305 Lecture 7.6 - Bayes's Theorem

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Plan

- This lecture will go over one of the most famous theorems in probability: Bayes's Theorem.

Associated Reading

Still chapter 8 of Odds and Ends

One Important Equivalence

$$\Pr(B|A)\Pr(A) = \Pr(A|B)\Pr(B)$$

We know that because these are two different ways of expressing $\Pr(A \wedge B)$.

One Important Equivalence

Divide both sides by Pr(B) (and flip sides around) and you get

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

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Divide both sides by Pr(B) (and flip sides around) and you get

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

This is the formula that's written in neon in the textbook.

Another Important Equivalence

Start again with our canonical formula for conditional probability.

$$\Pr(A|B) = \frac{\Pr(A \wedge B)}{\Pr(B)}$$

Another Important Equivalence

Replace $\Pr(A \wedge B)$ with its definition in terms of conditional probability.

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

Another Important Equivalence

Now replace Pr(B) with the formula we derived for it in an earlier lecture.

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B|A)\Pr(A) + \Pr(B|\neg A)\Pr(\neg A)}$$

And this is what is sometimes called Bayes's Theorem.

General Version

- Let $X_1, \ldots, X_n$ be a partition of possibility space.
- Then B is equivalent to $(B \wedge X_1) \vee \cdots \vee (B \wedge X_n)$.
- So $Pr(B) = Pr(B \wedge X_1) + \cdots + Pr(B \wedge X_n)$
- And we can use that to get the very general form of Bayes's theorem

General Version

$$\Pr(X_i|B) = \frac{\Pr(B|X_i)\Pr(X_i)}{\sum_{k=1}^{n}\Pr(B|X_k)\Pr(X_k)}$$

So if you know the prior probability of each cell in the partition, and the probability of B conditional on each cell, you can work out the probability of being in a particular cell given B.

General Version

- A lot of people make a big deal about this formula.
- I rarely find myself in situations where it is easier to use than something like the trees or tables.
- But the fact that so many people fuss so much about it suggests that for a lot of applications it is very helpful.
- For the assignment questions where this is relevant, it's totally up to you whether to use the formula, or trees, or tables.