444 Lecture 10

Probability

Brian Weatherson

2/9/23

Day Plan

Pure revision/introduction of basics of probability

Totally Not Compulsory

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- I bet many people in the class could *teach* all this stuff.
- That's fine - take a week off.
- But I really don't want anyone left behind before we dive into the more applied stuff.

Book

Odds & Ends

Introducing Probability & Decision with a Visual Emphasis

Jonathan Weisberg

Preface

THIS textbook is for introductory philosophy courses on probability and inductive logic. It is based on a typical such course I teach at the University of Toronto, where we offer "Probability & Inductive Logic" in the second year, alongside the usual deductive logic intro.

The book assumes no deductive logic. The early chapters introduce the little that's used. In fact almost no formal background is presumed, only very simple high school algebra.

Several well known predecessors inspired and shaped this book. Brian Skyrms' *Choice & Chance* and Ian Hacking's *An Introduction to Probability and Inductive Logic* were especially influential. Both texts are widely used with good reason—they are excellent. I've taught both myself many times, with great success. But this book blends my favourite aspects of each, organizing them in the sequence and style I prefer.

These slides are based off an open access textbook, Odds and Ends, available at https://jonathanweisberg.org/vip/

Day Plan

Basics of Probability

Features of Probabiility Functions

Tree Example

Conditional Probability

Independence

Associated Reading

Odds and Ends, Chapter 1

Basic Idea

- A probability function is a mapping from possibilities to numbers.

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- The numbers must sum to one.
- Intuitively, the numbers measure how likely the possibilities are.

Sum to One

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- What proportion of UM students are from North Carolina?

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- What proportion of UM students are from North Carolina?
- What proportion of UM undergraduates are Tigers fans?

Three Big Questions

1. What to do with these numbers?

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2. Where these numbers come from?

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1. What to do with these numbers?
2. Where these numbers come from?
3. What do the numbers even mean?

A Simple Case

- Imagine that it is basketball season, and UM has planned to have both the women's and men's teams play on the same night.

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- Imagine that it is basketball season, and UM has planned to have both the women's and men's teams play on the same night.
- So at the end of the night there are four possible outcomes.

Made Up Probabilities

I'll stipulate that the probabilities of the four possible outcomes are given by this table.

	Men Win	Men Lose
Women Win	0.45	0.25
Women Lose	0.20	0.10

Another Representation

Here are the same numbers written a different way.

Women	Men	Probability
Win	Win	0.45
Win	Lose	0.25
Lose	Win	0.20
Lose	Lose	0.10

Possibilities

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- But it tells us everything that's relevant to a particular inquiry.
- It is a lot like a line on a truth table.

Events

We will say an **event** is a proposition that can be defined using these possibilities. So here are some sample events.

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- The women's team wins.
- The men's team wins.
- At least one Michigan team wins.
- The two teams have the same result.

Probability of Events

- An event is true at some possibilities, false at others.

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- An event is true at some possibilities, false at others.
- Each possibility gets a probability.
- The probability of an event is the sum of the probabilities of the possibilities where it is true.

Examples - Probability Women's Team Wins

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Win	Win	0.45
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- The women's team wins at lines 1 and 2.

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- So its probability is
- $0.45 + 0.25 = 0.7$.

Examples - Probability Men's Team Wins

Women	Men	Probability
Win	Win	0.45
Win	Lose	0.25
Lose	Win	0.20
Lose	Lose	0.10

- The men's team wins at lines 1 and 3.

Examples - Probability Men's Team Wins

Women	Men	Probability
Win	Win	0.45
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Lose	Win	0.20
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- The men's team wins at lines 1 and 3.
- So its probability is 0.45 + 0.20 = **0.65**.

Examples - At Least One Team Wins

Women	Men	Probability
Win	Win	0.45
Win	Lose	0.25
Lose	Win	0.20
Lose	Lose	0.10

- At least one team wins at lines 1, 2 and 3.

Examples - At Least One Team Wins

Women	Men	Probability
Win	Win	0.45
Win	Lose	0.25
Lose	Win	0.20
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- At least one team wins at lines 1, 2 and 3.
- So its probability is 0.45 + 0.25 + 0.20 = **0.90**.

Examples - Same Result in Each Game

Women	Men	Probability
Win	Win	0.45
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Lose	Win	0.20
Lose	Lose	0.10

- It is the same result in each game at lines 1 and 4.

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- It is the same result in each game at lines 1 and 4.
- So its probability is 0.45 + 0.10 = {0.55}.

Day Plan

Basics of Probability

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Associated Reading

Odds and Ends, Chapter 5

Scale

$$0 \leq \Pr(A) \leq 1$$

Negation

$$\Pr(\neg A) = 1 - \Pr(A)$$

Excluded Middle

$$\Pr(A) + \Pr(\neg A) = 1$$

Partition

Some events $A_1, \ldots A_n$ form a partition if, necessarily, exactly one of them is true.

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Partition

Some events $A_1, \ldots A_n$ form a partition if, necessarily, exactly one of them is true.

- So they are **exclusive** - you can't have any two of them both be true.
- And they are **exhaustive** - you have to have at least one true.

Partition

If $A_1, \ldots A_n$ form a partition then

$$\Pr(A_1) + \cdots + \Pr(A_n) = 1$$

Exclusive

If A, B are exclusive

$$\Pr(A \vee B) = \Pr(A) + \Pr(B)$$

General Principle

$$\Pr(A) + \Pr(B) = \Pr(A \vee B) + \Pr(A \wedge B)$$

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$$\Pr(A) + \Pr(B) = \Pr(A \vee B) + \Pr(A \wedge B)$$

It's worth thinking through why this is true in terms of possibilities.

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- But we can write down, or at least make reasonable guesses about, numbers for certain events.
- And we can think about how things are likely to go given those events happen.
- This is the tree structure that is used in *Odds and Ends*.

Football Forecasting

Let's think about the probability that Michigan wins the men's football championship next year.

One Method

- Divide up the state space.

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- Work out the probability of being in one or other part of the space.

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- Work out the probability of winning given you are in one or other part of the space.

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- Work out the probability of being in one or other part of the space.
- Work out the probability of winning given you are in one or other part of the space.
- Add things up.

Nothing is Reliable

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- Or you could divide it up by whether we win one or other basketball championship.

Nothing is Reliable

- There are a lot of ways to do this.
- You could divide up the space by how much it snows for the next two months.
- Or you could divide it up by whether we win one or other basketball championship.
- Or, more reasonably, and this is what we'll use, you could divide it up by what Coach Harbaugh does over the next few months.

Coach Outcomes

- So assume (I'm writing these slides a fair way in advance of the actual class) that we don't know whether Coach Harbaugh will stay at Michigan or leave for an NFL job.

Coach Outcomes

- So assume (I'm writing these slides a fair way in advance of the actual class) that we don't know whether Coach Harbaugh will stay at Michigan or leave for an NFL job.
- But what he does makes a big difference to Michigan's chances.

Coach Outcomes

We need some numbers for illustration, so let's make some up.

- 60% chance he simply stays with no distraction.

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- 30% chance he goes to the NFL.

Coach Outcomes

We need some numbers for illustration, so let's make some up.

- 60% chance he simply stays with no distraction.
- 30% chance he goes to the NFL.
- 10% chance he dithers and is linked to NFL jobs all through spring before staying.

Team Outcomes

Again, let's make some numbers up. (Not realistic; not investment advice!)

- If the coach stays (with no distractions) we have a 30% chance of winning.

Team Outcomes

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- If the coach stays (with no distractions) we have a 30% chance of winning.
- If he leaves, we have a 20% chance of winning.

Team Outcomes

Again, let's make some numbers up. (Not realistic; not investment advice!)

- If the coach stays (with no distractions) we have a 30% chance of winning.
- If he leaves, we have a 20% chance of winning.
- But if he dithers, and there's no good team planning, we have a 0% chance of winning.

Two Step Process

1. Work out (or at least estimate) probability of each state.

This will involve a lot of guesswork – do not make investment decisions on the basis of the numbers I'm using

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1. Work out (or at least estimate) probability of each state.
2. Work out probability of the winning within each of those states.

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The Giant Table

	We win!	We don't win
Coach simply stays	$0.6 \times 0.3 = 0.18$	$0.6 \times 0.7 = 0.42$
Coach leaves	$0.3 \times 0.2 = 0.06$	$0.3 \times 0.8 = 0.24$
Coach dithers	$0.1 \times 0.0 = 0.00$	$0.1 \times 1.0 = 0.10$

So the probability that we win (given these literally made up assumptions) is $0.18 + 0.06 + 0.00 = 0.24$, or 76%.

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- The error bars on that calculation are massive.
- But it's a kind of sanity check on your reasoning
- Especially in situations where only a handful of paths lead to a salient result (like in playoffs in sports, or when thinking about the likelihood of a particular challenger winning), doing the tree even with favorable numbers can give you a conservative estimate of some probability.

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1. Probabilities are stipulated

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1. Probabilities are stipulated
2. Probabilities are due to well understood chance processes (like gambling devices)
3. Probabilities are derived from very large data sets

Day Plan

- Basics of Probability
- Features of Probabiility Functions
- Tree Example
- Conditional Probability
- Independence

Soccer Tournament

There is a big soccer tournament this weekend. The teams competing are

- Fireflies

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Soccer Tournament

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- Fireflies
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- Huskies

Soccer Tournament

There is a big soccer tournament this weekend. The teams competing are

- Fireflies
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Tournament Structure

There will be three games. Each game will have a winner one way or the other (maybe via penalty kicks or extra time).

1. Fireflies vs Penguins

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1. Fireflies vs Penguins
2. Huskies vs Bluebirds

Tournament Structure

There will be three games. Each game will have a winner one way or the other (maybe via penalty kicks or extra time).

1. Fireflies vs Penguins
2. Huskies vs Bluebirds
3. Winner of Game 1 vs Winner of Game 2

Team Strength

The teams are not all equally good. They each have a ‘strength’. Here is their respective strengths

Team	Strength
Fireflies	5
Penguins	4
Huskies	3
Bluebirds	1

Win Probabilities

If a team with strength x plays a team with strength y, the team with strength x will win with probability

$$\frac{x}{x+y}$$

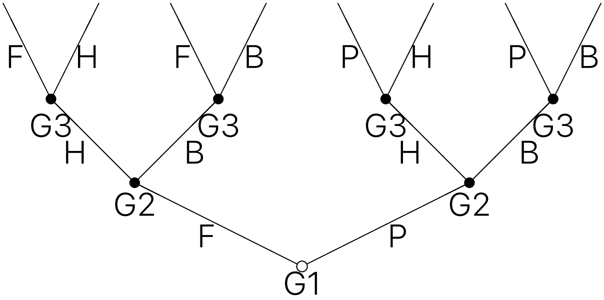
And the team with strength y will win with probability

$$\frac{y}{x+y}$$

Question

What is the probability that each team will win the tournament?

- We will answer this by doing a tree.


F
H
F
B
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G3
G3
G3
G3
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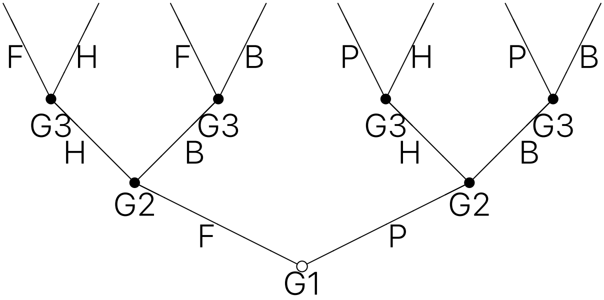
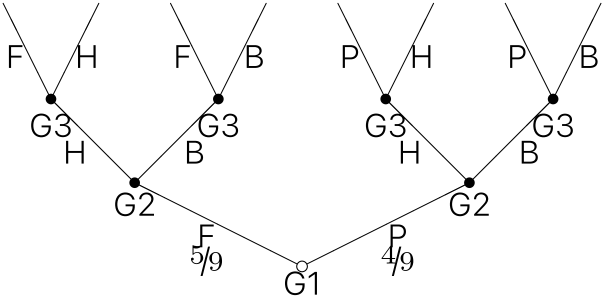
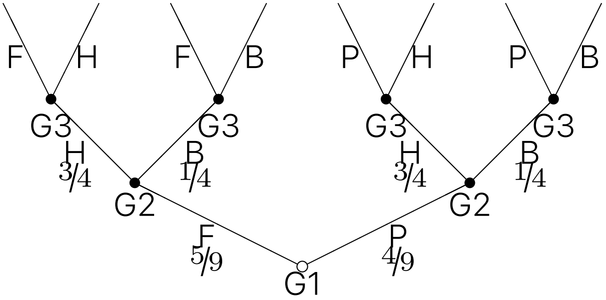
B
H
B
G2
G2
F
P
G1



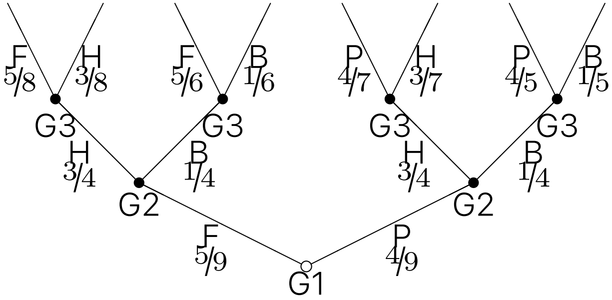
Now we have to add the probabilities to it.



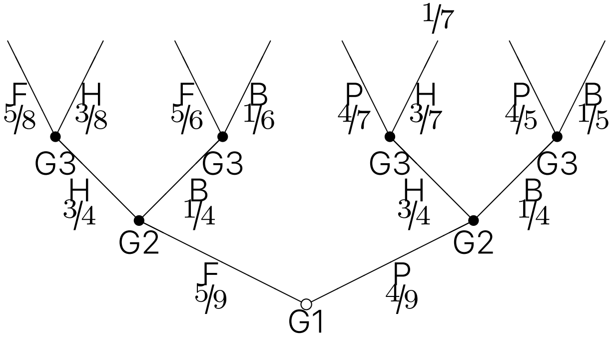
The first game is strength 5 vs strength 4, so the win probability for the stronger team is $5/5+4$, i.e., $5/9$.



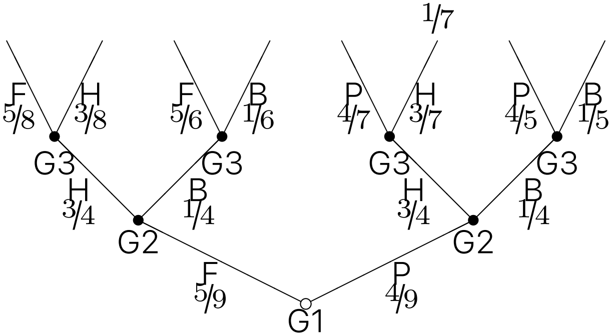
The second game is strength 3 vs strength 1, so the win probability for the stronger team is $3/3+1$, i.e., $3/4$. And it doesn't matter how the first game went – that's the probability for the second game.



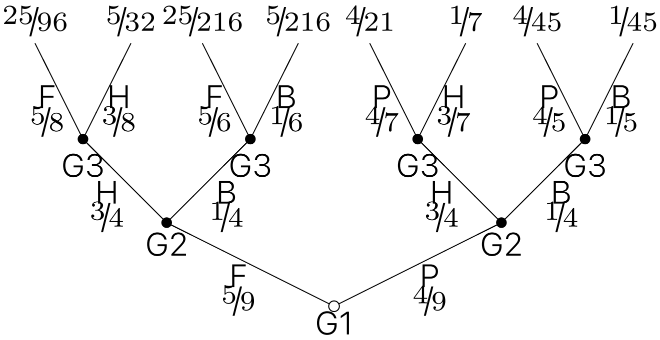
And now for each possible match up in game 3, we apply the formula to get the win probability for each team.



- The probability of each completed branch is the product of each of the smaller branches.



- The probability of each completed branch is the product of each of the smaller branches.
- So the one I've marked is $4/9 \times 3/4 \times 3/7 = 1/7$.



I've included all the others – they usually don't cancel as nicely as that one.

It might be easier to see the results in a table

Winner	Runner-Up	Probability	Approx
Fireflies	Huskies	$^{25}/_{96}$	0.260
Huskies	Fireflies	$^{5}/_{32}$	0.156
Fireflies	Bluebirds	$^{25}/_{216}$	0.116
Bluebirds	Fireflies	$^{5}/_{216}$	0.023
Penguins	Huskies	$^{4}/_{21}$	0.190
Huskies	Penguins	$^{1}/_{7}$	0.143
Penguins	Bluebirds	$^{4}/_{45}$	0.089
Bluebirds	Penguins	$^{1}/_{45}$	0.022

And we can rearrange that so the rows where each team wins are adjacent.

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Huskies	Penguins	$1/7$	0.143
Penguins	Huskies	$4/21$	0.190
Penguins	Bluebirds	$4/45$	0.089
Bluebirds	Fireflies	$5/216$	0.023
Bluebirds	Penguins	$1/45$	0.022

And then just adding up the probabilities for the two ways each team can win, we get the actual probabilities of each win. (I'm just doing the decimals now.)

Winner	Approx Probability
Fireflies	0.376
Huskies	0.299
Penguins	0.279
Bluebirds	0.045

(Those numbers don't sum to 1 precisely because of rounding.)

Day Plan

Associated Reading

Odds and Ends, Chapter 6

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- This might be because we want to plan.
- It might be because we want to compute overall probabilities.
- Or it might be because we've found something out, and want to know what it means for other likelihoods.

Prior Examples

We've already used some conditional probabilities.

- We already talked, for example, about the probability of the Fireflies winning conditional on them being in the final against the Bluebirds.

Inverting

But there are other questions we might want to ask as well.

- E.g., conditional on the Fireflies winning, how likely is it that they played the Huskies.

Intuitions

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- It is more likely that the Huskies will be actually in the final – because they are the better team.
- But it is more likely that the Fireflies will win against the Bluebirds – because they are weaker.
- It isn't always easy to intuitively balance these forces.

Formula

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$$\Pr(A|B) = \frac{Pr(A \wedge B)}{Pr(B)}$$

- The left-hand side means "The probability of A given B."
- And the right-hand side says that this is equal to the probability of $A \wedge B$ divided by the probability of B.

Fireflies

$$\Pr(HR|FC) = \frac{\Pr(HR \wedge FC)}{\Pr(FC)}$$

That is, the probability that the Huskies are runners-up (HR) given that the Fireflies are champions (FC), is given by the formula on the right.

Fireflies

$$\Pr(HR|FC) = \frac{0.26}{0.26 + 0.116} = \frac{0.26}{0.376} \approx 0.691$$

So conditional on the Fireflies winning, it's just under 70% likely they beat the Huskies.

Updating

We are going to mostly assume that this philosophical claim is true.

$$\Pr_B(A) = \Pr(A|B)$$

The way to read that is saying that the unconditional probability of A after learning B equals the conditional probability of A given B. This claim - and it is a philosophical claim not a mathematical one - is a big part of why we care about conditional probability.

Day Plan

Basics of Probability

Features of Probabiility Functions

Tree Example

Conditional Probability

Independence

Associated Reading

Odds and Ends, Chapter 6.5

Independence

A and B are independent if (and only if)

$$\Pr(A|B) = \Pr(A)$$

That is, taking things conditional on B doesn't change A.

Ways Independence can Fail

Causal

- B might be a possible cause of A.

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Ways Indepenence can Fail

Epistemic

- B being true could tell you that a source that also predicts A is more reliable than you thought.

Two Big Real World Facts about Independence

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These are consistent, but it does mean be careful. Sometimes assuming independence is like assuming that relativistic considerations aren't important to figuring out whether a bridge will stand up. And sometimes it is like assuming that friction isn't important to figuring out whether a bridge will stand up.

For Next Time

We'll get back to game theory, looking in more detail at what happens once strategies are probabilistic.