Honors Logic, Lecture 13 - Modal Logic

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Six New Steps

1. Every line has a world number.
2. The rules for non-modal connectives preserve world.
3. For negated modals, move negation inside and flip
4. For true $\Diamond$ sentences, introduce a new world.
5. For true $\Box$ worlds, do nothing at first, but make boxed sentence true everywhere accessible.
6. Only close a branch when a sentence is true and false at same world.

Step 1

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World Numbers

Lines now look like this.

$$p \wedge q, 1$$

Read this as saying that the conjunction $p \wedge q$ is true at world 1.

Step 2

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World Preservation

- All the old rules didn't have line numbers.
- But the way to apply them is just to keep the world numbers the same.

Example 1

$$p \wedge q, 3$$
$$p, 3$$
$$q, 3$$

Example 2

$$p \supset q, 4$$

$$\neg p, 4 \qquad\qquad q, 4$$

Step 3

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Negated Modals ($\Diamond$)

For each of them, the rule is move the negation inside, and invert.

$$\neg\Diamond A, n$$
$$\Box\neg A, n$$

Note that the world stays the same, as does what comes after the modal.

Negated Modals ($\Box$)

For each of them, the rule is move the negation inside, and invert.

$$\neg\Box A, n$$
$$\Diamond\neg A, n$$

Note that the world stays the same, as does what comes after the modal.

Step 4

1. Every line has a world number.
2. The rules for non-modal connectives preserve world.
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Example 3

Here is an instance of the true $\Diamond$ rule in action.

$$\Diamond(p \wedge \Box q), 4$$
$$4r5$$
$$p \wedge \Box q, 5$$

- This would only be ok if 5 had not been used on the branch before.

General Rule

When you have a true $Diamond$ sentence:

- On a new line, copy down the sentence;
- Delete the $\Diamond$;
- Change the world number to a number that didn't previously appear on the tree.
- Write that the world from the original sentence can access the new world.
- That's it; there are no more rules to apply.

Explanation

A true $\Diamond$ sentence says that at some accessible world, what's inside the $\Diamond$ is true.

- Since the world names are arbitrary, we're just giving whatever world that is an arbitrary name.
- And it's accessible, so we say that the original world can see it.
- You have two lines to write down; the order you write them in doesn't matter.

Step 5

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5. **For true $\Box$ worlds, do nothing at first, but make boxed sentence true everywhere accessible.**
6. Only close a branch when a sentence is true and false at same world.

Do Nothing

Here is a completed tableau showing that $\Box p \vdash p$ is not a theorem of the basic modal logic K.

$$\Box p, 0$$
$$\neg p, 0$$

There's nothing more to do.

Example 4 - $\Box p \vdash \Box\Box p$

$$\Box p, 0$$
$$\neg\Box\Box p, 0$$
$$\Diamond\neg\Box p, 0$$
$$0r1$$
$$\neg\Box p, 1$$
$$p, 1$$
$$\Diamond\neg p, 1$$
$$1r2$$
$$\neg p, 2$$

All the rules are applied. Crucially, because the only 0rx is for x=1, just apply line 1 to world 1.

Step 6

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5. For true $\Box$ worlds, do nothing at first, but make boxed sentence true everywhere accessible.
6. **Only close a branch when a sentence is true and false at same world.**

Don't do this!!!

A tableau that 'shows' the mistaken claim $\vdash \neg(\Diamond p \wedge \Diamond\neg p)$

$$\neg\neg(\Diamond p \wedge \Diamond\neg p), 0$$
$$\Diamond p \wedge \Diamond\neg p, 0$$
$$\Diamond p, 0$$
$$\Diamond\neg p, 0$$
$$0r1$$
$$p, 1$$
$$0r2$$
$$\neg p, 2$$

x (since p and $\neg p$)

More examples

We'll work through some more examples from the exercises at the end of chapter 2