Honors Logic, Lecture 16 - Modal Logic

Brian Weatherson

2022-10-26

Why Do Modal Logic?

- In particular, why not just use first order logic to quantify over possibilities?
- Instead of saying $\Box p$, say $\forall w : p(w)$ or something?
- This is a non-rhetorical question; in lots of situations we do say something more like $\forall w : p(w)$.

Why Do Modal Logic?

Historically, four reasons.

1. Scepticism about the w, such as Prior on time.

Why Do Modal Logic?

Historically, four reasons.

1. Scepticism about the w, such as Prior on time.
2. Scepticism about $p(w)$, as in worries about truth.

Why Do Modal Logic?

Historically, four reasons.

1. Scepticism about the w, such as Prior on time.
2. Scepticism about $p(w)$, as in worries about truth.
3. More natural/intuitive to talk about $\Box$ than about r.

Why Do Modal Logic?

Historically, four reasons.

1. Scepticism about the w, such as Prior on time.
2. Scepticism about $p(w)$, as in worries about truth.
3. More natural/intuitive to talk about $\Box$ than about r.
4. We want to understand $\Box$, and thinking about r helps us do that.

$\Diamond p \supset \Box\Diamond p$

We are going to do this in K, then in Kρ, then in K$\rho\tau$, then in K$\rho\sigma$, then finally in K$\rho\sigma\tau$, which is equivalent to Kυ.

Start with the tableau for K.

$$\neg(\Diamond p \supset \Box\Diamond p), 0$$
$$\Diamond p, 0$$
$$\neg\Box\Diamond p, 0$$
$$\Diamond\neg\Diamond p, 0$$
$$0r1$$
$$p, 1$$
$$0r2$$
$$\neg\Diamond p, 2$$
$$\Box\neg p, 2$$

Now let's extend the tableau for Kρ.

$$\neg(\Diamond p \supset \Box\Diamond p), 0$$
$$\Diamond p, 0$$
$$\neg\Box\Diamond p, 0$$
$$\Diamond\neg\Diamond p, 0$$
$$0r1$$
$$p, 1$$
$$0r2$$
$$\neg\Diamond p, 2$$
$$\Box\neg p, 2$$
$$0r0$$
$$1r1$$
$$2r2$$
$$\neg p, 2$$

That is also a tableau for K$\rho\tau$.

$\neg(\Diamond p \supset \Box\Diamond p)$, 0	$0r0$
$\Diamond p, 0$	$1r1$
$\neg\Box\Diamond p, 0$	$2r2$
$\Diamond\neg\Diamond p, 0$	$\neg p, 2$
$0r1$	
$p, 1$	
$0r2$	
$\neg\Diamond p, 2$	
$\Box\neg p, 2$	

To do the tableau for K$\rho\sigma$, we need a couple more lines, but it's still open.

$$
\begin{array}{cc}
\neg(\Diamond p \supset \Box\Diamond p), 0 & 0r0 \\
\Diamond p, 0 & 1r1 \\
\neg\Box\Diamond p, 0 & 2r2 \\
\Diamond\neg\Diamond p, 0 & \neg p, 2 \\
0r1 & 2r0 \\
p, 1 & 1r0 \\
0r2 & \neg p, 0 \\
\neg\Diamond p, 2 & \\
\Box\neg p, 2 &
\end{array}
$$

To do the tableau for K$\rho\sigma\tau$, we need more lines, and now it closes.

$$\neg(\Diamond p \supset \Box\Diamond p), 0$$
$$\Diamond p, 0$$
$$\neg\Box\Diamond p, 0$$
$$\Diamond\neg\Diamond p, 0$$
$$0r1$$
$$p, 1$$
$$0r2$$
$$\neg\Diamond p, 2$$
$$\Box\neg p, 2$$

$$0r0$$
$$1r1$$
$$2r2$$
$$\neg p, 2$$
$$2r0$$
$$1r0$$
$$\neg p, 0$$
$$2r1$$
$$\neg p, 1$$
$$x$$

The tableau for Kυ, is a bit simpler, because it doesn't include r lines.

$$\neg(\Diamond p \supset \Box\Diamond p), 0$$
$$\Diamond p, 0$$
$$\neg\Box\Diamond p, 0$$
$$\Diamond\neg\Diamond p, 0$$
$$p, 1$$
$$\neg\Diamond p, 2$$
$$\Box\neg p, 2$$
$$\neg p, 2$$
$$\neg p, 0$$
$$\neg p, 1$$
$$x$$

$\Box(\Box p \supset q) \vee \Box(\Box q \supset p)$

This is not very intuitive, but it is a little interesting mathematically.

- It's the **characteristic axiom** of models that are 'linear'.
- I don't *think* I'll have time today to say what a characteristic axiom is; the priority today is getting the formalism down.
- But hopefully we'll have some time for it next week.

Start with an open tableau in K.

$$\neg(\Box(\Box p \supset q) \vee \Box(\Box q \supset p)), 0$$
$$\neg\Box(\Box p \supset q), 0$$
$$\neg\Box(\Box q \supset p), 0$$
$$\Diamond\neg(\Box p \supset q), 0$$
$$\Diamond\neg(\Box q \supset p), 0$$
$$\neg(\Box p \supset q), 1$$
$$0r1$$
$$\neg(\Box q \supset p), 2$$
$$0r2$$
$$\Box p, 1$$
$$\neg q, 1$$
$$\Box q, 2$$
$$\neg p, 2$$

Now onto Kρ, which is still open.

$\neg(\Box(\Box p \supset q) \vee \Box(\Box q \supset p)), 0$	$0r0$
$\neg\Box(\Box p \supset q), 0$	$1r1$
$\neg\Box(\Box q \supset p), 0$	$2r2$
$\Diamond\neg(\Box p \supset q), 0$	$p, 1$
$\Diamond\neg(\Box q \supset p), 0$	$q, 2$
$\neg(\Box p \supset q), 1$	
$0r1$	
$\neg(\Box q \supset p), 2$	
$0r2$	
$\Box p, 1$	
$\neg q, 1$	
$\Box q, 2$	
$\neg p, 2$	

That’s also an open tableau for K$\rho\tau$.

$$\neg(\Box(\Box p \supset q) \vee \Box(\Box q \supset p)), 0$$
$$\neg\Box(\Box p \supset q), 0$$
$$\neg\Box(\Box q \supset p), 0$$
$$\Diamond\neg(\Box p \supset q), 0$$
$$\Diamond\neg(\Box q \supset p), 0$$
$$\neg(\Box p \supset q), 1$$
$$0r1$$
$$\neg(\Box q \supset p), 2$$
$$0r2$$
$$\Box p, 1$$
$$\neg q, 1$$
$$\Box q, 2$$
$$\neg p, 2$$

$$0r0$$
$$1r1$$
$$2r2$$
$$p, 1$$
$$q, 2$$

K$\rho\sigma$ needs a few more lines, but is still open.

$$
\begin{array}{cc}
\neg(\Box(\Box p \supset q) \vee \Box(\Box q \supset p)), 0 & 0r0 \\
\neg\Box(\Box p \supset q), 0 & 1r1 \\
\neg\Box(\Box q \supset p), 0 & 2r2 \\
\Diamond\neg(\Box p \supset q), 0 & p, 1 \\
\Diamond\neg(\Box q \supset p), 0 & q, 2 \\
\neg(\Box p \supset q), 1 & 1r0 \\
0r1 & 2r0 \\
\neg(\Box q \supset p), 2 & p, 0 \\
0r2 & q, 0 \\
\Box p, 1 & \\
\neg q, 1 & \\
\Box q, 2 & \\
\neg p, 2 &
\end{array}
$$

But K$\rho\sigma\tau$ closes

$\neg(\Box(\Box p \supset q) \vee \Box(\Box q \supset p)), 0$	$0r0$
$\neg\Box(\Box p \supset q), 0$	$1r1$
$\neg\Box(\Box q \supset p), 0$	$2r2$
$\Diamond\neg(\Box p \supset q), 0$	$p, 1$
$\Diamond\neg(\Box q \supset p), 0$	$q, 2$
$\neg(\Box p \supset q), 1$	$1r0$
$0r1$	$2r0$
$\neg(\Box q \supset p), 2$	$p, 0$
$0r2$	$q, 0$
$\Box p, 1$	$1r2$
$\neg q, 1$	$p, 2$
$\Box q, 2$	x
$\neg p, 2$	

$\Box p \supset \Box\Box p$

Next example is this one, which is the characteristic axiom of transitive models.

Let's skip K and go straight to Kρ, since it's open.

$$\neg(\Box p \supset \Box\Box p), 0$$
$$\Box p, 0$$
$$\neg\Box\Box p, 0$$
$$0r0$$
$$p, 0$$
$$\Diamond\neg\Box p, 0$$
$$0r1$$
$$\neg\Box p, 1$$
$$p, 1$$
$$1r1$$
$$\Diamond\neg p, 1$$
$$1r2$$
$$\neg p, 2$$
$$2r2$$

It takes just a bit more work to get K$\rho\sigma$.

$$\begin{array}{ccc}
\neg(\Box p \supset \Box\Box p), 0 & \qquad & \Diamond\neg p, 1 \\
\Box p, 0 & & 1r2 \\
\neg\Box\Box p, 0 & & \neg p, 2 \\
0r0 & & 2r2 \\
p, 0 & & \mathbf{2r1} \\
\Diamond\neg\Box p, 0 & & \mathbf{1r0} \\
0r1 & & \\
\neg\Box p, 1 & & \\
p, 1 & & \\
1r1 & &
\end{array}$$

I bolded the new lines, but they don't have any interesting implications.

But this closes in K$\rho\tau$, and hence in K$\rho\sigma\tau$ and in Kυ.

$\neg(\Box p \supset \Box\Box p), 0$	$\Diamond\neg p, 1$
$\Box p, 0$	$1r2$
$\neg\Box\Box p, 0$	$\neg p, 2$
$0r0$	$2r2$
$p, 0$	**0r2**
$\Diamond\neg\Box p, 0$	**p, 2**
$0r1$	x
$\neg\Box p, 1$	
$p, 1$	
$1r1$	

And we get p is both true and false at 2.

$\neg\Box\Diamond p$

This obviously isn't a logical truth!

- But the tableau for it in Kυ is annoying.

$$\neg\neg\Box\Diamond p, 0$$
$$\Box\Diamond p, 0$$
$$\Diamond p, 0$$
$$p, 1$$
$$\Diamond p, 1$$
$$p, 2$$
$$\Diamond p, 2$$
$$\ldots$$

A model

- There are infinitely many points.
- They can all access each other.
- p is true at all of them.