ChatGPT

Rationalizability by a Finite Set of Utilities

A choice function C defined on subsets of alternatives is *rationalizable* by a finite set of utility functions if there is a finite collection $L={u_1,\dots,u_k}$ such that for every feasible menu S, $$ C(S)=\bigcup_{u\in L}\arg\max_{x\in S}u(x)\,. $$ Equivalently, each $x\in C(S)$ is **maximal** in S according to at least one of the (complete) preference orders induced by the $u\in L$. Such choice functions are known in the literature as **pseudo-rationalizable** or **multi-preference rationalizable** choice functions [1] [2]. A classical result by Aizerman and Malishevski (1981), as presented by Moulin (1985), shows that *exactly* those choice functions satisfy two elementary axioms: *contraction consistency* (Chernoff's axiom, "α") and *Aizerman's axiom*. In modern terms, Stewart (2020) confirms that a choice function is representable as the union of maxima of a finite family of (linear) preferences if and only if it satisfies Sen's property α together with Aizerman's condition [1] [3]. No additional axiom like Sen's β (completeness consistency) is needed or possible: requiring only α and a suitable expansion consistency (a weakened γ) characterizes *acyclic* multi-utility choice [4]. Equivalently, these conditions imply **path-independence** (Plott's axiom): $$C(S\cup T)=C\bigl(C(S)\cup C(T)\bigr)$$ for all S,T [5], a property known to be necessary and sufficient for pseudo-rationalizability (path-independence itself being called "pseudo-rationalizable" choice by Chambers & Yenmez 2017 [5]).

- **Contraction Consistency (Sen's α, a.k.a. Chernoff).** If $x\in C(S)$ and $T\subseteq S$ with $x\in T$, then $x\in C(T)$. In words, removing unchosen alternatives cannot eliminate a chosen one. This axiom is necessary for *any* rationalization by utility or preference [6].
- **Aizerman's Axiom.** If $C(T)\subseteq S\subseteq T$, then $C(S)\subseteq C(T)$. Equivalently, deleting unchosen alternatives from a menu does not **expand** the choice set. Aizerman (1981, 1985) introduced this condition and showed that, together with α, it forces the existence of a set of linear orders rationalizing C [1] [3]. Intuitively, if all maxima of T lie in a smaller sub-menu S, then any maximal element of S (with respect to some utility in L) must also have been maximal in T. Formally, Stewart (2020) states: *"A choice function C is pseudo-rationalizable iff it satisfies α and Aizerman's axiom"* [1].
- **Path-Independence.** This single axiom, due to Plott, is equivalent to the conjunction of α and Aizerman's axiom (under nonemptiness). It requires $C(S\cup T)=C(C(S)\cup C(T))$ for all menus S,T. Path-independence ensures consistency of choices when aggregating or splitting menus. It has long been known that path independence exactly characterizes choice functions that are the union of some fixed family of linear-ranking maximizers [5] [7]. Indeed, Stewart notes that choice rules satisfying path independence are *"pseudo-rationalizable"* (multi-order rationalizable) [5].
- **Expansion Consistency (Sen's γ) vs. Weak γ.** Full expansion consistency (if $x\in C(S)$ and $x\in C(T)$ then $x\in C(S\cup T)$) is **not required** and typically fails for multi-utility choice. Instead, a *weakened* expansion axiom (called "weak γ") can be imposed: if $C(S)={x}$ is a singleton and $x\in C(T)$, then $x\in C(S\cup T)$. Stewart (2020) shows that α plus weak γ characterize exactly the broader class of *weakly pseudo-rationalizable* choice functions (those rationalizable by a set of **acyclic** relations, not necessarily complete) [4]. In the finite-linear case, however, one can stay with α and Aizerman's axiom alone.

These axioms have appeared in various forms in the choice/rationality literature. Sen's "weak axiom of revealed preference" (WARP) is the classical α–β pair; here α is contraction consistency and β is a completeness-linked condition that enforces transitivity of the underlying preference. In contrast, pseudo-rationalizability drops β (allowing preference incompleteness across criteria) and replaces it with Aizerman's axiom (or equivalently path-independence) [1] [5]. Moulin (1985) summarizes these results, noting that *"Chernoff plus Aizerman characterize pseudorationalizability"* [3]. (For a single rationalization by one linear order, the standard result is that C must satisfy α and expansion consistency, yielding a transitive preference – Sen's classical theorem.)

Related Results and Literature

The question of rationalizing choice by *sets* of utilities is closely linked to models of **incomplete or multi-criteria preferences**. If one views L as representing multiple independent objectives, then $C(S)$ is the set of Pareto-optimal alternatives under the incomplete preference "x is at least as good as y iff $u_i(x) \ge u_i(y)$ for **all** i". Thus $C(S)$ coincides with the set of *maximal elements* of the induced partial order. This connection appears in many works on multiple-criteria decision (e.g. *Pareto rational choice* versus *utilitarian maximality*).

In decision theory, **multi-utility representations** of incomplete preferences have been studied extensively. Early results (Debreu, 1964; Fishburn, 1970s) show that certain incomplete relations can be represented by finitely many utility functions under conditions like continuity or convexity. Ok (2000) provides axioms for representing partial orders by vector-valued utilities [8]. These representations ensure that the maximal elements (undominated choices) correspond to those with some maximal utility component. In this sense, choosing the union of L-maximizers is equivalent to choosing all *undominated* alternatives under an incomplete preference relation (Levi 1986, Dubra et al. 2004).

There is also a connection to **choice under uncertainty with imprecise probabilities or utilities**. For example, Seidenfeld et al. (2007) axiomatize *coherent choice functions* that select options maximizing expected utility for some (p,u) in a fixed set. In the certainty case (degenerate probabilities), their framework specializes to exactly our setting: admissible choices are those that maximize *some* utility in a family. They develop axioms (extensions of Hurwicz's criterion, Γ-maximin, etc.) for such choice correspondences [9] [6]. However, their focus is on continuity/mixings under uncertainty, whereas the finite deterministic case reduces to the simpler axioms above.

In social choice and game theory, the union-of-max model appears in "multiple-rationale" and "multi-preference" contexts. Kalai, Rubinstein and Spiegler (2002) study *rationalizing by multiple rationales*, where each rationale is a linear ordering and one outcome per menu is chosen; they show that almost all choice functions require many rationales (orders) to rationalize (for a finite set of alternatives) [10], though they do not give axioms for rationalizability beyond noting the IIA/IIA failures. Chambers and Yenmez (2017) use path-independence axioms for choice rules in matching markets, recognizing that path-independent choice functions are precisely those rationalizable by multiple linear orders [5].

Finally, the broader revealed-preference literature offers many related concepts. Aizerman's classic papers (1981, 1985) and subsequent surveys (Aizerman 1985; Aleskerov et al. 2007) laid out the "general theory of choice" including pseudo-rationalizability [11] [3]. Nehring and Puppe (1997, 1998, 2002) study non-binary choice and *hyper-relational* rationalizations, generalizing these ideas to cases where choices themselves have structure. In incomplete preferences, Gilboa–Schmeidler (1989) consider maximizing multiple expected utilities (maximin expected utility), and Bewley (1986) studies choices that cannot be dominated by any single criterion; these models likewise pick options that are best under some weighting. These all reinforce the view that *union-of-maxima* choice functions capture decision rules with multiple independent values or criteria.

In summary, the **exact axiomatic characterization** is known: a choice function is rationalizable by a finite set of utilities if and only if it satisfies contraction consistency (Chernoff's α) and Aizerman's deletion property [1] [3]. Equivalently, it is path-independent [5]. These results go back to Aizerman and Malishevski (1981) and have been rederived and popularized by Moulin (1985) and Stewart (2020) [1] [3]. If one allows arbitrary acyclic preferences (dropping completeness), the analogous result is that α plus a weak expansion axiom characterize the "weakly pseudo-rationalizable" class [4]. Thus, while no *new* axioms beyond α and variants of expansion are needed, the literature offers a rich context linking this problem to incomplete-preference theory, multi-criteria choice, and coherent choice under uncertainty.

Sources: For the precise axiomatic theorem see Stewart (2020, *Math. Soc. Sci.*) [1] [4]; Aizerman and Malishevski (1981) and Moulin (1985) provide the original proofs [3]. Related theory of incomplete/multi-utility preferences is discussed in Ok (2000), Dubra et al. (2004), and Seidenfeld et al. (2007) [8] [9]. Path-independence equivalence is noted in Plott (1973) and Chambers–Yenmez (2017) [5]. Other relevant reviews include Aizerman (1985) and Aleskerov et al. (2007) on multi-criteria choice, and Kalai et al. (2002) on multiple rationales [10] [5]. These sources cover both the classical characterization and its modern generalizations in decision theory.

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