444 Lecture 4

Equilibrium

Brian Weatherson

1/17/23

Day Plan

Trees

Backward Induction

Ties

Time

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- But few games are like that.
- We need a way to represent games that take time.

Trees

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- A tree represents all the ways that a game that takes place over time could go.

Nodes

- Trees have nodes.
- Some nodes are **terminal nodes**; they represent that the game has ended.
- Each terminal node has a payout for each of the players.
- At any other node, either a player moves, or Nature 'moves'.
- One of the non-terminal nodes is special: it is the node where the game starts.

Branches

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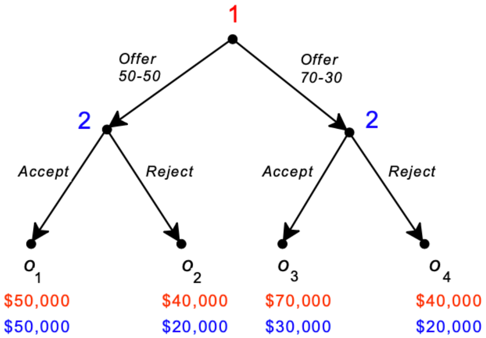
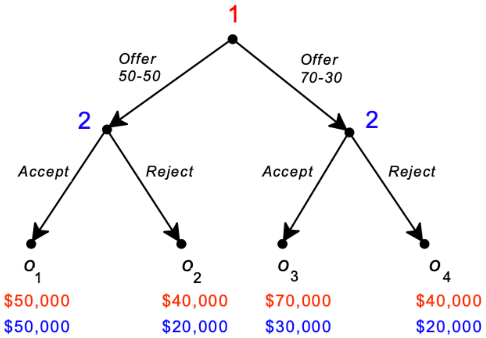
- A move at a node is always a choice of branches.



Example from Bonanno

- There are two players, 1 and 2.



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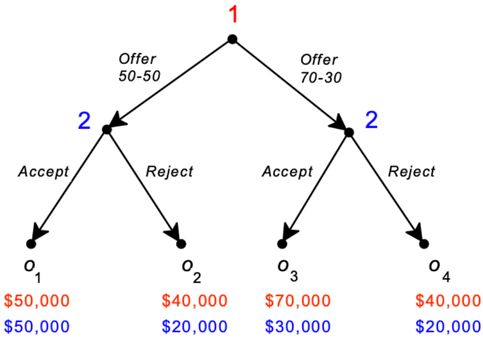
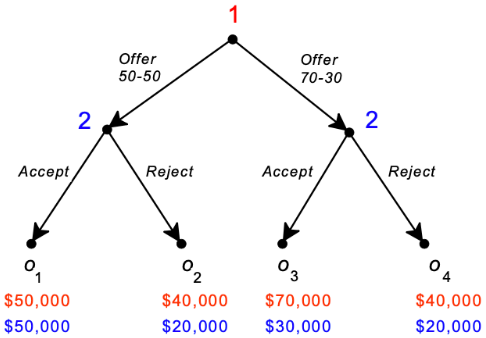


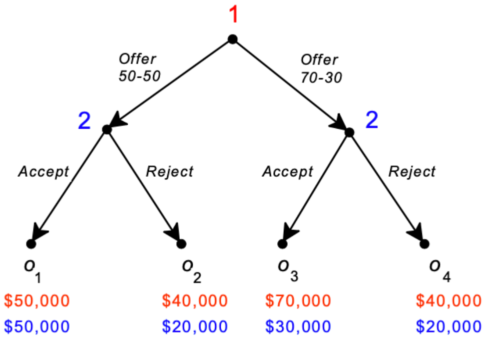
Example from Bonanno

- There are two players, 1 and 2.
- Each player moves once.
- First 1 moves, then 2 moves, then the game ends.



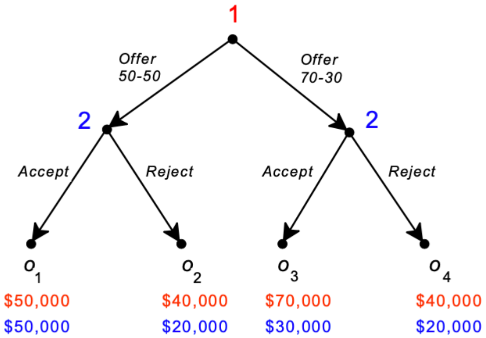
Example from Bonanno

- Some books use a special notation for the initial node, such as having an open circle rather than a closed circle.



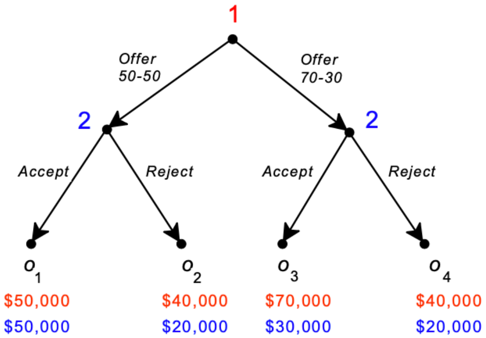
Example from Bonanno

- Some books use a special notation for the initial node, such as having an open circle rather than a closed circle.
- Bonanno doesn’t, but it’s clear in context what the initial node is.



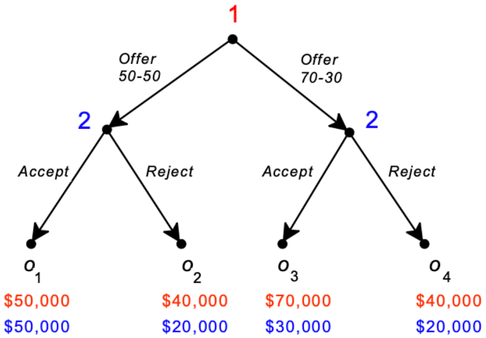
Example from Bonanno

- As he goes on to note, this isn’t really a tree yet.



Example from Bonanno

- As he goes on to note, this isn't really a tree yet.
- It describes the physical outcomes of the game at each terminal node, but not the **payoffs**.



Example from Bonanno

There is a natural function from outcomes to payoffs - more money equals more utility - but it is not a compulsory interpretation.

Future Additions

- Moves by Nature
- Moves under uncertainty

Day Plan

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Class of Games We're Discussing

- Two-player
- Turn-taking
- Finite
- No hidden facts
- No randomness
- We'll start with zero-sum games, though drop this later.

Five

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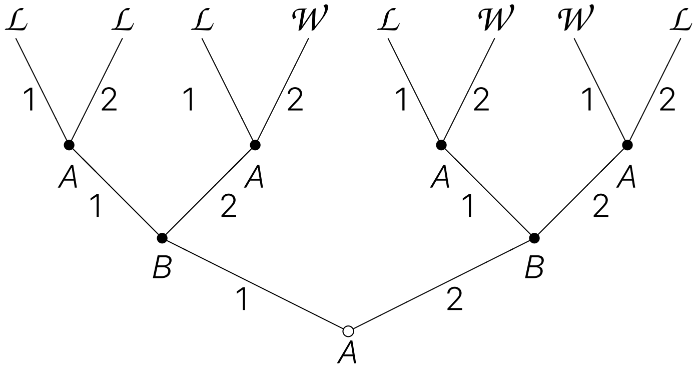
- There are two players, who we'll call A and B.
- First A moves, then B, then finally A moves again.
- Each move involves announcing a number, 1 or 2.
- A wins if after the three moves, the numbers announced sum to 5.
- B wins otherwise.

Five

Question: How should you play this game?

Game Tree for Five



$\mathcal{W}$ means that A wins, and $\mathcal{L}$ means that B wins.

How to Solve These Games

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- Assume that they will make the higher value for them choice.
- Mark that choice, e.g., by doubling the line (as the textbook does) or bolding the line (as I'll do).
- If there are ties, mark both of the lines. (This gets more complicated once we leave zero-sum games.)

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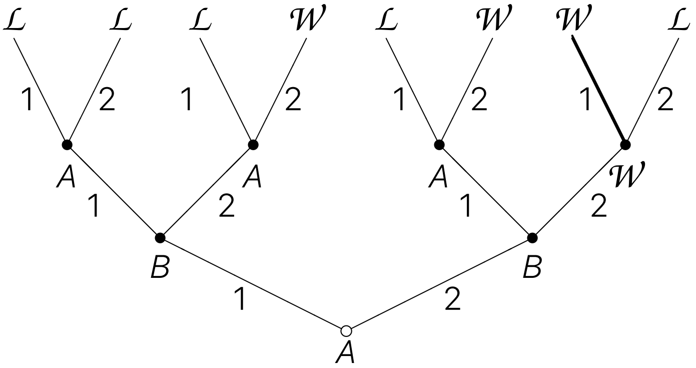
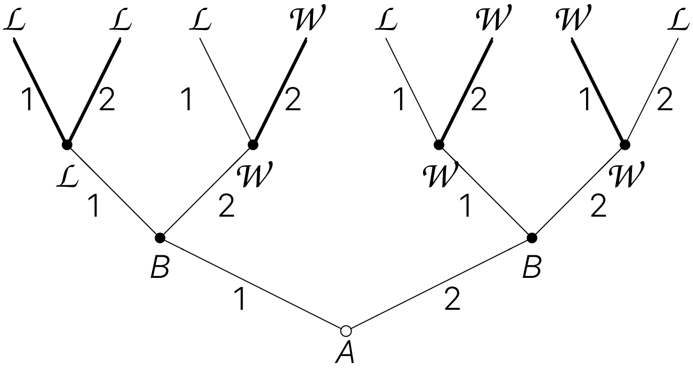
- So just the game assigns values to terminal nodes, we'll now assign value to choice nodes.
- In **Five**, we'll assign the value $\mathcal{W}$ to the top right node.

Five (after one step)



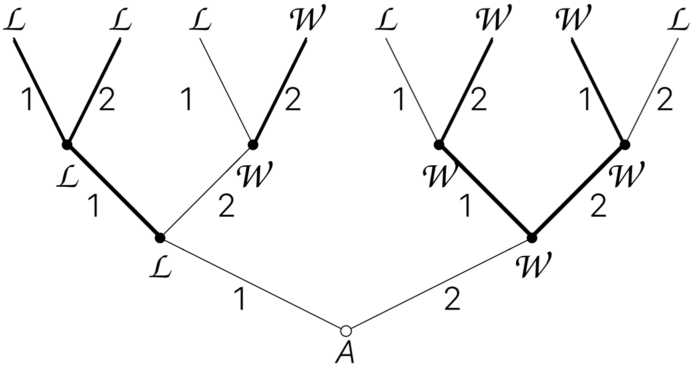
Five (after first level)



Next Steps Back

- Now we do the same thing for B.
- We act as if B is choosing between terminal nodes.
- It is as if A doesn't have a choice - they will just make the choice that is best for them (i.e., worst for B).
- So B knows what the outcome of each choice will be.

Five (After Two Rounds)



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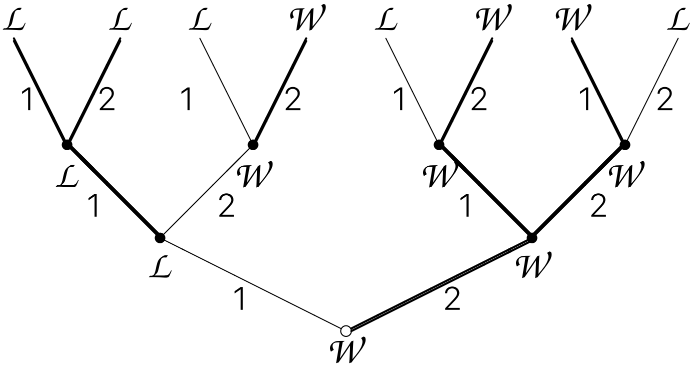
- So we act as if getting to the left hand node means B wins, and getting to the right hand node means A wins.
- And now we just have to make the choice for the initial node, using this fact.

Five (Full Graph)



Five - Full Analysis

- The equilibrium state of the game is that A wins.

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- The equilibrium state of the game is that A wins.
- A plays 2 first.
- Then B can play anything they line.
- But whatever they do, A will win, by playing the opposite number.

Backwards Induction

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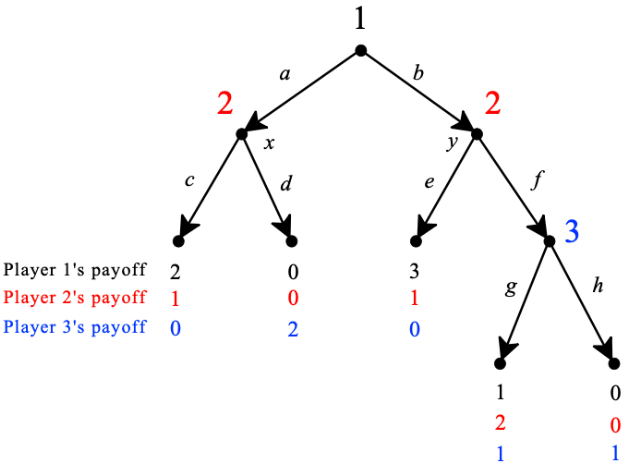
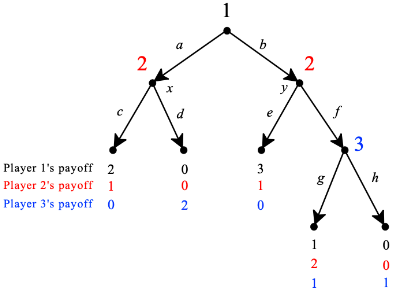
- We start at the possible ends of the game.
- At each step, we assume that each player makes the best decision they can, on the assumption that later players will do the same thing.

Day Plan

Backwards Induction in Positive Sum Games

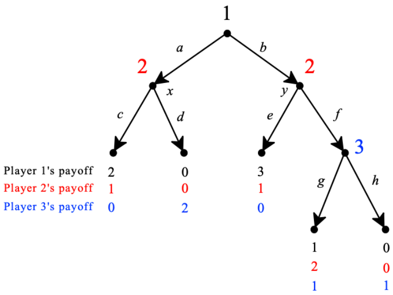


This is three player, but crucially, it is not zero sum.



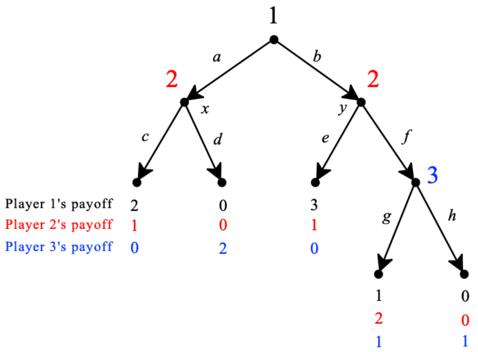
Three player game tree

- In the bottom right, Player 3 doesn’t care which choice is made.



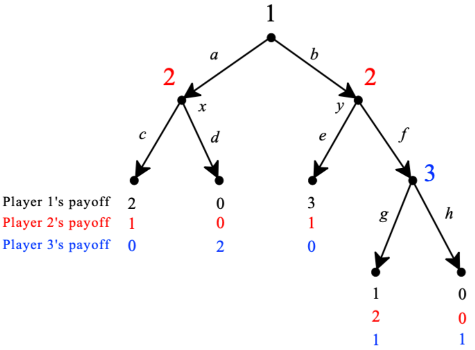
Three player game tree

- In the bottom right, Player 3 doesn't care which choice is made.
- So we can't infer what Player 3 will do.



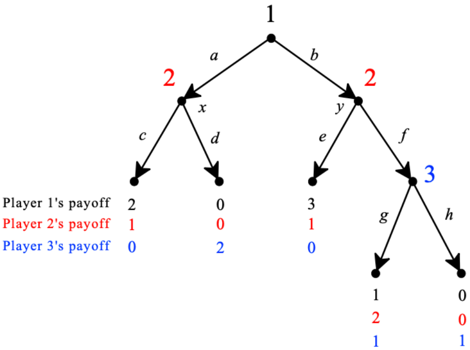
Three player game tree

- But the other players do care what Player 3 will do.



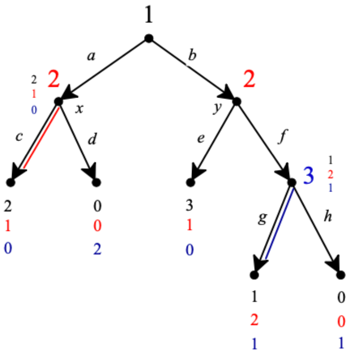
Three player game tree

- But the other players do care what Player 3 will do.
- So we can't just ignore this choice.



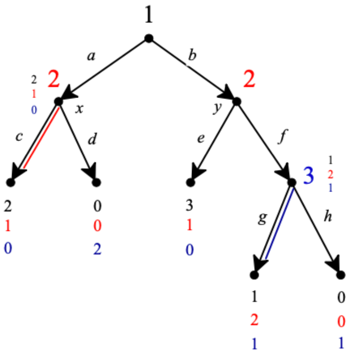
Three player game tree

- The solution is to build two trees, one for each of Player 3′s choices.



Solution One

- First, assume 3 plays g.



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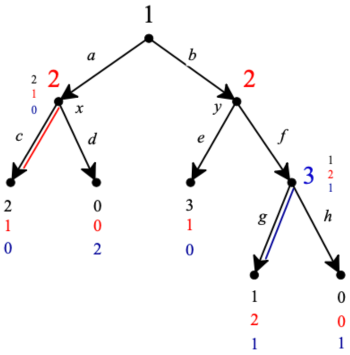
- First, assume 3 plays g.
- Then 2 would play f at node y.



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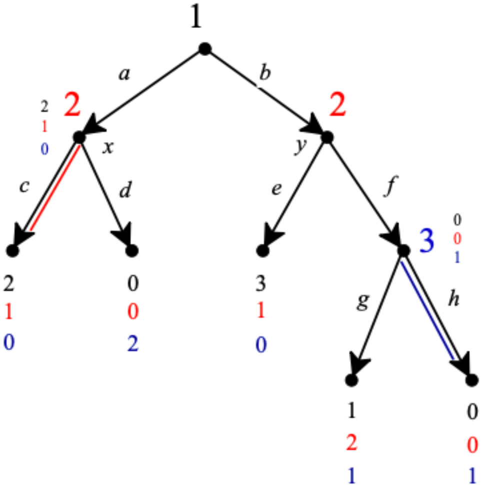
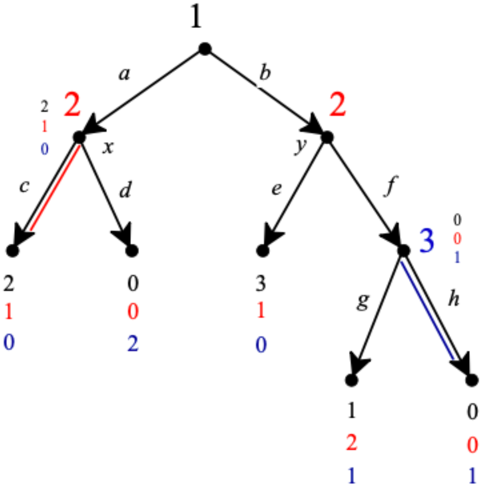


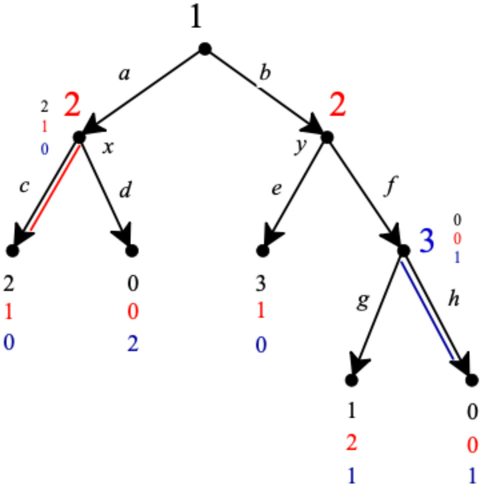
Solution Two

- Now, assume 3 plays h.



Solution Two

- Now, assume 3 plays h.
- Then 2 would play e at node y.



Solution Two

- Now, assume 3 plays h.
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Multiple Solutions

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- The solutions don't just differ in what Player 3, who faces the tie, plays.
- They differ in the very first move!

Multiple Solutions

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Multiple Solutions

- This is the totally general case; most solution concepts are like this.
- But it's a pain to deal with.
- And eventually we can solve the game.