Honors Logic, Lecture 09 - Modal Logic

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What Modal Logic Is

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- Why plural? Because we do not assume that these words have a single determinate meaning.

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To my ears, 1 is **logical** necessity, 2 is **metaphysical** necessity, 3 is **epistemic** necessity, 4 is **legal** necessity, 5 is **moral** (or **deontic**) necessity and 6 is **etiquette** necessity.

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To my ears, 1 is **logical** possibility, 2 is **metaphysical** possibility, 3 is **epistemic** possibility, 4 is **legal** possibility, 5 is **moral** (or **deontic**) possibility and 6 is **etiquette** possibility (though I'm not sure about any of these).

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So we want some logics where it is a logical truth, and some where it is not.

Language

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- These mean, respectively, that p must be true, and that p might be true.
- We interpret these somewhat similar to negations; they just bind what they are immediately next to.
- So $\Box p \to q$ means $(\Box p) \to q$, not $\Box(p \to q)$.

Truth

What does it take for these sentences to be true?

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- So instead of saying that each proposition simply has a truth value, we'll say that there are many **worlds**, and at each world each proposition has a truth value.
- But don't assume that propositions have the same truth value at each world.
- In one world I might be standing, and in another world I might be sitting.

What Are Worlds

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- Indeed, they need not even be anything like possible worlds in the sense that metaphysicians usually care about.
- They might, for instance, be different times.
- All we care about is that they are things at which propositions can be true or false.

Valuations

A valuation function tells us which worlds atomic sentences are true at.

- These can be completely arbitrary; we don't put any restrictions on them.

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- For sentences built up using $\wedge, \vee, \rightarrow, \neg$, this is relatively easy.
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- So if at world w, A is true and B is false, then $A \wedge B$ is false and $A \vee B$ is true.

Modal Values

We also need values for these sentences:

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- We don’t yet put any restrictions on it.
- Notably, we don’t assume that it is **reflexive**, **symmetric** or **transitive**

Properties of Relations

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- R is symmetric iff for all x, y, if xRy then yRx.
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A lot of relations we care about have one or more of these properties, but not all do. Consider, for example, **admires** as an example of a relation with none of them.

Truth of Modal Formulas

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A sentence $\Diamond A$ is true at a world x just in case the following condition is met:

- For some world y such that xRy, A is true at world y.

Modal Truth

- Something is necessarily true iff it is true everywhere that is accessible.

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We get back the Leibnizian idea that necessity is truth in all possible worlds if we assume the accessibility relation is the universal relation, i.e., xRy for all x, y.

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3. $\Diamond\Box A$

- That's because if $\Box A$ is true at any world, then it is true at all worlds. In the general case, where we do not assume that R is universal, these are not equivalent.

For Next Time

We'll talk about the relationship between boxes and diamonds.