

$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$  (in S5)

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Build a Tableau

To Check Whether it is Valid

# Hypothesis

$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$  is a theorem of S5.

- We are going to use simplified S5 because normal S5 is really painful for cases like this one.

$$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$$

1.                      1,  $\neg \Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$                       Assumption

Start with it being false at 1.

$$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$$

- |    |  |                     |
|----|--|---------------------|
| 1. | 1, $\mathbb{F}$ $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$ ✓ | Assumption          |
| 2. | 1, $\mathbb{F}$ $\Box(\Box A \rightarrow B)$                                   | $\vee\mathbb{F}, 1$ |
| 3. | 1, $\mathbb{F}$ $\Box(\Box B \rightarrow A)$                                   | $\vee\mathbb{F}, 1$ |

False  $\vee$  sentences have both sides false.

$$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$$

- |    |  |             |
|----|--|-------------|
| 1. | 1, $\neg \Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$ ✓ | Assumption  |
| 2. | 1, $\neg \Box(\Box A \rightarrow B)$ ✓                                 | $\vee F, 1$ |
| 3. | 1, $\neg \Box(\Box B \rightarrow A)$ ✓                                 | $\vee F, 1$ |
| 4. | 2, $\neg \Box A \rightarrow B$   | $\Box F, 2$ |
| 5. | 3, $\neg \Box B \rightarrow A$   | $\Box F, 3$ |

Both of these false  $\Box$  sentences have to be made false somehow.

$$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$$

1.	1, $\mathbb{F}$ $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A) \checkmark$	Assumption
2.	1, $\mathbb{F}$ $\Box(\Box A \rightarrow B) \checkmark$	$\vee\mathbb{F}, 1$
3.	1, $\mathbb{F}$ $\Box(\Box B \rightarrow A) \checkmark$	$\vee\mathbb{F}, 1$
4.	2, $\mathbb{F}$ $\Box A \rightarrow B \checkmark$	$\Box\mathbb{F}, 2$
5.	3, $\mathbb{F}$ $\Box B \rightarrow A \checkmark$	$\Box\mathbb{F}, 3$
6.	2, $\mathbb{T}$ $\Box A$	$\rightarrow\mathbb{F}, 4$
7.	2, $\mathbb{F}$ $B$	$\rightarrow\mathbb{F}, 4$
8.	3, $\mathbb{T}$ $\Box B$	$\rightarrow\mathbb{F}, 5$
9.	3, $\mathbb{F}$ $A$	$\rightarrow\mathbb{F}, 5$

- False  $\rightarrow$  sentences mean left false; right true.

$$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$$

1.	1, $\mathbb{F}$ $\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A) \checkmark$	Assumption
2.	1, $\mathbb{F}$ $\Box(\Box A \rightarrow B) \checkmark$	$\vee\mathbb{F}, 1$
3.	1, $\mathbb{F}$ $\Box(\Box B \rightarrow A) \checkmark$	$\vee\mathbb{F}, 1$
4.	1.1, $\mathbb{F}$ $\Box A \rightarrow B \checkmark$	$\Box\mathbb{F}, 2$
5.	1.2, $\mathbb{F}$ $\Box B \rightarrow A \checkmark$	$\Box\mathbb{F}, 3$
6.	2, $\mathbb{T}$ $\Box A$	$\rightarrow\mathbb{F}, 4$
7.	2, $\mathbb{F}$ $B$	$\rightarrow\mathbb{F}, 4$
8.	3, $\mathbb{T}$ $\Box B$	$\rightarrow\mathbb{F}, 5$
9.	3, $\mathbb{F}$ $A$	$\rightarrow\mathbb{F}, 5$
10.	3, $\mathbb{T}$ $A$	$\Box\mathbb{T}, 6$
	x	

- And if  $\Box A$  is true, then  $A$  is true everywhere, including 3.
- This closes the tree, so it is a theorem of S5.