Humberstone on Possibility Frames

BRIAN WEATHERSON

University of Michigan

Insert abstract here

In his 1981 paper, "From Worlds to Possibilities", Lloyd Humberstone shows a way to do modal logic without the apparatus of possible worlds. Instead of worlds he uses *possibilities*, which may, unlike worlds, be incomplete. The non-modal parts of the view are discussed again in section 6.44 of *The Connectives*, though the differences between the view there and the 1981 view are largely presentational. In this paper I'll set out this *possibility frame* approach to modal logic, make some notes about its logic, and end with a survey of the many possible applications it has.

Mathematically, possibilities are just points in a model, just like possible worlds are points in different kinds of models. But it helps to have a mental picture of what kind of thing they are. In "From Worlds to Possibilities", Humberstone notes that one picture you could have is that they are sets of possible worlds. This isn't a terrible picture, but it's not perfect for a couple of reasons. For one thing, as Humberstone notes, part of the point of developing possibilities is to do without the machinery of possible worlds. Understanding possibilities as sets of possible worlds wouldn't help with that project. For another, as Wesley Holliday (2025, 271–72) notes, the natural way to generate modal accessibility relations on sets of worlds from accessibility on the worlds themselves doesn't always work the way Humberstone wants accessibility to work. So let's start with a different picture.

Possibilities, as I'll think of them, are *stories*. To make things concrete, let's focus on a particular story: *A Study in Scarlet* (Conan Doyle (1995)), the story where Sherlock Holmes was introduced. That story settles some questions, both explicitly, e.g., that Holmes is a detective, and implicitly, e.g., that Holmes has never set foot on the moon. But it leaves several other questions open, e.g., how many (first) cousins Holmes has. It's not that *A Study in Scarlet* is a story. It has proper parts which are stories. The first chapter is a story, one which tells of the first meeting between Holmes and Watson. And arguably it is a proper part of larger story, made up of all of Conan Doyle's stories of Holmes and Watson. When a story x is a proper part of story y, what that means is that everything settled in x is still true in y, and more things besides are settled. When this happens, we'll call y a proper *refinement* of x. For most purposes it will be more convenient to use the more general notion of *refinement*, where each story counts as an improper refinement of itself.

Following Humberstone, I'll write $x \leqslant y$ to mean that y is a refinement of x. As he notes, this notation can be confusing if one things of x and y as sets, because in that case the refinement will typically be *smaller*.[1] But if we think of possibilities as stories, the notation becomes more intuitive. We have $x \leqslant y$ when y is created by adding new content to x. Keeping with this theme, I'll typically model stories not as worlds, but as finite sequences. (In the main example in **?@sec-proof**, they will be sequences of 0s and 1s.) In these models, $x \leqslant y$ means that x is an initial segment of y.

Formal Structure

To start with, assume we're working in a simple language that just has a countable set $\mathcal{P}$ countable infinity of propositional variables, and three connectives: $\neg$, $\wedge$ and $\vee$. We have a set of possibilities R, and a transitive refinement relation $\geqslant$ on them. The following rules show how to build what I'll call a *Humberstone possibility model* on $\langle R, \leqslant\rangle$. (I'll call this a *possibility frame* in most contexts, but a *Humberstone frame* when I'm comparing it to similar structures, especially in the context of discussing Holliday (2025).)

A Humberstone possibility model $\mathcal{M}$ is a triple $\langle R, \leqslant, V\rangle$, where V is a function from $\mathcal{P}$ to R, intuitively saying where each atomic proposition is true, satisfying these two constraints:

- For all x, if $x \in V(p)$ and $y \geqslant x$, then $y \in V(p)$. Intuitively, truth for atomics is **persistent** across refinements.
- For all x, if $\forall y \geqslant x \exists z \geqslant y : z \in V(p)$, then $x \in V(p)$. This is what Humberstone (2011, 900) calls **refinability**, and it means that p only fails to be true at x if there is some refinement of x where it is settled as being untrue.

Given these constraints, Humberstone suggests the following theory of truth at a possibility for all sentences in this language.

$$
\begin{aligned}
&[\text{Vbls}] \quad \mathcal{M} \vDash_x p_i \text{ iff } x \in V(p_i);\\
&[\neg] \quad \mathcal{M} \vDash_x \neg A \text{ iff } \forall y \geqslant x,\ \mathcal{M} \nvDash_y A;\\
&[\wedge] \quad \mathcal{M} \vDash_x A \wedge B \text{ iff } \mathcal{M} \vDash_x A \text{ and } \mathcal{M} \vDash_x B;\\
&[\vee] \quad \mathcal{M} \vDash_x A \vee B \text{ iff } \forall y \geqslant x\, \exists z \geqslant y : \mathcal{M} \vDash_z A \text{ or } \mathcal{M} \vDash_z B.
\end{aligned}
$$

In the models Humberstone develops, there are two important binary relations between points. One is the familiar accessibility relation R, where xRy means roughly that y is possible from x. The other is a new *refinement* relation $\geqslant$, where $y \geqslant x$ means that y is a refinement of x. If we think of the possibilities x and y as *stories*, which settle some things but leave others out, then $y \geqslant x$ means that you can create the story y by starting with x and adding some things, but subtracting nothing. Humberstone puts some fairly tight constraints both on $\geqslant$ itself, and on the interaction between $\geqslant$ and R, and between $\geqslant$ and the valuation function V, with the result that despite the incompleteness in the worlds, we get back classical modal logic(s).

Recently, Wesley Holliday (2025) has published a massive study of the possibilities approach to modal logic. His approach differs from Humberstone's in some respects. For one thing, he discusses

1. Holliday (2025) writes $y \sqsubseteq x$ when y is a refinement of x, mirroring this way of thinking about possibilities.

multi-modal logics, which Humberstone did not. (Though it isn't hard to generalise to them.) More importantly, he prefers slightly weaker constraints $\geqslant$. He calls frames satisfy Humberstone's original constraints *Humberstone frames*, and frames that satisfy his weaker conditions *full possibility frames*. Using an ingenious construction, he shows that there is a class of full possibility frames which determines a logic that cannot be determined by any class of Kripke frames. But the construction does not satisfy the more demanding constraints on Humberstone frames, and he leaves it as an open question whether there is a class of Humberstone frames

Conan Doyle, Arthur. 1995. *A Study in Scarlet*. Urbana, Illinois: Project Gutenberg.

Holliday, Wesley H. 2025. "Possibility Frames and Forcing for Modal Logic." *Australasian Journal of Logic* 22 (2): 44–288. https://doi.org/10.26686/ajl.v22i2.5680.

Humberstone, Lloyd. 1981. "From Worlds to Possibilities." *Journal of Philosophical Logic* 10 (3): 313–39. https://doi.org/10.1007/BF00293423.

———. 2011. *The Connectives*. Cambridge, MA: MIT Press.