Humberstone on Possibility Frames

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In his 1981 paper, “From Worlds to Possibilities”, Lloyd Humberstone shows a way to do modal logic without the apparatus of possible worlds. Instead of worlds he uses *possibilities*, which may, unlike worlds, be incomplete. The non-modal parts of the view are discussed again in section 6.44 of *The Connectives*, though the differences between the view there and the 1981 view are largely presentational. In this paper I’ll set out this *possibility frame* approach to modal logic, make some notes about its logic, and end with a survey of the many possible applications it has.

Mathematically, possibilities are just points in a model, just like possible worlds are points in different kinds of models. But it helps to have a mental picture of what kind of thing they are. In “From Worlds to Possibilities”, Humberstone notes that one picture you could have is that they are sets of possible worlds. This isn’t a terrible picture, but it’s not perfect for a couple of reasons. For one thing, as Humberstone notes, part of the point of developing possibilities is to do without the machinery of possible worlds. Understanding possibilities as sets of possible worlds wouldn’t help with that project. For another, as Wesley Holliday (2025, 271–72) notes, the natural way to generate modal accessibility relations on sets of worlds from accessibility on the worlds themselves doesn’t always work the way Humberstone wants accessibility to work. So let’s start with a different picture.

Possibilities, as I’ll think of them, are *stories*. To make things concrete, let’s focus on a particular story: *A Study in Scarlet* (Conan Doyle (1995)), the story where Sherlock Holmes was introduced. That story settles some questions, both explicitly, e.g., that Holmes is a detective, and implicitly, e.g., that Holmes has never set foot on the moon. But it leaves several other questions open, e.g., how many (first) cousins Holmes has. It’s not that *A Study in Scarlet* is a story. It has proper parts which are stories. The first chapter is a story, one which tells of the first meeting between Holmes and Watson. And arguably it is a proper part of larger story, made up of all of Conan Doyle’s stories of Holmes and Watson. When a story x is a proper part of story y, what that means is that everything settled in x is still true in y, and more things besides are settled. When this happens, we’ll call y a proper *refinement* of x. For most purposes it will be more convenient to use the more general notion of *refinement*, where each story counts as an improper refinement of itself.

Following Humberstone, I’ll write $x \leqslant y$ to mean that y is a refinement of x. As he notes, this notation can be confusing if one things of x and y as sets, because in that case the refinement will typically be *smaller*.[1] But if we think of possibilities as stories, the notation becomes more intuitive. We have $x \leqslant y$ when y is created by adding new content to x. Keeping with this theme, I'll typically model stories not as worlds, but as finite sequences. (In the main example in Section , they will be sequences of 0s and 1s.) In these models, $x \leqslant y$ means that x is an initial segment of y.

Formal Structure

To start with, assume we're working in a simple language that just has a countable set $\mathcal{P}$ countable infinity of propositional variables, and three connectives: $\neg$, $\wedge$ and $\vee$. We have a set of possibilities W, and a transitive refinement relation $\geqslant$ on them. The following rules show how to build what I'll call a *Humberstone possibility model* on $\langle W, \leqslant\rangle$. (I'll call this a *possibility frame* in most contexts, but a *Humberstone frame* when I'm comparing it to similar structures, especially in the context of discussing Holliday (2025).)

A Humberstone possibility model $\mathcal{M}$ is a triple $\langle W, \leqslant, V\rangle$, where V is a function from $\mathcal{P}$ to W, intuitively saying where each atomic proposition is true, satisfying these two constraints:

- For all x, if $x \in V(p)$ and $y \geqslant x$, then $y \in V(p)$. Intuitively, truth for atomics is **persistent** across refinements.
- For all x, if $\forall y \geqslant x \exists z \geqslant y : z \in V(p)$, then $x \in V(p)$. This is what Humberstone (2011, 900) calls **refinability**, and it means that p only fails to be true at x if there is some refinement of x where it is settled as being untrue.

Given these constraints, Humberstone suggests the following theory of truth at a possibility for all sentences in this language.

$$
\begin{aligned}
&[\text{Vbls}] \quad \mathcal{M} \vDash_x p_i \text{ iff } x \in V(p_i);\\
&[\neg] \quad \mathcal{M} \vDash_x \neg A \text{ iff } \forall y \geqslant x,\ \mathcal{M} \nvDash_y A;\\
&[\wedge] \quad \mathcal{M} \vDash_x A \wedge B \text{ iff } \mathcal{M} \vDash_x A \text{ and } \mathcal{M} \vDash_x B;\\
&[\vee] \quad \mathcal{M} \vDash_x A \vee B \text{ iff } \forall y \geqslant x\, \exists z \geqslant y : \mathcal{M} \vDash_z A \text{ or } \mathcal{M} \vDash_z B.
\end{aligned}
$$

Given these definitions, it's possible to prove three things. First, every sentence in the language is persistent. If $\mathcal{M} \vDash_x A$ and $x \leqslant y$, then $\mathcal{M} \vDash_x A$. For any sentence, truth is always preserved when moving to a refinement. Second, refinability holds for all sentences in the language. This is, as Humberstone notes, easier to state using this definition of $\neg$. It now becomes the claim, for arbitrary A, that if $\mathcal{M} \nvDash_x A$, there is some refinement y of x such that $\mathcal{M} \vDash_y \neg A$. Third, for any set of sentences Γ and sentence A, the truth at a point of all sentences in Γ guarantees the truth of A iff the sequent Γ entails A in classical propositional logic.

In this paper, I'm going to discuss three extensions of this language. I'll introduce them in reverse order of how much they are discussed in Humberstone, starting with one he does not discuss at all: infinitary disjunction.

1. Holliday (2025) writes $y \sqsubseteq x$ when y is a refinement of x, mirroring this way of thinking about possibilities.

We'll add to the language a new symbol $\bigvee$, which forms a new sentence out of any countable set of sentences not containing $\bigvee$. Intuitively, it is true when one of the sentences in the set is true. More formally, its definition of truth at a possibility is:

$$[\bigvee] \quad \mathcal{M} \vDash_x \bigvee(A_1, A_2, \ldots) \text{ iff } \forall y \geqslant x\, \exists z \geqslant y : \text{ for some } i\, \mathcal{M} \vDash_z A_i.$$

Again, it's fairly simple to show that this addition to the language will preserve persistence and refinability. But while this is simple, it is significant, because things could easily have been otherwise.

The second extension will be to add quantifiers, following a suggestion in Humberstone (1981, xxxx). Assume, as usual, that the language has a stock of names $c_1, \ldots$, and for each n, a stock of n-place predicates $F_1^n, F_2^n, \ldots$. A *first-order (Humberstone) possibility model* is a structure $\langle W, \leqslant, D, V\rangle$, where D assigns a non-empty domain of objects to each point, and V interprets the non-logical vocabulary. More precisely:

- D is a function assigning to each $x \in W$ a non-empty set $D(x)$, the **domain** at x.
- V assigns to each name c_i and each $x \in W$ either a designated element $V(c_i, x) \in D(x)$, or is undefined at x.
- V assigns to each n-place predicate F_j^n and each $x \in W$ a set $V(F_j^n, x) \subseteq D(x)^n$, the **extension** of F_j^n at x.

These must satisfy the following constraints:

Domain monotonicity If $x \leqslant y$, then $D(x) \subseteq D(y)$.

Name coverage For each name c_i and each $x \in W$, there exists some $y \geqslant x$ such that $V(c_i, y)$ is defined.

Persistence for names If $V(c_i, x)$ is defined and $x \leqslant y$, then $V(c_i, y)$ is defined and $V(c_i, y) = V(c_i, x)$.

Persistence for predicate extensions If $\langle o_1, \ldots, o_n\rangle \in V(F_j^n, x)$ and $x \leqslant y$, then $\langle o_1, \ldots, o_n\rangle \in V(F_j^n, y)$.

Refinability for predicate extensions If $\langle o_1, \ldots, o_n\rangle \notin V(F_j^n, x)$, then there exists some $y \geqslant x$ such that for all $z \geqslant y$, $\langle o_1, \ldots, o_n\rangle \notin V(F_j^n, z)$.

Given a model and a variable assignment g mapping variables to objects, truth at a point is defined as follows. Write $g[v/o]$ for the assignment that maps variable v to object o and otherwise agrees with g. For a term t, write $[\![t]\!]^{g,x}$ for the denotation of t under g at x: for a variable v this is $g(v)$, and for a name c_i this is $V(c_i, x)$ when defined, and undefined otherwise.

$$[=] \quad \mathcal{M}, g \vDash_x t_1 = t_2 \text{ iff } \forall y \geqslant x\, \exists z \geqslant y : [\![t_1]\!]^{g,z} \text{ and } [\![t_2]\!]^{g,z} \text{ are both defined and equal;}$$

$$[F^n] \quad \mathcal{M}, g \vDash_x F_j^n(t_1, \ldots, t_n) \text{ iff } \forall y \geqslant x\, \exists z \geqslant y : \langle [\![t_1]\!]^{g,z}, \ldots, [\![t_n]\!]^{g,z}\rangle \in V(F_j^n, z);$$

$$[\forall] \quad \mathcal{M}, g \vDash_x \forall v\, A \text{ iff } \forall y \geqslant x\, \forall o \in D(y) : \mathcal{M}, g[v/o] \vDash_y A;$$

$$[\exists] \quad \mathcal{M}, g \vDash_x \exists v\, A \text{ iff } \forall y \geqslant x\, \exists z \geqslant y\, \exists o \in D(z) : \mathcal{M}, g[v/o] \vDash_z A.$$

The Boolean connectives are handled exactly as in the propositional case.

The $\forall\exists$ pattern in the atomic clauses is necessary to ensure that persistence and refinability hold for all sentences, including atomic ones. Consider $c_i = c_i$: if a name has no denotation at x but acquires one at some refinement, then a simple "check the denotation at x" condition would leave $c_i = c_i$ neither true nor false at x, and no refinement of x could settle it as false either, violating refinability. The $\forall\exists$ condition handles this correctly: $c_i = c_i$ is true at x whenever c_i is covered at x (i.e., every refinement has a further refinement where c_i gets a referent), since once c_i gets a referent o, persistence of names ensures $o = o$ at all further refinements.

The atomic clauses simplify when names are fully defined. If t_1 and t_2 are variables, or names that already have denotations at x, then by persistence of names and predicate extensions the $\forall\exists$ quantifier prefix collapses: $\mathcal{M}, g \vDash_x t_1 = t_2$ iff $[\![t_1]\!]^{g,x} = [\![t_2]\!]^{g,x}$, and $\mathcal{M}, g \vDash_x F_j^n(t_1, \dots, t_n)$ iff $\langle [\![t_1]\!]^{g,x}, \dots, [\![t_n]\!]^{g,x} \rangle \in V(F_j^n, x)$. The more complex clauses above are needed only to handle the case where some name occurring in the formula lacks a denotation at x but is guaranteed to acquire one.

This is a possibilist treatment of the universal quantifier, in contrast to the actualist quantifiers discussed in (**?**). I'll return in **?@sec-quant** to the reasons we are best off using possibilist quantifiers, and the difficulties this raises for talking about just what's true in a possibility.

The third extension will be the introduction of modal operators. Here I'll follow Humberstone (1981) very closely, save just that I'll have a plurality of modal operators rather than just one. So I'll use these structures to define (as Holliday (2025) does) multi-modal logics. But I'll follow Humberstone, and not Holliday, in defining modal operators in terms of accessibility relations R_i satisfying these three conditions[2]:

UpR: If $x \leqslant x'$ and $x' R_i y$, then $x R_i y$.
RDown: If $x R_i y$ and $y \leqslant y'$, then $x R_i y'$.
RRef++: If $x R_i y$, then there exists $x' \geqslant x$ such that for all $x'' \geqslant x'$, $x'' R_i y$.

UpR says that if a refinement of x can access y, then x itself can already access y: accessibility is not something that can be gained by adding detail to the source. **RDown** is a converse of this; it says that accessibility cannot be gained by adding detail to the target. **RRef++** says that if x can access y, there is some refinement x' of x where it is settled that x' can access y. This last access can't be overturned by further refinement of x'.

Given these constraints, the truth conditions for the box and diamond operators are:

$$[\Box_i] \quad \mathcal{M} \vDash_x \Box_i A \text{ iff } \forall y\, (x R_i y \Rightarrow \mathcal{M} \vDash_y A);$$
$$[\Diamond_i] \quad \mathcal{M} \vDash_x \Diamond_i A \text{ iff } \forall y \geqslant x\, \exists z \geqslant y\, \exists w\, (z R_i w \text{ and } \mathcal{M} \vDash_w A).$$

The clause for $\Box$ is the standard one: $\Box_i A$ is true at x iff A is true at every R_i-accessible possibility. Officially, Humberstone treats $\Diamond$ as a defined connective, $\Diamond_i A$ just means $\neg\Box_i\neg A$. I've spelled out what that means using the truth condition for $\neg$. It says that no refinement of x can perma-

2. I'm using the names for these that Holliday uses, which are more evocative than Humberstone's original names.

nently rule out there being an accessible point where A holds. Equivalently, every refinement of x has a further (possibly improper) refinement that accesses some point where A is true.

Why should we impose these constraints? It is not hard to show that they guarantee that persistence and refinability hold for sentences generated using these new modal connectives. At least, it isn't hard as long as we remember that $\Diamond$ is being treated as defined, so the only new step in the inductive proofs involves $\Box$. And **UpR** guarantees persistence for $\Box$ sentences, while **RRef++** guarantees refinability.

But this is overkill. As Humberstone points out, we haven't used **RDown** in the proof, so this doesn't explain why we'd impose **RDown**. As Holliday points out [note to Claude, we need page number for this] **UpR** is stronger than we need for persistence. We could weaken it by making greater use of the fact that A is persistent. All we need is that if $x \leqslant x'$ and xRy then there is some $z \geqslant y$ such that xRy. That will guarantee the key fact if x' can access a world where A, then so can x.

So we need other arguments for these constraints other than their role in ensuring persistence and refinability. Humberstone offers two other arguments here. One anticipates the multi-modal setting that is being used here. It's that if we want to use this system for tense logic, then we want R_i^{-1} to satisfy all the constraints, so if we impose **UpR**, we should also impose **RDown**. This isn't convincing on its own though. For one thing, as already noted, we might not need **UpR**. For another, by these lights we should worry that the system is incomplete because we haven't put in a converse of **RRef++**.

The argument that Humberstone spends more time on, and which I think is more compelling, comes from rethinking the relationship between R_i and $\Box_i$. It's very tempting to read those truth conditions as being explanatory from right-to-left. On this way of thinking, facts about which $\Box_i$ sentences are true at a point are grounded in which non-modal sentences are true and which R_i relations obtain. But while this is tempting, it isn't compulsory.

We could instead take the modal facts as given, and ask what accessibility relations must obtain to be consistent with them. The process here is familiar from the construction of canonical models. We take the sets of consistent sentences as given, and say $s_1 R_i s_2$ iff whenever $\Box_i A \in s_1$, then $A \in s_2$. Humberstone's approach is, I think, similar. Start with the idea that some sentences are true in some model $\mathcal{M}$ at possibilities x and y, say xR_iy iff $\mathcal{M} \vDash_y A$ whenever $\mathcal{M} \vDash_x \Box_i A$, and ask what constraints R_i will thereby satisfy.

The answer is that, if all sentences are persistent and refinable, then R_i will satisfy these three constraints. If $\mathcal{M} \vDash_y A$ whenever $\mathcal{M} \vDash_x' \Box_i A$, then by the persistence of $\Box_i$, we know that $\mathcal{M} \vDash_y A$ whenever $\mathcal{M} \vDash_x \Box_i A$. A similar argument, using the persistence of A, justifies **RDown**. Finally, **RRef++** follows from the fact that $\Box_i A$ is refinable. If $\Box_i A$ is not true at x, that means that it is determinately not true at some refinement. So if $\mathcal{M} \nvDash_x A$, there must be some refinement x' such that for all further refinements x'', $\mathcal{M} \nvDash_x'' A$. If $\mathcal{M} \nvDash_x \Box_i A$, then there must be some possibility y such that xR_iy and $\mathcal{M} \nvDash_y A$. So as long as $x''R_iy$, refinability will be satisfied. I don't see how to prove there isn't a weaker condition that would also work, it's possible we could use the refinability of A to find some weaker condition, but I don't quite see how that would work. So I think **RRef++** also follows from this way of thinking about accessibility.

In the next section I'll discuss what logics can be defined using frames that satisfy all of these conditions.

Logics Determinable on Humberstone Frames

Holliday (2025) raises an interesting question. As well as the familiar Kripke frames most commonly used as a semantics for modal logic, and the Humberstone frames defined above, he introduces a class of 'full possibility' frames, which weaken some of Humberstone's constraints. It won't matter here exactly what these weakenings are, but what does matter is that he shows that using these weakened frames, we can determine logics which are not determinable on any class of Kripke frames. To state this more precisely, for any class of frames **F**, let L($**F**) be the set of sentences true at all points in all models definable some member of **F**. Then let ML(**F**) be the set $\{\mathrm{L}(\mathbf{X}) : \mathbf{X} \subseteq \mathbf{F}\}$

Since every Kripke frame is a Humberstone frame and every Humberstone frame is a full possibility frame, we have

$$\mathrm{ML}(\mathbf{K}) \subseteq \mathrm{ML}(\mathbf{H}) \subseteq \mathrm{ML}(\mathbf{FP}),$$

where **K** is the class of Kripke frames, **H** is the class of Humberstone frames, **FP** is the class of full possibility frames, and ML(**F**) denotes the set of modal formulas valid over every frame in the class **F**—the modal logic determined by that class. We know that at least one of these inclusions is strict—but which one?

We will not answer Holliday's question exactly as posed. But we can answer a closely related question. If we expand the language to include infinitary disjunction $\bigvee$, then the first inclusion is strict: there are logics definable on Humberstone frames that are not definable on Kripke frames. We will show this by constructing a single Humberstone frame that, in the infinitary language, defines a logic with no Kripke equivalent.

The Frame

The frame is built from two copies of the set of finite binary sequences—sequences of 0s and 1s of any finite length, including the empty sequence. We call one copy the **left-handed** sequences and the other the **right-handed** sequences. We write x^L for the left-handed copy of a sequence x and x^R for the right-handed copy.

The refinement relation is: $x \leqslant y$ iff x and y have the same handedness and x is an initial segment of y. So within each copy the frame is just the binary tree ordered by extension, and no left-handed sequence refines a right-handed sequence or vice versa.

The Accessibility Relations

We define two infinite families of accessibility relations. As we go, we verify that each relation satisfies the three Humberstone constraints **UpR**, **RDown**, and **RRef++**.

The left-to-right relations. For each $k \geqslant 0$, define $R_k^{\rightarrow}$ by: $xR_k^{\rightarrow}y$ iff x is left-handed, y is right-handed, y has length at least k, and either x is an initial segment of y, or the first k elements of

x are an initial segment of y.

The special case $k = 0$ is simply: x is left-handed and y is right-handed (the length condition and the initial-segment disjunction are trivially satisfied).

It helps to picture $R_k^{\rightarrow}$ as follows. Each of the two sets of sequences forms a binary tree in the usual way. Imagine bridges connecting left-handed nodes to right-handed nodes whenever the two nodes carry the same sequence of length exactly k. Then $xR_k^{\rightarrow}y$ holds precisely when there is a top-to-bottom path in the left tree starting at x, crossing one of these bridges, and then continuing along (and possibly beyond) y in the right tree.

We verify the three constraints for $R_k^{\rightarrow}$.

- **UpR**: Suppose $x \leqslant x'$ and $x' R_k^{\rightarrow} y$. Then x and x' are both left-handed and x is an initial segment of x'. Since x' is an initial segment of y or its first k elements are an initial segment of y, the same holds for x (being a shorter initial segment of x'). So $xR_k^{\rightarrow}y$.
- **RDown**: Suppose $xR_k^{\rightarrow}y$ and $y \leqslant y'$. Then y' is right-handed and is a refinement (extension) of y, so y' has length at least k, and x is still an initial segment of y' or its first k elements are still an initial segment of y'. So $xR_k^{\rightarrow}y'$.
- **RRef++**: Suppose $xR_k^{\rightarrow}y$. Let x' be the left-handed sequence consisting of the first k elements of x (or all of x if $|x| < k$). Then $x \leqslant x'$ and for every $x'' \geqslant x'$, the first k elements of x'' are an initial segment of y (since x' is an initial segment of x'' and the first k elements of x' are the first k elements of x, which by hypothesis form an initial segment of y). Hence $x'' R_k^{\rightarrow} y$ for all $x'' \geqslant x'$.

The right-to-left relations. For each $k > 0$, define $R_k^{\leftarrow}$ by: $xR_k^{\leftarrow}y$ iff x is right-handed, x does not have a 0 in its k-th position (either because x has length less than k, or because it has a 1 in position k), and y is left-handed.

We verify the three constraints for $R_k^{\leftarrow}$.

- **UpR**: Suppose $x \leqslant x'$ and $x' R_k^{\leftarrow} y$. Then x and x' are both right-handed and x is an initial segment of x'. Since x' does not have a 0 in position k, either $|x'| < k$ or position k of x' is 1. If $|x'| < k$ then also $|x| < k$, so x does not have a 0 in position k. If position k of x' is 1, then since x is an initial segment of x', either $|x| < k$ or position k of x is also 1. In either case x does not have a 0 in position k, so $xR_k^{\leftarrow}y$.
- **RDown**: Suppose $xR_k^{\leftarrow}y$ and $y \leqslant y'$. Then y' is left-handed, and the condition on x is unchanged. So $xR_k^{\leftarrow}y'$.
- **RRef++**: Suppose $xR_k^{\leftarrow}y$. We need some $x' \geqslant x$ such that for all $x'' \geqslant x'$, $x'' R_k^{\leftarrow} y$. Take $x' = x$. For any $x'' \geqslant x'$, x'' is right-handed and x' is an initial segment of x''. Since x' does not have a 0 in position k, neither does x'' (if $|x'| \geqslant k$ then position k of x'' equals position k of x', which is 1; if $|x'| < k$ then either $|x''| < k$ also, or position k of x'' is some bit which has nothing to do with x'—but wait, x' is an initial segment of x'', so if $|x''| \geqslant k > |x'|$, position k of x'' is freely determined and may be 0).

  To handle this correctly, take x' to be the left-handed sequence obtained by extending x to length k with 1s if $|x| < k$ (i.e., append 1s until we reach length k, keeping x's bits for the first $|x|$ positions). Then for all $x'' \geqslant x'$, x'' has length at least k and its k-th element is 1 (since x'

is an initial segment of x'' and the k-th element of x' is 1). So $x'' R_k^{\leftarrow} y$ for all $x'' \geqslant x'$. This establishes **RRef++**.

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