305 Lecture 3.4 - Features of Validity

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Plan

This lecture finishes our discussion of truth tables by looking some properties validity has in the truth table system.

Associated Reading

forall x, chapter 12, sections 12.5-12.7.

The Rules

- An argument is **invalid** if there is a row on the truth table where all the premises are true and the conclusion is false. (Roughly!)
- It is **valid** if all the rows where the premises are all true, the conclusion is true as well.

A Relevance Failure

Is this argument valid?

$$\begin{array}{rl} & A \\ \therefore & B \lor \neg B \end{array}$$

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Is this argument valid?

$$\begin{array}{rl} & A \\ \therefore & B \lor \neg B \end{array}$$

Yes!

- There is no line where the conclusion is false.
- So there are no lines where the premise is true and the conclusion false.
- So it is not invalid, i.e., it is valid.

Terminology

Say a **valuation** is a function v from sentences to $\{\mathbb{T}, \mathbb{F}\}$ satisfying these constraints.

1. $v(\neg A) = \mathbb{T}$ if $v(A) = \mathbb{F}$, and $v(\neg A) = \mathbb{F}$ otherwise.
2. $v(A \vee B) = \mathbb{T}$ if $V(A) = \mathbb{T}$ or $v(B) = \mathbb{T}$, and $v(A \vee B) = \mathbb{F}$ otherwise.
3. $v(A \wedge B) = \mathbb{T}$ if $V(A) = \mathbb{T}$ and $v(B) = \mathbb{T}$, and $v(A \wedge B) = \mathbb{F}$ otherwise.
4. $v(A \to B) = \mathbb{T}$ if $V(A) = \mathbb{F}$ or $v(B) = \mathbb{T}$, and $v(A \to B) = \mathbb{F}$ otherwise.

Restating

- An argument is valid relative to a class of valuations V iff any valuation $\mathsf{v} \in \mathsf{V}$ that makes all the premises $\mathbb{T}$ also makes the conclusion $\mathbb{T}$.

Restating

- An argument is valid relative to a class of valuations V iff any valuation $v \in V$ that makes all the premises $\mathbb{T}$ also makes the conclusion $\mathbb{T}$.
- An argument is truth functionally valid when the class V is the class of valuations satisfying the constraints on the previous slide.

Very Technical Terminology

- I'll use $\Gamma \vDash A$ to mean that the argument with premises Γ and conclusion A is valid in this sense - i.e., all valuations that make all of Γ come out $\mathbb{T}$ also make A come out $\mathbb{T}$.
- The double bar in $\vDash$ is to represent that this is a kind of validity defined in terms of valuations (or, as we'll start calling them, models), and not proofs.
- For purposes of 305, the difference between $\vdash$ and $\vDash$ is not important, and if this is the last logic/mathematical philosophy course you plan to take, you don't have to worry about this.
- But I like being pedantic even when it isn't relevant to the course.

Closure

If $\Gamma \models A$ and $\Gamma \models A \to B$ then $\Gamma \models B$

Closure

$$\text{If } \Gamma \vDash A \text{ and } \Gamma \vDash A \to B \text{ then } \Gamma \vDash B.$$

Proof: Assume this is false. So assume that $\Gamma \nvDash B$. So there is a valuation function v that makes everything in Γ come out $\mathbb{T}$ and B come out $\mathbb{F}$

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Proof: Assume this is false. So assume that $\Gamma \nvDash B$. So there is a valuation function v that makes everything in Γ come out $\mathbb{T}$ and B come out $\mathbb{F}$. Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$

Closure

If $\Gamma \vDash A$ and $\Gamma \vDash A \to B$ then $\Gamma \vDash B$.

Proof: Assume this is false. So assume that $\Gamma \nvDash B$. So there is a valuation function v that makes everything in Γ come out $\mathbb{T}$ and B come out $\mathbb{F}$. Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. If $v(A) = \mathbb{T}$, then $v(A \to B) = \mathbb{F}$, contradicting $\Gamma \vDash A \to B$

Closure

If $\Gamma \vDash A$ and $\Gamma \vDash A \rightarrow B$ then $\Gamma \vDash B$.

Proof: Assume this is false. So assume that $\Gamma \nvDash B$. So there is a valuation function v that makes everything in Γ come out $\mathbb{T}$ and B come out $\mathbb{F}$. Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. If $v(A) = \mathbb{T}$, then $v(A \rightarrow B) = \mathbb{F}$, contradicting $\Gamma \vDash A \rightarrow B$. If $v(A) = \mathbb{F}$, then v is a counterexample to $\Gamma \vDash A$, but we know $\Gamma \vDash A$ is true. Either way, such a v cannot exist, so $\Gamma \vDash B$ is true.

Monotony

If $\Gamma \vDash A$, and $\Gamma \subset \Delta$, then $\Delta \vDash A$.

That is, adding premises can't turn an argument from being valid to invalid.

Monotony Proof

- Assume that for all $B \in \Delta$, $v(B) = \mathbb{T}$.
- We need to prove that $v(A) = \mathbb{T}$.

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- Assume $C \in \Gamma$.
- Then $C \in \Delta$, since $\Gamma \subset \Delta$.

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- So by hypothesis, v(C) = $\mathbb{T}$, since everything in Δ is $\mathbb{T}$.

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- So v is such that everything in Γ is $\mathbb{T}$.

Monotony Proof

- Assume that for all $B \in \Delta$, $v(B) = \mathbb{T}$.
- We need to prove that $v(A) = \mathbb{T}$.
- Assume $C \in \Gamma$.
- Then $C \in \Delta$, since $\Gamma \subset \Delta$.
- So by hypothesis, $v(C) = \mathbb{T}$, since everything in Δ is $\mathbb{T}$.
- So v is such that everything in Γ is $\mathbb{T}$.
- And since $\Gamma \vDash A$, that implies $v(A) = \mathbb{T}$, as required.

Monotony Commentary

- This idea, that adding premises doesn't destroy validity, only works for logical arguments.
- It isn't true for good arguments in general.

Tweety the First

Tweety is a bird.

$\therefore$ Tweety flies.

That's a perfectly good, though not logically valid, argument.

Tweety the Second

Tweety is a bird.
Tweety is black and white, lives in Antarctica, and lays large eggs.
$\therefore$ Tweety flies.

That's not a very good argument!

Transitivity

If $\Gamma \vDash A$ and $\Delta \cup A \vDash B$ then $\Gamma \cup \Delta \vDash B$

If some premises entail A, and some other premises plus A entail B, then the two sets of premises between them entail B.

Transitivity

If $\Gamma \vDash A$ and $\Delta \cup A \vDash B$ then $\Gamma \cup \Delta \vDash B$

If some premises entail A, and some other premises plus A entail B, then the two sets of premises between them entail B. This is crucial for being able to chain together lines of reasoning.

Transitivity Proof

- Assume that for all C $\in \Gamma \cup \Delta$, v(C) = $\mathbb{T}$.
- We need to prove v(B) = $\mathbb{T}$.

Transitivity Proof

- Assume that for all $C \in \Gamma \cup \Delta$, $v(C) = \mathbb{T}$.
- We need to prove $v(B) = \mathbb{T}$.
- Since everything in Γ is $\mathbb{T}$ according to v, and $\Gamma \vDash A$, it follows that $v(A) = \mathbb{T}$.

Transitivity Proof

- Assume that for all $C \in \Gamma \cup \Delta$, $v(C) = \mathbb{T}$.
- We need to prove $v(B) = \mathbb{T}$.
- Since everything in Γ is $\mathbb{T}$ according to v, and $\Gamma \vDash A$, it follows that $v(A) = \mathbb{T}$.
- Since everything in Δ is $\mathbb{T}$ according to v, and A is $\mathbb{T}$ according to v, and $\Delta \cup A \vDash B$, it follows that $v(B) = \mathbb{T}$, as required.

Deduction Theorem

This is why we define $\rightarrow$ the way we do.

> $\Gamma \models A \rightarrow B$ if and only if $\Gamma \cup A \models B$.

Note that there are two claims here - one each direction. We need to prove each.

Deduction Theorem Left-to-Right

- Assume $\Gamma \vDash A \to B$, and prove $\Gamma \cup A \vDash B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $v(A) = \mathbb{T}$, and aim to prove $v(B) = \mathbb{T}$.

Deduction Theorem Left-to-Right

- Assume $\Gamma \vDash A \to B$, and prove $\Gamma \cup A \vDash B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $v(A) = \mathbb{T}$, and aim to prove $v(B) = \mathbb{T}$.
- Since $\Gamma \vDash A \to B$ and $v(C) = \mathbb{T}$ for all $C \in \Gamma$, it follows that $v(A \to B) = \mathbb{T}$.

Deduction Theorem Left-to-Right

- Assume $\Gamma \vDash A \rightarrow B$, and prove $\Gamma \cup A \vDash B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $v(A) = \mathbb{T}$, and aim to prove $v(B) = \mathbb{T}$.
- Since $\Gamma \vDash A \rightarrow B$ and $v(C) = \mathbb{T}$ for all $C \in \Gamma$, it follows that $v(A \rightarrow B) = \mathbb{T}$.
- Since $v(A \rightarrow B) = \mathbb{T}$ and $v(A) = \mathbb{T}$, it must be that $v(B) = \mathbb{T}$, since that's the only line on the truth table where $A \rightarrow B$ and A are both $\mathbb{T}$.

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \mathbb{T}$.

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \rightarrow B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and prove $v(A \rightarrow B) = \mathbb{T}$.
- Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. Take each case in turn.

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \mathbb{T}$.
- Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. Take each case in turn.
- If $v(A) = \mathbb{T}$, then since $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \mathbb{T}$

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \mathbb{T}$.
- Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. Take each case in turn.
- If $v(A) = \mathbb{T}$, then since $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \mathbb{T}$, so $v(A \to B) = \mathbb{T}$

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \rightarrow B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and prove $v(A \rightarrow B) = \mathbb{T}$.
- Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. Take each case in turn.
- If $v(A) = \mathbb{T}$, then since $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \mathbb{T}$, so $v(A \rightarrow B) = \mathbb{T}$.
- If $v(A) = \mathbb{F}$, it follows directly that $v(A \rightarrow B) = \mathbb{T}$

Deduction Theorem Right-to-Left

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \to B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and prove $v(A \to B) = \mathbb{T}$.
- Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. Take each case in turn.
- If $v(A) = \mathbb{T}$, then since $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \mathbb{T}$, so $v(A \to B) = \mathbb{T}$.
- If $v(A) = \mathbb{F}$, it follows directly that $v(A \to B) = \mathbb{T}$.
- Either way, $v(A \to B) = \mathbb{T}$ as required.

Deduction Theorem Comments

- This is a striking result.
- It shows that proving $A \to B$ is just the same as proving B, assuming you're allowed to add A as an extra assumption.
- And that's a good thing, intuitively. That is how we prove conditionals.
- But it only works if you have the (very odd looking) truth table that we're using for $\to$.
- This is the main reason for thinking, despite it's odd appearance, that this truth table is the right one for $\to$.

For Next Time

We will start working on a different way to analyse arguments: truth trees.