

444 Lecture 10

Probability

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February 9, 2023

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Day Plan

Pure revision/introduction of basics of probability

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Totally Not Compulsory

- If you've taken any college class using probability before, then probably 100% of what I say this whole week will be pure revision.
- I bet many people in the class could *teach* all this stuff.
- That's fine - take a week off.
- But I really don't want anyone left behind before we dive into the more applied stuff.

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Book

Odds & Ends

Introducing Probability & Decision with a Visual Emphasis
Jonathan Weisberg

Preface

Tim textbook is for introductory philosophy courses on probability and inductive logic. It is based on a typical such course I teach at the University of Toronto, where we offer "Probability & Inductive Logic" in the second year, alongside the usual inductive logic intro.

The book assumes no deductive logic. The only chapters introduce the little that's used. In fact almost no formal background is presumed, only very simple high school algebra.

Several well known professors inspired and shaped this book. Brian Skyrms' *Choice & Chance* and Ian Hacking's *An Introduction to Probability and Inductive Logic* were especially influential. Both texts are widely used with good reason—they are excellent. I've taught both myself many times, with great success. But this book blends my favourite aspects of each, organizing them in the sequence and style I prefer.

These slides are based off an open access textbook, Odds and Ends, available at <https://jonathanweisberg.org/vip/>

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Day Plan

Basics of Probability

Features of Probability Functions

Tree Example

Conditional Probability

Independence

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Associated Reading

Odds and Ends, Chapter 1

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Examples - Same Result in Each Game

Women	Men	Probability
Win	Win	0.45
Win	Lose	0.25
Lose	Win	0.20
Lose	Lose	0.10

- It is the same result in each game at lines 1 and 4.
- So its probability is $0.45 + 0.10 = 0.55$.

Day Plan

- Basics of Probability
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Associated Reading

Odds and Ends, Chapter 5

Scale

$$0 \leq \Pr(A) \leq 1$$

Negation

$$\Pr(\neg A) = 1 - \Pr(A)$$

Excluded Middle

$$\Pr(A) + \Pr(\neg A) = 1$$

Some events A_1, \dots, A_n form a partition if, necessarily, exactly one of them is true.

- So they are **exclusive** - you can't have any two of them both be true.
- And they are **exhaustive** - you have to have at least one true.

If A_1, \dots, A_n form a partition then

$$\Pr(A_1) + \dots + \Pr(A_n) = 1$$

If A, B are exclusive

$$\Pr(A \vee B) = \Pr(A) + \Pr(B)$$

$$\Pr(A) + \Pr(B) = \Pr(A \vee B) + \Pr(A \wedge B)$$

It's worth thinking through why this is true in terms of possibilities.

- Often, we can't just write down numbers for the possibilities.
- But we can write down, or at least make reasonable guesses about, numbers for certain events.
- And we can think about how things are likely to go given those events happen.
- This is the tree structure that is used in *Odds and Ends*.

Let's think about the probability that Michigan wins the men's football championship next year.

- Divide up the state space.
- Work out the probability of being in one or other part of the space.
- Work out the probability of winning given you are in one or other part of the space.
- Add things up.

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- There are a lot of ways to do this.
- You could divide up the space by how much it snows for the next two months.
- Or you could divide it up by whether we win one or other basketball championship.
- Or, more reasonably, and this is what we'll use, you could divide it up by what Coach Harbaugh does over the next few months.

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- So assume (I'm writing these slides a fair way in advance of the actual class) that we don't know whether Coach Harbaugh will stay at Michigan or leave for an NFL job.
- But what he does makes a big difference to Michigan's chances.

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We need some numbers for illustration, so let's make some up.

- 60% chance he simply stays with no distraction.
- 30% chance he goes to the NFL.
- 10% chance he dithers and is linked to NFL jobs all through spring before staying.

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Again, let's make some numbers up. (Not realistic; not investment advice!)

- If the coach stays (with no distractions) we have a 30% chance of winning.
- If he leaves, we have a 20% chance of winning.
- But if he dithers, and there's no good team planning, we have a 0% chance of winning.

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1. Work out (or at least estimate) probability of each state.
2. Work out probability of the winning within each of those states.

This will involve a lot of guesswork – do not make investment decisions on the basis of the numbers I'm using

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The Giant Table

	We win!	We don't win
Coach simply stays	$0.6 \times 0.3 = 0.18$	$0.6 \times 0.7 = 0.42$
Coach leaves	$0.3 \times 0.2 = 0.06$	$0.3 \times 0.8 = 0.24$
Coach dithers	$0.1 \times 0.0 = 0.00$	$0.1 \times 1.0 = 0.10$

So the probability that we win (given these literally made up assumptions) is $0.18 + 0.06 + 0.00 = 0.24$, or 76%.

Probabilities and Errors

- The error bars on that calculation are massive.
- But it's a kind of sanity check on your reasoning
- Especially in situations where only a handful of paths lead to a salient result (like in playoffs in sports, or when thinking about the likelihood of a particular challenger winning), doing the tree even with favorable numbers can give you a conservative estimate of some probability.

Three Cases Where This is More Precise

1. Probabilities are stipulated
2. Probabilities are due to well understood chance processes (like gambling devices)
3. Probabilities are derived from very large data sets

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Soccer Tournament

There is a big soccer tournament this weekend. The teams competing are

- Fireflies
- Penguins
- Huskies
- Bluebirds

Tournament Structure

There will be three games. Each game will have a winner one way or the other (maybe via penalty kicks or extra time).

1. Fireflies vs Penguins
2. Huskies vs Bluebirds
3. Winner of Game 1 vs Winner of Game 2

The teams are not all equally good. They each have a ‘strength’. Here is their respective strengths

Team	Strength
Fireflies	5
Penguins	4
Huskies	3
Bluebirds	1

If a team with strength x plays a team with strength y , the team with strength x will win with probability

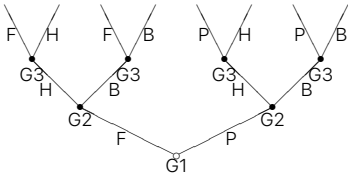
$$\frac{x}{x + y}$$

And the team with strength y will win with probability

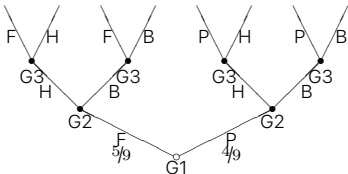
$$\frac{y}{x + y}$$

What is the probability that each team will win the tournament?

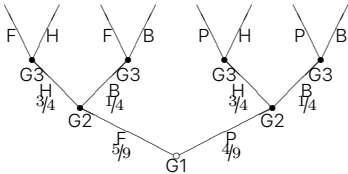
- We will answer this by doing a tree.



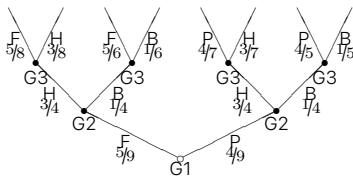
Now we have to add the probabilities to it.



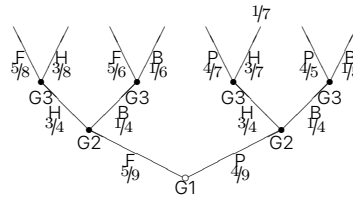
The first game is strength 5 vs strength 4, so the win probability for the stronger team is $\frac{5}{5+4}$, i.e., $\frac{5}{9}$.



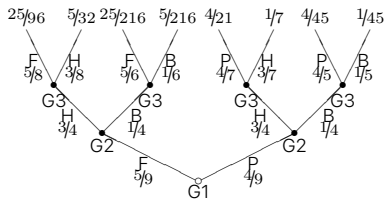
The second game is strength 3 vs strength 1, so the win probability for the stronger team is $\frac{3}{3+1}$, i.e., $\frac{3}{4}$. And it doesn't matter how the first game went - that's the probability for the second game.



And now for each possible match up in game 3, we apply the formula to get the win probability for each team.



- The probability of each completed branch is the product of each of the smaller branches.
- So the one I've marked is $\frac{4}{9} \times \frac{3}{4} \times \frac{3}{7} = \frac{1}{7}$.



I've included all the others - they usually don't cancel as nicely as that one.

It might be easier to see the results in a table

Winner	Runner-Up	Probability	Approx
Fireflies	Huskies	$\frac{25}{96}$	0.260
Huskies	Fireflies	$\frac{5}{32}$	0.156
Fireflies	Bluebirds	$\frac{25}{216}$	0.116
Bluebirds	Fireflies	$\frac{5}{216}$	0.023
Penguins	Huskies	$\frac{4}{21}$	0.190
Huskies	Penguins	$\frac{1}{7}$	0.143
Penguins	Bluebirds	$\frac{4}{45}$	0.089
Bluebirds	Penguins	$\frac{1}{45}$	0.022

And we can rearrange that so the rows where each team wins are adjacent.

Winner	Runner-Up	Probability	Approx
Fireflies	Huskies	$\frac{25}{96}$	0.260
Fireflies	Bluebirds	$\frac{25}{216}$	0.116
Huskies	Fireflies	$\frac{5}{32}$	0.156
Huskies	Penguins	$\frac{1}{7}$	0.143
Penguins	Huskies	$\frac{4}{21}$	0.190
Penguins	Bluebirds	$\frac{4}{45}$	0.089
Bluebirds	Fireflies	$\frac{5}{216}$	0.023
Bluebirds	Penguins	$\frac{1}{45}$	0.022

And then just adding up the probabilities for the two ways each team can win, we get the actual probabilities of each win. (I'm just doing the decimals now.)

Winner	Approx Probability
Fireflies	0.376
Huskies	0.299
Penguins	0.279
Bluebirds	0.045

(Those numbers don't sum to 1 precisely because of rounding.)

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Associated Reading

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Odds and Ends, Chapter 6

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What it is

Basics of Probability

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- Sometimes we don't just care about how likely something is.
- We care about how likely it is given some other thing has happened or will happen.
- This might be because we want to plan.
- It might be because we want to compute overall probabilities.
- Or it might be because we've found something out, and want to know what it means for other likelihoods.

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Prior Examples

Basics of Probability

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We've already used some conditional probabilities.

- We already talked, for example, about the probability of the Fireflies winning conditional on them being in the final against the Bluebirds.

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Inverting

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But there are other questions we might want to ask as well.

- E.g., conditional on the Fireflies winning, how likely is it that they played the Huskies.

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Intuitions

Basics of Probability

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This isn't an easy question to answer intuitively.

- It is more likely that the Huskies will be actually in the final - because they are the better team.
- But it is more likely that the Fireflies will win against the Bluebirds - because they are weaker.
- It isn't always easy to intuitively balance these forces.

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$$\Pr(A|B) = \frac{\Pr(A \wedge B)}{\Pr(B)}$$

- The left-hand side means "The probability of A given B."
- And the right-hand side says that this is equal to the probability of $A \wedge B$ divided by the probability of B.

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$$\Pr(HR|FC) = \frac{\Pr(HR \wedge FC)}{\Pr(FC)}$$

That is, the probability that the Huskies are runners-up (HR) given that the Fireflies are champions (FC), is given by the formula on the right.

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$$\Pr(HR|FC) = \frac{0.26}{0.26 + 0.116} = \frac{0.26}{0.376} \approx 0.691$$

So conditional on the Fireflies winning, it's just under 70% likely they beat the Huskies.

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We are going to mostly assume that this philosophical claim is true.

$$\Pr_B(A) = \Pr(A|B)$$

The way to read that is saying that the unconditional probability of A after learning B equals the conditional probability of A given B . This claim - and it is a philosophical claim not a mathematical one - is a big part of why we care about conditional probability.

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Odds and Ends, Chapter 6.5

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A and B are independent if (and only if)

$$\Pr(A|B) = \Pr(A)$$

That is, taking things conditional on B doesn't change A .

Causal

- B might be a possible cause of A .
- B might be a possible preventer of A .
- B might be a common effect of a frequent cause of A .
- B might be a common effect of a frequent preventer of A .

Epistemic

- B being true could tell you that a source that also predicts A is more reliable than you thought.

1. In reality, strict independence almost never obtains.
2. In practice, it's very often useful to assume independence for modelling purposes.

These are consistent, but it does mean be careful. Sometimes assuming independence is like assuming that relativistic considerations aren't important to figuring out whether a bridge will stand up. And sometimes it is like assuming that friction isn't important to figuring out whether a bridge will stand up.

We'll get back to game theory, looking in more detail at what happens once strategies are probabilistic.