444 Lecture 4

Equilibrium

Brian Weatherson

1/17/23

Day Plan

Trees

Backward Induction

Ties

Time

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- But few games are like that.
- We need a way to represent games that take time.

Trees

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- A tree represents all the ways that a game that takes place over time could go.

Nodes

- Trees have nodes.
- Some nodes are **terminal nodes**; they represent that the game has ended.
- Each terminal node has a payout for each of the players.
- At any other node, either a player moves, or Nature 'moves'.
- One of the non-terminal nodes is special: it is the node where the game starts.

Branches

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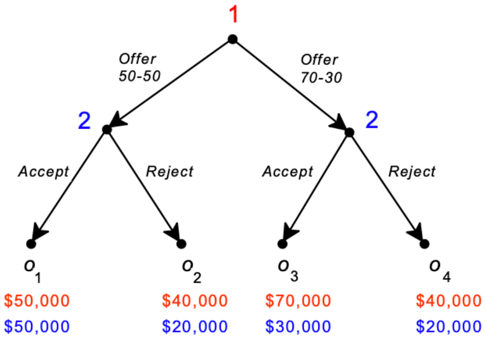
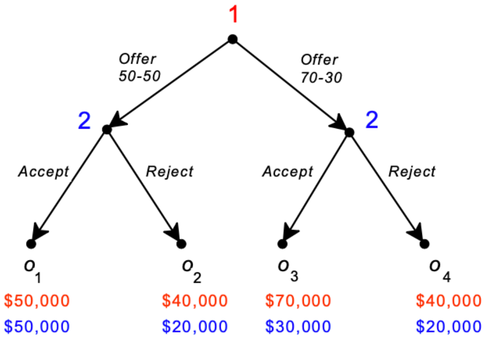
- A move at a node is always a choice of branches.



Example from Bonanno

- There are two players, 1 and 2.



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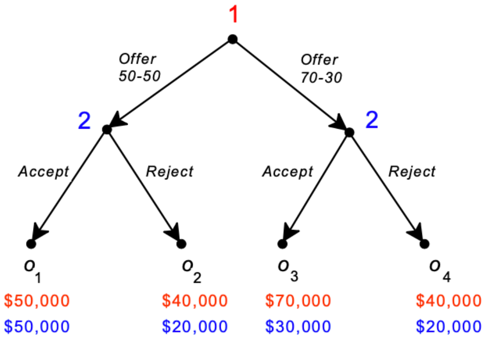
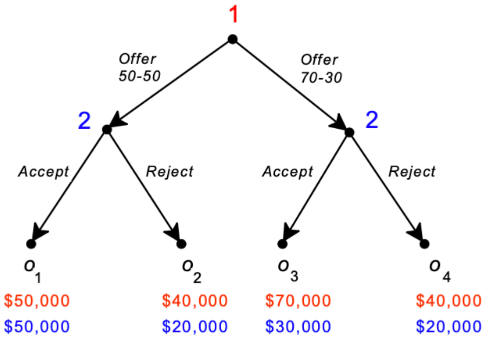


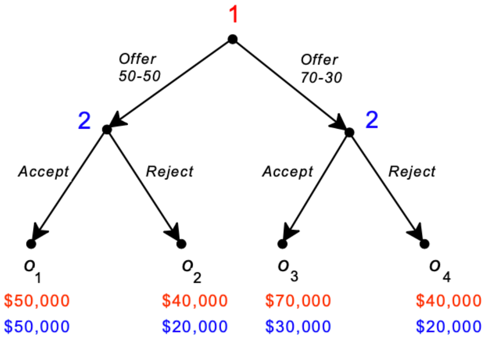
Example from Bonanno

- There are two players, 1 and 2.
- Each player moves once.
- First 1 moves, then 2 moves, then the game ends.



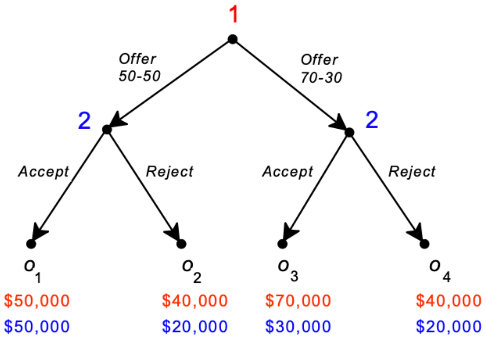
Example from Bonanno

- Some books use a special notation for the initial node, such as having an open circle rather than a closed circle.



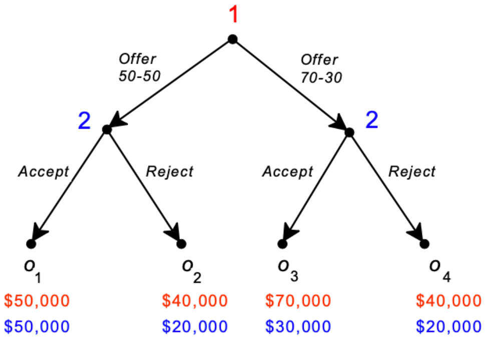
Example from Bonanno

- Some books use a special notation for the initial node, such as having an open circle rather than a closed circle.
- Bonanno doesn't, but it's clear in context what the initial node is.



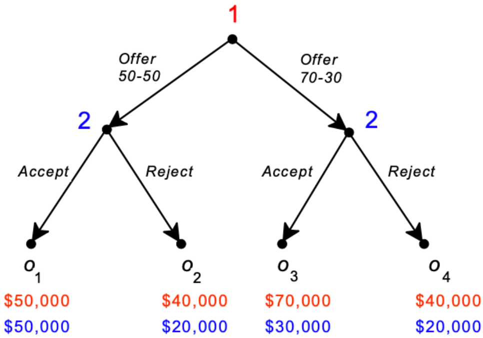
Example from Bonanno

- As he goes on to note, this isn't really a tree yet.



Example from Bonanno

- As he goes on to note, this isn’t really a tree yet.
- It describes the physical outcomes of the game at each terminal node, but not the **payoffs**.



Example from Bonanno

There is a natural function from outcomes to payoffs - more money equals more utility - but it is not a compulsory interpretation.

Future Additions

- Moves by Nature
- Moves under uncertainty

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Class of Games We're Discussing

- Two-player
- Turn-taking
- Finite
- No hidden facts
- No randomness
- We'll start with zero-sum games, though drop this later.

Five

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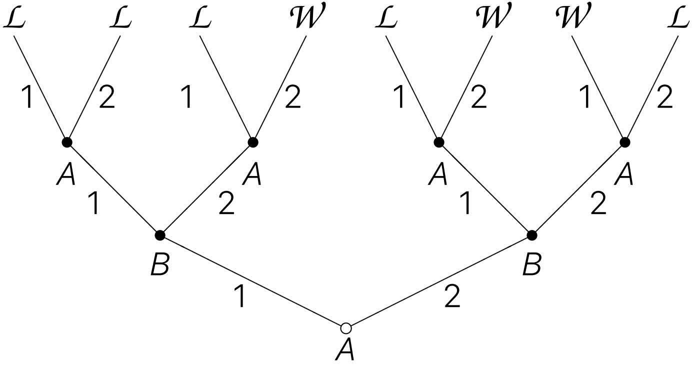
- There are two players, who we'll call *A* and *B*.
- First *A* moves, then *B*, then finally *A* moves again.
- Each move involves announcing a number, 1 or 2.
- *A* wins if after the three moves, the numbers announced sum to 5.
- *B* wins otherwise.

Five

Question: How should you play this game?

Game Tree for Five



$\mathcal{W}$ means that A wins, and $\mathcal{L}$ means that B wins.

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- First, find points where a player has a choice between two terminal nodes.
- Assume that they will make the higher value for them choice.
- Mark that choice, e.g., by doubling the line (as the textbook does) or bolding the line (as I'll do).
- If there are ties, mark both of the lines. (This gets more complicated once we leave zero-sum games.)

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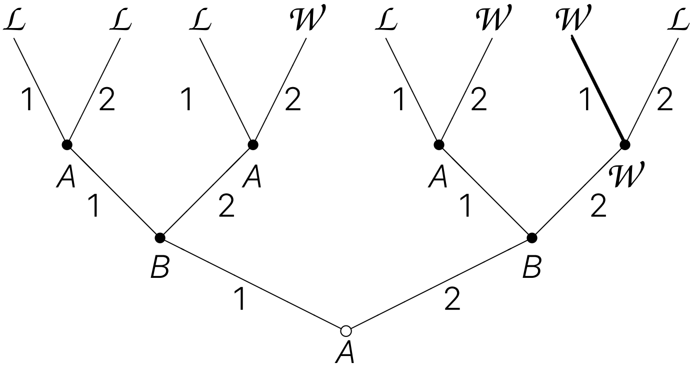
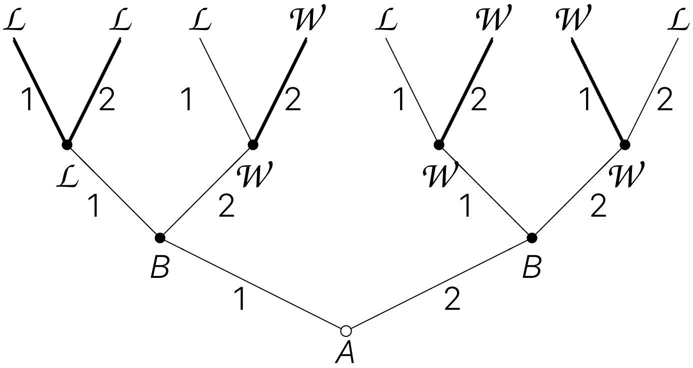
- So just the game assigns values to terminal nodes, we'll now assign value to choice nodes.
- In **Five**, we'll assign the value $\mathcal{W}$ to the top right node.

Five (after one step)



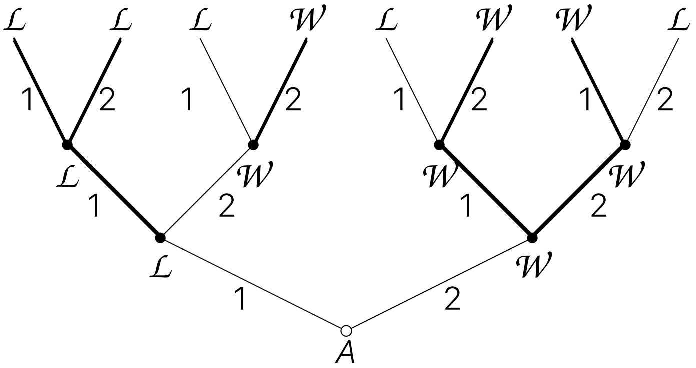
Five (after first level)



Next Steps Back

- Now we do the same thing for *B*.
- We act as if *B* is choosing between terminal nodes.
- It is as if *A* doesn't have a choice – they will just make the choice that is best for them (i.e., worst for *B*).
- So *B* knows what the outcome of each choice will be.

Five (After Two Rounds)



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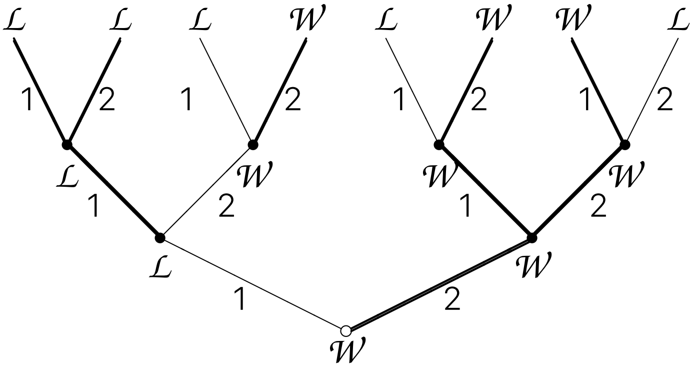
- So we act as if getting to the left hand node means B wins, and getting to the right hand node means A wins.
- And now we just have to make the choice for the initial node, using this fact.

Five (Full Graph)



Five - Full Analysis

- The equilibrium state of the game is that A wins.

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- Then B can play anything they line.

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- The equilibrium state of the game is that *A* wins.
- *A* plays 2 first.
- Then *B* can play anything they line.
- But whatever they do, *A* will win, by playing the opposite number.

Backwards Induction

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- We start at the possible ends of the game.
- At each step, we assume that each player makes the best decision they can, on the assumption that later players will do the same thing.

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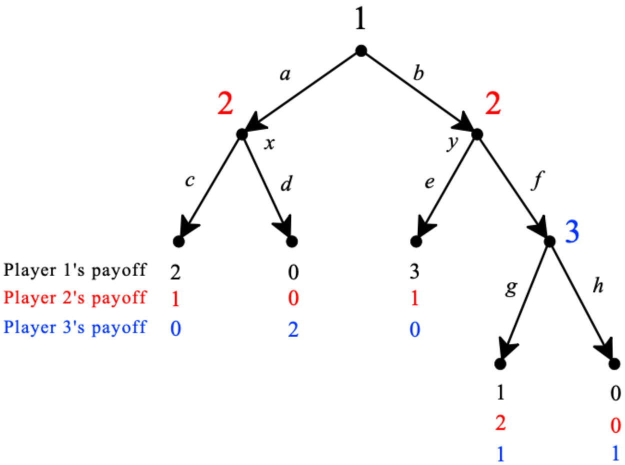
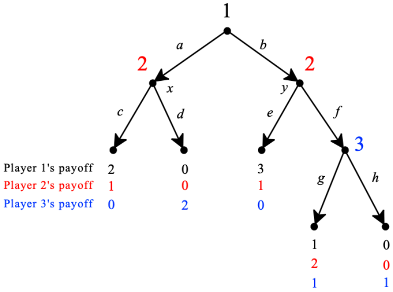
Backwards Induction in Positive Sum Games

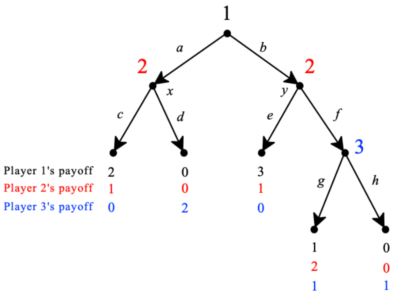


This is three player, but crucially, it is not zero sum.



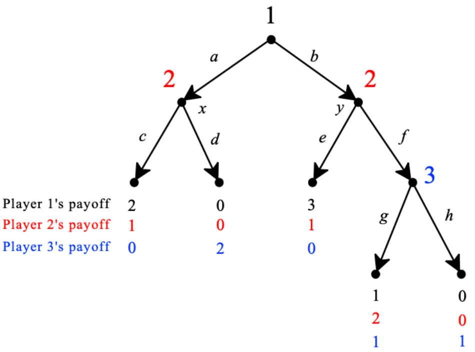
Three player game tree

- In the bottom right, Player 3 doesn’t care which choice is made.



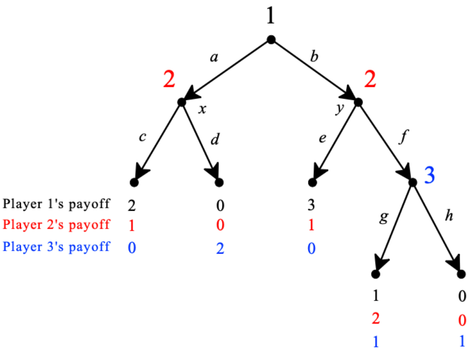
Three player game tree

- In the bottom right, Player 3 doesn't care which choice is made.
- So we can't infer what Player 3 will do.



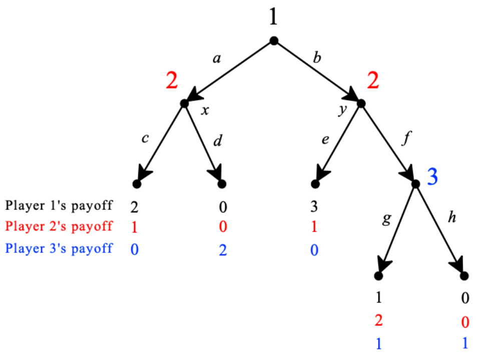
Three player game tree

- But the other players do care what Player 3 will do.



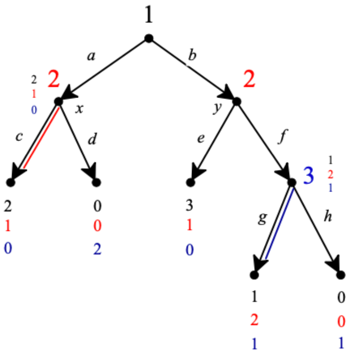
Three player game tree

- But the other players do care what Player 3 will do.
- So we can't just ignore this choice.



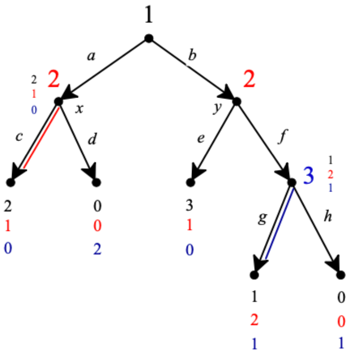
Three player game tree

- The solution is to build two trees, one for each of Player 3′s choices.



Solution One

- First, assume 3 plays g.



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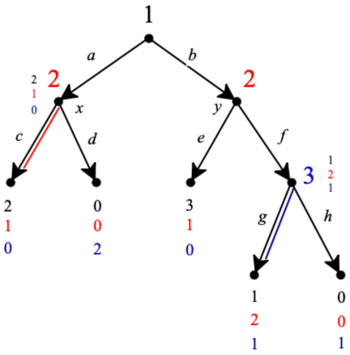
- First, assume 3 plays g.
- Then 2 would play f at node y.



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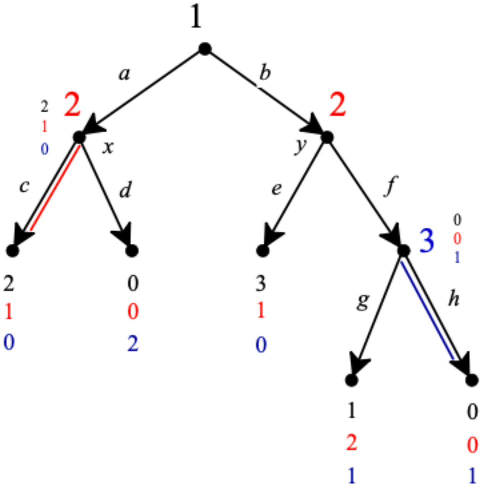
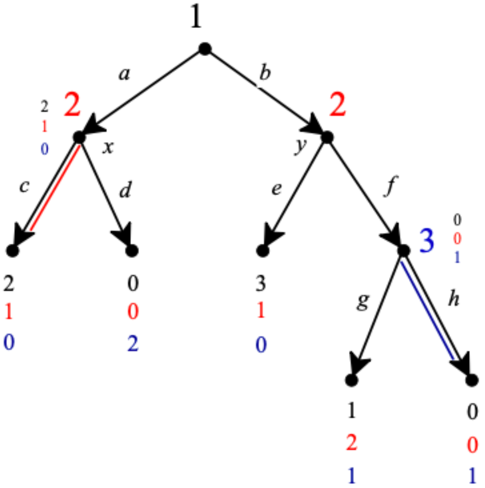


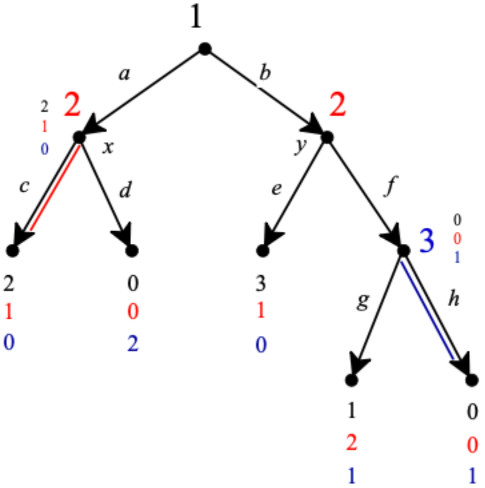
Solution Two

- Now, assume 3 plays h.



Solution Two

- Now, assume 3 plays h.
- Then 2 would play e at node y.



Solution Two

- Now, assume 3 plays h.
- Then 2 would play e at node y.
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Multiple Solutions

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- The solutions don't just differ in what Player 3, who faces the tie, plays.
- They differ in the very first move!

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- But it's a pain to deal with.

Multiple Solutions

- This is the totally general case; most solution concepts are like this.
- But it's a pain to deal with.
- And eventually we can solve the game.