305 Lecture 8.4 - Sampling without Replacement

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Plan

- To illustrate another special kind of updating on two data points: Sampling without Replacement

Associated Reading

Odds and Ends, Chapter 9

Dependence

What happens if the events B_1 and B_2 are dependent on one or other of the hypotheses?

- The typical case is that they will be dependent on none or all of the hypotheses.
- But it's possible in principle to have independence on some and dependence on others.
- And in that case we have to use the more complicated procedure I'm about to describe.

Sampling Without Replacement

The paradigm example of conditional dependence is sampling **without replacement**.

- Assume you know which urn I'm using.
- Then the draws without replacement won't be independent because every time you draw a marble, there are fewer marbles of that color to draw the next time.

Example

Assume that I am using urn A. (Or assume that we are working out conditional probabilities conditional on urn A.)

- For the first draw, the probability of red is 4 in 10, or 0.4.
- Conditional on the first draw being red, the probability of the second draw being red is 3 in 9, or $\frac{1}{3}$.
- That's because there are now 9 marbles left, and 3 of them are red.

Continuing the Example

So to work out the probability of some sequence of draws D_1, D_2 given a hypothesis X about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

Continuing the Example

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$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

For example

$$\Pr(Red_1 \wedge Red_2 | A) = \Pr(Red_1 | A) \Pr(Red_2 | A \wedge Red_1) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

Continuing the Example

So to work out the probability of some sequence of draws D_1, D_2 given a hypothesis X about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

For example

$$\Pr(Red_1 \wedge Red_2 | B) = \Pr(Red_1 | B) \Pr(Red_2 | B \wedge Red_1) = \frac{8}{10} \times \frac{7}{9} = \frac{28}{45}$$

Another Example

There are two urns in front of us.

- One of them - urn A - has 4 red marbles, 3 green marbles, and 3 blue marbles.
- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

Another Example

There are two urns in front of us.

- One of them - urn A - has 4 red marbles, 3 green marbles, and 3 blue marbles.
- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

One of the urns will be selected at random, and then two marbles drawn from it **without replacement**.

- If both draws are red, what is the probability that Urn A was selected?

	Red-Red
Urn A	$0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$
Urn B	$0.5 \times \frac{8}{10} \times \frac{7}{9} = \frac{14}{45}$
Total	$\frac{1}{15} + \frac{14}{45} = \frac{17}{45}$

	Red-Red
Urn A	$0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$
Urn B	$0.5 \times \frac{8}{10} \times \frac{7}{9} = \frac{14}{45}$
Total	$\frac{1}{15} + \frac{14}{45} = \frac{17}{45}$

$$\Pr(A|\text{Red}-\text{Red}) = \frac{\Pr(A \wedge \text{Red}-\text{Red})}{\Pr(\text{Red}-\text{Red})} = \frac{\frac{1}{15}}{\frac{17}{45}} = \frac{3}{17}$$

The probability of Urn A fell by a bit more.

Yet Another Example

There are two urns in front of us.

- One of them - urn A - has 4 red marbles, 3 green marbles, and 3 blue marbles.
- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

Yet Another Example

There are two urns in front of us.

- One of them - urn A - has 4 red marbles, 3 green marbles, and 3 blue marbles.
- The other - urn B- has 8 red marbles, 1 green marbles and 1 blue marbles.

One of the urns will be selected at random, and then two marbles drawn from it **with replacement**.

- If the first draw is red and the second green, what is the probability that Urn A was selected?

The General Conjunction Rule

To work out the probability of some sequence of draws D_1, D_2 given a hypothesis X about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

The General Conjunction Rule

To work out the probability of some sequence of draws D_1, D_2 given a hypothesis X about the urn, we need to use the more complicated rule.

$$\Pr(D_1 \wedge D_2 | X) = \Pr(D_1 | X) \Pr(D_2 | X \wedge D_1)$$

So in this case we get

$$\Pr(\text{Red}_1 \wedge \text{Green}_2 | A) = \Pr(\text{Red}_1 | A) \Pr(\text{Green}_2 | A \wedge \text{Red}_1) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

The General Conjunction Rule

To work out the probability of some sequence of draws D_1, D_2 given a hypothesis X about the urn, we need to use the more complicated rule.

$$Pr(D_1 \wedge D_2|X) = Pr(D_1|X)\, Pr(D_2|X \wedge D_1)$$

And for Urn B we get

$$Pr(Red_1 \wedge Green_2|B) = Pr(Red_1|B)\, Pr(Green_2|B \wedge Red_1) = \frac{8}{10} \times \frac{1}{9} = \frac{4}{45}$$

	Red-Green
Urn A	$0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$
Urn B	$0.5 \times \frac{8}{10} \times \frac{1}{9} = \frac{2}{45}$
Total	$\frac{1}{15} + \frac{2}{45} = \frac{5}{45}$

	Red-Green
Urn A	$0.5 \times \frac{4}{10} \times \frac{3}{9} = \frac{1}{15}$
Urn B	$0.5 \times \frac{8}{10} \times \frac{1}{9} = \frac{2}{45}$
Total	$\frac{1}{15} + \frac{2}{45} = \frac{5}{45}$

$$\Pr(A|\mathrm{Red}-\mathrm{Green}) = \frac{\Pr(A \wedge \mathrm{Red}-\mathrm{Green})}{\Pr(\mathrm{Red}-\mathrm{Green})} = \frac{\frac{1}{15}}{\frac{5}{45}} = \frac{3}{5}$$

Which, interestingly, is exactly the same as in the with replacement case.

For Next Time

We'll end the week with one last example.