$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$ **(in S4)**

Build a Tableau

To Check Whether it is Valid

Hypothesis

$\Box(\Box A \to B) \lor \Box(\Box B \to A)$ is a theorem of S4.

- So we can use all the rules, plus the special rules for T and 4.

$\Box(\Box A \to B) \lor \Box(\Box B \to A)$

1.	$1, \mathbb{F}$	$\Box(\Box A \to B) \lor \Box(\Box B \to A)$	Assumption

Start with it being false at 1.

$\Box(\Box A \to B) \lor \Box(\Box B \to A)$

1.	$1, \mathbb{F}$	$\Box(\Box A \to B) \lor \Box(\Box B \to A)$ ✓	Assumption
2.	$1, \mathbb{F}$	$\Box(\Box A \to B)$	$\lor\mathbb{F}$, 1
3.	$1, \mathbb{F}$	$\Box(\Box B \to A)$	$\lor\mathbb{F}$, 1

False $\lor$ sentences have both sides false.

$\Box(\Box A \to B) \lor \Box(\Box B \to A)$

1.	$1, \mathbb{F}$	$\Box(\Box A \to B) \lor \Box(\Box B \to A)$ ✓	Assumption
2.	$1, \mathbb{F}$	$\Box(\Box A \to B)$ ✓	$\lor\mathbb{F}$, 1
3.	$1, \mathbb{F}$	$\Box(\Box B \to A)$ ✓	$\lor\mathbb{F}$, 1
4.	$1.1, \mathbb{F}$	$\Box A \to B$	$\Box\mathbb{F}$, 2
5.	$1.2, \mathbb{F}$	$\Box B \to A$	$\Box\mathbb{F}$, 3

Both of these false $\Box$ sentences have to be made false somehow.

$\Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$

1.	$1, \mathbb{F} \quad \Box(\Box A \rightarrow B) \vee \Box(\Box B \rightarrow A)$ ✓	Assumption
2.	$1, \mathbb{F} \quad \Box(\Box A \rightarrow B)$ ✓	$\vee\mathbb{F}, 1$
3.	$1, \mathbb{F} \quad \Box(\Box B \rightarrow A)$ ✓	$\vee\mathbb{F}, 1$
4.	$1.1, \mathbb{F} \quad \Box A \rightarrow B$ ✓	$\Box\mathbb{F}, 2$
5.	$1.2, \mathbb{F} \quad \Box B \rightarrow A$ ✓	$\Box\mathbb{F}, 3$
6.	$1.1, \mathbb{T} \quad \Box A$	$\rightarrow\mathbb{F}, 4$
7.	$1.1, \mathbb{F} \quad B$	$\rightarrow\mathbb{F}, 4$
8.	$1.2, \mathbb{T} \quad \Box B$	$\rightarrow\mathbb{F}, 5$
9.	$1.2, \mathbb{F} \quad A$	$\rightarrow\mathbb{F}, 5$

- False $\rightarrow$ sentences mean left false; right true.
- Note we have no special rules for 4 or T triggered yet, because no true $\Box$ or false $\Diamond$, but that's about to change.

$\Box(\Box A \to B) \lor \Box(\Box B \to A)$

1.	$1, \mathbb{F} \quad \Box(\Box A \to B) \lor \Box(\Box B \to A)$ ✓	Assumption
2.	$1, \mathbb{F} \quad \Box(\Box A \to B)$ ✓	$\lor\mathbb{F}$, 1
3.	$1, \mathbb{F} \quad \Box(\Box B \to A)$ ✓	$\lor\mathbb{F}$, 1
4.	$1.1, \mathbb{F} \quad \Box A \to B$ ✓	$\Box\mathbb{F}$, 2
5.	$1.2, \mathbb{F} \quad \Box B \to A$ ✓	$\Box\mathbb{F}$, 3
6.	$1.1, \mathbb{T} \quad \Box A$	$\to\mathbb{F}$, 4
7.	$1.1, \mathbb{F} \quad B$	$\to\mathbb{F}$, 4
8.	$1.2, \mathbb{T} \quad \Box B$	$\to\mathbb{F}$, 5
9.	$1.2, \mathbb{F} \quad A$	$\to\mathbb{F}$, 5
10.	$1.1, \mathbb{T} \quad A$	$T\Box 6$
11.	$1.2, \mathbb{T} \quad B$	$T\Box 8$

T means true $\Box$ sentences are true unboxed.

$\Box(\Box A \to B) \lor \Box(\Box B \to A)$

1.	$1, \mathbb{F} \quad \Box(\Box A \to B) \lor \Box(\Box B \to A)$ ✓	Assumption
2.	$1, \mathbb{F} \quad \Box(\Box A \to B)$ ✓	$\lor\mathbb{F}$, 1
3.	$1, \mathbb{F} \quad \Box(\Box B \to A)$ ✓	$\lor\mathbb{F}$, 1
4.	$1.1, \mathbb{F} \quad \Box A \to B$ ✓	$\Box\mathbb{F}$, 2
5.	$1.2, \mathbb{F} \quad \Box B \to A$ ✓	$\Box\mathbb{F}$, 3
6.	$1.1, \mathbb{T} \quad \Box A$	$\to\mathbb{F}$, 4
7.	$1.1, \mathbb{F} \quad B$	$\to\mathbb{F}$, 4
8.	$1.2, \mathbb{T} \quad \Box B$	$\to\mathbb{F}$, 5
9.	$1.2, \mathbb{F} \quad A$	$\to\mathbb{F}$, 5
10.	$1.1, \mathbb{T} \quad A$	$T\Box$ 6
11.	$1.2, \mathbb{T} \quad B$	$T\Box$ 8

But now we’ve applied all the rules, and the tableau doesn’t close. So not a theorem.

A Model

- Three worlds, $w_1, w_{1.1}, w_{1.2}$
- The accessibility relations are $w_1 R w_{1.1}, w_1 R w1.2, w_1 R w_1, w_{1.1} R w_{1.1}, w_{1.2} R w_{1.2}$.
- The first two from the tree, the last three from T. Adding 4 doesn't require anything in this case.
- A is true at $w_{1.1}$ and false at $w_{1.2}$.
- B is true at $w_{1.2}$ and false at $w_{1.1}$.
- It doesn't matter what values the atomics take at w_1.
- So $\Box A$ will be true at $w_{1.1}$, while B is false.
- And $\Box B$ will be true at $w_{1.2}$, while A is false.
- So the original sentence fails at w_1.