

# Humberstone on Possibility Frames

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Insert abstract here

In his 1981 paper, “From Worlds to Possibilities”, Lloyd Humberstone shows a way to do modal logic without the apparatus of possible worlds. Instead of worlds he uses *possibilities*, which may, unlike worlds, be incomplete. The non-modal parts of the view are discussed again in section 6.44 of *The Connectives*, though the differences between the view there and the 1981 view are largely presentational. In this paper I’ll set out this *possibility frame* approach to modal logic, make some notes about its logic, and end with a survey of the many possible applications it has.

Mathematically, possibilities are just points in a model, just like possible worlds are points in different kinds of models. But it helps to have a mental picture of what kind of thing they are. In “From Worlds to Possibilities”, Humberstone notes that one picture you could have is that they are sets of possible worlds. This isn’t a terrible picture, but it’s not perfect for a couple of reasons. For one thing, as Humberstone notes, part of the point of developing possibilities is to do without the machinery of possible worlds. Understanding possibilities as sets of possible worlds wouldn’t help with that project. For another, as Wesley Holliday (2025, 271–72) notes, the natural way to generate modal accessibility relations on sets of worlds from accessibility on the worlds themselves doesn’t always work the way Humberstone wants accessibility to work. So let’s start with a different picture.

Possibilities, as I’ll think of them, are *stories*. To make things concrete, let’s focus on a particular story: *A Study in Scarlet* (Conan Doyle (1995)), the story where Sherlock Holmes was introduced. That story settles some questions, both explicitly, e.g., that Holmes is a detective, and implicitly, e.g., that Holmes has never set foot on the moon. But it leaves several other questions open, e.g., how many (first) cousins Holmes has. It’s not that *A Study in Scarlet* is a story. It has proper parts which are stories. The first chapter is a story, one which tells of the first meeting between Holmes and Watson. And arguably it is a proper part of larger story, made up of all of Conan Doyle’s stories of Holmes and Watson. When a story  $x$  is a proper part of story  $y$ , what that means is that everything settled in  $x$  is still true in  $y$ , and more things besides are settled. When this happens, we’ll call  $y$  a proper *refinement* of  $x$ . For most purposes it will be more convenient to use the more general notion of *refinement*, where each story counts as an improper refinement of itself.

Following Humberstone, I’ll write  $x \leq y$  to mean that  $y$  is a refinement of  $x$ . As he notes, this notation can be confusing if one thinks of  $x$  and  $y$  as sets, because in that case the refinement will

typically be *smaller*.<sup>1</sup> But if we think of possibilities as stories, the notation becomes more intuitive. We have  $x \leq y$  when  $y$  is created by adding new content to  $x$ . Keeping with this theme, I'll typically model stories not as worlds, but as finite sequences. (In the main example in ?@sec-proof, they will be sequences of 0s and 1s.) In these models,  $x \leq y$  means that  $x$  is an initial segment of  $y$ .

## Formal Structure

To start with, assume we're working in a simple language that just has a countable set  $\mathcal{P}$  countable infinity of propositional variables, and three connectives:  $\neg$ ,  $\wedge$  and  $\vee$ . We have a set of possibilities  $R$ , and a transitive refinement relation  $\geq$  on them. The following rules show how to build what I'll call a *Humberstone possibility model* on  $\langle R, \leq \rangle$ . (I'll call this a *possibility frame* in most contexts, but a *Humberstone frame* when I'm comparing it to similar structures, especially in the context of discussing Holliday (2025).)

A Humberstone possibility model  $\mathcal{M}$  is a triple  $\langle R, \leq, V \rangle$ , where  $V$  is a function from  $\mathcal{P}$  to  $R$ , intuitively saying where each atomic proposition is true, satisfying these two constraints:

- For all  $x$ , if  $x \in V(p)$  and  $y \geq x$ , then  $y \in V(p)$ . Intuitively, truth for atomics is **persistent** across refinements.
- For all  $x$ , if  $\forall y \geq x \exists z \geq y : z \in V(p)$ , then  $x \in V(p)$ . This is what Humberstone (2011, 900) calls **refinability**, and it means that  $p$  only fails to be true at  $x$  if there is some refinement of  $x$  where it is settled as being untrue.

Given these constraints, Humberstone suggests the following theory of truth at a possibility for all sentences in this language.

$$\begin{aligned}
 [\text{VbIs}] \quad & \mathcal{M} \models_x p_i \text{ iff } x \in V(p_i); \\
 [\neg] \quad & \mathcal{M} \models_x \neg A \text{ iff } \forall y \geq x, \mathcal{M} \not\models_y A; \\
 [\wedge] \quad & \mathcal{M} \models_x A \wedge B \text{ iff } \mathcal{M} \models_x A \text{ and } \mathcal{M} \models_x B; \\
 [\vee] \quad & \mathcal{M} \models_x A \vee B \text{ iff } \forall y \geq x \exists z \geq y : \mathcal{M} \models_z A \text{ or } \mathcal{M} \models_z B.
 \end{aligned}$$

Given these definitions, it's possible to prove three things. First, every sentence in the language is persistent. If  $\mathcal{M} \models_x A$  and  $x \leq y$ , then  $\mathcal{M} \models_y A$ . For any sentence, truth is always preserved when moving to a refinement. Second, refinability holds for all sentences in the language. This is, as Humberstone notes, easier to state using this definition of  $\neg$ . It now becomes the claim, for arbitrary  $A$ , that if  $\mathcal{M} \not\models_x A$ , there is some refinement  $y$  of  $x$  such that  $\mathcal{M} \models_y \neg A$ . Third, for any set of sentences  $\Gamma$  and sentence  $A$ , the truth at a point of all sentences in  $\Gamma$  guarantees the truth of  $A$  iff the sequent  $\Gamma$  entails  $A$  in classical propositional logic.

In this paper, I'm going to discuss three extensions of this language. I'll introduce them in reverse order of how much they are discussed in Humberstone, starting with one he does not discuss at all: infinitary disjunction.

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1. Holliday (2025) writes  $y \sqsubseteq x$  when  $y$  is a refinement of  $x$ , mirroring this way of thinking about possibilities.

We'll add to the language a new symbol  $\bigvee$ , which forms a new sentence out of any countable set of sentences not containing  $\bigvee$ . Intuitively, it is true when one of the sentences in the set is true. More formally, its definition of truth at a possibility is:

$$[\bigvee] \quad \mathcal{M} \models_x \bigvee (A_1, A_2, \dots) \text{ iff } \forall y \geq x \exists z \geq y : \text{ for some } i \mathcal{M} \models_z A_i.$$

Again, it's fairly simple to show that this addition to the language will preserve persistence and refinability. But while this is simple, it is significant, because things could easily have been otherwise.

The second extension will be to add quantifiers, following a suggestion in Humberstone (1981, xxxx). Assume, as usual, that the language has a stock of names  $c_1, \dots$ , and for each  $n$ , a stock of  $n$ -place predicates  $F_1^n, F_2^n, \dots$ . A *first-order (Humberstone) possibility model* is a structure  $\langle R, \leq, D, V \rangle$ , where  $D$  assigns a non-empty domain of objects to each point, and  $V$  interprets the non-logical vocabulary. More precisely:

- $D$  is a function assigning to each  $x \in R$  a non-empty set  $D(x)$ , the **domain** at  $x$ .
- $V$  assigns to each name  $c_i$  and each  $x \in R$  either a designated element  $V(c_i, x) \in D(x)$ , or is undefined at  $x$ .
- $V$  assigns to each  $n$ -place predicate  $F_j^n$  and each  $x \in R$  a set  $V(F_j^n, x) \subseteq D(x)^n$ , the **extension** of  $F_j^n$  at  $x$ .

These must satisfy the following constraints:

**Domain monotonicity** If  $x \leq y$ , then  $D(x) \subseteq D(y)$ .

**Name coverage** For each name  $c_i$  and each  $x \in R$ , there exists some  $y \geq x$  such that  $V(c_i, y)$  is defined.

**Persistence for names** If  $V(c_i, x)$  is defined and  $x \leq y$ , then  $V(c_i, y)$  is defined and  $V(c_i, y) = V(c_i, x)$ .

**Persistence for predicate extensions** If  $\langle o_1, \dots, o_n \rangle \in V(F_j^n, x)$  and  $x \leq y$ , then  $\langle o_1, \dots, o_n \rangle \in V(F_j^n, y)$ .

**Refinability for predicate extensions** If  $\langle o_1, \dots, o_n \rangle \notin V(F_j^n, x)$ , then there exists some  $y \geq x$  such that for all  $z \geq y$ ,  $\langle o_1, \dots, o_n \rangle \notin V(F_j^n, z)$ .

Given a model and a variable assignment  $g$  mapping variables to objects, truth at a point is defined as follows. Write  $g[v/o]$  for the assignment that maps variable  $v$  to object  $o$  and otherwise agrees with  $g$ . For a term  $t$ , write  $\llbracket t \rrbracket^{g,x}$  for the denotation of  $t$  under  $g$  at  $x$ : for a variable  $v$  this is  $g(v)$ , and for a name  $c_i$  this is  $V(c_i, x)$  when defined, and undefined otherwise.

- $[=]$   $\mathcal{M}, g \models_x t_1 = t_2$  iff  $\forall y \geq x \exists z \geq y : \llbracket t_1 \rrbracket^{g,z}$  and  $\llbracket t_2 \rrbracket^{g,z}$  are both defined and equal;
- $[F^n]$   $\mathcal{M}, g \models_x F_j^n(t_1, \dots, t_n)$  iff  $\forall y \geq x \exists z \geq y : \langle \llbracket t_1 \rrbracket^{g,z}, \dots, \llbracket t_n \rrbracket^{g,z} \rangle \in V(F_j^n, z)$ ;
- $[\forall]$   $\mathcal{M}, g \models_x \forall v A$  iff  $\forall y \geq x \forall o \in D(y) : \mathcal{M}, g[v/o] \models_y A$ ;
- $[\exists]$   $\mathcal{M}, g \models_x \exists v A$  iff  $\forall y \geq x \exists z \geq y \exists o \in D(z) : \mathcal{M}, g[v/o] \models_z A$ .

The Boolean connectives are handled exactly as in the propositional case.

The  $\forall\exists$  pattern in the atomic clauses is necessary to ensure that persistence and refinability hold for all sentences, including atomic ones. Consider  $c_i = c_i$ : if a name has no denotation at  $x$  but acquires one at some refinement, then a simple “check the denotation at  $x$ ” condition would leave  $c_i = c_i$  neither true nor false at  $x$ , and no refinement of  $x$  could settle it as false either, violating refinability. The  $\forall\exists$  condition handles this correctly:  $c_i = c_i$  is true at  $x$  whenever  $c_i$  is covered at  $x$  (i.e., every refinement has a further refinement where  $c_i$  gets a referent), since once  $c_i$  gets a referent  $o$ , persistence of names ensures  $o = o$  at all further refinements.

The atomic clauses simplify when names are fully defined. If  $t_1$  and  $t_2$  are variables, or names that already have denotations at  $x$ , then by persistence of names and predicate extensions the  $\forall\exists$  quantifier prefix collapses:  $\mathcal{M}, g \models_x t_1 = t_2$  iff  $\llbracket t_1 \rrbracket^{g,x} = \llbracket t_2 \rrbracket^{g,x}$ , and  $\mathcal{M}, g \models_x F_j^n(t_1, \dots, t_n)$  iff  $\langle \llbracket t_1 \rrbracket^{g,x}, \dots, \llbracket t_n \rrbracket^{g,x} \rangle \in V(F_j^n, x)$ . The more complex clauses above are needed only to handle the case where some name occurring in the formula lacks a denotation at  $x$  but is guaranteed to acquire one.

Conan Doyle, Arthur. 1995. *A Study in Scarlet*. Urbana, Illinois: Project Gutenberg.

Holliday, Wesley H. 2025. “Possibility Frames and Forcing for Modal Logic.” *Australasian Journal of Logic* 22 (2): 44–288. <https://doi.org/10.26686/ajl.v22i2.5680>.

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