

# 444 Lecture 9

## Mixed Strategies and Philosophy

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# Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

# Basic Example

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

This is a bit boring for Column, but let's focus on Row for now.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

At first it looks like there are no dominated strategies.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- Up does worse than Middle and Down if Column plays Right, so it doesn't dominate anything.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- Up does worse than Middle and Down if Column plays Right, so it doesn't dominate anything.
- Middle does worse than Up if Column plays Left, and worse than Down if Column plays Right.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- Up does worse than Middle and Down if Column plays Right, so it doesn't dominate anything.
- Middle does worse than Up if Column plays Left, and worse than Down if Column plays Right.
- Down does worse than both of them if Column plays Left.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

But compare these two strategies.

- Middle
- The mixed strategy of Up with probability 0.5, and Down with probability 0.5.



	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- Middle gets an actual return of 1 whatever Column does.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
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- The mixed strategy gets an expected return of 1.5 whatever Column does.

	Left	Right
Up	3, 0	0, 0
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- Middle gets an actual return of 1 whatever Column does.
- The mixed strategy gets an expected return of 1.5 whatever Column does.
- So it has a higher expected return given Left ( $1.5 > 1$ ), and a higher expected return given Right ( $1.5 > 1$ ).

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- If that happens, then we'll say that Middle is dominated by this mixture.

	Left	Right
Up	3, 0	0, 0
Middle	1, 0	1, 0
Down	0, 0	3, 0

- If that happens, then we'll say that Middle is dominated by this mixture.
- When we're deleting dominated strategies, we should delete it too.

# Nash and Dominance

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- A strategy that is dominated by a mixture like this can never be part of a Nash equilibrium.
- After all, the player would be better off playing the mixture than playing it, so it fails the test that there is nothing better to do.
- So being able to find these dominating mixtures can be very helpful in working out what the Nash equilibrium is.



# Rational Play and Dominance

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- But even beyond that, it seems wrong to play strategies that are dominated in this way.
- If you're thinking about playing Middle (as Row), you increase your expected return by simply flipping a coin to choose between Left and Right.
- So that's what you should do.

# Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Up is the best response to Left.

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Up is the best response to Left.
- Down is the best response to Right.

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Up is the best response to Left.
- Down is the best response to Right.
- Is Middle the best response to anything?

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

Yes!

- Middle is the best response to the mixed strategy Left with probability 0.5, Right with probability 0.5.



	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

Yes!

- Middle is the best response to the mixed strategy Left with probability 0.5, Right with probability 0.5.
- It gets 2, the other options have an expected return of 1.5.

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Middle is the best thing to do if you know Column is going to flip a coin to decide what to do.

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Middle is the best thing to do if you know Column is going to flip a coin to decide what to do.
- But it's also the best thing to do if you have no idea what Column is going to do, and the best you can do is say it's 50/50 what they are going to do.

	Left	Right
Up	3, 0	0, 0
Middle	2, 0	2, 0
Down	0, 0	3, 0

- Middle is the best thing to do if you know Column is going to flip a coin to decide what to do.
- But it's also the best thing to do if you have no idea what Column is going to do, and the best you can do is say it's 50/50 what they are going to do.
- So it's actually pretty easy to think of situations where Middle is the smart play.

# Best Response

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- A strategy  $S$  is a **best response** just in case...
- There is some probability distribution over the other player's strategies and ...
- No strategy has a higher expected return than  $S$  given that probability distribution.

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- Weakly dominated strategies can even be best responses in this sense.
- This definition also covers mixed strategies; they can also be best responses.

# A Surprising Theorem

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# A Surprising Theorem

- Say a strategy is **undominated** if no other strategy, pure or mixed, strongly dominates it.
- And it is a **best response** if it does as well as anything, given at least one probability distribution.
- Here's the surprising theorem:

The strategies that are best responses are just the same strategies as those that are undominated.



# Best Responses

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- The strategies that are dominated by mixtures didn't seem to make sense - you could just play the mixtures.
- But here's another property that they have - they are never best responses.

# Best Responses

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# Best Responses

- And if they are not best responses, no one can play them while maximising expected utility.
- Whatever probability you give to the other player's play, if you maximise expected utility you will play a best response.
- And you should maximise expected utility.

# Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

	Left	Right
Up	3, 0	0, 1
Middle	1, 1	1, 0
Down	0, 0	3, 1

In this game, the best responses are:

- Row can play Up (best response to Left) or Down (best response to Right);



	Left	Right
Up	3, 0	0, 1
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In this game, the best responses are:

- Row can play Up (best response to Left) or Down (best response to Right);
- Column can play Left (best response to Middle) or Right (best response to either Up or Down).

	Left	Right
Up	3, 0	0, 1
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- But Middle is not a best response.

	Left	Right
Up	3, 0	0, 1
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- But Middle is not a best response.
- It is dominated by the 50/50 mixture of Left and Right.

	Left	Right
Up	3, 0	0, 1
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Down	0, 0	3, 1

- So while Left is a best response...

	Left	Right
Up	3, 0	0, 1
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- So while Left is a best response...
- It is not a best response to a best response.

# Iterated Best Responses

- That makes it seem irrational to play Middle.

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- That makes it seem irrational to play Middle.
- I could build more complicated examples, where we had cases that are best responses to best responses, but not best responses to best responses to best responses.

# Iterated Best Responses

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- In the Ice Cream game, 2 is a best response to 1, which is a best response to 0.

# Iterated Best Responses

- Actually we've already seen such a case.
- In the Ice Cream game, 2 is a best response to 1, which is a best response to 0.
- But 2 is not a best response to any best response to a best response.

# Iterated Best Responses

- Some strategies are at the start of an infinite chain  $S_1, S_2, \dots$  where each strategy is a best response to the one that comes after it.

# Iterated Best Responses

- Some strategies are at the start of an infinite chain  $S_1, S_2, \dots$  where each strategy is a best response to the one that comes after it.
- Call these the **rationalizable** strategies.

# Infinite Chains

Here is one way to get an infinite chain like this.

- If the pair  $\langle S_1, S_2 \rangle$  is a Nash equilibrium, ...

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- If the pair  $\langle S_1, S_2 \rangle$  is a Nash equilibrium, ...
- Then  $S_1$  is a best response to  $S_2$ , which is a best response to  $S_1$ , which is a best response to  $S_2$ , which ...

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- If the pair  $\langle S_1, S_2 \rangle$  is a Nash equilibrium, ...
- Then  $S_1$  is a best response to  $S_2$ , which is a best response to  $S_1$ , which is a best response to  $S_2$ , which ...
- But you don't only need to use Nash equilibria.



# Infinite Chains

- Think about Rock, Paper, Scissors.

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- Rock is a best response to Scissors, which is a best response to Paper, which is a best response to Rock, which is...

# Infinite Chains

- Think about Rock, Paper, Scissors.
- Rock is a best response to Scissors, which is a best response to Paper, which is a best response to Rock, which is...
- But Rock is not part of a Nash equilibrium.

# Rationalizability

I'm not going to prove this, but the following turns out to be true.

- The strategies that can be at the start of these infinite chains ...

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# Rationalizability

I'm not going to prove this, but the following turns out to be true.

- The strategies that can be at the start of these infinite chains ...
- Are exactly those strategies that survive iterated deletion of strongly dominated strategies ...
- Provided we include dominance by mixtures when we're doing the deleting.

# Philosophical Payoff

Some economists, and a few philosophers, have argued that this is the key philosophical notion in game theory.

- They say that a strategy is rational to play if and only if it is rationalizable in this sense.

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# Philosophical Payoff

Some economists, and a few philosophers, have argued that this is the key philosophical notion in game theory.

- They say that a strategy is rational to play if and only if it is rationalizable in this sense.
- In economics, this is very much a **heterodox** view.
- Note that this view is more permissive than the view that rational players will choose Nash equilibria.

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- All Nash equilibria are rationalizable, but some rationalizable strategies (e.g., Rock!), are not Nash equilibria.

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- All Nash equilibria are rationalizable, but some rationalizable strategies (e.g., Rock!), are not Nash equilibria.
- Most economists think that if there is a key notion in game theory, it is **less permissive** than Nash equilibrium.

# Day Plan

Dominance by Mixture

Best Responses

Rationalizable Strategies

Why Nash?

# Reading

Bonanno, sections 2.6 (which we discussed earlier) and 6.4.

# Two Conjectures

1. It is rational to play any rationalizable strategy.

I'm going to end this week talking a bit about why people might prefer 2 over 1.

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1. It is rational to play any rationalizable strategy.
2. It is only rational to play Nash Equilibrium strategies

I'm going to end this week talking a bit about why people might prefer 2 over 1.

# One Intuitive Idea

- Don't just play Rock - the other person will figure it out.



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- Don't just play Rock - the other person will figure it out.
- Rock every time is rationalizable.

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- Don't just play Rock - the other person will figure it out.
- Rock every time is rationalizable.
- But you shouldn't do it.

# One Intuitive Idea

- Don't just play Rock - the other person will figure it out.
- Rock every time is rationalizable.
- But you shouldn't do it.
- Therefore principle 1 must be false.

# Response 1

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- But on any given occasion, it's fine.

# Response 1

- Yeah, you shouldn't play Rock every single time, that's dumb.
- But on any given occasion, it's fine.
- And we know, from e.g., Prisoners' Dilemma, that we shouldn't infer what to do in a single shot game from what happens in the repeated game.

## Response 2

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## Response 2

- The orthodox solution (i.e., principle 2) actually doesn't give you the result you might want here.
- It is possible that the randomising device will come up Rock every single time.
- So if you think it's always irrational to play Rock repeatedly, you have to think both of these are wrong.

## Response 3

- If principle 2 is right, all rational players will randomise every time.

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- So the expected return of Rock is just the same as the expected return of randomisation.

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- If principle 2 is right, all rational players will randomise every time.
- So the expected return of Rock is just the same as the expected return of randomisation.
- So it can't be wrong to play it.

## Mixing Response 2 and Response 3

- If principle 2 is right, and it's common knowledge that the players are rational, then the rational way to interpret the other player playing Rock every time is "Wow, their random device is having a freaky run."

## Mixing Response 2 and Response 3

- If principle 2 is right, and it's common knowledge that the players are rational, then the rational way to interpret the other player playing Rock every time is "Wow, their random device is having a freaky run."
- But if that's right, there isn't anything wrong with playing Rock every time.

## Other Direction

- As we'll see when we get to O'Connor's book, you mostly see people wanting more restrictions on moves than Nash.

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- As we'll see when we get to O'Connor's book, you mostly see people wanting more restrictions on moves than Nash.
- But Bonanno ends chapter 6 with an interesting reason for thinking even rationalisability (i.e., IDSDS) is too strong.



## Other Direction

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- It's really incredibly unrealistic to know the utility function that the other player has.
- You might know the physical outcomes of the game, but knowing what utility each player gets is a huge assumption.
- So in practice, you should probably not rely too heavily on theories or policies that rely on this knowledge.

# For Next Time

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- It's completely optional, and will be very familiar to many people here; if you want a day off, take a day off.
- After that, we'll look at how game theorists think about signals and messages.