444 Class Four

Brian Weatherson

September 16, 2019

Dynamic Games

Slides!

- There won’t normally be slides - but today is more lecture-y than normal.
- These are on Canvas; don’t copy anything down.

Class of Games We're Discussing

- Two-player

Class of Games We're Discussing

- Two-player
- Turn-taking

Class of Games We're Discussing

- Two-player
- Turn-taking
- Finite

Class of Games We're Discussing

- Two-player
- Turn-taking
- Finite
- No hidden facts

Class of Games We're Discussing

- Two-player
- Turn-taking
- Finite
- No hidden facts
- No randomness

Class of Games We're Discussing

- Two-player
- Turn-taking
- Finite
- No hidden facts
- No randomness
- We'll start with zero-sum games, though drop this later.

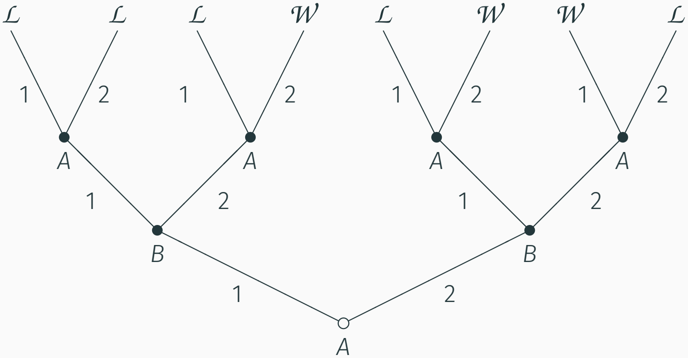
Five

- There are two players, who we'll call A and B.
- First A moves, then B, then finally A moves again.
- Each move involves announcing a number, 1 or 2.
- A wins if after the three moves, the numbers announced sum to 5.
- B wins otherwise.

Five

Question: How should you play this game?

Game Tree for Five



$\mathcal{W}$ means that A wins, and $\mathcal{L}$ means that B wins.

How to Solve These Games

- Work backwards.

How to Solve These Games

- Work backwards.
- First, find points where a player has a choice between two terminal nodes.

How to Solve These Games

- Work backwards.
- First, find points where a player has a choice between two terminal nodes.
- Assume that they will make the choice that has higher value for them.

How to Solve These Games

- Work backwards.
- First, find points where a player has a choice between two terminal nodes.
- Assume that they will make the choice that has higher value for them.

How to Solve These Games

- Work backwards.
- First, find points where a player has a choice between two terminal nodes.
- Assume that they will make the choice that has higher value for them.
- Mark that choice, e.g., by doubling the line (as the textbook does) or bolding the line (as I'll do).

How to Solve These Games

- Work backwards.
- First, find points where a player has a choice between two terminal nodes.
- Assume that they will make the choice that has higher value for them.
- Mark that choice, e.g., by doubling the line (as the textbook does) or bolding the line (as I'll do).
- If there are ties, mark both of the lines. (This gets more complicated once we leave zero-sum games.)

How to Solve These Games

- Assign the value they choose to the choice node.

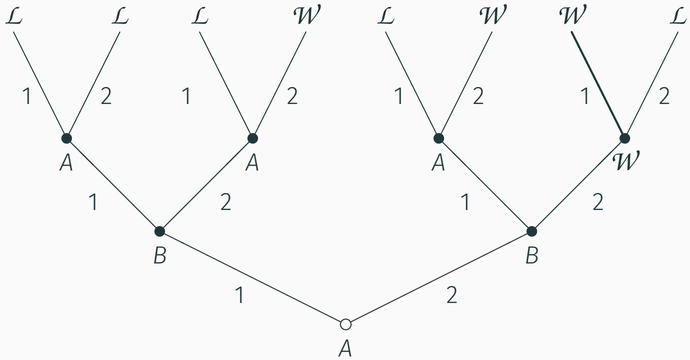
How to Solve These Games

- Assign the value they choose to the choice node.
- So just the game assigns values to terminal nodes, we'll now assign value to choice nodes.

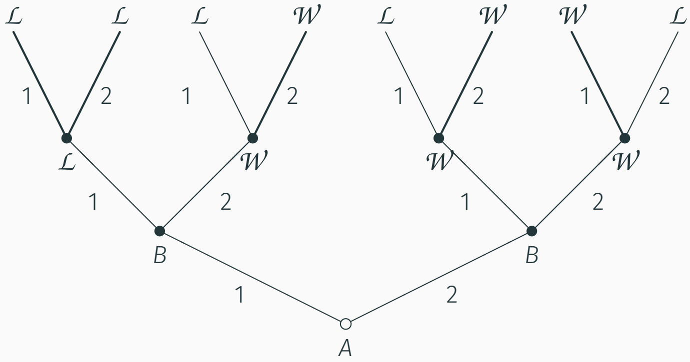
How to Solve These Games

- Assign the value they choose to the choice node.
- So just the game assigns values to terminal nodes, we'll now assign value to choice nodes.
- In **Five**, we'll assign the value $\mathcal{W}$ to the top right node.

Five (after one step)



Five (after first level)



Next Steps Back

- Now we do the same thing for *B*.

Next Steps Back

- Now we do the same thing for B.
- We act as if B is choosing between terminal nodes.

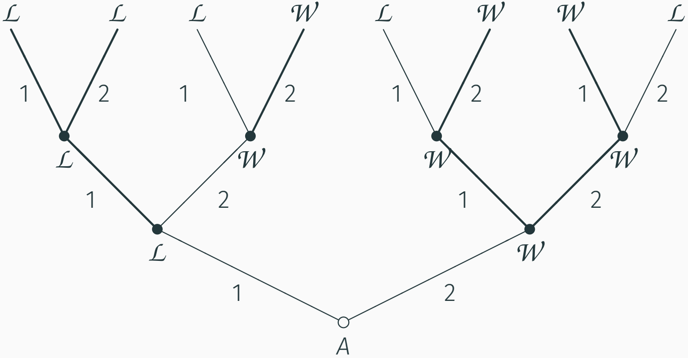
Next Steps Back

- Now we do the same thing for *B*.
- We act as if *B* is choosing between terminal nodes.
- It is as if *A* doesn't have a choice - they will just make the choice that is best for them (i.e., worst for *B*).

Next Steps Back

- Now we do the same thing for *B*.
- We act as if *B* is choosing between terminal nodes.
- It is as if *A* doesn't have a choice - they will just make the choice that is best for them (i.e., worst for *B*).
- So *B* knows what the outcome of each choice will be.

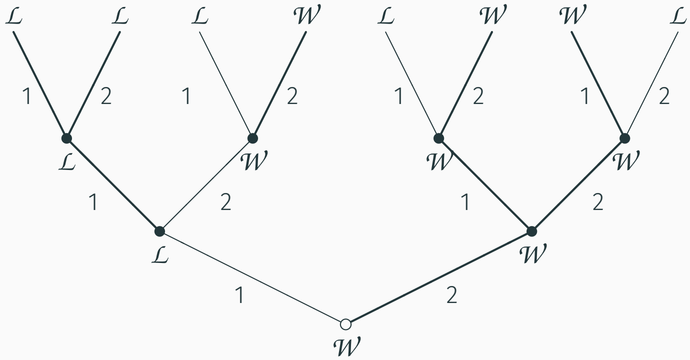
Five (After Two Rounds)



Five (After Two Rounds)

- So we act as if getting to the left hand node means *B* wins, and getting to the right hand node means *A* wins.
- And now we just have to make the choice for the initial node, using this fact.

Five (Full Graph)



Five - Full Analysis

- The equilibrium state of the game is that A wins.
- A plays 2 first.
- Then B can play anything they line.
- But whatever they do, A will win, by playing the opposite number.

Backwards Induction

- This process is called backwards induction.
- We start at the possible ends of the game.
- At each step, we assume that each player makes the best decision they can, on the assumption that later players will do the same thing.
- And eventually we can solve the game.

Value of Games

The Value of a Game

The value of a game is the outcome of the game if everyone does the backwards induction recommended move at every possible point.

The Value of a Game

Every zero-sum game (that satisfies the earlier constraints) has a value.

The Value of a Game

Every zero-sum game (that satisfies the earlier constraints) has a value.

The value is the value of one of the terminal nodes.

The Value of a Game

Every zero-sum game (that satisfies the earlier constraints) has a value.

The value is the value of one of the terminal nodes.

The proof is a fairly straightforward proof by induction, but I won't go through it here.

The Value of a Game

So for any such game that ends with either win to one player or a draw, either

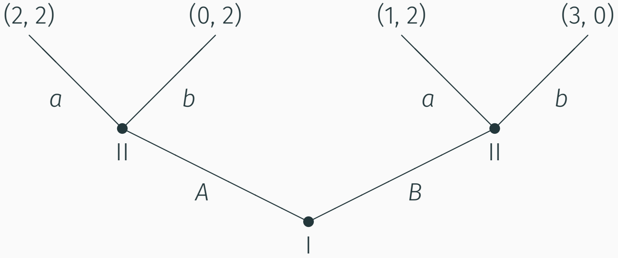
1. The first player can guarantee a win; or
2. The second player can guarantee a win; or
3. Both players can guarantee at least a draw.

Generalising

A Complication

In a non-zero-sum game, a player might have a choice between two nodes that are alike for them, but are different for the other player.

Indifferent Choices



Indifferent Choices

- If player I chooses B, player II will choose a, ending the game at $(1, 2)$.
- But what happens if player I chooses A?

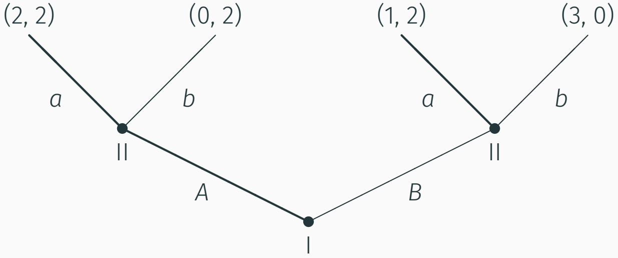
Duplicating the graph

- You can't just draw two lines, because we need a value of the middle-left node for player I, and we don't know what that is.

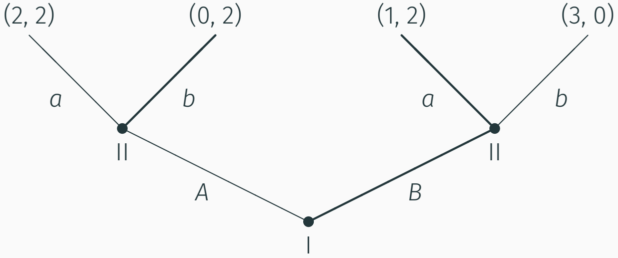
Duplicating the graph

- You can't just draw two lines, because we need a value of the middle-left node for player I, and we don't know what that is.
- The solution is to duplicate the graph.

One Backwards Induction Solution



Another Backwards Induction Solution

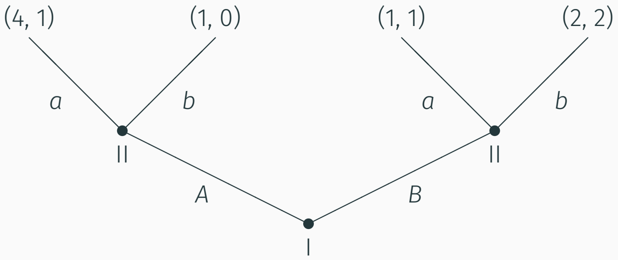


What Should Player I Do?

Good question!

Strategies and Threats

Threat Game



Strategies

- A **strategy** for a game is a set of instructions for what to do at each node of a game.
- Even very small game trees there are a lot of possible strategies.
- If there are k possible nodes a player could have a choice at, and m possible moves at each of these nodes, then there are m^k possible strategies.
- Note that a strategy has to say what to do at nodes that are ruled out by your own prior moves.

Threat Game Strategies

- For player I, there are just two strategies: *A* and *B*.
- For player II, there are two nodes, and two possible choices at each node. So there are $2^2 = 4$ possible strategies.
- We'll write *xy* for the strategy of doing *x* in response to *A*, and *y* in response to *B*.
- And note I'm capitalising player I's moves, and using lower case for player II's moves, to make things clearer.

Threat Game Strategies

Here are the four strategies for player II:

1. *aa* - Do *a* no matter what.
2. *ab* - Do whatever player I does.
3. *ba* - Do the opposite of what player I does.
4. *bb* - Do *b* no matter what.

Threat Game Strategy Tables

The strategies for the players determine the outcome. Here is the table for the game, given the strategies.

		Player II			
		aa	ab	ba	bb
Player I	A	4, 1	4, 1	1, 0	1, 0
	B	1, 1	2, 2	1, 1	2, 2

Threat Game Strategy Tables

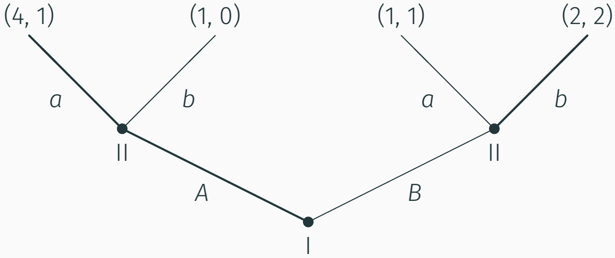
I've underlined the best responses.

		Player II			
		aa	ab	ba	bb
Player I	A	$\underline{4}, \underline{1}$	$\underline{4}, \underline{1}$	$\underline{1}, 0$	$1, 0$
	B	$1, 1$	$2, \underline{2}$	$\underline{1}, 1$	$\underline{2}, \underline{2}$

There are three Nash equilibria.

1. A, aa - with result 4, 1
2. A, ab - with result 4, 1
3. B, bb - with result 2, 2

Threat Game with Backward Induction



- I've bolded the best moves at each node, assuming backward induction.
- The path of best moves is the (in this case unique) backward induction solution.

Threat Game

- There are three Nash equilibria of the game: strategy pairs that no one can improve on by unilaterally changing strategy.
- There is just one backward induction solution of the game: a strategy pair where everyone does the best they can **at every node** assuming others play rationally at every node.

For Next Time

Examples!

We will talk over some examples where backward induction reasoning might help understand what's going on.