444 Lecture 9

Mixed Strategies and Philosophy

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Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

Basic Example

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

This is a bit boring for Column, but let’s focus on Row for now.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

At first it looks like there are no dominated strategies.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- Up does worse than Middle and Down if Column plays Right, so it doesn’t dominate anything.
- Middle does worse that Up if Column plays Left, and worse than Down if Column plays Right.
- Down does worse than both of them if Column plays Left.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

But compare these two strategies.

- Middle
- The mixed strategy of Up with probability 0.5, and Down with probability 0.5.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- Middle gets an actual return of 1 whatever Column does.
- The mixed strategy gets an expected return of 1.5 whatever Column does.
- So it has a higher expected return given Left (1.5 > 1), and a higher expected return given Right (1.5 > 1).

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- If that happens, then we'll say that Middle is dominated by this mixture.
- When we're deleting dominated strategies, we should delete it too.

Nash and Dominance

- A strategy that is dominated by a mixture like this can never be part of a Nash equilibrium.
- After all, the player would be better off playing the mixture than playing it, so it fails the test that there is nothing better to do.
- So being able to find these dominating mixtures can be very helpful in working out what the Nash equilibrium is.

Rational Play and Dominance

- But even beyond that, it seems wrong to play strategies that are dominated in this way.
- If you're thinking about playing Middle (as Row), you increase your expected return by simply flipping a coin to choose between Left and Right.
- So that's what you should do.

Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Up is the best response to Left.
- Down is the best response to Right.
- Is Middle the best response to anything?

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

Yes!

- Middle is the best response to the mixed strategy Left with probability 0.5, Right with probability 0.5.
- It gets 2, the other options have an expected return of 1.5.

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Middle is the best thing to do if you know Column is going to flip a coin to decide what to do.
- But it's also the best thing to do if you have no idea what Column is going to do, and the best you can do is say it's 50/50 what they are going to do.
- So it's actually pretty easy to think of situations where Middle is the smart play.

Best Response

- A strategy *S* is a **best response** just in case...
- There is some probability distribution over the other player's strategies and ...
- No strategy has a higher expected return than *S* given that probability distribution.

Best Response

- Note that this allows for ties.
- Weakly dominated strategies can even be best responses in this sense.
- This definition also covers mixed strategies; they can also be best responses.

A Surprising Theorem

- Say a strategy is **undominated** if no other strategy, pure or mixed, strongly dominates it.
- And it is a **best response** if it does as well as anything, given at least one probability distribution.
- Here's the surprising theorem:

  The strategies that are best responses are just the same strategies as those that are undominated.

Best Reponses

- This relates back to something I was saying in the last lecture.
- The strategies that are dominated by mixtures didn't seem to make sense - you could just play the mixtures.
- But here's another property that they have - they are never best responses.

Best Reponses

- And if they are not best responses, no one can play them while maximising expected utility.
- Whatever probability you give to the other player's play, if you maximise expected utility you will play a best response.
- And you should maximise expected utility.

Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

	Left	Right
Up	3, 0	0, 1
Middle	1, 1	1, 0
Down	0, 0	3, 1

In this game, the best responses are:

- Row can play Up (best response to Left) or Down (best response to Right);
- Column can play Left (best response to Middle) or Right (best response to either Up or Down).

	Left	Right
Up	3, 0	0, 1
Middle	1, 1	1, 0
Down	0, 0	3, 1

- But Middle is not a best response.
- It is dominated by the 50/50 mixture of Left and Right.

	Left	Right
Up	3, 0	0, 1
Middle	1, 1	1, 0
Down	0, 0	3, 1

- So while Left is a best response...
- It is not a best response to a best response.

Iterated Best Responses

- That makes it seem irrational to play Middle.
- I could build more complicated examples, where we had cases that are best responses to best responses, but not best responses to best responses to best responses.

Iterated Best Responses

- Actually we've already seen such a case.
- In the Ice Cream game, 2 is a best response to 1, which is a best response to 0.
- But 2 is not a best response to any best response to a best response.

Iterated Best Responses

- Some strategies are at the start of an infinite chain $S_1, S_2, \ldots$ where each strategy is a best response to the one that comes after it.
- Call these the **rationalizable** strategies.

Infinite Chains

Here is one way to get an infinite chain like this.

- If the pair $\langle S_1, S_2 \rangle$ is a Nash equilibrium, …
- Then S_1 is a best response to S_2, which is a best response to S_1, which is a best response to S_2, which …
- But you don't only need to use Nash equilibria.

Infinite Chains

- Think about Rock, Paper, Scissors.
- Rock is a best response to Scissors, which is a best response to Paper, which is a best response to Rock, which is…
- But Rock is not part of a Nash equilibrium.

Rationalizability

I'm not going to prove this, but the following turns out to be true.

- The strategies that can be at the start of these infinite chains …
- Are exactly those strategies that survive iterated deletion of strongly dominated strategies …
- Provided we include dominance by mixtures when we're doing the deleting.

Philosophical Payoff

Some economists, and a few philosophers, have argued that this is the key philosophical notion in game theory.

- They say that a strategy is rational to play if and only if it is rationalizable in this sense.
- In economics, this is very much a **heterodox** view.
- Note that this view is more permissive than the view that rational players will choose Nash equilibria.

Philosophical Payoff

- All Nash equilibria are rationalizable, but some rationalizable strategies (e.g., Rock!), are not Nash equilibria.
- Most economists think that if there is a key notion in game theory, it is **less permissive** than Nash equilibrium.

Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

Reading

Bonanno, sections 2.6 (which we discussed earlier) and 6.4.

Two Conjectures

1. It is rational to play any rationalizable strategy.
2. It is only rational to play Nash Equilibrium strategies

I'm going to end this week talking a bit about why people might prefer 2 over 1.

One Intuitive Idea

- Don't just play Rock - the other person will figure it out.
- Rock every time is rationalizable.
- But you shouldn't do it.
- Therefore principle 1 must be false.

Response 1

- Yeah, you shouldn't play Rock every single time, that's dumb.
- But on any given occasion, it's fine.
- And we know, from e.g., Prisoners' Dilemma, that we shouldn't infer what to do in a single shot game from what happens in the repeated game.

Response 2

- The orthodox solution (i.e., principle 2) actually doesn't give you the result you might want here.
- It is possible that the randomising device will come up Rock every single time.
- So if you think it's always irrational to play Rock repeatedly, you have to think both of these are wrong.

Response 3

- If principle 2 is right, all rational players will randomise every time.
- So the expected return of Rock is just the same as the expected return of randomisation.
- So it can't be wrong to play it.

Mixing Response 2 and Response 3

- If principle 2 is right, and it's common knowledge that the players are rational, then the rational way to interpret the other player playing Rock every time is "Wow, their random device is having a freaky run."
- But if that's right, there isn't anything wrong with playing Rock every time.

Other Direction

- As we'll see when we get to O'Connor's book, you mostly see people wanting more restrictions on moves than Nash.
- But Bonanno ends chapter 6 with an interesting reason for thinking even rationalisability (i.e., IDSDS) is too strong.

Other Direction

- It's really incredibly unrealistic to know the utility function that the other player has.
- You might know the physical outcomes of the game, but knowing what utility each player gets is a huge assumption.
- So in practice, you should probably not rely too heavily on theories or policies that rely on this knowledge.

For Next Time

- Next week we will do a bit of revision of probability theory.
- It's completely optional, and will be very familiar to many people here; if you want a day off, take a day off.
- After that, we'll look at how game theorists think about signals and messages.