$\Box \Diamond A \rightarrow A$ **(in S5)**

Build a Tableau

To Check Whether it is Valid

Hypothesis

$\Box \Diamond A \rightarrow A$ is a theorem of S5.

- We are going to use simplified S5.

Hypothesis

$\Box \Diamond A \rightarrow A$ is a theorem of S5.

- We are going to use simplified S5.
- Note this 'should' fail given what we have shown.
- We know that $\Diamond A$ implies $\Box \Diamond A$.
- So if $\Box \Diamond A$ implies A, we'll have $\Diamond A$ implies A, which is absurd.

$\Box \Diamond A \to A$

1.	$1, \mathbb{F}$	$\Box \Diamond A \to A$	Assumption

Start with it being false at 1.

$\Box \Diamond A \to A$

1.	$1, \mathbb{F} \quad \Box \Diamond A \to A$ ✓	Assumption
2.	$1, \mathbb{T} \quad \Box \Diamond A$	$\to\mathbb{F}, 1$
3.	$1, \mathbb{F} \quad A$	$\to\mathbb{F}, 1$

This feels familiar by now.

$\Box \Diamond A \to A$

1.	$1, \mathbb{F} \quad \Box \Diamond A \to A$ ✓	Assumption
2.	$1, \mathbb{T} \quad \Box \Diamond A$	$\to\mathbb{F}, 1$
3.	$1, \mathbb{F} \quad A$	$\to\mathbb{F}, 1$
4.	$1, \mathbb{T} \quad \Diamond A$	$\Box\mathbb{T}, 2$

The unboxed part of a box sentence is true everywhere, so it is true here. (But it doesn't get checked, because it's an open ended rule.)

$\Box \Diamond A \to A$

1.	$1, \mathbb{F} \quad \Box \Diamond A \to A$ ✓	Assumption
2.	$1, \mathbb{T} \quad \Box \Diamond A$	$\to \mathbb{F}, 1$
3.	$1, \mathbb{F} \quad A$	$\to \mathbb{F}, 1$
4.	$1, \mathbb{T} \quad \Diamond A$ ✓	$\Box \mathbb{T}, 2$
5.	$2, \mathbb{T} \quad A$	$\Diamond \mathbb{T}, 4$

True $\Diamond$ sentences have to be made true by some world. It can't be this one, since A is false here, and in any case we always use a new name.

$\Box \Diamond A \to A$

1.	1, $\mathbb{F}$ $\Box \Diamond A \to A$ ✓	Assumption
2.	1, $\mathbb{T}$ $\Box \Diamond A$	$\to\mathbb{F}$, 1
3.	1, $\mathbb{F}$ A	$\to\mathbb{F}$, 1
4.	1, $\mathbb{T}$ $\Diamond A$ ✓	$\Box\mathbb{T}$, 2
5.	2, $\mathbb{T}$ A	$\Diamond\mathbb{T}$, 4
6.	2, $\mathbb{T}$ $\Diamond A$	$\Box\mathbb{T}$, 2

True $\Box$ sentences are true, unboxed, everywhere, so we have to make them true in the new world we added.

$\Box \Diamond A \rightarrow A$

1.	$1, \mathbb{F} \quad \Box \Diamond A \rightarrow A$ ✓	Assumption
2.	$1, \mathbb{T} \quad \Box \Diamond A$	$\rightarrow\mathbb{F}, 1$
3.	$1, \mathbb{F} \quad A$	$\rightarrow\mathbb{F}, 1$
4.	$1, \mathbb{T} \quad \Diamond A$ ✓	$\Box\mathbb{T}, 2$
5.	$2, \mathbb{T} \quad A$	$\Diamond\mathbb{T}, 4$
6.	$2, \mathbb{T} \quad \Diamond A$	$\Box\mathbb{T}, 2$

Now if we are following the rules mechanically we will add a new world to make A true, call it world 3. And the cycle will continue.

$\Box \Diamond A \rightarrow A$

1.	$1, \mathbb{F} \quad \Box \Diamond A \rightarrow A$ ✓	Assumption
2.	$1, \mathbb{T} \quad \Box \Diamond A$	$\rightarrow \mathbb{F}, 1$
3.	$1, \mathbb{F} \quad A$	$\rightarrow \mathbb{F}, 1$
4.	$1, \mathbb{T} \quad \Diamond A$ ✓	$\Box \mathbb{T}, 2$
5.	$2, \mathbb{T} \quad A$	$\Diamond \mathbb{T}, 4$
6.	$2, \mathbb{T} \quad \Diamond A$	$\Box \mathbb{T}, 2$

- But if we think through what the lines are saying, we can see that we can stop here.
- Line 6 has to be made true somehow.
- But line 5 says that world 2 itself makes line 6 true.
- So there isn't anything extra to do.

$\Box \Diamond A \rightarrow A$

So here is a description of a model where this alleged theorem fails.

- There are two worlds: w_1 and w_2.
- At w_1, A is false.
- At w_2, A is true.
- Because w_2 exists, $\Diamond$A is true everywhere.
- So $\Box \Diamond A$ is true everywhere.
- So at w_1, $\Box \Diamond A$ is true and A is false.
- So at w_1, $\Box \Diamond A \rightarrow A$ is false.

For Next Time

Next week we'll end the course by discussing conditionals.