

444 Lecture 4 Equilibrium

Brian Weatherson

January 17, 2023

Day Plan

Trees

Backward Induction

Ties

Time

- The tables we discussed last week represent games where each player moves once, and those moves are simultaneous.
- But few games are like that.
- We need a way to represent games that take time.

Trees

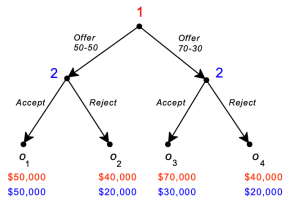
- We do that with trees.
- A tree represents all the ways that a game that takes place over time could go.

Nodes

- Trees have nodes.
- Some nodes are **terminal nodes**; they represent that the game has ended.
- Each terminal node has a payout for each of the players.
- At any other node, either a player moves, or Nature ‘moves’.
- One of the non-terminal nodes is special: it is the node where the game starts.

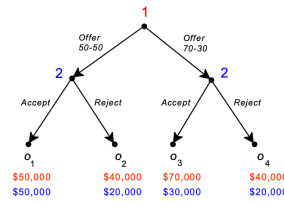
Branches

- Each non-terminal node has branches, leading to other nodes.
- A move at a node is always a choice of branches.



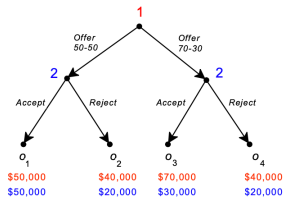
Example from Bonanno

- There are two players, 1 and 2.
- Each player moves once.
- First 1 moves, then 2 moves, then the game ends.



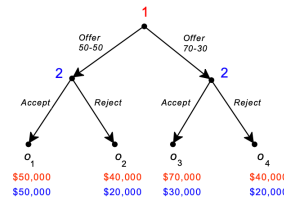
Example from Bonanno

- Some books use a special notation for the initial node, such as having an open circle rather than a closed circle.
- Bonanno doesn't, but it's clear in context what the initial node is.



Example from Bonanno

- As he goes on to note, this isn't really a tree yet.
- It describes the physical outcomes of the game at each terminal node, but not the **payoffs**.



Example from Bonanno

There is a natural function from outcomes to payoffs - more money equals more utility - but it is not a compulsory interpretation.

Future Additions

- Moves by Nature
- Moves under uncertainty

Day Plan

Trees

Backward Induction

Ties

Team Backward Induction Tie

Class of Games We're Discussing

- Two-player
- Turn-taking
- Finite
- No hidden facts
- No randomness
- We'll start with zero-sum games, though drop this later.

13 / 35

Team Backward Induction Tie

Five

- There are two players, who we'll call A and B .
- First A moves, then B , then finally A moves again.
- Each move involves announcing a number, 1 or 2.
- A wins if after the three moves, the numbers announced sum to 5.
- B wins otherwise.

14 / 35

Team Backward Induction Tie

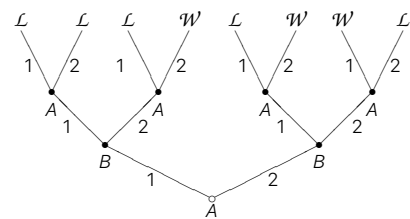
Five

Question: How should you play this game?

15 / 35

Team Backward Induction Tie

Game Tree for Five



W means that A wins, and L means that B wins.

16 / 35

Team Backward Induction Tie

How to Solve These Games

- Work backwards.
- First, find points where a player has a choice between two terminal nodes.
- Assume that they will make the higher value for them choice.
- Mark that choice, e.g., by doubling the line (as the textbook does) or bolding the line (as I'll do).
- If there are ties, mark both of the lines. (This gets more complicated once we leave zero-sum games.)

17 / 35

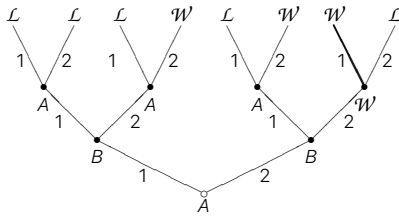
Team Backward Induction Tie

How to Solve These Games

- Assign the value they choose to the choice node.
- So just the game assigns values to terminal nodes, we'll now assign value to choice nodes.
- In **Five**, we'll assign the value W to the top right node.

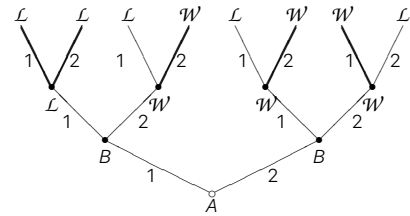
18 / 35

Five (after one step)



19 / 35

Five (after first level)



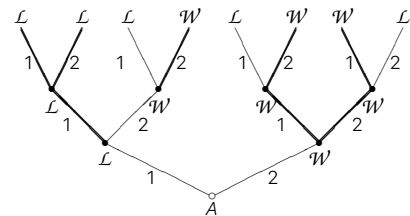
20 / 35

Next Steps Back

- Now we do the same thing for B .
- We act as if B is choosing between terminal nodes.
- It is as if A doesn't have a choice - they will just make the choice that is best for them (i.e., worst for B).
- So B knows what the outcome of each choice will be.

21 / 35

Five (After Two Rounds)



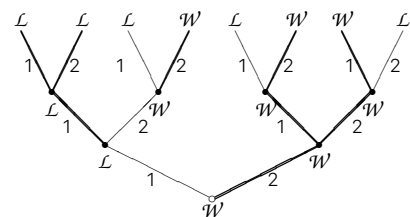
22 / 35

Five (After Two Rounds)

- So we act as if getting to the left hand node means B wins, and getting to the right hand node means A wins.
- And now we just have to make the choice for the initial node, using this fact.

23 / 35

Five (Full Graph)



24 / 35

- The equilibrium state of the game is that *A* wins.
- *A* plays 2 first.
- Then *B* can play anything they line.
- But whatever they do, *A* will win, by playing the opposite number.

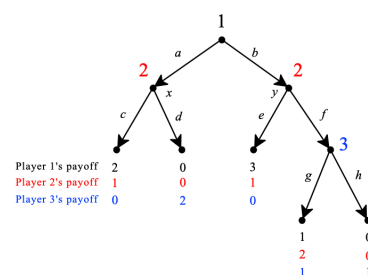
25 / 35

- This process is called backwards induction.
- We start at the possible ends of the game.
- At each step, we assume that each player makes the best decision they can, on the assumption that later players will do the same thing.

26 / 35

- Trees
- Backward Induction
- Ties

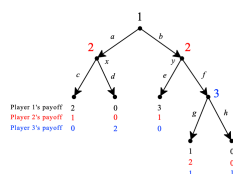
27 / 35



Three player game tree

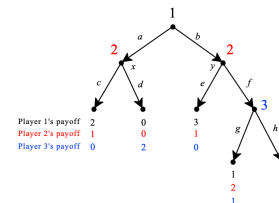
This is three player, but crucially, it is not zero sum.

28 / 35



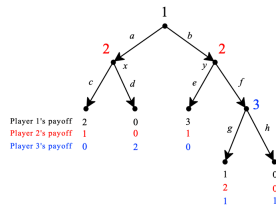
Three player game tree

- In the bottom right, Player 3 doesn't care which choice is made.
- So we can't infer what Player 3 will do.

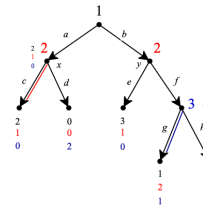


Three player game tree

- But the other players do care what Player 3 will do.
- So we can't just ignore this choice.



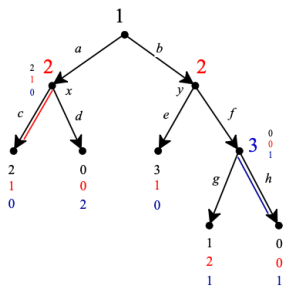
Three player game tree



Solution One

- First, assume 3 plays g.
- Then 2 would play f at node y.
- So 1 will actually play a.

- The solution is to build two trees, one for each of Player 3's choices.



Solution Two

- Now, assume 3 plays h.
- Then 2 would play e at node y.
- So 1 will actually play b triggering this play.

Multiple Solutions

- This is a game with multiple backwards induction solutions.
- The solutions don't just differ in what Player 3, who faces the tie, plays.
- They differ in the very first move!

Multiple Solutions

- This is the totally general case; most solution concepts are like this.
- But it's a pain to deal with.
- And eventually we can solve the game.