Humberstone on Possibility Frames

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In his 1981 paper "From Worlds to Possibilities", Lloyd Humberstone developed an approach to modal logic using *possibilities* rather than possible worlds. Possibilities, unlike worlds, may be incomplete. This paper sets out the possibility frame approach to modal logic, proves some results about its logic (including that some logics definable on Humberstone frames are not definable on Kripke frames), and surveys several applications, including to conditionals, vagueness, and fiction.

In his 1981 paper "From Worlds to Possibilities", Lloyd Humberstone shows a way to do modal logic without the apparatus of possible worlds. Instead of worlds he uses *possibilities*, which may, unlike worlds, be incomplete. The non-modal parts of the view are discussed again in section 6.44 of *The Connectives*, though the differences between the view there and the 1981 view are largely presentational. In this paper I'll set out this *possibility frame* approach to modal logic, make some notes about its logic, and end with a survey of its possible applications.

Mathematically, possibilities are points in a model, like possible worlds are points in different kinds of models. But it helps to have a mental picture of what kind of thing they are. In "From Worlds to Possibilities", Humberstone notes that one picture you could have is that they are sets of possible worlds. This isn't a terrible picture, but it's not perfect for a couple of reasons. For one thing, as Humberstone notes, part of the point of developing possibilities is to do without the machinery of possible worlds. Understanding possibilities as sets of possible worlds wouldn't help with that project. For another, as Wesley Holliday (2025, 271–72) notes, the natural way to generate modal accessibility relations on sets of worlds from accessibility on the worlds themselves doesn't always work the way Humberstone wants accessibility to work. So let's start with a different picture.

Possibilities, as I'll think of them, are *stories*. To make things concrete, let's focus on a particular story: *A Study in Scarlet* (Conan Doyle (1995)), the story in which Sherlock Holmes was introduced. That story settles some questions, both explicitly, e.g., that Holmes is a detective, and implicitly, e.g., that Holmes has never set foot on the moon. But it leaves several other questions open, e.g., how many cousins Holmes has. It's not just that *A Study in Scarlet* is a story. It has proper parts that are stories. The first chapter is a story, one that tells of the first meeting between Holmes and Watson. And arguably it is a proper part of a larger story, made up of all of Conan Doyle's stories of Holmes and Watson. When a story x is a proper part of story y, what that means is that everything settled in x is still true in y, and more things besides are settled. When this happens, we'll call y a proper *refinement* of x. For most purposes it will be more convenient to use the more general notion of *refinement*, where each story counts as an improper refinement of itself.

Following Humberstone, I'll write $x \leqslant y$ to mean that y is a refinement of x. As he notes, this notation can be confusing if one thinks of x and y as sets, because in that case the refinement will typically be *smaller*.[1] But if we think of possibilities as stories, the notation becomes more intuitive. We have $x \leqslant y$ when y is created by adding new content to x. Keeping with this theme, I'll typically model stories not as worlds, but as finite sequences. (In the main example in Section 2, they will be sequences of 0s and 1s.) In these models, $x \leqslant y$ means that x is an initial segment of y.

1. The Formal Structure

The Basic Language

To start with, assume we're working in a simple language that has a countable set $\mathcal{P}$ of propositional variables, and three connectives: $\neg$, $\wedge$, and $\vee$. We have a set of possibilities W, and a transitive refinement relation $\leqslant$ on them. The following rules show how to build what I'll call a *Humberstone possibility model* on $\langle W, \leqslant\rangle$. (I'll call this a *possibility frame* in most contexts, but a *Humberstone frame* when I'm comparing it to similar structures, especially in the context of discussing Holliday (2025).)

A Humberstone possibility model $\mathcal{M}$ is a triple $\langle W, \leqslant, V\rangle$, where V is a function from $\mathcal{P}$ to $\wp(W)$, intuitively saying where each atomic proposition is true, satisfying these two constraints:

- For all x, if $x \in V(p)$ and $y \geqslant x$, then $y \in V(p)$. Intuitively, truth for atomics is **persistent** across refinements.
- For all x, if $\forall y \geqslant x \exists z \geqslant y : z \in V(p)$, then $x \in V(p)$. This is what Humberstone (2011, 900) calls **refinability**, and it means that p only fails to be true at x if there is some refinement of x where it is settled as being untrue.

Given these constraints, Humberstone suggests the following theory of truth at a possibility for all sentences in this language. (We'll treat $\rightarrow$ as a defined connective, with $A \rightarrow B =_{df} \neg A \vee B$.)

$$\begin{aligned}
&[\text{Vbls}] && \mathcal{M} \vDash_x p_i \text{ iff } x \in V(p_i);\\
&[\neg] && \mathcal{M} \vDash_x \neg A \text{ iff } \forall y \geqslant x,\ \mathcal{M} \nvDash_y A;\\
&[\wedge] && \mathcal{M} \vDash_x A \wedge B \text{ iff } \mathcal{M} \vDash_x A \text{ and } \mathcal{M} \vDash_x B;\\
&[\vee] && \mathcal{M} \vDash_x A \vee B \text{ iff } \forall y \geqslant x\, \exists z \geqslant y : \mathcal{M} \vDash_z A \text{ or } \mathcal{M} \vDash_z B.
\end{aligned}$$

Given these definitions, it's possible to prove three things. First, every sentence in the language is persistent. If $\mathcal{M} \vDash_x A$ and $x \leqslant y$, then $\mathcal{M} \vDash_y A$. For any sentence, truth is always preserved when moving to a refinement. Second, refinability holds for all sentences in the language. This is, as Humberstone notes, easier to state using this definition of $\neg$. It now becomes the claim, for arbitrary A, that if $\mathcal{M} \nvDash_x A$, there is some refinement y of x such that $\mathcal{M} \vDash_y \neg A$. Third, for any set of sentences Γ and sentence A, the truth at a point of all sentences in Γ guarantees the truth of A iff

1. Holliday (2025) writes $y \sqsubseteq x$ when y is a refinement of x, mirroring this way of thinking about possibilities.

the sequent Γ entails A in classical propositional logic.

In this paper, I'm going to discuss three extensions of this language. I'll introduce them in reverse order of how much they are discussed in Humberstone, starting with one he does not discuss at all: infinitary disjunction.

Infinitary Disjunction

We'll add to the language a new symbol $\bigvee$, which forms a new sentence out of any countable set of sentences not containing $\bigvee$. Intuitively, it is true when one of the sentences in the set is true. More formally, its truth at a possibility is defined as follows:

$$[\bigvee] \quad \mathcal{M} \vDash_x \bigvee(A_1, A_2, \ldots) \text{ iff } \forall y \geqslant x\, \exists z \geqslant y : \text{ for some } i\, \mathcal{M} \vDash_z A_i.$$

Again, it's fairly simple to show that this addition to the language will preserve persistence and refinability. But while this is simple, it is significant, because things could easily have been otherwise.

Quantifiers

The second extension will be to add quantifiers, following a suggestion in Humberstone (1981, xxxx). Assume, as usual, that the language has a stock of names $c_1, \ldots$, and for each n, a stock of n-place predicates $F_1^n, F_2^n, \ldots$. A *first-order (Humberstone) possibility model* is a structure $\langle W, \leqslant, D, V\rangle$, where D assigns a non-empty domain of objects to each point, and V interprets the non-logical vocabulary. More precisely:

- D is a function assigning to each $x \in W$ a non-empty set $D(x)$, the **domain** at x.
- V assigns to each name c_i and each $x \in W$ either a designated element $V(c_i, x) \in D(x)$, or is undefined at x.
- V assigns to each n-place predicate F_j^n and each $x \in W$ a set $V(F_j^n, x) \subseteq D(x)^n$, the **extension** of F_j^n at x.

These must satisfy the following constraints:

Domain monotonicity If $x \leqslant y$, then $D(x) \subseteq D(y)$.

Name coverage For each name c_i and each $x \in W$, there exists some $y \geqslant x$ such that $V(c_i, y)$ is defined.

Persistence for names If $V(c_i, x)$ is defined and $x \leqslant y$, then $V(c_i, y)$ is defined and $V(c_i, y) = V(c_i, x)$.

Persistence for predicate extensions If $\langle o_1, \ldots, o_n\rangle \in V(F_j^n, x)$ and $x \leqslant y$, then $\langle o_1, \ldots, o_n\rangle \in V(F_j^n, y)$.

Refinability for predicate extensions If $\langle o_1, \ldots, o_n\rangle \notin V(F_j^n, x)$, then there exists some $y \geqslant x$ such that for all $z \geqslant y$, $\langle o_1, \ldots, o_n\rangle \notin V(F_j^n, z)$.

Given a model and a variable assignment g mapping variables to objects, truth at a point is defined as follows. Write $g[v/o]$ for the assignment that maps variable v to object o and otherwise agrees with g. For a term t, write $[\![t]\!]^{g,x}$ for the denotation of t under g at x: for a variable v this is $g(v)$, and for a name c_i this is $V(c_i, x)$ when defined, and undefined otherwise.

[=] $\mathcal{M}, g \vDash_x t_1 = t_2$ iff $\forall y \geqslant x\, \exists z \geqslant y : [\![t_1]\!]^{g,z}$ and $[\![t_2]\!]^{g,z}$ are both defined and equal;

$[F^n]$ $\mathcal{M}, g \vDash_x F^n_j(t_1, \dots, t_n)$ iff $\forall y \geqslant x\, \exists z \geqslant y : \langle [\![t_1]\!]^{g,z}, \dots, [\![t_n]\!]^{g,z} \rangle \in V(F^n_j, z)$;

[$\forall$] $\mathcal{M}, g \vDash_x \forall v\, A$ iff $\forall y \geqslant x\, \forall o \in D(y) : \mathcal{M}, g[v/o] \vDash_y A$;

[$\exists$] $\mathcal{M}, g \vDash_x \exists v\, A$ iff $\forall y \geqslant x\, \exists z \geqslant y\, \exists o \in D(z) : \mathcal{M}, g[v/o] \vDash_z A$.

The Boolean connectives are handled exactly as in the propositional case.

The $\forall\exists$ pattern in the atomic clauses is necessary to ensure that persistence and refinability hold for all sentences, including atomic ones. Consider $c_i = c_i$: if a name has no denotation at x but acquires one at some refinement, then a simple "check the denotation at x" condition would leave $c_i = c_i$ neither true nor false at x, and no refinement of x could settle it as false either, violating refinability. The $\forall\exists$ condition handles this correctly: $c_i = c_i$ is true at x whenever c_i is covered at x (i.e., every refinement has a further refinement where c_i gets a referent), since once c_i gets a referent o, persistence of names ensures $o = o$ at all further refinements.

The atomic clauses simplify when names are fully defined. If t_1 and t_2 are variables, or names that already have denotations at x, then by persistence of names and predicate extensions the $\forall\exists$ quantifier prefix collapses: $\mathcal{M}, g \vDash_x t_1 = t_2$ iff $[\![t_1]\!]^{g,x} = [\![t_2]\!]^{g,x}$, and $\mathcal{M}, g \vDash_x F^n_j(t_1, \dots, t_n)$ iff $\langle [\![t_1]\!]^{g,x}, \dots, [\![t_n]\!]^{g,x} \rangle \in V(F^n_j, x)$. The more complex clauses above are needed only to handle the case where some name occurring in the formula lacks a denotation at x but is guaranteed to acquire one.

This is a possibilist treatment of the universal quantifier, in contrast to the actualist quantifiers discussed in Harrison-Trainor (2019). I'll return in Section 3 to the reasons we are best off using possibilist quantifiers, and the difficulties this raises for talking about just what's true in a possibility.

Modal Operators

The third extension will be the introduction of modal operators. Here I'll follow Humberstone (1981) very closely, save that I'll have a plurality of modal operators rather than just one. So I'll use these structures to define (as Holliday (2025) does) multi-modal logics. But I'll follow Humberstone, and not Holliday, in defining modal operators in terms of accessibility relations R_i satisfying these three conditions[2]:

UpR: If $x \leqslant x'$ and $x' R_i y$, then $x R_i y$.
RDown: If $x R_i y$ and $y \leqslant y'$, then $x R_i y'$.
RRef++: If $x R_i y$, then there exists $x' \geqslant x$ such that for all $x'' \geqslant x'$, $x'' R_i y$.

UpR says that if a refinement of x can access y, then x itself can already access y: accessibility is not something that can be gained by adding detail to the source. **RDown** is a converse of this; it says that accessibility cannot be lost by adding detail to the target. **RRef++** says that if x can access y, there

2. I'm using the names for these that Holliday uses, which are more evocative than Humberstone's original names.

is some refinement x' of x where it is settled that x' can access y. This last access can't be overturned by further refinement of x'.

Given these constraints, the truth condition for the box operator is:

$$[\Box_i] \quad \mathcal{M} \vDash_x \Box_i A \text{ iff } \forall y\, (xR_iy \Rightarrow \mathcal{M} \vDash_y A);$$

This should be familiar: $\Box_i A$ is true at x iff A is true at every R_i-accessible possibility.

Humberstone treats $\Diamond$ as a defined connective, $\Diamond_i A$ just means $\neg\Box_i\neg A$, and I'll follow suit. If we spell out what it means for $\neg\Box_i\neg A$ to be true, we get the rule $[\Diamond_i]_{\text{Official}}$. But if we impose the above three constraints on R_i, we can see that this is equivalent to the more familiar $[\Diamond_i]_{\text{Simple}}$.

$$[\Diamond_i]_{\text{Official}} \quad \mathcal{M} \vDash_x \Diamond_i A \text{ iff } \forall y \geqslant x\, \exists z \geqslant y\, \exists w\, (zR_iw \text{ and } \mathcal{M} \vDash_w A).$$

$$[\Diamond_i]_{\text{Simple}} \quad \mathcal{M} \vDash_x \Diamond_i A \text{ iff } \exists y\, (xR_iy \text{ and } \mathcal{M} \vDash_y A);$$

If R_i obeys **UpR**, then $[\Diamond_i]_{\text{Official}}$ will imply $[\Diamond_i]_{\text{Simple}}$. For $\Diamond_i A$ to be true at x according to $[\Diamond_i]_{\text{Official}}$, there must be some refinement which can access a point where A is true, and so by **UpR**, x itself can access that point. If R_i obeys **RRef++**, then $[\Diamond_i]_{\text{Simple}}$ will imply $[\Diamond_i]_{\text{Official}}$. If there is some y such that xR_iy and A is true at y, then by **RRef++**, there is some refinement of x such that every refinement of it can access y, and hence can access a point where A is true. So these are equivalent.

From now on, I'll use $[\Diamond_i]_{\text{Simple}}$ when working out what's true at points in particular models. But when we are proving general facts about the language, it will help to remember that $\Diamond_i$ is a defined connective, so we don't need an extra part of inductive arguments to cover it.

Modal Constraints

Why should we impose the three constraints Humberstone proposes? It is not hard to show that they guarantee that persistence and refinability hold for sentences generated using these new modal connectives. At least, it isn't hard as long as we remember that $\Diamond$ is being treated as defined, so the only new step in the inductive proofs involves $\Box$. And **UpR** guarantees persistence for $\Box$ sentences, while **RRef++** guarantees refinability.

But this is overkill. As Humberstone points out, we haven't used **RDown** in the proof, so this doesn't explain why we'd impose **RDown**. As Holliday (2025, 62) points out **UpR** is stronger than we need for persistence. We could weaken it by making greater use of the fact that A is persistent. All we need is that if $x \leqslant x'$ and xRy then there is some $z \geqslant y$ such that $x'Rz$. That will guarantee the key fact if x' can access a world where A, then so can x.

So we need other arguments for these constraints other than their role in ensuring persistence and refinability. Humberstone offers two other arguments here. One anticipates the applications of possibilities to multi-modal logics. If we want to use possibilities for tense logic, then we want the converse of any accessibility relation to satisfy the constraints on an accessibility relation. So if all acceptable accessibility relations satisfy **UpR**, they should also satisfy **RDown**. This isn't convincing on its own though. For one thing, as already noted, we might not need **UpR**. For another, by these lights we should worry that the system is incomplete because we haven't put in a converse of **RRef++**.

The argument that Humberstone spends more time on, and which I think is more compelling, comes from rethinking the relationship between R_i and $\Box_i$. It's very tempting to read those truth conditions as being explanatory from right-to-left. On this way of thinking, facts about which $\Box_i$ sentences are true at a point are grounded in which non-modal sentences are true and which R_i relations obtain. But while this is tempting, it isn't compulsory.

We could instead take the modal facts as given, and ask what accessibility relations must obtain to be consistent with them. The process here is familiar from the construction of canonical models. We take the sets of consistent sentences as given, and say $s_1 R_i s_2$ iff whenever $\Box_i A \in s_1$, then $A \in s_2$. Humberstone's approach is similar. Start with the idea that some sentences are true in some model $\mathcal{M}$ at possibilities x and y, say xR_iy iff $\mathcal{M} \vDash_y A$ whenever $\mathcal{M} \vDash_x \Box_i A$, and ask what constraints R_i will thereby satisfy.

The answer is that, if all sentences are persistent and refinable, then R_i will satisfy these three constraints. If $\mathcal{M} \vDash_y A$ whenever $\mathcal{M} \vDash_x' \Box_i A$, then by the persistence of $\Box_i$, we know that $\mathcal{M} \vDash_y A$ whenever $\mathcal{M} \vDash_x \Box_i A$. A similar argument, using the persistence of A, justifies **RDown**. Finally, **RRef++** follows from the fact that $\Box_i A$ is refinable. If $\Box_i A$ is not true at x, that means that it is determinately not true at some refinement. So if $\mathcal{M} \nvDash_x A$, there must be some refinement x' such that for all further refinements x'', $\mathcal{M} \nvDash_x'' A$. If $\mathcal{M} \nvDash_x \Box_i A$, then there must be some possibility y such that xR_iy and $\mathcal{M} \nvDash_y A$. So as long as $x''R_iy$, refinability will be satisfied. I don't see how to prove there isn't a weaker condition that would also work, it's possible we could use the refinability of A to find some weaker condition, but I don't quite see how that would work. So I think **RRef++** also follows from this way of thinking about accessibility.

In the next section I'll discuss what logics can be defined using frames that satisfy all of these conditions.

2. Logics Determinable on Humberstone Frames

Holliday (2025, sec. 8.2) raises an interesting question. As well as the familiar Kripke frames most commonly used as a semantics for modal logic, and the Humberstone frames defined above, he introduces a class of 'full possibility' frames, which weaken some of Humberstone's constraints. It won't matter here exactly what these weakenings are, but what does matter is that he shows that using these weakened frames, we can determine logics that are not determinable on any class of Kripke frames. To state this more precisely, for any class of frames **F**, let L(**F**) be the set of sentences true at all points in all models definable on some member of **F**. Then let ML(**F**) be the set $\{\mathrm{L}(\mathbf{X}) : \mathbf{X} \subseteq \mathbf{F}\}$. That is, ML(**F**) is the class of logics that can be determined using just **F**.

If we let **K** denote the class of Kripke frames, and **FP** denote the class of full possibility frames, Holliday (2025, sec. 2.5) constructs a very clever argument to show that $\mathrm{ML}(\mathbf{K}) \subsetneq \mathrm{ML}(\mathbf{FP})$. But if we let **H** denote the class of Humberstone frames, it follows from the fact that every Kripke frame is a Humberstone frame and every Humberstone frame is a full possibility frame that ML(**K**) $\subseteq$ ML(**H**) $\subseteq$ ML(**FP**). And while ML(**K**) $\subsetneq$ ML(**FP**) implies that at least one of those inclusions is strict, it isn't clear which one. He leaves the question of which one it is as an open question.

I don't have an answer to that question as asked, since it is asked about languages whose sentences have finite length. I do have a proof that if we allow infinite disjunction, as discussed above, then ML(**H**) $\neq$ ML(**K**). If we expand the language like that, at least the first inclusion is strict. I will show this by constructing a single Humberstone frame that, in the infinitary language, defines a logic with no Kripke equivalent. The construction will follow Holliday's very closely, but differ just enough to ensure compliance with Humberstone's conditions.

The Frame

The frame is built from two copies of the set of finite binary sequences — sequences of 0s and 1s of any finite length, including the empty sequence. Call one copy the **left-handed** sequences and the other the **right-handed** sequences. The refinement relation is: $x \leqslant y$ iff x and y have the same handedness and x is an initial segment of y. So within each copy the frame is just the binary tree ordered by extension, and no left-handed sequence refines a right-handed sequence or vice versa. It will help to have some notation for referring to points in this model. When s is a finite binary sequence, I'll write s^L for the left-handed version of s, and s^R for the right-handed version.

The Accessibility Relations

Next I'll define an accessibility relation and a separate infinite family of accessibility relations. The single relation, which I'll write $R^{\rightarrow}$, is such that $xR^{\rightarrow}y$ iff x is left-handed and y is right-handed. The family of relations, each written $R_i^{\leftarrow}$ for $i \in \mathbb{N}$, is such that $xR_i^{\leftarrow}y$ iff x is right-handed, x does not have a 0 in its i-th position (either because x has length less than i, or because it has a 1 in position i), and y is left-handed.

That $R^{\rightarrow}$ satisfies **UpR**, **RDown**, and **RRef++** is obvious. It is also obvious that for each i, $R_i^{\leftarrow}$ satisfies **UpR** and **RDown**. It's only a little harder to show that it satisfies **RRef++**. Assume $xR_i^{\leftarrow}y$. So x is right-handed and y is left-handed. If x is of length at least i, then x itself can serve as the refinement such that every further refinement can access y. If x's length is less than i, extend x with enough 1s to create a sequence of length i. The result will be a refinement such that every further refinement can access y, as required.

The Example

Now consider the proposition (Splitting), a minor variant on Holliday's example (Split). (I'm using **T** for an arbitrary tautology.)

$$\neg\Diamond^{\rightarrow}p \vee \bigvee_{i\in\mathbb{N}} (\Diamond^{\rightarrow}(p \wedge \Diamond_i^{\leftarrow}\mathbf{T}) \wedge \Diamond^{\rightarrow}(p \wedge \neg\Diamond_i^{\leftarrow}\mathbf{T})) \qquad \text{(Splitting)}$$

I'm going to make three claims about (Splitting). First, it is true throughout the frame I just described. Second, $\neg\Diamond^{\rightarrow}p$ is not true on all models on that frame. Third, there is no class of Kripke frames throughout which (Splitting) is always true and $\neg\Diamond^{\rightarrow}p$ is not always true. From this it follows that $\mathrm{ML}(\mathbf{K}) \neq \mathrm{ML}(\mathbf{H})$.

For the first claim, I'll show something slightly stronger, namely that at each point one or other disjunct in (Splitting) is true. If the point is right-handed, then the first disjunct is true, since each right-handed point is a dead-end with respect to $R^{\rightarrow}$. So we just have to look at the left-handed points. Let x be an arbitrary left-handed point. If there is no y such that $xR^{\rightarrow}y$ and p is true at y, then again the first disjunct is true.

Now consider the case where x is left-handed, and there are right-handed points such that p is true at y. Here it will be helpful to write a right-handed point y as s_y^R. Among those s_y^R at which p is true, consider the ones where $|s_y^R|$, the length of s_y^R, is minimal. There must be some such points, since sequence lengths are non-negative integers and p is true at some accessible right-handed point. Let $i = |s_y^R| + 1$. Let $s_y^R \oplus \langle 0 \rangle$ and $s_y^R \oplus \langle 1 \rangle$ be the right-handed sequences generated by adding either a 0 or a 1, respectively, to s_y^R. Since p is true at s_y^R and truth is persistent, p will be true at both $s_y^R \oplus \langle 0 \rangle$ and $s_y^R \oplus \langle 1 \rangle$. Since $s_y^R \oplus \langle 0 \rangle$ has a 0 at its ith position, it is a dead-end with respect to $R_i^{\leftarrow}$. So $\neg\Diamond_i^{\leftarrow}\mathbf{T}$ is true there. So since x is left-handed, and $s_y^R \oplus \langle 0 \rangle$ is right-handed, $\Diamond^{\rightarrow}p \wedge \neg\Diamond_i^{\leftarrow}\mathbf{T}$ is true at x. A similar argument shows that $p \wedge \Diamond_i^{\leftarrow}\mathbf{T}$ is true at $s_y^R \oplus \langle 1 \rangle$, so $\Diamond^{\rightarrow}(p \wedge \Diamond_i^{\leftarrow}\mathbf{T})$ is true at x. And that implies that the ith disjunct of the right-hand disjunction of (Splitting) is true at x, as required.

The second claim, that $\neg\Diamond^{\rightarrow}p$ is not true on all models on the frame, is trivial, since it will fail at some left-handed points whenever p is true at some right-handed point.

For the third claim, we follow Holliday's argument particularly closely. In particular, we'll appeal to his insight that "Worlds cannot split, but possibilities can" (Holliday 2025, 95). Consider any class of Kripke frames such that $\neg\Diamond^{\rightarrow}p$ is not valid on that class. Look at the class of models on those frames where p is true at exactly one world. For any disjunct of the right-hand disjunction of (Splitting) to be true at a point, that point must access a world where p is true that is a dead-end with respect to $R_i^{\leftarrow}$, and also a world where p is true that is not a dead-end with respect to $R_i^{\leftarrow}$. That's impossible if p is true at just one world. So throughout this class of models, the right-hand disjunction of (Splitting) will be false. But if $\neg\Diamond^{\rightarrow}p$ is not valid on the frame, there will also be points in this class of models where $\Diamond^{\rightarrow}p$ is true. So the whole disjunction will be false at those points, and hence (Splitting) is not valid on the frame.

Putting these together, there is no class of Kripke frames that validates exactly the sentences valid on this particular Humberstone frame. So $\mathrm{ML}(\mathbf{K}) \neq \mathrm{ML}(\mathbf{H})$, and hence $\mathrm{ML}(\mathbf{K}) \subsetneq \mathrm{ML}(\mathbf{H})$.

3. Quantifiers and Necessitism

4. Conditionals

The only discussion of possibilities (as opposed to worlds) in *The Connectives* is in the chapter on disjunction. But there are several potential connections to conditionals, and in this section I'll go over a couple of them.

One connection concerns Conditional Excluded Middle (as discussed on pages 1008-1013 of

The Connectives), and more generally the relationship between $A > (B \vee C)$ and $(A \mathrel{\Box\!\!\rightarrow} B) \vee (A \mathrel{\Box\!\!\rightarrow} C)$. On a Stalnaker-Lewis style approach to conditionals, these are equivalent iff there is a nearest possible world in which A is true, for any possble A. It is natural to think about whether that is true in largely metaphysical terms, asking whether there really is guaranteed to be a single nearest world where A is true. And as Lewis (1973) argued, it is natural to answer that question negatively.

To take one striking example of that, consider the discussion by Jeremy Goodman (2018) of the example, originally due to Max Black (1952), of the two spheres alone in space. Black says that the spheres are really two, so this is a counterexample to the Principle of Identity of Indiscernibles. Let's assume, for now, that Black is right, and we can call one sphere a and the other sphere b. Goodman asks what we should say about the counterfactual possibility that one of the spheres is heavier. On Lewis's picture, rejecting Conditional Excluded Middle, both (1) and (2) are false.

(1) If one of the spheres were heavier, it would be a.
(2) If one of the spheres were heavier, it would be b.

A common thought at this point is that this verdict really does follow from Lewis's 'nearest possible world' semantics for conditionals, but that data about the inferential role of conditionals shows that Conditional Excluded Middle must be correct.[3] This is, many think, a problem for the Lewisian view.

One move here, discussed by Humberstone (2011, 1011), is to use supervaluations. Perhaps it is in some sense indeterminate whether the world where a is heavier or the world where b is heavier is more like actuality. A related, but I think, preferable, move is to analyse conditionals not in terms of possible worlds, but in terms of possibilities.

Here is one possible way to analyse conditionals, mixing Stalnaker's approach with Humberstone's possibilities. (The particular formulation I'm going to use draws heavily on the theory presented by Andrew Bacon (2023, 382). The four conditions are directly quotes from his paper, though I mean something different by them since on my version the variables pick out possibilities not worlds. I'll have more to say about Bacon's paper presently.) Extend a possibility model $\langle W, \geqslant, V \rangle$ to a conditional possibility model by adding a selection function f. This is a function $\mathcal{P}(W) \times W \rightarrow \mathcal{P}(W)$, intuitively picking out the 'nearest' possibilities to a world where a particular proposition is true, satisfying these constraints.[4]

MP $x \in f(A, x)$ whenever $x \in A$
ID $f(A, x) \subseteq A$
CEM $|f(A, x)| \leq 1$
AB If $f(A, x) \subseteq B$ and $f(B, x) = \emptyset$ then $f(A, x) = \emptyset$

The cardinality constraint **CEM** guarantees that Conditional Excluded Middle will hold. But we don't have to make invidious choices about whether the nearest possibility where one of the spheres is heavier makes a or b heavier. Rather, we just say that the nearest possibility is an unrefined pos-

3. For a recent statement of this last view, with many more citations to similar statements, see Cariani and Goldstein (2020).

4. Humberstone uses R rather than f for the same idea; I'm going to follow Bacon, who in turn follows Stalnaker, to highlight the connection to theories which validate Conditional Excluded Middle.

sibility that makes the disjunctive proposition *One of them is heavier* true, without making either disjunct true. It will have refinements where each is true, but the nearest possibility will not validate either disjunct. This seems like an intuitive treatment of the case.

Goodman uses this example to argue that Black was incorrect, and the spheres are in fact discernible. His argument is that one but not the other will have the property of being the one which would be heavier if they were different. I don't think this argument goes through on the possibilities framework, but settling that would require saying more about how higher-order quantification works on the possibilities framework, and that would take us too far afield. Instead I'll turn to the puzzle that Bacon introduces them to solve. That is a puzzle, introduced by Kit Fine (2012b, 2012a), which is a counterfactual version of a paradox from José Bernadete (1964).

There is a room that is very dangerous to cross. A man is thinking of crossing it, but he is warned off when he learns that it contains an infinity of gods. God$_1$ will kill him if he makes it half-way across the room. God$_2$ will kill him if he makes it a quarter of the way across, $_3$ will kill him if he makes it one-eighth of the way across. More generally, God$_n$ will kill him if he makes it $(1/2)^n$ of the way across the room. Does he enter? Of course not; he'd be killed! But who would kill him? Presumably not God$_1$; how would he make it that far? This generalises. God$_n$ can't kill him, because God$_{n+1}$ would already have done the job. So he would be killed by the gods, but not by any God. This doesn't sound very plausible.

The case looks like the kind that motivated Lewis to reject what he called the *Limit Assumption*. This says that if A is possible, then relative to any world w there are some closest worlds where A is true. Humberstone (2011, 1014–15) discusses Lewis's rejection of the Limit Assumption, and adopts the position that we shouldn't impose it in general, but can freely talk as if it is true, because it doesn't make a difference to the logic. This is right in the context Humberstone is writing in, but possibly misleading. The Limit Assumption does make a big difference to the logic if we have either quantifiers or infinitary connectives in the language. This fact is what Fine's puzzle turns on.

Stated without the Limit Assumption, Lewis's view is that $A \mathrel{\Box\!\!\rightarrow} B$ is true at w if there is some world where A is true such that there is no closer world where $A \wedge \neg B$ is true. If we assume that for any n, the world where the man enters and God$_{n+1}$ kills him is closer than the world where he enters and God$_n$ kills him, then Lewis is committed to both (3) and (4).

(3) If the man were to enter the room, he would be killed by either God$_1$ or God$_2$ or ....
(4) For each n, it is not the case that if the man were to enter the room, he would be killed by God$_n$.

As Fine notes, Lewis's theory of counterfactuals is committed to denying a principle he calls **Infinite Conjunction**.

Infinite Conjunction If $A \mathrel{\Box\!\!\rightarrow} C_i$ is true for each i, then $A \mathrel{\Box\!\!\rightarrow} (C_1 \wedge C_2 \wedge \dots)$ is true.

Without the Limit Assumption, Lewis's semantics would endorse **Infinite Conjunction**. But it would also have a problem. Which of the gods would kill the man? Any choice seems not only arbitrary, but mistaken. Let's spell this out a bit more carefully. Fine's way of spelling out the paradox makes heavy use of a principle he calls **Disjunction**. (This is called **Subj. Dilemma** by Humberstone (2011, 1015).)

Disjunction If $A \boxright C$ and $B \boxright C$ are true, so is $(A \vee B) \boxright C$.

Then both (5) and (6) seem like they should be true.

(5) If the man had entered the room, and been killed by one of God_1 through God_{k+1}, he wouldn't have been killed by God_k (because God_{k+1} would have killed him first).

(6) If the man had entered the room, and not been killed by one of God_1 through God_{k+1}, he wouldn't have been killed by God_k.

Putting these together using **Disjunction**, we get the following sentence. It's a mouthful, but it's important to spell it out for what comes next.

(7) If either the man had entered the room, and been killed by one of God_1 through God_{k+1}, or he had entered the room, and not been killed by one of God_1 through God_{k+1}, he wouldn't have been killed by God_k.

To finish off the paradox, let's add a new principle **Antecedent Substitution**.

Antecedent Substitution If A and B are provably equivalent in classical logic, and $A \boxright C$ is true, so is $B \boxright C$.

Then using **Antecedent Substitution** we can get from (7) to (8).

(8) If the man had entered the room, he would not have been killed by God_k.

And since k is arbitrary in (8), we can derive (4), without any appeal to the metaphysics of counterfactuals. It looks like our only options, short of abandoning classical logic, are to give up one **Infinite Conjunction**, **Disjunction**, or **Antecedent Substitution**. In Fine's original discussion of the paradox he introduces several more principles that could in theory be given up, but Brian Embry (2014) convincingly argues that really it has to be one of these three that go, and I'm following his lead.

If these are the three options, it isn't obvious which one to take. All three paths forward have their proponents, or at least are consequences of otherwise plausible views. I've already noted that Lewis (1973) is committed to rejecting **Infinite Conjunction**, because he does not endorse the Limit Assumption. This does not look great; it's crucial to reasoning with counterfactuals that if some things would each be true were A the case, then were A the case they would each be true.

Fine (2012b) recommends giving up **Antecedent Substitution**. He develops a theory of conditionals that doesn't use possible worlds, but instead uses incomplete states. These are not entirely unlike Humberstone's possibilities, but the resulting theory is quite different. I suspect the key distinction, the one that drives all of the rest of the results, is that Fine takes disjunctions to be true at a state only if a disjunct is true at that state. Anyway, Fine thinks that the misstep in the trilemma above is the derivation of (8) from (7). That step requires substituting A for $(A \wedge B) \vee (A \wedge \neg B)$ in an antecedent, which Fine takes to be illegitimate.

There are a couple of reasons to be unhappy with this way of getting out of the problem. One is that this did not feel like the most controversial step when we were developing the problem. But a bigger one is that Fine's resolution of the trilemma ends up endorsing not just **Disjunction**, but also its converse. This is the principle that Humberstone (2011, 1016) calls *Conv. Subj. Dilemma*, the key being that from $(A \vee B) \boxright C)$ one can infer $A \boxright C$. The criticisms of Fine in Embry (2014) largely centre on this aspect of Fine's view, and its consequences. But the key problems with the principle are already pretty clear in Humberstone (2011, 1016–22). So I think we shouldn't go that way.

So that leaves **Disjunction**. This is the step that Andrew Bacon (2023) rejects, and it's what I'll reject as well. I think there are two key reasons to worry about giving up **Disjunction**, and they are pretty hard worries to address in the possible worlds framework. But they both seem more or less manageable in the possibilities framework.

The first worry is that without **Disjunction**, we have to give up the idea that the selection function f is in any sense a measure of similarity, or really any kind of nearness. Here is how Bacon puts it,

> The second consideration in favour of **Disjunction** is that its validity is predicted by the now dominant account of counterfactuals, prominently defended by Lewis and Stalnaker, based on similarity semantics. For roughly, if the closest A worlds are C worlds, and the closest B worlds are C worlds, then the closest $A \vee B$ worlds are C worlds. (Bacon 2023, 374)

Now for Bacon, this isn't a worry, because he thinks there are independent reasons to reject the picture of similarity or nearness as being foundational to counterfactuals. I don't find those reasons convincing, and basically agree with the response to them that Fine (2023) gives. But it's not just critics of **Disjunction** who connect it to the similarity picture. The same connection is drawn by Humberstone, who does endorse **Disjunction**. He first notes that, by analogy with other maximal relations, we should expect that the nearest $A \vee B$-world will be either the nearest A-world or the nearest B-world (Humberstone 2011, 1015). He then argues more formally that if the set of nearest A-worlds is generated by an underlying three place similarity relation $S_w xy$, meaning x is at least as similar to w as y is, and if for any w, S_w is a total preorder on worlds, then **Disjunction** is guaranteed to hold (Humberstone 2011, 1025–26).

That argument seems irrefutable if, *but only if*, we're working in a possible worlds framework. If we're in a possibilities framework, it doesn't look right. It could be that the nearest possibility in which $A \vee B$ is not identical to either the nearest possibility in which A, or the nearest possibility in which B, but is instead a coarsening of one of those possibilities.

How could **Disjunction** fail on the possible worlds picture? We must fail to have the nearest $A \vee B$-world be a C-world. But that's impossible if the nearest A-world is a C-world, and the nearest B-world is a C-world, and all $A \vee B$-worlds are A-worlds or B-worlds. On the possibilities picture, that last clause fails. It might be that the nearest possibility which makes $A \vee B$ true does not make either A true or B true, it just guarantees that sequence of refinements will eventually make one or the other true. Also note that we don't require that the nearest $A \vee B$ possibility makes $\neg C$ true; it could be that **Disjunction** fails because both C and $\neg C$ are true at different refinements of the nearest $A \vee B$ possibility.

That's what happens with our man who wisely doesn't enter the room. What's the nearest possibility in which he does enter the room? It's the incomplete possibility where he is killed by one of the gods. For each *i, j:* $i > j$, the possibility where he is killed by God$_i$ is closer than the one where he is killed by God$_j$. But the indeterminate possibility where he is killed, but the possibility does not specify which god he is killed by, is closer to actuality than any complete possibility which specifies the homicidal divinity.

So rejecting **Disjunction** is compatible with the similarity approach to counterfactuals, as long as we use possibilities. The other worry with rejecting disjunction, one Fine (2023) stresses in his response to Bacon, is that we use **Disjunction** a lot in ordinary counterfactual reasoning. We should be cautious about giving it up. Of course, there are plenty of rules that we use in ordinary reasoning that work in all but a few edge cases. If we could show that **Disjunction** was truth-preserving in all but some rare exception cases, we could justify using it as a rule of inference. After all, we don't think that the failure of Axiom V in full generality means that it's always a mistake to infer from the existence of some things to the existence of a set containing all and only them.

As Fine stresses, it's hard to see how on Bacon's view **Disjunction** would even count as typically fine. The reason Bacon says that it fails in the paradoxical case from Bernadete generalises to more humdrum uses.

But that's not true for the possibilities model. You need to have some very unusual relationships between Humberstone's family of similarity relations S_w and $\geqslant$ in order for **Disjunction** to fail. For one thing, you need the nearest possibility x where $A \vee B$ holds to be an incomplete possibility where neither A nor B holds. Clearly x has to have refinements where A is true, and refinements where B is true. To get a **Disjunction** failure, you need one of the refinements of x where one of the disjuncts is true to not be (a refinement of) the closest possibility to actuality where that disjunct obtains. I can't construct an intuitive case where that happens that doesn't involve infinite sequences like in Bernadete's case, though I also don't have a proof that no such case can be constructed. This is all far from conclusive, but it seems plausible that on the possibilities model, failures of **Disjunction** will be rare. And that would be enough to explain the fact Fine appeals to, that we are usually happy using it in everyday inferences.

5. Conclusion

I've gone over two more uses of the possibilties framework, but there are many more things that we could imagine using it for. I'll end by briefly mentioning two of them.

As I briefly alluded to in Section 4, possibilities can do a lot of work that philosophers have tried to make supervaluations do. As well as using possibilities instead of supervaluations in preserving Conditional Excluded Middle, it's worth exploring whether they are useful in thinking about vagueness, or about open future.

Humberstone (1981) mentions that possibilities do a better job than possible worlds at making sense of talk about 'belief worlds'. We could say the same thing about the worlds of fiction. David Lewis (1978) ends up treating the operator *In this fiction* as box-like, because he thinks otherwise we'd have to make arbitrary choices about some details of how the story is to be filled out. Using possibilities here seems smoother. Any (coherent) fiction, I conjecture, picks out a particular possibility. That will always be a less than fully refined possibility, but a possibility nonetheless. On this approach we don't get left with the unfortunate triple, which Lewis is committed to, of it being true in the story that $A \vee B$, but false that it's true in the story that A, and false that it's true in the story that B. I also think there might be some uses of possibilities in characterising the distinctive relationship between a story and its sequel.

But I'll leave those tasks for another day. The main point of this paper is to remind the reader how many uses Humberstone's notion of a possibility has, and to explore what happens to the logic of possibilities when we add quantifiers or infinitary connectives.

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