each proposition simply has a truth value, we'll say that there are many **worlds**, and at each world each proposition has a truth value.
- But don't assume that propositions have the same truth value at each world.
- In one world I might be standing, and in another world I might be sitting.

What Are Worlds

We are well and truly not going to get into the metaphysics of worlds here.

- Indeed, they need not even be anything like possible worlds in the sense that metaphysicians usually care about.
- They might, for instance, be different times.
- All we care about is that they are things at which propositions can be true or false.

Valuations

A valuation function tells us which worlds atomic sentences are true at.

- These can be completely arbitrary; we don't put any restrictions on them.

Truth at a World

We want more generally a function that tells us whether a sentence is true at a particular world.

- For sentences built up using $\wedge$, $\vee$, $\rightarrow$, $\neg$, this is relatively easy.
- We just keep on using truth tables.
- So if at world w, A is true and B is false, then $A \wedge B$ is false and $A \vee B$ is true.

Modal Values

We also need values for these sentences:

- $\Box A$
- $\Diamond A$

It turns out these are more complicated - but not much more complicated.

Accessibility

The last part of our model is an **accessibility** relation between worlds.

- Again, this can be completely arbitrary.
- We don’t yet put any restrictions on it.
- Notably, we don’t assume that it is **reflexive**, **symmetric** or **transitive**

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A lot of relations we care about have one or more of these properties, but not all do. Consider, for example, **admires** as an example of a relation with none of them.

Truth of Modal Formulas

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We get back the Leibnizian idea that necessity is truth in all possible worlds if we assume the accessibility relation is the universal relation, i.e., xRy for all x, y.

Metaphysical Necessity

On this Leibnizian model, where all worlds can access all worlds, iterated modalities are rather uninteresting. These three sentences are true in the same worlds/models.

1. $\Box A$
2. $\Box \Box A$
3. $\Diamond \Box A$

That's because if $\Box A$ is true at any world, then it is true at all worlds. In the general case, where we do not assume that R is universal, these are not equivalent.

For Next Time

We'll talk about the relationship between boxes and diamonds.