444 Lecture 6

Games and Uncertainty

Brian Weatherson

1/24/23

Day Plan

Information Sets

Structure of Information Sets

Strategies

Subgames

Subgame Perfect Equilibrium

Basic Idea

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- Sometimes when a player has to make a choice, they know they are at one of a set of nodes in the tree, but they don't know which one.
- We will illustrate this by drawing a circle around the nodes.
- The circle means that the player making a choice knows that they are in that circle somewhere, but the rules of the game don't guarantee that they know which point they are in.

Cheating Game

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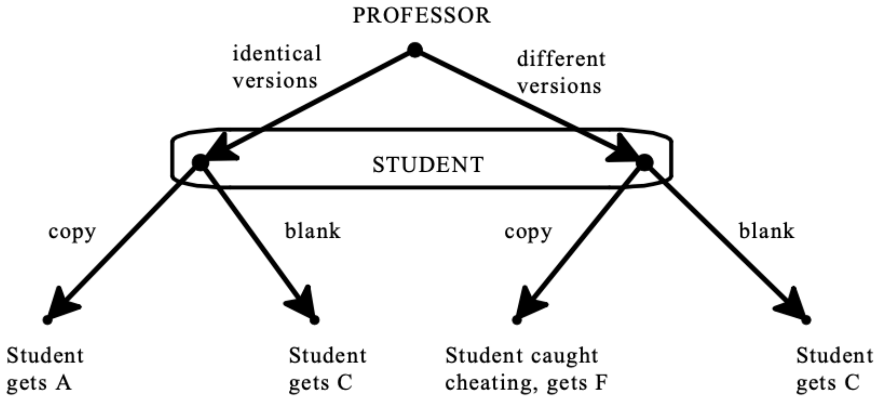
- Professor decides to either give every student the same exam, or give different exams to different students.
- Student doesn't know what professor did, and has to decide whether to copy off (known to be good at the course) neighbour.



Payoff tree for cheating game

How To Read Tree

- Professor makes a choice.

How To Read Tree

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- Then student makes a choice.

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- When student chooses, there is a fact about where in the tree we are.

How To Read Tree

- Professor makes a choice.
- Then student makes a choice.
- When student chooses, there is a fact about where in the tree we are.
- But student isn't told that fact - they are just told that we are at one of the nodes in the circled set.

Circles Everywhere

- We don’t normally draw them, but you should imagine these circles everywhere on the tree.

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- We don’t normally draw them, but you should imagine these circles everywhere on the tree.
- If a node doesn’t have a circle around it, that means that its circle just contains itself.

Information Set

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- Each node is in precisely one information set.
- That set may be a singleton; it might just be that node.
- But that's not the general case.

Constraints on Information Sets - Outputs

Every node in an information set must have the same outputs.

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- You can’t have an information set where the Player has three options from one node, but only two from another.
- The player knows how many options they have.
- So if the options were different, they could figure out which node they were at.

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Constraints on Information Sets - Inputs

Every node in an information set has the same history of moves by the player whose turn it is.

- We assume everyone knows what moves they have made.
- It is an interesting fact that some real life board games rely on the falsity of this assumption.
- But as on previous slide, if the nodes have a different history for this player, that means the player knows which node they are at.

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Disclaimer

This isn't in the books; it's not something game theorists discuss. And it really isn't on the test. But this is a philosophy class, and this is philosophically important, so we're going to pause over it for 10–15 minutes.

Three Features of Information Sets

1. Reflexive

Three Features of Information Sets

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2. Symmetric

Three Features of Information Sets

1. Reflexive
2. Symmetric
3. Transitive

Reflexive

Each point is in its own information set.

- This seems fair enough; if you're somewhere, then for all you know, you are there.

Symmetric

- If when you’re at a you might be at b, then

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- If when you're at a you might be at b, then
- When you're at b you might be at a.

Symmetric

- If when you're at a you might be at b, then
- When you're at b you might be at a.
- In information set terms, if b is in a's information set, then a is in b's information set.

Symmetry

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- This seems like an inappropriate assumption in, e.g., drinking games.
- At least in some games, one's ability to discriminate between some options will be dependent on the path taken to get to those options.
- The standard treatment of partial information doesn't allow us to represent this.

Transitive

- If when you're at *a* you might be at *b*, and

Transitive

- If when you're at a you might be at b, and
- When you're at b you might be at c, then

Transitive

- If when you're at a you might be at b, and
- When you're at b you might be at c, then
- When you're at a you might be at c.

Transitive

In information set terms, if b is in a's information set, and c is in b's information set, then it must be that c is in a's information set.

Transitivity

This rules out games where players can tell that they are in a certain 'neighbourhood'. For example,

- Player 1 puts some jelly beans in a jar and gives it to Player 2.

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- Player 1 puts some jelly beans in a jar and gives it to Player 2.
- It matters to Player 2 how many jelly beans are in the jar, but she doesn't have direct access to that.

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Transitivity

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- Still, she isn't totally ignorant. She can see the jar and guess the number to the nearest, say 10.
- So if there are 160 jelly beans in the jar, she knows that there are between 150 and 170.
- And in fact, though Player 2 doesn't know this, that's true – there are 160 jelly beans.

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- And her information set should include n_{160} - reflexivity guarantees that.

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- Player 2 knows she's not at n_{140}; her information set should exclude that.
- And her information set should include n_{160} - reflexivity guarantees that.
- Should it include n_{150}?

A Challenge

- On the one hand, it should - since from n_{160} all she knows is that there are between 150 and 170 jelly beans in the jar.

A Challenge

- On the one hand, it should - since from n_{160} all she knows is that there are between 150 and 170 jelly beans in the jar.
- On the other hand, it should not - since she can rule out n_{140}, and if n_{150} were actual, she could not rule out n_{140}.

What To Do

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What To Do

- In theory, there is an opening here for someone working out a theory of games with imperfect information that drops either the symmetry or transitivity assumption.
- In practice, no one has actually worked out that theory, and I'm not going to try teaching a non-existent theory.
- There is a little work on games involving "unawareness", which gets close to this, but it's way too novel a field to know where it will go.

Day Plan

Back to Orthodoxy

We're back onto what normal game theorists talk about, and up to 4.2 of Bonanno.

Definition

A **strategy** is a plan for what to do at each information set where the rules of the game give the player a choice.

Inputs to Strategy

- Strategies are for information sets, not nodes.

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- Strategies are for information sets, not nodes.
- Can't have a plan to do different things at different points in the information set.

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- Reason 2: Understanding decisions by other players

Strategic Form

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- We can turn any game, even ones with substantive information sets, into a single simultaneous move game.
- Just take the possible moves to be the strategies.
- In any realistic game, there will be a lot of these.

Strategic Form

- But the strategic form can obscure things, as we saw in the case of non-credible threats.

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- But the strategic form can obscure things, as we saw in the case of non-credible threats.
- Next section, we'll start looking at how to remove the obscurity.

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Subgames

Subgame Perfect Equilibrium

Definition

Roughly, a subgame is a part of the game that could be a complete game.

First Constraint

A subgame has an initial node.

- All nodes in the subgame are downstream of that node.

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A subgame has an initial node.

- All nodes in the subgame are downstream of that node.
- Remember that in general, there is only one way to get to a node.

Histories

So don't think of nodes as like positions in a chess game.

- You can get to the same position multiple ways.

Histories

So don't think of nodes as like positions in a chess game.

- You can get to the same position multiple ways.
- Rather, think of them as like the history of the moves.

Second Constraint

The subgame consists of all the nodes downstream of that initial node.

- If a is the initial node, and in the original game you can get from a to b, then b is in the subgame.

Third Constraint

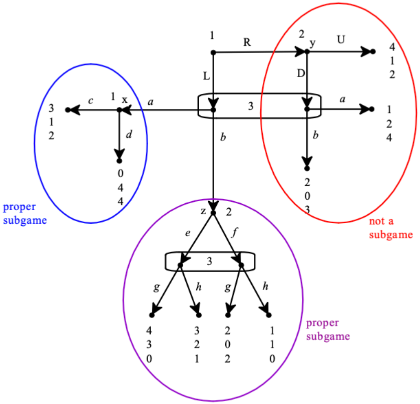
The subgame does not 'cut' any information sets.

- If b is in the subgame, and c is in the information set that includes b, then c is in the subgame.

Common Knowledge

Put another way, *a* has to be a point such that when you get to it, it is common knowledge you are there.

- It being in a singleton information set is necessary for that (if not, the player who has to play doesn't know you are there) but not sufficient.



A violation of third constraint

Summing Up

- A subgame consists of all and only the points that are ‘downstream’ of some initial node.

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- A subgame consists of all and only the points that are 'downstream' of some initial node.
- That initial node has to be such that if/when it is reached, it is common knowledge among the players that it is reached.

Day Plan

Definition

A set of strategies for each of the players is a subgame perfect equilibrium if and only if

- The set forms a Nash equilibrium.

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A set of strategies for each of the players is a subgame perfect equilibrium if and only if

- The set forms a Nash equilibrium.
- In every subgame, the set forms a Nash equilibrium.

Subgame Perfect and Nash

The second clause is non-trivial.

- It rules out players doing certain kinds of odd things at nodes that are not reached.

Subgame Perfect and Nash

The second clause is non-trivial.

- It rules out players doing certain kinds of odd things at nodes that are not reached.
- At subgame perfect equilibrium, each player's strategies make sense given the other player's strategies, and they are disposed to keep making sense under different assumptions about what they might do.

Finding Subgame Perfect Equilibrium

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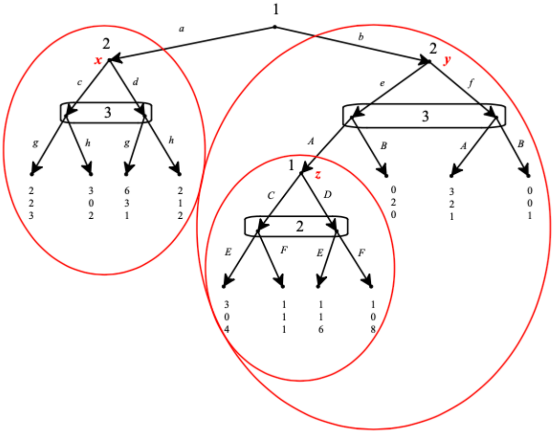
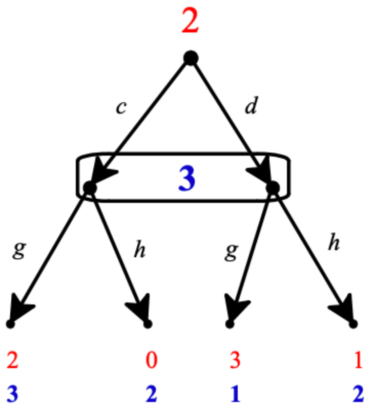
- If there are multiple equilibria, duplicate the tree enough times to cover each of them - you'll have multiple subgame perfect equilibria.
- Repeat until you've covered the whole tree.

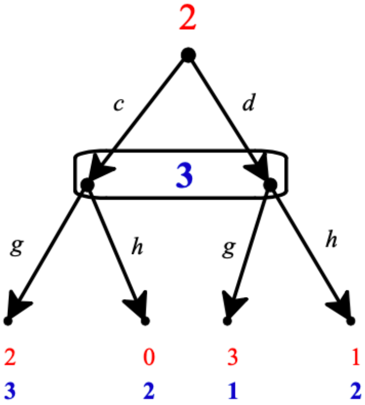


The large game



		Player 3	
		g	*h*
Player 2	*c*	2 3	0 2
	d	3 1	1 2

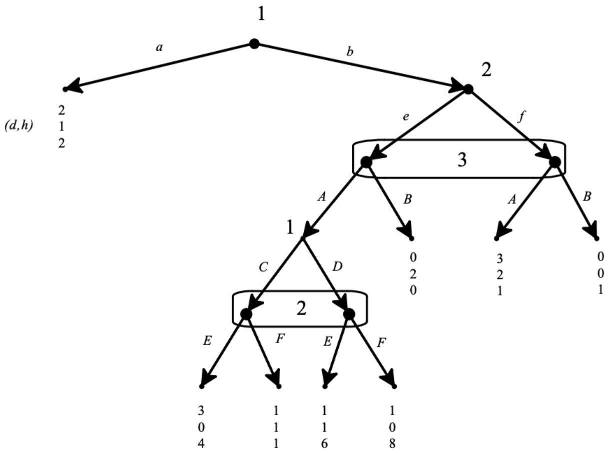
The left subgame



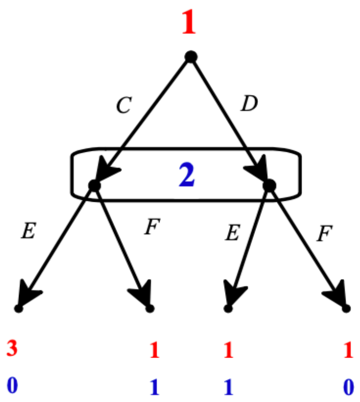
Player 3

Player 2		g		h	
	c	2	3	0	2
	d	3	1	1	2

The left subgame with labeled best responses

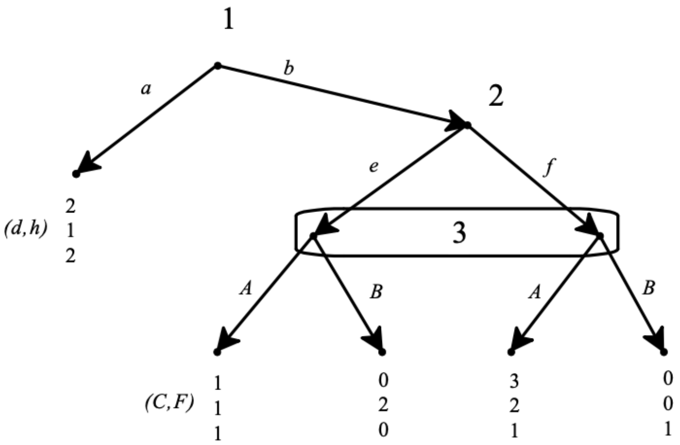


The big game with reduced left subgame



		Player 2	
		E	F
Player 1	C	3 0	1 1
	D	1 1	1 0

The middle subgame with labeled best responses



The big game with reduced middle subgame

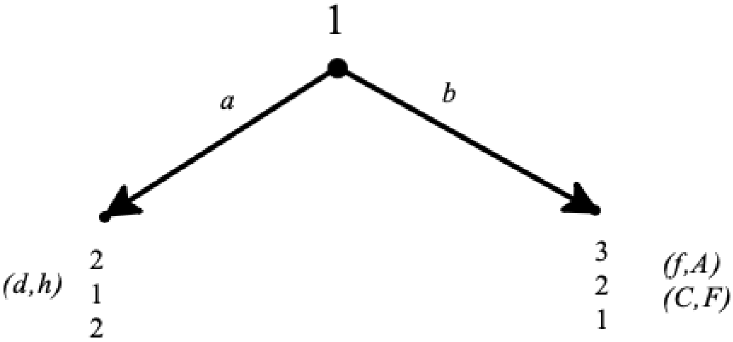
The right subgame

	A	B
e	1, 1	2, 0
f	2, 1	0, 1

- Player 2 is row
- Player 3 is column
- Player 1 is ignored, because they have no moves

The right subgame with labeled best responses

	A	B
e	1, $\boxed{1}$	$\boxed{2}$, 0
f	$\boxed{2}$, $\boxed{1}$	0, $\boxed{1}$



The big game with reduced right subgame

- Only Nash equilibrium is Player 1 plays *b*.

Summary

So the subgame perfect equilibrium is:

- Player 1 plays b, C.

And the payouts are reverse order of their names: 3, 2, 1.

Summary

So the subgame perfect equilibrium is:

- Player 1 plays b, C.
- Player 2 plays d, f, F.

And the payouts are reverse order of their names: 3, 2, 1.

Summary

So the subgame perfect equilibrium is:

- Player 1 plays b, C.
- Player 2 plays d, f, F.
- Player 3 plays h, A.

And the payouts are reverse order of their names: 3, 2, 1.

For Next Time

We will start looking at games with cardinal utility.

Remember

- No class Thursday; I'm away at a conference.

Remember

- No class Thursday; I'm away at a conference.
- Happy Republic Day to all who celebrate it.