444 Lecture 2

Introducing Games

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1/12/23

Game Outcomes

There are two natural ways to specify the outcome of a game.

1. Describe the physical situation that results.
2. Describe how much **utility** each player gets from that result.

Utility

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- That's because we want to know what makes sense from the players' perspectives.
- And just knowing the physical outcomes doesn't tell us that.

What is Utility

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- The players are aiming to maximise their own number, not maximise the difference between the numbers.

Figure: A memorable scoreboard

What is Utility

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- So this is very much unlike soccer, even though the numbers will often feel a lot like soccer scores.

What is Utility

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What is Utility

- It’s not money, for two distinct reasons.
- First, the players might care how much money the other players get.

Utility and Altruism

Consider these three situations

1. Billy gets \$90, Suzy gets \$100.
2. Billy gets \$100, Suzy gets nothing.
3. Billy gets \$110, Suzy gets \$100.

How do you order these in terms of utility to Billy, from highest to lowest?

Utility and Altruism

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Utility and Altruism

- We don’t know given just this description.
- If Billy wants Suzy to get money, he might prefer option 1 to option 2.
- If Billy wants Suzy to not have money, he might prefer option 2 to option 3.

What is Utility

- It’s not money, for two distinct reasons.

What is Utility

- It’s not money, for two distinct reasons.
- Second, getting twice as much money typically doesn’t produce twice as much utility.

What is Utility

It is, more or less, desirability.

- Outcome O_1 has more utility for player X than outcome O_2 iff X prefers to be in O_1 than O_2.

Utility and Numbers

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- Now you might have noticed something odd there.
- We are trying to define this numerical quantity, but we've just told you something about when it is bigger or smaller.
- Surely we need to say something more, like how much bigger or smaller it is in different situations.

Utility

A utility function (for a particular agent) is a mapping U from situations to numbers satsifying this constraint.

- $U(S_1) > U(S_2)$ iff the agent is better off in S_1 than in S_2.

Welfare

This isn't part of the formal theory, but we usually implicitly assume (at least in our narratives), the following principle.

> The agent is better off in S_1 than in S_2 iff, given a choice and assuming they are fully informed, they prefer being in S_1 to S_2.

That is, we'll usually speak as if a radically subjectivist view of welfare is correct. I've been doing this already, and I'm going to keep doing it.

Ordinal Utility

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- When we say that we're working with **ordinal** utility functions, really the only principle that applies is the one from two slides back.
- Higher utilities are better, i.e., are preferred.
- The term **ordinal** should make you think of 'orders'; all an ordinal utility function does is provide a rank **ordering** of the outcomes.

Two Functions

So if we're working in ordinal utility, these two functions describe the same underlying reality.

	U_1	U_2
O_1	1	1
O_2	2	10
O_3	3	500
O_4	4	7329

Cardinal Utility

- In cardinal utility theory, the differences between the numbers matter.

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- In cardinal utility theory, the differences between the numbers matter.
- The numbers now express quantities, and the two functions from the previous slide do not represent the same underlying reality.

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- There is a fussy point here that's worth going over.
- Even cardinal utility functions don't come with a scale.
- So two functions with different numbers in them can still express the same underlying reality.

Cardinal Utility (Detail)

The standard way to put this is that (cardinal) utility is defined only up to a **positive, affine transformation**. That means that if U_1 and U_2 are related by the following formula, then they represent the same state of affairs.

$$U_2(o) = aU_1(o) + b \text{ where } a > 0$$

Celsius and Farenheit

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- The main real world cases of scales that are related in this way are temperature scales.
- To convert between Celsius and Farenheit you use the formula $F = 1.8C + 32$.
- But the scales are just two ways of representing the same physical reality.

Cardinal Utility (Detail)

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- So there is no such thing as one outcome being *twice as good* as another.
- But we can say a lot of things about differences.

Cardinal Utility (Detail)

- If the difference between O_1 and O_2 is the same as the difference between O_2 and O_3, that will stay the same under any positive affine transformation.

Cardinal Utility (Detail)

- If the difference between O_1 and O_2 is the same as the difference between O_2 and O_3, that will stay the same under any positive affine transformation.
- Indeed, for any k, if the difference between O_1 and O_2 is k times the difference between O_2 and O_3, that will stay the same under any positive affine transformation.

A Simple Game

	Left	Right
Up	4, 1	2, 0
Down	3, 0	1, 1

Here's how to read this table.

1. Two players, call them Row and Column.
2. Row chooses the row, Column chooses the column – between them they choose a cell.
3. Each cell has two numbers – the first is Row's payout, the second is Column's payout.

Strong Dominance

	Left	Right
Up	4, 1	2, 0
Down	3, 0	1, 1

- Whatever Column does, Row is better off playing Up rather than Down.

Strong Domination

	Left	Right
Up	4, 1	2, 0
Down	3, 0	1, 1

- Whatever Column does, Row is better off playing Up rather than Down.
- We say that Up **strongly dominates** Down.

Strong Domination

	Left	Right
Up	4, 1	2, 0
Middle	5, 0	0, 0
Down	3, 0	1, 1

- Adding options doesn't change things.

Strong Dominance

	Left	Right
Up	4, 1	2, 0
Middle	5, 0	0, 0
Down	3, 0	1, 1

- Adding options doesn't change things.
- Up still dominates Down, even if it isn't always best.

Strong Dominance

	Left	Right
Up	3, 1	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 1

- This is **not** a case of dominance.

Strong Dominance

	Left	Right
Up	3, 1	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 1

- This is **not** a case of dominance.
- Even though Middle is never the highest value, it isn't dominated by any one option.

Strong Dominance

Strategy S_1 strongly dominates strategy S_2 if for any strategy S by the other player(s), if S is played, then S_1 returns a higher payoff than S_2.

Weak Dominance

Strategy S_1 weakly dominates strategy S_2 if for any strategy S by the other player(s), if S is played, then S_1 returns a payoff that is at least as high S_2, and for some strategy by the other player(s), S_1 returns a higher payoff than S_2.

- The difference is that weak dominance allows for **ties**.

Two Dominance Notions

Strong Dominance

- Always better.

Weak Dominance

- Never worse.
- Sometimes better.

Weak Dominance

	Left	Right
Up	4, 1	2, 0
Down	3, 0	**2**, 1

- I've changed the payoffs in the bottom right cell.
- Now Up does not strongly dominate Down.
- But it does weakly dominate Down.

Utility

Ordinal and Cardinal Utility

Dominance Arguments

Some Famous Games

Prisoners' Dilemma

	Coop	Defect
Coop	3, 3	0, 5
Defect	5, 0	1, 1

Generic Symmetric Game

	X	Y
X	a, a	b, c
Y	c, b	d, d

Prisoners' Dilemma

	X	Y
X	a, a	b, c
Y	c, b	d, d

Ordinal constraints

- $c > a, d > b$
- $a > d$

Cardinal constraints

- $2a > b + c$

Stag Hunt

	Coop	Defect
Coop	5, 5	0, 4
Defect	4, 0	2, 2

Stag Hunt

	X	Y
X	a, a	b, c
Y	c, b	d, d

Ordinal constraints

- $a > c, d > b$
- $a > d$

Cardinal constraints

- $a + b < c + d$

Battle of the Sexes

	Row	Col
Row	4, 1	0, 0
Col	0, 0	1, 4

Battle of the Sexes (relabelled)

	Self	Other
Self	0, 0	4, 1
Other	1, 4	0, 0

Chicken

	Attack	Retreat
Attack	-99, -99	2, 0
Retreat	0, 2	1, 1

Rock Paper Scissors

	Rock	Paper	Scissors
Rock	0, 0	–1, 1	1, –1
Paper	1, –1	0, 0	–1, 1
Scissors	–1, 1	1, –1	0, 0

For Next Time

We're jumping ahead to section 2.5 of Bonanno.