444 Lecture 3.7 - Zero Sum Turn Taking Games

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Plan

To discuss a remarkable result - that every turn taking zero sum game has a **value**.

Reading

Bonanno, section 3.5

The Class of Games Under Discussion

- Two-player
- Turn-taking
- Finite
- No hidden facts
- No randomness
- Zero-sum

That is a lot of restrictions, but it includes classic games like chess, checkers, go, othello and more.

Theorem

Every one of these games has a **value**.

- The value of the game is an outcome that each player can guarantee that they get at least that good a result.
- Since the game is zero-sum, it follows that no player can guarantee that they do better.

Proof Schema

- We prove this by induction on the nodes.
- The value of a terminal node is given by the definition.
- Say a penultimate node (at any stage of the process) is a node such that whatever move is made, the next node has a value.
- The value of that node is the best value of the subsequent nodes, by the lights of the player who has to play.
- So if player 1 has to play, and one nodes leads to a draw, and the other to Player 2 winning, the value of that node is draw.

Proof Schema

- Since the game is finite, a finite number of steps of this process will assign a value to every node in the tree.
- So eventually the initial node will get a value.
- And that's the value of the game.
- And the way we've constructed that value shows that each player always has a path to ensure they never do worse.
- If it is their move, there will be a move that preserves value.
- And if it is the other players' move, the value is by definition the most harm they can do with that move.

Consequences

- Consider any game with just two outcomes, one player wins or the other.
- Whatever the tree is, we know the table will have one of the following two features.

1. There is a row where every result is that Player 1 wins; or
2. There is a column where every result is that Player 2 wins.

Consequences

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- For some chess like games this is not a surprise - if player 2 had a winning strategy, player 1 could as it were execute it first. This isn't quite right for chess, but no one really thinks that player 2 has a winning strategy in chess.
- Probably the value of chess is a draw, but this isn't known yet.

Solved Games

Solved games [edit]

Awari (a game of the Mancala family)

The variant of Oware allowing game ending "grand slams" was strongly solved by Henri Bal and John Romein at the Vrije Universiteit in Amsterdam, Netherlands (2002). Either player can force the game into a draw.

Chopsticks

The second player can always force a win.[citation needed]

Connect Four

Solved first by James D. Allen on October 1, 1988, and independently by Victor Allis on October 16, 1988.[6] The first player can force a win. Strongly solved by John Tromp's 8-ply database[7] (Feb 4, 1995). Weakly solved for all boardsizes where width+height is at most 15 (as well as 8×8 in late 2015)[6] (Feb 18, 2006).

English draughts (checkers)

This 8×8 variant of draughts was **weakly solved** on April 29, 2007, by the team of Jonathan Schaeffer. From the standard starting position, both players can guarantee a draw with perfect play.[8] Checkers is the largest game that has been solved to date, with a search space of 5×10^{20}.[9] The number of calculations involved was 10^{14}, which were done over a period of 18 years. The process involved from 200 desktop computers at its peak down to around 50.[10]

Fanorona

Weakly solved by Maarten Schadd. The game is a draw.[citation needed]

Free gomoku

Solved by Victor Allis (1993). The first player can force a win without opening rules.

Ghost

Solved by Alan Frank using the *Official Scrabble Players Dictionary* in 1987.[citation needed]

Guess Who?

Strongly solved by Mihai Nica in 2016.[11] The first player has a 63% chance of winning under optimal play by both sides.

Hex

- A strategy-stealing argument (as used by John Nash) shows that all square board sizes cannot be lost by the first player. Combined with a proof of the impossibility of a draw this shows that the game is ultra-weak solved as a first player win.
- Strongly solved by several computers for board sizes up to 6×6.
- Jing Yang has demonstrated a winning strategy (weak solution) for board sizes 7×7, 8×8 and 9×9.
- A winning strategy for Hex with swapping is known for the 7×7 board.
- Strongly solving Hex on an $N \times N$ board is unlikely as the problem has been shown to be PSPACE-complete.
- If Hex is played on an $N \times (N+1)$ board then the player who has the shorter distance to connect can always win by a simple pairing strategy, even with the disadvantage of playing second.
- A weak solution is known for all opening moves on the 8×8 board.[12]

Hexapawn

3×3 variant solved as a win for black, several other larger variants also solved.[13]

Kalah

Most variants solved by Geoffrey Irving, Jeroen Donkers and Jos Uiterwijk (2000) except Kalah (6/6). The (6/6) variant was solved by Anders Carstensen (2011). Strong first-player advantage was proven in most cases.[14][15] Mark Rawlings, of Gaithersburg, MD, has quantified the magnitude of the first player win in the (6/6) variant (2015). After creation of 39 GB of endgame databases, searches totaling 106 days of CPU time and over 55 trillion nodes, it was proven that, with perfect play, the first player wins by 2. Note that all these results refer to the Empty-pit Capture variant and therefore are of very limited interest for the standard game. Analysis of the standard rule game has now been posted for Kalah(6,4), which is a win by 8 for the first player, and Kalah(6,5), which is a win by 10 for the first player. Analysis of Kalah(6,6) with the standard rules is on-going, however, it has been proven that it is a win by at least 4 for the first player.

L game

Easily solvable. Either player can force the game into a draw.

Losing chess

Weakly solved as a win for white beginning with 1. e3.[16]

Maharajah and the Sepoys

This asymmetrical game is a win for the sepoys player with correct play.

Nim

Strongly solved.

Wikipedia page on solved games

For Next Week

We will start chapter 4, on games involving hidden information.