305 Lecture 5.6 - Two Special Cases

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Plan

This lecture looks at two tricky cases, and how the proofs for them work.

Associated Reading

forall x, chapter 17.

Disjunctive Syllogism

1. $A \vee B$
2. $\neg A$
3. So, B

That looks pretty good; let's try proving it.

A ∨ B, ¬A ⊢ B

No Rule Found

1. A \/ B	:PR	+
2. ~A	:PR	+
3.		⚠
4.		⚠
5.		⚠
6.		⚠
7.		⚠
8. B		⚠

1.	A ∨ B	PR
2.	¬A	PR
3.	""	
4.	""	
5.	""	
6.	""	
7.	""	
8.	B	

List premises and conclusion

$A \vee B, \neg A \vdash B$

No Rule Found

```
 1. A \/ B                :PR          +
 2. ~A                    :PR          +
 3.      A                :AS          +
 4.                                    ⚠
 5.                                    ⚠
 6.                                    ⚠
 7.                                    ⚠
 8.      B                             ⚠
 9. --
10.      B                             ⚠
11.      B                :R           ?
12. B                                  ⚠
```

Expand

1.	$A \vee B$	PR
2.	$\neg A$	PR
3.	A	AS
4.	""	
5.	""	
6.	""	
7.	""	
8.	B	
10.	B	
11.	B	R
12.	B	

We have a $\vee$ premise, so set up $\vee$E

$A \vee B, \neg A \vdash B$

No Rule Found

```
 1. A \/ B                :PR          +
 2. ~A                    :PR          +
 3.      A                :AS          +
 4.                                    ⚠
 5.                                    ⚠
 6.                                    ⚠
 7.                                    ⚠
 8.      B                             ⚠
 9. --
10.      B                             ⚠
11.      B                :R           ?
12. B                                  ⚠
```

Expand

1.	$A \vee B$	PR
2.	$\neg A$	PR
3.	A	AS
4.	""	
5.	""	
6.	""	
7.	""	
8.	B	
10.	B	
11.	B	R
12.	B	

I've already added the rule for the second half

A ∨ B, ¬A ⊢ B

No Rule Found

```
 1. A \/ B                  :PR                +
 2. ~A                      :PR                +
 3.      A                  :AS                +
 4.          ~B                                ⚠
 5.                                            ⚠
 6.                                            ⚠
 7.          !?                                ⚠
 8.      B                  :IP                ?
 9. --
10.      B                                     ⚠
11.      B                  :R                 ?
12. B                                          ⚠
```

Expand

1.	A ∨ B	PR
2.	¬A	PR
3.	A	AS
4.	¬B	
5.	""	
6.	""	
7.	⊥	
8.	B	IP
10.	B	
11.	B	R
12.	B	

A tricky move - we have a ¬ premise so we'll need Indirect Proof

$A \vee B, \neg A \vdash B$

No Rule Found

```
 1. A \/ B                :PR                +
 2. ~A                    :PR                +
 3.      A                :AS                +
 4.           ~B          :AS                +
 5.           !?          :~E 2, 3           +
 6.      B                :IP                ?
 7. --
 8.      B                                   ⚠
 9.      B                :R                 ?
10. B                                        ⚠
```

Expand

1.	$A \vee B$	PR
2.	$\neg A$	PR
3.	A	AS
4.	$\neg B$	AS
5.	$\bot$	$\neg$E 2, 3
6.	B	IP
8.	B	
9.	B	R
10.	B	

And the contradiction comes very easily

$A \vee B, \neg A \vdash B$

No Rule Found

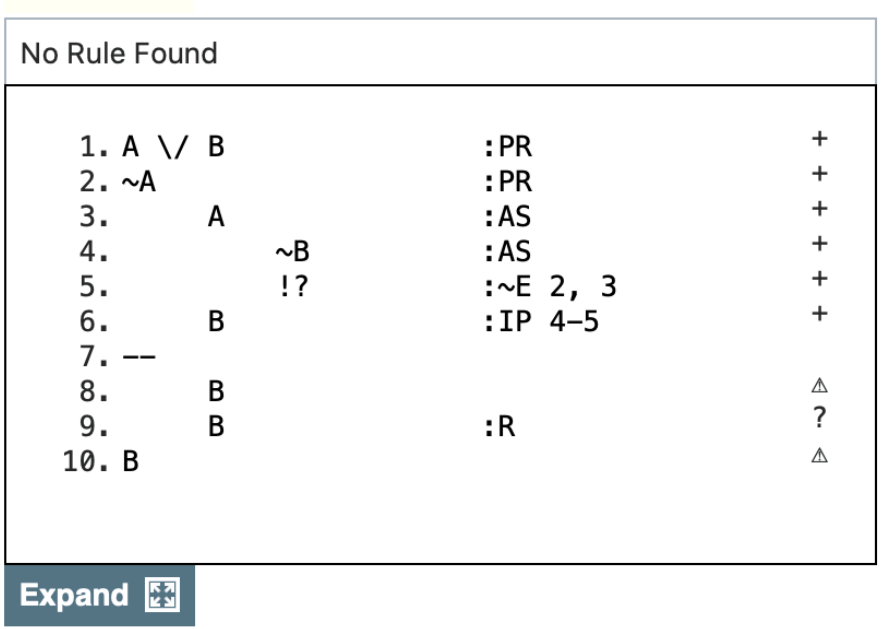
```
 1. A \/ B                :PR                +
 2. ~A                    :PR                +
 3.      A                :AS                +
 4.           ~B          :AS                +
 5.           !?          :~E 2, 3           +
 6.      B                :IP                ?
 7. --
 8.      B                                   ⚠
 9.      B                :R                 ?
10. B                                        ⚠
```

Expand

1.	$A \vee B$	PR
2.	$\neg A$	PR
3.	A	AS
4.	$\neg B$	AS
5.	$\bot$	$\neg$E 2, 3
6.	B	IP
8.	B	
9.	B	R
10.	B	

Maybe too easily? Should we have been forced to use B?

$A \vee B, \neg A \vdash B$



1.	$A \vee B$	PR
2.	$\neg A$	PR
3.	A	AS
4.	$\neg B$	AS
5.	$\bot$	$\neg$E 2, 3
6.	B	IP 4-5
8.	B	
9.	B	R
10.	B	

Since $\neg B$ got to contradiction, we can infer B

$A \vee B, \neg A \vdash B$

No Rule Found

```
 1. A \/ B                :PR                +
 2. ~A                    :PR                +
 3.      A                :AS                +
 4.          ~B           :AS                +
 5.          !?           :~E 2, 3           +
 6.      B                :IP 4-5            +
 7. --
 8.      B                :AS                +
 9.      B                :R 8               +
10. B                                        ⚠
```

1.	$A \vee B$	PR
2.	$\neg A$	PR
3.	A	AS
4.	$\neg B$	AS
5.	$\bot$	$\neg$E 2, 3
6.	B	IP 4-5
8.	B	AS
9.	B	R 8
10.	B	

Now fill in line number on second subproof

A ∨ B, ¬A ⊢ B

A ∨ B , ¬A ⊢ B

```
 1. A \/ B                  :PR                    +
 2. ~A                      :PR                    +
 3.      A                  :AS                    +
 4.          ~B             :AS                    +
 5.          !?             :~E 2, 3               +
 6.      B                  :IP 4-5                +
 7. --
 8.      B                  :AS                    +
 9.      B                  :R 8                   +
10. B                       :\/E 1, 3-6, 8-9       +
```

1.	A ∨ B	PR
2.	¬A	PR
3.	A	AS
4.	¬B	AS
5.	⊥	¬E 2, 3
6.	B	IP 4-5
8.	B	AS
9.	B	R 8
10.	B	∨E 1, 3-6, 8-9

And now we're done.

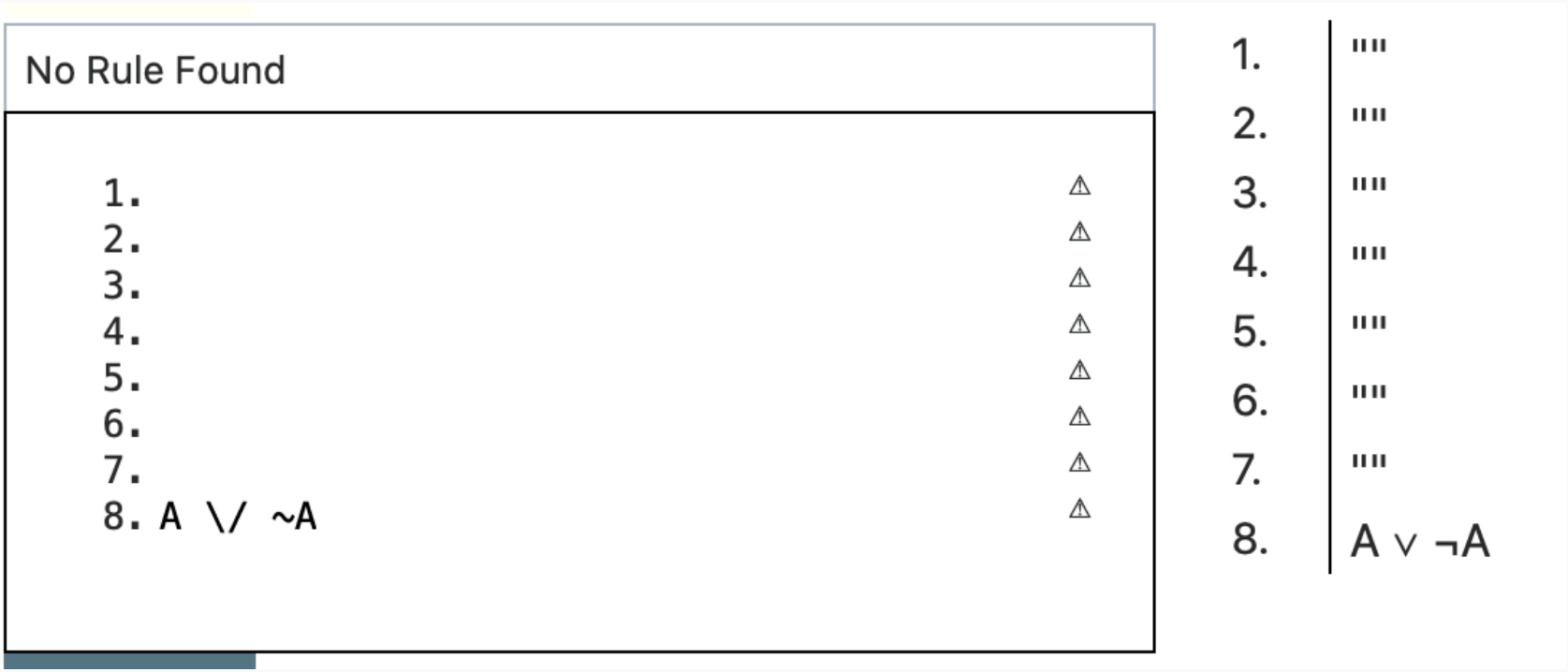
Second example

$\vdash A \vee \neg A$

How to prove things from zero premises

1. If they are a conditional, set up →I. The left hand side will work just like a premise.
2. If they are not a conditional, go for Indirect Proof.

⊢ A ∨ ¬A



Writing out conclusion - but not premise because there is none

⊢ A ∨ ¬A

No Rule Found

```
1.     ~(A \/ ~A)     :AS                    +
2.                                           ⚠
3.                                           ⚠
4.                                           ⚠
5.                                           ⚠
6.                                           ⚠
7.     !?                                    ⚠
8. A \/ ~A            :IP                    ?
```

1.	¬(A ∨ ¬A)	AS
2.	"	
3.	"	
4.	"	
5.	"	
6.	"	
7.	⊥	
8.	A ∨ ¬A	IP

Setting up indirect proof

⊢ A ∨ ¬A

No Rule Found

```
 1.     ~(A \/ ~A)     :AS                    +
 2.         A          :AS                    +
 3.         A \/ ~A    :\/I 2                 +
 4.         !?         :~E 1, 3               +
 5.     ~A             :~I 2-4                +
 6.                                           ⚠
 7.                                           ⚠
 8.                                           ⚠
 9.     !?                                    ⚠
10. A \/ ~A            :IP                    ?
```

Expand

1.	¬(A ∨ ¬A)	AS
2.	A	AS
3.	A ∨ ¬A	∨I 2
4.	⊥	¬E 1, 3
5.	¬A	¬I 2-4
6.	""	
7.	""	
8.	""	
9.	⊥	
10.	A ∨ ¬A	IP

A very tricky move - extracting something from negated disjunction

Negated Disjunctions

- The move from the previous slide is more or less compulsory.
- The only way to get something out of $\neg(X \vee Y)$ is to assume X, get a contradiction (via deriving $X \vee Y$), and then use $\neg$I to get $\neg X$.
- It's a pain, and it's just a move you have to learn.
- It might be my single least favorite part of this system.

⊢ A ∨ ¬A

No Rule Found

```
1.     ~(A \/ ~A)     :AS                +
2.         A          :AS                +
3.         A \/ ~A    :\/I 2             +
4.         !?         :~E 1, 3           +
5.     ~A             :~I 2-4            +
6.     A \/ ~A        :\/I 5             +
7.     !?             :~E 1, 6           +
8. A \/ ~A            :IP                ?
```

1.	¬(A ∨ ¬A)	AS
2.	A	AS
3.	A ∨ ¬A	∨I 2
4.	⊥	¬E 1, 3
5.	¬A	¬I 2-4
6.	A ∨ ¬A	∨I 5
7.	⊥	¬E 1, 6
8.	A ∨ ¬A	IP

If we have ¬A, we are basically home

⊢ A ∨ ¬A

```
∅ ⊢ A ∨ ¬A

 1.     ~(A \/ ~A)     :AS                    +
 2.         A          :AS                    +
 3.         A \/ ~A    :\/I 2                 +
 4.         !?         :~E 1, 3               +
 5.     ~A             :~I 2-4                +
 6.     A \/ ~A        :\/I 5                 +
 7.     !?             :~E 1, 6               +
 8. A \/ ~A            :IP 1-7                +
```

1.	¬(A ∨ ¬A)	AS
2.	A	AS
3.	A ∨ ¬A	∨I 2
4.	⊥	¬E 1, 3
5.	¬A	¬I 2-4
6.	A ∨ ¬A	∨I 5
7.	⊥	¬E 1, 6
8.	A ∨ ¬A	IP 1-7

Filling in line numbers - note the subproof is the entire proof to this point

Challenge Problem

1. Prove $\vdash ((A \to B) \to A) \to A$
2. See if you can complete that proof in under 25 lines.

That one, like $\vdash A \vee \neg A$, is a sign that the strategies in 17.1 and 17.2 work 98% of the time, but not 100% of the time.

For Next Time

No recorded lectures next week. We'll just go over the proofs in the weekly assignment, and do some revision.