Honors Logic, Lecture 12 - Modal Logic

Brian Weatherson

2022-10-10

Access

We can think, a little picturesquely, as the accessibility relation being a 'step' between worlds.

- If wRy, then you can 'step' from w to y.

Access

We can think, a little picturesquely, as the accessibility relation being a ‘step’ between worlds.

- If wRy, then you can ‘step’ from w to y.
- $\Box A$ means that anywhere you can step to from w is a world where A is true.

Access

We can think, a little picturesquely, as the accessibility relation being a 'step' between worlds.

- If wRy, then you can 'step' from w to y.
- $\Box A$ means that anywhere you can step to from w is a world where A is true.
- And $\Box\Box A$ means that anywhere you can get to in two steps from w is a world where A is true.

Iterated Modalities

We can run the rules in sequence.

- What does it take for $\Diamond\Diamond A$ to be true at w?

Iterated Modalities

We can run the rules in sequence.

- What does it take for $\Diamond\Diamond A$ to be true at w?
- It is for $\Diamond A$ to be true at some y such that wRy.

Iterated Modalities

We can run the rules in sequence.

- What does it take for $\Diamond\Diamond A$ to be true at w?
- It is for $\Diamond A$ to be true at some y such that wRy.
- And that means that A has to be true at some world z such that yRz (for some y such that wRy).

Iterated Modalities

We can run the rules in sequence.

- What does it take for $\Diamond\Diamond A$ to be true at w?
- It is for $\Diamond A$ to be true at some y such that wRy.
- And that means that A has to be true at some world z such that yRz (for some y such that wRy).
- In the picturesque terms, you can get from w to an A-world in two steps.

Mixed Modalities

What does it mean for $\Diamond\Box A$ to be true at w?

- There is some accessible world where $\Box A$ is true.

Mixed Modalities

What does it mean for $\Diamond\Box A$ to be true at w?

- There is some accessible world where $\Box A$ is true.
- That is, there is some accessible world such that everywhere you can go from there, A is true.

Mixed Modalities

What does it mean for $\Box\Diamond A$ to be true at w?

- At all accessible worlds, $\Diamond A$ is true.

Mixed Modalities

What does it mean for $\Box\Diamond A$ to be true at w?

- At all accessible worlds, $\Diamond A$ is true.
- That is, wherever you go, you can get to there is some accessible world such that everywhere you can go from there, A is true.

Longer Sentences

What does it mean for $\Box(p \vee (q \supset \Diamond r))$ to be true at w?

- It's for $p \vee (q \supset \Diamond r)$ to be true everywhere you can get to (in one step) from w.

Longer Sentences

What does it mean for $\Box(p \vee (q \supset \Diamond r))$ to be true at w?

- It's for $p \vee (q \supset \Diamond r)$ to be true everywhere you can get to (in one step) from w.
- That is, at every one of those worlds, either p is true or $q \supset \Diamond r$ is true.

Longer Sentences

What does it mean for $\Box(p \vee (q \supset \Diamond r))$ to be true at w?

- It's for $p \vee (q \supset \Diamond r)$ to be true everywhere you can get to (in one step) from w.
- That is, at every one of those worlds, either p is true or $q \supset \Diamond r$ is true.
- That is, at every one of those worlds, either p is true, or q is false, or $\Diamond r$ is true.

Longer Sentences

What does it mean for $\Box(p \vee (q \supset \Diamond r))$ to be true at w?

- It's for $p \vee (q \supset \Diamond r)$ to be true everywhere you can get to (in one step) from w.
- That is, at every one of those worlds, either p is true or $q \supset \Diamond r$ is true.
- That is, at every one of those worlds, either p is true, or q is false, or $\Diamond r$ is true.
- That is, at every one of those worlds, either p is true, or q is false, or there is some world you can get to where r is true.

Box and connectives

The general rule is just to apply the rules for sentences inside the brackets at each world in W, and then apply the rule for $\Box$ or $\Diamond$. But there are three special cases worth thinking about.

- $\Box(A \land B)$ means that all accessible worlds are A and B worlds.

We'll use that last one a lot.

Box and connectives

The general rule is just to apply the rules for sentences inside the brackets at each world in W, and then apply the rule for $\Box$ or $\Diamond$. But there are three special cases worth thinking about.

- $\Box(A \wedge B)$ means that all accessible worlds are A and B worlds.
- $\Box(A \vee B)$ means that all accessible worlds make at least one of A and B true.

We'll use that last one a lot.

Box and connectives

The general rule is just to apply the rules for sentences inside the brackets at each world in W, and then apply the rule for $\Box$ or $\Diamond$. But there are three special cases worth thinking about.

- $\Box(A \wedge B)$ means that all accessible worlds are A and B worlds.
- $\Box(A \vee B)$ means that all accessible worlds make at least one of A and B true.
- $\Box(A \supset B)$ means that all accessible A-worlds are B-worlds.

We'll use that last one a lot.

A Weird Special Case

If for some w there is no x such that wRx, then at w:

1. Every box sentence is true.

A Weird Special Case

If for some w there is no x such that wRx, then at w:

1. Every box sentence is true.
2. Every diamond sentence is false.

A Weird Special Case

If for some w there is no x such that wRx, then at w:

1. Every box sentence is true.
2. Every diamond sentence is false.

- This is weird; normally box something is a much **stronger** claim than diamond, but this is a weird exception.

A Weird Special Case

I normally wouldn't mention this, because it's not something that's of particular philosophical significance.

- Except, when we're doing trees/tableau, we start out with no R relations at all, so we end up in this special case a lot. It's somewhat annoying to spend so much time doing something with no philosophical relevance, but it is mathematicaly convenient to have these cases around.

I’m going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$

I'm going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$

I'm going to go through this table and show how each of them can be false.

1. $\Box(A \lor B) \supset (\Box A \lor \Box B)$
2. $(\Diamond A \land \Diamond B) \supset \Diamond(A \land B)$
3. $A \supset \Box A$

I'm going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$
3. $A \supset \Box A$
4. $\Box A \supset A$

I’m going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$
3. $A \supset \Box A$
4. $\Box A \supset A$
5. $\Box\Diamond A \supset B$

I'm going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$
3. $A \supset \Box A$
4. $\Box A \supset A$
5. $\Box\Diamond A \supset B$
6. $\Box\Diamond A \supset A$

I’m going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$
3. $A \supset \Box A$
4. $\Box A \supset A$
5. $\Box\Diamond A \supset B$
6. $\Box\Diamond A \supset A$
7. $\Box\Box A \supset \Box A$

I'm going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$
3. $A \supset \Box A$
4. $\Box A \supset A$
5. $\Box\Diamond A \supset B$
6. $\Box\Diamond A \supset A$
7. $\Box\Box A \supset \Box A$
8. $\Box A \supset \Box\Box A$

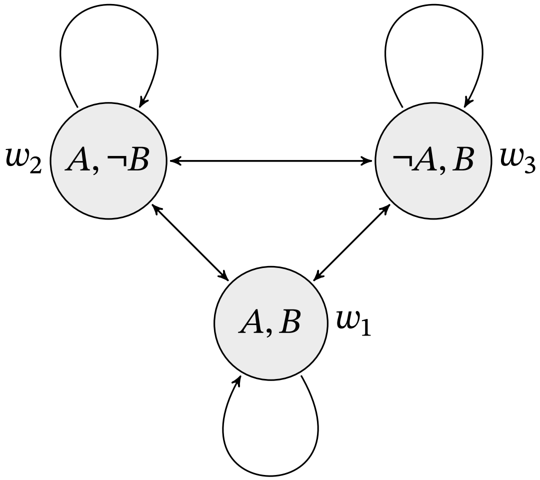
I'm going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$
3. $A \supset \Box A$
4. $\Box A \supset A$
5. $\Box\Diamond A \supset B$
6. $\Box\Diamond A \supset A$
7. $\Box\Box A \supset \Box A$
8. $\Box A \supset \Box\Box A$
9. $\Box\Diamond A \supset \Diamond\Box A$

I'm going to go through this table and show how each of them can be false.

1. $\Box(A \vee B) \supset (\Box A \vee \Box B)$
2. $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$
3. $A \supset \Box A$
4. $\Box A \supset A$
5. $\Box\Diamond A \supset B$
6. $\Box\Diamond A \supset A$
7. $\Box\Box A \supset \Box A$
8. $\Box A \supset \Box\Box A$
9. $\Box\Diamond A \supset \Diamond\Box A$
10. $\Box A \supset \Diamond A$

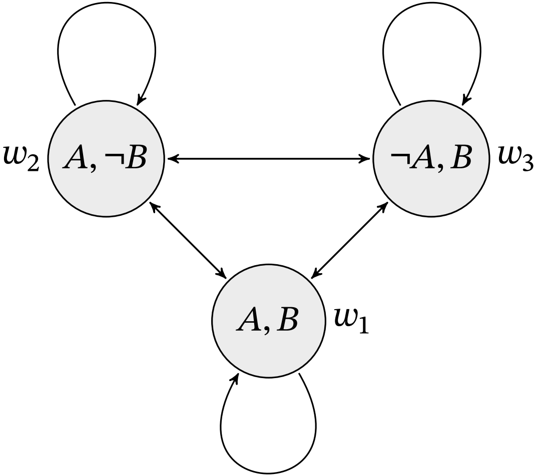
$\Box(A \vee B) \supset (\Box A \vee \Box B)$



At all points, either A or B is true, so $\Box(A \vee B)$ is true.

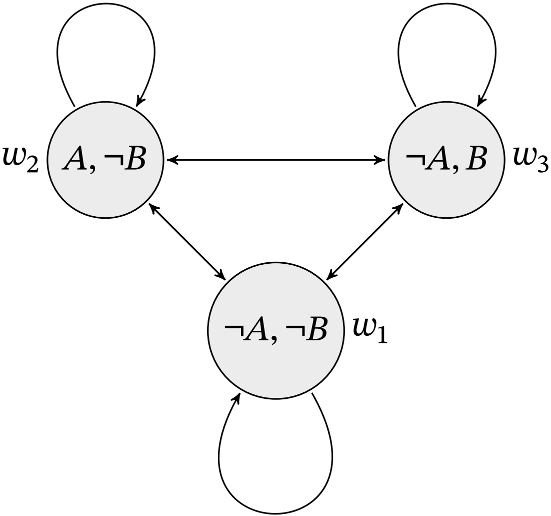
But $\Box A$ and $\Box B$ are false everywhere. So the conditional is false everywhere.

$\Box(A \vee B) \supset (\Box A \vee \Box B)$



Note that this is overkill. We just need to show that the formula can be false somewhere in order to show that it is not a theorem.

$(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$

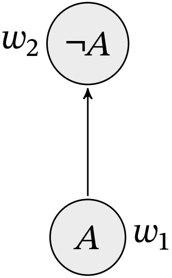


At w_1, we have $\Diamond A \wedge \Diamond B$ true.

But nowhere is $A \wedge B$ true, so $\Diamond(A \wedge B)$ is false at w_1. So the conditional is false.

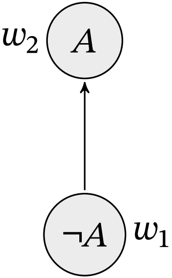
Again, this is overkill.

$A \supset \Box A$



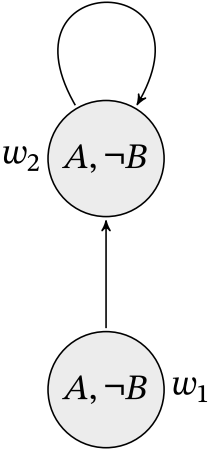
- At w_1 A is true.
- But $\Box A$ is false, since w_1 can access w_2, and A is false there.
- So $A \supset \Box A$ is false.

$\Box A \supset A$



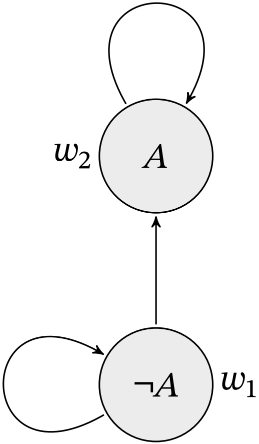
- At w_1 $\Box A$ is true. The only accessible world is w_2, and A is true there.
- But A is false there.
- So $\Box A \supset A$ is false.

$\Box\Diamond A \supset B$



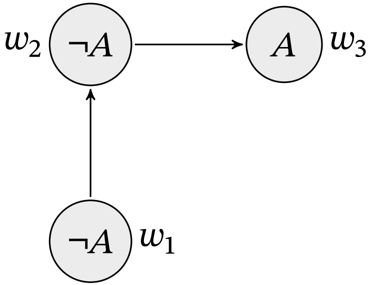
- At w_1 $\Box\Diamond A$ is true. The only accessible world is w_2, and $\Diamond A$ is true there. (Why?)
- But B is false at w_1.
- So $\Box\Diamond A \supset B$ is false.

$\Box\Diamond A \supset A$



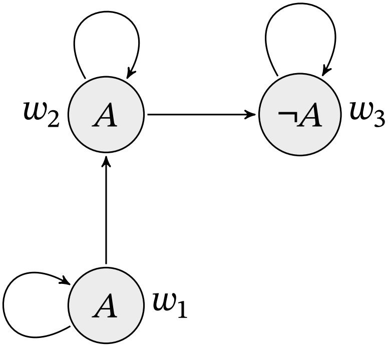
- At w_1 $\Box\Diamond A$ is true. At every world, w_2 is accessible, and A is true there.
- But A is false at w_1.
- So $\Box\Diamond A \supset A$ is false at w_1.

$\Box\Box A \supset \Box A$



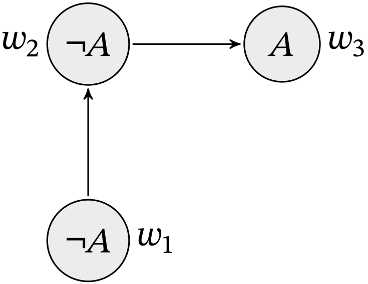
- The only world w_2 can access is w_3, and A is true there, so $\Box A$ is true at w_2.
- The only world w_1 can access is w_2, and $\Box A$ is true there, so $\Box\Box A$ is true at w_1.
- But $\Box A$ is false at w_1.
- So $\Box\Box A \supset \Box A$ is false at w_1.

$\Box A \supset \Box\Box A$



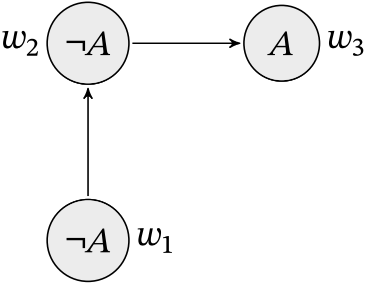
- Since A is false at w_3, and w_2 can access w_3, $\Box A$ is false at w_2.
- Since $\Box A$ is false at w_2, and w_1 can access w_2, $\Box\Box A$ is false at w_1.
- But $\Box A$ is true at w_1.
- So $\Box A \supset \Box\Box A$ is false at w_1.

$\Box A \supset \Diamond A$



- Focus on w_3.
- There is no accessible world where A is false, so $\Box A$ is true there.
- But there is no accessible world where A is true, so $\Diamond A$ is false there.
- So $\Box A \supset \Diamond A$ is false there.

$\Box A \supset \Diamond A$



Whenever there are no accessible worlds, the following two weird things happen.

1. All $\Box$-sentences (i.e., sentences that start with a $\Box$ that takes scope over the whole sentence) are true.
2. All $\Diamond$-sentences (i.e., sentences that start with a $\Diamond$ that takes scope over the whole sentence) are false.