444 Lecture 8

Mixed Strategies

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Handout

There is a handout for today. It kind of duplicates the slides, so I haven't made copies of it, but it is on Canvas, under Handouts in files. If you prefer seeing things in paragraphs rather than bullet points, you could follow along on that.

Day Plan

Games with Lotteries

Here is one thing we can do with cardinal utilities - include lotteries in the payoffs.

- We can treat the lottery ticket as having a value equal to the **expected value** of the lottery.

Games with Lotteries

Bonanno illustrates this with a game that involves an actual lottery - an auction where tied bids are resolved by a chance mechanism.

- But philosophically, lots of things in life look like lottery ticket.

Life is a Series of Lotteries

- How much is $1 million worth?
- It depends a bit on whether there is lots of inflation in the near future.
- It also depends on whether there is a revolution soon and millionaires are in danger.

Everything's a Gamble

The orthodox treatment of these questions, which I totally endorse, is that a quantity of money is just as much a gamble as a lottery ticket.

- It's a relatively safe gamble; there hasn't been hyperinflation or anti-capitialist revolution in America in a long time.
- But it's a gamble.
- So even games with monetary payouts are gambles - gambles on the future value of money.

Chicken 1

Here is a version of chicken using ordinal utility.

	swerve	drive
Swerve	3, 3	2, 4
Drive	4, 2	1, 1

Chicken 2

	swerve	drive
Swerve	1, 1	0, 2
Drive	2, 0	-5, -5

I guess you mostly swerve in this game, but you think about driving.

Chicken 3

	swerve	drive
Swerve	1, 1	0, 2
Drive	2, 0	-5000, -5000

Please swerve!

- But (Swerve, swerve) is not Nash. We'll come back to this.

Cardinal Utility Matters

- The last two games were alike in ordinal utility.
- But they were unlike in how you should play them.
- So more than ordinal utility matters for how you should play.

Day Plan

Core Idea

Flip a coin!

Motivation

Rock! Paper!! Scissors!!!

Payout

If Row plays 60% Up and 40% Down, and Column plays Left, then Row's expected payout will be

- 0.6 times the payout for Up/Left
- 0.4 times the payout for Down/Left

When we evaluate a mixed strategy, we evaluate it by its expected payout, not its actual payout.

Payout

If Row plays 60% Up and 40% Down, and Column plays Left, then Column's payout will be

- 0.6 times the payout for Left/Up
- 0.4 times the payout for Left/Down

We have to use expected values for evaluating even **pure** strategies like Left, once one player plays mixtures.

Both Mixing

What if Column also mixes, playing 70% Left and 30% Right.

- Then Row's expected payout is 0.7 times the **expected** payout of their mixed strategy when Column plays Left, plus 0.3 times the expected payout of their mixed strategy when Column plays right.
- We have to use expected values twice over to get the philosophically important value of this strategy.

Independence

- What we're assuming here is that the probabilities of each player are **independent**.
- If they roll 20-sided dice or whatever, they roll different ones.
- There is an interesting part of game theory where we drop that assumption, but I'm not going to cover it.
- This is the part that deals with so-called **correlated equilibrium**.
- Bonanno doesn't cover this, and I won't either.

What is a Mixed Strategy

At least three interpretations.

1. Random device
2. Frequencies
3. Epistemic

Random Device

- I mean, you literally roll a die, flip a coin, use something on your phone or something.
- This is the traditional interpretation, but it's become less popular.

Frequencies

- In the long-run, your frequency of each play tracks the probabilities in the strategy.
- And more specifically, your frequency of each play in each identifiable situation tracks the probabilities.

Frequencies

- Alternating rock-then-paper-then-scissors isn't a way to play the mixed strategy 1/3 for each move.
- Relatedly, you will lose.
- On the frequency interpretation of mixed strategies, playing that mixed strategy means that after rock, you'll play 1/3 rock, 1/3 paper, 1/3 scissors, and after paper-then-scissors, you'll play ..., and so on.

Epistemic

- A rational onlooker will not be able to figure out what you will do/are doing, beyond saying that you're playing each strategy with the specified probability.
- It doesn't matter how you get there, as long as that's all that an onlooker (or co-player) can figure out.

Mixtures Always Available

- Game theorists standardly assume that if some moves are available, so is any mixed strategy built out of them.
- Philosophers do not assume this, and often think it's the least plausible thing about game theory.
- Is this assume plausible?

What's a Mixed Theory

- On the random device interpretation, it's very implausible.
- Not all game players have arbitrary random devices available.
- But on the other interpretations, it is more plausible.
- A good player should be able to hide their strategy, even in repeated plays.

Day Plan

Cardinal Importance

Mixed Strategies

Mixed Strategies in Equilibria

Finding Mixed Strategy Equilibria

Key Theorem

In any finite game in which all mixtures of strategies are available, there is at least one Nash equilibria.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

Let's discuss this game for a bit. Does it have an equilibrium?

	Left	Right
Up	$\boxed{2}$, 0	0, $\boxed{3}$
Down	0, $\boxed{1}$	$\boxed{1}$, 0

No Nash equilibrium in pure strategies.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

Consider what happens if Row plays

- Up with probability $1/4$;
- Down with probability $3/4$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

Column's expected return from playing Left is

- 0 with probability $1/4$ plus
- 1 with probability $3/4$, i.e.,
- $3/4$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

Column's expected return from playing Right is

- 3 with probability $1/4$ plus
- 0 with probability $3/4$, i.e.,
- $3/4$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

- So either way, Column's expected return from playing a pure strategy is $3/4$.
- And hence Column's expected return from playing any mixture of the two pure strategies is $3/4$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

Consider what happens if Column plays

- Left with probability $1/3$;
- Right with probability $2/3$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

Row's expected return from playing Up is

- 2 with probability $1/3$ plus
- 0 with probability $2/3$, i.e.,
- $2/3$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

Row's expected return from playing Down is

- 0 with probability $1/3$ plus
- 1 with probability $2/3$, i.e.,
- $2/3$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

- So either way, Row's expected return from playing a pure strategy is $2/3$.
- And hence Row's expected return from playing any mixture of the two pure strategies is $2/3$.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

- What happens if they both play the mixed strategies we've been discussing?
- It's an equilibria.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

- Whatever Column does, their expected return is $3/4$.
- So they can't do better than play this mixed strategy.
- They can't do worse either, but the definition of equilibrium just requires that they can't do better.

	Left	Right
Up	2, 0	0, 3
Down	0, 1	1, 0

- Whatever Row does, their expected return is $2/3$.
- So they can't do better than play this mixed strategy.
- They can't do worse either, but the definition of equilibrium just requires that they can't do better.

Two General Points

1. There is always some equilibria like this (at least in finite games), even if it doesn't look like it at first.
2. Typically, the way we find equilibria is making the other player indifferent between a bunch of options.

Day Plan

Basic Idea

- In equilibria, the other player is willing to play a mixed strategy.
- That requires that they be indifferent between other strategies.
- So we find the equilibria by finding the mixture that makes them indifferent.

	Left	Right
Up	5, 2	1, 3
Down	2, 2	3, 0

- You can see fairly quicklythat there is no pure strategy equilibria.
- So the equilibria must be a mixed strategy equilibria.

Theory

What does it take for Column to play a mixed strategy in equilibria?

- Assume that Left has a higher expected return than Right.
- The expected return of a mixed strategy is a weighted average of the expected returns of Left and Right.
- If Left has a higher expected return than Right, that weighted average will be strictly between the expected returns of Left and Right.

Theory

- And that means it can't be an equilibrium, since in equilibrium there is no alternative with a higher expected return.
- And the same reasoning shows Right can't have a higher expected return than Left.

Theory

So we are trying to find the mixture such that Column is indifferent between Left and Right.

- The other crucial thing to remember is that probabilities add to 1. (We'll do a review lecture on probability next time.)
- So when working out Row's strategy, there is only one variable.
- Once we set the probability of Row playing Up to x, that sets all the probabilities, because the probablity of playing Down is $1 - x$.

A Note

I'm only going to go over cases where the mixed strategy equilibrium involves a mixture of two pure strategies.

- There are cases where the mixed strategy equilibrium involves mixtures of 3 or more pure strategies.
- Rock, Paper, Scissors is the simplest such example.
- But in general the math of calculating these is considerably fancier than what we'll be doing, and I'll stick to cases where the mixed strategy equilibrium only involves 2 pure strategies.

	Left	Right
Up	5, 2	1, 3
Down	2, 2	3, 0

- Assume Row plays Up with probability x, and Down with probability $1 - x$.
- Our job is to find an x such that the expected return of Left and Right is the same.

	Left	Right
Up	5, 2	1, 3
Down	2, 2	3, 0

- The expected return of Left is $2x + 2(1 - x)$, i.e., 2.
- The expected return of Right is $3x + 0(1 - x)$, i.e., $3x$.

	Left	Right
Up	5, 2	1, 3
Down	2, 2	3, 0

- So $2 = 3x$, so $x = 2/3$.
- So Row's strategy is to play Up with probability $2/3$, and hence Down with probability $1/3$.
- The expected return of Right is $3x + 0(1 - x)$, i.e., $3x$.

	Left	Right
Up	5, 2	1, 3
Down	2, 2	3, 0

- Assume Column plays Left with probability x, and Right with probability $1 - x$.
- Our job is to find an x such that the expected return of Up and Down is the same.

	Left	Right
Up	5, 2	1, 3
Down	2, 2	3, 0

- The expected return of Up is $5x + 1(1 - x)$, i.e., $4x + 1$.
- The expected return of Down is $2x + 3(1 - x)$, i.e., $3 - x$.

$$\begin{aligned} 4x + 1 &= 3 - x \\ 5x + 1 &= 3 \\ 5x &= 2 \\ x &= 2/5 \end{aligned}$$

So Column's strategy is to play Left with probability $2/5$, and hence Right with probability $3/5$.

Takeaways

- To find a player's move probabilities in equilibria, look to the other player's payouts.
- Try to make the other player indifferent between their choices.

Extra Steps

- I'm not going to go over more complicated examples on the slides, but there is an extra step you can do (and which we can discuss in class if you're interested).
- Sometimes you can find the mixed strategy equilibria of a game with more than 2 moves by first deleting **strongly** dominated strategies.
- Bonanno works through an example like this.

For Next Time

I'm going to start on an idea I want to work through very slowly, and spend a bit of time on - the idea that a mixture of strategies can dominate another strategy.