305 Lecture 5.5 - Strategies 2: Working Forwards

Brian Weatherson



This lecture discusses strategies for constructing proofs that involve working forwards.

Associated Reading

forall x, section 17.2.



Big Idea: Plan to use the Elimination rules on the connectives in the premises.



When one of the premises is of the form X \wedge y, you'll almost certainly need to apply $\wedge E$ to get X and Y.

Slightly Trickier: If

 When one of the premises is of the form X → Y, you'll almost certainly need to apply →E.

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- And that means you'll need X.

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- When one of the premises is of the form X → Y, you'll almost certainly need to apply →E.
- And that means you'll need X.
- But in practice it's hard to tell in advance whether you'll prove X, or have it as the start of a subproof, or something else.

Working forward from Or

When one of the premises is $X \vee Y$ there is a clear(ish) strategy.

- 1. Find a target conclusion C.
- 2. Do a subproof from X to C.
- 3. Do a subproof from Y to C.
- 4. Conclude C by ∨E.

Working forward from Or

Why clear-ish?

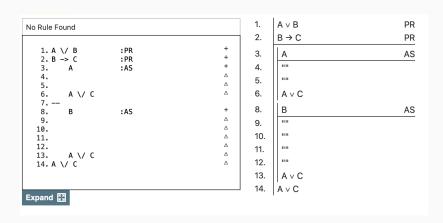
- Because it isn't always true that the target here should be the conclusion of the whole argument.
- Sometimes it is optimal to do a step or two of working backwards first.
- But if you want a simple rule to go by, the best is to do what's on the previous slide with C as the conclusion of the whole argument.

$\mathsf{A} \vee \mathsf{B}, \mathsf{B} \to \mathsf{C} \vdash \mathsf{A} \vee \mathsf{C}$

No Rule Found			1.	A v B	PR
			2.	$B \rightarrow C$	PR
1. A \/ B	:PR	+	3.	ш	
2. B -> C 3.	:PR	+ A	4.	nn	
4.		Δ	5.		
5. 6.		Δ.	6.		
7.		Δ	7.	nn	
8. 9. A \/ C		Δ.	8.		
3.4 () 4			9.	A v C	
				1	

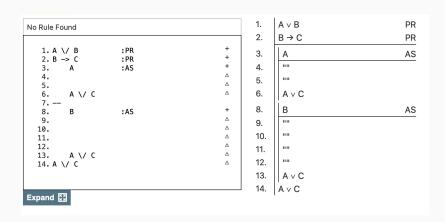
Write Out Premises and Conclusion

$A \vee B, B \rightarrow C \vdash A \vee C$



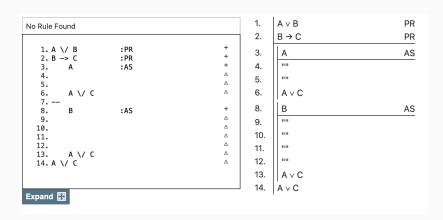
Set Up ∨E

$A \vee B, B \rightarrow C \vdash A \vee C$



Note what happens on line 7

$A \vee B, B \rightarrow C \vdash A \vee C$



There are indents on all the blank lines

$\mathsf{A} \vee \mathsf{B}, \mathsf{B} \to \mathsf{C} \vdash \mathsf{A} \vee \mathsf{C}$

No Rule Found			1.	A v B	PR
			2.	$B \rightarrow C$	PR
1. A \/ B 2. B -> C	:PR :PR	+ +	3.	A	AS
3. A	:AS	+	4.	A v C	vI 3
3. A 4. A \/ C 5	:\/I 3	+	6.	В	AS
6. B	:AS	+ _	7.		
7. 8. 9.		Δ	8.	nn	
9. 10.		Δ.	9.	IIII	
11. A \/ C		Δ	10.	""	
12. A \/ C		Δ	11.	A v C	
			12.	A v C	
Expand 🔯					

The left-hand subproof

$A \lor B, B \to C \vdash A \lor C$

1. A \/ B	:PR	+
2. B -> C	:PR	+
3. A	:AS	+
	:\/I 3	+
5	. 0 - 0	
6. B	:AS	+
7. C	:->E 2, 6	+
		+
	: (/1 /	Δ.
		213
51 /		
8. A \/ C 9. A \/ C	:\/I 7	

1.	A v B	PF
2.	$B \rightarrow C$	PR
3.	A	AS
4.	A v C	VI 3
6.	В	AS
7.	С	→E 2, 6
8.	A v C	VI 7
9	AVC	

The right-hand subproof

$A \lor B, B \to C \vdash A \lor C$

```
A ∨ B , B → C ⊢ A ∨ C
1.
A ∨ B

1. A ∨ B
:PR
+

2. B →
:PR
+

3. A
:AS
+

4. A ∨ C
:\/I 3
+

5. --
6. B
:AS
+

6. B
:AS
+
7.

7. C
:->E 2, 6
+
7.

8. A ∨ C
:\/I 7
+
8.
A ∨ C

9. A ∨ C
:\/E 1, 3-4, 6-8
+
9.
A ∨ C
```

1.	A v B	PR
2.	B → C	PR
3.	A	AS
4.	A v C	∨I 3
6.	В	AS
7.	С	→E 2, 6
8.	A v C	∨I 7
9.	AVC	∨E 1, 3-4, 6-8

Finishing the proof - note the justifications

Proofs From Disjunctions

- That's the basic structure.
- They are a bit of a pain; I've illustrated almost the easiest one I could find.
- But it's really important to keep track of what your goal is at every point.
- For almost everyone, that's impossible if you try to just start at line 1 and work to line 9.
- You have to bounce forward and backward in these proofs; just like I've done here.

Working Forward from Not

- It's going to be some kind of proof involving \bot .
- Whether that's Indirect Proof or ¬E isn't always clear, but that's going to be the structure.

A Simple Strategy

- If any of the premises is negated, then assume the opposite of the conclusion and try to derive ⊥.
- If the conclusion is positive, its opposite is adding a negation.
- If the conclusion is already negated, its opposite is deleting the negation.

A More Complicated Strategy

- Sometimes the simple strategy won't be optimal.
- Sometimes it will be quicker to do some working forward from the other premises, or backwards from the conclusion.
- But the simple strategy is going to work, even in those cases.

Rule Found		1.	¬A	PR
		2.	mm .	
1. ~A	:PR	+ 3.	ш	
2.		Δ Δ 4.		
3. 4.		<u>A</u> 5.		
5. 6. 7. ~(A /\ B)		Δ 6		
6.		-		
7. ~(A /\ B)		△ 7.	¬(A ∧ B)	

Premise and Conclusion

No Rule Found			1.	¬А	PR
1. ~A 2. A /\ B 3. 4. !? 5. ~(A /\ B)	:PR :AS :~I	+ + & & ?	2. 3. 4. 5.	_ A ∧ B "" 	AS

Set up $\neg I$ - note how \bot is written

+		1. ~A
_	:AS	2. A /\ B
т	:/\E 2	3. A
+	:~E 1, 3	4. !?
?	:~I	5. ~(A /\ B)
	:~I	5. ~(A /\ B)

1.	¬A	PR
2.	A∧B	AS
3.	Α	∧E 2
4. 5.	1	¬E 1, 3
5.	¬(A ∧ B)	٦l

Derive the contradiction

1. ~A	:PR	+
2. A /\ B	:AS	+
3. A	:/\E 2	+
4. !?	:~E 1, 3	+
5. ~(A /\ B)	:~I 2-4	+

۱.	¬А	PR
2.	A ∧ B	AS
3.	Α	∧E 2
4.	Т	¬E 1, 3
5.	¬(A ∧ B)	¬I 2-4

Finish the proof



• I'll end with two special techniques